

CHAPTER A.3. ADJUNCTIONS IN $(\infty, 2)$ -CATEGORIES

CONTENTS

Introduction	1
0.1. What is done in this Chapter?	1
0.2. How is this done?	2
1. Adjunctions	3
1.1. Adjointable arrows and functors	3
1.2. The universal adjointable functor	5
1.3. The case $\mathbf{C} = \mathbb{S}^{1\text{-Cat}}$	6
1.4. Proof of Theorem 1.2.4 for ordinary 2-categories	7
2. Proof of Theorem 1.2.4	10
2.1. The swapping procedure	11
2.2. Proof of Theorem 1.2.4 for $\mathbb{T} = \mathbf{1}\text{-Cat}$	11
2.3. Swapping procedure: relative version	13
2.4. Adjunctions: relative version	15
2.5. Proofs of Theorem 1.2.4, the general case	15
3. Adjunction with parameters	17
3.1. The set-up for adjunction with parameters	18
3.2. Proof of Theorem 3.1.6	20
4. An alternative proof	22
4.1. An alternative proof of Corollary 3.1.7 for $\mathbb{T} = \mathbf{1}\text{-Cat}$	22
4.2. An alternative proof of Corollary 3.1.7 for a general \mathbb{T}	23
4.3. An alternative proof of Theorem 1.2.4	25

INTRODUCTION

0.1. What is done in this Chapter? This Chapter contains the only piece of original mathematics pertaining to $(\infty, 2)$ -categories that we develop in this book. It has to do with pairs of functors obtained from one another by the procedure of *passage adjoints*.

0.1.1. First, we note that if \mathbb{T} is an ordinary 2-category, and $t \xrightarrow{\alpha} t'$ is a 1-morphism, there exists an (elementary) notion of right (resp., left) adjoint 1-morphism. For example, if $\mathbb{T} = (\mathbf{1}\text{-Cat})^{2\text{-ordn}}$, this is the usual notion of adjunction for functors between $(\infty, 1)$ -categories.

If \mathbb{T} is a $(\infty, 2)$ -category, we will say that a 1-morphism admits a right (resp., left) adjoint, if it does so when considered as a 1-morphism in $\mathbb{T}^{2\text{-ordn}}$.

0.1.2. Consider the following situation: let \mathbb{S} and \mathbb{T} be $(\infty, 2)$ -categories, and let $F : \mathbb{S} \rightarrow \mathbb{T}$.

We shall say that F is right-adjointable (resp., left-adjointable) if for every 1-morphism $s \xrightarrow{\alpha} s'$ in \mathbb{S} , the corresponding 1-morphism $F(\alpha)$ admits a right (resp., left) adjoint.

We let

$$\text{Maps}_{2\text{-Cat}}(\mathbb{S}, \mathbb{T})^R \subset \text{Maps}_{2\text{-Cat}}(\mathbb{S}, \mathbb{T})$$

denote the full subspace spanned by left-adjointable functors (the superscript ‘‘R’’ is because 1-morphisms generated by these functors are right adjoints).

Let

$$\text{Maps}_{2\text{-Cat}}(\mathbb{S}, \mathbb{T})^L \subset \text{Maps}_{2\text{-Cat}}(\mathbb{S}, \mathbb{T})$$

denote the full subspace spanned by right-adjointable functors.

0.1.3. Assume for a moment that \mathbb{T} is an ordinary 2-category. In this case, the procedure of *passage to adjoint 1-morphisms* defines a canonical isomorphism

$$(0.1) \quad \text{Maps}_{2\text{-Cat}}(\mathbb{S}, \mathbb{T})^R \simeq \text{Maps}_{2\text{-Cat}}(\mathbb{S}^{1\&2\text{-op}}, \mathbb{T})^L.$$

The main result of this Chapter (stated in Corollary 1.3.4) is that the isomorphism (0.1) holds for an arbitrary $(\infty, 2)$ -category.

0.1.4. Moreover, we note that the spaces $\text{Maps}_{2\text{-Cat}}(\mathbb{S}, \mathbb{T})^R$ and $\text{Maps}_{2\text{-Cat}}(\mathbb{S}^{1\&2\text{-op}}, \mathbb{T})^L$ can naturally be extended to $(\infty, 2)$ -categories

$$\text{Funct}(\mathbb{S}, \mathbb{T})_{\text{right-lax}}^R \text{ and } \text{Funct}(\mathbb{S}^{1\&2\text{-op}}, \mathbb{T})_{\text{left-lax}}^L,$$

where we allow right-lax (resp., left-lax) natural transformations.

We will show (see Corollary 3.1.9) that the isomorphism (0.1) extends to an equivalence of $(\infty, 2)$ -categories

$$\text{Funct}(\mathbb{S}, \mathbb{T})_{\text{right-lax}}^R \simeq \text{Funct}(\mathbb{S}^{1\&2\text{-op}}, \mathbb{T})_{\text{left-lax}}^L.$$

0.2. How is this done? To simplify the discussion, we will focus on the construction of the isomorphism of spaces (0.1).

0.2.1. It is not difficult to see that for a given $(\infty, 2)$ -category \mathbb{S} , there exists a *universal left-adjointable functor*

$$F_{\text{univ}} : \mathbb{S} \rightarrow \mathbb{S}^R.$$

I.e., any left-adjointable functor $F : \mathbb{S} \rightarrow \mathbb{T}$ uniquely factors as

$$G \circ F_{\text{univ}}, \quad G : \mathbb{S}^R \rightarrow \mathbb{T}.$$

Similarly, we have the universal right-adjointable functor $\mathbb{S} \rightarrow \mathbb{S}^L$.

0.2.2. The isomorphism (0.1), functorial in \mathbb{T} , is equivalent to the existence of a canonical equivalence of $(\infty, 2)$ -categories

$$(0.2) \quad (\mathbb{S}^{2\text{-op}})^R \simeq (\mathbb{S}^{1\text{-op}})^L.$$

The construction of the equivalence (0.2) is based on the explicit description of the $(\infty, 2)$ -category \mathbb{S}^R (resp., \mathbb{S}^L). Such a description is given by Theorem 1.2.4 and is the key idea of this Chapter.

0.2.3. Namely, we prove that the $(\infty, 2)$ -category \mathbb{S}^R is obtained by applying the functor \mathfrak{L}^{Sq} , left adjoint to

$$\text{Sq}_{\bullet, \bullet} : 2\text{-Cat} \rightarrow \text{Spc}^{\Delta^{\text{op}} \times \Delta^{\text{op}}},$$

to a particular object of $\text{Spc}^{\Delta^{\text{op}} \times \Delta^{\text{op}}}$.

That object of $\text{Spc}^{\Delta^{\text{op}} \times \Delta^{\text{op}}}$ is the following: we consider

$$\text{Sq}_{\bullet, \bullet}(\mathbb{S}^{2\text{-op}}),$$

and then we *invert the vertical arrows*, i.e., apply the involution

$$\text{rev} : \Delta^{\text{op}} \rightarrow \Delta^{\text{op}}$$

along the *first factor* in $\Delta^{\text{op}} \times \Delta^{\text{op}}$.

We note that this object *does not* belong to the essential image of the functor $\text{Sq}_{\bullet, \bullet}$, so the above procedure is doing something non-trivial.

0.2.4. By unwinding the construction, we see that the procedures for obtaining $(\mathbb{S}^{2\text{-op}})^R$ and $(\mathbb{S}^{1\text{-op}})^L$ are exactly the same, thereby leading to the isomorphism (0.2).

1. ADJUNCTIONS

In this section we study the following situation: let \mathbb{S} be a $(\infty, 2)$ -category equipped with a 1-full subcategory $\mathbf{C} \subset \mathbb{S}^{1\text{-Cat}}$ with the same class of objects. We will consider functors $F : \mathbb{S} \rightarrow \mathbb{T}$ such that for every 1-morphism α in \mathbf{C} , the corresponding 1-morphism $F(\alpha)$ in \mathbb{T} *admits a left adjoint*. We will state a theorem to the effect that there exists a universal such functor

$$F_{\text{univ}} : \mathbb{S} \rightarrow \mathbb{S}^{R\mathbf{C}},$$

i.e., any functor $F : \mathbb{S} \rightarrow \mathbb{T}$ with the above property uniquely factors as

$$G \circ F_{\text{univ}}, \quad G : \mathbb{S}^{R\mathbf{C}} \rightarrow \mathbb{T}.$$

However more importantly, the $(\infty, 2)$ -category $\mathbb{S}^{R\mathbf{C}}$ can be described explicitly.

This explicit description will allow us to show that to a functor $F : \mathbb{S} \rightarrow \mathbb{T}$ that maps *all* 1-morphisms in \mathbb{S} to left-adjointable 1-morphisms in \mathbb{T} , there canonically corresponds a functor

$$\mathbb{S}^{1\&2\text{-op}} \rightarrow \mathbb{T}$$

obtained from F by *passing to left adjoints* along 1-morphisms.

1.1. Adjointable arrows and functors. In this subsection we will define what it means for a 1-morphism in an $(\infty, 2)$ -category to admit an adjoint, and the related notion of a functor to be *adjointable*.

The main feature of these notions is that they *do not* depend on the ∞ -categorical structure, i.e., these are conditions on arrows/functors between the underlying *ordinary* 2-categories.

1.1.1. Let \mathbb{T} be an *ordinary* 2-category, and let $t \xrightarrow{\alpha} t'$ be a 1-morphism. In this case there exists the notion of *right adjoint 1-morphism*.

Namely, a 1-morphism $t' \xrightarrow{\beta} t$ is said to be the *right adjoint* of α if we are given 2-morphisms

$$\text{co-unit} : \alpha \circ \beta \rightarrow \text{id}_{t'} \quad \text{and} \quad \text{unit} : \text{id}_t \rightarrow \beta \circ \alpha$$

such that the compositions

$$\alpha \xrightarrow{\alpha \circ \text{unit}} \alpha \circ \beta \circ \alpha \xrightarrow{\text{co-unit} \circ \alpha} \alpha \quad \text{and} \quad \beta \xrightarrow{\text{unit} \circ \beta} \beta \circ \alpha \circ \beta \xrightarrow{\beta \circ \text{co-unit}} \beta$$

are the identity 2-morphisms.

It is easy to see that if a right adjoint 1-morphism exists, it is defined up to a unique isomorphism.

1.1.2. Replacing \mathbb{T} by $\mathbb{T}^{2\text{-op}}$, we obtain the notion of *left adjoint 1-morphism*.

It is easy to see that the data defining β as a right adjoint of α is equivalent to the data defining α as a left adjoint of β .

1.1.3. Let now \mathbb{T} be an $(\infty, 2)$ -category, and let $t \xrightarrow{\alpha} t'$ be a 1-morphism.

Definition 1.1.4. *We shall say that α admits a right (resp., left) adjoint, if it does so in the ordinary 2-category $\mathbb{T}^{2\text{-ordn}}$.*

Let $\text{Seq}_1(\mathbb{T})^R \subset \text{Seq}_1(\mathbb{T})$ be the full subcategory spanned by 1-morphisms that admit a *left adjoint*¹.

Let $\text{Seq}_1(\mathbb{T})^L \subset \text{Seq}_1(\mathbb{T})$ be the full subcategory spanned by 1-morphisms that admit a *right adjoint*.

The procedure of passage to the adjoint 1-morphism defines an equivalence

$$(\text{Seq}_1(\mathbb{T})^R)^{\text{ordn}} \simeq \left(\left((\text{Seq}_1(\mathbb{T})^L)^{\text{ordn}} \right)^{\text{op}} \right).$$

Remark 1.1.5. In Corollary 3.1.9 we will see that the above equivalence of ordinary categories in fact lifts to an equivalence of $(\infty, 1)$ -categories

$$\text{Seq}_1(\mathbb{T})^R \simeq \left((\text{Seq}_1(\mathbb{T})^L)^{\text{op}} \right).$$

1.1.6. Let \mathbb{S} be an $(\infty, 2)$ -category, and $\mathbf{C} \subset \mathbb{S}^{1\text{-Cat}}$ be a 1-full subcategory with the same class of objects. Let

$$F : \mathbb{S} \rightarrow \mathbb{T}$$

be a functor, where \mathbb{T} is another $(\infty, 2)$ -category.

Definition 1.1.7. *We shall say that F is right adjointable with respect to \mathbf{C} , if for every 1-morphism $s \xrightarrow{\alpha} s'$ in \mathbf{C} , the 1-morphism*

$$F(s) \xrightarrow{F(\alpha)} F(s')$$

admits a right adjoint.

In a similar way we define the notion of functor *left adjointable* with respect to \mathbf{C} .

¹The superscript “R” means the 1-morphisms in question are themselves right adjoint to something.

1.1.8. We denote by

$$\text{Funct}(\mathbb{S}, \mathbb{T})_{\text{right-lax}}^{R\mathbf{C}} \subset \text{Funct}(\mathbb{S}, \mathbb{T})_{\text{right-lax}}$$

the full subcategory corresponding to functors that are *left* adjointable with respect to \mathbf{C} .

Let

$$\text{Funct}(\mathbb{S}, \mathbb{T})_{\text{left-lax}}^{L\mathbf{C}} \subset \text{Funct}(\mathbb{S}, \mathbb{T})_{\text{right-lax}}$$

denote the full subcategory corresponding to functors *right* adjointable with respect to \mathbf{C} .

Let

$$\text{Maps}_{2\text{-Cat}}(\mathbb{S}, \mathbb{T})^{R\mathbf{C}} \subset \text{Maps}_{2\text{-Cat}}(\mathbb{S}, \mathbb{T}) \supset \text{Maps}_{2\text{-Cat}}(\mathbb{S}, \mathbb{T})^{L\mathbf{C}}$$

the be the corresponding full subspaces.

Clearly, under the isomorphism

$$\text{Funct}(\mathbb{S}, \mathbb{T})_{\text{left-lax}} \simeq \text{Funct}(\mathbb{S}^{2\text{-op}}, \mathbb{T}^{2\text{-op}})_{\text{right-lax}}$$

we have:

$$\text{Funct}(\mathbb{S}, \mathbb{T})_{\text{left-lax}}^{L\mathbf{C}} \simeq \text{Funct}(\mathbb{S}^{2\text{-op}}, \mathbb{T}^{2\text{-op}})_{\text{right-lax}}^{R\mathbf{C}}.$$

Consider the particular case when \mathbf{C} is all of $\mathbb{S}^{1\text{-Cat}}$. In this case we will simply write

$$\text{Funct}(\mathbb{S}, \mathbb{T})_{\text{right-lax}}^R \text{ and } \text{Funct}(\mathbb{S}, \mathbb{T})_{\text{left-lax}}^L$$

and

$$\text{Maps}_{2\text{-Cat}}(\mathbb{S}, \mathbb{T})^R \text{ and } \text{Maps}_{2\text{-Cat}}(\mathbb{S}, \mathbb{T})^L,$$

respectively.

1.1.9. Assume now that \mathbb{T} is ordinary. In this case one shows that there is a canonical equivalence of (ordinary) 2-categories

$$(1.1) \quad \text{Funct}(\mathbb{S}, \mathbb{T})_{\text{right-lax}}^R \simeq \text{Funct}(\mathbb{S}^{1\&2\text{-op}}, \mathbb{T})_{\text{left-lax}}^L,$$

given by *passage* to left adjoint 1-morphisms.

The goal of this chapter is to generalize the equivalence (1.1) to the case when \mathbb{T} is an $(\infty, 2)$ -category. This will ultimately be achieved in Corollary 3.1.9.

1.2. The universal adjointable functor. It is fairly easy to see that, given a pair (\mathbb{S}, \mathbf{C}) , where \mathbb{S} is an $(\infty, 2)$ -category and $\mathbf{C} \subset \mathbb{S}^{1\text{-Cat}}$ is a 1-full subcategory with the same class of objects, there exists a *universal* recipient, denoted $\mathbb{S}^{R\mathbf{C}}$, of functors left-adjointable with respect to \mathbf{C} .

The point is that this $(\infty, 2)$ -category $\mathbb{S}^{R\mathbf{C}}$ can be described explicitly in terms of the adjoint functors

$$\text{Sq}_{\bullet, \bullet} : 2\text{-Cat} \rightleftarrows \text{Spc}^{\Delta^{\text{op}} \times \Delta^{\text{op}}} : \mathfrak{L}^{\text{Sq}}.$$

This description is given by Theorem 1.2.4. In fact, the $(\infty, 2)$ -category $\mathbb{S}^{R\mathbf{C}}$ is obtained from \mathbb{S} as a combination of the following three steps:

- Starting from \mathbb{S} , we pass to $\mathbb{S}^{2\text{-op}}$ and form the bi-simplicial groupoid $\text{Sq}_{\bullet, \bullet}^{\text{Pair}}(\mathbb{S}^{2\text{-op}}, \mathbf{C})$, see [Chapter A.1, Sect. 4.3] for the notation.
- We take $\text{Sq}_{\bullet, \bullet}^{\text{Pair}}(\mathbb{S}^{2\text{-op}}, \mathbf{C})$ and *reverse* its vertical arrows. Note that the resulting bi-simplicial groupoid will *not* be in the essential image of the functor $\text{Sq}_{\bullet, \bullet}$.
- We take $(\text{Sq}_{\bullet, \bullet}^{\text{Pair}}(\mathbb{S}^{2\text{-op}}, \mathbf{C}))^{\text{vert-op}}$ and apply to it the functor \mathfrak{L}^{Sq} .

The idea of this construction is that the reversed vertical arrows will supply the data of left adjoints for 1-morphisms in \mathbf{C} .

1.2.1. Let \mathbb{S} be an $(\infty, 2)$ -category, and $\mathbf{C} \subset \mathbb{S}^{1\text{-Cat}}$ be a 1-full subcategory with the same space of objects.

Consider the bi-simplicial category $(\text{Sq}_{\bullet, \bullet}^{\text{Pair}}(\mathbb{S}^{2\text{-op}}, \mathbf{C}))^{\text{vert-op}}$, where the notation $(-)^{\text{vert-op}}$ is as in [Chapter A.1, Sect. 4.1.5], and $\text{Sq}_{\bullet, \bullet}^{\text{Pair}}$ is as in [Chapter A.1, Sect. 4.3.3].

We define the $(\infty, 2)$ -category $\mathbb{S}^{R\mathbf{C}}$ to be

$$\mathfrak{L}^{\text{Sq}}((\text{Sq}_{\bullet, \bullet}^{\text{Pair}}(\mathbb{S}^{2\text{-op}}, \mathbf{C}))^{\text{vert-op}}),$$

where \mathfrak{L}^{Sq} is as in [Chapter A.1, Sect. 4.4.4].

I.e., by definition, for $\mathbb{T} \in 2\text{-Cat}$,

$$\text{Maps}_{2\text{-Cat}}(\mathbb{S}^{R\mathbf{C}}, \mathbb{T}) = \text{Maps}_{\text{Funct}(\Delta^{\text{op}}, 1\text{-Cat})}((\text{Sq}_{\bullet, \bullet}^{\text{Pair}}(\mathbb{S}^{2\text{-op}}, \mathbf{C}))^{\text{vert-op}}, \text{Sq}_{\bullet, \bullet}(\mathbb{T})).$$

1.2.2. We claim that we have a canonically defined functor

$$(1.2) \quad \mathbb{S} \rightarrow \mathbb{S}^{R\mathbf{C}}.$$

Namely, it is obtained via the isomorphism

$$\mathbb{S} \simeq \mathfrak{L}^{\text{Sq}} \circ \text{Sq}_{\bullet, \bullet}^{\sim}(\mathbb{S})$$

(of [Chapter A.1, Corollary 4.4.5]) by applying \mathfrak{L}^{Sq} to the tautological bi-simplicial functor

$$(1.3) \quad \text{Sq}_{\bullet, \bullet}^{\sim}(\mathbb{S}) \simeq (\text{Sq}_{\bullet, \bullet}^{\sim}(\mathbb{S}^{2\text{-op}}))^{\text{vert-op}} \hookrightarrow (\text{Sq}_{\bullet, \bullet}^{\text{Pair}}(\mathbb{S}^{2\text{-op}}, \mathbf{C}))^{\text{vert-op}}.$$

1.2.3. Recall the notation

$$\text{Maps}_{2\text{-Cat}}(\mathbb{S}, \mathbb{T})^{R\mathbf{C}} \subset \text{Maps}_{2\text{-Cat}}(\mathbb{S}, \mathbb{T}),$$

see Sect. 1.1.8.

We will prove the following result:

Theorem 1.2.4. *Restriction along (1.2) defines an isomorphism*

$$\text{Maps}_{2\text{-Cat}}(\mathbb{S}^{R\mathbf{C}}, \mathbb{T}) \rightarrow \text{Maps}_{2\text{-Cat}}(\mathbb{S}, \mathbb{T})^{R\mathbf{C}}.$$

1.3. **The case $\mathbf{C} = \mathbb{S}^{1\text{-Cat}}$.** Let us consider a particular case of Theorem 1.2.4 when $\mathbf{C} = \mathbb{S}^{1\text{-Cat}}$. In this case we shall simply write \mathbb{S}^R . The point is that in this case we will have a canonical equivalence

$$(\mathbb{S}^R)^{2\text{-op}} \simeq (\mathbb{S}^{1\text{-op}})^R,$$

which will allow to realize the *passage to adjoints* construction.

1.3.1. Recall that for $\mathbb{X} \in 2\text{-Cat}$ we have

$$(\text{Sq}_{\bullet, \bullet}(\mathbb{X}))^{\text{vert}\&\text{horiz-op}} \simeq \text{Sq}_{\bullet, \bullet}(\mathbb{X}^{1\&2\text{-op}}),$$

see [Chapter A.1, Sect. 4.1.6].

Recall also the involution *reflect* on $\text{Spc}^{\Delta^{\text{op}} \times \Delta^{\text{op}}}$, see [Chapter A.1, Sect. 4.1.5]. For $\mathbb{X} \in 2\text{-Cat}$ we have

$$(1.4) \quad (\text{Sq}_{\bullet, \bullet}(\mathbb{X}))^{\text{reflect}} \simeq \text{Sq}_{\bullet, \bullet}(\mathbb{X}^{2\text{-op}}).$$

Hence, for $\mathbb{S} \in 2\text{-Cat}$ we have obtain a canonical isomorphism

$$(1.5) \quad \begin{aligned} \left((\text{Sq}_{\bullet, \bullet}(\mathbb{S}^{2\text{-op}}))^{\text{vert-op}} \right)^{\text{reflect}} &\simeq \left((\text{Sq}_{\bullet, \bullet}(\mathbb{S}^{2\text{-op}}))^{\text{reflect}} \right)^{\text{horiz-op}} \simeq (\text{Sq}_{\bullet, \bullet}(\mathbb{S}))^{\text{horiz-op}} \simeq \\ &\simeq \left((\text{Sq}_{\bullet, \bullet}(\mathbb{S}))^{\text{vert}\&\text{horiz-op}} \right)^{\text{vert-op}} \simeq \left((\text{Sq}_{\bullet, \bullet}(\mathbb{S}^{1\&2\text{-op}}))^{\text{vert-op}} \right). \end{aligned}$$

1.3.2. Note that from (1.4) it follows that for $E_{\bullet, \bullet} \in \text{Spc}^{\Delta^{\text{op}} \times \Delta^{\text{op}}}$ we have

$$(1.6) \quad \mathfrak{L}^{\text{Sq}}((E_{\bullet, \bullet})^{\text{reflect}}) \simeq (\mathfrak{L}^{\text{Sq}}(E_{\bullet, \bullet}))^{2\text{-op}}.$$

Hence, we obtain an identification

$$(1.7) \quad (\mathbb{S}^R)^{2\text{-op}} = \left(\mathfrak{L}^{\text{Sq}} \left((\text{Sq}_{\bullet, \bullet}(\mathbb{S}^{2\text{-op}}))^{\text{vert-op}} \right) \right)^{2\text{-op}} \stackrel{(1.6)}{\simeq} \mathfrak{L}^{\text{Sq}} \left(\left((\text{Sq}_{\bullet, \bullet}(\mathbb{S}^{2\text{-op}}))^{\text{vert-op}} \right)^{\text{reflect}} \right) \stackrel{(1.5)}{\simeq} \\ \simeq \mathfrak{L}^{\text{Sq}} \left((\text{Sq}_{\bullet, \bullet}(\mathbb{S}^{1\&2\text{-op}}))^{\text{vert-op}} \right) = (\mathbb{S}^{1\text{-op}})^R.$$

1.3.3. Combining (1.7) and (1.2), we obtain a canonically defined map

$$(1.8) \quad \mathbb{S}^{1\text{-op}} \rightarrow (\mathbb{S}^R)^{2\text{-op}}$$

and hence a map

$$(1.9) \quad \mathbb{S}^{1\&2\text{-op}} \rightarrow \mathbb{S}^R$$

Applying Theorem 1.2.4 we obtain:

Corollary 1.3.4. *The functors*

$$\mathbb{S} \rightarrow \mathbb{S}^R \text{ and } \mathbb{S}^{1\&2\text{-op}} \rightarrow \mathbb{S}^R$$

define isomorphisms

$$\text{Maps}_{2\text{-Cat}}(\mathbb{S}^R, \mathbb{T}) \rightarrow \text{Maps}_{2\text{-Cat}}(\mathbb{S}, \mathbb{T})^R$$

and

$$\text{Maps}_{2\text{-Cat}}(\mathbb{S}^R, \mathbb{T}) \simeq \text{Maps}_{2\text{-Cat}}(((\mathbb{S}^{1\text{-op}})^R)^{2\text{-op}}, \mathbb{T}) \simeq \text{Maps}_{2\text{-Cat}}((\mathbb{S}^{1\text{-op}})^R, \mathbb{T}^{2\text{-op}}) \rightarrow \\ \rightarrow \text{Maps}_{2\text{-Cat}}(\mathbb{S}^{1\text{-op}}, \mathbb{T}^{2\text{-op}})^R \simeq \text{Maps}_{2\text{-Cat}}(\mathbb{S}^{1\&2\text{-op}}, \mathbb{T})^L.$$

In particular, we obtain a canonical identification

$$(1.10) \quad \text{Maps}_{2\text{-Cat}}(\mathbb{S}, \mathbb{T})^R \simeq \text{Maps}_{2\text{-Cat}}(\mathbb{S}^{1\&2\text{-op}}, \mathbb{T})^L.$$

We will refer to the isomorphism of (1.10) as the procedure of *passing to right adjoints*.

1.3.5. *The adjoint 1-morphism.* Let us specialize the above discussion further to the case $\mathbb{S} = [1]$. For $\mathbb{T} \in 2\text{-Cat}$, let

$$((\text{Seq}_1(\mathbb{T}))^{\text{Spc}})^R \subset (\text{Seq}_1(\mathbb{T}))^{\text{Spc}} \supset ((\text{Seq}_1(\mathbb{T}))^{\text{Spc}})^L,$$

be the subspaces of 1-morphisms that admit right and left adjoints, respectively.

As a particular case of Corollary 1.3.4 we obtain:

Corollary 1.3.6. *There exists a canonical isomorphism of spaces*

$$((\text{Seq}_1(\mathbb{T}))^{\text{Spc}})^R \simeq ((\text{Seq}_1(\mathbb{T}))^{\text{Spc}})^L$$

that induces the isomorphism

$$\pi_0 \left(((\text{Seq}_1(\mathbb{T}))^{\text{Spc}})^R \right) \simeq \pi_0 \left(((\text{Seq}_1(\mathbb{T}))^{\text{Spc}})^L \right),$$

given by passage to the adjoint 1-morphism.

1.4. Proof of Theorem 1.2.4 for ordinary 2-categories. In this subsection we take \mathbb{T} to be an *ordinary* 2-category. In this case Theorem 1.2.4 can be proved by explicit analysis.

1.4.1. Let us first show that the image of the restriction functor

$$\text{Maps}_{2\text{-Cat}}(\mathbb{S}^{R\mathbf{C}}, \mathbb{T}) \rightarrow \text{Maps}_{2\text{-Cat}}(\mathbb{S}, \mathbb{T})$$

belongs to $\text{Maps}_{2\text{-Cat}}(\mathbb{S}, \mathbb{T})^{R\mathbf{C}} \subset \text{Maps}_{2\text{-Cat}}(\mathbb{S}, \mathbb{T})$.

Let $F : \mathbb{S} \rightarrow \mathbb{T}$ be a functor, such that the map

$$\text{Sq}_{\bullet, \bullet}^{\sim}(\mathbb{S}) \rightarrow \text{Sq}_{\bullet, \bullet}(\mathbb{T})$$

has been extended to a map

$$F_{\bullet, \bullet} : (\text{Sq}_{\bullet, \bullet}^{\text{Pair}}(\mathbb{S}^{2\text{-op}}, \mathbf{C}))^{\text{vert-op}} \rightarrow \text{Sq}_{\bullet, \bullet}(\mathbb{T}).$$

Let $s \xrightarrow{\alpha} s'$ be a morphism in \mathbf{C} , and let $F(s) \xrightarrow{F(\alpha)} F(s')$ be its image in \mathbb{T} . We wish to show that $F(\alpha)$ admits a left adjoint.

Consider α as a $(1, 0)$ -simplex in $\text{Sq}_{\bullet, \bullet}^{\text{Pair}}(\mathbb{S}, \mathbf{C})$. Let us now vertically invert it, and thus consider it as a $(1, 0)$ -simplex in $(\text{Sq}_{\bullet, \bullet}^{\text{Pair}}(\mathbb{S}^{2\text{-op}}, \mathbf{C}))^{\text{vert-op}}$. The image of the latter under $F_{\bullet, \bullet}$ is a $(1, 0)$ -simplex in $\text{Sq}_{\bullet, \bullet}(\mathbb{T})$, i.e., a 1-morphism

$$F(s') \xrightarrow{\beta} F(s)$$

(note the direction of the arrow!).

Let us show that β is the left adjoint of $F(\alpha)$.

1.4.2. Consider the following point in $\text{Sq}_{1,1}^{\text{Pair}}(\mathbb{S}^{2\text{-op}}, \mathbf{C})$:

$$(1.11) \quad \begin{array}{ccc} s & \xrightarrow{\alpha} & s' \\ \alpha \downarrow & \swarrow & \downarrow \text{id} \\ s' & \xrightarrow{\text{id}} & s' \end{array}$$

where the 2-morphism is the identity map $\alpha \Rightarrow \alpha$.

Let us vertically invert it, and thus consider it as a $(1, 1)$ -simplex in $(\text{Sq}_{\bullet, \bullet}^{\text{Pair}}(\mathbb{S}^{2\text{-op}}, \mathbf{C}))^{\text{vert-op}}$. Take the image of the latter under $F_{\bullet, \bullet}$. We obtain a $(1, 1)$ -simplex in $\text{Sq}_{\bullet, \bullet}(\mathbb{T})$, i.e., a diagram

$$(1.12) \quad \begin{array}{ccc} F(s') & \xrightarrow{\text{id}} & F(s') \\ \beta \downarrow & \swarrow & \downarrow \text{id} \\ F(s) & \xrightarrow{F(\alpha)} & F(s') \end{array}$$

The $(1, 1)$ -simplex (1.12) represents a 2-morphism

$$(1.13) \quad \text{id} \rightarrow F(\alpha) \circ \beta,$$

which will be the unit of the adjunction $(\beta, F(\alpha))$ -adjunction.

1.4.3. Consider the following point in $\text{Sq}_{1,1}^{\text{Pair}}(\mathbb{S}^{2\text{-op}}, \mathbf{C})$:

$$(1.14) \quad \begin{array}{ccc} s & \xrightarrow{\text{id}} & s \\ \text{id} \downarrow & \swarrow & \downarrow \alpha \\ s & \xrightarrow{\alpha} & s' \end{array}$$

where the 2-morphism is the identity map $\alpha \Rightarrow \alpha$.

Let us vertically invert it, and thus consider it as a $(1, 1)$ -simplex in $(\text{Sq}_{\bullet,\bullet}^{\text{Pair}}(\mathbb{S}^{2\text{-op}}, \mathbf{C}))^{\text{vert-op}}$. Take the image of the latter under $F_{\bullet,\bullet}$. We obtain a $(1, 1)$ -simplex in $\text{Sq}_{\bullet,\bullet}(\mathbb{T})$, i.e., a diagram

$$(1.15) \quad \begin{array}{ccc} F(s) & \xrightarrow{F(\alpha)} & F(s') \\ \text{id} \downarrow & \swarrow & \downarrow \beta \\ F(s) & \xrightarrow{\text{id}} & F(s) \end{array}$$

The $(1, 1)$ -simplex (1.12) represents a 2-morphism

$$(1.16) \quad \beta \circ F(\alpha) \rightarrow \text{id},$$

which will be the co-unit of the adjunction $(\beta, F(\alpha))$ -adjunction.

1.4.4. The adjunction identities for (1.16) and (1.13) follow by concatenating the diagrams (1.11) and (1.14) first vertically, and then horizontally.

1.4.5. Let us now be given a functor $F : \mathbb{S} \rightarrow \mathbb{T}$ such that for each arrow $s \xrightarrow{\alpha} s'$ in \mathbf{C} , the corresponding 1-morphism $F(\alpha)$ admits a left adjoint. Let us construct the corresponding map

$$F_{\bullet,\bullet} : (\text{Sq}_{\bullet,\bullet}^{\text{Pair}}(\mathbb{S}^{2\text{-op}}, \mathbf{C}))^{\text{vert-op}} \rightarrow \text{Sq}_{\bullet,\bullet}(\mathbb{T}).$$

With no restriction of generality, we can assume that \mathbb{S} is also ordinary. We will define the map in question for $(0, 0)$, $(0, 1)$, $(1, 0)$ and $(1, 1)$ simplices, and it will be clear that it extends to a map of bi-simplicial *sets* by associativity.

1.4.6. At the level of $(0, 0)$ simplices, $F_{\bullet,\bullet}$ sends a vertex

$$s \in \mathbb{S}^{\text{Spc}},$$

thought of the space of $(0, 0)$ -simplices in $(\text{Sq}_{\bullet,\bullet}^{\text{Pair}}(\mathbb{S}^{2\text{-op}}, \mathbf{C}))^{\text{vert-op}}$, to

$$F(s) \in \mathbb{T}^{\text{Spc}} = \text{Sq}_{0,0}(\mathbb{T}).$$

1.4.7. At the level of $(0, 1)$ simplices, $F_{\bullet,\bullet}$ sends

$$(s_0 \xrightarrow{\alpha} s_1) \in (\text{Seq}_1(\mathbb{S}))^{\text{Spc}},$$

thought of the space of $(0, 1)$ -simplices in $(\text{Sq}_{\bullet,\bullet}^{\text{Pair}}(\mathbb{S}^{2\text{-op}}, \mathbf{C}))^{\text{vert-op}}$, to

$$\left(F(s_0) \xrightarrow{F(\alpha)} F(s_1) \right) \in (\text{Seq}_1(\mathbb{T}))^{\text{Spc}} = \text{Sq}_{0,1}(\mathbb{T}).$$

1.4.8. At the level of $(1, 0)$ simplices, $F_{\bullet, \bullet}$ sends

$$(s_0 \xrightarrow{\alpha} s_1) \in \text{Seq}_1(\mathbf{C}),$$

thought of the space of $(1, 0)$ -simplices in $(\text{Sq}_{\bullet, \bullet}^{\text{Pair}}(\mathbb{S}^{2\text{-op}}, \mathbf{C}))^{\text{vert-op}}$, to

$$\left(F(s_1) \xrightarrow{F(\alpha)^L} F(s_0) \right) \in (\text{Seq}_1(\mathbb{T}))^{\text{SpC}} = \text{Sq}_{1,0}(\mathbb{T}),$$

where $F(\alpha)^L$ is the left adjoint of $F(\alpha)$.

1.4.9. At the level of $(1, 1)$ simplices, $F_{\bullet, \bullet}$ sends a point

$$(1.17) \quad \begin{array}{ccc} s_{0,0} & \xrightarrow{\alpha_0} & s_{0,1} \\ \beta_0 \downarrow & \nearrow \phi & \downarrow \beta_1 \\ s_{1,0} & \xrightarrow{\alpha_1} & s_{1,1} \end{array}$$

in $\text{Sq}_{1,1}^{\text{Pair}}(\mathbb{S}^{2\text{-op}}, \mathbf{C})$, thought of as a $(1, 1)$ -simplex in $(\text{Sq}_{\bullet, \bullet}^{\text{Pair}}(\mathbb{S}^{2\text{-op}}, \mathbf{C}))^{\text{vert-op}}$ to the element of $\text{Sq}_{1,1}(\mathbb{T})$, given by the diagram

$$\begin{array}{ccc} F(s_{1,0}) & \xrightarrow{F(\alpha_1)} & F(s_{1,1}) \\ F(\beta_0)^L \downarrow & \nearrow \psi & \downarrow F(\beta_1)^L \\ F(s_{0,0}) & \xrightarrow{F(\alpha_0)} & F(s_{0,1}) \end{array}$$

where the 2-morphism

$$\psi : F(\beta_1)^L \circ F(\alpha_1) \rightarrow F(\alpha_0) \circ F(\beta_0)^L,$$

is obtained from

$$\phi : F(\alpha_1) \circ F(\beta_0) \rightarrow F(\beta_1) \circ F(\alpha_0)$$

by adjunction. (Note the direction in which ϕ goes—this is due to the fact that (1.17) was a $(1, 1)$ -simplex in $\text{Sq}_{1,1}^{\text{Pair}}(\mathbb{S}^{2\text{-op}}, \mathbf{C})$, i.e., we inverted the 2-morphisms in \mathbb{S} .)

2. PROOF OF THEOREM 1.2.4

We will first prove Theorem 1.2.4 in the case when the target $(\infty, 2)$ -category is $\mathbf{1-Cat}$, and then reduce to this case using the Yoneda embedding of [Chapter A.2, Sect. 6.3].

The idea of the proof in the case of $\mathbb{T} = \mathbf{1-Cat}$ is the following.

Consider the 2-category $[m, n] := [m] \otimes [n]$. Let \mathbf{C} be the 1-full subcategory in $[m, n]^{1\text{-Cat}}$, corresponding to the horizontal direction (i.e., the 2nd coordinate). The main observation is that the corresponding space $\text{Maps}_{2\text{-Cat}}(\mathbb{S}, \mathbf{1-Cat})^{R\mathbf{C}}$ can be described very explicitly.

Namely, this is the space of functors

$$[m] \rightarrow \text{biCart}_{/[n]^{\text{op}}},$$

where $\text{biCart}_{/\mathbf{I}}$ denoted the category of *bi-Caretsian fibrations* over a given $(\infty, 1)$ -category \mathbf{I} , i.e.,

$$\text{biCart}_{/\mathbf{I}} = \text{Cart}_{/\mathbf{I}} \cap \text{coCart}_{/\mathbf{I}} \subset \mathbf{1-Cat}_{/\mathbf{I}}.$$

2.1. The swapping procedure. In this subsection we will make a (relatively elementary) observation pertaining to $(\infty, 1)$ -categories that lies in the heart of the proof of Theorem 1.2.4 for the target $\mathbb{T} = \mathbf{1-Cat}$.

2.1.1. Let \mathbf{I} and \mathbf{J} be $(\infty, 1)$ -categories. Consider the following $(\infty, 1)$ -category, denoted

$$\text{Cart-coCart}_{\mathbf{I}, \mathbf{J}}.$$

This is the full subcategory of $1\text{-Cat}_{/\mathbf{I} \times \mathbf{J}}$, that consists of those $(\infty, 1)$ -categories \mathbf{C} over $\mathbf{I} \times \mathbf{J}$ that satisfy:

- The composite functor $\mathbf{C} \rightarrow \mathbf{I} \times \mathbf{J} \rightarrow \mathbf{I}$ is a Cartesian fibration;
- The functor $\mathbf{C} \rightarrow \mathbf{I} \times \mathbf{J}$, viewed as a functor between Cartesian fibrations over \mathbf{I} , sends Cartesian arrows to Cartesian arrows (i.e., belongs to $(\text{Cart}_{/\mathbf{I}})_{\text{strict}}$);
- The composite functor $\mathbf{C} \rightarrow \mathbf{I} \times \mathbf{J} \rightarrow \mathbf{J}$ is a coCartesian fibration;
- The functor $\mathbf{C} \rightarrow \mathbf{I} \times \mathbf{J}$, viewed as a functor between coCartesian fibrations over \mathbf{J} , sends coCartesian arrows to coCartesian arrows (i.e., belongs to $(\text{coCart}_{/\mathbf{J}})_{\text{strict}}$);

2.1.2. The first two conditions define a 1-fully faithful embedding

$$\text{Cart-coCart}_{\mathbf{I}, \mathbf{J}} \hookrightarrow \text{Func}(\mathbf{I}^{\text{op}}, 1\text{-Cat}_{/\mathbf{J}}),$$

and the second two conditions imply that it factors as

$$\text{Cart-coCart}_{\mathbf{I}, \mathbf{J}} \hookrightarrow \text{Func}(\mathbf{I}^{\text{op}}, \text{coCart}_{/\mathbf{J}}).$$

Similarly, we have a 1-fully faithful embedding

$$\text{Cart-coCart}_{\mathbf{I}, \mathbf{J}} \hookrightarrow \text{Func}(\mathbf{J}, \text{Cart}_{/\mathbf{I}}).$$

We claim:

Proposition 2.1.3. *The induced maps*

$$(\text{Cart-coCart}_{\mathbf{I}, \mathbf{J}})^{\text{Spc}} \rightarrow \text{Maps}(\mathbf{I}^{\text{op}}, \text{coCart}_{/\mathbf{J}})$$

and

$$(\text{Cart-coCart}_{\mathbf{I}, \mathbf{J}})^{\text{Spc}} \rightarrow \text{Maps}(\mathbf{J}, \text{Cart}_{/\mathbf{I}})$$

are isomorphisms.

Proof. We will prove the first isomorphism, the second being similar. The inverse map is constructed as follows: given

$$\mathbf{I}^{\text{op}} \rightarrow 1\text{-Cat}_{/\mathbf{J}},$$

we tautologically construct a Cartesian fibration $\mathbf{C} \rightarrow \mathbf{I}$, equipped with a functor $\mathbf{C} \rightarrow \mathbf{I} \times \mathbf{J}$ that takes Cartesian arrows to Cartesian arrows.

We need to show that if the initial map takes values in $\text{coCart}_{/\mathbf{J}} \subset 1\text{-Cat}_{/\mathbf{J}}$, then the resulting functor $\mathbf{C} \rightarrow \mathbf{J}$ is a coCartesian fibration, and the functor $\mathbf{C} \rightarrow \mathbf{I} \times \mathbf{J}$, viewed as a functor between coCartesian fibrations over \mathbf{J} , sends coCartesian arrows to coCartesian arrows. This is a straightforward verification. □

Corollary 2.1.4. *There exists a canonical isomorphism*

$$\text{Maps}_{1\text{-Cat}}(\mathbf{I}^{\text{op}}, \text{coCart}_{/\mathbf{J}}) \simeq \text{Maps}_{1\text{-Cat}}(\mathbf{J}, \text{Cart}_{/\mathbf{I}}).$$

2.2. Proof of Theorem 1.2.4 for $\mathbb{T} = \mathbf{1-Cat}$.

2.2.1. The datum of a functor

$$\mathbb{S} \rightarrow \mathbf{1-Cat},$$

is equivalent to that of a functor

$$\mathbb{S}^{2\text{-op}} \rightarrow (\mathbf{1-Cat})^{2\text{-op}},$$

which by [Chapter A.1, Corollary 4.4.5], is equivalent to the datum of a bi-simplicial map

$$\mathrm{Sq}_{\bullet, \bullet}^{\mathrm{Pair}}(\mathbb{S}^{2\text{-op}}, \mathbf{C}) \rightarrow \mathrm{Sq}_{\bullet, \bullet}((\mathbf{1-Cat})^{2\text{-op}}),$$

and finally a map

$$(2.1) \quad \mathrm{Sq}_{\bullet, \bullet}^{\mathrm{Pair}}(\mathbb{S}^{2\text{-op}}, \mathbf{C}) \rightarrow (\mathrm{Sq}_{\bullet, \bullet}(\mathbf{1-Cat}))^{\mathrm{reflect}}.$$

2.2.2. The datum of a functor

$$\mathbb{S}^{R\mathbf{C}} \rightarrow \mathbf{1-Cat}$$

is equivalent to that of a bi-simplicial map

$$(2.2) \quad \mathrm{Sq}_{\bullet, \bullet}^{\mathrm{Pair}}(\mathbb{S}^{2\text{-op}}, \mathbf{C}) \rightarrow (\mathrm{Sq}_{\bullet, \bullet}(\mathbf{1-Cat}))^{\mathrm{vert-op}}.$$

2.2.3. Recall that the space $\mathrm{Sq}_{m,n}(\mathbf{1-Cat})$ is described as

$$\mathrm{Maps}_{\mathbf{1-Cat}}([m], \mathrm{Cart}_{/[n]^{\mathrm{op}}}).$$

Hence, the space of (m, n) -simplices of $(\mathrm{Sq}_{\bullet, \bullet}(\mathbf{1-Cat}))^{\mathrm{reflect}}$ is described as

$$\mathrm{Maps}_{\mathbf{1-Cat}}([n], \mathrm{Cart}_{/[m]^{\mathrm{op}}}).$$

We claim that a functor

$$\mathbb{S} \rightarrow \mathbf{1-Cat},$$

belongs to $\mathrm{Maps}(\mathbb{S}, \mathbf{1-Cat})^{R\mathbf{C}}$ if and only if each of the maps

$$\mathrm{Sq}_{m,n}^{\mathrm{Pair}}(\mathbb{S}^{2\text{-op}}, \mathbf{C}) \rightarrow \mathrm{Maps}_{\mathbf{1-Cat}}([n], \mathrm{Cart}_{/[m]^{\mathrm{op}}})$$

takes values in

$$\mathrm{Maps}_{\mathbf{1-Cat}}([n], \mathrm{biCart}_{/[m]^{\mathrm{op}}}) \subset \mathrm{Maps}_{\mathbf{1-Cat}}([n], \mathrm{Cart}_{/[m]^{\mathrm{op}}}),$$

where for $\mathbf{I} \in \mathbf{1-Cat}$, we let $\mathrm{biCart}_{\mathbf{I}}$ denote the full subcategory of $\mathbf{1-Cat}_{/\mathbf{I}}$ equal to

$$\mathrm{Cart}_{/\mathbf{I}} \cap \mathrm{coCart}_{/\mathbf{I}}.$$

Indeed, this assertion can be checked at the level of the underlying ordinary 1-categories, in which case it is a straightforward verification.

2.2.4. The space of (m, n) -simplices of $(\mathrm{Sq}_{\bullet, \bullet}(\mathbf{1-Cat}))^{\mathrm{vert-op}}$ is described as

$$\mathrm{Maps}_{\mathbf{1-Cat}}([m]^{\mathrm{op}}, \mathrm{Cart}_{/[n]^{\mathrm{op}}}).$$

We claim that for every functor

$$\mathbb{S}^{R\mathbf{C}} \rightarrow \mathbf{1-Cat},$$

then each of the maps

$$\mathrm{Sq}_{m,n}^{\mathrm{Pair}}(\mathbb{S}^{2\text{-op}}, \mathbf{C}) \rightarrow \mathrm{Maps}_{\mathbf{1-Cat}}([m]^{\mathrm{op}}, \mathrm{Cart}_{/[n]^{\mathrm{op}}}) \stackrel{\text{Corollary 2.1.4}}{\simeq} \mathrm{Maps}_{\mathbf{1-Cat}}([n], \mathrm{coCart}_{/[m]^{\mathrm{op}}})$$

takes values in

$$\mathrm{Maps}_{\mathbf{1-Cat}}([n], \mathrm{biCart}_{/[m]^{\mathrm{op}}}) \subset \mathrm{Maps}_{\mathbf{1-Cat}}([n], \mathrm{coCart}_{/[m]^{\mathrm{op}}}).$$

Indeed, this assertion can be checked at the level of the underlying ordinary 1-categories, in which case it follows from the validity of Theorem 1.2.4 with values in $\mathbf{1-Cat}^{\mathrm{ordn}}$.

2.2.5. Hence, when considering the bi-simplicial maps (2.1) and (2.2) we can replace the bi-simplicial spaces

$$(\mathrm{Sq}_{\bullet, \bullet}(\mathbf{1-Cat}))^{\mathrm{reflect}} \text{ and } (\mathrm{Sq}_{\bullet, \bullet}(\mathbf{1-Cat}))^{\mathrm{vert-op}}$$

by their common full bi-simplicial subspace

$$(2.3) \quad '(\mathrm{Sq}_{\bullet, \bullet}(\mathbf{1-Cat}))^{\mathrm{reflect}} \simeq '(\mathrm{Sq}_{\bullet, \bullet}(\mathbf{1-Cat}))^{\mathrm{vert-op}}, \\ (m, n) \mapsto \mathrm{Maps}_{1\text{-Cat}}([n], \mathrm{biCart}_{/[m]^{\mathrm{op}}}).$$

Thus, we obtain that the datum of a bi-simplicial map in (2.1) is equivalent to the datum of a bi-simplicial map in (2.2), thereby establishing an isomorphism

$$\mathrm{Maps}_{2\text{-Cat}}(\mathbb{S}, \mathbf{1-Cat})^{R\mathbf{C}} \simeq \mathrm{Maps}_{2\text{-Cat}}(\mathbb{S}^{R\mathbf{C}}, \mathbf{1-Cat}).$$

2.2.6. Finally, it follows from the construction, that the composed map

$$\mathrm{Maps}_{2\text{-Cat}}(\mathbb{S}, \mathbf{1-Cat})^{R\mathbf{C}} \simeq \mathrm{Maps}_{2\text{-Cat}}(\mathbb{S}^{R\mathbf{C}}, \mathbf{1-Cat}) \xrightarrow{(1.2)} \mathrm{Maps}_{2\text{-Cat}}(\mathbb{S}, \mathbf{1-Cat})$$

is the tautological embedding

$$\mathrm{Maps}_{2\text{-Cat}}(\mathbb{S}, \mathbf{1-Cat})^{R\mathbf{C}} \hookrightarrow \mathrm{Maps}_{2\text{-Cat}}(\mathbb{S}, \mathbf{1-Cat}).$$

This shows that the map

$$\mathrm{Maps}_{2\text{-Cat}}(\mathbb{S}^{R\mathbf{C}}, \mathbf{1-Cat}) \xrightarrow{(1.2)} \mathrm{Maps}_{2\text{-Cat}}(\mathbb{S}, \mathbf{1-Cat})$$

is an isomorphism onto $\mathrm{Maps}_{2\text{-Cat}}(\mathbb{S}, \mathbf{1-Cat})^{R\mathbf{C}}$, as required.

2.3. Swapping procedure: relative version. The contents of this subsection are needed in order to generalize the contents of Sect. 2.2 to the case when instead of the target $(\infty, 2)$ -category $\mathbf{1-Cat}$, we are dealing with

$$\mathrm{Funct}(\mathbf{I} \otimes \mathbf{J}, \mathbf{1-Cat}), \quad \mathbf{I}, \mathbf{J} \in 1\text{-Cat}.$$

2.3.1. For $\mathbf{I}, \mathbf{J} \in 1\text{-Cat}$ we let

$$\mathrm{Cart}\text{-}\mathrm{Cart}_{\mathbf{I}, \mathbf{J}}$$

be the full subcategory of $1\text{-Cat}_{/\mathbf{I} \times \mathbf{J}}$ that consists of objects $\mathbf{C} \rightarrow \mathbf{I} \times \mathbf{J}$ satisfying the following conditions:

- The composite functor $\mathbf{C} \rightarrow \mathbf{I} \times \mathbf{J} \rightarrow \mathbf{I}$ is a Cartesian fibration;
- The functor $\mathbf{C} \rightarrow \mathbf{I} \times \mathbf{J}$, viewed as a functor between Cartesian fibrations over \mathbf{I} sends coCartesian arrows to Cartesian ones;
- For every $i \in \mathbf{I}$, the resulting functor $\mathbf{C}_i \rightarrow \mathbf{J}$ is a Cartesian fibration.

Unstraightening over \mathbf{I} defines an equivalence

$$\mathrm{Maps}_{1\text{-Cat}}(\mathbf{I}^{\mathrm{op}}, \mathrm{Cart}_{/\mathbf{J}}) \simeq (\mathrm{Cart}\text{-}\mathrm{Cart}_{\mathbf{I}, \mathbf{J}})^{\mathrm{Spc}}.$$

2.3.2. For a triplet of $(\infty, 1)$ -categories $\mathbf{J}, \mathbf{K}, \mathbf{L}$ let

$$\mathrm{Cart}\text{-}\mathrm{Cart}\text{-}\mathrm{Cart}_{\mathbf{J}, \mathbf{K}, \mathbf{L}}$$

denote the full subcategory of $1\text{-Cat}_{/\mathbf{J} \times \mathbf{K} \times \mathbf{L}}$ that consists of objects $\mathbf{C} \rightarrow \mathbf{J} \times \mathbf{K} \times \mathbf{L}$ satisfying the following conditions:

- When viewed as a category over $\mathbf{K} \times (\mathbf{J} \times \mathbf{L})$, it belongs to $\mathrm{Cart}_{\mathbf{K}, \mathbf{J} \times \mathbf{L}}$;
- For every fixed $l \in \mathbf{L}$, the category \mathbf{C}_l is a Cartesian fibration over $\mathbf{J} \times \mathbf{K}$.

Let

$$(\text{Cart-Cart-Cart}_{\mathbf{J},\mathbf{K},\mathbf{L}})_{\text{strict}} \subset \text{Cart-Cart-Cart}_{\mathbf{J},\mathbf{K},\mathbf{L}}$$

be the following 1-full subcategory: Given two objects $\mathbf{C}, \mathbf{C}' \in \text{Cart-Cart-Cart}_{\mathbf{J},\mathbf{K},\mathbf{L}}$, we restrict 1-morphisms to those functors $F : \mathbf{C} \rightarrow \mathbf{C}'$ over $\mathbf{J} \times \mathbf{K} \times \mathbf{L}$ that:

- For every fixed $j \in \mathbf{J}$ and $l \in \mathbf{L}$, the corresponding functor $F_{j,l} : \mathbf{C}_{j,l} \rightarrow \mathbf{C}'_{j,l}$ carries arrows Cartesian over \mathbf{K} to arrows Cartesian over \mathbf{K} .
- For every fixed $j \in \mathbf{J}$ and $k \in \mathbf{K}$, the corresponding functor $F_{j,k} : \mathbf{C}_{j,k} \rightarrow \mathbf{C}'_{j,k}$ carries arrows Cartesian over \mathbf{L} to arrows Cartesian over \mathbf{L} .

For another category \mathbf{I} , consider the goupoid

$$\text{Maps}(\mathbf{I}, (\text{Cart-Cart-Cart}_{\mathbf{J},\mathbf{K},\mathbf{L}})_{\text{strict}}).$$

We shall now describe it in several different ways.

2.3.3. Let $\text{coCart-Cart-Cart}_{\mathbf{J},\mathbf{K},\mathbf{L}}$ denote the full subcategory of $1\text{-Cat}_{/\mathbf{J} \times \mathbf{K} \times \mathbf{L}}$ that consists of objects $\mathbf{C} \rightarrow \mathbf{J} \times \mathbf{K} \times \mathbf{L}$ satisfying the following conditions:

- The composite functor $\mathbf{C} \rightarrow \mathbf{J} \times \mathbf{K} \times \mathbf{L} \rightarrow \mathbf{J}$ is a coCartesian fibration;
- The functor $\mathbf{C} \rightarrow \mathbf{J} \times \mathbf{K} \times \mathbf{L}$, when viewed as a map between coCartesian fibrations over \mathbf{J} , sends coCartesian arrows to coCartesian ones;
- For every $j \in \mathbf{J}$, the resulting object $\mathbf{C}_j \in 1\text{-Cat}_{\mathbf{K} \times \mathbf{L}}$ belongs to $\text{Cart-Cart}_{/\mathbf{K} \times \mathbf{L}}$;
- For every fixed $k \in \mathbf{K}$ and an arrow $j_0 \rightarrow j_1$ in \mathbf{J} , the resulting functor $\mathbf{C}_{j_0,k} \rightarrow \mathbf{C}_{j_1,k}$ carries arrows Cartesian over \mathbf{L} to arrows Cartesian over \mathbf{L} .
- For every fixed $l \in \mathbf{L}$ and an arrow $j_0 \rightarrow j_1$ in \mathbf{J} , the resulting functor $\mathbf{C}_{j_0,l} \rightarrow \mathbf{C}_{j_1,l}$ carries arrows Cartesian over \mathbf{K} to arrows Cartesian over \mathbf{K} .

Let

$$(\text{coCart-Cart-Cart}_{\mathbf{J},\mathbf{K},\mathbf{L}})_{\text{strict}} \subset \text{coCart-Cart-Cart}_{\mathbf{J},\mathbf{K},\mathbf{L}}$$

be the following 1-full subcategory: Given two objects $\mathbf{C}, \mathbf{C}' \in \text{Cart-Cart-Cart}_{\mathbf{J},\mathbf{K},\mathbf{L}}$, we restrict 1-morphisms to those functors $F : \mathbf{C} \rightarrow \mathbf{C}'$ over $\mathbf{J} \times \mathbf{K} \times \mathbf{L}$ that:

- For any fixed $j \in \mathbf{J}$ and $l \in \mathbf{L}$, the corresponding functor $F_{j,l} : \mathbf{C}_{j,l} \rightarrow \mathbf{C}'_{j,l}$ carries arrows Cartesian over \mathbf{K} to arrows Cartesian over \mathbf{K} .
- For any fixed $j \in \mathbf{J}$ and $k \in \mathbf{K}$ the corresponding functor $F_{j,k} : \mathbf{C}_{j,k} \rightarrow \mathbf{C}'_{j,k}$ carries arrows Cartesian over \mathbf{L} to arrows Cartesian over \mathbf{L} .

For another category \mathbf{I} , consider the goupoid

$$\text{Maps}(\mathbf{I}, (\text{coCart-Cart-Cart}_{\mathbf{J},\mathbf{K},\mathbf{L}})_{\text{strict}}).$$

2.3.4. Now, we claim that as in Corollary 2.1.4 we have a canonical isomorphism:

$$(2.4) \quad \text{Maps}(\mathbf{J}, (\text{Cart-Cart-Cart}_{\mathbf{I},\mathbf{K},\mathbf{L}})_{\text{strict}}) \simeq \text{Maps}(\mathbf{I}^{\text{op}}, (\text{coCart-Cart-Cart}_{\mathbf{J},\mathbf{K},\mathbf{L}})_{\text{strict}}).$$

This is obtained by identifying both sides with the *space* of the full subcategory

$$\text{Cart-coCart-Cart-Cart}_{\mathbf{I},\mathbf{J},\mathbf{K},\mathbf{L}} \subset 1\text{-Cat}_{\mathbf{I} \times \mathbf{J} \times \mathbf{K} \times \mathbf{L}},$$

consisting of \mathbf{C} over $\mathbf{I} \times \mathbf{J} \times \mathbf{K} \times \mathbf{L}$ that satisfy:

- The composite functor $\mathbf{C} \rightarrow \mathbf{I} \times \mathbf{J} \times \mathbf{K} \times \mathbf{L} \rightarrow \mathbf{J}$ is a coCartesian fibration;
- The functor $\mathbf{C} \rightarrow \mathbf{I} \times \mathbf{J} \times \mathbf{K} \times \mathbf{L}$, when viewed as a map between coCartesian fibrations over \mathbf{J} , sends coCartesian arrows to coCartesian ones;
- For every $j \in \mathbf{J}$, the resulting object $\mathbf{C}_j \in 1\text{-Cat}_{\mathbf{I} \times \mathbf{K} \times \mathbf{L}}$ belongs to $\text{Cart-Cart-Cart}_{/\mathbf{I} \times \mathbf{K} \times \mathbf{L}}$;
- For every fixed $k \in \mathbf{K}$, $i \in \mathbf{I}$ and an arrow $j_0 \rightarrow j_1$ in \mathbf{J} , the resulting functor $\mathbf{C}_{i,j_0,k} \rightarrow \mathbf{C}_{i,j_1,k}$ carries arrows Cartesian over \mathbf{L} to arrows Cartesian over \mathbf{L} .

- For every fixed $l \in \mathbf{L}$, $i \in \mathbf{I}$ and an arrow $j_0 \rightarrow j_1$ in \mathbf{J} , the resulting functor $\mathbf{C}_{i,j_0,l} \rightarrow \mathbf{C}_{i,j_1,l}$ carries arrows Cartesian over \mathbf{K} to arrows Cartesian over \mathbf{K} .

2.3.5. Finally, we claim that the space

$$\text{Maps}(\mathbf{I}, (\text{Cart-Cart-Cart}_{\mathbf{J},\mathbf{K},\mathbf{L}})_{\text{strict}})$$

can be also identified with the *space* of the following full subcategory

$$\text{Cart-Cart-Cart-Cart}_{\mathbf{I}^{\text{op}},\mathbf{J},\mathbf{K},\mathbf{L}} \subset 1\text{-Cat}_{\mathbf{I}^{\text{op}} \times \mathbf{J} \times \mathbf{K} \times \mathbf{L}}.$$

Namely, for quadruplet of $(\infty, 1)$ -categories $\mathbf{I}, \mathbf{J}, \mathbf{K}, \mathbf{L}$, we define $\text{Cart-Cart-Cart-Cart}_{\mathbf{I},\mathbf{J},\mathbf{K},\mathbf{L}}$ to consist of those \mathbf{C} over $\mathbf{I} \times \mathbf{J} \times \mathbf{K} \times \mathbf{L}$ that satisfy:

- When viewed as a category over $(\mathbf{I} \times \mathbf{K}) \times (\mathbf{J} \times \mathbf{L})$, it belongs to $\text{Cart-Cart}_{\mathbf{I} \times \mathbf{K}, \mathbf{J} \times \mathbf{L}}$;
- For every fixed $i \in \mathbf{I}$ and $l \in \mathbf{L}$, the resulting functor $\mathbf{C}_{i,l} \rightarrow \mathbf{J} \times \mathbf{K}$ is a Cartesian fibration;
- For every fixed $j \in \mathbf{J}$ and $k \in \mathbf{K}$, the resulting functor $\mathbf{C}_{j,k} \rightarrow \mathbf{I} \times \mathbf{L}$ is a Cartesian fibration.

2.4. **Adjunctions: relative version.** Let us fix another pair of $(\infty, 1)$ -categories \mathbf{K} and \mathbf{L} . We will now explain the modifications needed to adapt the above proof of Theorem 1.2.4 for $\mathbb{T} = \mathbf{1-Cat}$ to the case when $\mathbb{T} = \text{Funct}(\mathbf{K} \otimes \mathbf{L}, \mathbf{1-Cat})$.

2.4.1. First, we claim that the bi-simplicial space $\text{Sq}(\text{Funct}(\mathbf{K} \otimes \mathbf{L}, \mathbf{1-Cat}))$ is described as follows:

$$(2.5) \quad \text{Sq}_{m,n}(\text{Funct}(\mathbf{K} \otimes \mathbf{L}, \mathbf{1-Cat})) \simeq \text{Maps}([m], (\text{Cart-Cart-Cart}_{[n]^{\text{op}},\mathbf{K}^{\text{op}},\mathbf{L}^{\text{op}}})_{\text{strict}}),$$

where the notation $(\text{Cart-Cart-Cart}_{[n]^{\text{op}},\mathbf{K}^{\text{op}},\mathbf{L}^{\text{op}}})_{\text{strict}}$ is as in Sect. 2.3.2.

Assuming that, the proof in Sect. 2.2 goes through, once we substitute the isomorphism of Corollary 2.1.4 by that of (2.4).

2.4.2. To establish (2.5), we proceed as follows. We rewrite

$$\text{Sq}_{m,n}(\text{Funct}(\mathbf{K} \otimes \mathbf{L}, \mathbf{1-Cat})) = \text{Maps}([m] \otimes [n] \times (\mathbf{K} \otimes \mathbf{L}), \mathbf{1-Cat}),$$

and further by [Chapter A.2, Corollary 2.2.6] as

$$(\mathbf{1-Cat})_{/((([m] \otimes [n]) \times (\mathbf{K} \otimes \mathbf{L}))^{1\text{-op}})}^{\text{Spc}} \simeq (\mathbf{1-Cat})_{/([n]^{\text{op}} \otimes [m]^{\text{op}}) \times (\mathbf{L}^{\text{op}} \otimes \mathbf{K}^{\text{op}})}^{\text{Spc}}.$$

Now, using [Chapter A.2, Lemmas 2.2.5 and 2.2.8 and Corollary 4.6.5], we obtain

$$(\mathbf{1-Cat})_{/([n]^{\text{op}} \otimes [m]^{\text{op}}) \times (\mathbf{L}^{\text{op}} \otimes \mathbf{K}^{\text{op}})}^{\text{Spc}} \simeq (\text{Cart-Cart-Cart-Cart}_{[m]^{\text{op}},[n]^{\text{op}},\mathbf{K}^{\text{op}},\mathbf{L}^{\text{op}}})^{\text{Spc}},$$

and finally, using Sect. 2.3.5, we identify

$$(\text{Cart-Cart-Cart-Cart}_{[m]^{\text{op}},[n]^{\text{op}},\mathbf{K}^{\text{op}},\mathbf{L}^{\text{op}}})^{\text{Spc}} \simeq \text{Maps}([m], (\text{Cart-Cart-Cart}_{[n]^{\text{op}},\mathbf{K}^{\text{op}},\mathbf{L}^{\text{op}}})_{\text{strict}}),$$

as required.

2.5. **Proofs of Theorem 1.2.4, the general case.** The proof will amount to deducing Theorem 1.2.4 from the particular case of $\mathbb{T} = \text{Funct}([m] \otimes [n], \mathbf{1-Cat})$ using the 2-categorical Yoneda embedding.

2.5.1. For a target category \mathbb{T} we consider its Yoneda embedding

$$\mathbb{T} \xrightarrow{\text{Yon}_{\mathbb{T}}} \text{Func}(\mathbb{T}^{1\text{-op}}, \mathbf{1}\text{-Cat}).$$

Consider the commutative diagram

$$\begin{array}{ccc} \text{Maps}_{2\text{-Cat}}(\mathbb{S}^{Rc}, \mathbb{T}) & \longrightarrow & \text{Maps}_{2\text{-Cat}}(\mathbb{S}, \mathbb{T})^{Rc} \\ \downarrow & & \downarrow \\ \text{Maps}_{2\text{-Cat}}(\mathbb{S}^{Rc}, \text{Func}(\mathbb{T}^{1\text{-op}}, \mathbf{1}\text{-Cat})) & \longrightarrow & \text{Maps}_{2\text{-Cat}}(\mathbb{S}, \text{Func}(\mathbb{T}^{1\text{-op}}, \mathbf{1}\text{-Cat}))^{Rc}, \end{array}$$

where the vertical arrows are fully faithful embeddings.

We claim that it is sufficient to show that the bottom arrow in the above diagram, i.e.,

$$(2.6) \quad \text{Maps}_{2\text{-Cat}}(\mathbb{S}^{Rc}, \text{Func}(\mathbb{T}^{1\text{-op}}, \mathbf{1}\text{-Cat})) \rightarrow \text{Maps}_{2\text{-Cat}}(\mathbb{S}, \text{Func}(\mathbb{T}^{1\text{-op}}, \mathbf{1}\text{-Cat}))^{Rc},$$

is an isomorphism.

2.5.2. Indeed, if this is the case, we obtain that the functor

$$\text{Maps}_{2\text{-Cat}}(\mathbb{S}^{Rc}, \mathbb{T}) \rightarrow \text{Maps}_{2\text{-Cat}}(\mathbb{S}, \mathbb{T})^{Rc}$$

is fully faithful, and it remains to show that it is essentially surjective.

This follows from the next general assertion:

Lemma 2.5.3. *Let $\mathbb{T}_1 \hookrightarrow \mathbb{T}_2$ be a fully faithful functor. Then the diagram of spaces*

$$\begin{array}{ccc} \text{Maps}_{2\text{-Cat}}(\mathbb{S}^{Rc}, \mathbb{T}_1) & \longrightarrow & \text{Maps}_{2\text{-Cat}}(\mathbb{S}^{Rc}, \mathbb{T}_2) \\ \downarrow & & \downarrow \\ \text{Maps}_{2\text{-Cat}}(\mathbb{S}, \mathbb{T}_1) & \longrightarrow & \text{Maps}_{2\text{-Cat}}(\mathbb{S}, \mathbb{T}_2) \end{array}$$

is a pull-back square.

Proof. Let us be given a functor $\mathbb{S}^{Rc} \rightarrow \mathbb{T}_2$, such that the composition

$$\mathbb{S} \rightarrow \mathbb{S}^{Rc} \rightarrow \mathbb{T}_2$$

factors through $\mathbb{T}_1 \subset \mathbb{T}_2$. We wish to show that the initial functor also factors through $\mathbb{T}_1 \subset \mathbb{T}_2$.

Consider the corresponding map of bi-simplicial spaces

$$(\text{Sq}_{\bullet, \bullet}^{\text{Pair}}(\mathbb{S}^{2\text{-op}}, \mathbf{C}))^{\text{vert-op}} \rightarrow \text{Sq}_{\bullet, \bullet}(\mathbb{T}_2).$$

We wish to show that it takes values in $\text{Sq}_{\bullet, \bullet}(\mathbb{T}_1) \subset \text{Sq}_{\bullet, \bullet}(\mathbb{T}_2)$.

The latter is enough to check on $(0, 0)$ -simplicies, and the assertion follows from the assumption as

$$\text{Sq}_{0,0}^{\sim}(\mathbb{S}) \rightarrow (\text{Sq}_{0,0}^{\text{Pair}}(\mathbb{S}^{2\text{-op}}, \mathbf{C}))^{\text{vert-op}}$$

is an isomorphism. □

2.5.4. Thus, we wish to show that for $\mathbb{S}, \mathbb{T} \in 2\text{-Cat}$, the map

$$\text{Maps}_{2\text{-Cat}}(\mathbb{S}^{Rc}, \text{Func}(\mathbb{T}, \mathbf{1}\text{-Cat})) \rightarrow \text{Maps}_{2\text{-Cat}}(\mathbb{S}, \text{Func}(\mathbb{T}, \mathbf{1}\text{-Cat}))$$

is an isomorphism with essential image

$$\text{Maps}_{2\text{-Cat}}(\mathbb{S}, \text{Func}(\mathbb{T}, \mathbf{1}\text{-Cat}))^{Rc} \subset \text{Maps}_{2\text{-Cat}}(\mathbb{S}, \text{Func}(\mathbb{T}, \mathbf{1}\text{-Cat})).$$

2.5.5. For any $\mathbb{S}' \in 2\text{-Cat}$, the space $\text{Maps}_{2\text{-Cat}}(\mathbb{S}', \text{Func}(\mathbb{T}, \mathbf{1}\text{-Cat}))$ can be described as that of bi-simplicial functors

$$\text{Sq}_{\bullet, \bullet}(\mathbb{T}) \rightarrow \text{Sq}_{\bullet, \bullet}(\text{Func}(\mathbb{S}', \mathbf{1}\text{-Cat})).$$

Note that the bi-simplicial space $\text{Sq}_{\bullet, \bullet}(\text{Func}(\mathbb{S}', \mathbf{1}\text{-Cat}))$ identifies with

$$\text{Maps}_{2\text{-Cat}}(\mathbb{S}', \text{Func}([\bullet] \otimes [\bullet], \mathbf{1}\text{-Cat})),$$

where the bi-simplicial $(\infty, 2)$ -category $\text{Func}([\bullet] \otimes [\bullet], \mathbf{1}\text{-Cat})$ attaches to m, n the $(\infty, 2)$ -category

$$\text{Func}([m] \otimes [n], \mathbf{1}\text{-Cat}).$$

2.5.6. Note that under the above identification

$$\begin{aligned} \text{Maps}_{2\text{-Cat}}(\mathbb{S}, \text{Func}(\mathbb{T}, \mathbf{1}\text{-Cat})) &\simeq \\ &\simeq \text{Maps}_{\text{SpC}\Delta^{\text{op}} \times \Delta^{\text{op}}}(\text{Sq}_{\bullet, \bullet}(\mathbb{T}), \text{Maps}_{2\text{-Cat}}(\mathbb{S}, \text{Func}([\bullet] \otimes [\bullet], \mathbf{1}\text{-Cat}))), \end{aligned}$$

the subspace

$$\text{Maps}_{2\text{-Cat}}(\mathbb{S}, \text{Func}(\mathbb{T}, \mathbf{1}\text{-Cat}))^{Rc} \subset \text{Maps}(\mathbb{S}, \text{Func}(\mathbb{T}, \mathbf{1}\text{-Cat}))$$

maps to

$$(2.7) \quad \begin{aligned} &\text{Maps}_{\text{SpC}\Delta^{\text{op}} \times \Delta^{\text{op}}}(\text{Sq}_{\bullet, \bullet}(\mathbb{T}), \text{Maps}_{2\text{-Cat}}(\mathbb{S}, \text{Func}([\bullet] \otimes [\bullet], \mathbf{1}\text{-Cat}))^{Rc}) \subset \\ &\subset \text{Maps}_{\text{SpC}\Delta^{\text{op}} \times \Delta^{\text{op}}}(\text{Sq}_{\bullet, \bullet}(\mathbb{T}), \text{Maps}_{2\text{-Cat}}(\mathbb{S}, \text{Func}([\bullet] \otimes [\bullet], \mathbf{1}\text{-Cat}))). \end{aligned}$$

Hence, we obtain that it is enough to show that the map

$$\begin{aligned} &\text{Maps}_{\text{SpC}\Delta^{\text{op}} \times \Delta^{\text{op}}}(\text{Sq}_{\bullet, \bullet}(\mathbb{T}), \text{Maps}_{2\text{-Cat}}(\mathbb{S}^{Rc}, \text{Func}([\bullet] \otimes [\bullet], \mathbf{1}\text{-Cat}))) \rightarrow \\ &\rightarrow \text{Maps}_{\text{SpC}\Delta^{\text{op}} \times \Delta^{\text{op}}}(\text{Sq}_{\bullet, \bullet}(\mathbb{T}), \text{Maps}_{2\text{-Cat}}(\mathbb{S}, \text{Func}([\bullet] \otimes [\bullet], \mathbf{1}\text{-Cat}))) \end{aligned}$$

is an isomorphism onto the subspace (2.7).

2.5.7. To prove the latter, it is sufficient to show that for every m, n , the map

$$\text{Maps}_{2\text{-Cat}}(\mathbb{S}^{Rc}, \text{Func}([m] \otimes [n], \mathbf{1}\text{-Cat})) \rightarrow \text{Maps}_{2\text{-Cat}}(\mathbb{S}, \text{Func}([m] \otimes [n], \mathbf{1}\text{-Cat}))$$

is an isomorphism onto

$$\text{Maps}_{2\text{-Cat}}(\mathbb{S}, \text{Func}([m] \otimes [n], \mathbf{1}\text{-Cat}))^{Rc} \subset \text{Maps}_{2\text{-Cat}}(\mathbb{S}, \text{Func}([m] \otimes [n], \mathbf{1}\text{-Cat})).$$

2.5.8. However, the latter statement is the assertion of Theorem 1.2.4 for the target category $\text{Func}([m] \otimes [n], \mathbf{1}\text{-Cat})$, and it holds due to Sect. 2.4.

3. ADJUNCTION WITH PARAMETERS

Our current goal is to lift the isomorphism of spaces

$$\text{Maps}_{2\text{-Cat}}(\mathbb{S}, \mathbb{T})^R \simeq \text{Maps}_{2\text{-Cat}}(\mathbb{S}^{1\&2\text{-op}}, \mathbb{T})^L,$$

which is part of the statement of Corollary 1.3.4 to an equivalence of $(\infty, 2)$ -categories

$$(3.1) \quad \text{Func}(\mathbb{S}, \mathbb{T})_{\text{right-lax}}^R \simeq \text{Func}(\mathbb{S}^{1\&2\text{-op}}, \mathbb{T})_{\text{left-lax}}^L.$$

3.1. The set-up for adjunction with parameters. By definition, for an $(\infty, 2)$ -category \mathbb{X} , we have

$$\text{Maps}_{2\text{-Cat}}(\mathbb{X}, \text{Func}(\mathbb{S}, \mathbb{T})_{\text{right-lax}}) = \text{Maps}_{2\text{-Cat}}(\mathbb{X} \otimes \mathbb{S}, \mathbb{T})$$

and

$$\text{Maps}_{2\text{-Cat}}(\mathbb{X}, \text{Func}(\mathbb{S}^{1\&2\text{-op}}, \mathbb{T})_{\text{left-lax}}) = \text{Maps}_{2\text{-Cat}}(\mathbb{S}^{1\&2\text{-op}} \otimes \mathbb{X}, \mathbb{T}).$$

Hence, we have the subspaces

$$(3.2) \quad \text{Maps}_{2\text{-Cat}}(\mathbb{X} \otimes \mathbb{S}, \mathbb{T})^{\mathbb{S}_R} \subset \text{Maps}_{2\text{-Cat}}(\mathbb{X} \otimes \mathbb{S}, \mathbb{T})$$

and

$$(3.3) \quad \text{Maps}_{2\text{-Cat}}(\mathbb{S}^{1\&2\text{-op}} \otimes \mathbb{X}, \mathbb{T})^{\mathbb{S}_L} \subset \text{Maps}_{2\text{-Cat}}(\mathbb{S}^{1\&2\text{-op}} \otimes \mathbb{X}, \mathbb{T}),$$

corresponding to

$$\text{Maps}_{2\text{-Cat}}(\mathbb{X}, \text{Func}(\mathbb{S}, \mathbb{T})_{\text{right-lax}}^R) \subset \text{Maps}_{2\text{-Cat}}(\mathbb{X}, \text{Func}(\mathbb{S}, \mathbb{T})_{\text{right-lax}})$$

and

$$\text{Maps}_{2\text{-Cat}}(\mathbb{X}, \text{Func}(\mathbb{S}^{1\&2\text{-op}}, \mathbb{T})_{\text{left-lax}}^L) \subset \text{Maps}_{2\text{-Cat}}(\mathbb{X}, \text{Func}(\mathbb{S}^{1\&2\text{-op}}, \mathbb{T})_{\text{left-lax}}),$$

respectively.

We will interpret the subspaces (3.2) and (3.3) via the *universal adjointable functors* of Theorem 1.2.4, which will allow to construct the desired equivalence

$$\text{Maps}_{2\text{-Cat}}(\mathbb{X} \otimes \mathbb{S}, \mathbb{T})^{\mathbb{S}_R} \simeq \text{Maps}_{2\text{-Cat}}(\mathbb{S}^{1\&2\text{-op}} \otimes \mathbb{X}, \mathbb{T})^{\mathbb{S}_L}.$$

3.1.1. We start with a pair of $(\infty, 2)$ -categories \mathbb{S}_1 and \mathbb{S}_2 , and consider their Gray product $\mathbb{S}_1 \otimes \mathbb{S}_2$

We let \mathbf{C}_1 be the 1-full subcategory in $(\mathbb{S}_1 \otimes \mathbb{S}_2)^{1\text{-Cat}}$, corresponding to 1-morphisms of the form

$$(s_1, s_2) \xrightarrow{(\alpha, \text{id})} (s'_1, s_2), \quad \alpha \in \text{Maps}_{\mathbb{S}}(s_1, s'_1).$$

We let \mathbf{C}_2 be the 1-full subcategory in $(\mathbb{S}_1 \otimes \mathbb{S}_2)^{1\text{-Cat}}$, corresponding to 1-morphisms of the form

$$(s_1, s_2) \xrightarrow{(\text{id}, \beta)} (s_1, s'_2), \quad \beta \in \text{Maps}_{\mathbb{S}}(s_2, s'_2).$$

Consider the corresponding $(\infty, 2)$ -categories

$$(\mathbb{S}_1 \otimes \mathbb{S}_2)^{R_{\mathbb{S}_2}} := (\mathbb{S}_1 \otimes \mathbb{S}_2)^{R_{\mathbf{C}_2}} \text{ and } (\mathbb{S}_1 \otimes \mathbb{S}_2, \mathbb{T})^{L_{\mathbb{S}_1}} := (\mathbb{S}_1 \otimes \mathbb{S}_2)^{R_{\mathbf{C}_1}},$$

see Sect. 1.2.1.

3.1.2. For another $(\infty, 2)$ -category, let

$$\text{Maps}(\mathbb{S}_1 \otimes \mathbb{S}_2, \mathbb{T})^{R_{\mathbb{S}_2}} := \text{Maps}(\mathbb{S}_1 \otimes \mathbb{S}_2, \mathbb{T})^{R_{\mathbf{C}_2}}$$

and

$$\text{Maps}(\mathbb{S}_1 \otimes \mathbb{S}_2, \mathbb{T})^{L_{\mathbb{S}_1}} := \text{Maps}(\mathbb{S}_1 \otimes \mathbb{S}_2, \mathbb{T})^{L_{\mathbf{C}_1}}$$

denote the corresponding full subspaces, see Sect. 1.1.8.

3.1.3. Consider the bi-simplicial space

$$(\mathrm{Sq}_{\bullet, \bullet}^{\sim}(\mathbb{S}_2^{1\text{-op}}))^{\mathrm{reflect}} \times \mathrm{Sq}_{\bullet, \bullet}^{\sim}(\mathbb{S}_1).$$

Recall (see [Chapter A.1, Formula (4.6)]), there is a canonically defined map

$$(3.4) \quad (\mathrm{Sq}_{\bullet, \bullet}^{\sim}(\mathbb{S}_2^{1\text{-op}}))^{\mathrm{reflect}} \times \mathrm{Sq}_{\bullet, \bullet}^{\sim}(\mathbb{S}_1) \rightarrow \mathrm{Sq}_{\bullet, \bullet}(\mathbb{S}_2^{1\&2\text{-op}} \otimes \mathbb{S}_1).$$

Moreover, by [Chapter A.1, Proposition 4.5.4], the map

$$(3.5) \quad \mathcal{L}^{\mathrm{Sq}}((\mathrm{Sq}_{\bullet, \bullet}^{\sim}(\mathbb{S}_2^{1\text{-op}}))^{\mathrm{reflect}} \times \mathrm{Sq}_{\bullet, \bullet}^{\sim}(\mathbb{S}_1)) \rightarrow \mathbb{S}_2^{1\&2\text{-op}} \otimes \mathbb{S}_1,$$

obtained from (3.4) by adjunction, is an isomorphism.

3.1.4. We claim that there is a canonically defined map

$$(3.6) \quad \mathcal{L}^{\mathrm{Sq}}((\mathrm{Sq}_{\bullet, \bullet}^{\sim}(\mathbb{S}_2^{1\text{-op}}))^{\mathrm{reflect}} \times \mathrm{Sq}_{\bullet, \bullet}^{\sim}(\mathbb{S}_1)) \rightarrow (\mathbb{S}_1 \otimes \mathbb{S}_2)^{R_{\mathbb{S}_2}}.$$

Indeed, using [Chapter A.1, Formula (4.6)] again, we obtain a map

$$(\mathrm{Sq}_{\bullet, \bullet}^{\sim}(\mathbb{S}_2))^{\mathrm{reflect}} \times \mathrm{Sq}_{\bullet, \bullet}^{\sim}(\mathbb{S}_1^{2\text{-op}}) \rightarrow \mathrm{Sq}_{\bullet, \bullet}(\mathbb{S}_2^{2\text{-op}} \otimes \mathbb{S}_1^{2\text{-op}}).$$

However, it follows by unwinding the construction, that the essential image of the latter map belongs to the full subspace

$$\mathrm{Sq}_{\bullet, \bullet}^{\mathrm{Pair}}(\mathbb{S}_2^{2\text{-op}} \otimes \mathbb{S}_1^{2\text{-op}}, \mathbf{C}_2) \subset \mathrm{Sq}_{\bullet, \bullet}(\mathbb{S}_2^{2\text{-op}} \otimes \mathbb{S}_1^{2\text{-op}})$$

thereby giving rise to a map

$$(3.7) \quad (\mathrm{Sq}_{\bullet, \bullet}^{\sim}(\mathbb{S}_2))^{\mathrm{reflect}} \times \mathrm{Sq}_{\bullet, \bullet}^{\sim}(\mathbb{S}_1^{2\text{-op}}) \rightarrow \mathrm{Sq}_{\bullet, \bullet}^{\mathrm{Pair}}(\mathbb{S}_2^{2\text{-op}} \otimes \mathbb{S}_1^{2\text{-op}}, \mathbf{C}_2)$$

Now, the sought-for map (3.6) is given by applying $\mathcal{L}^{\mathrm{Sq}}$ to the composite map

$$(3.8) \quad \begin{aligned} (\mathrm{Sq}_{\bullet, \bullet}^{\sim}(\mathbb{S}_2^{1\text{-op}}))^{\mathrm{reflect}} \times \mathrm{Sq}_{\bullet, \bullet}^{\sim}(\mathbb{S}_1) &\simeq ((\mathrm{Sq}_{\bullet, \bullet}^{\sim}(\mathbb{S}_2))^{\mathrm{horiz-op}})^{\mathrm{reflect}} \times (\mathrm{Sq}_{\bullet, \bullet}^{\sim}(\mathbb{S}_1^{2\text{-op}}))^{\mathrm{vert-op}} \simeq \\ &\simeq ((\mathrm{Sq}_{\bullet, \bullet}^{\sim}(\mathbb{S}_2))^{\mathrm{reflect}} \times \mathrm{Sq}_{\bullet, \bullet}^{\sim}(\mathbb{S}_1^{2\text{-op}}))^{\mathrm{vert-op}} \xrightarrow{(3.7)} ((\mathrm{Sq}_{\bullet, \bullet}^{\mathrm{Pair}}(\mathbb{S}_2^{2\text{-op}} \otimes \mathbb{S}_1^{2\text{-op}}, \mathbf{C}_2))^{\mathrm{vert-op}} \simeq \\ &\simeq ((\mathrm{Sq}_{\bullet, \bullet}^{\mathrm{Pair}}((\mathbb{S}_1 \otimes \mathbb{S}_2)^{2\text{-op}}, \mathbf{C}_2))^{\mathrm{vert-op}}. \end{aligned}$$

3.1.5. We claim:

Theorem 3.1.6. *The composition*

$$\begin{aligned} \mathrm{Maps}(\mathbb{S}_1 \otimes \mathbb{S}_2, \mathbb{T})^{R_{\mathbb{S}_2}} &\simeq \mathrm{Maps}((\mathbb{S}_1 \otimes \mathbb{S}_2)^{R_{\mathbb{S}_2}}, \mathbb{T}) \xrightarrow{(3.6)} \\ &\rightarrow \mathrm{Maps}(\mathcal{L}^{\mathrm{Sq}}((\mathrm{Sq}_{\bullet, \bullet}^{\sim}(\mathbb{S}_2^{1\text{-op}}))^{\mathrm{reflect}} \times \mathrm{Sq}_{\bullet, \bullet}^{\sim}(\mathbb{S}_1)), \mathbb{T}) \xrightarrow{(3.5)} \mathrm{Maps}(\mathbb{S}_2^{1\&2\text{-op}} \otimes \mathbb{S}_1, \mathbb{T}) \end{aligned}$$

is fully faithful with essential image equal to

$$\mathrm{Maps}(\mathbb{S}_2^{1\&2\text{-op}} \otimes \mathbb{S}_1, \mathbb{T})^{L_{\mathbb{S}_2^{1\&2\text{-op}}}} \subset \mathrm{Maps}(\mathbb{S}_2^{1\&2\text{-op}} \otimes \mathbb{S}_1, \mathbb{T}).$$

As a corollary, we obtain:

Corollary 3.1.7. *There exists a canonical isomorphism*

$$\mathrm{Maps}(\mathbb{S}_1 \otimes \mathbb{S}_2, \mathbb{T})^{R_{\mathbb{S}_2}} \simeq \mathrm{Maps}(\mathbb{S}_2^{1\&2\text{-op}} \otimes \mathbb{S}_1, \mathbb{T})^{L_{\mathbb{S}_2^{1\&2\text{-op}}}}.$$

3.1.8. Since the equivalence of Corollary 3.1.7 is by construction functorial in $\mathbb{S}_1 \in 2\text{-Cat}$, we obtain:

Corollary 3.1.9. *For $\mathbb{S}, \mathbb{T} \in 2\text{-Cat}$, the isomorphism of (1.10) upgrades to an equivalence of $(\infty, 2)$ -categories*

$$\mathrm{Funct}(\mathbb{S}, \mathbb{T})_{\mathrm{right-lax}}^R \simeq \mathrm{Funct}(\mathbb{S}^{1\&2\text{-op}}, \mathbb{T})_{\mathrm{left-lax}}^L.$$

3.2. Proof of Theorem 3.1.6. We will give a proof in the particular case when the target category \mathbb{T} is $\mathbf{1-Cat}$. The general case is deduced by the same procedure as one employed in Sect. 2.5.

3.2.1. First, we claim that the assertion holds for $\mathbf{1-Cat}$ replaced by $\mathbf{1-Cat}^{\text{ordn}}$, i.e., when we consider functors from $\mathbb{S}^{2\text{-op}}$ (resp., \mathbb{S}^L) with values in ordinary 2-categories. This can be checked directly as in Sect. 1.4.

3.2.2. The datum of a map

$$\mathfrak{L}^{\text{Sq}} \left((\text{Sq}_{\bullet,\bullet}^{\sim}(\mathbb{S}_2^{1\text{-op}}))^{\text{reflect}} \times \text{Sq}_{\bullet,\bullet}^{\sim}(\mathbb{S}_1) \right) \rightarrow \mathbf{1-Cat}$$

is equivalent to the datum of a map of bi-simplicial spaces

$$(\text{Sq}_{\bullet,\bullet}^{\sim}(\mathbb{S}_2^{1\text{-op}}))^{\text{reflect}} \times \text{Sq}_{\bullet,\bullet}^{\sim}(\mathbb{S}_1) \rightarrow \text{Sq}_{\bullet,\bullet}(\mathbf{1-Cat}),$$

or equivalently

$$\left((\text{Sq}_{\bullet,\bullet}^{\sim}(\mathbb{S}_2^{1\text{-op}}))^{\text{reflect}} \times \text{Sq}_{\bullet,\bullet}^{\sim}(\mathbb{S}_1) \right)^{\text{vert-op}} \rightarrow (\text{Sq}_{\bullet,\bullet}(\mathbf{1-Cat}))^{\text{vert-op}}.$$

3.2.3. Let us describe the subspace

$$\begin{aligned} & \text{Maps}_{\text{SpC}\Delta^{\text{op}} \times \Delta^{\text{op}}} \left(\left((\text{Sq}_{\bullet,\bullet}^{\sim}(\mathbb{S}_2^{1\text{-op}}))^{\text{reflect}} \times \text{Sq}_{\bullet,\bullet}^{\sim}(\mathbb{S}_1) \right)^{\text{vert-op}}, (\text{Sq}_{\bullet,\bullet}(\mathbf{1-Cat}))^{\text{vert-op}} \right)_{\mathbb{S}_2^{1\&2\text{-op}}}^{L_{\mathbb{S}_2^{1\&2\text{-op}}} \subset} \\ & \subset \text{Maps}_{\text{SpC}\Delta^{\text{op}} \times \Delta^{\text{op}}} \left(\left((\text{Sq}_{\bullet,\bullet}^{\sim}(\mathbb{S}_2^{1\text{-op}}))^{\text{reflect}} \times \text{Sq}_{\bullet,\bullet}^{\sim}(\mathbb{S}_1) \right)^{\text{vert-op}}, (\text{Sq}_{\bullet,\bullet}(\mathbf{1-Cat}))^{\text{vert-op}} \right) \end{aligned}$$

that corresponds to

$$\begin{aligned} & \text{Maps}_{2\text{-Cat}}(\mathbb{S}_2^{1\&2\text{-op}} \otimes \mathbb{S}_1, \mathbf{1-Cat})_{\mathbb{S}_2^{1\&2\text{-op}}}^{L_{\mathbb{S}_2^{1\&2\text{-op}}} \subset} \text{Maps}_{2\text{-Cat}}(\mathbb{S}_2^{1\&2\text{-op}} \otimes \mathbb{S}_1, \mathbf{1-Cat}) \stackrel{(3.5)}{\simeq} \\ & \text{Maps}_{\text{SpC}\Delta^{\text{op}} \times \Delta^{\text{op}}} \left((\text{Sq}_{\bullet,\bullet}^{\sim}(\mathbb{S}_2^{1\text{-op}}))^{\text{reflect}} \times \text{Sq}_{\bullet,\bullet}^{\sim}(\mathbb{S}_1), \text{Sq}_{\bullet,\bullet}(\mathbf{1-Cat}) \right) \simeq \\ & \simeq \text{Maps}_{\text{SpC}\Delta^{\text{op}} \times \Delta^{\text{op}}} \left(\left((\text{Sq}_{\bullet,\bullet}^{\sim}(\mathbb{S}_2^{1\text{-op}}))^{\text{reflect}} \times \text{Sq}_{\bullet,\bullet}^{\sim}(\mathbb{S}_1) \right)^{\text{vert-op}}, (\text{Sq}_{\bullet,\bullet}(\mathbf{1-Cat}))^{\text{vert-op}} \right) \end{aligned}$$

Namely, we claim that

$$\begin{aligned} & \text{Maps}_{\text{SpC}\Delta^{\text{op}} \times \Delta^{\text{op}}} \left(\left((\text{Sq}_{\bullet,\bullet}^{\sim}(\mathbb{S}_2^{1\text{-op}}))^{\text{reflect}} \times \text{Sq}_{\bullet,\bullet}^{\sim}(\mathbb{S}_1) \right)^{\text{vert-op}}, (\text{Sq}_{\bullet,\bullet}(\mathbf{1-Cat}))^{\text{vert-op}} \right)_{\mathbb{S}_2^{1\&2\text{-op}}}^{L_{\mathbb{S}_2^{1\&2\text{-op}}} =} \\ & = \text{Maps}_{\text{SpC}\Delta^{\text{op}} \times \Delta^{\text{op}}} \left(\left((\text{Sq}_{\bullet,\bullet}^{\sim}(\mathbb{S}_2^{1\text{-op}}))^{\text{reflect}} \times \text{Sq}_{\bullet,\bullet}^{\sim}(\mathbb{S}_1) \right)^{\text{vert-op}}, '(\text{Sq}_{\bullet,\bullet}(\mathbf{1-Cat}))^{\text{vert-op}} \right), \end{aligned}$$

where

$$'(\text{Sq}_{\bullet,\bullet}(\mathbf{1-Cat}))^{\text{vert-op}} \subset (\text{Sq}_{\bullet,\bullet}(\mathbf{1-Cat}))^{\text{vert-op}}$$

is as in (2.3).

Indeed, this assertion can be checked at the level of ordinary categories, where it is a straightforward verification.

3.2.4. To summarize, we obtain a canonical identification

$$\begin{aligned} (3.9) \quad & \text{Maps}_{2\text{-Cat}}(\mathbb{S}_2^{1\&2\text{-op}} \otimes \mathbb{S}_1, \mathbf{1-Cat})_{\mathbb{S}_2^{1\&2\text{-op}}}^{L_{\mathbb{S}_2^{1\&2\text{-op}}} \simeq} \\ & \simeq \text{Maps}_{\text{SpC}\Delta^{\text{op}} \times \Delta^{\text{op}}} \left(\left((\text{Sq}_{\bullet,\bullet}^{\sim}(\mathbb{S}_2^{1\text{-op}}))^{\text{reflect}} \times \text{Sq}_{\bullet,\bullet}^{\sim}(\mathbb{S}_1) \right)^{\text{vert-op}}, '(\text{Sq}_{\bullet,\bullet}(\mathbf{1-Cat}))^{\text{vert-op}} \right). \end{aligned}$$

3.2.5. According to [Chapter A.1, Proposition 4.5.4], the datum of a map

$$\mathbb{S}_1 \otimes \mathbb{S}_2 \rightarrow \mathbf{1-Cat}$$

is equivalent to the datum of a bi-simplicial map

$$(\mathrm{Sq}_{\bullet,\bullet}(\mathbb{S}_1^{2\text{-op}}))^{\mathrm{reflect}} \times \mathrm{Sq}_{\bullet,\bullet}(\mathbb{S}_2) \rightarrow \mathrm{Sq}_{\bullet,\bullet}(\mathbf{1-Cat}),$$

or, equivalently,

$$((\mathrm{Sq}_{\bullet,\bullet}(\mathbb{S}_1^{2\text{-op}}))^{\mathrm{reflect}} \times \mathrm{Sq}_{\bullet,\bullet}(\mathbb{S}_2))^{\mathrm{reflect}} \rightarrow (\mathrm{Sq}_{\bullet,\bullet}(\mathbf{1-Cat}))^{\mathrm{reflect}}.$$

3.2.6. Let us describe the subspace

$$\begin{aligned} & \mathrm{Maps}_{\mathrm{Spc}\Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}}} \left(((\mathrm{Sq}_{\bullet,\bullet}(\mathbb{S}_1^{2\text{-op}}))^{\mathrm{reflect}} \times \mathrm{Sq}_{\bullet,\bullet}(\mathbb{S}_2))^{\mathrm{reflect}}, (\mathrm{Sq}_{\bullet,\bullet}(\mathbf{1-Cat}))^{\mathrm{reflect}} \right)^{R_{\mathbb{S}_2}} \subset \\ & \mathrm{Maps}_{\mathrm{Spc}\Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}}} \left(((\mathrm{Sq}_{\bullet,\bullet}(\mathbb{S}_1^{2\text{-op}}))^{\mathrm{reflect}} \times \mathrm{Sq}_{\bullet,\bullet}(\mathbb{S}_2))^{\mathrm{reflect}}, (\mathrm{Sq}_{\bullet,\bullet}(\mathbf{1-Cat}))^{\mathrm{reflect}} \right) \end{aligned}$$

that corresponds to

$$\mathrm{Maps}_{2\text{-Cat}}(\mathbb{S}_1 \otimes \mathbb{S}_2, \mathbf{1-Cat})^{R_{\mathbb{S}_2}} \subset \mathrm{Maps}_{2\text{-Cat}}(\mathbb{S}_1 \otimes \mathbb{S}_2, \mathbf{1-Cat}).$$

Namely, we claim that

$$\begin{aligned} & \mathrm{Maps}_{\mathrm{Spc}\Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}}} \left(((\mathrm{Sq}_{\bullet,\bullet}(\mathbb{S}_1^{2\text{-op}}))^{\mathrm{reflect}} \times \mathrm{Sq}_{\bullet,\bullet}(\mathbb{S}_2))^{\mathrm{reflect}}, (\mathrm{Sq}_{\bullet,\bullet}(\mathbf{1-Cat}))^{\mathrm{reflect}} \right)^{R_{\mathbb{S}_2}} = \\ & \mathrm{Maps}_{\mathrm{Spc}\Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}}} \left(((\mathrm{Sq}_{\bullet,\bullet}(\mathbb{S}_1^{2\text{-op}}))^{\mathrm{reflect}} \times \mathrm{Sq}_{\bullet,\bullet}(\mathbb{S}_2))^{\mathrm{reflect}}, '(\mathrm{Sq}_{\bullet,\bullet}(\mathbf{1-Cat}))^{\mathrm{reflect}} \right), \end{aligned}$$

where

$$'(\mathrm{Sq}_{\bullet,\bullet}(\mathbf{1-Cat}))^{\mathrm{reflect}} \subset (\mathrm{Sq}_{\bullet,\bullet}(\mathbf{1-Cat}))^{\mathrm{reflect}}$$

is an in (2.3).

Indeed, this assertion can be checked at the level of ordinary categories, where it is a straightforward verification.

3.2.7. To summarize, we obtain a canonical identification

$$(3.10) \quad \begin{aligned} & \mathrm{Maps}_{2\text{-Cat}}(\mathbb{S}_1 \otimes \mathbb{S}_2, \mathbf{1-Cat})^{R_{\mathbb{S}_2}} \simeq \\ & \simeq \mathrm{Maps}_{\mathrm{Spc}\Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}}} \left(((\mathrm{Sq}_{\bullet,\bullet}(\mathbb{S}_1^{2\text{-op}}))^{\mathrm{reflect}} \times \mathrm{Sq}_{\bullet,\bullet}(\mathbb{S}_2))^{\mathrm{reflect}}, '(\mathrm{Sq}_{\bullet,\bullet}(\mathbf{1-Cat}))^{\mathrm{reflect}} \right) \end{aligned}$$

3.2.8. Note that we have a tautological identification:

$$((\mathrm{Sq}_{\bullet,\bullet}(\mathbb{S}_2^{1\text{-op}}))^{\mathrm{reflect}} \times \mathrm{Sq}_{\bullet,\bullet}(\mathbb{S}_1))^{\mathrm{vert-op}} \simeq ((\mathrm{Sq}_{\bullet,\bullet}(\mathbb{S}_1^{2\text{-op}}))^{\mathrm{reflect}} \times \mathrm{Sq}_{\bullet,\bullet}(\mathbb{S}_2))^{\mathrm{reflect}}.$$

Hence, from the isomorphisms (3.9) and (3.10) and the isomorphism

$$'(\mathrm{Sq}_{\bullet,\bullet}(\mathbf{1-Cat}))^{\mathrm{vert-op}} \simeq '(\mathrm{Sq}_{\bullet,\bullet}(\mathbf{1-Cat}))^{\mathrm{reflect}},$$

we obtain an identification

$$(3.11) \quad \mathrm{Maps}_{2\text{-Cat}}(\mathbb{S}_2^{1\&2\text{-op}} \otimes \mathbb{S}_1, \mathbf{1-Cat})^{L_{\mathbb{S}_2}^{1\&2\text{-op}}} \simeq \mathrm{Maps}_{2\text{-Cat}}(\mathbb{S}_1 \otimes \mathbb{S}_2, \mathbf{1-Cat})^{R_{\mathbb{S}_2}}.$$

3.2.9. Consider now the map

$$\begin{aligned} \mathrm{Maps}_{2\text{-Cat}}(\mathbb{S}_1 \otimes \mathbb{S}_2, \mathbf{1}\text{-Cat})^{R_{\mathbb{S}_2}} &\xrightarrow{(1.2)} \mathrm{Maps}_{2\text{-Cat}}((\mathbb{S}_1 \otimes \mathbb{S}_2)^{R_{\mathbb{S}_2}}, \mathbf{1}\text{-Cat}) \xrightarrow{(3.6)} \\ &\rightarrow \mathrm{Maps}_{\mathrm{Spc}\Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}}}((\mathrm{Sq}_{\bullet, \bullet}(\mathbb{S}_2^{1\text{-op}}))^{\mathrm{reflect}} \times \mathrm{Sq}_{\bullet, \bullet}(\mathbb{S}_1), \mathbf{1}\text{-Cat}) \end{aligned}$$

that appears in Theorem 3.1.6.

It follows from the construction that it equals the composition

$$\begin{aligned} \mathrm{Maps}_{2\text{-Cat}}(\mathbb{S}_1 \otimes \mathbb{S}_2, \mathbf{1}\text{-Cat})^{R_{\mathbb{S}_2}} &\stackrel{(3.10)}{\simeq} \\ &\simeq \mathrm{Maps}_{\mathrm{Spc}\Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}}}\left(\left(\left(\mathrm{Sq}_{\bullet, \bullet}(\mathbb{S}_1^{2\text{-op}})\right)^{\mathrm{reflect}} \times \mathrm{Sq}_{\bullet, \bullet}(\mathbb{S}_2)\right)^{\mathrm{reflect}}, \left(\mathrm{Sq}_{\bullet, \bullet}(\mathbf{1}\text{-Cat})\right)^{\mathrm{reflect}}\right) = \\ &= \mathrm{Maps}_{\mathrm{Spc}\Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}}}\left(\left(\left(\mathrm{Sq}_{\bullet, \bullet}(\mathbb{S}_1^{2\text{-op}})\right)^{\mathrm{reflect}} \times \mathrm{Sq}_{\bullet, \bullet}(\mathbb{S}_2)\right)^{\mathrm{reflect}}, \left(\mathrm{Sq}_{\bullet, \bullet}(\mathbf{1}\text{-Cat})\right)^{\mathrm{vert-op}}\right) \hookrightarrow \\ &\hookrightarrow \mathrm{Maps}_{\mathrm{Spc}\Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}}}\left(\left(\left(\mathrm{Sq}_{\bullet, \bullet}(\mathbb{S}_1^{2\text{-op}})\right)^{\mathrm{reflect}} \times \mathrm{Sq}_{\bullet, \bullet}(\mathbb{S}_2)\right)^{\mathrm{reflect}}, \left(\mathrm{Sq}_{\bullet, \bullet}(\mathbf{1}\text{-Cat})\right)^{\mathrm{vert-op}}\right) \simeq \\ &\simeq \mathrm{Maps}_{\mathrm{Spc}\Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}}}\left(\left(\left(\left(\mathrm{Sq}_{\bullet, \bullet}(\mathbb{S}_1^{2\text{-op}})\right)^{\mathrm{reflect}} \times \mathrm{Sq}_{\bullet, \bullet}(\mathbb{S}_2)\right)^{\mathrm{reflect}}\right)^{\mathrm{vert-op}}, \mathrm{Sq}_{\bullet, \bullet}(\mathbf{1}\text{-Cat})\right) \simeq \\ &\simeq \mathrm{Maps}_{\mathrm{Spc}\Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}}}\left(\left(\mathrm{Sq}_{\bullet, \bullet}(\mathbb{S}_2^{1\text{-op}})\right)^{\mathrm{reflect}} \times \mathrm{Sq}_{\bullet, \bullet}(\mathbb{S}_1), \mathrm{Sq}_{\bullet, \bullet}(\mathbf{1}\text{-Cat})\right), \end{aligned}$$

thus implying the assertion of the theorem.

4. AN ALTERNATIVE PROOF

In this section we will give an alternative proof of Theorem 1.2.4 and Corollary 3.1.7.

4.1. **An alternative proof of Corollary 3.1.7 for $\mathbb{T} = \mathbf{1}\text{-Cat}$.** This proof will make a substantial use of the unstraightening equivalence of [Chapter A.2, Corollary 1.2.6].

We need to show that for $\mathbb{S}_1, \mathbb{S}_2 \in \mathbf{2}\text{-Cat}$ there exists a canonical isomorphism

$$(4.1) \quad \mathrm{Maps}(\mathbb{S}_1 \otimes \mathbb{S}_2, \mathbf{1}\text{-Cat})^{R_{\mathbb{S}_2}} \simeq \mathrm{Maps}(\mathbb{S}_2^{1\&2\text{-op}} \otimes \mathbb{S}_1, \mathbf{1}\text{-Cat})^{L_{\mathbb{S}_2}^{1\&2\text{-op}}}.$$

4.1.1. For a given $(\infty, 2)$ -category \mathbb{S} , recall the full subcategory

$$\mathbf{1}\text{-Cart}_{/\mathbb{S}} \subset \mathbf{2}\text{-Cat}_{/\mathbb{S}},$$

see [Chapter A.2, Sect. 1.2.3].

We let $\mathbf{1}\text{-biCart}_{/\mathbb{S}}$ be the full subcategory of $\mathbf{1}\text{-Cart}_{/\mathbb{S}}$ that consists of objects that under the equivalence

$$\mathbf{2}\text{-Cat}_{/\mathbb{S}} \simeq (\mathbf{2}\text{-Cat}_{/\mathbb{S}^{1\text{-op}}})^{2\text{-op}}, \quad \mathbb{T} \mapsto \mathbb{T}^{1\text{-op}}$$

corresponds to

$$\mathbf{1}\text{-Cart}_{/\mathbb{S}^{1\text{-op}}} \subset \mathbf{2}\text{-Cat}_{/\mathbb{S}^{1\text{-op}}}.$$

We have a tautological fully faithful embedding

$$(4.2) \quad \mathbf{1}\text{-biCart}_{/\mathbb{S}} \hookrightarrow (\mathbf{1}\text{-Cart}_{/\mathbb{S}^{1\text{-op}}})^{2\text{-op}}.$$

4.1.2. By definition, the left-hand side in (4.1) is a full subspace in

$$\text{Maps}(\mathbb{S}_1 \otimes \mathbb{S}_2, \mathbf{1-Cat}) \simeq \text{Maps}(\mathbb{S}_1, \text{Func}(\mathbb{S}_2, \mathbf{1-Cat})_{\text{right-lax}}),$$

which we rewrite, using [Chapter A.2, Corollary 1.2.6], as

$$\text{Maps}(\mathbb{S}_1, \mathbf{1-Cart}_{/\mathbb{S}_2^{1\text{-op}}}).$$

Now, it follows from the definitions that the full subspace in question corresponds to

$$\text{Maps}(\mathbb{S}_1, \mathbf{1-biCart}_{/\mathbb{S}_2^{1\text{-op}}}).$$

4.1.3. Similarly, the right-hand side in (4.1) is a full subspace in

$$\begin{aligned} \text{Maps}(\mathbb{S}_2^{1\&2\text{-op}} \otimes \mathbb{S}_1, \mathbf{1-Cat}) &\simeq \text{Maps}(\mathbb{S}_1^{2\text{-op}} \otimes \mathbb{S}_2^{1\text{-op}}, \mathbf{1-Cat}^{2\text{-op}}) \simeq \\ &\simeq \text{Maps}(\mathbb{S}_1^{2\text{-op}} \otimes \mathbb{S}_2^{1\text{-op}}, \mathbf{1-Cat}), \end{aligned}$$

where the last isomorphism comes from the identification $\mathbf{1-Cat} \simeq \mathbf{1-Cat}^{2\text{-op}}$ given by $\mathbb{T} \mapsto \mathbb{T}^{1\text{-op}}$.

We rewrite

$$\text{Maps}(\mathbb{S}_1^{2\text{-op}} \otimes \mathbb{S}_2^{1\text{-op}}, \mathbf{1-Cat}) \simeq \text{Maps}(\mathbb{S}_1^{2\text{-op}}, \text{Func}(\mathbb{S}_2^{1\text{-op}}, \mathbf{1-Cat})_{\text{right-lax}}),$$

and further, using [Chapter A.2, Corollary 1.2.6], as

$$\text{Maps}(\mathbb{S}_1^{2\text{-op}}, \mathbf{1-Cart}_{/\mathbb{S}_2}) \simeq \text{Maps}(\mathbb{S}_1, (\mathbf{1-Cart}_{/\mathbb{S}_2})^{2\text{-op}}).$$

It again follows from the definitions that the right-hand side of (4.1), viewed as a full subspace in $\text{Maps}(\mathbb{S}_1, (\mathbf{1-Cart}_{/\mathbb{S}_2})^{2\text{-op}})$ equals to

$$\text{Maps}(\mathbb{S}_1, (\mathbf{1-biCart}_{/\mathbb{S}_2^{1\text{-op}}})) \subset \text{Maps}(\mathbb{S}_1, (\mathbf{1-Cart}_{/\mathbb{S}_2})^{2\text{-op}})$$

with respect to the fully faithful embedding (4.2).

4.1.4. Comparing the descriptions of the left-hand side and the right-hand side of (4.1), given in Sects. 4.1.2 and 4.1.3 above, we obtain the desired isomorphism. \square

4.2. An alternative proof of Corollary 3.1.7 for a general \mathbb{T} . We need to show that for $\mathbb{S}_1, \mathbb{S}_2, \mathbb{T} \in \mathbf{2-Cat}$ there exists a canonical isomorphism

$$(4.3) \quad \text{Maps}(\mathbb{S}_1 \otimes \mathbb{S}_2, \mathbb{T})^{R_{\mathbb{S}_2}} \simeq \text{Maps}(\mathbb{S}_2^{1\&2\text{-op}} \otimes \mathbb{S}_1, \mathbb{T})^{L_{\mathbb{S}_2^{1\&2\text{-op}}}}.$$

4.2.1. Note that the above isomorphism is equivalent to Corollary 3.1.9:

$$\text{Func}(\mathbb{S}, \mathbb{T})_{\text{right-lax}}^R \simeq \text{Func}(\mathbb{S}^{1\&2\text{-op}}, \mathbb{T})_{\text{left-lax}}^L.$$

In particular, we recover the isomorphism of spaces

$$\text{Maps}_{\mathbf{2-Cat}}(\mathbb{S}, \mathbb{T})_{\text{right-lax}}^R \simeq \text{Maps}_{\mathbf{2-Cat}}(\mathbb{S}^{1\&2\text{-op}}, \mathbb{T})_{\text{left-lax}}^L$$

of (1.10).

4.2.2. We start with the following lemma:

Lemma 4.2.3. *Let \mathbb{T} be an $(\infty, 2)$ -category, and let $\alpha : t_0 \rightarrow t_1$ be a 1-morphism. Then α admits a left adjoint if and only if the following two conditions hold:*

(i) *For every $s \in \mathbb{T}$, the resulting functor of $(\infty, 1)$ -categories*

$$\mathbf{Maps}_{\mathbb{T}}(s, t) \xrightarrow{\alpha \circ -} \mathbf{Maps}_{\mathbb{T}}(s, t')$$

admits a left adjoint;

(ii) *The Beck-Chevalley condition is satisfied. I.e., for every 1-morphism $\beta : s_0 \rightarrow s_1$, the corresponding natural transformation*

$$(\alpha \circ -)^L \circ (- \circ \beta) \rightarrow (- \circ \beta) \circ (\alpha \circ -)^L,$$

arising by adjunction from the isomorphism

$$(- \circ \beta) \circ (\alpha \circ -) \simeq (\alpha \circ -) \circ (- \circ \beta),$$

is an isomorphism.

Proof. The assertion reduces to the case when \mathbb{T} is an *ordinary* 2-category, and the latter is a straightforward verification. □

4.2.4. The Yoneda embedding for \mathbb{T} gives rise to a fully faithful map

$$\mathbf{Maps}(\mathbb{S}_1 \otimes \mathbb{S}_2, \mathbb{T}) \hookrightarrow \mathbf{Maps}(\mathbb{S}_1 \otimes \mathbb{S}_2, \mathbf{Func}(\mathbb{T}^{1\text{-op}}, \mathbf{1-Cat})) \simeq \mathbf{Maps}((\mathbb{S}_1 \otimes \mathbb{S}_2) \times \mathbb{T}^{1\text{-op}}, \mathbf{1-Cat}),$$

which we further compose with the fully faithful embedding

$$\begin{aligned} \mathbf{Maps}((\mathbb{S}_1 \otimes \mathbb{S}_2) \times \mathbb{T}^{1\text{-op}}, \mathbf{1-Cat}) &\hookrightarrow \mathbf{Maps}(\mathbb{S}_1 \otimes \mathbb{T}^{1\text{-op}} \otimes \mathbb{S}_2, \mathbf{1-Cat}) \simeq \\ &\simeq \mathbf{Maps}((\mathbb{S}_1 \otimes \mathbb{T}^{1\text{-op}}) \otimes \mathbb{S}_2, \mathbf{1-Cat}). \end{aligned}$$

It is easy to see that the image of

$$\mathbf{Maps}(\mathbb{S}_1 \otimes \mathbb{S}_2, \mathbb{T})^{R_{\mathbb{S}_2}} \subset \mathbf{Maps}(\mathbb{S}_1 \otimes \mathbb{S}_2, \mathbb{T})$$

belongs to

$$\mathbf{Maps}((\mathbb{S}_1 \otimes \mathbb{T}^{1\text{-op}}) \otimes \mathbb{S}_2, \mathbf{1-Cat})^{R_{\mathbb{S}_2}} \subset \mathbf{Maps}((\mathbb{S}_1 \otimes \mathbb{T}^{1\text{-op}}) \otimes \mathbb{S}_2, \mathbf{1-Cat}).$$

Applying the isomorphism (4.1), we rewrite

$$\mathbf{Maps}((\mathbb{S}_1 \otimes \mathbb{T}^{1\text{-op}}) \otimes \mathbb{S}_2, \mathbf{1-Cat})^{R_{\mathbb{S}_2}}$$

as

$$\mathbf{Maps}(\mathbb{S}_2^{1\&2\text{-op}} \otimes (\mathbb{S}_1 \otimes \mathbb{T}^{1\text{-op}}), \mathbf{1-Cat})^{L_{\mathbb{S}_2}^{1\&2\text{-op}}}.$$

4.2.5. Consider the resulting fully faithful embedding

$$\mathbf{Maps}(\mathbb{S}_1 \otimes \mathbb{S}_2, \mathbb{T})^{R_{\mathbb{S}_2}} \hookrightarrow \mathbf{Maps}(\mathbb{S}_2^{1\&2\text{-op}} \otimes (\mathbb{S}_1 \otimes \mathbb{T}^{1\text{-op}}), \mathbf{1-Cat}) \simeq \mathbf{Maps}(\mathbb{S}_2^{1\&2\text{-op}} \otimes \mathbb{S}_1 \otimes \mathbb{T}^{1\text{-op}}, \mathbf{1-Cat}).$$

It follows from Lemma 4.2.3 that the essential image of the latter map belongs to the full subspace

$$\begin{aligned} \mathbf{Maps}((\mathbb{S}_2^{1\&2\text{-op}} \otimes \mathbb{S}_1) \times \mathbb{T}^{1\text{-op}}, \mathbf{1-Cat}) &\subset \mathbf{Maps}((\mathbb{S}_2^{1\&2\text{-op}} \otimes \mathbb{S}_1) \otimes \mathbb{T}^{1\text{-op}}, \mathbf{1-Cat}) \simeq \\ &\simeq \mathbf{Maps}(\mathbb{S}_2^{1\&2\text{-op}} \otimes \mathbb{S}_1 \otimes \mathbb{T}^{1\text{-op}}, \mathbf{1-Cat}). \end{aligned}$$

4.2.6. Finally, it is easy to see that the essential image of the resulting fully faithful map

$$\mathrm{Maps}(\mathbb{S}_1 \otimes \mathbb{S}_2, \mathbb{T})^{R_{\mathbb{S}_2}} \rightarrow \mathrm{Maps}((\mathbb{S}_2^{1\&2\text{-op}} \otimes \mathbb{S}_1) \times \mathbb{T}^{1\text{-op}}, \mathbf{1}\text{-Cat})$$

equals the essential image of

$$\begin{aligned} \mathrm{Maps}(\mathbb{S}_2^{1\&2\text{-op}} \otimes \mathbb{S}_1, \mathbb{T})^{L_{\mathbb{S}_2}^{1\&2\text{-op}}} &\hookrightarrow \mathrm{Maps}(\mathbb{S}_2^{1\&2\text{-op}} \otimes \mathbb{S}_1, \mathbb{T}) \hookrightarrow \\ &\hookrightarrow \mathrm{Maps}(\mathbb{S}_2^{1\&2\text{-op}} \otimes \mathbb{S}_1, \mathrm{Func}(\mathbb{T}^{1\text{-op}}, \mathbf{1}\text{-Cat})) \simeq \mathrm{Maps}((\mathbb{S}_2^{1\&2\text{-op}} \otimes \mathbb{S}_1) \times \mathbb{T}^{1\text{-op}}, \mathbf{1}\text{-Cat}), \end{aligned}$$

as desired. \square

4.3. **An alternative proof of Theorem 1.2.4.** Suppose we have a pair (\mathbb{S}, \mathbf{C}) and target $(\infty, 2)$ -category \mathbb{T} . We will establish a canonical isomorphism

$$(4.4) \quad \mathrm{Maps}_{2\text{-Cat}}(\mathbb{S}, \mathbb{T})^{R_{\mathbf{C}}} \simeq \mathrm{Maps}_{2\text{-Cat}}(\mathbb{S}^{R_{\mathbf{C}}}, \mathbb{T}).$$

It will follow (see Sect. 4.3.3 below) that the map \rightarrow in (4.4) is the same as the one given by restriction along (1.2).

4.3.1. Let $\mathbf{D} \subset \mathbb{T}^{1\text{-Cat}}$ be the 1-full subcategory consisting of 1-morphisms that admit a left adjoint. We have

$$(4.5) \quad \mathrm{Maps}_{2\text{-Cat}}(\mathbb{S}, \mathbb{T})^{R_{\mathbf{C}}} \simeq \mathrm{Maps}_{2\text{-Cat}^{\mathrm{Pair}}}((\mathbb{S}, \mathbf{C}), (\mathbb{T}, \mathbf{D})).$$

Since the functor $\mathrm{Sq}_{\bullet, \bullet}^{\mathrm{Pair}}$ is fully faithful, we can rewrite the right-hand side in (4.5) as

$$(4.6) \quad \mathrm{Maps}_{\mathrm{SpC}^{\Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}}}}(\mathrm{Sq}_{\bullet, \bullet}^{\mathrm{Pair}}(\mathbb{S}, \mathbf{C}), \mathrm{Sq}_{\bullet, \bullet}^{\mathrm{Pair}}(\mathbb{T}, \mathbf{D})).$$

4.3.2. It is easy to see that the full subspace

$$\mathrm{Sq}_{m, n}^{\mathrm{Pair}}(\mathbb{T}, \mathbf{D}) \subset \mathrm{Sq}_{m, n}(\mathbb{T}) = \mathrm{Maps}_{2\text{-Cat}}([m, n], \mathbb{S}) = \mathrm{Maps}_{2\text{-Cat}}([m] \otimes [n], \mathbb{T})$$

identifies with

$$\mathrm{Maps}_{2\text{-Cat}}([m] \otimes [n], \mathbb{T})^{R_{[m]}} \subset \mathrm{Maps}_{2\text{-Cat}}([m] \otimes [n], \mathbb{T}),$$

where the superscript $R_{[m]}$ follows the notational convention of (3.2).

Applying the isomorphism of Corollary 3.1.7, we rewrite

$$\mathrm{Maps}_{2\text{-Cat}}([m] \otimes [n], \mathbb{T})^{R_{[m]}} \simeq \mathrm{Maps}_{2\text{-Cat}}([n]^{\mathrm{op}} \otimes [m], \mathbb{T})^{L_{[m]}}.$$

Thus, $\mathrm{Maps}_{2\text{-Cat}}(\mathbb{S}, \mathbb{T})^{R_{\mathbf{C}}}$ identifies with the full subspace of

$$(4.7) \quad \mathrm{Maps}_{\mathrm{SpC}^{\Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}}}}(\mathrm{Sq}_{\bullet, \bullet}^{\mathrm{Pair}}(\mathbb{S}, \mathbf{C}), ((\mathrm{Sq}_{\bullet, \bullet}(\mathbb{T}))^{\mathrm{vert-op}})^{\mathrm{reflect}}),$$

that corresponds to the bi-simplicial subspace

$$'((\mathrm{Sq}_{\bullet, \bullet}(\mathbb{T}))^{\mathrm{vert-op}})^{\mathrm{reflect}} \subset ((\mathrm{Sq}_{\bullet, \bullet}(\mathbb{T}))^{\mathrm{vert-op}})^{\mathrm{reflect}}$$

given by

$$\mathrm{Maps}_{2\text{-Cat}}([n]^{\mathrm{op}} \otimes [m], \mathbb{T})^{L_{[m]}} \subset \mathrm{Maps}_{2\text{-Cat}}([n]^{\mathrm{op}} \otimes [m], \mathbb{T}).$$

4.3.3. Note that the expression in (4.7) identifies tautologically with

$$\begin{aligned} \text{Maps}_{\mathbb{S}^{\text{pc}}\Delta^{\text{op}}\times\Delta^{\text{op}}}(((\text{Sq}_{\bullet,\bullet}^{\text{Pair}}(\mathbb{S}, \mathbf{C}))^{\text{reflect}})^{\text{vert-op}}, \text{Sq}_{\bullet,\bullet}(\mathbb{T})) &\simeq \\ &\simeq \text{Maps}_{2\text{-Cat}}(\mathcal{L}^{\text{Sq}}(((\text{Sq}_{\bullet,\bullet}^{\text{Pair}}(\mathbb{S}, \mathbf{C}))^{\text{reflect}})^{\text{vert-op}}), \mathbb{T}), \end{aligned}$$

while the latter is

$$\text{Maps}_{2\text{-Cat}}(\mathbb{S}^{R\mathbf{C}}, \mathbb{T}),$$

by the construction of $\mathbb{S}^{R\mathbf{C}}$.

Thus, we have obtained a fully faithful embedding

$$(4.8) \quad \text{Maps}_{2\text{-Cat}}(\mathbb{S}, \mathbb{T})^{R\mathbf{C}} \hookrightarrow \text{Maps}_{2\text{-Cat}}(\mathbb{S}^{R\mathbf{C}}, \mathbb{T}).$$

It follows from the construction that the composite map

$$(4.9) \quad \text{Maps}_{2\text{-Cat}}(\mathbb{S}, \mathbb{T})^{R\mathbf{C}} \hookrightarrow \text{Maps}_{2\text{-Cat}}(\mathbb{S}^{R\mathbf{C}}, \mathbb{T}) \xrightarrow{(1.2)} \text{Maps}_{2\text{-Cat}}(\mathbb{S}, \mathbb{T})$$

is the tautological embedding $\text{Maps}_{2\text{-Cat}}(\mathbb{S}, \mathbb{T})^{R\mathbf{C}} \hookrightarrow \text{Maps}_{2\text{-Cat}}(\mathbb{S}, \mathbb{T})$.

4.3.4. It remains to show that the essential image of (4.8) is everything. I.e., we need to show that for any functor

$$\mathbb{S}^{R\mathbf{C}} \rightarrow \mathbb{T},$$

the corresponding map of bi-simplicial spaces

$$\text{Sq}_{\bullet,\bullet}^{\text{Pair}}(\mathbb{S}, \mathbf{C}) \rightarrow ((\text{Sq}_{\bullet,\bullet}(\mathbb{T}))^{\text{vert-op}})^{\text{reflect}}$$

has the property that its essential image belongs to

$$'((\text{Sq}_{\bullet,\bullet}(\mathbb{T}))^{\text{vert-op}})^{\text{reflect}} \subset ((\text{Sq}_{\bullet,\bullet}(\mathbb{T}))^{\text{vert-op}})^{\text{reflect}}.$$

However, the latter assertion can be checked at the level of the *ordinary* 2-category underlying \mathbb{T} . And in the latter case, the assertion follows from Sect. 1.4:

Indeed we already know that the second map in (4.9) is an isomorphism onto $\text{Maps}_{2\text{-Cat}}(\mathbb{S}, \mathbb{T})^{R\mathbf{C}} \subset \text{Maps}_{2\text{-Cat}}(\mathbb{S}, \mathbb{T})$. □