GEOMETRIC REPRESENTATION THEORY, FALL 2005

Some references

1) J. -P. Serre, Complex semi-simple Lie algebras.
2) T. Springer, Linear algebraic groups.
3) A course on D-modules by J. Bernstein, availiable at www.math.uchicago.edu/∼arinkin/langlands.
4) J. Dixmier, Enveloping algebras.

1. Basics of the category $\mathcal{O}$

1.1. **Refresher on semi-simple Lie algebras.** In this course we will work with an algebraically closed ground field of characteristic 0, which may as well be assumed equal to $\mathbb{C}$

Let $\mathfrak{g}$ be a semi-simple Lie algebra. We will fix a Borel subalgebra $\mathfrak{b} \subset \mathfrak{g}$ (also sometimes denoted $\mathfrak{b}^+$) and an opposite Borel subalgebra $\mathfrak{b}^-$. The intersection $\mathfrak{b}^+ \cap \mathfrak{b}^-$ is a Cartan subalgebra, denoted $\mathfrak{h}$. We will denote by $\mathfrak{n}$, and $\mathfrak{n}^-$ the unipotent radicals of $\mathfrak{b}$ and $\mathfrak{b}^-$, respectively. We have $\mathfrak{n} = [\mathfrak{b}, \mathfrak{b}]$, and

$$\mathfrak{h} \simeq \mathfrak{b}/\mathfrak{n}.$$

(I.e., $\mathfrak{h}$ is better to think of as a quotient of $\mathfrak{b}$, rather than a subalgebra.)

The eigenvalues of $\mathfrak{h}$ acting on $\mathfrak{n}$ are by definition the positive roots of $\mathfrak{g}$; this set will be denoted by Δ^+. We will denote by Q^+ the sub-semigroup of $\mathfrak{h}^*$ equal to the positive span of Δ^+ (i.e., Q^+ is the set of eigenvalues of $\mathfrak{h}$ under the adjoint action on $U(\mathfrak{n})$). For $\lambda, \mu \in \mathfrak{h}^*$ we shall say that $\lambda \geq \mu$ if $\lambda - \mu \in Q^+$. We denote by P^+ the sub-semigroup of dominant integral weights, i.e., those λ that satisfy $\langle \lambda, \check{\alpha} \rangle \in \mathbb{Z}^+$ for all $\alpha \in \Delta^+$.

For $\alpha \in \Delta^+$, we will denote by $\mathfrak{n}_\alpha$ the corresponding eigen-space. If $0 \neq e_\alpha \in \mathfrak{n}_\alpha$ and $0 \neq f_\alpha \in \mathfrak{n}^-_\alpha$, then

$$[e_\alpha, f_\alpha] \in \mathfrak{h}$$

is proportionate to the coroot $h_{\check{\alpha}}$.

The half-sum of the positive roots, denoted ρ, is an element of $\mathfrak{h}^*$, and the half-sum of positive coroots will be denoted by $\check{\rho}$; the latter is en element of $\mathfrak{h}$.

Let $U(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$. As a vector space and a $(\mathfrak{n}^-, \mathfrak{n})$-bimodule, it is isomorphic to

$$U(\mathfrak{n}^-) \otimes U(\mathfrak{h}) \otimes U(\mathfrak{n}). \tag{1.1}$$

Modules over $U(\mathfrak{g})$ as an associative algebra are the same as $\mathfrak{g}$-modules. We will denote this category by $\mathfrak{g}$-mod.

1.2. **Verma modules.** Let λ be a weight of $\mathfrak{g}$, i.e., an element of $\mathfrak{h}^*$. We introduce the Verma module $M_\lambda \in \mathfrak{g}$-mod as follows. For any object $\mathcal{M} \in \mathfrak{g}$-mod,

$$Hom_{\mathfrak{g}}(M_\lambda, \mathcal{M}) = Hom_{\mathfrak{b}}(\mathbb{C}^\lambda, \mathcal{M}),$$

where $\mathbb{C}^\lambda$ is the 1-dimensional $\mathfrak{b}$-module, on which $\mathfrak{b}$ acts through the character $\mathfrak{b} \to \mathfrak{h} \xrightarrow{\lambda} \mathbb{C}$.

By definition, we have:

Lemma 1.3. $M_\lambda \simeq U(\mathfrak{g}) \underset{U(\mathfrak{b})}{\otimes} \mathbb{C}^\lambda$.

By construction, M_λ is generated over $\mathfrak{g}$ by a vector, denoted v_λ, which is annihilated by $\mathfrak{n}$, and on which $\mathfrak{h}$ acts via the character λ.

Corollary 1.4. *The vector v_λ freely generates M_λ over $\mathfrak{n}^-$.*

(The latter means that the action of $\mathfrak{n}^-$ on v_λ defines an isomorphism $U(\mathfrak{n}^-) \to M_\lambda$.)

Proof. This follows from the decomposition $U(\mathfrak{g}) \simeq U(\mathfrak{n}^-) \otimes U(\mathfrak{b})$, similar to (1.1). □

1.5. **The action of $\mathfrak{h}$.**

Proposition 1.6. *The action of $\mathfrak{h}$ on M_λ is locally finite and semi-simple. The eigenvalues* [1] *of $\mathfrak{h}$ on M_λ are of the form*

$$\lambda - \sum_{\alpha \in \Delta^+} n_\alpha \cdot \alpha,\ n_\alpha \in \mathbb{Z}^+.$$

Proof. We have

$$M_\lambda = \bigcup_i U(\mathfrak{n}^-)_i \cdot v_\lambda,$$

where $U(\mathfrak{n}^-)_i$ is the i-th term of the PBW filtration on $U(\mathfrak{n}^-)$. Each $U(\mathfrak{n}^-)_i \cdot v_\lambda$ is a finite-dimensional subspace of M_λ, and we claim that it is $\mathfrak{h}$-stable (this would prove the locally finiteness assertion of the proposition).

Indeed, $U(\mathfrak{n}^-)_i$ is spanned by elements of the form

$$f_{\alpha_1} \cdot ... \cdot f_{\alpha_j},\ j \leq i,$$

where $f_\alpha \in \mathfrak{n}^-_\alpha$. The action of an element $h \in \mathfrak{h}$ on $(f_{\alpha_1} \cdot ... \cdot f_{\alpha_j}) \cdot v_\lambda$ is given by

$$\sum_{1 \leq k \leq j} (f_{\alpha_1} \cdot ... \cdot [h, f_{\alpha_k}] \cdot ... \cdot f_{\alpha_j}) \cdot v_\lambda + (f_{\alpha_1} \cdot ... \cdot f_{\alpha_j}) \cdot h(v_\lambda) = (\lambda - \sum_{1 \leq k \leq j} \alpha_k) \cdot (f_{\alpha_1} \cdot ... \cdot f_{\alpha_j}) \cdot v_\lambda.$$

The above formula also computes the weights of $\mathfrak{h}$ on M_λ. □

Corollary 1.7. *The multiplicities of weights of $\mathfrak{h}$ on M_λ are finite.*

Proof. Let μ be a weight of $\mathfrak{h}$ on M_λ, and let us write

$$\lambda - \mu = \Sigma\, n_\alpha \cdot \alpha,\ n_\alpha \in \mathbb{Z}^+.$$

We claim that

$$\sum_\alpha n_\alpha \leq \langle \lambda - \mu, \check{\rho} \rangle.$$

Indeed, this follows from the fact that $\langle \alpha, \check{\rho} \rangle \geq 1$ (the equality being achieved for simple roots).

Hence, the number of possibilities for n_α's to produce a given μ is bounded. □

In the sequel, for an element $\nu \in Q^+$ we will refer to the integer $\langle \nu, \check{\rho} \rangle$ as its length, and denote it by $|\nu|$.

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[1] If $\mathfrak{h}$ acts locally finitely and semi-simply on a vector space, its eigenvalues are called weights. We will adopt this terminology.

1.8. **The case of $\mathfrak{sl}_2$.** Let us see what Verma modules look like explicitly in the simplest case of $\mathfrak{g} = \mathfrak{sl}_2$. The parameter λ amounts to a complex number $l := \langle \lambda, \check{\alpha} \rangle$, where $\check{\alpha}$ is the unique coroot of $\mathfrak{sl}_2$.

Then the weights of $\mathfrak{h}$ on M_λ are of the form $\lambda - n \cdot \alpha$, i.e., $l - 2n$; we will denote by $v_{l'}$ the weight vector $f^n \cdot v_l$.

Proposition 1.9. *The $\mathfrak{sl}_2$-module M_l is irreducible unless $l \in \mathbb{Z}^+$. In the latter case, it fits into a short exact sequence*

$$0 \to M_{-l-2} \to M_l \to V_l \to 0,$$

where V_l is a finite-dimensional $\mathfrak{sl}_2$-module of highest weight l.

Proof. Suppose M_l contains a proper submodule, call it N. Then N is $\mathfrak{h}$-stable, and in particular, $\mathfrak{h}$-diagonalizable. Let l' be the maximal weight of $\mathfrak{h}$ appearing in N, i.e., $l' = l - 2n$, where n is the minimal integer such that $l - 2n$ is a weight of N. We have $l' \neq l$, since otherwise v_l would belong to N, but we know that v_l generates the entire M_l.

Since $e \cdot v_{l'}$ is either 0, or has weight $l' + 2$, the maximality assumption on l' forces the former option. Thus, we obtain:

$$e \cdot f^n \cdot v_l = \sum_{1 \le i \le n} f^{n-i} \cdot [e,f] \cdot f^{i-1} \cdot v_l + f^n \cdot e \cdot v_l = \sum_{1 \le i \le n} -2(i-1) \cdot f^{n-1} \cdot h \cdot v_l = n(l-(n-1)) \cdot f^{n-1} \cdot v_l,$$

which implies that $l = n - 1$.

The same computation shows that if $l = n - 1$ for a positive integer n, then $e \cdot v_{l-2n} = 0$, i.e., the vector $v_{l-2n} \in M_l$ generates a module isomorphic to M_{l-2n}.

□

1.10. **Irreducible quotients of Verma modules.** We return to the case of a general $\mathfrak{g}$.

Theorem 1.11. *The Verma module M_λ admits a unique irreducible quotient module.*

We will denote the resulting irreducible quotient of M_λ by L_λ.

Proof. Let $N \subset M_\lambda$ be a proper submodule. Since N is $\mathfrak{h}$-stable, it's a direct sum of weight spaces, i.e.,

$$N \simeq \bigoplus_{\mu} N_\lambda(\mu).$$

Note that since λ appears with multilicity 1 in M_λ, it is is not among the weights of N, for otherwise N would contain v_λ, and hence the entire M_λ.

Let ${}^0M_\lambda$ be the union over all N as above. It is still a direct sum of weight spaces:

$${}^0M_\lambda \simeq \bigoplus_{\mu} {}^0M_\lambda(\mu),$$

and ${}^0M_\lambda(\lambda) = 0$. Hence, ${}^0M_\lambda$ is still a proper submodule of M_λ, and by construction, it is maximal.

Therefore, $L_\lambda := M_\lambda / {}^0M_\lambda$ does not contain proper submodules, and hence is irreducible.

□

Lemma 1.12. *For $\lambda \neq \lambda'$, the modules L_λ and $L_{\lambda'}$ are non-isomorphic.*

Proof. Suppose the contrary. Then λ' must appear among the weights of L_λ, and in particular, among those of M_λ. I.e., λ' must be of the form $\lambda - Q^+$. By the same logic, $\lambda \in \lambda' - Q^+$, which is a contradiction. □

1.13. **Definition of the Category $\mathcal{O}$.** We define the Category $\mathcal{O}$ to be the full subcategory of $\mathfrak{g}$-mod, consisting of representations $\mathcal{M}$, satisfying the following three properties:

- The action of $\mathfrak{n}$ on $\mathcal{M}$ is locally finite. (I.e., for every $v \in \mathcal{M}$, the subspace $U(\mathfrak{n}) \cdot v \subset \mathcal{M}$ is finite-dimensional.)
- The action of $\mathfrak{h}$ on $\mathcal{M}$ is locally finite and semi-simple.
- $\mathcal{M}$ is finitely generated as a $\mathfrak{g}$-module.

Lemma 1.14. *Verma modules belong to the category $\mathcal{O}$.*

Proof. The only thing to check is that $\mathfrak{n}$ acts locally finitely on M_λ. Let $U(\mathfrak{g})_i$ be the i-th term of the PBW filtration on M_λ. It is enough to check that the finite-dimensional subspace subspace $U(\mathfrak{g})_i \cdot v_\lambda \subset M_\lambda$ if $\mathfrak{n}$-stable.

For $u \in U(\mathfrak{g})_i$ and $x \in \mathfrak{g}$ we have:

$$x \cdot (u \cdot v_\lambda) = [x, u] \cdot v_\lambda + u \cdot (x \cdot v_\lambda),$$

where the second term is 0 if $x \in \mathfrak{n}$. Hence, our assertion follows from the fact that $[\mathfrak{g}, U(\mathfrak{g})_i] \subset U(\mathfrak{g})_i$. □

Lemma 1.15. *If $\mathcal{M} \in \mathcal{O}$ and $\mathcal{M}'$ is a $\mathfrak{g}$-submodule of $\mathcal{M}$, then $\mathcal{M}' \in \mathcal{O}$, and similarly for quotient modules. The category $\mathcal{O}$ is Noetherian.*

Proof. The algebra $U(\mathfrak{g})$ is Noetherian, i.e., a submodule of a f.g. module is f.g. This makes the first assertion of the lemma evident.

If $\mathcal{M} \in \mathcal{O}$ and $\mathcal{M}_i \subset \mathcal{M}$ is an increasing chain of submodules, its union is f.g. as a $\mathfrak{g}$-module, and hence the chain stabilizes.

The assertion about quotient modules is evident.

□

We shall shortly prove that the category $\mathcal{O}$ is in fact Artinian, i.e., every object has a finite length.

By Lemma 1.12, the irreducible objects of $\mathcal{O}$ are exactly the modules L_λ for $\lambda \in \mathfrak{h}^*$.

1.16. **Some properties of $\mathcal{O}$.**

Lemma 1.17. *The action of $\mathfrak{n}$ on every object of $\mathcal{O}$ is locally nilpotent.*

Proof. Given $\mathcal{M}$, we have to show that every vector $v \in \mathcal{M}$ is contained in a finite-dimensional subspace W, stable under $\mathfrak{n}$, and which admits an $\mathfrak{n}$-stable filtration, on whose subquotients the action of $\mathfrak{n}$ is trivial.

Take $W = U(\mathfrak{b}) \cdot v$. This is a finite-dimensional $\mathfrak{b}$-module, by assumption. We claim that any $\mathfrak{b}$-module admits a filtration with the required properties.

Indeed, by Lie's theorem, W admits a filtration with 1-dimensional subquotients. Now the assertion follows from the fact that every 1-dimensional $\mathfrak{b}$-module is acted on via $\mathfrak{b} \twoheadrightarrow \mathfrak{h}$, since $\mathfrak{n} = [\mathfrak{b}, \mathfrak{b}]$.

We can also argue using Engel's theorem rather than Lie's theorem. We have to show that every element of $\mathfrak{n}$ acts nilpotently on W. Let us show that for any string of basis elements $e_{\alpha_1}, ..., e_{\alpha_n}$, which is long enough and any $v' \in W$, the vector

$$e_{\alpha_1} \cdot ... \cdot e_{\alpha_n} \cdot v' \in W$$

is zero.

Assume that v' has weight λ, and let n' be the maximum of $|\mu - \lambda|$ as μ ranges over the weights that appear in W that are bigger than λ. Then, as in the proof of Corollary 1.7, we obtain that any string as above of length $n > n'$ annihilates v'.

□

We shall now prove the following useful technical assertion:

Proposition 1.18. *Every object in the category $\mathcal{O}$ is a quotient of a finite successive extension of Verma modules.*

Proof. Let $\mathcal{M}$ be an object of $\mathcal{O}$, and let W be a finite-dimensional vector space that generates it. By assumption $W' := U(\mathfrak{b}) \cdot W$ is still finite-dimensional.

Let us regard W' is a $\mathfrak{b}$-module, and consider the $\mathfrak{g}$-module

$$\mathcal{M}' := U(\mathfrak{g}) \underset{U(\mathfrak{b})}{\otimes} W'.$$

As in the proof of Lemma 1.14, one easily shows that $\mathcal{M}'$ belongs to $\mathcal{O}$. We have an evident surjection $\mathcal{M}' \twoheadrightarrow \mathcal{M}$.

We claim now that $\mathcal{M}'$ is a successive extension of modules, each isomorphic to a Verma module. For that, it is enough to show that W' is a successive extension of 1-dimensional $\mathfrak{b}$-modules. But this has been established in the course of the proof of the previous lemma.

□

Corollary 1.19. *For $\mathcal{M} \in \mathcal{O}$, let $\underset{\mu}{\oplus} \mathcal{M}(\mu)$ be the decomposition into the weight saces. Then each $\mathcal{M}(\mu)$ is finite-dimensional.*

Proof. Proposition 1.18 reduces the assertion to the case $\mathcal{M} = \mathcal{M}_\lambda$, and we are done by Corollary 1.7.

□

Corollary 1.20. *Every object of $\mathcal{O}$ admits a non-zero map from some M_λ.*

Proof. Let $\mathcal{M}' \twoheadrightarrow \mathcal{M}$ be a surjection given by Proposition 1.18, and let $\mathcal{M}'_i$ be the corresponding filtration on $\mathcal{M}'$. Let i be the minimal index, such that the map $\mathcal{M}'_i \to \mathcal{M}$ is non-zero. Hence, we obtain a non-zero map $\mathcal{M}'_i/\mathcal{M}'_{i-1} \to \mathcal{M}$, and $\mathcal{M}'_i/\mathcal{M}'_{i-1}$ is isomorphic to a Verma module, by assumption.

□

2. Chevalley and Harish-Chandra isomorphisms

2.1. **The Chevalley isomorphism.** Consider the space $\mathrm{Fun}(\mathfrak{g}) := \mathrm{Sym}(\mathfrak{g}^*)$ of polynomial functions on $\mathfrak{g}$. The group G acts on this space by conjugation. We are interested in the space of G-invariants, $\mathrm{Fun}(\mathfrak{g})^G$. Since G is connected, this is the same as the space of invariants for the Lie algebra.

Lemma 2.2.

(1) *The restriction of any element $a \in \mathrm{Fun}(\mathfrak{g})$ to $\mathfrak{b} \subset \mathfrak{g}$ comes by means of pull-back from a polynomial function on $\mathfrak{h} \simeq \mathfrak{b}/\mathfrak{n}$.*

(2) *The resulting functionon $\mathfrak{h}$ is W-invariant.*

Proof. It is convenient to identify $\mathfrak{g} \simeq \mathfrak{g}^*$. Under this identification $\mathfrak{b}^* \simeq \mathfrak{g}/\mathfrak{n} \simeq \mathfrak{b}^-$, and the projection $\mathfrak{b} \to \mathfrak{h}$ corresponds to the embedding of algebras $\mathrm{Sym}(\mathfrak{h}) \to \mathrm{Sym}(\mathfrak{b}^-)$.

Since $\mathrm{Sym}(\mathfrak{b}^-) \simeq \mathrm{Sym}(\mathfrak{h}) \otimes \mathrm{Sym}(\mathfrak{n}^-)$, the H-invariance of a implies that it belongs to $\mathrm{Sym}(\mathfrak{h})$ (for otherwise its weight will be a non-zero element of $-Q^+$).

Since $\mathfrak{h} \subset \mathfrak{b}$ projects isomorphically onto $\mathfrak{b}/\mathfrak{n}$, the resulting homomorphism $\mathrm{Fun}(\mathfrak{g})^G \to \mathrm{Fun}(\mathfrak{h})$ can be also realized as restriction under $\mathfrak{h} \hookrightarrow \mathfrak{g}$.

Let $N(H)$ be the normalizer of $\mathfrak{h}$ in G. Since H is commutative, $H \subset N(H)$, and we have $N(H)/H \simeq W$. The restriction maps

$$\mathrm{Fun}(\mathfrak{g})^G \to \mathrm{Fun}(\mathfrak{h})^{N(H)} \simeq \mathrm{Fun}(\mathfrak{h})^W.$$

□

Let us denote the resulting map $\mathrm{Fun}(\mathfrak{g})^G \to \mathrm{Fun}(\mathfrak{h})^W$ by ϕ_{cl}.

Theorem 2.3. *The map ϕ_{cl} is an isomorphism.*

Proof. Let us first show that the map in question is injective. Suppose the contrary: let a be a G-invariant polynomial function on $\mathfrak{g}$ that vanishes on $\mathfrak{h}$. Since every semi-simple element of $\mathfrak{g}$ can be conjugated into $\mathfrak{h}$, we obtain that a vanishes on all semi-simple elements. Since the latter are Zariski dense in $\mathfrak{g}$, we obtain that $a = 0$.

The proof of surjectivity is based on considering trace functions, corresponding to finite-dimensional representations of $\mathfrak{g}$. Namely, for $\lambda \in P^+$ and a natural number n consider the function on $\mathfrak{g}$ equal to

$$a_{\lambda,n} := Tr(x^n, V^\lambda),$$

where V^λ is the finite-dimensional representation with highest weight λ. It is clear that $a_{\lambda,n}$ belongs to $\mathrm{Fun}(\mathfrak{g})^G$.

In addition, for $\lambda \in P^+$ and $n \in \mathbb{Z}^+$ we can consider the function on $\mathfrak{h}$ defined as

$$b_{\lambda,n}(x) := \sum_{w \in W} \lambda(x)^n.$$

Lemma 2.4. *The functions $b_{\lambda,n}(x)$ span $\mathrm{Fun}(\mathfrak{h})^W \cap \mathrm{Sym}^n(\mathfrak{h}^*)$ as a vector space.*

Proof. Since the category of finite-dimensional representations of W is semi-simple, it is enough to see that the elements λ^n span the vector space $\mathrm{Sym}^n(\mathfrak{h}^*)$. But this is a general assertion from linear algebra:

Let $v_1, ..., v_n$ be a basis of a vector space V. Then the monomials of the form $\left(\Sigma\, m_i \cdot v_i\right)^n$ with $m_i \in \mathbb{Z}^+$ span $\mathrm{Sym}^n(V)$. □

The surjectivity assertion of the theorem follows by induction on $|\lambda|$ from the next statement:

Lemma 2.5. $a_{\lambda,n}|_{\mathfrak{h}} = \frac{1}{|\,\mathrm{stab}_W(\lambda)|} \cdot b_{\lambda,n} + \sum_{\lambda' < \lambda} c' \cdot b_{\lambda',n}$*, where c' are scalars.*

Proof. For $x \in \mathfrak{h}$, we have:

$$a_{\lambda,n}(x) = \sum_{\mu} \mu(x)^n,$$

where the sum is taken over the set of weights of V^λ with multiplicities. Recall that this set is W-invariant.

All weights of V^λ are of the form $w(\lambda')$, where $w \in W$ and $\lambda' \leq \lambda$, and λ itself appears with multiplicity 1.

□

□

Let us denote by $\mathfrak{h}//W$ the variety $\mathrm{Spec}(\mathrm{Fun}(\mathfrak{h})^W)$. The map ϕ_{cl} can be interpreted as a map $\mathfrak{g} \to \mathfrak{h}//W$. Let us see what this map amounts to in the case $\mathfrak{g} = \mathfrak{sl}_n$. In this case $\mathfrak{h} \simeq \ker(\mathbb{C}^n \to \mathbb{C})$, and the algebra $\mathrm{Fun}(\mathfrak{h})^W$ is generated by $n-1$ elementary symmetric functions:

$$a_i := x \mapsto Tr(\Lambda^i(x)),\ i = 2, ..., n.$$

The map $\mathfrak{sl}_n \to \mathrm{Spec}(a_2, ..., a_n)$ assigns to a matrix x its characteristic polynomial.

Later we will prove the following:

Theorem 2.6. $\mathrm{Fun}(\mathfrak{h})^W$ *is isomorphic to a polynomial algebra on r generators, where $r = \dim(\mathfrak{h})$.*

2.7. **A deviation: invariant functions on the group.** Let $\mathrm{Fun}(G)$ denote the space of polynomial functions on the group G. It is acted on by G on the left and on the right, and in particular, by conjugation. Consider the space $\mathrm{Fun}(G)^G$ of conjugation-invariant functions.

Proceeding as in the case of the Lie algebra, the restriction defines a map

$$\mathrm{Fun}(G)^G \hookrightarrow \mathrm{Fun}(G) \twoheadrightarrow \mathrm{Fun}(B),$$

and we show that its image in fact belongs to $\mathrm{Fun}(H)$, which maps to $\mathrm{Fun}(B)$ by means of the pull-back under $B \twoheadrightarrow H$.

The map $\mathrm{Fun}(G)^G \to \mathrm{Fun}(H)$ can be also defined directly as the restriction under $H \hookrightarrow B \hookrightarrow G$, which implies that the image of this map lies in $\mathrm{Fun}(H)^W$.

As in the case of $\mathfrak{g}$, we show the following:

Theorem 2.8. *There above map $\mathrm{Fun}(G)^G \to \mathrm{Fun}(H)^W$ is an isomorphism.*

Proof. The injectivity statement follows as in the Lie algebra case. To prove the surjectivity we will again use finite-dimensional representations:

Let $\mathrm{Rep}(G)$ be the category of finite-dimensional representations of G. Let $K(\mathrm{Rep}(G))$ be its Grothendieck group. Since representations can be tensored, $K(\mathrm{Rep}(G))$ has a natural commutative ring string structure. By taking traces of elements of G on finite-dimensional representations, we obtain a ring homomorphism

$$K(\mathrm{Rep}(G)) \underset{\mathbb{Z}}{\otimes} \mathbb{C} \to \mathrm{Fun}(G)^G.$$

We claim that the composed map $K(\mathrm{Rep}(G)) \underset{\mathbb{Z}}{\otimes} \mathbb{C} \to \mathrm{Fun}(H)^W$ is an isomorphism. Indeed, both algebras have bases, parametrized by elements of P^+: to each dominant integral weight λ we assign $[V^\lambda] \in K(\mathrm{Rep}(G))$ this is $[V^\lambda]$,

$$B_\lambda := \sum_{w \in W} w(\lambda) \in \mathrm{Fun}(H)^W.$$

The same proof as in the case of Lie algebras shows that the image of $[V^\lambda]$ in $\mathrm{Fun}(H)^W$ is a linear combination of B_λ (with the same non-zero coefficient) and $B_{\lambda'}$ with $\lambda' < \lambda$, implying the isomorphism statement.

In particular, we obtain that the map $\mathrm{Fun}(G)^G \to \mathrm{Fun}(H)^W$ is surjective.

□

Note that $K(\mathrm{Rep}(G))$ is isomorphic to a polynomial algebra on r variables. Indeed, it is generated by V^{ω_i}, where ω_i are the fundamental weights:

$$\langle \omega_i, \check{\alpha}_j \rangle = \delta_{i,j}$$

for the simple coroots α_j.

To prove that these elements freely generate $K(\mathrm{Rep}(G))$ one has to use the fact that

$$V^\lambda \otimes V^\mu \simeq V^{\lambda+\mu} \oplus \bigoplus_\nu V^\nu,\ \nu < \lambda + \mu.$$

2.9. **The Harish-Chandra homomorphism.** Let $Z(\mathfrak{g})$ denote the center of the universal enveloping algebra $U(\mathfrak{g})$.

Lemma 2.10. $Z(\mathfrak{g}) = \{u \in U(\mathfrak{g}), \mid [x,u] = 0, \forall x \in \mathfrak{g}\}$.

Let $Z(\mathfrak{g})_i$ be the intersection of $Z(\mathfrak{g})$ with the i-th term of the PBW filtration on $U(\mathfrak{g})$. For each i we have a short exact sequence of $\mathfrak{g}$-modules (under conjugation):

$$0 \to U(\mathfrak{g})_{i-1} \to U(\mathfrak{g})_i \to \mathrm{Sym}^i(\mathfrak{g}) \to 0. \tag{2.1}$$

The functor of $\mathfrak{g}$-invariants is clearly left-exact. Hence, for each i we obtain an embedding

$$Z(\mathfrak{g})_i / Z(\mathfrak{g})_{i-1} \hookrightarrow \mathrm{Sym}^i(\mathfrak{g})^{\mathfrak{g}}. \tag{2.2}$$

However, since the category of finite-dimensional representations of $\mathfrak{g}$ is semi-simple, the short exact sequence (2.1) splits. In particular, the map of (2.2) is an isomorphism. This implies that we have an isomorphism of algebras:

$$gr(Z(\mathfrak{g})) \simeq \mathrm{Sym}(\mathfrak{g})^{\mathfrak{g}}.$$

Our present goal is to construct a homomorphism $\phi : Z(\mathfrak{g}) \to U(\mathfrak{h}) \simeq \mathrm{Sym}(\mathfrak{h})$. Consider the $\mathfrak{g}$-module

$$M_{univ} := U(\mathfrak{g})/U(\mathfrak{g}) \cdot \mathfrak{n} \simeq U(\mathfrak{g}) \underset{U(\mathfrak{n})}{\otimes} \mathbb{C} \simeq U(\mathfrak{g}) \underset{U(\mathfrak{b})}{\otimes} U(\mathfrak{h}).$$

The last interpretaion makes it clear that M_{univ} is naturally a $(\mathfrak{g}, \mathfrak{h})$-bimodule; in particular, we have an action on it of $U(\mathfrak{h})$ by endomorphisms of the $\mathfrak{g}$-module structure.

For $\lambda \in \mathfrak{h}^*$,

$$M_{univ} \underset{U(\mathfrak{h})}{\otimes} \mathbb{C}^{\lambda} \simeq M_{\lambda}.$$

Note that we have a canonical embedding of $U(\mathfrak{h})$ into M_{univ} as a vector space, given by

$$U(\mathfrak{h}) \simeq U(\mathfrak{b}) \underset{U(\mathfrak{b})}{\otimes} U(\mathfrak{h}) \to U(\mathfrak{g}) \underset{U(\mathfrak{b})}{\otimes} U(\mathfrak{h}).$$

In fact, as a $(\mathfrak{n}^-, \mathfrak{h})$-bimodule, M_{univ} is isomorphic to $U(\mathfrak{n}^-) \otimes U(\mathfrak{h})$.

Lemma 2.11. *For $a \in Z(\mathfrak{g})$ its image in M_{univ} belongs to the image of $U(\mathfrak{h})$. The resulting map $Z(\mathfrak{g}) \to U(\mathfrak{h})$ is an algebra homomorphism.*

Proof. The first assertion follows just as in Lemma 2.2 by looking at the adjoint action of $\mathfrak{h}$. Let us denote the resulting map by ϕ.

The map of $Z(\mathfrak{g})$ to M_{univ}, which we will denote by ϕ, can be interpreted as the action on the generating vector $v_{univ} \in M_{univ}$, using the $\mathfrak{g}$-module structure. The map of $U(\mathfrak{h})$ to M_{univ} equals the action on the same vector using the $\mathfrak{h}$-module structure. I.e., for $a \in Z(\mathfrak{g})$, we have:

$$a \cdot v_{univ} \simeq v_{univ} \cdot \phi(a).$$

For $a_1, a_2 \in Z(\mathfrak{g})$ we have:

$$a_1 \cdot (a_2 \cdot v_{univ}) = a_1 \cdot (v_{univ} \cdot \phi(a_2)) = (a_1 \cdot v_{univ}) \cdot \phi(a_2) = (v_{univ} \cdot \phi(a_1)) \cdot \phi(a_2),$$

implying that the map ϕ respects products.

□

Corollary 2.12. *An element $a \in Z(\mathfrak{g})$ acts on any M_λ as a multiplication by the scalar equal to $\phi(a)(\lambda)$, where we view $\phi(a)$ as an element of*

$$U(\mathfrak{h}) \simeq \mathrm{Sym}(\mathfrak{h}) \simeq \mathrm{Fun}(\mathfrak{h}^*).$$

2.13. **The dotted action.** We introduce a new action of W on $\mathfrak{h}^*$ as follows:

$$w \cdot \lambda = w(\lambda + \rho) - \rho.$$

(Of course, the automorphism of $\mathfrak{h}^*$ given by $\lambda \mapsto \lambda - \rho$ intertwines this action with the usual one.) This action induces a new W-action on $\mathrm{Fun}(\mathfrak{h}^*) \simeq \mathrm{Sym}(\mathfrak{h}) \simeq U(\mathfrak{h})$.

Theorem 2.14. *The map ϕ defines an isomorphism*

$$Z(\mathfrak{g}) \to \mathrm{Sym}(\mathfrak{h})^{W,\cdot}.$$

The rest of this section is devoted to the proof of this theorem. Let us first show that the image of ϕ indeed lands in $\mathrm{Sym}(\mathfrak{h})^{W,\cdot}$. Let a be an element of $Z(\mathfrak{g})$. By Corollary 2.12, and since W is generated by its simple reflections, it is sufficient to show that the action of any a on M_λ and $M_{s_i \cdot \lambda}$ is given by the same scalar.

Furthermore, instead of all λ's, it sufficient to consider a Zariski-dense subset, which we take to be P^+. In this case, our assertion follows from the next result:

Lemma 2.15. *Let λ be any weight and α_i a simple root, such that $\langle \lambda + \rho, \check{\alpha}_i \rangle \in \mathbb{Z}^+$. Then the Verma module M_λ contains $M_{s_i \cdot \lambda}$ as a submodule.*

Proof. Set $n = \langle \lambda + \rho, \check{\alpha}_i \rangle$ and note that $s_i \cdot \lambda = \lambda - n \cdot \alpha_i$. The proof will be a straightforward generalization of the analysis used in the proof of Proposition 1.9.

Consider the vector $f_i^n \cdot v_\lambda \in M_\lambda$. It is non-zero and it has weight $\lambda - n \cdot \alpha_i$. We claim that it generates a Verma module with this highest weight. This amounts to checking that

$$\mathfrak{n} \cdot (f_i^n \cdot v_\lambda) = 0.$$

Since $\mathfrak{n}$ is generated as a Lie algebra by simple roots, it is enough to show that $e_j \cdot f_i^n \cdot v_\lambda = 0$. If $j \neq i$, this is evident, since $[f_i, e_j] = 0$. If $j = i$, this is the same computation as we did in the $\mathfrak{sl}_2$-case.

□

2.16. **Passage to the associated graded.** Thus, ϕ is indeed a homomorphism $Z(\mathfrak{g}) \to U(\mathfrak{h})^{W,\cdot}$. Let us consider the PBW filtartions on both sides: $Z(\mathfrak{g}) = \cup Z(\mathfrak{g})_i$ and $U(\mathfrak{h})^{W,\cdot} = U(\mathfrak{h})_i^{W,\cdot}$, where the latter is by definition the intersection of $U(\mathfrak{h})^{W,\cdot}$ with $U(\mathfrak{h})_i$. It is clear that ϕ is compatible with the filtartions, i.e., that it sends $Z(\mathfrak{g})_i$ to $U(\mathfrak{h})_i^{W,\cdot}$, thereby inducing a map

$$gr(Z(\mathfrak{g})) \to gr(U(\mathfrak{h})^{W,\cdot}) \tag{2.3}$$

We claim that the map of (2.3) essentially coincides with the Chevalley map ϕ_{cl}, and hence is an isomorphism. Granting the last assertion, the proof of Theorem 2.14 follows from the next statement:

Lemma 2.17. *Let V^1, V^2 be two (positively) filtered vector spaces, and $V^1 \to V^2$ a map, compatible with filtrations. Assume that $gr(V^1) \to gr(V^2)$ is an isomorphism. Then the initial map is an isomorphism.*

Let us now analyze the map (2.3). As we have seen already, $gr(Z(\mathfrak{g}))$ is isomorphic to $\mathrm{Sym}(\mathfrak{g})^{\mathfrak{g}}$. Since W is a finite group (and hence its category of representations in char. 0 is semi-simple), we have also the isomorphism $gr(U(\mathfrak{h})^{W,\cdot}) \simeq \mathrm{Sym}(\mathfrak{h})^W$. (Note that the W-action induced on $\mathrm{Sym}(\mathfrak{h}) \simeq gr(U(\mathfrak{h}))$ is the usual, the undotted one.)

Finally, the resulting map $\mathrm{Sym}(\mathfrak{g})^{\mathfrak{g}} \to \mathrm{Sym}(\mathfrak{h})$ is described as follows. Taking an element $a \in \mathrm{Sym}(\mathfrak{g})^{\mathfrak{g}}$ we consider it modulo the ideal generated by $\mathfrak{n}$, and it turns out to be an element in $\mathrm{Sym}(\mathfrak{h}) \subset \mathrm{Sym}(\mathfrak{g}/\mathfrak{n})$. But this coincides with the definition of ϕ_{cl}, once we identify $\mathfrak{g} \simeq \mathfrak{g}^*$.

3. Further properties of the Category $\mathcal{O}$

3.1. Decomposition of the Category $\mathcal{O}$ with respect to the center.

Proposition 3.2. *The action of $Z(\mathfrak{g})$ on every object $\mathcal{M} \in \mathcal{O}$ factors though an ideal of finite codimension.*

Proof. By Proposition 1.18, the assertion reduces to the case when $\mathcal{M} = M_\lambda$ for some $\lambda \in \mathfrak{h}^*$. In the latter case, the action is given by the evaluation at the maximal ideal corresponding to

$$Z(\mathfrak{g}) \simeq \mathrm{Sym}(\mathfrak{h})^{W,\cdot} \hookrightarrow \mathrm{Sym}(\mathfrak{h}) \xrightarrow{\lambda} \mathbb{C}.$$

□

Corollary 3.3. *Every object $\mathcal{M}$ of $\mathcal{O}$ splits as a direct sum*

$$\mathcal{M} \simeq \underset{\chi \in \mathrm{Spec}(Z(\mathfrak{g}))}{\oplus} \mathcal{M}_\chi,$$

where $Z(\mathfrak{g})$ acts on each $\mathcal{M}_\chi$ via some power of the maximal ideal corresponding to χ. For a morphism $\mathcal{M} \to \mathcal{N}$, the components $\mathcal{M}_\chi \to \mathcal{N}_{\chi'}$ for $\chi \neq \chi'$ are 0.

One can reformulate the above corollary by saying that category $\mathcal{O}$ splits into a direct sum of blocks, denoted $\mathcal{O}_\chi$, parametrized by points of $\mathrm{Spec}(Z(\mathfrak{g}))$. By definition, $\mathcal{O}_\chi$ is the subcategory of $\mathcal{O}$, consisting of modules, on which the center $Z(\mathfrak{g})$ acts by the generalized character χ.

Let us denote by $\mathfrak{h}//W$ the algebraic variety $\mathrm{Spec}\Big(\mathrm{Sym}(\mathfrak{h})^{W,\cdot}\Big)$, which we identify with $\mathrm{Spec}(Z(\mathfrak{g}))$ by the Harish-Chandra isomorphism, and let ϖ denote the natural map $\mathfrak{h} \twoheadrightarrow \mathfrak{h}//W$. Let us recall the following general assertion from algebraic geometry/Galois theory:

Lemma 3.4. *Let Γ be a finite group acting on an affine scheme $X := \mathrm{Spec}(A)$. Denote by $X//\Gamma$ the scheme $\mathrm{Spec}(A^\Gamma)$. We have:*

(1) *The morphism $X \to X//\Gamma$ is finite.*

(2) *$\mathbb{C}$-points of $X//\Gamma$ are in bijection with Γ-orbits on the set of $\mathbb{C}$-points of X.*

Hence, we obtain that points of $\mathfrak{h}//W$ are in bijection with W-orbits on $\mathfrak{h}$ under the dotted action. I.e.,

$$\varpi(\lambda) = \varpi(\mu) \Leftrightarrow \mu \in W \cdot \lambda. \tag{3.1}$$

Proposition 3.5. *If L_μ is is isomorphic to a subquotient of M_λ, then $\mu = w \cdot \lambda$ for some $w \in W$.*

Proof. Being a quotient of M_μ, the module L_μ belongs to $\mathcal{O}_{\varpi(\mu)}$. If it is also a subquotient of M_λ, then it also belongs to $\mathcal{O}_{\varpi(\lambda)}$. By (3.1) we arrive to the assertion of the corollary.

□

Finally, we are ready to prove the following:

Theorem 3.6. *Every object of $\mathcal{O}$ has a finite length.*

Proof. By Proposition 1.18, it is enough to show that the Verma module M_λ has a finite length for every λ. By Corollary 3.5 its only possible irreducible subquotients are of the form $L_{w\cdot\lambda}$ for $w \in W$, in particular there is only a finite number of them.

Hence, it suffices to show that for each μ the multiplicity $[L_\mu, M_\lambda]$ is *a priori bounded.* The latter means by definition that whenever we have a filtration $M_\lambda = \underset{i}{\cup} \mathcal{M}_i$, then the number of indices i, for which L_μ is isomorphic to a subquotient of $\mathcal{M}_i/\mathcal{M}_{i-1}$, is bounded.

Indeed, since $L_\mu(\mu) \simeq \mathbb{C}$, the above multiplicity cannot exceed $\dim(\mathcal{M}_\lambda(\mu))$, and the latter is finite by Corollary 1.7.

□

3.7. **Dominance and anti-dominance.**

Definition 3.8. *We shall say that a weight $\lambda \in \mathfrak{h}^*$ is dominant if*

$$w(\lambda) - \lambda \notin \{Q^+ - 0\} \tag{3.2}$$

For any $w \in W$.

We shall say that λ is anti-dominant if $-\lambda$ is dominant. Here is an explicit combinatorial description of the dominace/anti-dominance condition:

Theorem 3.9. *A weight λ is dominant if and only if for all $\alpha \in \Delta^+$,*

$$\langle \lambda, \check{\alpha} \rangle \neq -1, -2, ... \tag{3.3}$$

The proof relies on the combinatorics of the affine Weyl group, and will be omitted. (We will not use this theorem explicitly.) Nonetheless, let us analyze some particular cases. First, we claim that the condition stated in the theorem is necessary for dominance:

Indeed, if for some $\alpha \in \Delta^+$,

$$\langle \lambda, \check{\alpha} \rangle = n \in \mathbb{N},$$

then

$$s_\alpha(\lambda) = \lambda + n \cdot \alpha,$$

contradicting the dominance assumption. Hence, the statement of the theorem is that it is enough to check (3.2) only for the reflections in W.

Assume now that λ is integral, i.e., $\langle \lambda, \check{\alpha} \rangle \in \mathbb{Z}$ for all roots α. Since every positive root is a sum of simple roots with non-negative integral coefficients, it is enough to check (3.3) for the simple roots, in which case it becomes equivalent to λ being inside the dominant Weyl chamber. Let us prove that in the latter case (3.2) holds:

Proof. Suppose that $w(\lambda) = \lambda + \nu$ with $\nu \in Q^+$. Consider the W-invariant scalar product $(\cdot, \cdot)$ on $\mathfrak{h}$. We have:

$$(\lambda, \lambda) = (w(\lambda), w(\lambda)) = (\lambda + \nu, \lambda + \nu) = (\lambda, \lambda) + (\nu, \nu) + 2(\lambda, \nu).$$

However, $(\nu, \nu) > 0$, and $(\lambda, \nu) \geq 0$, since

$$(\lambda, \alpha) = \langle \lambda, \check{\alpha} \rangle \cdot \frac{(\alpha, \alpha)}{2}.$$

This is a contradiction.

□

Finally, to get a hint of what the argument proving Theorem 3.9 in the general case is, assume that λ is such that

$$\langle \lambda, \check{\alpha} \rangle \notin \mathbb{Z}$$

for any α. In this case we claim that $w(\lambda) - \lambda \notin Q$ (here Q denotes the root lattice, i.e., the integral span of simple roots).

Proof. Since we are dealing with a linear algebra problem defined over $\mathbb{Q}$, we can replace the field $\mathbb{C}$ by $\mathbb{R}$, i.e., we can assume that λ is real.

Consider the affine Weyl group $W_{aff} = W \ltimes Q$, which acts naturally on $\mathfrak{h}^*$. This is a group generated by reflections parametrized by pairs $(\alpha \in \Delta^+, n \in \mathbb{Z})$, where each such reflection acts as follows:

$$\lambda \mapsto \lambda - \alpha \cdot \Big(\langle \lambda, \check{\alpha} \rangle - n \Big).$$

The fixed loci of these reflections are affine hyperplanes in $\mathfrak{h}^*$. The connected components of the complement to their union in $\mathfrak{h}^*_{\mathbb{R}}$ are called open alcoves. Their closures are called alcoves.

The condition on λ implies that it belongs to the interior of one of the alcoves. The claim now follows from the fact that each alcove is a fundamental domain for W_{aff}.

□

Proposition 3.10.

(1) *Assume that $\lambda \in \mathfrak{h}^*$ is such that $\lambda+\rho$ is dominant. Then the Verma module M_λ is projective as an object of $\mathcal{O}$.*

(2) *Assume that $\lambda \in \mathfrak{h}^*$ is such that $\lambda + \rho$ is anti-dominant. Then the Verma module M_λ is irreducible.*

Proof. Assume first that $\lambda + \rho$ is anti-dominant. If M_λ wasn't irreducible, it would contain at least one irredicuble submodule, i.e., we have a map $L_\mu \to V_\lambda$, for some $\mu \neq \lambda$. Since L_μ and M_λ must belong to the same $\mathcal{O}_\chi$, we obtain that $\mu \in W \cdot \lambda$, or, equivalently, $\mu + \rho = w(\lambda + \rho)$ for some $w \in W$.

However, since μ is among the weights of M_λ, we obtain $\mu \in \lambda - (Q^+ - 0)$, and hence $(\mu + \rho) \in (\lambda + \rho) - (Q^+ - 0)$. This contradicts the assumption on $\lambda + \rho$.

Assume now that $\lambda+\rho$ is dominant, and let $\mathcal{M} \twoheadrightarrow M_\lambda$ be a surjection. To show that it splits, we need to find a vector $v \in \mathcal{M}$ of weight λ, and which is annihilated by $\mathfrak{n}^+$, and which maps to v_λ under the above map.

First, we may assume that $\mathcal{M}$ is acted on by the same generalized central character as M_λ, i.e., that $\mathcal{M} \in \mathcal{O}_{\varpi(\lambda)}$. Secondly, since the action of $\mathfrak{h}$ on $\mathcal{M}$ is semi-simple, there always exists some vector v of weight λ mapping to v_λ. We claim that v is automatically annihilated by $\mathfrak{n}^+$.

Indeed, let us look at the vector space $U(\mathfrak{n}^+) \cdot v \subset \mathcal{M}$. By Lemma 1.17, this vector space contains a vector, denote it w, annihilated by $\mathfrak{n}^+$. The weight of this vector, call it μ, belongs to $\lambda + (Q^+ - 0)$. We obtain that there exists a non-zero map $M_\mu \to \mathcal{M}$, and we claim that this is a contradiction:

On the one hand, $(\mu + \rho) \in (\lambda + \rho) + (Q^+ - 0)$, and on the other hand $(\mu + \rho) \in W(\lambda + \rho)$, contradicting the assumption on $\lambda + \rho$.

□

3.11. **Behavior of $\mathcal{O}_\chi$ for various χ.** Consider first the case when $\chi = \varpi(-\rho)$. The Verma module $M_{-\rho}$ is both irreducible and projective, and it is the only irreducible object in $\mathcal{O}_\chi$. Hence, this category is equivalent to that of finite-dimensional vector spaces.

Let now λ be such that $\langle \lambda, \check{\alpha} \rangle \notin \mathbb{Z}$ for any $\alpha \in \Delta^+$, and take $\chi = \varpi(\lambda)$. Since

$$\langle s_i \cdot \lambda, \check{\alpha} \rangle = \langle \lambda, s_i(\check{\alpha}) \rangle - \langle \alpha_i, \check{\alpha} \rangle,$$

we obtain that the for all $\mu \in W \cdot \lambda$,

$$\langle \mu, \check{\alpha} \rangle \notin \mathbb{Z},$$

in particular, all these weights are distinct, and $\mu + \rho$ are both dominant and anti-dominant.

Hence, we have a collection of $|W|$ distinct objects $M_\mu \in \mathcal{O}_\chi$, $\mu \in W \cdot \lambda$, all of which are irreducible and projective. Hence, $\mathcal{O}_\chi$ is still semi-simple and is equivalent to the direct sum of $|W|$ many copies of the category of finite-dimensional vector spaces.

Let us now consider another extreme (and the most interesting) case, when $\chi = \varpi(\lambda)$, where λ is such that

$$\langle \lambda, \check{\alpha} \rangle \in \mathbb{Z}^{\geq 0}.$$

For example, $\lambda = 0$ is such a weight.

The weight $\lambda + \rho$ is regular and belongs to the dominant Weyl chamber, i.e., all $w \cdot \lambda$ are all distinct. In particular, the irreducibles L_μ, $\mu \in W \cdot \lambda$ are pairwise non-isomorphic.

Proposition 3.12. *Under the above circumstances the category $\mathcal{O}_\chi$ is indecomposable, i.e., it can't be non-trivially represented as a direct sum of abelian subcategories.*

Proof. Let $\mathcal{C}$ be an Artinian abelian category, which is equivalent to a direct sum $\mathcal{C}_1 \oplus \mathcal{C}_2$. Then the set $Irr(\mathcal{C})$ of isomorphism classes of irreducible objects in $\mathcal{C}$ can be decomposed as a union $Irr(\mathcal{C}_1) \cup Irr(\mathcal{C}_2)$, since no irreducible can belong to both $\mathcal{C}_1$ and $\mathcal{C}_2$. Moreover, no indecomposable object of $\mathcal{C}$ can contain elements of both $Irr(\mathcal{C}_1)$ and $Irr(\mathcal{C}_2)$ in its Jordan-Hölder series.

Hence, the proposition follows from either point (1) or point (2) of the next lemma:

Lemma 3.13.

(1) *The object M_λ contains every other M_μ as a submodule.*

(2) *Every M_μ contains $M_{w_0 \cdot \lambda}$ as a submodule, where w_0 is the longest element in the Weyl group.*

Indeed, using point (1), the assertion follows from the fact that M_λ is indecomposable (because it has a unique irreducible quotient by Theorem 1.11), and that L_μ is a quotient of M_μ. Hence, if we had a decomposition $\mathcal{O}_\chi = \mathcal{C}_1 \oplus \mathcal{C}_2$ with $L_\lambda \in Irr(\mathcal{C}_1)$, we would obtain that all other L_μ also belong to $Irr(\mathcal{C}_1)$.

Using point (2) of the lemma, we obtain that if we had a decomposition $\mathcal{O}_\chi = \mathcal{C}_1 \oplus \mathcal{C}_2$ with $L_{w_0 \cdot \lambda} \in Irr(\mathcal{C}_1)$, then $L_\mu \in Irr(\mathcal{C}_1)$ for all other μ.

□

Proof. (of the lemma)

Let $s_{i_n} \cdot s_{i_{n-1}} \cdot ... \cdot s_{i_2} \cdot s_{i_1}$ be a reduced expression of some element $w \in W$. Let us denote by w^k, $k = 0, ..., n$ the elements

$$s_{i_k} \cdot s_{i_{k-1}} \cdot ... \cdot s_{i_2} \cdot s_{i_1},$$

i.e., $w^n = w$ and $w^0 = 1$.

We claim, by induction, that for each k, there exists an embedding

$$M_{w^k \cdot \lambda} \hookrightarrow M_{w^{k-1} \cdot \lambda}.$$

Indeed, by Lemma 2.15, it is enough to show that

$$\langle (w^{k-1} \cdot \lambda) + \rho, \check{\alpha}_{i_k} \rangle = \langle w^{k-1}(\lambda + \rho), \check{\alpha}_{i_k} \rangle = \langle \lambda + \rho, (w^{k-1})^{-1}(\check{\alpha}_{i_k}) \rangle \in \mathbb{N}.$$

Note for each k the string $s_{i_1} \cdot s_{i_2} \cdot ... \cdot s_{i_{k-1}} \cdot s_{i_k}$ is *a reduced expression* for $(w^{k-1})^{-1} \cdot s_k = (w^k)^{-1}$. In particular $(w^{k-1})^{-1}(\alpha_{i_k}) \in \Delta^+$. Hence, our assertion follows from the assumption on $\lambda + \rho$.

□

Let us now present a slightly different argument proving point (2) of the lemma:

Proof. Let $L_{\mu'}$ be an irreducible submodule of M_μ. This implies that there is a non-zero map $M_{\mu'} \to M_\mu$. But any map between Verma modules is an injection, since $U(\mathfrak{n}^-)$ has no zero-divisors. Hence, $M_{\mu'} \simeq L_{\mu'}$, in particular, $M_{\mu'}$ is irreducible. We claim that $\mu' + \rho$ is anti-dominant, and, hence, must equal $w_0(\lambda + \rho)$, since the latter is the only anti-dominant weight in the orbit.

Indeed, consider $\langle \mu' + \rho, \check{\alpha}_i \rangle$ for all simple roots α_i. All these numbers are non-zero integers, and if one of them was positive, we would have that $M_{\mu'}$ contains $M_{s_i \cdot \mu'}$ as a submodule, by Lemma 2.15, contradicting the irreducibility.

□

3.14. **Contragredient duality.** Let $\mathfrak{g}\text{-mod}^{\mathfrak{h}-ss}$ be the full subcategory of $\mathfrak{g}$-mod consisting of objects, on which the action of $\mathfrak{h}$ is semi-simple. If $\mathcal{M}$ is such a module we will denote by $\underset{\mu}{\oplus}\, \mathcal{M}(\mu)$ its decomposition into weight spaces.

Consider the full linear dual $\mathcal{M}^*$ of $\mathcal{M}$; it is naturally a $\mathfrak{g}$-module.

Lemma 3.15.

(1) *For a $\mathfrak{g}$-module $\mathcal{M}$ and a Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$, the maximal subspace of $\mathcal{M}$, on which the action of $\mathfrak{h}$ is locally finite, is $\mathfrak{g}$-stable.*

(2) *For $\mathcal{M} \in \mathfrak{g}\text{-mod}^{\mathfrak{h}-ss}$, the maximal subspace of $\mathcal{M}^*$ on which the action of $\mathfrak{h}$ is locally finite is $\underset{\mu}{\oplus}\, \mathcal{M}(\mu)^*$.*

Proof. For point (1), if $V \subset \mathcal{M}$ is a finite-dimensional $\mathfrak{h}$-stable subspace, then $\mathfrak{g} \cdot V$ is still finite-dimensional and $\mathfrak{h}$-stable.

Point (2) is obvious.

□

Let τ be the Cartan involution on $\mathfrak{g}$. This is the unique automorphism of $\mathfrak{g}$, which acts as -1 on $\mathfrak{h}$, and maps $\mathfrak{b}$ to $\mathfrak{b}^-$. For $\mathcal{M} \in \mathfrak{g}\text{-mod}^{\mathfrak{h}-ss}$, we define another object $\mathcal{M}^\vee \in \mathfrak{g}\text{-mod}^{\mathfrak{h}-ss}$ to be $\underset{\mu}{\oplus}\, \mathcal{M}(\mu)^*$ as a vector space, with the action of $\mathfrak{g}$ (which is well-defined by the previous lemma), twisted by τ.

Evidently, $\mathcal{M} \mapsto \mathcal{M}^\vee$ is a well-defined, contravariant and exact functor

$$\mathfrak{g}\text{-mod}^{\mathfrak{h}-ss} \to \mathfrak{g}\text{-mod}^{\mathfrak{h}-ss}.$$

Theorem 3.16. *If $\mathcal{M}$ belongs to $\mathcal{O}$, then so does $\mathcal{M}^\vee$.*

Before giving a proof, let us make the following observation. Let $\mathfrak{g}\text{-mod}^{\mathfrak{h}-ss,fd}$ be the full subcategory of $\mathfrak{g}\text{-mod}^{\mathfrak{h}-ss}$, defined by the condition that the weight spaces $\mathcal{M}(\mu)$ are finite-dimensional. Evidently, this subcategory is preserved by $\mathcal{M} \mapsto \mathcal{M}^\vee$; in fact $\mathcal{M}^\vee(\mu) \simeq \mathcal{M}(\mu)^*$; moreover,

$$(\mathcal{M}^\vee)^\vee \simeq \mathcal{M}. \tag{3.4}$$

Therefore, the above functor is a contravariant self-equivalence of $\mathfrak{g}\text{-mod}^{\mathfrak{h}-ss,fd}$. Since $\mathcal{O} \subset \mathfrak{g}\text{-mod}^{\mathfrak{h}-ss,fd}$, Theorem 3.16 implies that the contragredient duality induces a contravariant self-equivalence of $\mathcal{O}$ as well.

Proof. (of the theorem)

By Theorem 3.6, the assertion of the theorem follows from the next proposition:

Proposition 3.17. *For any $\lambda \in \mathfrak{h}^*$, there exists an isomorphism $L_\lambda^\vee \simeq L_\lambda$.*

□

Proof. (of the proposition)

By (3.4), the $\mathfrak{g}$-module $L_\lambda^\vee$ is irreducible. Hence, by Theorem 1.11, it suffices to construct a non-trivial map $M_\lambda \to L_\lambda^\vee$. The latter amounts to finding a vector of weight λ, which is annihilated by $\mathfrak{n}^+$.

Consider the functional $M_\lambda \to \mathbb{C}$, given by the projection on the λ-weight space. In fact, it factors through

$$M_\lambda \to M_\lambda/\mathfrak{n}^- \cdot M_\lambda \to \mathbb{C}. \tag{3.5}$$

By the construction of L_λ, the above functional also factors through

$$M_\lambda \to L_\lambda \to \mathbb{C}.$$

By (3.5), the resulting functional on L_λ is 0 on the subspace $\mathfrak{n}^- \cdot L_\lambda \subset L_\lambda$.

When we view this functional as an element of L_λ^*, it belongs to $L_\lambda(\lambda)^*$, and hence to $L_\lambda^\vee(\lambda)$. Moreover, since

$$(\mathcal{M}/\mathfrak{n}^- \cdot \mathcal{M})^* \simeq (\mathcal{M}^*)^{\mathfrak{n}^-},$$

from the definition of τ, we obtain that the resulting vector in $L_\lambda^\vee$ is annihilated by $\mathfrak{n}^+$.

□

3.18. **Dual Verma modules.** We shall now study the modules $M_\lambda^\vee$ for $\lambda \in \mathfrak{h}^*$. First, let us characterize these modules functorially:

Lemma 3.19. *For $\mathcal{M} \in \mathfrak{g}\text{-}mod^{\mathfrak{h}-ss,fd}$,* [2] *the space $Hom(\mathcal{M}, M_\lambda^\vee)$ is canonically isomorphic to the space of functionals $\mathcal{M}_{\mathfrak{n}^-}(\lambda) \to \mathbb{C}$.*

In the above formula, the subscript $\mathfrak{n}^-$ designates $\mathfrak{n}^-$-coinvariants, i.e., the space $\mathcal{M}/\mathfrak{n}^- \cdot \mathcal{M}$.

Proof. The space of all functionals $\mathcal{M}(\lambda) \to \mathbb{C}$ is of course isomorphic to $\mathcal{M}^\vee(\lambda)$. The condition that such a functional factors through $\mathcal{M} \to \mathcal{M}/\mathfrak{n}^- \cdot \mathcal{M} \to \mathbb{C}$ is equivalent to the fact that the corresonding vector of $\mathcal{M}^\vee$ is annihilated by $\mathfrak{n}^+$. The latter space identifies by the definition of Verma modules with $Hom(M_\lambda, \mathcal{M}^\vee)$.

Since contragredient duality is a self-equivalence of $\mathfrak{g}\text{-mod}^{\mathfrak{h}-ss,fd}$, we have:

$$Hom(\mathcal{M}, M_\lambda^\vee) \simeq Hom(M_\lambda, \mathcal{M}^\vee),$$

implying the assertion of the lemma.

□

We shall now study how Verma modules interact with contragredient Verma modules.

Theorem 3.20.

(1) *The module $M_\lambda^\vee$ has L_λ as its unique irreducible submodule.*

(2) *$Hom(M_\lambda, M_\lambda^\vee) \simeq \mathbb{C}$, such that $1 \in \mathbb{C}$ corresponds to the composition*

$$M_\lambda \twoheadrightarrow L_\lambda \hookrightarrow M_\lambda^\vee.$$

(3) *For $\lambda \neq \mu$,*

$$Hom(M_\lambda, M_\mu^\vee) = 0.$$

(4) *$Ext^1_{\mathcal{O}}(M_\lambda, M_\mu^\vee) = 0$ for all λ, μ.*

---

[2]A slightly more careful analysis shows that the assertion remains valid more generally for any $\mathcal{M} \in \mathfrak{g}\text{-mod}^{\mathfrak{h}-ss}$.

Proof. Point (1) follows from Proposition 3.17, combined with Theorem 1.11, since $\mathcal{M} \mapsto \mathcal{M}^\vee$ is an equivalence. To calculate $Hom(M_\lambda, M_\mu^\vee)$ we will use Lemma 3.19. We obtain that the above Hom equals the space of functionals on M_λ, which are non-zero only on the weight component $M_\lambda(\mu)$, and which are 0 on $\mathfrak{n}^- \cdot M_\lambda$.

However, since M_λ is free over $\mathfrak{n}^-$,

$$M_\lambda \simeq (\mathfrak{n}^- \cdot M_\lambda) \oplus \mathbb{C}^\lambda,$$

implying both (2) and (3).

To prove (4), let

$$0 \to M_\mu^\vee \to \mathcal{N} \to M_\lambda \to 0 \tag{3.6}$$

be a short exact sequence, and we need to show that it splits.

From Lemma 3.19, we have a canonical $\mathfrak{b}^-$-invariant functional $M_\mu^\vee \to \mathbb{C}^\mu$, and let $\mathcal{N}'$ denote its kernel. Consider also the short exact sequence of $\mathfrak{b}^-$-modules:

$$0 \to \mathbb{C}^\mu \to \mathcal{N}/\mathcal{N}' \to M_\lambda \to 0. \tag{3.7}$$

We claim that splitting (3.6) as $\mathfrak{g}$-modules is equivalent to splitting (3.7) as $\mathfrak{b}^-$-modules. Indeed, this follows immediately from Lemma 3.19.

Since M_λ is free as a $\mathfrak{n}^-$-module generated by v_λ, splitting (3.7) as $\mathfrak{b}^-$-modules amounts to finding in $\mathcal{N}/\mathcal{N}'$ a vector of weight λ. This is possible, since the action of $\mathfrak{h}$ on $\mathcal{N}$, and hence on $\mathcal{N}'$, is semi-simple.

□

We would now like to describe how dual Verma modules look as vector spaces with a $\mathfrak{n}^+$-action , parallel to the description of M_λ as $U(\mathfrak{n}^-)$ as a $\mathfrak{n}^-$-module.

Let N^+ be the algebraic group, corresponding to $\mathfrak{n}^+$. The category of its algebraic representations is (more or less by definition) equivalent to the subcategory of $\mathfrak{n}^+$-mod, consisting of locally nilpotent representations.

Let $\text{Fun}(N^+)$ be the space of regular functions on N^+; this is an N^+-representation under the action of N^+ on itself by left translations. For any other $V \in \text{Rep}(N^+)$, we have:

$$Hom_{N^+}(V, \text{Fun}(N^+)) \simeq Hom_{Vect}(V, \mathbb{C}). \tag{3.8}$$

Proposition 3.21. *For any λ, we have an isomorphism of $\mathfrak{n}^+$-modules:*

$$M_\lambda^\vee \simeq \text{Fun}(N^+).$$

Proof. The action of $\mathfrak{n}^+$ on every object of $\mathcal{O}$ is locally nilpotent by Lemma 1.17, hence we can regard it as an object of $\text{Rep}(N^+)$.

Consider the canonical map $v_\lambda : M_\lambda^\vee \to \mathbb{C}$, and let

$$M_\lambda^\vee \to \text{Fun}(N^+) \tag{3.9}$$

be the map that corresponds to it by (3.8). We claim that the latter is an isomorphism.

First, we claim that it is injective. Suppose not, and let $\mathcal{N}$ be its kernel. Then $\mathcal{N}$ is the collection of all vectors $\psi \in M_\lambda^\vee$, such that $v_\lambda\Big(U(\mathfrak{n}^+) \cdot \psi\Big) = 0$. This subspace is clearly $\mathfrak{h}$-stable. It identifies with the set of functionals $\psi : M_\lambda \to \mathbb{C}$, which are non-zero on finitely many weight subspaces, such that

$$\psi\Big(U(\mathfrak{n}^-) \cdot v_\lambda\Big) = 0.$$

However, since v_λ generates M_λ over $\mathfrak{n}^-$, the latter space is zero.

To prove that the map (3.9) is surjective we will count dimensions. Consider the adjoint action of H on N^+. It is easy to see that (3.9) maps $M^\vee_\lambda(\mu+\lambda)$ to $\mathrm{Fun}(N^+)(\mu)$, so it is enough to check that

$$\dim(\mathrm{Fun}(N^+)(\mu)) = \dim(M^\vee_\lambda(\mu+\lambda)).$$

Note that the latter equals

$$\dim(M_\lambda(\mu+\lambda)) = \dim(U(\mathfrak{n}^-)(\mu)) = \dim(\mathrm{Sym}(\mathfrak{n}^-)(\mu)).$$

As an H-scheme, N^+ is isomorphic to its Lie algebra $\mathfrak{n}^+$ under the exponential map. Hence,

$$\dim(\mathrm{Fun}(N^+)(\mu)) = \dim(\mathrm{Fun}(\mathfrak{n}^+)(\mu)) = \dim(\mathrm{Sym}(\mathfrak{n}^-)(\mu)).$$

□

Working out the details of the proof of the following theorem (which will be proved by another method later) is a good exercise:

Theorem 3.22. $Ext^2_{\mathcal{O}}(M_\lambda, M^\vee_\mu) = 0$ *for all* λ, μ.

Proof.

Step 1. Since $\mathcal{O}$ is a full subcategory of $\mathfrak{g}\text{-mod}^{\mathfrak{h}-ss}$, which is *stable under extensions*, the natural map

$$Ext^2_{\mathcal{O}}(M_\lambda, M^\vee_\mu) \to Ext^2_{\mathfrak{g}\text{-mod}^{\mathfrak{h}-ss}}(M_\lambda, M^\vee_\mu)$$

is an embedding.

Step 2. We claim that for any $\mathcal{M} \in \mathfrak{g}\text{-mod}^{\mathfrak{h}-ss}$ and any i,

$$Ext^i_{\mathfrak{g}\text{-mod}^{\mathfrak{h}-ss}}(M_\lambda, \mathcal{M}) \simeq \Big(H^i(\mathfrak{n}^+, \mathcal{M})\Big)(\lambda),$$

where we are using the fact that the Lie algebra cohomology with respect to $\mathfrak{n}^+$ of a module, endowed with an action of $\mathfrak{b}$, carries an action of $\mathfrak{h}$; and if the initial action of $\mathfrak{h} \subset \mathfrak{b}$ was semi-simple, then so is the action on cohomology.

Step 3. The Lie algebra cohomology $H^i(\mathfrak{n}^+, \mathrm{Fun}(N^+))$ vanishes for $i > 0$.

□

Remark As we shall see later, $Ext^i_{\mathcal{O}}(M_\lambda, M^\vee_\mu) = 0$ for all $i > 0$, but the above proof doesn't give that, since the maps

$$Ext^i_{\mathcal{O}}(M_\lambda, M^\vee_\mu) \to Ext^i_{\mathfrak{g}\text{-mod}^{\mathfrak{h}-ss}}(M_\lambda, M^\vee_\mu)$$

are not *a priori* embeddings for $i \geq 3$.

4. Projective objects in $\mathcal{O}$

4.1. **Construction of projectives.** Recall that an object $\mathcal{P}$ of an abelian category $\mathcal{C}$ is called projective if the functor $\mathcal{C} \to \mathrm{Ab} : \mathcal{M} \mapsto Hom(\mathcal{P}, \mathcal{M})$ is exact. We say that $\mathcal{C}$ has enough projectives if every object of $\mathcal{C}$ admits a surjection from a projective one.

Theorem 4.2. *The category* $\mathcal{O}$ *has enough projectives.*

Proof. It is enough to show that each of the categories $\mathcal{O}_\chi$ has enough projectives, so from now on we will fix χ.

Consider the functor on $\mathcal{O}_\chi$ that attaches to a module $\mathcal{M}$ the vector space $\mathcal{M}(\mu)$ for some weight μ. We claim that this functor is representable (by an object that we will denote $\mathcal{P}_{\mu,\chi}$). We claim that this implies the theorem:

Indeed, since the above functor is evidently exact, $\mathcal{P}_{\mu,\chi}$ is projective. For any object $\mathcal{M} \in \mathcal{O}_\chi$ we have a surjection of $\mathfrak{g}$-modules

$$\underset{\mu}{\oplus}\, \mathcal{P}_{\mu,\chi} \otimes \mathcal{M}(\mu) \to \mathcal{M}.$$

Since $\mathcal{M}$, being an object of $\mathcal{O}$, is finitely generated, in the above infinite direct sum one can find a direct sum over finitely many indices, such that its map to $\mathcal{M}$ will still be surjective.

Let us now prove the representability assertion. (This will basically repeat the proof of Proposition 3.10(1).) For an integer n, consider the quotient of $U(\mathfrak{g})$ be the left ideal, generated by elements of the form $x - \mu(x)$, $x \in \mathfrak{h}$, and $x_1 \cdot ... \cdot x_n$, $x_i \in \mathfrak{n}^+$. Let us denote the resulting module by $M_{\mu,n}$; for $n = 1$ we recove the Verma module M_μ. As in the case of Verma modules, we show that $M_{\mu,n}$ belongs to $\mathcal{O}$. For a given χ, let $M_{\mu,n,\chi}$ be the corresponding direct summand of $M_{\mu,n}$.

We claim that for fixed μ and χ and n large enough, the module $M_{\mu,n,\chi}$ will represent the functor $\mathcal{M} \mapsto \mathcal{M}(\mu)$ on $\mathcal{O}_\chi$.

Indeed, for any object $\mathcal{M} \in \mathcal{O}$, the set $Hom(M_{\mu,n}, \mathcal{M})$ is isomorphic to the set of elements of $\mathcal{M}(\mu)$, which are annihilated by any $x_1 \cdot ... \cdot x_n$, $x_i \in \mathfrak{n}^+$. We claim that for a fixed μ and χ, *any* vector in $\mathcal{M}(\mu)$ has this property, provided that n is large enough.

Let $\lambda_1, ..., \lambda_k$ be the set of weights such that $\varpi(\lambda_i) = \chi$. Let N a positive integer such that for no index i

$$N < \langle \lambda_i - \mu, \check{\rho} \rangle,$$

where we regard the inequality as empty unless the RHS is an integer. We claim that any $n \geq N$ will do.

Indeed, suppose that $v' := x_1 \cdot ... \cdot x_n \cdot v \neq 0$ for some $v \in \mathcal{M}(\mu)$. Then there exists a vector $v'' \in U(\mathfrak{n}^+) \cdot v'$, which is annihilated by $\mathfrak{n}^+$. Let μ' (resp., μ'') denote the weight of v' (resp., v''). We have $\mu'' - \mu' \in Q^+ - 0$, and

$$\langle \mu'' - \mu', \check{\rho} \rangle > \langle \mu' - \mu', \check{\rho} \rangle \geq n.$$

However, by assumption, μ'' must be one of the weights $\lambda_1, ..., \lambda_k$, which is a contradiction. □

4.3. **Standard filtrations.** We shall say that an object of $\mathcal{O}$ admits a standard filtration if it it admits a filtration whose subquotients are isomorphic to Verma modules. Later we shall give an intrinsic characterization of objects that admit a standard filtration.

Theorem 4.4. *Every projective object of $\mathcal{O}$ admits a standard filtration.*

Proof. Let us recall the objects $M_{\mu,n}$ introduced in the course of the proof of Theorem 4.2. (Note that these modules are not in general projective.) However, every projective object of $\mathcal{O}$ is a direct summand of a direct sum of some $M_{\mu,n}$. The assertion of the theorem follows, therefore, from the next two assertions:

Lemma 4.5. *Each $M_{\mu,n}$ admits a standard filtration.*

Lemma 4.6. *A direct summand of an object admitting a standard filtration itself admits a standard filtration.*

□

Proof. (of Lemma 4.5)

Let $U(\mathfrak{n})^+$ denote the augmentation ideal of $U(\mathfrak{n})$, i.e., $\mathfrak{n} \cdot U(\mathfrak{n}) = U(\mathfrak{n}) \cdot \mathfrak{n}$. We have:

$$M_{\mu,n} \simeq U(\mathfrak{g})/U(\mathfrak{g}) \cdot I_n,$$

where $I_n \subset U(\mathfrak{b})$ is the left (in fact two-sided) ideal equal to

$$(U(\mathfrak{n})^+)^n \cdot \ker(U(\mathfrak{h}) \to \mathbb{C}),$$

where the map $U(\mathfrak{h}) \to \mathbb{C}$ is the homomorphism corresponding to the character μ. For each n we have an inclusion $I_n \subset I_{n-1}$, and the quotient is a $\mathfrak{b}$-module, on which $\mathfrak{n}$ acts trivially, and $\mathfrak{h}$ acts as on $\mathrm{Sym}^n(\mathfrak{n}) \otimes \mathbb{C}^\mu$.

We have a sequence of surjections,

$$M_{\mu,n} \twoheadrightarrow M_{\mu,n-1} \twoheadrightarrow M_{\mu,n-2} \twoheadrightarrow \ldots \twoheadrightarrow M_\mu,$$

whose kernels, by the above, are isomorphic to direct sums of modules of the form $M_{\mu+\nu}$, $\nu \in Q^+$.

□

Proof. (of Lemma 4.6)

We shall first prove the following auxiliary assertion:

Let $\mathcal{M}$ be an object of $\mathcal{O}$ that admits a standard filtration. Let λ be a *maximal* weight with respect to the order relation $<$ on $\mathfrak{h}^*$ among the weights of $\mathcal{M}$ (I.e., for no μ with $\mathcal{M}(\mu) \neq 0$ we have $\mu - \lambda \in Q^+ - 0$.) Let v be a vector of weight λ. (The above condition implies that $\mathfrak{n}^+ \cdot v = 0$.) We claim that the resulting map $M_\lambda \to \mathcal{M}$ is an embedding, and the quotient $\mathcal{M}/M_\lambda$ also admits a standard filtration.

Indeed, let $\mathcal{M}_i$ be the standard filtration, and let i be the minimal index for which the image of M_λ belongs to $\mathcal{M}_i$. Hence, the map $M_\lambda \to \mathcal{M}^i := \mathcal{M}_i/\mathcal{M}_{i-1}$ is non-trivial. However, $\mathcal{M}^i$ is isomorphic to some M_μ. The condition on λ implies that $\mu = \lambda$; in particular, the map $M_\lambda \to \mathcal{M}^i$ is an isomorphism. Hence, we have a short exact sequence

$$0 \to \mathcal{M}_{i-1} \to \mathcal{M}/M_\lambda \to \mathcal{M}/\mathcal{M}_i \to 0,$$

implying our assertion.

Returning to the situation of the lemma, let $\mathcal{M} = \mathcal{M}^1 \oplus \mathcal{M}^2$ admit a standard filtration. We shall argue by a decreasing induction on the length of $\mathcal{M}$. Let λ be the maximal among the weights of $\mathcal{M}$. With no restriction of generality, we can assume that $\mathcal{M}^1(\lambda) \neq 0$, and let $M_\lambda \to \mathcal{M}^1$ be the corresponding map.

Consider the composition $M_\lambda \to \mathcal{M}^1 \to \mathcal{M}$, and let us apply the above assertion. We obtain that $M_\lambda \to \mathcal{M}^1$ is an injection and that $\mathcal{M}/M_\lambda \simeq \mathcal{M}^1/M_\lambda \oplus \mathcal{M}^2$ admits a standard filtration. This completes the induction step.

□

Corollary 4.7. *(of the proof) Let $\mathcal{M} \to M_\mu$ be a surjection, where $\mathcal{M}$ is a module that admits a standard filtration. Then the kernel of this map also admits a standard filtration.*

Proof. Let λ be the maximal among the weights of $\mathcal{M}$, and let $M_\lambda \subset \mathcal{M}$ be the corresponding submodule. If $\lambda \neq \mu$, then the composed map $M_\lambda \to \mathcal{M} \to M_\mu$ is 0, by the maximality assumption. Hence, we have a surjection $\mathcal{M}/M_\lambda \to M_\mu$, and we argue by induction.

If $\lambda = \mu$, we have $\ker(\mathcal{M} \to M_\mu) \simeq \mathcal{M}/M_\lambda$, and we are done.

□

4.8. **More on dual Verma modules.**

Proposition 4.9. *$Ext^i(M_\lambda, M_\mu^\vee) = 0$ for all $i > 0$ and all λ, μ.*

Proof. We will argue by induction on i. By Theorem 3.20, the assertion holds for $i = 1$. Let us perform the induction step. By the long exact sequence, $Ext^i_{\mathcal{O}}(\mathcal{M}, M_\mu^\vee) = 0$ for any $\mathcal{M}$ that admits a standard filtration.

Let $\mathcal{P}$ be a projective module that surjects on M_λ. By Corollary 4.7, the kernel $\mathcal{M}$ of this surjection admits a standard filtration. We have a long exact sequence:

$$... \to Ext^i_{\mathcal{O}}(\mathcal{M}, M^\vee_\mu) \to Ext^{i+1}_{\mathcal{O}}(M_\lambda, M^\vee_\mu) \to Ext^{i+1}_{\mathcal{O}}(\mathcal{P}, M^\vee_\mu) \to ...$$

However, $Ext^i_{\mathcal{O}}(\mathcal{M}, M^\vee_\mu) = 0$ by the induction hypothesis, and $Ext^{i+1}_{\mathcal{O}}(\mathcal{P}, M^\vee_\mu) = 0$, since $\mathcal{P}$ is projective. Hence, $Ext^{i+1}_{\mathcal{O}}(M_\lambda, M^\vee_\mu) = 0$.

□

Here is an intrinsic characterization of objects that admit a standard filtration:

Proposition 4.10. *For an object $\mathcal{M} \in \mathcal{O}$ the following conditions are equivalent:*
(1) $\mathcal{M}$ *admits a standard filtration.*
(2) $Ext^i_{\mathcal{O}}(\mathcal{M}, M^\vee_\mu) = 0$ *for any μ and $i > 0$.*
(3) $Ext^1_{\mathcal{O}}(\mathcal{M}, M^\vee_\mu) = 0$ *for any μ.*

Remark. Note that the proof given below uses only the following information from the previous proposition: namely, that $Ext^i(M_\lambda, M^\vee_\mu) = 0$ for $i = 1, 2$

Proof. The previous proposition implies that (1) $\Rightarrow$ (2). The fact that (2) implies (3) is a tautology. Let us show that (3) implies (1). We will argue by induction on the length of $\mathcal{M}$.

Let λ be the maximal among the weights that appear in $\mathcal{M}$. Then any vector of weight λ is annihilated by $\mathfrak{n}^+$. Hence, we obtain a well-defined map

$$M_\lambda \otimes \mathcal{M}(\lambda) \to \mathcal{M},$$

which induces an isomorphism on the λ-weight spaces. Let $\mathcal{N}_1$ and $\mathcal{N}_2$ be, respectively, the kernel and cokernel of the above map.

Lemma 4.11. *The module $\mathcal{N}_2$ also satisfies $Ext^1_{\mathcal{O}}(\mathcal{N}_2, M^\vee_\mu) = 0$ for any μ.*

Proof. We have the following long exact sequence:

$$... \to R^i Hom_{\mathcal{O}}(\mathcal{M}, M^\vee_\mu) \to R^i Hom_{\mathcal{O}}(M_\lambda \otimes \mathcal{M}(\lambda), M^\vee_\mu) \to$$
$$R^{i+1} Hom_{\mathcal{O}}\Big((M_\lambda \otimes \mathcal{M}(\lambda) \to \mathcal{M}), M^\vee_\mu\Big) \to R^{i+1} Hom_{\mathcal{O}}(\mathcal{M}, M^\vee_\mu) \to ...$$

Consider this sequence for $i = 0$. We claim that

$$Hom_{\mathcal{O}}(\mathcal{M}, M^\vee_\mu) \to Hom_{\mathcal{O}}(M_\lambda \otimes \mathcal{M}(\lambda), M^\vee_\mu)$$

is a surjection. Indeed, if $\mu \neq \lambda$, then $Hom(M_\lambda, M^\vee_\mu) = 0$, and we are done. For $\mu = \lambda$ we claim that the map

$$Hom_{\mathcal{O}}(\mathcal{M}, M^\vee_\lambda) \to Hom_{\mathcal{O}}(M_\lambda \otimes \mathcal{M}(\lambda), M^\vee_\lambda)$$

is an isomorphism. Indeed, the RHS is isomorphic to $\mathcal{M}(\lambda)^*$. The LHS is isomorphic to $\Big(\mathcal{M}(\lambda)^*\Big)^{\mathfrak{n}^-}$, by Lemma 3.19. But $\Big(\mathcal{M}(\lambda)^*\Big)^{\mathfrak{n}^-} \to \mathcal{M}(\lambda)^*$ is an isomorphism, by the maximality assumption on λ.

Therefore, since $Ext^1_{\mathcal{O}}(\mathcal{M}, M^\vee_\mu) = 0$, from the above long exact sequence we obtain that

$$R^1 Hom_{\mathcal{O}}\Big((M_\lambda \otimes \mathcal{M}(\lambda) \to \mathcal{M}), M^\vee_\mu\Big) = 0.$$

Consider now another long exact sequence:

$$.. \to R^{i-2} Hom_{\mathcal{O}}(\mathcal{N}_1, M^\vee_\mu) \to R^i Hom_{\mathcal{O}}(\mathcal{N}_2, M^\vee_\mu) \to$$
$$R^i Hom_{\mathcal{O}}\Big((M_\lambda \otimes \mathcal{M}(\lambda) \to \mathcal{M}), M^\vee_\mu\Big) \to R^{i-1} Hom_{\mathcal{O}}(\mathcal{N}_1, M^\vee_\mu) \to R^{i+1} Hom_{\mathcal{O}}(\mathcal{N}_2, M^\vee_\mu) \to ...$$

Applying it for $i = 1$ we obtain that $Ext^1_{\mathcal{O}}(\mathcal{N}_2, M^\vee_\mu)$ injects into $R^1Hom_{\mathcal{O}}\Big((M_\lambda \otimes \mathcal{M}(\lambda) \to \mathcal{M}), M^\vee_\mu\Big) = 0$, implying the assertion of the lemma.

□

By the induction hypothesis, we obtain that $\mathcal{N}_2$ admits a standard filtration; in particular, $Ext^2(\mathcal{N}_2, M^\vee_\mu) = 0$ for any μ. To finish the proof of the proposition, it suffices to show that $Hom(\mathcal{N}_1, M^\vee_\mu) = 0$ for all μ.

Consider the second long exact sequence appearing in the above lemma for $i = 1$. Our assertion follows from:

$$Ext^2(\mathcal{N}_2, M^\vee_\mu) = 0 \text{ and } R^1Hom_{\mathcal{O}}\Big((M_\lambda \otimes \mathcal{M}(\lambda) \to \mathcal{M}), M^\vee_\mu\Big) = 0,$$

both of which have been established above.

□

4.12. **Formal properties of $\mathcal{O}$.** One can formalize much of the above discussion as follows. Let $\mathcal{C}$ be an Artinian category, whose set of irreducibles $\mathrm{Irr}(\mathcal{C})$ is endowed with a partial ordering that we denote by $\leq$.

Suppose that for each $\lambda \in \mathrm{Irr}(\mathcal{C})$ we are given two objects M_λ, and $M^\vee_\lambda$ and a diagram

$$M_\lambda \twoheadrightarrow L_\lambda \hookrightarrow M^\vee_\lambda,$$

where L_λ denotes the corresponding irreducible. Moreover, we assume the following:

- (1) L_λ is the unique irreducible quotient object (resp., sub-object) of M_λ (resp., $M^\vee_\lambda$).
- (2) The Jordan-Hölder series $\ker(M_\lambda \to L_\lambda)$ (resp., $\mathrm{coker}(L_\lambda \to M^\vee_\lambda)$) consists of L_μ with $\mu < \lambda$ in the above order relation.
- (3) $Ext^i(M_\lambda, M^\vee_\mu) = 0$ for $i = 1, 2$ and all $\lambda, \mu \in \mathrm{Irr}(\mathcal{C})$.

Then the following assertions hold:

Proposition 4.13.

- *(1) $Ext^i(M_\lambda, L_\mu) = 0$ and $Ext^i(L_\mu, M^\vee_\lambda) = 0$ for $i > 0$ and $\mu \leq \lambda$.*
- *(2) $Ext^i(M_\lambda, M^\vee_\mu) = 0$ for $i > 0$ and all λ, μ.*
- *(3) An object $\mathcal{M} \in \mathcal{C}$ admits a standard (resp., co-standard) filtration if and only if $Ext^1(\mathcal{M}, M^\vee_\mu) = 0$ (resp., $Ext^1(M_\mu, \mathcal{M}) = 0$) for all μ.*

Henceforth, we will continue to work with the usual category $\mathcal{O}$, but all the assertions of categorical nature will be valid in this more general framework.

4.14. **BGG reciprocity.** Let us fix $\chi \in \mathfrak{h}//W$ and consider the category $\mathcal{O}_\chi$. This is an Artinian category with finitely many irreducibles and enough projectives.

Lemma 4.15. *Any such category $\mathcal{C}$ is equivalent to that of finite-dimensional modules over a finite-dimensional associative algebra.*

Proof. Let L_α, $\alpha \in A$ be the irreducibles of $\mathcal{C}$. For each α let $\mathcal{P}_\alpha$ be a projective that maps non-trivially to L_α. Then $\mathcal{P} := \bigoplus_\alpha \mathcal{P}_\alpha$ is a projective generator of $\mathcal{C}$, i.e., the functor $\mathcal{C} \to \mathrm{Vect}$:

$$\mathcal{M} \mapsto Hom(\mathcal{P}, \mathcal{M}) \tag{4.1}$$

is exact and faithful. Note also that $\dim(Hom(\mathcal{M}_1, \mathcal{M}_2)) < \infty$, by Artinianness. In particular, the algebra $End(\mathcal{P})$ is finite-dimensional.

Under the above circumstances, it is easy to see that the functor (4.1) defines an equivalence between $\mathcal{C}$ and the category of finite-dimensional *right* modules over $End(\mathcal{P})$.

□

As in any Artinian category with enough projectives, every irreducible L_λ admits a projective cover. I.e., there exists an indecomposable projective object P_λ, which surjects onto L_λ. Moreover,

$$Hom(P_\lambda, L_\mu) = 0 \text{ for } \mu \neq \lambda. \tag{4.2}$$

Note that P_λ is defined up to (a non-unique !) isomorphism.

By Theorem 4.4, P_λ admits a filtration, whose subquotients are isomorphic to Verma modules. Let us denote by $\mathrm{mult}(M_\mu, P_\lambda)$ the number of occurences of M_μ as a subquotient in any such filtration.

Theorem 4.16. *(BGG reciprocity)* $\mathrm{mult}(M_\mu, P_\lambda) = [L_\lambda : M_\mu^\vee]$.

The notation $[L_\lambda : \mathcal{M}]$ means the multiplicity of the irreducible L_λ in the Jordan-Hölder series of a module $\mathcal{M}$. Before giving a proof, let us make several remarks.

Consider the Grothendieck group of the category $\mathcal{O}$. On the one hand, this is a free abelian group with a basis given by the classes of L_λ. On the other hand, since

$$[M_\lambda] = [L_\lambda] + \sum_{\lambda' < \lambda} [L_{\lambda'}],$$

we obtain that the elements $[M_\lambda]$ also form a basis for $K(\mathcal{O})$. The transition matrix between these two bases, i.e., the matrix

$$(\lambda, \mu) \mapsto [L_\lambda : M_\mu]$$

is called the Kazhdan-Lusztig matrix of the category.

In addition, since $L_\lambda^\vee \simeq L_\lambda$, we obtain that $[\mathcal{M}^\vee] = [\mathcal{M}]$ for any $\mathcal{M}$. In particular, $M_\lambda^\vee$ has the same Jordan-Hölder series as M_λ.

Finally, let us note that from Theorem 4.16 we obtain that

$$[P_\lambda] = [M_\lambda] + \sum_{\lambda' > \lambda} [M_{\lambda'}],$$

i.e., the elements $[P_\lambda]$ also form a basis for $K(\mathcal{O})$, numbered by the same set. Thus, Theorem 4.16 can be viewed as expressing a certain relation between the two transition matrices.

Proof. (of the theorem)

For λ and μ as in the theorem, consider the vector space

$$Hom(P_\lambda, M_\mu^\vee).$$

We claim that its dimension equals both $\mathrm{mult}(M_\mu, P_\lambda)$ and $[L_\lambda : M_\mu^\vee]$. This follows from the next two lemmas:

Lemma 4.17. *Let $\mathcal{M}$ be an object of $\mathcal{O}$ that admits a standard filtartion. Then* $\mathrm{mult}(M_\mu, \mathcal{M}) = \dim(Hom(\mathcal{M}, M_\mu^\vee))$.

Lemma 4.18. *For any $\mathcal{N} \in \mathcal{O}$,*

$$\dim(Hom(P_\lambda, \mathcal{N})) = [L_\lambda : \mathcal{N}].$$

The first lemma uses Proposition 4.9, and the second lemma follows from (4.2).

∎

4.19. **Some examples.** Note that $[L_\mu : M_{\mu'}]$ can be non-zero only when $\mu \leq \mu'$ and $\varpi(\mu) = \chi = \varpi(\mu')$. Let us consider the case when $\chi = \varpi(\lambda)$ with λ dominant integral.

Proposition 4.20. *For any $w \in W$,*

$$[L_{w_0\cdot\lambda}, M_{w\cdot\lambda}] = 1 \text{ and } [L_{w\cdot\lambda}, M_\lambda] \geq 1.$$

Proof. Let us return to the proof of Lemma 3.13. The fact that there exist embeddings

$$L_{w_0\cdot\lambda} \simeq M_{w_0\cdot\lambda} \hookrightarrow M_{w\cdot\lambda} \hookrightarrow M_\lambda$$

implies that

$$[L_{w_0\cdot\lambda}, M_{w\cdot\lambda}] \geq 1 \text{ and } [L_{w\cdot\lambda}, M_\lambda] \geq 1.$$

It remains to show that $M_{w\cdot\lambda}/M_{w_0\cdot\lambda}$ does not contain $M_{w_0\cdot\lambda}$ as a subquotient. This follows from an argument involving the notion of *functional dimension*:

We claim that to any finitely generated $\mathfrak{g}$-module $\mathcal{M}$ we can assign an integer, called its functional dimension. Namely, we can choose a filtration on $\mathcal{M}$ by vector subspaces $\mathcal{M}_i$, $i \in \mathbb{Z}^{\geq 0}$, such that $\mathfrak{g} \cdot \mathcal{M}_i \in \mathcal{M}_{i+1}$, and such that $\mathrm{gr}(\mathcal{M})$, regarded as a module over $\mathrm{gr}(U(\mathfrak{g})) \simeq \mathrm{Sym}(\mathfrak{g})$, is finitely generated. Such a filtration is called a "good" filtration. We set $\dim(\mathcal{M})$ to be $\dim(\mathrm{gr}(\mathcal{M}))$ in the algebro-geometric sense, i.e., the dimension of its support.

Theorem 4.21. $\dim(\mathcal{M})$ *is independent of the choice of a "good" filtration.*

If $\mathcal{M}'$ is a $\mathfrak{g}$-submodule of $\mathcal{M}$ and $\mathcal{M}_i$ is a good filtration on $\mathcal{M}$. Since the algebra $\mathrm{Sym}(\mathfrak{g})$ is Noetherian, $\mathcal{M}'_i := \mathcal{M}' \cap \mathcal{M}_i$ is a good filtration on $\mathcal{M}'$. From this it follows that

$$\dim(\mathcal{M}) = \max\Big(\dim(\mathcal{M}'), \dim(\mathcal{M}/\mathcal{M}')\Big).$$

Take $\mathcal{M} = M_\lambda$, realized as $U(\mathfrak{g}) \underset{U(\mathfrak{b})}{\otimes} \mathbb{C}^\lambda$. The PBW filtration on $U(\mathfrak{g})$ induces a filtration on M_λ, such that $\mathrm{gr}(M_\lambda) \simeq \mathrm{Sym}(\mathfrak{g}/\mathfrak{b}) \simeq \mathrm{Sym}(\mathfrak{n}^-)$. Hence, $\dim(M_\lambda) = \dim(\mathfrak{n}^-)$ for any λ.

To prove the assertion of the proposition it remains to show that $\dim(M_\lambda/\mathcal{M}') < \dim(\mathfrak{n}^-)$ for any proper submodule $\mathcal{M}' \subset M_\lambda$.

Indeed, for any such $\mathcal{M}'$ and the above choice of a "good" filtration on M_λ, the $\mathrm{Sym}(\mathfrak{g})$-module $\mathrm{gr}(M_\lambda/\mathcal{M}')$ is a proper quotient of $\mathrm{Sym}(\mathfrak{g}/\mathfrak{b})$. But since $\mathrm{Spec}(\mathrm{Sym}(\mathfrak{g}/\mathfrak{b})) \simeq \mathfrak{n}$ is irreducible as an algebraic variety, any of its proper sub-schemes has a strictly smaller dimension. $\square$

Exercise. Deduce from the above argument that any Verma module M_λ contains a unique irreducible submodule, which is itself isomorphic to a Verma module. (Do not confuse this fact with the uniqueness of an irreducible quotient of a Verma module.)

This following result, which we shall neither prove nor use gives a necessary and sufficient condition for $[L_\mu, M_\lambda]$ to be non-zero.

Theorem 4.22. *The following conditions are equivalent:*

(1) *M_λ contains M_μ as a submodule.*

(2) *M_λ contains L_μ as a subquotient.*

(3) *There exists a sequence of weights $\lambda = \mu_0, \mu_1, ..., \mu_{n-1}, \mu_n = \mu$, such that $\mu_{i+1} = s_{\beta_i} \cdot \mu_i$ for some $\beta \in \Delta^+$ and $\langle \mu_i, \check{\beta}_i \rangle \in \mathbb{Z}^{\geq 0}$.*

Exercise. Combine this theorem with Theorem 4.16 to show that the conditions stated in Proposition 3.10 are in fact "if and only if".

4.23. **Translation functors.** Let V be a finite dimensional $\mathfrak{g}$-module. We can consider the functor $T_V : \mathfrak{g}\text{-mod} \to \mathfrak{g}\text{-mod}$ given by

$$\mathcal{M} \mapsto \mathcal{M} \otimes V.$$

Evidently, this functor sends $\mathcal{O}$ to itself. This functor is exact, and its (both left and right) adjoint is given by T_{V^*}, where V^* is the dual representations. In particular, T_V sends projectives to projectives and injectives to injectives.

Lemma 4.24. *The module $M_\lambda \otimes V$ admits a filtration, whose subquotients are isomorphic to $M_{\lambda+\mu}$. Moreover,*

$$\mathrm{mult}(M_{\lambda+\mu}, M_\lambda \otimes V) = \dim(V(\mu)).$$

Proof. We have:

$$M_\lambda \otimes V \simeq U(\mathfrak{g}) \underset{U(\mathfrak{b})}{\otimes} (V \otimes \mathbb{C}^\lambda),$$

where V is regarded as a $\mathfrak{b}$-module. (This is a general fact about the induction functor.)

There exists a $\mathfrak{b}$-stable filtration $F_i(V)$ on V with 1-dimensional quotients. The occurence of $\mathbb{C}^\mu$ as a subquotient of this filtration equals $\dim(V(\mu))$. Hence, the induced filtration $U(\mathfrak{g}) \underset{U(\mathfrak{b})}{\otimes} (F_i(V) \otimes \mathbb{C}^\lambda)$ on $M_\lambda \otimes V$ has the required properties.

$\square$

Let χ_1, χ_2 be two points of $\mathfrak{h}^*//W$. Consider the composition:

$$T_{\chi_1,V,\chi_2} : \mathcal{O}_{\chi_1} \xrightarrow{\imath_{\chi_1}} \mathcal{O} \xrightarrow{T_V} \mathcal{O} \overset{\mathfrak{p}_{\chi_2}}{\to} \mathcal{O}_{\chi_2},$$

where $\imath_{\chi_1}$ and $\mathfrak{p}_{\chi_2}$ denote the embedding of $\mathcal{O}_{\chi_1}$ into $\mathcal{O}$ and the projection onto $\mathcal{O}_{\chi_2}$, respectively. This functor is also exact, and its (left and right) adjoint is given by T_{χ_2,V^*,χ_1}.

Lemma 4.25. *Let $\chi_i = \varpi(\lambda_i)$, $i = 1, 2$. Then $T_{\chi_1,V,\chi_2} = 0$ unless there exist $w_1, w_2 \in W$ and $\mu \in \mathfrak{h}^*$ with $V(\mu) \neq 0$, such that $w_1 \cdot \lambda_1 = w_2 \cdot \lambda_2 + \mu$.*

Proof. We claim that unless the condition of the lemma is satisfied, then $T_{\chi_1,V,\chi_2}(M_\lambda) = 0$ for any λ, such that $M_\lambda \in \mathcal{O}_{\chi_1}$. This follows from Lemma 4.24.

Hence, in this case $T_{\chi_1,V,\chi_2}(L_\lambda) = 0$ for all irreducibles $L_\lambda \in \mathcal{O}_{\chi_1}$, since T_{χ_1,V,χ_2} is exact. Again, by exactness, this implies that $T_{\chi_1,V,\chi_2}(\mathcal{M}) = 0$ for all $\mathcal{M} \in \mathcal{O}_{\chi_1}$.

$\square$

We shall now use the functors T_V to compare the categories $\mathcal{O}_\chi$ for different χ's.

Let λ be dominant and let μ be a dominant integral weight. Set $\chi_1 := \varpi(\lambda)$ and $\chi_2 = \varpi(\lambda + \mu)$. Let V^μ be the irreducible finite-dimensional $\mathfrak{g}$-module with highest weight μ.

Theorem 4.26. *Under the above circumstances, the functors T_{χ_1,V^μ,χ_2} and $T_{\chi_2,(V^\mu)^*,\chi_1}$ define mutually quasi-inverse equivalences*

$$\mathcal{O}_{\chi_1} \leftrightarrows \mathcal{O}_{\chi_2}.$$

Proof.

Lemma 4.27. *Let $\mathsf{F}, \mathsf{G} : \mathcal{C}_1 \leftrightarrows \mathcal{C}_2$ be mutually adjoint exact functors between Artinian abelian categories. Then F and G are mutually quasi-inverse equivalences if and only if they define mutually inverse isomorphisms on the level of Grothendieck groups.*

Proof. Let G be the right adjoint of F, and consider the adjunction map

$$\mathcal{M} \to \mathsf{F}(\mathsf{G}(\mathcal{M})).$$

We want to show that this map is an isomorphism. First, we claim that it is injective. Indeed, since G is conservative (i.e., sends no non-zero object to zero), it is enough to check that $\mathsf{G}(\mathcal{M}) \to \mathsf{G}(\mathsf{F}(\mathsf{G}(\mathcal{M})))$ is injective. But the composition

$$\mathsf{G}(\mathcal{M}) \to \mathsf{G}(\mathsf{F}(\mathsf{G}(\mathcal{M}))) \to \mathsf{G}(\mathcal{M})$$

is the identity map by the adjunction property.

Since now $[\mathcal{M}] = [\mathsf{F}(\mathsf{G}(\mathcal{M}))]$, we obtain that $\mathcal{M} \to \mathsf{F}(\mathsf{G}(\mathcal{M}))$ is an isomorphism. The fact that the second adjunction $\mathsf{G}(\mathsf{F}(\mathcal{N})) \to \mathcal{N}$ is an isomorphism follows by the same argument.

□

Thus, to prove the proposition it is sufficient to show that for any $w \in W$ we have:

$$[T_{\chi_1,V^\mu,\chi_2}(M_{w\cdot\lambda})] = [M_{w\cdot(\lambda+\mu)}] \text{ and } [T_{\chi_2,(V^\mu)^*,\chi_1}(M_{w\cdot(\lambda+\mu)})] = [M_{w\cdot\lambda}]. \tag{4.3}$$

By Lemma 4.24, the first equality is equivalent to showing that

$$w \cdot \lambda + \mu' = w' \cdot (\lambda + \mu) \tag{4.4}$$

with $V^\mu(\mu') \neq 0$ implies $w' = w$ and $\mu' = w(\mu)$.

For μ' as above we have $\mu - (w')^{-1}(\mu') \in Q^+$. Hence, from (4.4) and the fact that $\lambda + \rho$ is dominant we obtain that $\mu' = w'(\mu)$ and $w' \cdot \lambda = w \cdot \lambda$.

But the fact that λ is dominant implies that $\langle \lambda + \rho, \check{\alpha} \rangle \neq 0$ for $\alpha \in \Delta^+$. Hence, $w' \cdot \lambda = w \cdot \lambda$ implies $w' = w$.

The analysis of the second equality in (4.3) is similar.

□

We shall now consider the translation functor T_{χ_1,V,χ_2} for $\chi_1 = \varpi(-\rho)$ and $\chi_2 = \varpi(\lambda)$ with λ being dominant integral. Set $V = V^{\lambda+\rho}$.

Recall that $\mathcal{O}_{\chi_1}$ contains a unique irreducible object, $M_{-\rho}$, which is both projective and injective. Let us denote $T_{\chi_1,V,\chi_2}(M_{-\rho}) \in \mathcal{O}_{\chi_2}$ by Ξ_λ. We obtain that Ξ_λ is both injective and projective. This is "the most interesting" object of $\mathcal{O}_{\varpi(\lambda)}$.

Proposition 4.28. *Ξ_λ is indecomposable and is isomorphic to $P_{w_0\cdot\lambda}$.*

Proof. First, we claim that Ξ_λ contains $P_{w_0\cdot\lambda}$ as a direct summand. To see this, it is sufficient to show that $Hom(\Xi_\lambda, L_{w_0\cdot\lambda}) \neq 0$. However, by the construction of the filtration on $M_{-\rho} \otimes V^{\lambda+\rho}$ in Lemma 4.24, this module admits $M_{-\rho+w_0(\lambda+\rho)} = M_{w_0\cdot\lambda}$ as a quotient.

Hence, it remains to see that for all $w \in W$.

$$\text{mult}(M_{w\cdot\lambda}, \Xi_\lambda) = \text{mult}(M_{w\cdot\lambda}, P_{w_0\cdot\lambda}).$$

By Theorem 4.16 and Proposition 4.20, it is sufficient to see that the LHS of the above equation equals 1 for all $w \in W$. However, the latter fact follows from Lemma 4.24.

□

Exercise. Show that $w = w_0$ is the only element of W for which the projective module $P_{w\cdot\lambda}$ is also injective.

5. Basics of D-modules

The best existing reference for D-modules is a course by J. Bernstein that can be downloaded from www.math.uchicago.edu/∼arinkin/langlands.

5.1. **Differential operators.** Let X be a smooth affine algebraic variety over $\mathbb{C}$ (or any other field of char. 0). We define the notion of differential operator on X inductively:

A linear map $D : \mathcal{O}_X \to \mathcal{O}_X$ is a differential operator of order k ($k \geq 0$) if its commutator with the operation of multiplication by any function $f \in \mathcal{O}_X$, i.e., $[D, f]$, is a differential operator of order $k - 1$.

We will denote the vector space of differential operators of order k on X by $\mathfrak{D}(X)_k$. We have the natural inclusions $\mathfrak{D}(X)_{k-1} \hookrightarrow \mathfrak{D}(X)_k$. The union $\bigcup_k \mathfrak{D}(X)_k$ will be denoted by $\mathfrak{D}(X)$.

For example, the operator given by multiplication by a function is a differential operator of order 0. For a vector field ξ, the operator $f \mapsto \xi(f)$ is a differential operator of order 1. A product of differential operators of orders k_1 and k_2 is a differential operator of order $k_1 + k_2$; hence $\mathfrak{D}(X)$ is a filtered $\mathbb{C}$-algebra. For $D_i \in \mathfrak{D}(X)_{k_i}$, $i = 1, 2$, the commutator $[D_1, D_2]$ belongs to $\mathfrak{D}(X)_{k_1+k_2-1}$. Hence, $\mathrm{gr}(\mathfrak{D}(X))$ is a commutative $\mathcal{O}_X$-algebra.

Proposition 5.2.

(1) *The above map* $\mathcal{O}_X \to \mathfrak{D}(X)_0$ *is an isomorphism.*

(2) *The map* $\mathcal{O}_X \oplus T_X \to \mathfrak{D}(X)_1$ *is also an isomorphism.*

(3) *(Here we need that* X *to be smooth.) The associated graded algebra* $\mathrm{gr}(\mathfrak{D}_X)$ *is isomorphic to* $\mathrm{Sym}_{\mathcal{O}_X}(T_X)$.

Proof. We construct the inverese map $\mathfrak{D}(X)_0 \to \mathcal{O}_X$ by sending $D \mapsto D(1) \in \mathcal{O}_X$.

Given a differential operator D of order 1, we associate to it an element of $\mathcal{O}_X \oplus T_X$ as follows. The $\mathcal{O}_X$-component equals $D(1)$. The T_X-component is supposed to be a derivation $\mathcal{O}_X \to \mathcal{O}_X$ and we set it to be

$$f \mapsto [D, f] \in \mathfrak{D}(X)_0 \simeq \mathcal{O}_X.$$

Thus, we have a map $T_X \to \mathrm{gr}^1(\mathfrak{D}(X))$, which is easily seen to be $\mathcal{O}_X$-linear. This gives rise to a map

$$\mathrm{Sym}_{\mathcal{O}_X}(T_X) \to \mathrm{gr}(\mathfrak{D}_X).$$

We construct the inverse map $\mathrm{gr}^i(\mathfrak{D}(X)) \to \mathrm{Sym}^i_{\mathcal{O}_X}(T_X)$ as follows. Given $D \in \mathfrak{D}(X)_i$, the commutator $f \mapsto [D, f]$ defines a $\mathbb{C}$-linear map $\mathcal{O}_X \to \mathfrak{D}(X)_{i-1}$. Moreover, the composed map

$$\mathcal{O}_X \to \mathfrak{D}(X)_{i-1} \to \mathrm{gr}^{i-1}(\mathfrak{D}(X))$$

is easily seen to be a derivation.

Thus, we obtain a map

$$\mathfrak{D}(X)_i \to \mathit{Hom}_{\mathcal{O}_X}\Big(\Omega^1(X), \mathrm{gr}^{i-1}(\mathfrak{D}(X))\Big),$$

which is easily seen to factor through an $\mathcal{O}_X$-linear map

$$\mathrm{gr}^i(\mathfrak{D}(X)) \to \mathit{Hom}_{\mathcal{O}_X}\Big(\Omega^1(X), \mathrm{gr}^{i-1}(\mathfrak{D}(X))\Big) \simeq T_X \underset{\mathcal{O}_X}{\otimes} \mathrm{gr}^{i-1}(\mathfrak{D}(X)).$$

By induction on i we can assume that $\mathrm{gr}^{i-1}(\mathfrak{D}(X)) \simeq \mathrm{Sym}^{i-1}_{\mathcal{O}_X}(T_X)$. Then the desired map is the composition

$$\mathrm{gr}^i(\mathfrak{D}(X)) \to\simeq T_X \underset{\mathcal{O}_X}{\otimes} \mathrm{Sym}^{i-1}_{\mathcal{O}_X}(T_X) \to \mathrm{Sym}^i_{\mathcal{O}_X}(T_X),$$

where the last arrow is given by multiplication.

$\square$

Proposition 5.3. *As a ring, $\mathfrak{D}(X)$ is generated by the elements $f \in \mathcal{O}_X$, $\xi \in T_X$, subject to the following relations:*

$$f_1 \star f_2 = f_1 \cdot f_2,\ f \star \xi = f \cdot \xi,\ \xi_1 \star \xi_2 - \xi_2 \star \xi_1 = [\xi_1, \xi_2],\ \xi \star f - f \star \xi = \xi(f), \tag{5.1}$$

where $\star$ (temporarily) denotes the multiplication in $\mathfrak{D}(X)$.

Proof. Let us write down more explicitly how the ring with the above generators and relations looks like. Consider the Lie algebra, which is $\mathcal{O}_X \oplus T_X$ as a vector space, and the bracket, denoted, $[\cdot,\cdot]_\star$ is defined by

$$[f_1, f_2]_\star = 0,\ [\xi, f]_\star = \xi(f),\ [\xi_1, \xi_2]_\star = [\xi_1, \xi_2].$$

Consider its universal enveloping algebra $\mathcal{A}' := U(\mathcal{O}_X \oplus T_X)$, and let $\star$ denote its associative product. Consider the quotient of $\mathcal{A}'$ by the left (and automatically two-sided) ideal, generated by the elements of the form

$$1_{\mathcal{A}'} - 1_{\mathcal{O}_X},\ f_1 \star f_2 - f_1 \cdot f_2,\ f \star \xi = f \cdot \xi.$$

(Here $1_{\mathcal{A}'}$ denotes the unit of $\mathcal{A}'$, and $1_{\mathcal{O}_X}$ the unit of $\mathcal{O}_X$, considered as a subspace of $\mathcal{A}'$.) The quotient algebra by this ideal, denoted $\mathcal{A}$, is the one appearing in the statement of the proposition.

Evidently, we have a map $\mathcal{A} \to \mathfrak{D}(X)$, and we claim that it is an isomorphism. Consider the filtartion on $\mathcal{A}$, obtained by declaring that $\mathcal{A}_0$ equals the image of $\mathcal{O}_X$, and $\mathcal{A}_1$ equals the image of $\mathcal{O}_X \oplus T_X$. The relations (5.1) imply that $\mathrm{gr}(\mathcal{A})$ is commutative and that there exists a surjection

$$\mathrm{Sym}_{\mathcal{O}_X}(T_X) \twoheadrightarrow \mathrm{gr}(\mathcal{A}).$$

The map $\mathcal{A} \to \mathfrak{D}(X)$ is easily seen to be compatible with filtartions, and the composed map

$$\mathrm{Sym}_{\mathcal{O}_X}(T_X) \twoheadrightarrow \mathrm{gr}(\mathcal{A}) \to \mathrm{gr}(\mathfrak{D}(X))$$

is the map of (5.2), and hence is an isomorphism. Therefore, the map $\mathrm{gr}(\mathcal{A}) \to \mathrm{gr}(\mathfrak{D}(X))$ is an isomorphism, implying that $\mathcal{A} \simeq \mathfrak{D}(X)$

□

Let $f_1, ..., f_n$ be an étale coordinate system on X, i.e., a collection of functions, whose differentials span $T^*_x(X)$ for every $x \in X$. Let ∂_i be the vector fields, defined by $\langle \partial_i, dx_j \rangle = \delta^i_j$. Then, these vector fields commute among themselves, and

$$\mathfrak{D}_X \simeq \mathcal{O}_X \otimes \mathbb{C}[\partial_1, ..., \partial_n],$$

as a left $\mathcal{O}_X$-module, and

$$\mathfrak{D}_X \simeq \mathbb{C}[\partial_1, ..., \partial_n] \otimes \mathcal{O}_X,$$

as a right $\mathcal{O}_X$-module.

For example, let X be the affine space $\mathbb{A}^n = \mathrm{Spec}(\mathbb{C}[x_1, ..., x_n])$. From Proposition 5.3 we obtain that $\mathfrak{D}(\mathbb{A}^n)$ is generated by the elements $x_1, ..., x_n, \partial_1, ..., \partial_n$ with the relations

$$[x_i, x_j] = 0,\ [\partial_i, \partial_j],\ [\partial_i, x_j] = \delta^i_j.$$

The latter algebra is also referred to as the Weyl algebra, and denoted W_n.

5.4. **Localization of differential operators.** By construction, we have a homomorphism of rings $\mathcal{O}_X \to \mathfrak{D}(X)$; in fact, each term of the filtration $\mathfrak{D}(X)_i$ is a $\mathcal{O}_X$-bimodule.

Proposition 5.5. *Let f be a non-nilpotent function on X, and let X_f be the corresponding basic open subset. Then:*

$$\mathcal{O}_{X_f} \underset{\mathcal{O}_X}{\otimes} \mathfrak{D}(X)_i \simeq \mathfrak{D}(X_f)_i \simeq \mathfrak{D}(X)_i \underset{\mathcal{O}_X}{\otimes} \mathcal{O}_{X_f}.$$

Proof. We construct the maps

$$\mathcal{O}_{X_f} \underset{\mathcal{O}_X}{\otimes} \mathfrak{D}(X)_i \to \mathfrak{D}(X_f)_i \leftarrow \mathfrak{D}(X)_i \underset{\mathcal{O}_X}{\otimes} \mathcal{O}_{X_f} \tag{5.2}$$

inductively. Suppose that this has been done for indices $j < i$. Since $\mathfrak{D}(X_f)_i$ is an $\mathcal{O}_{X_f}$-bimodule, to define the maps of (5.2) it is sufficient to construct a map of $\mathcal{O}_X$-bimodules

$$\mathfrak{D}(X)_i \to \mathfrak{D}(X_f)_i. \tag{5.3}$$

Given a differential operator $D \in \mathfrak{D}(X)_i$ we need to be able to act by it on an element $g' = \frac{g}{f^n} \in \mathcal{O}_{X^f}$. We set

$$D(g') = f^{-n} \cdot f^n \cdot D(g') = f^{-n} \cdot D(f^n \cdot g') - f^{-n} \cdot [D, f^n](g'),$$

where both terms on the RHS are well-defined, since $f^n \cdot g' \in \mathcal{O}_X$ and $[D, f^n] \in \mathfrak{D}(X)_{i-1}$ (here we are using the induction hypothesis).

It is easy to see that the resulting endomorphism of $\mathcal{O}_{X_f}$ is a differential operator of order i, which does not depend on the choice of n. Moreover, the map of (5.3), defined in this way, respects the $\mathcal{O}_X$-bimodule structure.

To prove that the maps in (5.2) are isomorphisms, it is enough to show that

$$\mathcal{O}_{X_f} \underset{\mathcal{O}_X}{\otimes} \mathrm{gr}(\mathfrak{D}(X)) \to \mathrm{gr}(\mathfrak{D}(X_f)) \leftarrow \mathrm{gr}(\mathfrak{D}(X)) \underset{\mathcal{O}_X}{\otimes} \mathcal{O}_{X_f}$$

are isomorphisms. But this is evident from Proposition 5.2.

□

Exercise. Show that the statement of the above proposition is valid without the assumption that X be smooth.

Thus, if X is an arbitrary (not necessarily affine) smooth algebraic variety, we can define a sheaf of algebras $\mathfrak{D}(X)$ by setting for an affine $U \subset X$,

$$\Gamma(U, \mathfrak{D}(X)) := \mathfrak{D}(U).$$

It is quasi-coherent with respect to both (left and right) structures of sheaf of $\mathcal{O}_X$-modules on $\mathfrak{D}(X)$.

5.6. **D-modules.** To simplify the notation, we will assume that X is affine, but Proposition 5.5 guarantees that all the notions make sense for any smooth variety.

A left (resp., right) D-module $\mathcal{M}$ on X is by definition the same as a left (resp., right) module over $\mathfrak{D}(X)$. By Proposition 5.3, a left D-module can be thought of as an $\mathcal{O}_X$-module, endowed with an action of the Lie algebra of vector fields, and such that for $m \in \mathcal{M}$, $f \in \mathcal{O}_X$, $\xi \in T_X$,

$$f \cdot (\xi \cdot m) = (f \cdot \xi) \cdot m \text{ and } \xi \cdot (f \cdot m) - f \cdot (\xi \cdot m) = \xi(f) \cdot m,$$

and similarly for right D-modules.

Let us consider some examples.

1) $\mathfrak{D}(X)$ is evidently both a left and a right D-module.

2) $\mathcal{O}_X$ is a left D-module under $D \cdot f = D(f)$.

3) Consider Ω^n_X, where $n = \dim(X)$. We claim that this is naturally a right D-module. Namely, we set

$$\omega \cdot f = f \cdot \omega \text{ and } \omega \cdot \xi = -\operatorname{Lie}_\xi(\omega).$$

Exercise. Prove that this is indeed a right D-module!

More generally, for a left D-module $\mathcal{M}$ we can define a right D-module $\mathcal{M}^r$ by $\mathcal{M}^r := \mathcal{M} \underset{\mathcal{O}_X}{\otimes} \Omega^n_X$ and

$$(m \otimes \omega) \cdot f = m \otimes (f \cdot \omega) \text{ and } (m \otimes \omega) \cdot \xi = -(\xi \cdot m) \otimes \omega - m \otimes \operatorname{Lie}_\xi(\omega).$$

This defines an equivalence between the categories of left and right D-modules on X. Note that this equivalence acts non-trivially on the forgetful functor to $\mathcal{O}_X$-modules.

4) Take $X = \mathbb{A}^1$, and define the D-module "e^x" to be isomorphic to $\mathcal{O}_X \simeq \mathbb{C}[x]$ as an $\mathcal{O}_X$-module, with the vector field ∂_x acting by

$$\partial_x \cdot 1_{"e^x"} = 1_{"e^x"}$$

where $1_{"e^x"} \in "e^x"$ is the element corresponding to $1 \in \mathcal{O}_X$. (Recognize the differential equation, satisfied by the exponential function.)

5) Take $X = \mathbb{A}^1 - 0$ and for $\lambda \in \mathbb{C}$ define the D-module "x^λ" to be isomorphic to $\mathcal{O}_X \simeq \mathbb{C}[x, x^{-1}]$ with the vector field ∂_x acting by

$$\partial_x \cdot 1_{"x^\lambda"} = \lambda \cdot x^{-1} \cdot 1_{"x^\lambda"},$$

where $1_{"x^\lambda"} \in "x^\lambda"$ is the element corresponding to $1 \in \mathcal{O}_X$.

Exercise. Show that "x^λ" is isomorphic to $\mathcal{O}_X$ as a D-module if and only if $\lambda \in \mathbb{Z}$.

6) Take $\mathcal{M}$ to be the (huge) vector space of generalized functions on the C^∞ manifold underlying X. Define the left action of $\mathfrak{D}(X)$ by

$$f \cdot \mathfrak{d} = f \cdot \mathfrak{d} \text{ and } \xi \cdot \mathfrak{d} = \operatorname{Lie}_\xi(\mathfrak{d}).$$

Exercise. Work out the relation between the notion of solution of a system of linear differential equations on X and that of D-modules. Hint: given a system of linear differential equations on X construct a left D-module on X and study its Hom into the above $\mathcal{M}$.

7) Let $x \in X$ be a point. We define the right D-module δ_x to be generated by a single element $1_x \in \delta_x$ with the relations being

$$1_x \cdot f = f(x) \cdot 1_x, \;\; f \in \mathcal{O}_X$$

Alternatively,

$$\delta_x \simeq \mathbb{C}_x \underset{\mathcal{O}_X}{\otimes} \mathfrak{D}(X),$$

where $\mathbb{C}_x$ is the sky-scraper coherent sheaf at x.

8) Let E be a locally free sheaf of finite rank (i.e., vector bundle) on X. Then a structure of D-module on E amounts to that of integrable connection.

5.7. **Pull-back of D-modules.** In what follows by a left (resp., right) D-module on a variety X we shall mean a quasi-coherent sheaf of left (resp., right) $\mathfrak{D}(X)$-modules. We will denote the corresponding category by $\mathfrak{D}(X)$-mod (resp., $\mathfrak{D}(X)^r$-mod). But let us remember that the categories $\mathfrak{D}(X)$-mod and $\mathfrak{D}(X)^r$-mod are equivalent by means of $\mathcal{M} \mapsto \mathcal{M} \otimes \Omega^n_X$.

Let $\phi : Y \to X$ be a map of algebraic varieties. We shall now construct a functor

$$\phi^* : \mathfrak{D}(X)\text{-mod} \to \mathfrak{D}(Y)\text{-mod}.$$

For $\mathcal{M} \in \mathfrak{D}(X)$-mod we let $\phi^*(\mathcal{M})$ to be $\phi^*(\mathcal{M})$ as a $\mathcal{O}_Y$-module. We define the action of vector fields on it by:

$$\xi \cdot (f \otimes m) = \xi(f) \otimes m + f \cdot d\phi(\xi)(m),$$

where $\xi \mapsto d(\phi)(\xi)$ is the differential of ϕ, thought of as a map $T_Y \to \mathcal{O}_Y \underset{\mathcal{O}_X}{\otimes} T_Y$, and $d\phi(\xi)(m)$ in the above formula makes sense as an element of $\mathcal{O}_Y \underset{\mathcal{O}_X}{\otimes} \mathcal{M} =: \phi^*(\mathcal{M})$.

As in the case of quasi-coherent sheaves, the functor ϕ^* is only right exact and needs to be derived, by replacing $\mathcal{M}$ by a complex of $\mathfrak{D}(X)$-modules, which are flat as $\mathcal{O}_X$-modules. It is fairly easy to show that such resolution always exists.

Exercise. Construct a resolution as above by showing that any D-module admits a surjection from a D-module of the form $\mathfrak{D}_X \underset{\mathcal{O}_X}{\otimes} \mathcal{F}$, where $\mathcal{F}$ is a quasi-coherent sheaf on X.

Consider now the case when Y is a closed subvariety of X. We shall now consider another functor $\phi^! : \mathfrak{D}(X)^r\text{-mod} \to \mathfrak{D}(Y)^r\text{-mod}$:

For $\mathcal{M} \in \mathfrak{D}(X)^r$-mod, we let $\phi^!(\mathcal{M})$ to be the same-named object in the quasi-coherent category, i.e., as a quasi-coherent sheaf, $\phi^!(\mathcal{M})$ consists of sections that are annihilated by the ideal $I_Y \subset \mathcal{O}_X$ that cuts Y in X.

We define the action of T_Y on $\phi^!(\mathcal{M})$ as follows. For a vector field ξ_Y on Y choose a vector field ξ on X, which is tangent to Y, and whose restriction to Y equals ξ_Y. We set:

$$m \cdot \xi_Y := m \cdot \xi \in \mathcal{M}.$$

The fact that the RHS does not depend on the choice of ξ follows from the fact that $m \cdot I_Y = 0$. Moreover, for $f \in I_Y$,

$$(m \cdot \xi) \cdot f = (m \cdot f) \cdot \xi + m \cdot \xi(f) = 0,$$

since $\xi(f) \in I_Y$. Hence, $m \cdot \xi_Y$ defined above is an element of $\phi^!(\mathcal{M})$.

The functor $\phi^!$ is left exact, and one can consider its right derived functor $R\phi^!$.

One can show the following:

Lemma 5.8. *The functors ϕ^* and $\phi^!$ are related as follows: for $\mathcal{M} \in \mathfrak{D}(X)$-mod there exists a canonical isomorphism:*

$$\Big(L\phi^*(\mathcal{M})\Big)^r[-k] = R\phi^!(\mathcal{M}^r),$$

where k is the codimension of Y in X.

We shall now prove the following easy, but fundamental result, due to Kashiwara. Let $\mathfrak{D}(X)^r_Y$-mod be the full subcategory of $\mathfrak{D}(X)^r$, consisting of D-modules, which, as quasi-coherent sheaves, are *set-theoretically* supported on Y.

(We recall that a quasi-coherent sheaf $\mathcal{F}$ is said to be set-theoretically supported on a subscheme if every section of $\mathcal{F}$ is annihilated by some power of the ideal of this subscheme.)

Theorem 5.9. *The functor $\phi^!$ defines an equivalence $\mathfrak{D}(X)^r_Y$-mod $\to \mathfrak{D}(Y)^r$.*

Before giving a proof we will describe explicitly the functor in the opposite direction $\phi_\star : \mathfrak{D}(Y)^r\text{-mod} \to \mathfrak{D}(X)^r\text{-mod}$. Consider the *left* $\mathfrak{D}(Y)$-module

$$\phi^*(\mathfrak{D}(X)) \simeq \mathcal{O}_Y \underset{\mathcal{O}_X}{\otimes} \mathfrak{D}(X).$$

The right multiplication of $\mathfrak{D}(X)$ acts by endomorphisms of the left D-module structure. Hence, $\phi^*(\mathfrak{D}(X))$ carries a commuting right $\mathfrak{D}(X)$-action.

Therefore, given a *right* $\mathfrak{D}(Y)$-module $\mathcal{N}$, we can consider $\mathcal{N} \underset{\mathfrak{D}(Y)}{\otimes} \phi^*(\mathfrak{D}(X))$ as a right $\mathfrak{D}(X)$-module. This is our $\phi_\star(\mathcal{N})$.

Proof. (of Kashiwara's theorem)

Let us first show that the functors $\phi^! : \mathfrak{D}(X)^r \to \mathfrak{D}(Y)^r$ and $\phi_\star : \mathfrak{D}(Y)^r\text{-mod} \to \mathfrak{D}(X)^r\text{-mod}$ are mutually adjoint.

For $\mathcal{N} \in \mathfrak{D}(X)^r_Y$ we have a natural map

$$\mathcal{N} \simeq \mathcal{N} \underset{\mathcal{O}_Y}{\otimes} \Big(\mathcal{O}_Y \underset{\mathcal{O}_X}{\otimes} \mathcal{O}_X\Big) \to \mathcal{N} \underset{\mathfrak{D}(Y)}{\otimes} \Big(\mathcal{O}_Y \underset{\mathcal{O}_X}{\otimes} \mathfrak{D}(X)\Big) \simeq \phi_\star(\mathcal{N}). \tag{5.4}$$

Moreover, its image is annihilated by I_Y.

Therefore, given $\mathcal{M} \in \mathfrak{D}(X)^r$ and a map $\phi_\star(\mathcal{N}) \to \mathcal{M}$ we can restrict it to $\mathcal{N}$ under (5.4), and the resulting map will have its image in $\phi^!(\mathcal{M})$. Moreover, it is easy to check that the $\mathcal{O}_Y$-module map $\mathcal{N} \to \phi^!(\mathcal{M})$, thus obtained, respects the right $\mathfrak{D}(Y)$-module structure.

Vice versa, given a map $\mathcal{N} \to \phi^!(\mathcal{M})$, and in particular an $\mathcal{O}_X$-module map $\mathcal{N} \to \mathcal{M}$, consider the map

$$\mathcal{N} \underset{\mathcal{O}_X}{\otimes} \mathfrak{D}(X) \to \mathcal{M} \underset{\mathcal{O}_X}{\otimes} \mathfrak{D}(X) \to \mathcal{M},$$

where the last arrow is given by the right action of $\mathfrak{D}(X)$ on $\mathcal{M}$.

It is easy to see that the map

$$\mathcal{N} \underset{\mathcal{O}_Y}{\otimes} \Big(\mathcal{O}_Y \underset{\mathcal{O}_X}{\otimes} \mathfrak{D}(X)\Big) \simeq \mathcal{N} \underset{\mathcal{O}_X}{\otimes} \mathfrak{D}(X) \to \mathcal{M}$$

thus obtained factors through

$$\mathcal{N} \underset{\mathfrak{D}(Y)}{\otimes} \Big(\mathcal{O}_Y \underset{\mathcal{O}_X}{\otimes} \mathfrak{D}(X)\Big) \to \mathcal{M},$$

and that the latter map is compatible with the right action of $\mathfrak{D}(X)$.

It is straightforward to check that the maps

$$Hom_{\mathfrak{D}(Y)}(N, \phi^!(\mathcal{M})) \leftrightarrows Hom_{\mathfrak{D}(X)}(\phi_\star(\mathcal{N}), \mathcal{M}),$$

constructed above are mutually inverse.

Next, let us show that the image of $\phi_\star(\mathcal{N})$ lies in $\mathfrak{D}(X)^r_Y$-mod. In fact we claim that every section of

$$\mathcal{N} \underset{\mathcal{O}_X}{\otimes} \mathfrak{D}(X)$$

is annihilated by some power of I_Y under the action of $\mathcal{O}_X$ by right multiplication on $\mathfrak{D}_X$. This follows from the fact that for $D \in \mathfrak{D}(X)_n$, the commutator map $f \mapsto [D, f]$ sends I_Y^{n+k} to I_Y^k.

Finally, let us show that the adjunction maps

$$\phi_\star(\phi^!(\mathcal{M})) \to \mathcal{M} \text{ and } \mathcal{N} \to \phi^!(\phi_\star(\mathcal{N}))$$

are isomorphisms for $\mathcal{N} \in \mathfrak{D}(Y)^r$-mod and $\mathcal{M} \in \mathfrak{D}(X)^r_Y$-mod.

We will use the following general assertion:

Lemma 5.10. *Let* $\mathsf{G} : \mathcal{C}_1 \to \mathcal{C}_2$ *be a functor between two abelian categories, and let* F *be its right adjoint. Assume that (1)* F *is conservative, (2) the adjunction morphism* $\mathcal{N} \to \mathsf{F}(\mathsf{G}(\mathcal{N}))$ *is an isomorphism and (3)* $\mathsf{G}(\mathsf{F}(\mathcal{M})) \to \mathcal{M}$ *is surjective.*

Then G *and* F *are mutually quasi-inverse equivalences.*

Proof. We have to show that for $\mathcal{M} \in \mathcal{C}_2$ the map $\mathsf{G}(\mathsf{F}(\mathcal{M})) \to \mathcal{M}$ is an injection. Let $\mathcal{M}'$ be its kernel. Being a right adjoint, F is left exact. Hence,

$$\mathsf{F}(\mathcal{M}') \simeq \ker(\mathsf{F}(\mathsf{G}(\mathsf{F}(\mathcal{M}))) \to \mathsf{F}(\mathcal{M})).$$

But since the composition

$$\mathsf{F}(\mathcal{M}) \to \mathsf{F}(\mathsf{G}(\mathsf{F}(\mathcal{M}))) \to \mathsf{F}(\mathcal{M})$$

is the identity map and $? \to \mathsf{F}(\mathsf{G}(?))$ is an isomorphism, we obtain that the map $\mathsf{F}(\mathsf{G}(\mathsf{F}(\mathcal{M}))) \to \mathsf{F}(\mathcal{M})$ is an isomorphism. Hence, $\mathsf{F}(\mathcal{M}') = 0$, and since F is conservative, this implies that $\mathcal{M}'=0$. $\square$

Thus, we have to check that conditions (1)-(3) of the lemma hold in our situation. Note that (1) is automatic by the definition of $\mathfrak{D}(X)^r_Y$-mod.

The assertion is local and since both X and Y are smooth, we can assume that I_Y is generated by a regular sequence $f_1, ..., f_n$. By induction, we can assume that $n = 1$, i.e., $X = Y \times \mathbb{A}^1$, with ϕ corresponding to $0 \hookrightarrow \mathbb{A}^1$.

For $\mathcal{N} \in \mathfrak{D}(Y)^r$-mod,

$$\phi_\star(\mathcal{N}) \simeq \mathcal{N} \otimes \delta_0,$$

where δ_0 is as in Example 7 above. The map $\mathcal{N} \to \phi^!(\phi_\star(\mathcal{N}))$ corresponds to the canonical map $\mathbb{C} \to \delta_0$.

We have:

$$\delta_0 \simeq \mathbb{C}[\partial_x]$$

with

$$\partial_x^i \cdot x = i \cdot \partial_x^{i-1}.$$

Hence, $\phi^!(\delta_0) := \ker(x : \delta_0 \to \delta_0) = \mathbb{C}$, as required. This proves that condition (1) of the above lemma is satisfied.

Condition (2) of the lemma is equivalent to the fact that every $\mathcal{M} \in \mathfrak{D}(X)^r_Y$-mod is generated as a right D-module by its $\mathcal{O}_X$-submodule, consisting of sections that are annihilated by I_Y.

For $\mathcal{M}$ as above introduce a filtration by $\mathcal{O}_X$-submodules by declaring that $F_k(\mathcal{M})$ consists of sections annihilated by I_Y^{k+1}.

We will be working in the set-up of $X = Y \times \mathbb{A}^1$ as above. Then $F_k(\mathcal{M})$ is the kernel of x^{k+1} acting on $\mathcal{M}$. We have $F_0(\mathcal{M}) = \phi^!(\mathcal{M})$, the multiplication by x acts as $F_k(\mathcal{M}) \to F_{k-1}(\mathcal{M})$ and the multiplication by ∂_x as $F_k(\mathcal{M}) \to F_{k+1}(\mathcal{M})$.

To prove the assertion is it is enough to show that ∂_x defines a surjection $\mathrm{gr}^k(\mathcal{M}) \to \mathrm{gr}^{k+1}(\mathcal{M})$. We will show by induction on k that the vector field $x\partial_x$ acts as multiplication by k on $\mathrm{gr}^k(\mathcal{M})$. I.e., we need to show that for $m \in F_k(\mathcal{M})$

$$(m \cdot x \cdot \partial_x - k \cdot m) \cdot x^k = 0.$$

But the above expression equals

$$((m \cdot x) \cdot x \cdot \partial_x + m \cdot x - k \cdot m \cdot x) \cdot x^{k-1} = (m' \cdot x \cdot \partial_x - (k-1) \cdot m') \cdot x^{k-1},$$

where $m' = m \cdot x \in F_{k-1}(\mathcal{M})$ and the last expression vanishes by the induction hypothesis. $\square$

5.11. **D-modules on singular varieties.** Kashiwara's theorem enables us to define the category of D-modules on arbitrary (not necessarily smooth) schemes over $\mathbb{C}$. Let S be such a scheme.

First we shall assume that S is affine. Choose a closed embedding $S \hookrightarrow X$, where X is a non-singular variety, e.g., $\mathbb{A}^n$. Define the category $\mathfrak{D}(S)^r$-mod to be $\mathfrak{D}(X)^r_S$-mod, i.e., the full subcategory of $\mathfrak{D}(X)^r$-mod, consisting of objects that are set-theoretically supported on S.

We claim that this category is well-defined, i.e., that it is independent, up to a canonical equivalence, of the choice of X. Indeed, let $S \hookrightarrow Y$ be another embedding into a smooth variety. We can find a third variety Z that contains both X and Y as closed subvarieties, such that the diagram

$$\begin{array}{ccc} X & \longrightarrow & Z \\ \uparrow & & \uparrow \\ S & \longrightarrow & Y \end{array}$$

commutes.

By Kashiwara's theorem, we have the equivalences,

$$\mathfrak{D}(X)^r\text{-mod} \simeq \mathfrak{D}(Z)^r_X\text{-mod and } \mathfrak{D}(Z)^r_Y\text{-mod} \simeq \mathfrak{D}(Y)^r\text{-mod},$$

and by construction, they induces the equivalences

$$\mathfrak{D}(X)^r_S\text{-mod} \simeq \mathfrak{D}(Z)^r_S\text{-mod} \simeq \mathfrak{D}(Y)^r_S\text{-mod}.$$

I.e., we obtain a canonical equivalence $\mathfrak{D}(X)^r_S$-mod $\simeq \mathfrak{D}(Y)^r_S$-mod. It is easy to see that this equivalence does not depend, up to a canonical isomorphism, of the choice of Z.

If W is yet another smooth variety with $S \hookrightarrow W$, the composition of the equivalences $\mathfrak{D}(X)^r_S$-mod $\simeq \mathfrak{D}(Y)^r_S$-mod and $\mathfrak{D}(Y)^r_S$-mod $\simeq \mathfrak{D}(W)^r_S$-mod is isomorphic to $\mathfrak{D}(X)^r_S$-mod $\simeq$ $\mathfrak{D}(W)^r_S$-mod. Moreover, a natural compatibility relation concerning X, Y, W, U is satisfied.

This implies that $\mathfrak{D}(S)^r$-mod is well-defined as a category. Note, however, that the above construction does not give us any natural functor from $\mathfrak{D}(S)^r$-mod to $\mathcal{O}_S$-mod.

Let now S be arbitrary, i.e., not necessarily affine. Then the category $\mathfrak{D}(S)^r$-mod is obtained by gluing:

Let S_i be a cover of S by affines. For each S_i we have a well-defined category $\mathfrak{D}(S_i)^r$-mod, and for each pair i, j we have exact functors

$$\mathrm{Res}^i_{i,j} : \mathfrak{D}(S_i)^r\text{-mod} \to \mathfrak{D}(S_i \cap S_j)^r\text{-mod} \leftarrow \mathfrak{D}(S_j)^r : \mathrm{Res}^j_{i,j}\,.$$

We define $\mathfrak{D}(S)^r$-mod to have as objects collections $\{\mathcal{M}_i \in \mathfrak{D}(S_i)^r\text{-mod}\}$, endowed with isomorphisms for each i, j:

$$\alpha_{i,j} : \mathrm{Res}^i_{i,j}(\mathcal{M}_i) \simeq \mathrm{Res}^j_{i,j}(\mathcal{M}_j),$$

such that for each triple of indices i, j, k the two isomorphisms

$$\alpha_{j,k}|_{S_i\cap S_j\cap S_k} \circ \alpha_{i,j}|_{S_i\cap S_j\cap S_k} \text{ and } \alpha_{i,k}|_{S_i\cap S_j\cap S_k} : \mathrm{Res}^i_{i,j,k}(\mathcal{M}_i) \to \mathrm{Res}^k_{i,j,k}(\mathcal{M}_k)$$

coincide.

Morphisms in this category between $\{\mathcal{M}_i, \alpha_{i,j}\}$ and $\{\mathcal{M}'_i, \alpha'_{i,j}\}$ are collections of maps $\phi_i :$ $\mathcal{M}_i \to \mathcal{M}'_i$, such that the diagrams

$$\begin{array}{ccc} \mathrm{Res}^i_{i,j}(\mathcal{M}_i) & \xrightarrow{\alpha_{i,j}} & \mathrm{Res}^j_{i,j}(\mathcal{M}_j) \\ {\scriptstyle \phi_i}\downarrow & & {\scriptstyle \phi_j}\downarrow \\ \mathrm{Res}^i_{i,j}(\mathcal{M}'_i) & \xrightarrow{\alpha'_{i,j}} & \mathrm{Res}^j_{i,j}(\mathcal{M}'_j) \end{array}$$

commute for all i, j.

It is fairly easy to show that this category is well-defined, i.e., that it is canonically independent of the choice of the affine cover S_i.

From now on, unless specified otherwise, we will work with smooth varieties.

5.12. **$\mathcal{O}$-coherent D-modules.** As another application of Kashiwara's theorem, let us classify D-modules on a (smooth) variety X, which are *coherent* as $\mathcal{O}_X$-modules.

Proposition 5.13. *Any $\mathfrak{D}(X)$-module, which is coherent as an $\mathcal{O}_X$-module, is locally free.*

Proof. Note that a coherent sheaf $\mathcal{M}$ on X is locally free if and only if this is true for $\mathcal{M}|_{X'}$, for all smooth curves X' mapping to X. But each $\mathcal{M}|_{X'}$ is a left D-module on X', so we have reduced the assertion to the case when X is a curve.

Recall that a coherent sheaf on a curve is locally free if and only if it is torsion free. Assume by contradiction that $\mathcal{M}^r$ has torsion at some point $\imath : x \in X$. Then $\imath^!(\mathcal{M}^r) \neq 0$, and by adjunction we have a map

$$\imath_\star\big(\imath^!(\mathcal{M}^r)\big) \to \mathcal{M}^r.$$

We claim that is it injective. Indeed, if $\mathcal{M}_1$ is its kernel, the we an exact sequence

$$0 \to \imath^!(\mathcal{M}_1) \to \imath^!\Big(\imath_\star\big(\imath^!(\mathcal{M}^r)\big)\Big) \to \imath^!(\mathcal{M}^r).$$

But by Kashiwara's theorem, the last arrow is an isomorphism, hence $\imath^!(\mathcal{M}_1) = 0$, hence, $\mathcal{M}_1 = 0$, since it is set-theoretically supported at x.

Thus, $\mathcal{M}$ contains $\imath_\star\big(\imath^!(\mathcal{M}^r)\big)$ as a sub-module. But this is impossible, since $\imath_\star(\mathcal{N}) \simeq \delta_x^{\oplus \dim(\mathcal{N})}$ for any vector space $\mathcal{N}$ (i.e., a D-module on pt), and as we have seen, δ_x is not coherent.

□

5.14. **Open embeddings.** Let $j : \overset{o}{X} \hookrightarrow X$ be the embedding of an open subvariety, and consider the inverse image functor $j^* : \mathfrak{D}(X)\text{-mod} \to \mathfrak{D}(\overset{o}{X})\text{-mod}$. By construction, this functor amounts to usual restriction on the level of underlying $\mathcal{O}_X$-modules; in particular, it is exact.

Lemma 5.15. *The functor j^* admits a right adjoint.*

Proof. For $\mathcal{F} \in \mathfrak{D}(\overset{o}{X})$-mod we define $j_*(\mathcal{F})$ to be the same-named object as a quasi-coherent sheaf. The action of vector fields is defined in a straightforward way:

Given a vector field ξ (defined on some affine open subvariety $U \subset X$), we can restrict it to $\overset{o}{X}$ and act by it on sections of $\mathcal{F}$ over $U \cap \overset{o}{X}$. The latter are, by definition, the same as sections of $j_*(\mathcal{F})$ over U.

□

Note that the corresponding functors $\mathfrak{D}(X)^r\text{-mod} \leftrightarrows \mathfrak{D}(\overset{o}{X})^r\text{-mod}$ are also given by j_*, j^* on the level of $\mathcal{O}_X$-modules. This is because $j^*(\Omega^n(X)) \simeq \Omega^n(\overset{o}{X})$.

Note that if the embedding $\overset{o}{X} \hookrightarrow X$ is not affine, the functor j_* is not exact. We will denote by Rj_* the corresponding derived functor. Explicitly, it can be written down as follows. Let $\overset{o}{X}_i$ be a covering of $\overset{o}{X}$ by open affine subvarieties; in particular, the embedding of each $\overset{o}{X}_i$ into X is affine. Let us denote by $j^{i_1,\ldots,i_k}$ the embedding of the k-fold intersection $\overset{o}{X}_{i_1} \cap \ldots \cap \overset{o}{X}_{i_k}$. Then $Rj_*(\mathcal{M})$ can be represented by the complex

$$\underset{i}{\oplus}\, j^i_*(\mathcal{M}|_{\overset{o}{X}_i}) \to \ldots \to \underset{i_1,\ldots,i_k}{\oplus} j^{i_1,\ldots,i_k}_*(\mathcal{M}|_{\overset{o}{X}_{i_1}\cap\ldots\cap\overset{o}{X}_{i_k}}) \to \ldots.$$

Assume now that $\overset{o}{X} = X - Y$, where Y is a smooth closed subvariety of X.

Proposition 5.16. *For every $\mathcal{M} \in \mathfrak{D}(X)^r$-mod we have an exact triangle*

$$\imath_\star\big(Ri^!(\mathcal{M})\big) \to \mathcal{M} \to Rj_*\big(j^*(\mathcal{M})\big),$$

where $\imath$ denotes the embedding of Y into X.

5.17. **De Rham complex.** Let $\mathcal{M}$ be a left D-module on X. We can functorially attach to it a complex of sheaves on X, called the De Rham complex:

$$DR(\mathcal{M}) = \mathcal{M} \to \Omega^1(X) \underset{\mathcal{O}_X}{\otimes} \mathcal{M} \to \Omega^2(X) \underset{\mathcal{O}_X}{\otimes} \mathcal{M} \to ... \to \Omega^n(X) \underset{\mathcal{O}_X}{\otimes} \mathcal{M}.$$

The differential $\mathfrak{d}$ is constructed as follows: given $\omega^k \otimes m \in \Omega^k(X) \underset{\mathcal{O}_X}{\otimes} \mathcal{M}$, we have a map $T(X) \to \mathcal{M}$ given by $\xi \mapsto \xi \cdot m$, i.e., a section $m' \in \Omega^1(X) \underset{\mathcal{O}_X}{\otimes} \mathcal{M}$, and the sought-for section of $\Omega^{k+1}(X) \underset{\mathcal{O}_X}{\otimes} \mathcal{M}$ equals

$$d(\omega^k) \cdot m + (-1)^k \cdot \omega^k \wedge m'.$$

It is easy to see that this is well-defined, i.e., that $f \cdot \omega^k \otimes m$ and $\omega^k \otimes f \cdot m$ give the same result. (Here we use the fact that $\xi \cdot f \cdot m - f \cdot \xi \cdot m = \xi(f) \cdot m$ for $\xi \in T(X)$.) The condition that $\xi_1 \cdot \xi_2 \cdot m - \xi_2 \cdot \xi_1 \cdot m = [\xi_1, \xi_2] \cdot m$ implies that $\mathfrak{d}^2 = 0$.

Note that the terms of the complex $DR(\mathcal{M})$ are quasi-coherent as sheaves on X, but the differential $\mathfrak{d}$ is not $\mathcal{O}_X$-linear. In fact

$$\mathfrak{d}(f \cdot m) = f \cdot \mathfrak{d}(m) + df \wedge m.$$

Lemma 5.18. *A structure of left D-module on a quasi-coherent sheaf $\mathcal{M}$ is equivalent to that of a differential on $\Omega^\bullet(X) \underset{\mathcal{O}_X}{\otimes} \mathcal{M}$, which satisfies $\mathfrak{d}(\omega^k \cdot m) = d(\omega^k) \cdot m + (-1)^k \cdot \omega^k \cdot \mathfrak{d}(m)$.*

Let $\mathcal{M}^r = \Omega^n(X) \underset{\mathcal{O}_X}{\otimes} \mathcal{M}$ be the right D-module corresponding to $\mathcal{M}$. Let us interpret $DR(\mathcal{M})$ from this point of view. We obtain that $DR(\mathcal{M}^r) := DR(\mathcal{M})[n]$ identifies with the complex

$$... \to \mathcal{M}^r \underset{\mathcal{O}_X}{\otimes} \Lambda^k_{\mathcal{O}_X}(T(X)) \to ... \to \mathcal{M}^r \underset{\mathcal{O}_X}{\otimes} T(X) \to \mathcal{M}^r,$$

where the differential is defined as follows:

$$\mathfrak{d}(m \otimes \xi_1 \wedge ... \wedge \xi_k) = \sum_i (-1)^{i-1} \cdot (m \cdot \xi_i) \otimes (\xi_1 \wedge ... \wedge \widehat{\xi_i} \wedge ... \wedge \xi_k) +$$
$$\sum_{i<j} (-1)^{i+j-1} \cdot m \otimes ([\xi_i, \xi_j] \wedge \xi_1 \wedge ... \wedge \widehat{\xi_i} \wedge ... \wedge \widehat{\xi_j} \wedge ... \wedge \xi_k).$$

The De Rham cohomology, denoted $H^\bullet_{DR}(X, \mathcal{M})$, of a left D-module $\mathcal{M}$ on X is defined as the hypercohomology of the complex $DR(\mathcal{M})[n]$. By Serre's theorem that assures that $H^i(X, \mathcal{F}) = 0$ if X is affine, $\mathcal{F}$ quasi-coherent and $i > 0$, the De Rham cohomology can be computed explicitly as follows.

Let X_i be an open cover of X by affines. We can form a bi-complex, whose k, l-th term is

$$\underset{i_1,...,i_l}{\oplus} \Omega^k(X_{i_1} \cap ... \cap X_{i_l}) \underset{\mathcal{O}_{X_{i_1} \cap ... \cap X_{i_l}}}{\otimes} \mathcal{M}|_{X_{i_1} \cap ... \cap X_{i_l}}.$$

Then $H^\bullet_{DR}(X, \mathcal{M})[-n]$ is the cohomology of the complex associated to this bi-complex.

Exercise. Let $\imath : Y \hookrightarrow X$ be a closed embedding and let $\mathcal{M}^r$ be a right D-module on Y. Then

$$H^\bullet_{DR}(Y, \mathcal{M}^r) \simeq H^\bullet_{DR}(X, \imath_\star(\mathcal{M}^r)).$$

Exercise. Let X be affine. Show that $\mathcal{M}^r \mapsto H^\bullet_{DR}(X, \mathcal{M}^r)$ is the left derived functor of

$$\mathcal{M}^r \mapsto \mathcal{M}^r \underset{\mathfrak{D}(X)}{\otimes} \mathcal{O}_X \simeq \mathcal{M}^r / \mathcal{M}^r \cdot T(X).$$

5.19. **Relative De Rham complex and direct image.** Let $\phi : Y \to X$ be a smooth morphism. We are going to construct a functor $\phi_\star : D(\mathfrak{D}(Y)^r\text{-mod}) \to D(\mathfrak{D}(X)^r\text{-mod})$, called the direct image. If $X = \mathrm{pt}$ the direct image will be the same as De Rham cohomology.

Assume first that ϕ is affine. Consider $\phi^*(\mathfrak{D}(X))$. This is a $(\mathfrak{D}(Y), \mathfrak{D}(X))$-bimodule. For $\mathcal{M}^r \in \mathfrak{D}(Y)^r$-mod consider $\mathcal{M}^r \underset{\mathcal{O}_Y}{\otimes} \phi^*(\mathfrak{D}(X))$ as right $\mathfrak{D}(Y)$-module, which has an additional right $\mathfrak{D}(X)$-action. Then $DR\big(\mathcal{M}^r \underset{\mathcal{O}_Y}{\otimes} \phi^*(\mathfrak{D}(X))\big)$ is naturally a complex of $\mathfrak{D}(X)$-modules.

The above construction extends to a functor from the category of complexes of D-modules on Y to that on X; morover, this functor is easily seen to send acyclic complexes to acyclic ones; hence, it gives rise to a functor $D(\mathfrak{D}(Y)^r\text{-mod}) \to D(\mathfrak{D}(X)^r\text{-mod})$. This is the sought-for functor $\phi_\star$ for affine morphisms.

When ϕ is not affine, we use the Chech complex as in the definition of $H^\bullet_{DR}(X, ?)$.

Exercise. Show that for ϕ being the identity morphism $X \to X$, there exists a canonical quasi-isomorphism $\phi_\star(\mathcal{M}) \to \mathcal{M}$.

Suppose now that $\phi : Y \to X$ is an arbitrary morphism between smooth varietie. We can always factor it as $Y \xrightarrow{\imath} Z \xrightarrow{\pi} X$, where $\imath$ is a closed embedding, and π is smooth. (E.g. take $Z = X \times Y$, with $\imath$ being the graph map.)

Define $\phi_\star : D(\mathfrak{D}(Y)^r\text{-mod}) \to D(\mathfrak{D}(X)^r\text{-mod})$ to be the composition $\pi_\star \circ \imath_\star$. One shows that this functor is, on the level of derived categories, canonically independent of the choice of the factorization. (For that one uses a generalization of the exercise about the behavior of $H^\bullet_{DR}(X, ?)$ under closed embeddings.)

One also shows that for two morphisms $\phi : Y \to X$ and $\psi : Z \to Y$ there exists a canonical isomorphism $(\phi \circ \psi)_\star \simeq \phi_\star \circ \psi_\star$.

5.20. **Summary of functors and adjunctions.** For any map $\phi : Y \to X$ we always have the functor $L\phi^* : D(\mathfrak{D}(X)\text{-mod}) \to D(\mathfrak{D}(Y)\text{-mod})$. The functor $\phi^! : D(\mathfrak{D}(X)^r\text{-mod}) \to D(\mathfrak{D}(Y)^r\text{-mod})$ is defined by

$$\phi^!(\mathcal{M}^r) \simeq \big(L\phi^*(\mathcal{M})\big)^r[\dim(Y) - \dim(X)].$$

On the level of quasi-coherent sheaves, $\phi^!$ goes over to the same-named operation.

For ϕ being a closed embedding, this is consistent with the (derived version) of the definition given earlier. If ϕ is smooth, we define also the functor $\phi^\star : D(\mathfrak{D}(X)^r\text{-mod}) \to D(\mathfrak{D}(Y)^r\text{-mod})$ by

$$\phi^\star(\mathcal{M}) = \phi^!(\mathcal{M})[2 \cdot (\dim(X) - \dim(Y))].$$

For any map $\phi : Y \to X$ we always have the functor $\phi_\star : D(\mathfrak{D}(Y)^r\text{-mod}) \to D(\mathfrak{D}(X)^r\text{-mod})$.

If ϕ is a closed embedding, this is the same functor as was introduced earlier. For ϕ being an open embedding, on the level of quasi-coherent sheaves, $\phi_\star$ coincides with $R\phi_*$. If ϕ is proper (in particular, a closed embedding), define $\phi_!$ as $\phi_\star$.

We have the adjunction

$$Hom_{D(\mathfrak{D}(X)^r\text{-mod})}(\phi_!(\mathcal{M}^r), \mathcal{N}^r)) \simeq Hom_{D(\mathfrak{D}(Y)^r\text{-mod})}(\mathcal{M}^r, \phi^!(\mathcal{N}^r)),$$

which is valid whenever the LHS is defined, i.e., when ϕ is proper.

We also have the adjunction

$$Hom_{D(\mathfrak{D}(Y)^r\text{-mod})}(\phi^\star(\mathcal{N}^r), \mathcal{M}^r) \simeq Hom_{D(\mathfrak{D}(X)^r\text{-mod})}(\mathcal{N}^r, \phi_\star(\mathcal{M}^r)),$$

valid when the LHS is defined, i.e., when ϕ is smooth.

6. Localization theory

6.1. **Groups acting on schemes.** Let X be a scheme endowed with an action of an algebraic group G:

$$G \times X \overset{act}{\to} X.$$

Then we have a map $a : \mathfrak{g} \to \Gamma(X, T(X))$, defined as follows:

Consider the first infinitesimal neighborhood $G^{(1)}$ of $1 \in G$, i.e., $G^{(1)} \simeq \mathrm{Spec}(\mathbb{C} \oplus \epsilon \cdot \mathfrak{g}^* \mid \epsilon^2 = 0)$. The action map defines a morphism of schemes $G^{(1)} \times X \to X$, and since $G^{(1)}$ is nilpotent, for every open affine $U \subset G$ we have a map

$$\mathcal{O}_U \to \mathcal{O}_U \otimes \left(\mathbb{C} \oplus \epsilon \cdot \mathfrak{g}^*\right).$$

The projection $\mathcal{O}_U \to \mathcal{O}_U \otimes \mathfrak{g}^*$ is a derivation, i.e., we obtain an element of $Hom_{\mathcal{O}_U}(\Omega^1(U), \mathcal{O}_U \otimes \mathfrak{g}^*)$, and, dualizing, a map $\mathfrak{g} \to T(U)$.

Recall that a quasi-coherent sheaf $\mathcal{M}$ on X is called equivariant if we are given an isomorphism

$$\phi_{\mathcal{M}} : act^*(\mathcal{M}) \simeq p_2^*(\mathcal{M}),$$

where p_2 and act are the projection and the actions maps $G \times X \rightrightarrows X$. The following axioms must hold:

- (i) The restriction of $\phi_{\mathcal{M}}$ to $1 \times X \subset G \times X$ is the identity map $\mathcal{M} \to \mathcal{M}$.
- (ii) The diagram of maps of sheaves on $G \times G \times X$

$$\begin{tikzcd}
(\mathrm{id} \times act)^* \circ act^*(\mathcal{M}) \arrow[r, "(\mathrm{id} \times act)^*(\phi_{\mathcal{M}})"] \arrow[d, "\sim"'] & (\mathrm{id} \times act)^* \circ p_2^*(\mathcal{M}) \arrow[r, "\mathrm{id} \times \phi_{\mathcal{M}}"] & p_3^*(\mathcal{M}) \arrow[d, "="'] \\
(mult \times \mathrm{id})^* \circ act^*(\mathcal{M}) \arrow[r, "\phi_{\mathcal{M}}"] & (mult \times \mathrm{id})^* \circ p_2^*(\mathcal{M}) \arrow[r, "\sim"] & p_3^*(\mathcal{M})
\end{tikzcd}$$

  must commute, where p_3 is the projection $G \times G \times X \to X$.

Exercise. Show that every map $\phi_{\mathcal{F}}$ as above, which satisfies condition (i) is automatically an isomorphism. Show also that one can equivalently define equivariant sheaves using a map in the opposite direction, i.e., $p_2^*(\mathcal{M}) \to act^*(\mathcal{M})$, satisfying analogous two conditions, and that such a map will also automatically be an isomorphism.

Equivariant sheaves form a category: morphisms between $(\mathcal{M}_1, \phi_{\mathcal{M}_1})$ and $(\mathcal{M}_2, \phi_{\mathcal{M}_2})$ are maps of sheaves $\mathcal{M}_1 \to \mathcal{M}_2$, such that the diagram

$$\begin{tikzcd}
act^*(\mathcal{M}_1) \arrow[r, "\phi_{\mathcal{M}_1}"] \arrow[d] & p_2^*(\mathcal{M}_1) \arrow[d] \\
act^*(\mathcal{M}_2) \arrow[r, "\phi_{\mathcal{M}_2}"] & p_2^*(\mathcal{M}_2)
\end{tikzcd}$$

commutes.

A typical example of an equivariant sheaf is $\mathcal{M} = \mathcal{O}_X$.

Lemma 6.2. *If $\mathcal{M}$ is a G-equivariant quasi-coherent sheaf on X, there exists a canonical map of Lie algebras $a^\sharp : \mathfrak{g} \to End(\mathcal{M})$, such that for $\xi \in \mathfrak{g}$, a local section $m \in \Gamma(U, \mathcal{M})$ and $f \in \mathcal{O}_U$ we have:*

$$a^\sharp(\xi) \cdot f \cdot m = f \cdot a^\sharp(\xi) \cdot m + \langle a(\xi), df\rangle \cdot m.$$

Proof. Let us restrict the isomorphism $\phi_{\mathcal{F}}$ to $G^{(1)} \times X$. For every open affine U we obtain a map

$$\Gamma(U, \mathcal{M}) \to \Gamma(G^{(1)} \times X, act^*(\mathcal{M})|_{G^{(1)} \times X}) \overset{\phi_{\mathcal{F}}}{\simeq} \Gamma(U, \mathcal{M}) \otimes \left(\mathbb{C} \oplus \epsilon \cdot \mathfrak{g}^*\right) \twoheadrightarrow \Gamma(U, \mathcal{M}) \otimes \mathfrak{g}^*.$$

The dual map $\mathfrak{g} \otimes \Gamma(U, \mathcal{M}) \to \Gamma(U, \mathcal{M})$ is the sought-for action $a^\sharp$.

□

Let us consider the example of $X = G$ acting on itself by, say, left translations.

Proposition 6.3. *The category of G-equivariant quasi-coherent sheaves on G is equivalent to that of vector spaces.*

Proof. The functors in both directions are defined as follows. For a vector space V we consider the sheaf $\mathcal{M} := V \otimes \mathcal{O}_X$, with the equivariant structure induced by that of $\mathcal{O}_G$.

The functor in the opposite directions is given by $\mathcal{M} \mapsto \mathcal{M}_1$.

□

Let $\mathcal{M}$ be a G-equivariant sheaf on X. We say that a global section $m \in \Gamma(X, \mathcal{M})$ is G-invariant if its image under

$$\Gamma(X, \mathcal{M}) \to \Gamma(G \times X, act^*(\mathcal{M})) \overset{\phi_{\mathcal{M}}}{\simeq} \Gamma(G, \mathcal{O}_G) \otimes \Gamma(X, \mathcal{M})$$

equals $1 \otimes m$.

Exercise. Show that for $X = G$ the functor $\mathcal{M} \mapsto \mathcal{M}_1$ is isomorphic to the functor that associates to $\mathcal{M}$ the vector space of its G-invariant sections.

6.4. **Differential operators as an equivaraint sheaf.** Before we discuss the equivariant structure on the sheaf of differential operators, we need to make the following digression.

Let X be a scheme, and let $\mathcal{M}$ be a sheaf of $\mathcal{O}_X$-bimodules on it, quasi-coherent, with respect to one of the structures. Assume also that the following holds:

For every local section $m \in \mathcal{M}$ there exists an integer k, such that $\mathrm{ad}_{f_1} \cdot \ldots \cdot \mathrm{ad}_{f_k}(m) = 0$ for any k-tuple of local sections of $\mathcal{O}$, where $\mathrm{ad}_f(m) := f \cdot m - m \cdot f$.

Lemma 6.5.

(1) *Under the above circumstances, there exists a quasi-coherent sheaf $\widetilde{\mathcal{M}}$ on $X \times X$, supported set-theoretically on the diagonal $\Delta_X \subset X \times X$, such that $\mathcal{M} \simeq p_{1*}(\widetilde{\mathcal{M}})$ and $\mathcal{M} \simeq p_{2*}(\widetilde{\mathcal{M}})$. In particular, $\mathcal{M}$ is quasi-coherent with respect to the other $\mathcal{O}_X$-module structure too.*

(2) *For another algebraic variety Y we have a canonical isomorphism*

$$p_{1,2*}\left(p_{2,3}^*(\widetilde{\mathcal{M}})\right) \simeq p_2^*(\mathcal{M}),$$

where $p_{i,j}$ denotes the projection on the i and j factors of $Y \times X \times X$.

Proof. The assertion immediately reduces to the affine situation, where $X = \mathrm{Spec}(A)$ and $\mathcal{M}$ corresponds to an A-bimodule.

We have to prove the following: if f is a non-nilpotent element of A, then

$$A_f \underset{A}{\otimes} \mathcal{M} \simeq (A_f \otimes A_f) \underset{A \otimes A}{\otimes} \mathcal{M}.$$

We have an evident map from the LHS to the RHS. To prove that it is an isomorphism, we can assume that as an $A \otimes A$-module, $\mathcal{M}$ is annihilated by I^k, where $I = \ker(A \otimes A \to A)$. Then the assertion becomes obvious, since the open subsets

$$\mathrm{Spec}(A \otimes A/I^k)_{f\otimes 1, 1\otimes f} \text{ and } \mathrm{Spec}(A \otimes A/I^k)_{f\otimes 1}$$

of $\mathrm{Spec}(A \otimes A/I^k)$ coincide, since the underlying topological spaces of both of them are equal to $\mathrm{Spec}^{top}(A_f) \subset \mathrm{Spec}^{top}(A) \simeq \mathrm{Spec}^{top}(A \otimes A/I^k)$.

□

We will use this lemma for $\mathcal{M}$ being the sheaf of differential operators $\mathfrak{D}(X)$. Note that the assertion of the lemma implies, in particular, Proposition 5.5.

Proposition 6.6. *The sheaf $\mathfrak{D}(X)$, as a quasi-coherent sheaf on $X \times X$, is naturally G-equivariant.*

Note that by point (2) of the above lemma, we also obtain that $\mathfrak{D}(X)$ acquires a G-equivariant structure, when viewed as a quasi-coherent sheaf in each of the two structures.

Proof. Let U_X be an affine open subset of X and let V denote its preimage in $U_G \times X$, under the map *act*, where U_G is some affine open subset of G. Let us denote by $\widetilde{V}$ the open subset $V \underset{U_G}{\times} V \subset G \times X \times X$. We need to specify a map

$$\phi : \Gamma(U_X, \mathfrak{D}(X)) \to \Gamma\big(\widetilde{V}, p^*_{2,3}(\mathfrak{D}(X))\big),$$

which has the required commutation properties with respect to multiplication by elements of $\mathcal{O}_{U_X}$.

Lemma 6.7. *Let Y be an affine algebraic variety, and V be an open subset of $Y \times X$. Let $\widetilde{V} := V \underset{Y}{\times} V$ be the corresponding open subset of $Y \times X \times X$. Then $\Gamma\big(\widetilde{V}, p^*_{2,3}(\mathfrak{D}(X))\big)$ is isomorphic to the set of $\mathcal{O}_Y$-linear differential operators on V.*

Thus, to each $D \in \mathfrak{D}(U)$ we need to assign an $\mathcal{O}_G$-linear differential operator $\phi(D)$ on V. Note that the map

$$exch : G \times X \to G \times X : (g, x) \mapsto (g, g^{-1} \cdot x)$$

defines an isomorphism $V \to U_G \times U_X$. Given a function $f \in \Gamma(V, \mathcal{O}_{G\times X})$, we set

$$\phi(D) := exch^{-1}\big(D(exch(f))\big),$$

where $exch(f)$ is a function on $exch(V) \simeq U_G \times U_X$, and hence $D(exch(f))$ makes sense.

□

Thus, in particular, we obtain an action of $\mathfrak{g}$ on the sheaf $\mathfrak{D}(X)$.

Lemma 6.8. *For $\xi \in \mathfrak{g}$ and D a local section of $\mathfrak{D}(X)$,*

$$a^\sharp(\xi) \cdot D = a(\xi) \cdot D - D \cdot a(\xi).$$

6.9. **Differential operators on the group G.** Let us consider the group G as acting on itself by left and right translations. In particular, we obtain Lie algebra homomorphisms

$$a_l : \mathfrak{g} \to \Gamma(G, \mathfrak{D}(G)) \leftarrow \mathfrak{g} : a_r,$$

whose images evidently commute. In addition, the image of a_l is G-invariant under the right equivariant structure, and the image of a_r is G-invariant under the left-equivariant structure.

Thus, we obtain the maps

$$a_l : \mathcal{O}_G \otimes U(\mathfrak{g}) \to \mathfrak{D}(G) \leftarrow \mathcal{O}_G \otimes U(\mathfrak{g}) : a_r. \tag{6.1}$$

Note that a_r is expressible in terms of a_l as follows:

$$a_r(u) = (\mathrm{id}_{\mathcal{O}_G} \otimes a_l)(\Delta(u')),$$

where Δ is the map

$$U(\mathfrak{g}) \to \mathcal{O}_G \otimes U(\mathfrak{g}),$$

and $u \mapsto u'$ is the anti-involution on $U(\mathfrak{g})$ induced by $\xi \mapsto -\xi$ on $\mathfrak{g}$.

corresponding to the adjoint action of G on $U(\mathfrak{g})$.

Proposition 6.10. *The maps a_l and a_r are isomorphisms.*

Proof. Let us treat the case of a_l. As this map is compatible with filtrations, it is sufficient to check that

$$\mathcal{O}_G \otimes \mathrm{gr}(U(\mathfrak{g})) \to \mathrm{gr}(U(\mathfrak{g}))$$

is an isomorphism. We have

$$\mathcal{O}_G \otimes \mathrm{Sym}(\mathfrak{g}) \twoheadrightarrow \mathcal{O}_G \otimes \mathrm{gr}(U(\mathfrak{g})) \to \mathrm{Sym}_{\mathcal{O}_G}(T(X)),$$

where the composed map is evidently an isomorphism. This implies that both above arrows are isomorphisms, i.e., the assertion of the proposition, and, as a bonus, the PBW theorem. □

Corollary 6.11. *We have canonical isomorphisms:*

$$U(\mathfrak{g}) \simeq \mathfrak{D}(G)_1 \simeq \varinjlim_k (\mathcal{O}_G/\mathfrak{m}^k)^*,$$

where $\mathfrak{m} \subset \mathcal{O}_G$ is the maximal ideal, corresponding to $1 \in G$.

Proof. The first isomorphism follows immediately from Proposition 6.3. The second isomorphism is valid for any smooth algebraic variety X:

$$\delta_x \simeq \mathbb{C}_x \underset{\mathcal{O}_X}{\otimes} \mathfrak{D}(X) \simeq \varinjlim_k (\mathcal{O}_X/\mathfrak{m}_x^k)^*.$$

□

Remark. Note that the last corollary gives a nice interpretation of the formal group-law on $\widehat{\mathcal{O}_G} \simeq \varprojlim_k \mathcal{O}_G/\mathfrak{m}^k$. Indeed, by Corollary 6.11, $\widehat{\mathcal{O}_G}$ identifies with the full linear dual of $U(\mathfrak{g})^*$. The latter is naturally a commutative topological Hopf algebra, since $U(\mathfrak{g})$ is a commutative Hopf algebra.

In this terms the exponential map, which is an algebra isomorphism $\widehat{\mathcal{O}_G} \simeq \widehat{\mathcal{O}_\mathfrak{g}}$ is given by the symmetrization map

$$\mathrm{Sym}(\mathfrak{g}) \to U(\mathfrak{g}) : \xi^n \mapsto \frac{\xi^i}{i!}.$$

Let us discuss some other corollaries of Proposition 6.10.

Corollary 6.12. *The subset $a_l(U(\mathfrak{g})) \subset \Gamma(G, \mathfrak{D}(G))$ (resp., $a_l(U(\mathfrak{g})) \subset \Gamma(G, \mathfrak{D}(G))$) coincides with the set of G-invariant differential operators under the right (resp., left) action.*

Corollary 6.13. *The images of $Z(\mathfrak{g}) \subset U(\mathfrak{g})$ under a_l and a_r coincide. The corresponding maps are intertwined by the involution of $Z(\mathfrak{g})$, induced by the anti-involution $\xi \mapsto -\xi$ of $U(\mathfrak{g})$.*

Finally, let us see how the actions $a_l^\sharp$ and $a_r^\sharp$ look in terms of (6.1):

Lemma 6.14. *For $\xi \in \mathfrak{g}$, $u \in U(\mathfrak{g})$ and f a local section of $\Gamma(G, \mathcal{O}_G)$,*

$$a_l^\sharp(\xi) \cdot (f \otimes u) = \langle a_l(\xi), df \rangle \otimes u + f \otimes [a_l(\xi), u].$$

6.15. Let X be a smooth algebraic variety, and let G be a group acting on it. We have a homomorphism of Lie algebras $a : \mathfrak{g} \to \Gamma(X, T(X))$, and hence a homomorphism of associative algebras $a : U(\mathfrak{g}) \to \Gamma(X, \mathfrak{D}(X))$.

Hence, if $\mathcal{F}$ is a left D-module on X, its space of global sections is naturally a $\mathfrak{g}$-module. Thus, we obtain a functor:

$$\Gamma : \mathfrak{D}(X)\text{-mod} \to \mathfrak{g}\text{-mod}.$$

We shall now construct a left adjoint of this functor, called "the localization functor".

For $\mathcal{M} \in \mathfrak{g}$-mod, set

$$\mathrm{Loc}(\mathcal{M}) := \mathfrak{D}(X) \underset{\underline{U(\mathfrak{g})}}{\otimes} \underline{\mathcal{M}},$$

where $\underline{U(\mathfrak{g})}$ is the constant sheaf of algebras in the Zariski topology of X with fiber $U(\mathfrak{g})$ and $\underline{\mathcal{M}}$ is the corresponding sheaf of modules over it.

Lemma 6.16. *For $\mathcal{M} \in \mathfrak{g}$-mod and $\mathcal{F} \in \mathfrak{D}(X)$-mod we have a canonical isomorphism*

$$Hom_{\mathfrak{D}(X)\text{-}mod}(\mathrm{Loc}(\mathcal{M}), \mathcal{F}) \simeq Hom_{\mathfrak{g}\text{-}mod}(\mathcal{M}, \Gamma(X, \mathcal{F})).$$

6.17. The flag variety. We shall now specialize to the case when G is a semi-simple affine algebraic group, and X is its flag variety, i.e., $X = G/B$, where $B \subset G$ is the Borel subgroup. Our goal is to prove the following theorem:

Theorem 6.18.

(1) *The homomorphism $Z(\mathfrak{g}) \to \Gamma(G/B, \mathfrak{D}(G/B))$ factors through the character χ_0, corresponding to the trivial $\mathfrak{g}$-module.*

(2) *The resulting homomorphism $U(\mathfrak{g})_{\chi_0} \to \Gamma(G/B, \mathfrak{D}(G/B))$ is an isomorphism.*

(3) *The functor of sections $\Gamma : \mathfrak{D}(G/B)$-mod $\to U(\mathfrak{g})_{\chi_0}$-mod is exact and faithful.*

(4) *The functor Γ and its adjoint $\mathrm{Loc} : U(\mathfrak{g})_{\chi_0}$-mod $\to \mathfrak{D}(G/B)$-mod are mutually quasi-inverse equivalences of categories.*

Our goal from now on will be to prove this theorem.

6.19. Proof of (4) modulo (2) and (3). We will do this in the following general context: let $\mathcal{B}$ be a sheaf of associative algebras over a scheme X equipped with a homomorphism $\mathcal{O}_X \to \mathcal{B}$, and such that $\mathcal{B}$ is quasi-coherent as a left $\mathcal{O}_X$-module. Denote $A := \Gamma(X, \mathcal{B})$.

Let $\mathcal{B} - mod$ be the category of sheaves of $\mathcal{B}$-modules, which are quasi-coherent as $\mathcal{O}_X$-modules. We have a natural functor:

$$\Gamma : \mathcal{B}\text{-mod} \to A\text{-mod} : \mathcal{F} \mapsto \Gamma(X, \mathcal{F}),$$

and its left adjoint

$$\mathrm{Loc} : \mathcal{M} \mapsto \mathcal{B} \underset{\underline{A}}{\otimes} \underline{\mathcal{M}},$$

where $\underline{A}$ is the constant sheaf of algebras in the Zariski topology with fiber A, and $\underline{\mathcal{M}}$ is the corresponding sheaf of modules over it.

Assume that the functor Γ is exact and faithful. We claim that in this case Γ and Loc are mutually quasi-inverse equivalences of categories.

Let us first show that the adjunction morphism $\mathrm{Id}_{A\text{-mod}} \to \Gamma \circ \mathrm{Loc}$ is an isomorphism. Note that this map is an isomorphism when evaluated on the A-module equal to A itself: $\mathrm{Loc}(A) \simeq \mathcal{B}$, and the above adjunction morphism is the identity map $A \to A := \Gamma(X, \mathcal{B})$.

Hence, the above adjunction morphism is an isomorphism for every free A-module. Let us show that the exactness assumption on Γ implies that $\mathcal{M} \to \Gamma(X, \mathrm{Loc}(\mathcal{M}))$ is an isomorphism for any $\mathcal{M} \in A$-mod.

Indeed, for $\mathcal{M}$ as above, let

$$P_1 \to P_0 \to M \to 0$$

be an exact sequence with P_0 and P_1 free. We have a commutative diagram:

$$\begin{tikzcd} P_1 \arrow[r] \arrow[d] & P_1 \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & 0 \\ \Gamma(X, \mathrm{Loc}(P_1)) \arrow[r] & \Gamma(X, \mathrm{Loc}(P_0)) \arrow[r] & \Gamma(X, \mathrm{Loc}(M)) \arrow[r] & 0. \end{tikzcd}$$

We claim that the bottom row is exact. This is so because the functor Loc is tautologically right exact, and the functor Γ is exact by assumption. Since for $i = 0, 1$ the vertical maps $P_i \to \Gamma(X, \mathrm{Loc}(P_i))$ are known to be isomorphisms, then so is the map $\mathcal{M} \to \Gamma(X, \mathrm{Loc}(\mathcal{M}))$.

Finally, let us show that the adjunction map $\mathrm{Loc}(\Gamma(X, \mathcal{F})) \to \mathcal{F}$ is an isomorphism for any $\mathcal{F} \in \mathcal{B}$-mod. Since Γ was is faithful and exact, it is enough to show that this morphism becomes an isomorphism after applying the functor Γ. However, the composition

$$\Gamma(X, \mathcal{F}) \to \Gamma\big(X, \mathrm{Loc}(\Gamma(X, \mathcal{F}))\big) \to \Gamma(X, \mathcal{F})$$

is the identity map, and the first map is an isomorphism, by what we have shown above. Hence, the second map is an isomorphism too.

6.20. Fibers of localization. Let us first describe the fibers of D-modules $\mathrm{Loc}(\mathcal{M})$ for any X, which is a homegeneous space. For $x \in X$, let $\mathfrak{g}_x$ be its stabilizer in $\mathfrak{g}$, i.e., the kernel of the map $\mathfrak{g} \to \Gamma(X, T(X)) \to T_x(X)$.

Proposition 6.21. *For $\mathcal{M} \in \mathfrak{g}$-mod, we have a canonical isomorphism*

$$\big(\mathrm{Loc}(\mathcal{M})\big)_x \simeq (\mathcal{M})_{\mathfrak{g}_x}.$$

Proof. By definition,

$$\big(\mathrm{Loc}(\mathcal{M})\big)_x \simeq \mathbb{C}_x \underset{\mathcal{O}_X}{\otimes} \mathfrak{D}(X) \underset{U(\mathfrak{g})}{\otimes} \mathcal{M}.$$

Note that the tensor product $\mathbb{C}_x \underset{\mathcal{O}_X}{\otimes} \mathfrak{D}(X)$ is the right D-module δ_x. So, the statement of the proposition is equivalent to the following:

There exists a canonical isomorphism of $\mathfrak{g}$-modules:

$$U(\mathfrak{g}) \underset{U(\mathfrak{g}_x)}{\otimes} \mathbb{C} \to \Gamma(X, \delta_x). \tag{6.2}$$

Recall that δ_x has a canonical generator $\mathbf{1}_x$. It is annihilated by all vector fields that stabilize x. This gives a map in one direction ($\Rightarrow$) in (6.2).

Moreover, both sides of (6.2) are naturally filtered with

$$\mathrm{gr}(U(\mathfrak{g}) \underset{U(\mathfrak{g}_x)}{\otimes} \mathbb{C}) \simeq \mathrm{Sym}(\mathfrak{g}/\mathfrak{g}_x) \text{ and } \mathrm{gr}(\delta_x) \simeq \mathrm{Sym}(T_x(X)).$$

The map constructed above is compatible with filtrations, and it is easy to see that on the associated graded level it gives rise to the map

$$\mathrm{Sym}(\mathfrak{g}/\mathfrak{g}_x) \to \mathrm{Sym}(T_x(X)),$$

which is an isomorphism, since $\mathfrak{g}$ maps surjectively onto $T_x(X)$.

□

Let us see how the above proposition implies point (1) of Theorem 6.18:

Proof. It is enough to show that for any $u \in \ker(\chi_0)$ the image of u in the fiber of $\mathfrak{D}(G/B)$ at any point $x \in G/B$ is zero.

Note that the above image equals the result of the action of u on the generator of $\mathbf{1}_x \in \delta_x$ as a right D-module. Hence, by the previous proposition, it can be thought of as the image of the generator of $U(\mathfrak{g}) \underset{U(\mathfrak{g}_x)}{\otimes} \mathbb{C}$ under the action of u.

However, $U(\mathfrak{g}) \underset{U(\mathfrak{g}_x)}{\otimes} \mathbb{C} \simeq M_0$ for the choice of the Borel subalgebra, corresponding to x. The assertion follows now from the fact (Harish-Chandra's map) that $Z(\mathfrak{g})$ acts on M_0 by the same character as on $\mathbb{C}$.

□

Exercise. Let δ_x^l be the left D-module corresponding to δ_x under the equivalence $\mathfrak{D}(X)$-mod $\simeq$ $\mathfrak{D}(X)^r$-mod. Show that $\Gamma(G/B, \delta_x^l)$ is (non-canonically) isomorphic to the module $M_{-2\rho}$, where the latter is again the Verma module with highest weight -2ρ for the Borel subalgebra, corresponding to x.

6.22. Point (2) of Theorem 6.18. The proof of the second point of the LocalizationTheorem involves some serious (but fun) discussion of certain aspects of algebraic geometry related to $\mathfrak{g}$.

The map in question is evidently compatible with filtartions. Let us analyze its behavior at the associated graded level.

Lemma 6.23. *For any smooth (but not necessarily affine) algebraic variety X there is a natural embedding*

$$\operatorname{gr}\big(\Gamma(X, \mathfrak{D}(X))\big) \hookrightarrow \Gamma\big(X, \operatorname{Sym}_{\mathcal{O}_X}(T(X))\big) \simeq \Gamma(T^*(X), \mathcal{O}_{T^*(X)}).$$

Proof. The asserion follows from the fact that the functor of global sections is left-exact. Indeed, the short exact sequence

$$0 \to \mathfrak{D}(X)_{i-1} \to \mathfrak{D}(X)_i \to \operatorname{Sym}^i_{\mathcal{O}_X}(T(X)) \to 0$$

gives rise to an exact sequence

$$0 \to \Gamma(X, \mathfrak{D}(X)_{i-1}) \to \Gamma(X, \mathfrak{D}(X)_i) \to \Gamma\big(X, \operatorname{Sym}^i_{\mathcal{O}_X}(T(X))\big).$$

□

Recall that $\operatorname{gr}(Z(\mathfrak{g})) \simeq \operatorname{Sym}(\mathfrak{g})^G$, and let $\operatorname{Sym}(\mathfrak{g})^G_+$ be the kernel of the homomorphism

$$\operatorname{Sym}(\mathfrak{g})^G \hookrightarrow \operatorname{Sym}(\mathfrak{g}) \to \mathbb{C}.$$

Consider the map

$$\operatorname{Sym}(\mathfrak{g}) \to \Gamma\big(X, \operatorname{Sym}_{\mathcal{O}_X}(T(X))\big), \tag{6.3}$$

obtained from $a : \mathfrak{g} \to \Gamma(X, T(X))$.

Theorem 6.24. *(Kostant)*

The map (6.3) *for $X = G/B$ annihilates* $\operatorname{Sym}(\mathfrak{g})^G_+$ *and the resulting map*

$$\operatorname{Sym}(\mathfrak{g})/\operatorname{Sym}(\mathfrak{g}) \cdot \operatorname{Sym}(\mathfrak{g})^G_+ \to \Gamma\big(G/B, \operatorname{Sym}_{\mathcal{O}_{G/B}}(T(G/B))\big)$$

is an isomorphism.

Let us show how this theorem implies point (2) of Theorem 6.18:

Proof. It is enough to show that the map $\mathrm{gr}(U(\mathfrak{g})_{\chi_0}) \to \mathrm{gr}\big(\Gamma(G/B, \mathfrak{D}(G/B))\big)$ is an isomorphism. We have an evident surjection $\mathrm{Sym}(\mathfrak{g})/\mathrm{Sym}(\mathfrak{g}) \cdot \mathrm{Sym}(\mathfrak{g})^G_+ \to \mathrm{gr}(U(\mathfrak{g})_{\chi_0})$ and let us consider the composition

$$\mathrm{Sym}(\mathfrak{g})/\mathrm{Sym}(\mathfrak{g}) \cdot \mathrm{Sym}(\mathfrak{g})^G_+ \twoheadrightarrow \mathrm{gr}(U(\mathfrak{g})_{\chi_0}) \to \mathrm{gr}\big(\Gamma(G/B, \mathfrak{D}(G/B))\big) \hookrightarrow \tag{6.4}$$

$$\Gamma\big(G/B, \mathrm{Sym}_{\mathcal{O}_{G/B}}(T(G/B))\big). \tag{6.5}$$

By Theorem 6.24, the composite map is an isomorphism. Hence, so are all three arrows appearing in (6.4).

□

7. Some algebraic geometry related to $\mathfrak{g}$

7.1. More on Chevalley's map. Our goal in this section is to prove Kostant's result, Theorem 6.24. First, we will need to revisit Chevalley's map $\mathfrak{g} \to \mathfrak{h}//W$.

Proposition 7.2.

(1) *The variety* $\mathfrak{h}//W$ *is smooth.*

(2) *The map* $\varpi : \mathfrak{h} \to \mathfrak{h}//W$ *is flat.*

Proof. Both facts are proven in Bourbaki in a greater generality: they hold for any finite group, generated by reflections, acting on a vector space. Note that point (2) follows immediately from point (1), since any finite map between regular schemes of the same dimension is flat.

To show (1) let us note that the natural $\mathbb{G}_m$-action on $\mathfrak{h}$ by homotheties descends to $\mathfrak{h}//W$, making it a cone with the vertex being the point $\varpi(0) \in \mathfrak{h}//W$. Hence, it is enough to show that the completed local ring $\widehat{\mathcal{O}}_{\mathfrak{h}//W,\varpi(0)}$ is regular.

Since the map ϖ is finite, we have an isomorphism:

$$\widehat{\mathcal{O}}_{\mathfrak{h}//W,\varpi(0)} \simeq \big(\widehat{\mathcal{O}}_{\mathfrak{h},0}\big)^W.$$

Let us consider also the completed local ring of the point $\varpi(1) \in H//W := \mathrm{Spec}((\mathcal{O}_H)^W)$; we have:

$$\widehat{\mathcal{O}}_{H//W,\varpi(1)} \simeq \big(\widehat{\mathcal{O}}_{H,1}\big)^W.$$

However, the exponential map defines an isomorphism $\widehat{\mathcal{O}}_{H,1} \simeq \widehat{\mathcal{O}}_{\mathfrak{h},0}$, which is functorial, and, hence, W-invariant. Hence,

$$\widehat{\mathcal{O}}_{\mathfrak{h}//W,\varpi(0)} \simeq \widehat{\mathcal{O}}_{H//W,\varpi(1)}.$$

But in Sect. 2.7 we saw that when H corresponds to simply-connected G, the algebra $\mathcal{O}_H^W$ is isomorphic to a polynomial algebra. In particular, it is regular.

□

Corollary 7.3. *The map* $\phi_{cl} : \mathfrak{g} \to \mathfrak{h}//W$ *is flat.*

Proof. We need to show that $\mathrm{Sym}(\mathfrak{g})$ is free as a module over $\mathrm{Sym}(\mathfrak{g})^G \simeq \mathrm{Sym}(\mathfrak{h})^W$. (Note that in the case of non-negatively graded modules over a positively graded commutative algebra, the notions of freeness and flatness are equivalent.)

Let us choose a Borel subalgebra $\mathfrak{b} \subset \mathfrak{g}$. We claim that $\mathrm{Sym}(\mathfrak{g})$ is free as a module over $\mathrm{Sym}(\mathfrak{n}) \underset{\mathbb{C}}{\otimes} \mathrm{Sym}(\mathfrak{g})^G$. For that it is enough to show that $\mathrm{Sym}(\mathfrak{g}/\mathfrak{n})$ is free as a $\mathrm{Sym}(\mathfrak{h})^W$-module. However, $\mathrm{Sym}(\mathfrak{g}/\mathfrak{n})$ is evidently free over $\mathrm{Sym}(\mathfrak{b}/\mathfrak{n}) \simeq \mathrm{Sym}(\mathfrak{h})$, and the latter is flat (and, hence, free) over $\mathrm{Sym}(\mathfrak{h})^W$ by Proposition 7.2.

□

Corollary 7.4. $U(\mathfrak{g})$ *is flat as a module over* $Z(\mathfrak{g})$.

7.5. Grothendieck's alteration. Let us identify $\mathfrak{g}$ with $\mathfrak{g}^*$ by means of a non-degenerate G-invariant form. Note that in this case $T^*(G/B)$ can be interpreted as a fibration over the flag variety $X = G/B$, whose fiber at $x \in X$, corresponding to a Borel subalgebra $\mathfrak{b} \subset \mathfrak{g}$, is its unipotent radical $\mathfrak{n}$. Let us denote this variety by $\widetilde{\mathcal{N}}$. It is endowed with the natural forgetful map $\mathfrak{p} : \widetilde{\mathcal{N}} \to \mathfrak{g}$.

We will consider a bigger algebraic variety, denoted $\widetilde{\mathfrak{g}}$, which is also fibered over $X = G/B$, and whose fiber over $x \in X$ is the corresponding Borel subalgebra $\mathfrak{b}$. We have a natural smooth map

$$\mathfrak{q} : \widetilde{\mathfrak{g}} \to \mathfrak{h},$$

and $\widetilde{\mathcal{N}}$ is the preimage of 0 under this map. We will denote by $\mathfrak{p}$ the forgetful map $\widetilde{\mathfrak{g}} \to \mathfrak{g}$. We can think of $\widetilde{\mathfrak{g}}$ as a closed subvariety in $X \times \mathfrak{g}$, with the map $\mathfrak{p}$ being the projection on the second factor. In particular, this map is proper.

Lemma 7.6. *The square:*

$$\begin{array}{ccc} \widetilde{\mathfrak{g}} & \xrightarrow{\mathfrak{q}} & \mathfrak{h} \\ {\scriptstyle\mathfrak{p}}\big\downarrow & & {\scriptstyle\varpi}\big\downarrow \\ \mathfrak{g} & \xrightarrow{\phi_{cl}} & \mathfrak{h}//W. \end{array}$$

is commutative (but not Cartesian).

The lemma follows from the following picture:

Let $\mathfrak{g}_{reg,ss} \subset \mathfrak{g}$ be the locus of regular semi-simple elements, and let $\widetilde{\mathfrak{g}}_{reg,ss}$ be its preimage in $\widetilde{\mathfrak{g}}$. Let also $\overset{o}{\mathfrak{h}} \subset \mathfrak{h}$ be the complement to all the roots hyperplanes, and let $\overset{o}{\mathfrak{h}}//W$ be the corresponding open subset in $\mathfrak{h}//W$. It is easy to see that we have a Cartesian square:

$$\begin{array}{ccc} \widetilde{\mathfrak{g}}_{reg,ss} & \xrightarrow{\mathfrak{q}} & \overset{o}{\mathfrak{h}} \\ {\scriptstyle\mathfrak{p}}\big\downarrow & & {\scriptstyle\varpi}\big\downarrow \\ \mathfrak{g}_{reg,ss} & \xrightarrow{\phi_{cl}} & \overset{o}{\mathfrak{h}}//W. \end{array}$$

Moreover, the vertical arrows are étale Galois covers with the group W.

Let $\mathcal{N}$ denote the scheme-theoretic preimage of $\varpi(0) \in \mathfrak{h}//W$ under the map ϕ_{cl}. From the diagram in the lemma we obtain that the map $\mathfrak{p} : \widetilde{\mathcal{N}} \to \mathfrak{g}$ factors through $\mathcal{N}$. The resulting map on the level of functions

$$\mathrm{Fun}(\mathcal{N}) \to \mathrm{Fun}(\widetilde{\mathcal{N}})$$

is the one appearing in Theorem 6.24.

7.7. The regular locus. Let $\mathfrak{g}_{reg} \subset \mathfrak{g}$ be the locus of regular elements (we remind that an element $\xi \in \mathfrak{g}$ is called regular if its centralizer in $\mathfrak{g}$ is r-dimensional, where $r = \dim(\mathfrak{h})$). Let $\widetilde{\mathfrak{g}}_{reg}$ denote the preimage of $\mathfrak{g}_{reg}$ in $\widetilde{\mathfrak{g}}$.

Proposition 7.8. *The diagram*

$$\begin{array}{ccc} \widetilde{\mathfrak{g}}_{reg} & \xrightarrow{\mathfrak{q}} & \mathfrak{h} \\ {\scriptstyle\mathfrak{p}}\big\downarrow & & {\scriptstyle\varpi}\big\downarrow \\ \mathfrak{g}_{reg} & \xrightarrow{\phi_{cl}} & \mathfrak{h}//W. \end{array}$$

is Cartesian.

Proof. Since the map $\widetilde{\mathfrak{g}}_{reg} \to \mathfrak{g}_{reg} \underset{\mathfrak{h}//W}{\times} \mathfrak{h}$ is proper, to prove that it is an isomorphism, it is enough to show that the tangent spaces to the fibers vanish.

Thus, let (x, ξ) be a point in $\widetilde{\mathfrak{g}}$, where x corresponds to a Borel subalgebra $\mathfrak{b}$ and ξ is an element in $\mathfrak{b}$. A tangent vector to its fiber over $\mathfrak{g}_{reg} \underset{\mathfrak{h}//W}{\times} \mathfrak{h}$ can be represented by an element $\eta \in \mathfrak{g}$, defined modulo $\mathfrak{b}$, such that $[\eta, \xi] \in \mathfrak{n}$. We claim that if ξ is regular, then η necessarily belongs to $\mathfrak{b}$.

Indeed, regular elements in $\mathfrak{b}$ can be described as follows. Choose a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{b}$. Let $J \subset I$ be a subset of vertices of the Dynkin diagram, and let $\mathfrak{g}_J \subset \mathfrak{g}$ be the corresponding Levi subalgebra.

Then every regular element is conjugate to one of the form $h + n$, where $h \in \mathfrak{h}$ is such that $\alpha_j(h) = 0$ for $j \in J$, and $\beta(h) \neq 0$ for all other roots, and n is a regular nilpotent element in $\mathfrak{g}_J \cap \mathfrak{b}$. This reduces the assertion to the case when $J = I$, i.e., to the case of a regular nilpotent element.

Every regular nilpotent E element in $\mathfrak{b}$ is conjugate to one of the form $\sum\limits_{i \in I} E_i$, where E_i are the Chevalley generators of $\mathfrak{n}$. Let us complete it to an $\mathfrak{sl}_2$-triple E, H, F, where $H \in \mathfrak{h}$ is such that $\langle \alpha_i, H \rangle = 1$ for every $i \in I$, and $F = \sum\limits_{i \in I} c_i \cdot F_i$, where c_i are uniquely determined non-zero scalars.

Then the sub-spaces $\mathfrak{n}^-, \mathfrak{h}$ and $\mathfrak{n}$ are the sub-spaces of $\mathfrak{g}$, corresponding to negative, zero and positive eigenvalues of H, respectively. This makes it clear that $[E, \eta] \in \mathfrak{n} \Rightarrow \eta \in \mathfrak{b}$.

□

Corollary 7.9. *The map $\phi_{cl} : \mathfrak{g}_{reg} \to \mathfrak{h}//W$ is smooth.*

Proof. This follows from the fact that the map $\mathfrak{q} : \widetilde{\mathfrak{g}}_{reg} \to \mathfrak{h}$ is smooth, and the morphism ϖ is flat. □

7.10. First proof of Kostant's theorem. First, from Corollary 7.3 and Theorem 7.2, we obtain that the scheme $\mathcal{N}$ is a complete intersection, and hence Cohen-Macauley.

Secondly, from Corollary 7.9 we obtain that the intersection $\mathcal{N}_{reg} := \mathcal{N} \cap \mathfrak{g}_{reg}$ is smooth. Hence, $\mathcal{N}$ is reduced. We claim that $\mathcal{N}$ is normal. Knowing that it is Cohen-Macaulay, we have to check that it is regular in codimension 1. Assuming that, Theorem 6.24 follows from Zariski's Main Theorem, since the morphism $\widetilde{\mathcal{N}} \to \mathcal{N}$ is birational.

To show that $\mathcal{N}$ is regular in codimension it is sufficient to show that

$$\operatorname{codim}(\mathcal{N} - \mathcal{N}_{reg}) \geq 2.$$

This follows from the next result:

Theorem 7.11. *$\mathcal{N}$ consists of finitely many G-orbits.*

Assuming this theorem, the above inequality follows from the fact that co-adjoint orbits (i.e., orbits of G on $\mathfrak{g}^*$) are symplectic with respect to the natural Poisson structure on $\mathfrak{g}$, and hence are even-dimensional.

7.12. Another proof of Theorem 6.24 is based on the following:

Theorem 7.13.

(1) *There exists a canonical isomorphism*

$$\mathfrak{p}_*(\mathcal{O}_{\widetilde{\mathfrak{g}}}) \simeq \mathcal{O}_{\mathfrak{g}} \underset{\mathcal{O}_{\mathfrak{h}//W}}{\otimes} \mathcal{O}_{\mathfrak{h}}.$$

(2) *The higher direct images $R^i\mathfrak{p}_*(\mathcal{O}_{\widetilde{\mathfrak{g}}})$ vanish.*

Proof. (of point (1))

Consider the scheme $\widetilde{\mathfrak{g}}' := \mathrm{Spec}(\mathrm{Fun}(\mathfrak{g}) \underset{\mathrm{Fun}(\mathfrak{h}//W)}{\otimes} \mathrm{Fun}(\mathfrak{h}))$. Since all the varieties involved are smooth, and the morphisms flat, this scheme is a complete intersection, and hence Cohen-Macaulay. Moreover, $\widetilde{\mathfrak{g}}'$ is smooth in codimension 1, by Corollary 7.9, hence it is normal.

Since the map $\widetilde{\mathfrak{g}}_{reg} \to \widetilde{\mathfrak{g}}'_{reg}$ is an isomorphism, by Zariski's Main Theorem,

$$(\mathfrak{p} \times \mathfrak{q})_!(\mathcal{O}_{\widetilde{\mathfrak{g}}}) \simeq \mathcal{O}_{\widetilde{\mathfrak{g}}'},$$

implying the first point of the theorem.

□

7.14. Proof of point (2) of Theorem 7.13. Consider the full direct image $R\mathfrak{p}_*(\mathcal{O}_{\widetilde{\mathfrak{g}}})$ as an object in the derived category of coherent sheaves on $\mathfrak{g}$. From the $\mathbb{G}_m$-action and Nakayama's lemma we conclude that it is sufficient to show that

$$R\mathfrak{p}_*(\mathcal{O}_{\widetilde{\mathfrak{g}}}) \overset{L}{\underset{\mathcal{O}_{\mathfrak{g}}}{\otimes}} \mathbb{C}_0 \tag{7.1}$$

is acyclic in the cohomological degrees > 0, where $\mathbb{C}_0$ denotes the sky-scraper at $0 \in \mathfrak{g}$.

Note that the full assertion of Theorem 7.13 should imply that the object (7.1) is acyclic also in negative cohomological degrees, and its 0-th cohomology is $|W|$-dimensional. We will see all these facts explicitly.

We will calculate (7.1) using the base change theorem. Let us view $\mathcal{O}_{\widetilde{\mathfrak{g}}}$ as a coherent sheaf on $X \times \mathfrak{g}$, and let us consider the following general set-up:

Let $\pi : Y_1 \to Y_2$ be a projective and flat map, and let $\mathcal{F}$ be a quasi-coherent sheaf on Y_1. Let $f : Y_2' \hookrightarrow Y_2$ be a map of algebraic varieties, and let $f' : Y_1' \to Y_1$ be its base change. We will denote by π' the resulting morphism $Y_1' \to Y_2'$.

Theorem 7.15. *Under the above circumstances, we have a canonical isomorphism in the derived category ofquasi-coherent sheaves on Y_2':*

$$R\pi'_*(Lf'^*(\mathcal{F})) \simeq Lf^*(R\pi_*(\mathcal{F})).$$

We will apply this theorem for $Y_1 = X \times \mathfrak{g}$, $Y_2 = \mathfrak{g}$, $Y_2' = \{0\}$ and $\mathcal{F} = \mathcal{O}_{\widetilde{\mathfrak{g}}}$. Note that Y_1' is isomorphic to X. Thus, in order to calculate (7.1), we need first to calculate the derived pull-back of $\mathcal{O}_{\widetilde{\mathfrak{g}}}$ under the closed embedding $X \simeq X \times 0 \hookrightarrow X \times \mathfrak{g}$.

However, from the Koszul complex, we obtain that the $-i$-th cohomology of the resulting complex on X is isomorphic to $\Omega^i(X)$. Hence, (7.1) is an extension of complexes

$$R\Gamma(X, \Omega^i(X))[i].$$

However, $H^i(X, \Omega^j(X)) = 0$ for $i \neq j$ for X being the flag variety, and we are done.

7.16. Second proof of Kostant's theorem. We have show that the map

$$\mathcal{O}_{\mathcal{N}} \to \mathfrak{p}_*(\mathcal{O}_{\widetilde{\mathcal{N}}})$$

is an isomorphism. We will show, moreover, that the higher direct images $R^i\mathfrak{p}_*(\mathcal{O}_{\widetilde{\mathcal{N}}})$ vanish.

Let us consider the following version of the base change set-up. Let $\pi : Y_1 \to Y_2$ be a projective (but not flat morphism), and let $\mathcal{F}$ be a quasi-coherent sheaf on Y. Let $Y_2 \to Z$ be a flat morphism, such that $\mathcal{F}$ is Z-flat. Let $f : Z' \to Z$ be a map, and let

$$\pi' : Y_1' \to Y_2', f_1 : Y_1' \to Y_1 \text{ and } f_2 : Y_2' \to Y_2$$

be the corresponding base changed maps.

Theorem 7.15 implies that under the above circumstances we have:

$$R\pi'_*(f_1^*(\mathcal{F})) \simeq Lf_2^*(R\pi_*(\mathcal{F})). \tag{7.2}$$

Let us apply this to $Y_1 = \widetilde{\mathfrak{g}}$, $Y_2 = \widetilde{\mathfrak{g}}'$, $Z = \mathfrak{h}$ and $Z' = \{0\}$. We have $Y_1' \simeq \widetilde{\mathcal{N}}$ and $Y_2' \simeq \mathcal{N}$. From (7.2) and Theorem 7.13, we obtain that

$$R\mathfrak{p}_*(\mathcal{O}_{\widetilde{\mathcal{N}}}) \simeq \mathcal{O}_{\widetilde{\mathfrak{g}}'} \underset{\mathcal{O}_{\mathfrak{h}}}{\overset{L}{\otimes}} \mathbb{C}_0 \simeq \mathcal{O}_{\mathfrak{g}} \underset{\mathcal{O}_{\mathfrak{h}//W}}{\overset{L}{\otimes}} \mathbb{C} \simeq \mathcal{O}_{\mathcal{N}},$$

which is what we had to show.

7.17. Orbits of G on $\mathcal{N}$. We have the following :

Proposition 7.18. *For any G-orbit $\mathbf{O}$ on $\mathfrak{g}$,*

$$\dim(\mathbf{O} \cap \mathfrak{b}) \geq \frac{1}{2}\dim(\mathbf{O}).$$

Proof. We regard $\mathbf{O}$ as a Poisson variety, acted on by N. Then $\mathbf{O} \cap \mathfrak{b}$ identifies with the preimage of $0 \in \mathfrak{n}^*$ under the moment map. Since the group N is unipotent, this preimage is always anisotropic (which can be shown by induction). This implies the dimension estimate, since $\mathbf{O}$ is in fact symplectic.

□

Let us show how this proposition implies Theorem 7.11. Consider the variety $\mathrm{St} := \widetilde{\mathcal{N}} \underset{\mathcal{N}}{\times} \widetilde{\mathcal{N}}$.

Lemma 7.19. $\dim(\mathrm{St}) = 2\dim(G/B)$.

Proof. By Bruhat's decomposition, St is the union of locally closed subvarieties St_w, numbered by elements of the Weyl group, where each St_w classifies triples (x, x', ξ), where $(x, x') \in X \times X$ is a pair of Borel subalgebras in relative position w, and ξ is an element of $\mathfrak{n}_x \cap \mathfrak{n}_{x'}$.

The dimension of such St_w is manifestly $2\dim(G/B)$. □

For a G-orbit $\mathbf{O} \in \mathcal{N}$ consider its preimage $\widetilde{\mathbf{O}}$ in $\widetilde{\mathcal{N}}$. Its dimension equals $\dim(G/B) + \dim(\mathbf{O} \cap \mathfrak{b})$. Hence,

$$\dim(\widetilde{\mathbf{O}} \underset{\mathbf{O}}{\times} \widetilde{\mathbf{O}}) = 2(\dim(G/B) + \dim(\mathbf{O} \cap \mathfrak{b})) - \dim(\mathbf{O}) \geq 2\dim(G/B),$$

by Proposition 7.18

Hence, $\widetilde{\mathbf{O}} \underset{\mathbf{O}}{\times} \widetilde{\mathbf{O}}$ is a union of irreducible components of St. (As a by-product we see that the inequality in Proposition 7.18 is in fact an equality.) Since St has finitely many irreducible components, we obtain that $\mathcal{N}$ consists of finitely many G-orbits.

8. Proof of the localization theorem

8.1. Equivariant D-modules. Let X be a scheme and Y be the total space of a principal bundle over X with respect to some algebraic group B. We would like to express D-modules on X in terms of D-modules on Y, endowed with a certain equivariance structure. (The discussion will be local with respect to X, so with no restriction of generality we can assume that it is affine.) In practice, we will take $X = G/B$ and $Y = G$.

Let first Y be any variety endowed with an action of B. We say that a D-module $\mathcal{F}$ is *weakly* B-equivariant, if $\mathcal{F}$ is equipped with a B-equivariant structure as a quasi-coherent sheaf, and the action map

$$\mathfrak{D}(Y) \underset{\mathcal{O}_Y}{\otimes} \mathcal{F} \to \mathcal{F}$$

respects the equivariant structures. Weakly equivariant D-modules naturally form a category, where morphisms $\mathcal{F}_1 \to \mathcal{F}_2$ are by definition D-module morphisms, which are compatible with B-equivariant structures as quasi-coherent sheaves.

Recall that for any B-equivariant quasi-coherent sheaf there exists a map $a^\sharp : \mathfrak{b} \to End_{\mathbb{C}}(\mathcal{F})$. In addition, the action of vector fields $a(\xi)$, $\xi \in \mathfrak{b}$ defines another action of $\mathfrak{b}$ on $\mathcal{F}$.

Set $a^\flat(\xi) = a^\sharp(\xi) - a(\xi)$.

Lemma 8.2. *The assignment $\xi \mapsto a^\flat(\xi)$ is a homomorphism of Lie algebras $\mathfrak{b} \to End_{\mathfrak{D}(Y)}(\mathcal{F})$.*

We say that $\mathcal{F}$ is a *strongly* equivariant $\mathfrak{D}(Y)$-module if $a^\flat$ is identically equal to 0. Strongly B-equivariant D-modules form a full subcategory in the category of weakly equivariant D-modules; we shall denote it by $\mathfrak{D}(Y)\text{-mod}^B$.

Example 1. Take $\mathcal{F} = \mathfrak{D}(Y)$. As was discussed before, it has a natural equivariance structure as a quasi-coherent sheaf, compatible with the algebra structure. Hence, it is a weakly equivariant D-module.

Recall that for $D \in \mathfrak{D}(Y)$,

$$a^\sharp(\xi)(D) = a(\xi) \cdot D - D \cdot a(\xi).$$

Hence, $a^\flat(\xi)(D) = -D \cdot a(\xi)$.

Example 2. Take $\mathcal{F} = \mathcal{O}_Y$. Then $a(\xi) = a^\sharp(\xi)$, by definition. Hence, $\mathcal{O}_Y$ is strongly B-equivariant.

8.3. D-modules on principal bundles. Let us return to the situation, when Y is a B-principal bundle over X. Let π denote the projection $Y \to X$.

Proposition-Construction 8.4. *The pull-back functor $\mathcal{F}' \mapsto \pi^*(\mathcal{F}')$ defines an equivalence between the category of D-modules on X and that of strongly B-equivariant D-modules on Y.*

The rest of this subsection is devoted to the proof of this theorem. First, let us note that if $\mathcal{F}'$ is any quasi-coherent sheaf on X, then $\pi^*(\mathcal{F}')$ is naturally B-equivariant. Indeed, the diagram

$$\begin{array}{ccc} B \times X & \xrightarrow{act} & X \\ {\scriptstyle p_2}\downarrow & & \downarrow{\scriptstyle \pi} \\ X & \xrightarrow{\pi} & Y \end{array}$$

commutes, hence we have a natural isomorphism

$$act^*(\pi^*(\mathcal{F}')) \simeq p_2^*(\pi^*(\mathcal{F}'),$$

which is clearly associative.

We claim now that if $\mathcal{F}'$ is a D-module, the above equivariant structure on $\pi^*(\mathcal{F}')$ is compatible with the $\mathfrak{D}(Y)$-action, i.e., that $\pi^*(\mathcal{F}')$ is indeed a weakly equivariant D-module.

This amounts to checking the commutativity of the diagarm

$$\begin{array}{ccc} act^*(\mathfrak{D}(Y) \underset{\mathcal{O}_Y}{\otimes} \pi^*(\mathcal{F}')) & \longrightarrow & p_2^*(\mathfrak{D}(Y) \underset{\mathcal{O}_Y}{\otimes} \pi^*(\mathcal{F}')) \\ \downarrow & & \downarrow \\ act^*(\pi^*(\mathcal{F}')) & \longrightarrow & p_2^*(\pi^*(\mathcal{F}')). \end{array}$$

The question is local, hence we can assume that Y is isomorphic to the direct product $X \times B$. In this case $\pi^(\mathcal{F}') \simeq \mathcal{F}' \boxtimes \mathcal{O}_B$, with the natural D-module and B-equivariant structures. Hence, the situation reduces to Example 2 above. This also shows that the weak equivariant structure on $\pi^*(\mathcal{F}')$ is in fact strong.

Let us show now that the functor that we have just constructed $\mathfrak{D}(X)$-mod $\to \mathfrak{D}(Y)$-mod^B is fully-faithful. The question is local with respect to X, and it suffices to see that

$$Hom_{\mathfrak{D}(X)}(\mathcal{F}'_1, \mathcal{F}'_2) \simeq Hom_{\mathfrak{D}(X)}(\mathcal{F}'_1, \mathcal{F}'_2) \otimes Hom_{\mathfrak{D}(B)\text{-mod}^B}(\mathcal{O}_B, \mathcal{O}_B),$$

which is evident.

Finally, let us show that π^* is an equivalence. Since D-modules can be glued locally, we can once again assume that $Y = X \times B$. Let $\mathcal{F}$ be a weakly B-equivariant D-module on Y.

By Proposition 6.3, $\mathcal{F}$ is isomorphic, as a quasi-coherent sheaf to, $\mathcal{F}' \boxtimes \mathcal{O}_B$, where $\mathcal{F}'$ is some $\mathcal{O}$-module on X. In fact, $\mathcal{F}'$ can be recovered as $\Gamma(Y, \mathcal{F})^B$.

Since $\mathfrak{D}(X) \subset \mathfrak{D}(X) \otimes \mathfrak{D}(B)$ belongs to the subspace of B-invariants, we obtain that its action preserves $\mathcal{F}' \subset \mathcal{F}' \boxtimes \mathcal{O}_B \simeq \mathcal{F}$. Hence, $\mathcal{F}'$ is naturally a D-module on X.

Thus, it remains to analyze the action of $\mathfrak{D}(B)$ on $\mathcal{O}_B$. It suffices to calculate the action of the vector fields $a_l(\xi)$ for $\xi \in \mathfrak{b}$. We claim that there exists a character $\lambda : \mathfrak{b} \to \mathbb{C}$, such that for $f \in \mathcal{O}_B$,

$$a_l(\xi) \cdot f = Lie_{a_l(\xi)}(f) - \lambda(\xi) \cdot f,$$

and that this character is zero if and only if $\mathcal{F}$ is strongly B-equivariant.

Indeed, λ is reconstructed as

$$a^\flat : \mathfrak{b} \to End_{\mathfrak{D}(B)}(\mathcal{O}_B) \simeq \mathbb{C}.$$

Thus, the proof of Proposition-Construction 8.4 is complete. Let us, however, give a more explicit interpretation of the functor in the opposite direction: $\mathfrak{D}(Y)$-mod$^B \to \mathfrak{D}(X)$-mod. As was explained above, on the level of $\mathcal{O}$-modules, it is given by $\mathcal{F} \mapsto \mathcal{F}^B$. Let us describe the action of the algebra $\mathfrak{D}(X)$ on $\mathcal{F}^B$.

Consider the D-module on Y equal to $\mathfrak{D}(Y)_\mathfrak{b} := \mathfrak{D}(Y)/\mathfrak{D}(Y) \cdot a(\mathfrak{b})$. The weak B-equivariance structure on $\mathfrak{D}(Y)$ gives rise to one on $\mathfrak{D}(Y)_\mathfrak{b}$. However, by construction, the map $a^\flat$ is zero for $\mathfrak{D}(Y)_\mathfrak{b}$; hence it is strongly B-equivariant.

Lemma 8.5.

(1) $\mathfrak{D}(Y)_\mathfrak{b}$ *is canonically isomorphic, as an object of* $\mathfrak{D}(Y)$*-mod*^{B}*, to* $\pi^*(\mathfrak{D}(X))$.

(2) *We have natural isomorphisms:*

$$\mathfrak{D}(X)^{op} \simeq End_{\mathfrak{D}(Y)\text{-}mod^B}(\mathfrak{D}(Y)_\mathfrak{b}) \simeq \big(\mathfrak{D}(Y)/\mathfrak{D}(Y) \cdot a(\mathfrak{b})\big)^B.$$

Proof. The isomorphism $\pi^*(\mathfrak{D}(X)) \simeq \mathfrak{D}(Y)/\mathfrak{D}(Y) \cdot T(Y/X)$ holds for any smooth morphism, where $T(Y/X) \subset T(Y)$ denotes the subsheaf of vertical vector fields. This implies point (1), since $T(Y/X)$ is generated over $\mathcal{O}_Y$ by vector fields of the form $a(\xi)$, $\xi \in \mathfrak{b}$.

Point (2) of the lemma is a corollary of point (1) in view of Proposition-Construction 8.4. □

The description of $\mathfrak{D}(X)$ as $\big(\mathfrak{D}(Y)/\mathfrak{D}(Y) \cdot a(\mathfrak{b})\big)^B$ makes explicit its action on $\mathcal{F} \mapsto \mathcal{F}^B$ for a strongly equivariant $\mathfrak{D}(Y)$-module $\mathcal{F}$.

8.6. Proof of the exactness statement in Theorem 6.18(3).

In view of Proposition-Construction 8.4, the functor

$$\Gamma : \mathfrak{D}(X)\text{-mod} \to Vect$$

can be written down as a composition

$$\mathfrak{D}(X)\text{-mod} \overset{\pi^*}{\to} \mathfrak{D}(G)\text{-mod}^B \overset{B-\text{inv}}{\longrightarrow} Vect. \tag{8.1}$$

Recall that for any D-module $\mathcal{F}$ on G, the space $\Gamma(G,\mathcal{F})$ is naturally a $\mathfrak{g}$-bimodule. Let us regard it as a $\mathfrak{g}$-module via the map $a_r : \mathfrak{g} \to \mathfrak{D}(G)$.

By Proposition-Construction 8.4, for $\mathcal{F}' \in \mathfrak{D}(X)$-mod, the action of $\mathfrak{g}$ on $\Gamma(G,\pi^*(\mathcal{F}'))$ is such that the action of $\mathfrak{b}$ comes from an action of the algebraic group B.

Let us denote by $\overline{\mathcal{O}}$ the category of $\mathfrak{g}$-modules that can be represented as unions of modules from category $\mathcal{O}$. Thus, we obtain that the composition (8.1) can be written as

$$\mathfrak{D}(X)\text{-mod} \overset{\pi^*}{\to} \mathfrak{D}(G)\text{-mod}^B \overset{\Gamma}{\to} \overline{\mathcal{O}} \overset{\mathfrak{b}-\mathrm{inv}}{\longrightarrow} Vect,$$

where the first two functors are exact. However, the functor $\mathcal{M} \mapsto \mathcal{M}^{\mathfrak{b}}$ on $\overline{\mathcal{O}}$ is by definition the same as

$$\mathcal{M} \mapsto Hom(M_0, \mathcal{M}).$$

However, by Proposition 3.10, the object $\mathcal{M}_0$ is projective in $\mathcal{O}$, and, hence, in $\overline{\mathcal{O}}$. I.e., the functor $\overline{\mathcal{O}} \overset{\mathfrak{b}-\mathrm{inv}}{\longrightarrow} Vect$ is also exact.

8.7. A proof of faithfulness via Lie algebra cohomology. Given a D-module $\mathcal{F}$ on X we are going to look at the cohomology

$$H^\bullet(\mathfrak{n}, \Gamma(X,\mathcal{F}))$$

for some choice of the Borel subalgebra $\mathfrak{b}$.

Recall that the N (or B)-orbits on X are in a canonical bijection with W; for $w \in W$ let $X_w \overset{\imath_w}{\hookrightarrow} X$ denote the embedding of the corresponding locally closed subvariety into X.

Recall also that if $\mathcal{M}$ is a $\mathfrak{b}$-representation, then the cohomology groups $H^i(\mathfrak{n},\mathcal{M})$ are acted on by $\mathfrak{h}$.

Lemma 8.8.

(1) *For a D-module $\mathcal{F}$ on X_w there exists a canonical isomorphism*

$$H^i(\mathfrak{n}, \imath_{w\star}(\mathcal{F})) \simeq H^{i+\ell(w)}_{DR}(X_w, \mathcal{F}).$$

(2) *The $\mathfrak{h}$-action on the LHS is given by the character $-w(\rho) - \rho$.*

Let us prove the faithfilness part of Theorem 6.18(3), assuming this lemma. We have to show that if $\mathcal{F} \in \mathfrak{D}(X)$-mod is non-zero, then $R\Gamma(X,\mathcal{F}) \simeq \Gamma(X,\mathcal{F}) \neq 0$.

It is easy to see that for any quasi-coherent sheaf $\mathcal{F}$ there exists a point x (with values, perhaps, in an extension of the ground field), such that the derived fiber $L\imath_x^*(\mathcal{F})$ is non-zero.

Let $\mathfrak{b} = \mathfrak{b}_x$ for the above point x. We will show that $H^\bullet(\mathfrak{n}, \Gamma(X,\mathcal{F})) \neq 0$.

Consider the Cousin complex corresponding to the stratification $X = \underset{w}{\cup} X_w$.

We obtain that $R\Gamma(X,\mathcal{F})$, as an object of the derived category of $\mathfrak{b}$-modules, is filtered by objects of the form $R\Gamma\left(X, \imath_{w\star}(R\imath_w^!(\mathcal{F}))\right)$. Hence, there exists a spectral sequence, converging to $H^\bullet(\mathfrak{n}, \Gamma(X,\mathcal{F})) \neq 0$, and whose second term is

$$\underset{\ell(w)=j}{\oplus} H^i\left(\mathfrak{n}, \imath_{w\star}(R\imath_w^!(\mathcal{F}))\right).$$

Moreover, the terms of this spectral sequence are acted on by $\mathfrak{h}$, and the differential respects the $\mathfrak{h}$-action. However the characters of $\mathfrak{h}$, corresponding to different values of w, are distinct, by Lemma 8.8(2).

Hence, the differential is zero and we obtain a direct sum decomposition

$$H^i(\mathfrak{n}, \Gamma(X,\mathcal{F})) \underset{w}{\oplus} H^{i+\ell(w)}_{DR}(X_w, R\imath_w^!(\mathcal{F})).$$

However, the term corresponding to $w = 1$ is isomorphic to $R\imath_x^!(\mathcal{F})$, and by assumption, it is non-zero.

9. Twisted D-modules

9.1. Let us return to the set-up of Proposition 8.4. Let us fix a character $\lambda : \mathfrak{b} \to \mathbb{C}$. We shall now define another sheaf of rings on X, called λ-twisted differential operators, and denoted $\mathfrak{D}(X)^\lambda$.

Namely, consider the left D-module $\mathfrak{D}(Y)^\lambda_\mathfrak{b}$ on Y equal to the quotient of $\mathfrak{D}(Y)$ be the left ideal generated by sections of the form $\xi - \lambda(\xi)$, $\xi \in \mathfrak{b}$. This D-module is naturally weakly G-equivariant and the map $a^\flat$ is given by the character $-\lambda$. Let us denote the category of weakly B-equivariant D-modules on Y, for which the map $a^\flat$ is given by this character by $\mathfrak{D}(Y)\text{-mod}^{B,\lambda}$.

Set

$$\mathfrak{D}(X)^\lambda = End_{\mathfrak{D}(Y)\text{-mod}^{B,\lambda}}(\mathfrak{D}(Y)^\lambda_\mathfrak{b})^{op}.$$

The ring $\mathfrak{D}(X)^\lambda$ is naturally a $\mathcal{O}_X$-bimodule. It can also rewritten as $\left(\mathfrak{D}(Y)^\lambda_\mathfrak{b}\right)^B$.

Lemma 9.2.

(1) *For every choice of a trivialization of Y as a B-bundle over X, there exists an isomorphism $\mathfrak{D}(X)^\lambda \simeq \mathfrak{D}(X)$.*

(2) *When a trivialization is changed by a function $\phi : X \to B$, the resulting automorphism of $\mathfrak{D}(X)$ is induced by*

$$f \in \mathcal{O}_X \mapsto f,\ \mathbf{v} \in T(X) \mapsto \mathbf{v} + \lambda(d\phi(\mathbf{v})) \in T(X) \oplus \mathcal{O}(X).$$

Proof. For point (1) it suffices to note to analyze the D-module on B equal to $\mathfrak{D}(B)^\lambda_\mathfrak{b} := \mathfrak{D}(B)/\mathfrak{D}(B) \cdot (\xi - \lambda(\xi))$. As an $\mathcal{O}$-module is it isomorphic to $\mathcal{O}_B$, but the action of the vector fields $a_l(\xi)$ is given by

$$f \mapsto Lie_{a_l(\xi)} - \lambda(\xi) \cdot f.$$

Evidently, endomorphisms of such a D-module, commuting with the B-equivariant structure, are given by scalars.

If we change the trivialization by $\phi : X \to B$, the corresponding automorphism of $\mathfrak{D}(X) \otimes \mathfrak{D}(B)^\lambda_\mathfrak{b}$ is given by

$$\mathbf{v} \to \mathbf{v} + d\phi(\mathbf{v}) = \mathbf{v} + \lambda(d\phi(\mathbf{v})).$$

□

Generalizing Proposition-Construction 8.4 we have:

Proposition-Construction 9.3. *There exists a canonical equivalence of categories*

$$\mathfrak{D}(Y)\text{-}mod^{B,\lambda} \simeq \mathfrak{D}(X)^\lambda\text{-}mod.$$

Proof. We define the functor $\mathfrak{D}(Y)\text{-mod}^{B,\lambda} \to \mathfrak{D}(X)^\lambda\text{-mod}$ as follows:

For $\mathcal{F} \in \mathfrak{D}(Y)\text{-mod}^{B,\lambda}$ we set the corresponding object $\mathcal{F}' \in \mathfrak{D}(X)^\lambda\text{-mod}$ to be $\mathcal{F}^B$ as an $\mathcal{O}$-module. This can be rewritten as

$$Hom_{\mathfrak{D}(Y)\text{-mod}^{B,\lambda}}(\mathfrak{D}(Y)^\lambda_\mathfrak{b}, \mathcal{F}),$$

which makes the action of $\mathfrak{D}(X)^\lambda$-mod on it manifest.

To show that this functor is an equivalence, we can work locally. In this case the assertion becomes is equivalent to the fact that

$$V \mapsto V \otimes \mathcal{O}_B$$

is an equivalence between the category of vector spaces and $\mathfrak{D}(B)$-mod$^{B,\lambda}$.

Note that the inverse functor on the level of $\mathcal{O}$-modules is again given by $\mathcal{F}' \mapsto \pi^*(\mathcal{F}')$.

□

9.4. The case of an "integral" twisting. Assume now that the Lie algebra character λ is such that it comes from a character of the group $B \to \mathbb{G}_m$. Note that in this case we can construct a line bundle $\mathcal{L}^\lambda$ on X, by setting

$$\Gamma(U, \mathcal{L}^\lambda) = \{f \in \mathcal{O}_{\pi^{-1}(U)} \,|\, f(b \cdot y) = \lambda(b^{-1}) \cdot f(y)\}.$$

Proposition 9.5. *The $\mathcal{O}_X$-module $\mathcal{L}^\lambda$ has a natural structure of $\mathfrak{D}(X)^\lambda$-module. The functor $\mathcal{F}' \mapsto \mathcal{F}' \underset{\mathcal{O}_X}{\otimes} \mathcal{L}^\lambda$ defines an equivalence $\mathfrak{D}(X)$-mod $\to \mathfrak{D}(X)^\lambda$-mod.*

Proof. Let $\mathcal{F}$ be a B-equivariant quasi-coherent sheaf on Y. We can modify the equivariant structure on it by multiplying the isomorphism

$$act^*(\mathcal{F}) \simeq p_2^*(\mathcal{F})$$

by the function λ along the B-factor. I.e., for a section m of $\mathcal{F}$ we have:

$$a^\sharp_{new}(\xi)(m) = a^\sharp_{old}(\xi)(m) - \lambda(\xi) \cdot m.$$

If $\mathcal{F}$ was a weakly B-equivariant D-module, then it will be still weakly equivariant in this new structure. If, however, $\mathcal{F}$ was an object of $\mathfrak{D}(Y)$-mod^B with the old equivariant structure, it will be an object of $\mathfrak{D}(Y)$-mod$^{G,\lambda}$ with the new one.

Applying this to $\mathcal{F} = \mathcal{O}_Y$, we obtain an object of $\mathfrak{D}(Y)$-mod$^{G,\lambda}$, and using Proposition-Construction 9.3, we produce an object of $\mathfrak{D}(X)^\lambda$. Its underlying $\mathcal{O}$-module is by construction $\mathcal{L}^\lambda$.

The second statement of the proposition follows from the relationship between $\mathfrak{D}(Y)$-mod^B and $\mathfrak{D}(Y)$-mod$^{G,\lambda}$, mentioned above.

□

Note that if in the above set-up we choose a trivialization $Y = X \times B$, thereby trivializing the line bundle $\mathcal{L}^\lambda$, and identifying the rings $\mathfrak{D}(X) \simeq \mathfrak{D}(X)^\lambda$, the equivalence

$$\mathfrak{D}(X)\text{-mod} \to \mathfrak{D}(X)^\lambda\text{-mod}$$

becomes the identity functor.

Let us now give a slightly different interpretation of the ring $\mathfrak{D}(X)^\lambda$. In fact, we claim that $\mathfrak{D}(X)^\lambda$ can be naturally identified with the ring of differential operators $Diff(\mathcal{L}^\lambda, \mathcal{L}^\lambda)$ that act on the line bundle $\mathcal{L}^\lambda$, i.e., the space of $\mathbb{C}$-linear maps $D : \mathcal{L}^\lambda \to \mathcal{L}^\lambda$, such that $[[...[[D, f_1], f_2], ...], f_k] = 0$ for any k-tuple of functions $f_1, ..., f_k$ for a sufficiently large k.

Indeed, by construction $\mathfrak{D}(X)^\lambda$ does act on $\mathcal{L}^\lambda$, and to show that its image in $End_{\mathbb{C}}(\mathcal{L}^\lambda)$ indeed belongs to $Diff(\mathcal{L}^\lambda, \mathcal{L}^\lambda)$ and is isomorphic to this ring, it is enough to work locally, in which case we reduce to the situation when $\mathcal{L}^\lambda$ is trivial.

9.6. The case of the flag variety. Let us now specify to the case when $Y = G$ with B acting on the right, and $X = G/B$. Given a character $\lambda : \mathfrak{h} \to \mathbb{C}$, which we can regard as a character of $\mathfrak{b}$, we obtain the sheaf of rings $\mathfrak{D}(X)^\lambda$ on the flag variety.

Proposition-Construction 9.7.

(1) *There exists a natural map*

$$a^\lambda : \mathfrak{g} \to \Gamma(X, \mathfrak{D}(X)^\lambda).$$

(2) *The map*

$$Z(\mathfrak{g}) \hookrightarrow U(\mathfrak{g}) \xrightarrow{a^\lambda} \Gamma(X, \mathfrak{D}(X)^\lambda)$$

factors through the character

$$Z(\mathfrak{g}) \overset{\tau}{\simeq} Z(\mathfrak{g}) \xrightarrow{\phi} \mathrm{Sym}(\mathfrak{h}) \xrightarrow{\lambda} \mathbb{C},$$

where τ is the involution of $Z(\mathfrak{g})$, induced by the anti-involution $\xi \mapsto -\xi$ of $U(\mathfrak{g})$.

Proof. By construction, (local) sections of $\mathfrak{D}(X)^\lambda$ identify with B-invariant endomorphisms of the left D-module

$$\mathfrak{D}(G)^\lambda_{\mathfrak{b}} := \mathfrak{D}(G)/\mathfrak{D}(G) \cdot (a_r(\xi) - \lambda(\xi)),\ \xi \in \mathfrak{b}$$

on G.

However, since the images of the homomorphisms a_l and a_r from $\mathfrak{g}$ to $\mathfrak{D}(G)$ commute, the operation of *right* multiplication by $a_l(\eta), eta \in \mathfrak{g}$ on $\mathfrak{D}(X)$ descends to a well-defined endomorphism of $\mathfrak{D}(G)^\lambda_{\mathfrak{b}}$. Moreover, these endomorphisms are B-invariant with respect to the B-equivariant structure on $\mathfrak{D}(G)^\lambda_{\mathfrak{b}}$, since the vector fields $a_l(\xi)$ are B-invariant under the action of B on G by right multilication.

This proves point (1) of the proposition. To prove point (2) we need to calculate the endomorphism of $\mathfrak{D}(G)^\lambda_{\mathfrak{b}}$ induced by elements $a_l(u)$, $u \in Z(\mathfrak{g})$.

Recall that we have an identification $\mathfrak{D}(G) \simeq \mathcal{O}_G \otimes U(\mathfrak{g})$, under which $U(\mathfrak{g}) \hookrightarrow \mathcal{O}_G \otimes U(\mathfrak{g})$ corresponds to the homomorphism a_r. Then

$$\mathfrak{D}(G)^\lambda_{\mathfrak{b}} \simeq \mathcal{O}_G \otimes M_\lambda,$$

as modules over $U(\mathfrak{g})$ via a_r, where the action of $U(\mathfrak{g})$ on the first factor is trivial.

Hence, the second assertion of the proposition follows from Corollary 6.13.

□

From the above proposition, we obtain a functor

$$\Gamma : \mathfrak{D}(X)^\lambda\text{-mod} \to U(\mathfrak{g})_\chi\text{-mod},$$

where $\chi \in \mathrm{Spec}(Z(\mathfrak{g}))$ equals $\varpi(\tau(\lambda))$.

The following generalizes Theorem 6.18:

Theorem 9.8.

(1) *The map*

$$a_l^\lambda : U(\mathfrak{g})_\chi\textit{-mod} \to \Gamma(X, \mathfrak{D}(X)^\lambda)$$

is an isomorphism.

(2) *If λ is such that $\lambda + \rho$ is dominant, the the functor $\Gamma : \mathfrak{D}(X)^\lambda$-mod $\to Vect$ is exact.*

(3) *If λ itself is dominant, then the functor Γ is, in addition, faithful.*

(4) *Under the assumption of* 3)*, the functor Γ, viewed as $\mathfrak{D}(X)^\lambda$-mod $\to U(\mathfrak{g})_\chi$-mod is an equivalence.*

Let us analyze which parts of this theorem we know already:

Point (1) follows from Kostant's theorem by passing to the associated graded level, as in the non-twisted case.

The fact that (1) and (3) imply (4) follows in the same way as in the case $\lambda = 0$.

Point (2) can be proved in the same way as the exactness assertion in the non-twisted case, using the fact that the Verma module M_λ is projective in the category $\mathcal{O}$, if $\lambda + \rho$ is dominant.

Finally, point (3) can be proved by the same Lie algebra cohomology technique: the assumtion that λ is dominant will be used for the fact that all the weights $-w(\rho+\lambda)-\rho$, $w \in W$ are distinct.

We shall give, however, an alternative proof of both (2) and (3) using the translation principle.

9.9. Let the weight λ be integral, and consider the corresponding line bundle $\mathcal{L}^\lambda$.

Proposition-Construction 9.10.

(1) *The line bundle $\mathcal{L}^\lambda$ has a natural structure of G-equivariant coherent sheaf on $X = G/B$.*

(2) *The $a^\sharp$-action of $\mathfrak{g}$ on $\mathcal{L}^\lambda$, corresponding to this equivariant structure, equals the action corresponding to $a_l^\lambda : \mathfrak{g} \to \mathfrak{D}(X)^\lambda$.*

(3) *For any $\mu \in \mathfrak{h}^*$, the functor of tensor product $\mathcal{F}' \mapsto \mathcal{F}' \underset{\mathcal{O}_X}{\otimes} \mathcal{L}^\lambda$ defines an equivalence $\mathfrak{D}(X)^\mu$-mod $\to \mathfrak{D}(X)^{\mu+\lambda}$-mod.*

(4) *Under the above functor for $m_1 \in \mathcal{F}'$ and $m_2 \in \mathcal{L}^\lambda$ we have:*

$$a_l^{\lambda+\mu}(\xi)(m_1 \otimes m_2) = a_l^\mu(\xi)(m_1) \otimes m_2 + m_1 \otimes a^\sharp(\xi)(m_2).$$

Proof. Recall that $\mathcal{L}^\lambda$ was obtained from the structure sheaf $\mathcal{O}_G$ on G by modifying its equivariant structure with respect to B acting on G by right multiplication. This structure is compatible with the natural G-equivariant structure on $\mathcal{O}_G$ corresponding to the action of G on itself by left-multiplication. Hence, the latter descends to $\mathcal{L}^\lambda$.

All points of the proposition follows essentially from the constructions.

□

Assume now that λ is dominant. Let $V^{-w_0(\lambda)}$ be the irreducible finite-dimensional representation of G with highest weight $-w_0(\lambda)$. Consider the coherent sheaf on X equal to $V^{-w_0(\lambda)} \otimes \mathcal{O}_X$.

We define the G-equivariant coherent sheaf $\mathcal{V}_X^{-w_0(\lambda)}$ on X to be $V^{-w_0(\lambda)} \otimes \mathcal{O}_X$ with the equivariant structure twisted by the G-action on $V^{-w_0(\lambda)}$, i.e., we change the evident isomorphism

$$act^*(V^{-w_0(\lambda)} \otimes \mathcal{O}_X) \simeq p_2^*(V^{-w_0(\lambda)} \otimes \mathcal{O}_X),$$

by multiplying it by the map

$$\Delta : V^{-w_0(\lambda)} \to V^{-w_0(\lambda)} \otimes \mathcal{O}_G,$$

corresponding to the action of G on $V^{-w_0(\lambda)}$.

For $v \otimes f \in \mathcal{V}_X^{-w_0(\lambda)}$ we have:

$$a^\sharp(\xi) = \xi(v) \otimes f + v \otimes Lie_{a(\xi)}(f).$$

Proposition 9.11.

(1) *There exists a natural G-equivariant morphism*

$$\mathcal{V}_X^{-w_0(\lambda)} \to \mathcal{L}^\lambda.$$

(2) *The G-equivariant coherent sheaf $\mathcal{V}^{-w_0(\lambda)}$ admits a filtartion, whose subquotients are isomorphic to $\mathcal{L}^{\lambda'}$, each appearing the number of times equal to $\dim(V^{-w_0(\lambda)})(-\lambda')$, and the map of (1) above being the projection on the last quotient.*

Proof. Defining a map $\mathcal{V}_X^{-w_0(\lambda)} \to \mathcal{L}^\lambda$ is equivalent to giving a map

$$V^{-w_0(\lambda)} \to \Gamma(X, \mathcal{L}^\lambda), \tag{9.1}$$

compatible with the G-actions.

By definition,

$$\Gamma(X, \mathcal{L}^\lambda) = \{f \in \mathrm{Fun}(G) \,|\, f(g \cdot b) = \lambda(b) \cdot f(g).\}$$

Let $\psi : V^{-w_0(\lambda)} \to \mathbb{C}^{-\lambda}$ be the lowest weight covector. The map in (9.1) is given by

$$v \mapsto f_v(g) := \psi(g^{-1} \cdot v).$$

This proves point (1) of the proposition. To prove point (2) let us recall that G-equivariant quasi-coherent on G/B are in bijection with B-modules:

For a B-module W we attach a G-equivariant quasi-coherent sheaf $\mathcal{W}_X$ by setting for an open $U \subset X$ $\Gamma(U, \mathcal{W}_X)$ to be the set of W-valued functions on $\pi^{-1}(U)$ such that

$$f(g \cdot b) = b^{-1} \cdot f(g) \in W.$$

Taking the B-module to be $\mathbb{C}^{-\lambda}$ we obtain the line bundle $\mathcal{L}^\lambda$ on X.

Taking the B-module to be $\mathrm{Res}_B^G(V^{-w_0(\lambda)})$, we claim that we obtain $\mathcal{V}_X^{-w_0(\lambda)}$. Indeed, to any f as above we attach a W-valued function f' on U by $f'(g) = g \cdot f(g)$.

The assertion of the proposition follows now from the fact that there exists a B-stable filtration on $V^{-w_0(\lambda)}$ with 1-dimensional subquotients. Each such subquotient is isomorphic to $\mathbb{C}^{\lambda'}$ as a B-representation, and it appears as many times as is the multiplicity of λ' in $V^{-w_0(\lambda)}$. □

9.12. Proof of the localization theorem. Let λ be as in Theorem 9.8, and let μ be a dominant integral weight. Let $\mathcal{F}$ be an object of $\mathfrak{D}(X)^\lambda$.

We have a map of quasi-coherent sheaves

$$\mathcal{F} \underset{\mathcal{O}_X}{\otimes} \mathcal{V}^{-w_0(\mu)} \to \mathcal{F} \underset{\mathcal{O}_X}{\otimes} \mathcal{L}^\mu, \tag{9.2}$$

and by adjunction a map

$$\mathcal{F} \to \mathcal{F} \underset{\mathcal{O}_X}{\otimes} \mathcal{L}^\mu \underset{\mathcal{O}_X}{\otimes} (\mathcal{V}^{-w_0(\mu)})^*. \tag{9.3}$$

These maps are compatible with the $\mathfrak{g}$-actions, where on $\mathcal{F}$ it is given by a_l^λ, and on $\mathcal{V}^{-w_0(\mu)}$, $(\mathcal{V}^{-w_0(\mu)})^*$ and $\mathcal{L}^\mu$ by $a^\sharp$.

Now comes the crucial step in the proof of the localization theorem:

Proposition 9.13.

(1) *If $\mu + \rho$ is dominant, then the map* (9.3) *admits a $\mathbb{C}$-linear splitting.*

(2) *If $\mu + \rho$ is dominant and regular, then the map* (9.2) *admits a $\mathbb{C}$-linear splitting.*

Let us first show how points (1) and (2) of this proposition imply points (2) and (3), respectively, of Theorem 9.8.

Proof. (of point (2))

Let α be a non-zero class in $H^i(X, \mathcal{F})$. Then there exists a *coherent* subsheaf $\mathcal{M} \subset \mathcal{F}$ and a class $\alpha_{\mathcal{M}} \in H^i(X, \mathcal{M})$, such that α is its image.

Since for every regular dominant μ the line bundle $\mathcal{L}^\mu$ is ample, by Serre's theorem we can choose μ so that $H^i(X, \mathcal{M} \otimes \mathcal{L}^\mu) = 0$. Since $(\mathcal{V}^{-w_0(\mu)})^* \simeq \mathcal{O}_X \otimes (V^{-w_0(\mu)})^*$ as a coherent sheaf.

$$H^i(X, \mathcal{M} \otimes \mathcal{L}^\mu \otimes (\mathcal{V}^{-w_0(\mu)})^*) = 0.$$

Consider the commutative diagram of quasi-coherent sheaves:

$$\begin{array}{ccc} \mathcal{M} & \longrightarrow & \mathcal{M}\otimes\mathcal{L}^\mu\otimes(\mathcal{V}^{-w_0(\mu)})^* \\ \downarrow & & \downarrow \\ \mathcal{F} & \longrightarrow & \mathcal{F}\otimes\mathcal{L}^\mu\otimes(\mathcal{V}^{-w_0(\mu)})^*. \end{array}$$

By the above, the image of $\alpha_{\mathcal{M}}$ in $H^i(X, \mathcal{M}\otimes\mathcal{L}^\mu\otimes(\mathcal{V}^{-w_0(\mu)})^*) = 0$ is zero. Hence, the image of α in $H^i(X, \mathcal{F}\otimes\mathcal{L}^\mu\otimes(\mathcal{V}^{-w_0(\mu)})^*)$ is zero. But this is a contradiction, since by Proposition 9.13(1), the lower horizontal arrow in the above diagram is a split embedding, and hence induces an injection on cohomology.

□

Proof. (of point (3))

Note that

$$\Gamma(X, \mathcal{F}\otimes\mathcal{V}^{-w_0(\mu)}) \simeq \Gamma(X, \mathcal{F})\otimes V^{-w_0(\mu)}$$

for any dominant integral μ. Hence, it is enough to show that the latter is non-zero for some μ. However, by Proposition 9.13(2) the map

$$\Gamma(X, \mathcal{F}\otimes\mathcal{V}^{-w_0(\mu)}) \to \Gamma(X, \mathcal{F}\otimes\mathcal{L}^\mu)$$

is a split surjection. Therefore, we are done by Serre's theorem: we can always find μ large enough so that the RHS of the above equation be non-zero.

□

9.14. Proof of Proposition 9.13. For point (1) consider the filtration on

$$\mathcal{F} \to \mathcal{F} \underset{\mathcal{O}_X}{\otimes} \mathcal{L}^\mu \underset{\mathcal{O}_X}{\otimes} (\mathcal{V}^{-w_0(\mu)})^*,$$

given by the filtration on $\mathcal{V}^{-w_0(\mu)}$ of Proposition 9.11. The subquotients of this filtration are isomorphic to $\mathcal{F}\otimes\mathcal{L}^{\mu-\mu'}$ for μ' being a weight of $(V^{-w_0(\mu)})^* \simeq V^\mu$.

The action of $Z(\mathfrak{g}) \subset U(\mathfrak{g})$ on each such subquotient is given by the character equal to $\tau(\varpi(\lambda+\mu-\mu'))$, where ϖ denotes the projection $\mathfrak{h}^* \to \mathrm{Spec}(Z(\mathfrak{g}))$, corresponding to the Harish-Chandra isomorphism.

To prove the assertion of the proposition it suffices to show that for $\mu' \neq \mu$,

$$\varpi(\lambda+\mu-\mu') \neq \varpi(\lambda),$$

provided that $\lambda+\rho$ is dominant. But we have seen that in the course if the proof of Theorem 4.26.

Similarly, to prove point (2), it suffices to show that for $\lambda+\rho$ is dominant and regular, and $\mu \neq \mu'$

$$\varpi(\lambda+\mu') \neq \varpi(\lambda+\mu).$$

This we have also seen in the proof of Theorem 4.26.

9.15. As one application of the localization theorem, let us establish the following generalization of Theorem 4.26.

Let $\chi_i = \varpi(\lambda_i)$, $i = 1, 2$ be two characters of $Z(\mathfrak{g})$, such that both λ_1 and λ_2 are dominant. Assume also that $\lambda_2 - \lambda_1 = \mu$ is dominant and integral.

Let $\mathfrak{g}\text{-mod}_{\widetilde{\chi_i}}$ be the full subcategory of $\mathfrak{g}$-mod, consisting of modules on which $Z(\mathfrak{g})$ acts by the generalized character χ_i.

Consider the functor $T_\mu : \mathfrak{g}\text{-mod}_{\widetilde{\chi_1}} \to \mathfrak{g}\text{-mod}_{\widetilde{\chi_2}}$ that sends $\mathcal{M}$ to the maximal submodule of $\mathcal{M}\otimes V^\mu$, which belongs to $\mathfrak{g}\text{-mod}_{\widetilde{\chi_2}}$.

Theorem 9.16. *The functors T_μ is an equivalence.*

Proof. First, we will establish a partial result. Namely, let $\mathfrak{g}\text{-mod}_{\chi_i} \subset \mathfrak{g}\text{-mod}_{\widetilde{\chi_i}}$ be the full subcategory consisting of modules, on which $Z(\mathfrak{g})$ acts by the character χ_i.

Let us show first that the functor

$$T_\mu : \mathfrak{g}\text{-mod}_{\chi_1} \to \mathfrak{g}\text{-mod}_{\chi_2}$$

is an equivalence. Namely, we claim that the following diagram of functors

$$\begin{array}{ccc} \mathfrak{D}(X)^{\lambda_1} & \xrightarrow{\otimes \mathcal{L}^{-w_0(\mu)}} & \mathfrak{D}(X)^{\lambda_2} \\ \Gamma\Big\downarrow & & \Gamma\Big\downarrow \\ \mathfrak{g}\text{-mod}_{\chi_1} & \xrightarrow{T_\mu} & \mathfrak{g}\text{-mod}_{\chi_1} \end{array} \tag{9.4}$$

commutes.

This follows from the proof of the localization theorem discussed above.

Moreover, we obtain that $T_\mu(\mathcal{M})$ is canonically a direct summand in $\mathcal{M} \otimes V^\mu$. This implies that $T_\mu(\mathcal{M})$ is a direct summand of $\mathcal{M} \otimes V^\mu$ for any $\mathcal{M} \in \mathfrak{g}\text{-mod}_{\widetilde{\chi_1}}$.

In particular, the functor T_μ is exact on $\mathfrak{g}\text{-mod}_{\widetilde{\chi_1}}$. Its (both left and right) adjoint, denoted T_μ^*, is given by sending $\mathcal{M}' \in \text{-mod}_{\widetilde{\chi_2}}$ to the direct summand of $\mathcal{M}' \otimes (V^\mu)^*$, which belongs to $\mathfrak{g}\text{-mod}_{\widetilde{\chi_1}}$. This functor is also exact.

To prove that T_μ and T_μ^* are mutually quasi-inverse equivalences of categories, we have to show that the adjunction morphisms

$$\mathrm{Id} \to T^\mu \circ T_\mu^* \text{ and } T_\mu^* \circ T_\mu \to \mathrm{Id}$$

are isomorphisms.

By exactness, it is enough to do so for objects of $\mathfrak{g}\text{-mod}_{\chi_1}$ and $\mathfrak{g}\text{-mod}_{\chi_2}$, respectively, in which case the assertion follows from the equivalence of these categories, that has been already established.

□

As a corollary let us deduce the following:

Theorem 9.17 (BBW). *For λ dominant, $H^i(X, \mathcal{L}^\lambda) = 0$ for $i > 0$ and $\Gamma(X, \mathcal{L}^\lambda)$ is isomorphic to $V^{-w_0(\lambda)}$.*

Proof. The vanishing of higher cohomologies follows from point (2) if Theorem 9.8. The statement about $\Gamma(X, \mathcal{L}^\lambda)$ follows using (9.4) from the case $\lambda = 0$.

□

10. Identification of (dual) Verma modules

10.1. Let λ be such that $\lambda + \rho$ is dominant. Let $\chi \in \mathrm{Spec}(Z(\mathfrak{g}))$ be $\varpi(\lambda)$.

We consider the category $\mathcal{O}'_\chi \subset \mathfrak{g}\text{-mod}_\chi$ to be the full subcategory, consisting of finitely generated modules, on which the action of $\mathfrak{n}$ is locally nilpotent.

Much of the discussion about $\mathcal{O}_\chi$ applies to $\mathcal{O}'_\chi$. In particular, the modules $M_{w\cdot\lambda}$, $M^\vee_{w\cdot\lambda}$, $L_{w\cdot\lambda}$ belong to $\mathcal{O}_\chi$, and the results from Sect. 4.12 hold.

Lemma 10.2. *The action of $\mathfrak{h}$ on any object of $\mathcal{O}'_\chi$ is locally finite.*

Our goal is to describe this subcategory, and the above modules in terms of the localization theorem. To simplify the discussion we will assume $\lambda = 0$ and thus deal with usual (i.e., non-twisted) D-modules. However, everything goes through in the twisted case just as well.

Proposition 10.3. *For an object $\mathcal{F} \in \mathfrak{D}(X)$-mod, the $\mathfrak{g}$-module $\Gamma(X, \mathcal{F})$ belongs to $\mathcal{O}'_{\chi_0}$ if and only if $\mathcal{F}$ is finitely generated and strongly N-equivariant.*

10.4. Proof of Proposition 10.3. First, from the equivalence of categories Theorem 6.18, we obtain formally that $\mathcal{F}$ is finitely generated as a D-module if and only if $\Gamma(X, \mathcal{F})$ is finitely generated as an object of $\mathfrak{g}\text{-mod}_{\chi_0}$.

(We remind that an object $\mathcal{F}$ of an abelian category is called finitely generated if and only the functor $Hom(\mathcal{F}, ?)$ commutes with direct sums. For the category of modules over an algebra, this is equivalent for the usual finite-generation condition.)

Let now $\mathcal{F} \in \mathfrak{D}(X)$-mod be strongly N-equivariant. The equivariant structure on $\mathcal{F}$ as a quasi-coherent sheaf defines an action of the algebraic group N on $\Gamma(X, \mathcal{F})$. The assumption that $a^\flat = 0$ implies that the resulting action of the Lie algebra $\mathfrak{n}$ coincides with that obtained by restriction from the $\mathfrak{g}$-action on $\Gamma(X, \mathcal{F})$.

Instead of proving the implication in the opposite direction, we will show that if $\mathcal{M} \in \mathfrak{g}$-mod is such that the action of $\mathfrak{n}$ on it is locally nilpotent (which is equivalent to this action coming from an action of N), then $\mathrm{Loc}(\mathcal{M}) \in \mathfrak{D}(X)$-mod is strongly N-equivariant.

Recall that $\mathrm{Loc}(\mathcal{M}) = \mathfrak{D}(X) \underset{\underline{U(\mathfrak{g})}}{\otimes} \underline{\mathcal{M}}$, and it is the quotient of the "naive" tensor product $\mathfrak{D}(X) \otimes \mathcal{M}$, both being left D-modules on X.

We endow the latter with a structure of weakly N-equivariant D-modules, by twisting the natural weak N-equivariant structure on $\mathfrak{D}(X)$ by the given algebraic action of N on $\mathcal{M}$.

We claim that this weak equivariant structure descends to the quotient $\mathrm{Loc}(\mathcal{M})$, where it becomes strong. Since the group in questions are connected, it suffices to see that for $\xi \in \mathfrak{n}$, a local sections D of $\mathfrak{D}(X)$, and $u \in U(\mathfrak{g})$, $m \in \mathcal{M}$,

$$a^\sharp(\xi)\Big(D \cdot a(u) \otimes m\Big) = a^\sharp(\xi)\Big(D \otimes a(u) \cdot m\Big) \in \mathrm{Loc}(\mathcal{M})$$

and

$$a(\xi) \cdot \Big(D \otimes m\Big) = a^\sharp(\xi)\Big(D \otimes m\Big) \in \mathrm{Loc}(\mathcal{M}).$$

For the second identity we have:

$$a^\sharp(\xi)\Big(D \otimes m\Big) = a(\xi) \cdot D \otimes m - D \cdot a(\xi) \otimes m + D \otimes \xi \cdot m,$$

which equals $a(\xi) \cdot D \otimes m$ in $\mathrm{Loc}(\mathcal{M})$.

The first identity follows from the fact that

$$\begin{aligned} &a^\sharp(\xi)\Big(D \cdot a(u) \otimes m\Big) = a(\xi) \cdot D \cdot a(u) \otimes m - D \cdot a(u) \cdot a(\xi) \otimes m + D \cdot a(u) \otimes \xi \cdot m = \\ &a(\xi) \cdot D \cdot a(u) \otimes m - D \cdot a(\xi) \cdot a(u) \otimes m - D \cdot a([u, \xi]) \otimes m + D \cdot a(u) \otimes \xi \cdot m, \end{aligned}$$

which in $\mathrm{Loc}(\mathcal{M})$ is the same as

$$\begin{aligned} &a(\xi) \cdot D \otimes u \cdot m - D \cdot a(\xi) \otimes u \cdot m - D \otimes a([u, \xi]) \cdot m + D \otimes u \cdot \xi \cdot m = \\ &a^\sharp(\xi)(D) \otimes u \cdot m + D \otimes \xi \cdot (u \cdot m) = a^\sharp(\xi) \cdot \Big(D \otimes u \cdot m\Big). \end{aligned}$$

10.5. For an element $w \in W$ let $N \cdot w =: X_w \overset{\imath_w}{\hookrightarrow} X$ denote the embedding of the corresponding Schubert cell. Consider the D-module $(\imath_w)_*(\mathcal{O}_{X_w})$.

Theorem 10.6.

$$\Gamma\Big(X, (\imath_w)_\star(\mathcal{O}_{X_w})\Big) \simeq M^\vee_{-w(\rho)-\rho}.$$

The proof of the theorem will proceed by induction with respect to the $\ell(w)$. Namely, we assume that for all w' with $\ell(w') < \ell(w)$, have already shown that

$$\Gamma\Big(X, (\imath_{w'})_\star(\mathcal{O}_{X_{w'}})\Big) \simeq M^\vee_{\mu'},$$

where $\mu' = -w'(\rho) - \rho$. (Let us note that $\ell(w)$ equals the number of positive roots that are turned negative by w, which is the same as $\langle \rho - w(\rho), \check{\rho}\rangle$.)

To carry out the induction step we will use the following observation:

Lemma 10.7. *Let $\widetilde{M}^\vee_\mu$ be on object of $\mathcal{O}_{\chi_0}$, such that $[\widetilde{M}^\vee_\mu] = [M^\vee_\mu]$ and*

$$Hom(L_{\mu'}, \widetilde{M}^\vee_\mu) = 0$$

for $\mu' = -w'(\rho) - \rho$ with $\ell(w') < \ell(w)$. Then $\widetilde{M}^\vee_\mu \simeq M^\vee_\mu$.

Proof. It will be more convenient to dualize and prove the corresponding statement for Verma modules.

By the assumption on $[\widetilde{M}_\mu]$, the weight μ is maximal among the weights of $\widetilde{M}_\mu$. Hence, there is a map

$$M_\mu \to \widetilde{M}_\mu,$$

which we claim is an isomorphism.

Indeed, if it was not surjective, we would have a non-zero map from $\widetilde{M}_\mu$ to some irreducible $L_{\mu'}$. By the assumption on $[\widetilde{M}_\mu]$, this irreducible appears also in the Jordan-Hölder series of M_μ. Hence, $\mu' < \mu$, which implies that $\ell(w') < \ell(w)$ for w' such that $\mu' = -w'(\rho) - \rho$. However, this contradicts the assumption in the lemma, concerning $Hom(L_{\mu'}, \widetilde{M}^\vee_\mu)$.

The injectivity of the map in question follows from its surjectivity and the fact that the lengths of the two modules coincide.

□

We claim that the assumptions of the above lemma are satisfied for

$$\widetilde{M}^\vee_\mu = \Gamma\Big(X, (\imath_w)_\star(\mathcal{O}_{X_w})\Big).$$

Indeed, by the induction hypothesis, the D-module on X, corresponding to $L_{-w'(\rho)-\rho}$ is a submodule of $(\imath_{w'})_\star(\mathcal{O}_{X_{w'}})$ with $\ell(w') < \ell(w)$. In particular, it is supported on the closure of $X_{w'}$ in X, which is contained in

$$\bigcup_{\ell(w'') \leq \ell(w')} X_{w''}.$$

In particular, this closure does not intersect X_w. By adjunction

$$Hom(\mathcal{F}, \imath_w)_\star(\mathcal{O}_{X_w})) = 0$$

for any $\mathcal{F}$ supported on the closure of X_w.

Hence, to complete the induction step we need to show that

$$[\Gamma\Big(X, (\imath_{w'})_\star(\mathcal{O}_{X_{w'}})\Big)] = [M^\vee_\mu].$$

This will be done by computing the formal characters of both sides:

Recall that Λ denotes the weight lattice of H, so that the group algebra $\mathbb{C}[\Lambda]$ identifies with $\mathrm{Fun}(H)$. Let $\widehat{\mathbb{C}[\Lambda]}$ denote the completion of $\mathbb{C}[\Lambda]$, where we allow infinite sums $\Sigma\, c_n \cdot e^{\lambda_n}$, where λ_n is such that $\langle \lambda_n, \check{\rho}\rangle \to -\infty$.

We claim that there exists a well-defined "formal character" map $ch : K(\mathcal{O}_{\chi_0}) \to \widehat{\mathbb{C}[\Lambda]}$. Namely, to $\mathcal{M} \in \mathcal{O}_{\chi_0}$ we assign the formal sum

$$\sum_{\lambda \in \Lambda} \dim(\mathcal{M}(\lambda)) \cdot e^{\lambda},$$

where $\mathcal{M}(\lambda)$ denotes the corresponding generalized eigenspace of $\mathfrak{h}$.

Lemma 10.8. *The map ch is injective.*

Proof. Since $[M_\mu]$, $\mu \in W \cdot \rho$ form a basis for $K(\mathcal{O}_{\chi_0})$, it suffices to see that the elements $ch(M_\mu)$ are linearly independent in $\widehat{\mathbb{C}[\Lambda]}$.

Assume that

$$\sum c_\mu \cdot ch(M_\mu) = 0 \in \widehat{\mathbb{C}[\Lambda]}.$$

Let μ the maximal element (w.r. to $<$) in the above set with $c_\mu \neq 0$.

But then the coefficient next to e^μ in the above sum equals c_μ, which is a contradiction. $\square$

Thus, it remains to prove the following:

Proposition 10.9.

$$ch\left(\Gamma\Big(X, (\imath_w)_\star(\mathcal{O}_{X_w})\Big)\right) = \sum_{\lambda} \dim(U(\mathfrak{n}^-)_\lambda) \cdot e^{\lambda - w(\rho) - \rho}.$$

Proof. By construction, $(\imath_w)_\star(\mathcal{O}_{X_w})$ is strongly B-equivariant in a natural way. Thus, we have to determine the weights of $T \subset B$ acting on $(\imath_w)_\star(\mathcal{O}_{X_w})$ as a quasi-coherent sheaf.

Recall that as a $\mathcal{O}$-module, $(\imath_w)_\star(\mathcal{O}_{X_w})$ is equipped with a canonical filtration, such that the associated graded is isomorphic to

$$\mathrm{Sym}_{\mathcal{O}_{X_w}}\Big(N_{X_w/X}\Big) \underset{\mathcal{O}_{X_w}}{\otimes} \Lambda^{top}_{\mathcal{O}_{X_w}}\Big(N_{X_w/X}\Big),$$

where $N_{X_w/X}$ denotes the normal bundle to X_w in X.

We can identify X_w with the affine space $\mathfrak{n}/\mathfrak{n} \cap \mathrm{Ad}_w(\mathfrak{n})$, and the normal bundle with the constant vector bundle with fiber $\mathfrak{n}^-/\mathfrak{n}^- \cap \mathrm{Ad}_w(\mathfrak{n})$, with the natural action of H. Hence, as an H-module, $\mathrm{gr}\Big((\imath_w)_\star(\mathcal{O}_{X_w})\Big)$ is isomorphic to

$$\mathrm{Sym}\Big((\mathfrak{n}/\mathfrak{n} \cap \mathrm{Ad}_w(\mathfrak{n}))^*\Big) \otimes \mathrm{Sym}\Big(\mathfrak{n}^-/\mathfrak{n}^- \cap \mathrm{Ad}_w(\mathfrak{n})\Big) \otimes \Lambda^{top}\Big(\mathfrak{n}^-/\mathfrak{n}^- \cap \mathrm{Ad}_w(\mathfrak{n})\Big).$$

Since $\mathfrak{n}^* \simeq \mathfrak{n}^-$, we obtain $(\mathfrak{n}/\mathfrak{n} \cap \mathrm{Ad}_w(\mathfrak{n}))^* \simeq \mathfrak{n}^- \cap \mathrm{Ad}_w(\mathfrak{n})$, hence, the tensor product of symmetric algebras appearing above is isomorphic simply to $\mathrm{Sym}(\mathfrak{n}^-)$.

Finally, the character of H on $\Lambda^{top}\Big(\mathfrak{n}^-/\mathfrak{n}^- \cap \mathrm{Ad}_w(\mathfrak{n})\Big)$ equals the sum of negative roots which remain negative after applying w, which is the same as $-w(\rho) - \rho$. $\square$