Knot invariants from homotopy theory

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Online Joint Paris 7 & 13 Topology Seminars
Knot invariants from homotopy theory

Goal:

\( \text{Emb}_0(V, M) \)

space (Whitney \( C^\infty \) topology)

compact manifolds with boundary

Using:

- homotopy theory
- homotopy limits
- Whitehead/Samelson products
- configuration spaces
- operads (little disks)
  - formality, graph complexes

**Goodwillie - Weiss Embedding Calculus**

Motivation:

for a geometric topologist:

- classical unknot theory \( \text{Emb}_0(I, I^3) \)
- generalized unknot theory \( \dim V = 2 \)
  - or \( \dim V = 2k - 1 \)
  - \( \dim M = 4 \)
  - \( \dim M = 3k \)

- diffeomorphism groups \( \text{Diff}_0 M = \text{Emb}_0(M, M) \)

\( \text{Emb}_0(I^1, M) \) as a tool to study other geometric questions

- e.g. Budney-Gabai prove \( S^2 \to S^4 \) unknot bounds a non-unique \( B^3 \to S^4 \) (up to isotopy rel \( 3 \))

using \( M = S^3 \times B^3 \)

for a homotopy theorist:

Study \( \text{Emb}_0(V, M) = \text{Map}(V, M) \)

\( \text{V} \) injectivity is a non-local condition

Outline:

1. Introduce the main object of GW calculus:
   - the Taylor tower \( P_n(M) \).
2. State the main theorem.
3. Explain consequences for Vassiliou theory.

In this talk:

Give a geometric interpretation of GW calculus

for \( \text{Emb}_0(I^1, M) \) for \( \dim M = 3 \).

Relate to Vassiliou finite type invariants

In particular:

- \( M = I^3 \)

\( \text{Emb}(S^1, S^3) \) injective isotopy

\( \text{Emb}(I^1, I^3) \) in particular

\( \text{Comm. Monoid} \)
§1. The embedding calculus

**Theorem** [Goodwillie - Klein '15]

If $(\dim V, \dim M) \neq (1,3)$ then

$$w_n \text{ is } (3 - \dim M + (n+1)(\dim M - \dim V - 2)) \text{- connected.}$$

**Corollary**

For $\dim M - \dim V > 2$ $w_n$ is a weak equivalence.

**Note:**
- One can show $w_n$ not a w.e. for $\text{Emb}_3(I, I^3)$.
- However, the formula predicts

**Theorem A** [K]

For $\dim M = 3$ $w_n : \text{Emb}_3(I, M) \to P_n(I, M)$ is 0-connected.

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Outline:

1) compute $\pi_0 F_n(M)$: generated by trees
2) construct explicit points in $H_{n-1}(M)$ using Gropes: modelled on trees
3) MAIN THM:
   - $\pi_0$ maps a ‘grope point’ to its underlying tree

**Proof of Thm A**

Any tree can be realized by a grope

$\Rightarrow \pi_0 \text{en is surjective}$

Diagram + induction.

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Recall: a map is $k$-connected if it is an isomorphism on $\pi_k(V_{>k})$ and a surjection on $\pi_{k+1}$. 

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This will follow from a stronger result which considers:

$$\text{hofib}(w_n) : H_{n-1}(M) \xrightarrow{\text{en}} F_n(M) = \text{fibr.}_U(f_n)$$

$p_n$  

punctured umots model

punctured filtration

Diagram + induction.
§2. Results

1) **Theorem B** $[K]$

\[ \pi_0 \mathcal{F}_n(M) \cong \frac{\mathbb{Z}[\text{Tree}_{n-1}(n-1)]}{\text{AS, IHX}} \]

2) **Theorem C** $[\text{Joint w/ Y. Shi & P. Teichner}]

\[ \exists \ Grop_{n-1}(M; \mathcal{U}) \xrightarrow{\psi} H_{n-1}(M) \]

- a space of thin grafts on $\mathcal{U}$ in $M$

**Definition** Two knots $K, K' \in \text{Emb}_3(\mathcal{I}, M)$ are $n$-equivalent if $\exists G \in \text{Grop}_n(M; K)$ whose output is knot $K'$.

3) **Theorem D (main)** $[K]$

\[ \pi_0 \text{Grop}_{n-1}(M; \mathcal{U}) \xrightarrow{\pi_0 \psi} \pi_0 H_{n-1}(M) \xrightarrow{\pi_0 \text{em}} \pi_0 \mathcal{F}_n(M) \]

\[ \mathbb{Z}[\text{Tree}_{n-1}(n-1)] \xrightarrow{\text{mod AS, IHX}} \]

- Ufn-1 maps a thick graft to its underlying tree

\[ \begin{align*}
\text{AS} : & \quad \Gamma_2 + \Gamma_1 = 0 \\
\text{IHX} : & \quad \Gamma_3 - \Gamma_1 + \Gamma_2 = 0
\end{align*} \]

(50) grafts or claspers are objects used in the geometric approach to Vassiliev's theory of finite type knot invariants

Correct now!
§3 Consequences

Corollary of Thm C

\[ \pi_0 \mathcal{E}_{\mathcal{W}_n} \] is an invariant of geometric \( n \)-equivalence, i.e. it factors through

\[ \pi_0 \text{Emb}_2(I, M) \xrightarrow{\mathcal{W}_n} \pi_0 P_n(M) \]

Corollary of Corollary

Using [Habiro'00]

For \( M = I^3 \) \( \pi_0 \mathcal{E}_{\mathcal{W}_n} \) is a Vassiliev invariant of type \( \leq n-1 \).

Remark:

Shown by Budney-Conant-Koytcheff-Sinha '17

They also show: \( \pi_0 \mathcal{E}_{\mathcal{W}_n} : \pi_0 K(I^3) \rightarrow \pi_0 P_n(I^3) \) is a monoid map.

Conjecture

For \( M = I^3 \) \( \pi_0 \mathcal{E}_{\mathcal{W}_n} \) is a universal additive invariant of type \( \leq n-1 \).

Remark: Such invariants constructed so far only over \( \mathbb{Q} \): Kontsevich/Bott-Taubes integrals.

Corollary of Thm D

For \( M = I^3 \)

1. \( \mathcal{W}_n \) is surjective
2. Conjecture is TRUE over \( \mathbb{Q} \): \( \pi_0 \text{Emb}_2(I, I^3) \otimes \mathbb{Q} \xrightarrow{\mathcal{W}_n} \pi_0 P_n(I^3) \otimes \mathbb{Q} \)
3. Conjecture is TRUE over \( \mathbb{Z}_p \) in a RANGE: \( n \leq p+2 \).

[Boavida de Brito - Horel]
If group elements trivial get:

This is isotopic to:

Hence: trefoil is 2-equivalent to the unknot