Lagrangian Field Theory

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Contents

Chapter 1

What is a lagrangian field theory?

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1.1 Fields and actions

1.1.1 The set of fields

In physics, a field describes the state of a classical system by assigning to every point of a geometric space or object the value of some physical quantity at that point. An example for a field is the function that assigns to every point of a solid the temperature at that point. Another example is the field that assigns the wind velocity to every point on the surface of the earth. Such assignments are generally assumed to be smooth maps. This is an idealization, of course, as the two examples show, in which the physical systems consist of discrete atoms. But it has led to very accurate descriptions of physical phenomena. In mathematics, the idealization is promoted to a definition.

Definition 1.1.1. A **field** is a smooth section of a smooth fiber bundle $F \to M$. The set of all fields is denoted by $\mathcal{F} := \Gamma^\infty(M, F)$.

Remark 1.1.2. In the example of the temperature field the fibre bundle is $F = M \times [0, \infty) \to M$, where M is the manifold describing the solid. This shows that F is generally not a vector bundle. In the example of the air velocity field the fibre bundle is the tangent bundle $F = TS^2 \to S^2$ of the sphere, which shows that F is generally not a trivial bundle.

Terminology 1.1.3. In physics, the base manifold of the fibre bundle is called the **background** geometry or the **spacetime**, the latter especially in fundamental theories such as gauge theory or general relativity. F is sometimes called the **configuration bundle**, and the typical fiber of F the **configuration space** or the **field content**. $\mathcal{F}$ is usually called the **space of fields**, although it often remains unclear or implicit what "space" means mathematically.

Example 1.1.4. Let $M = \mathbb{R}$ and $F := Q \times \mathbb{R}$ be a trivial bundle. Then $\mathcal{F} = C^\infty(\mathbb{R}, Q)$ is the space of smooth paths in Q. If we replace $\mathbb{R}$ with S^1 then $\mathcal{F}$ is the free loop space of Q.

1.1.2 The action principle in its "mythological" form

In a field theory, the fields are usually subject to a **field equation** $f(\varphi) = 0$, where $f : \mathcal{F} \to V$ is a map to a vector space V. The solutions of the field equation are those fields that are governed by the laws of physics or that possess some desired mathematical properties. Typically, f is a differential operator.

Example 1.1.5. Let $M \subset \mathbb{R}^3$ be a submanifold with boundary ∂M. Let $F := \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}^3 =: M$, so that $\mathcal{F} = C^\infty(\mathbb{R}^3)$. In electrostatics, $\varphi \in C^\infty(\mathbb{R}^3)$ is viewed as the electrostatic potential. The field equation is $\Delta\varphi = 0$, where Δ is the Laplace operator. The solutions of the field equation are harmonic functions subject to boundary conditions on ∂M.

Terminology 1.1.6. In physics, the fields that solve the field equations are often called **on-shell** and those that do not **off-shell**. This terminology comes from from the so-called mass-shell (German: Massen*schale*), which is the positive energy *sheet* of the hyperboloid of the 4-momentum $(p_0, p_1, p_2, p_3) \in \mathbb{R}^4$ of a relativistic particle of rest mass $m^2 = (p_0)^2 - (p_1)^2 - (p_2)^2 - (p_3)^2$. In this sense "shell" is a mistranslation of "Schale". In early quantum field theory, where the momenta are represented by partial derivatives on the wave functions, the mass-shell became to denote the space of solutions of the equation of motion $\Box\varphi = m^2$ of the free relativistic particle.

The set of solutions of the field equation will be denoted by $\mathcal{F}_{\text{shell}} := f^{-1}(0)$. In general, $\mathcal{F}_{\text{shell}} \subset \mathcal{F}$ is not a submanifold. The field equations are often quite complicated. The main tool to study them is the **action principle**. In its ideal form it is stated as follows.

Action principle 1.1.7. There is a smooth function

$$S : \mathcal{F} \longrightarrow \mathbb{R},$$

called the **action**, such that $\varphi \in \mathcal{F}$ is a solution of the field equation if and only if it is a critical point of S.

The value of this principle is that it is usually much easier to construct and study a field theory via its action than via its field equations. For example, a diffeomorphism $\Phi \in \mathrm{Diff}(\mathcal{F})$ acts naturally on functions on $\mathcal{F}$ by pullback. So Φ is a symmetry of the field theory given by an action S if $\Phi^* S = S$. It follows that Φ maps critical points of S to critical points, i.e. $\Phi(\mathcal{F}_{\text{shell}}) = \mathcal{F}_{\text{shell}}$. Conversely, if the symmetries are known, like the Lorentz transformations of special relativity, the requirement for S to be invariant restricts the possible actions of any theory. For such reasons, the action principle is one of the most important guiding principles in both classical and quantum field theory.

Mathematically, however, the action principle 1.1.7 is often not rigorously true. In his 2011 Felix Klein lectures Graeme Segal called it the "mythological picture" of field theory. One of the main goal of these notes is to explain how the action principle can be restated so that it is rigorously true, sufficiently general to cover the most relevant field theories, such as General Relativity, and compatible with the current mathematical tools used in field theory.

1.1.3 The action principle in classical mechanics

What is the action? And how do we get from the action to the field equations? The basic example is a classical mechanical system, where $M = \mathbb{R}$ is time and $F = Q \times \mathbb{R}$, so that a field is a smooth path $q : \mathbb{R} \to Q$. Let us assume for simplicity that $Q = \mathbb{R}^n$. When the system is at rest, it will have to be at a critical point of the potential energy V. When the system moves, the kinetic energy has to be taken into account as well. The action turns out to be given by the difference of kinetic and potential energy,

$$S(q) := \int_{\mathbb{R}} \big\{ \tfrac{1}{2}\dot{q}^i(t)\dot{q}^i(t) - V\big(q(t)\big) \big\}\, dt\,,$$

where $q^i(t)$ are the components of the path, where repeated indices are being summed over so that $\dot{q}^i(t)\dot{q}^i(t) = \sum_{i=1}^n \dot{q}^i(t)\dot{q}^i(t)$, and where we have chosen units in which the mass is $m = 1$.

Problem 1.1.8. The integral over $\mathbb{R}$ that defines the action is generally divergent.

In a first attempt to avoid problem 1.1.8, we could consider only those q that have a finite action, but even the solutions of the field equation may not satisfy this condition. For example, consider the case of a free particle where $V(q) = 0$. The solutions of the equations of motion are paths of constant velocity. So only if the velocity is zero the action is finite.

In a second attempt, we can restrict the domain of integration to a compact interval $[a, b]$ for the action to be finite. We will denote this action by $S_{[a,b]}$. Following the action principle 1.1.7, we now have to compute the critical points of $S_{[a,b]}$. Let $q : [a, b] \to Q$ be a smooth path. Since we have assumed for simplicity that Q is a vector space, $T_q\mathcal{F} \cong \mathcal{F}$. Therefore, a tangent vector $\xi \in T_q\mathcal{F}$ can be represented by smooth family of paths $\mathbb{R} \ni \varepsilon \mapsto q_\varepsilon \in C^\infty(\mathbb{R}, Q)$ given by $q_\varepsilon = q + \varepsilon\xi$. The derivative of $S_{[a,b]}$ in the direction of ξ is obtained by inserting $q + \varepsilon\xi$ and expanding the result to first order in ε.

$$\begin{aligned}
&S_{[a,b]}(q + \varepsilon\xi) - S_{[a,b]}(q)\\
&= \varepsilon \int_a^b \Big\{ \dot{q}^i(t)\dot{\xi}^i(t) - \frac{\partial V}{\partial q^i}\big(q(t)\big)\, \xi^i(t) \Big\} dt + \mathcal{O}(\varepsilon^2)\\
&= \varepsilon \int_a^b \Big\{ \frac{d}{dt}\big(\dot{q}^i(t)\xi^i(t)\big) - \ddot{q}^i(t)\xi^i(t) - \frac{\partial V}{\partial q^i}\big(q(t)\big)\, \xi^i(t) \Big\} dt + \mathcal{O}(\varepsilon^2)\\
&= -\,\varepsilon \int_a^b \Big\{ \ddot{q}^i(t) + \frac{\partial V}{\partial q^i}\big(q(t)\big) \Big\} \xi^i(t)\, dt + \varepsilon \int_a^b \frac{d}{dt}\big(\dot{q}^i(t)\xi^i(t)\big)\, dt + \mathcal{O}(\varepsilon^2)\,.
\end{aligned}$$

Let us first consider variations ξ^i that have compact support in $[a, b]$, so that the second integral vanishes. The first integral vanishes for all ξ^i if and only if q^i satisfies the field equation

$$\ddot{q}^i = -\frac{\partial V}{\partial q^i}\,,$$

which is the equation of motions of a point particle in a potential V. The second integral is given by

$$\int_a^b \frac{d}{dt}\big(\dot{q}^i(t)\xi^i(t)\big)\, dt = \dot{q}^i(b)\,\xi^i(b) - \dot{q}^i(a)\,\xi^i(a)\,.$$

Now we consider variations ξ^i that have their support concentrated in small neighborhoods around the boundary points a and b. By keeping $\xi^i(a)$ and $\xi^i(b)$ constant while shrinking the support, we can make the first integral arbitrarily small. The conclusion is that the second integral has to vanish for all ξ^i independently of the first, which is the case if and only if

$$\dot{q}^i(a) = 0 \quad \text{and} \quad \dot{q}^i(b) = 0\,.$$

This is certainly not a condition we want to impose on q. We can modify the action principle by requiring $\xi^i(a) = 0 = \xi^i(b)$. But then the solutions of the field equation are not the critical points of S but rather points where the derivative of S vanishes on a subset of vectors in $T_q\mathcal{F}$. Moreover, we have to require this for all compact intervals $[a, b]$.

In a third attempt to solve problem 1.1.8, we as mathematicians could assume M to be closed, i.e. compact without boundary [Abb01]. In the case of classical mechanics this would mean, however, that time is S^1 so that we would only consider periodic solutions. The assumption that M is closed will also exclude some of the most interesting spacetimes, like Minkowski spacetime or many realistic physical models for the curved spacetime of the universe we live in.

1.1.4 Lagrangians

In the example of classical mechanics we have seen that the action is obtained by integrating for every field q a volume form over the spacetime manifold $\mathbb{R}$.

Definition 1.1.9. A smooth function $L : \mathcal{F} \to \Omega^{\mathrm{top}}(M)$, where $\mathrm{top} = \dim M$, is called a **lagrangian**.

Remark 1.1.10. For simplicity, we shall assume that M is oriented. If M is non-orientable, we have to tensor before integration with the determinant bundle of M as it is done in [DF99].

Given a lagrangian L, we tentatively define the action by

$$S(\varphi) := \int_M L(\varphi)\,.$$

But, as we have seen, even for classical mechanics the action is generally not finite, so it is certainly not a smooth map to $\mathbb{R}$. The issues come from the integration over the non-closed manifold $\mathbb{R}$.

When we review the derivation of the equation of motion carefully, we see that we did not need to compute any integrals. All we did is to discard exact terms under the integral. This means that we can just as well study the cohomology class of the integrand without ever pairing it with the fundamental class $[M]$. We will return to this idea in Chap. 7.

Definition 1.1.11. A **lagrangian field theory** (LFT) consists of a smooth fiber bundle $F \to M$ and a lagrangian $L : \mathcal{F} \to \Omega^{\mathrm{top}}(M)$.

1.2 Examples of lagrangian field theories

1.2.1 Classical mechanics

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1.2.2 Maxwell theory

Minkowski space Maxwell theory is the classical theory of electromagnetic fields. Its background geometry is physical spacetime, i.e. a lorentzian 4-manifold M. The most basic choice for M is Minkowski space, i.e. $M = \mathbb{R}^4$ equipped with the metric

$$\begin{aligned}\eta &= \tfrac{1}{2}\eta_{ij}dx^i dx^j \\ &= \tfrac{1}{2}\big(-(dx^0)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2\big)\,,\end{aligned}$$

where x^0 is the time-coordinate and x^1, x^2, x^3 the space-coordinates.

Remark 1.2.1. We define lorentzian metrics to have the signature $(-1, 1, 1, 1)$, which is sometimes called the "east coast" convention, the signature $(1, -1, -1, -1)$ being called the "west coast" convention. The advantage of the east coast convention is that the metric induces the usual euclidean scalar product on 3-space $\mathrm{Span}\{x^1, x^2, x^3\}$.

Terminology 1.2.2. A tangent vector $v \in TM$ on a lorentzian manifold is called **space-like** if $\eta(v, v) > 0$, **light-like** if $\eta(v, v) = 0$, and **time-like** if $\eta(v, v) < 0$. A submanifold $S \subset M$ is called space-like, light-like, or time-like, if all tangent vectors in TS are.

Recall that every bilinear form $\langle\ ,\ \rangle$ on a vector space V can be extended to a bilinear form $\langle\ ,\ \rangle : \wedge^k V \times \wedge^k V \to \mathbb{R}$ on the k-th exterior power by

$$\langle v_1 \wedge \ldots \wedge v_k, w_1 \wedge \ldots \wedge w_k\rangle := \det\big(\langle v_i, w_j\rangle_{1\leq i,j\leq k}\big)\,. \tag{1.1}$$

We consider the fibre-wise scalar product given by the inverse of η,

$$\begin{aligned}\langle\ ,\ \rangle : T^*M \times_M T^*M &\longrightarrow \mathbb{R} \\ \langle \alpha_i dx^i, \beta_j dx^j\rangle &:= \eta^{ij}\alpha_i\beta_j\,,\end{aligned}$$

where η^{ij} denotes the inverse matrix of η_{ij}, i.e. $\eta^{ij}\eta_{jk} = \delta^i_k$. By (1.1) this induces a bilinear form on differential k-forms,

$$\langle\ ,\ \rangle : \Omega^k(M) \times \Omega^k(M) \longrightarrow C^\infty(M)\,.$$

Let us equip M with the standard orientation for which (x^0, x^1, x^2, x^3) is an oriented chart. Then there is a unique oriented volume form $\mathrm{vol} \in \Omega^4(M)$ that is normalized, $\langle \mathrm{vol}, \mathrm{vol}\rangle = 1$. In terms of coordinate 1-forms, it is given by

$$\mathrm{vol} = dx^0 \wedge \ldots \wedge dx^3\,.$$

The volume form is used to define a **Hodge structure** (see e.g. Sec. 3.3 of [Jos17]), i.e. a $C^\infty(M)$-linear map

$$\star : \Omega^k(M) \longrightarrow \Omega^{\dim M - k}(M)\,,$$

which is uniquely determined by the defining equation

$$\alpha \wedge \star\beta = \langle \alpha, \beta \rangle \operatorname{vol},$$

for all $\alpha, \beta \in \Omega^k(M)$ and all k. Note that $\operatorname{vol} = \star 1$. The Hodge-$\star$ satisfies

$$\star\,(\star\alpha) = (\det \eta)(-1)^{(\dim M - |\alpha|)|\alpha|}\alpha\,, \tag{1.2}$$

where $\det \eta$ is the determinant of the metric in any orthonormal basis. For a metric of signature $(-1, 1, 1, 1)$ we have $\det \eta = -1$.

Charges and currents Electric charges and currents generate the electromagnetic field. In physics, a time-dependent charge density is a smooth function ρ on Minkowski space and a current density a vector field $v = v^1 \frac{\partial}{\partial x^1} + v^2 \frac{\partial}{\partial x^2} + v^3 \frac{\partial}{\partial x^3}$ on M with components only in the space directions.

The total charge $q_{S,t}$ contained in a submanifold $S \subset \mathbb{R}^3$ of space at time t is given by the integral

$$q_{S,t} = \int_{\{t\}\times S} \rho\, dx^1 \wedge dx^2 \wedge dx^3\,.$$

The flux of the current through the surface ∂S at time t is given by

$$\begin{aligned}
\Phi_{S,t} &:= \int_{\{t\}\times \partial S} \iota_v (dx^1 \wedge dx^2 \wedge dx^3)\\
&= \int_{\{t\}\times S} d\iota_v (dx^1 \wedge dx^2 \wedge dx^3)\\
&= \int_{\{t\}\times S} (\operatorname{div} v)\, dx^1 \wedge dx^2 \wedge dx^3\,,
\end{aligned}$$

where we have used Stokes' theorem and $\operatorname{div} v = \frac{\partial v^i}{\partial x^i}$.

The current density describes the flow of charge through space, so if the charge is conserved, then the rate of change of the charge in every space-region S must be equal to the negative flux through the surface of S, $\frac{d}{dt} q_{S,t} = -\Phi_{S,t}$. This is the case if and only if

$$\frac{\partial \rho}{\partial t} = -\operatorname{div} v\,. \tag{1.3}$$

We obtain a form of condition (1.3) that does not rely on the splitting of the manifold M into time and space directions by combining charge density and current density into the 4-vector field

$$J := \rho \frac{\partial}{\partial x^0} + v^1 \frac{\partial}{\partial x^1} + v^2 \frac{\partial}{\partial x^2} + v^3 \frac{\partial}{\partial x^3}\,.$$

The de Rham differential of $\iota_J \operatorname{vol}$ is

$$d\,\iota_J \operatorname{vol} = \Big(\frac{\partial \rho}{\partial x^0} + \operatorname{div} v\Big) \operatorname{vol}\,.$$

The conclusion is that Eq. (1.3) holds if and only if $j := \iota_J \operatorname{vol}$ is closed. This suggests the following definition.

Definition 1.2.3. Let M be an n-dimensional manifold. A form $j \in \Omega^{n-1}(M)$ is called a **current**. A current is **conserved** if it is closed, $dj = 0$.

Terminology 1.2.4. In physics, it is usually the vector field J that is called the 4-current. For our purposes, Def. 1.2.3 is more convenient. Unlike for J, the condition in Def. 1.2.3 for a current to be conserved does not involve the volume form.

Lagrangian and field equations The fields for Maxwell theory on Minkowski space are 1-forms. That is, the configuration bundle is $T^*M \to M$ and the space of fields

$$\mathcal{F} = \Omega^1(M)\,.$$

In Maxwell theory it is customary to denote the fields by the letter A. The lagrangian for the electromagnetic field generated by a current $j = \iota_J \mathrm{vol}$ is

$$\begin{aligned} L(A) &= \big(\tfrac{1}{2}\langle dA, dA\rangle + \iota_J A\big)\,\mathrm{vol} \\ &= \tfrac{1}{2}\, dA \wedge \star dA + j \wedge A\,. \end{aligned} \tag{1.4}$$

The Euler-Lagrange equation is

$$d \star dA = j\,. \tag{1.5}$$

The equation $d(dA) = 0$, which is satisfied for any field A, is also part of the Maxwell equations. Note that Eq. (1.5) implies that $dj = 0$, that is, j is conserved.

Terminology 1.2.5. In physics, A is usually called the **gauge field**, in order to distinguish it from the **electromagnetic field** $F := dA$. Denoting the electromagnetic field with F is so standard in physics, that we could not resolve to use a different letter in order to distinguish it from our notation for the configuration bundle.

If we view Eq. (1.5) as equation $d \star F = j$ for the electromagnetic field F, not assuming that the field is the differential of a 1-form A, we have to add the equation

$$dF = 0 \tag{1.6}$$

to the field equations. Eqs. (1.5) and (1.6) together are the **Maxwell equations**.

The Maxwell equations are invariant under the Lorentz group, the group of linear transformations of $\mathbb{R}^4$ that leave the bilinear form η invariant. A careful study of these symmetries led Einstein in 1905 to the development of special relativity [Ein05]. In addition to this **external symmetry** group that acts on the spacetime manifold, there is the **internal symmetry** group $\big(C^\infty(M), +, 0\big)$ that acts on the fields by

$$\begin{aligned} C^\infty(M) \times \Omega^1(M) &\longrightarrow \Omega^1(M) \\ (f, A) &\longmapsto A + df\,. \end{aligned}$$

A careful study of this symmetry, called **local gauge symmetry**, led to the development of gauge theories.

1.2.3 Review of connections on principal bundles

In Yang-Mills gauge theory the fields are connections on a principal bundle. We will first review this concept.

Definition of connections on principal bundles Let G be a Lie group and $\pi : P \to M$ a right principal G-bundle. We denote the free and proper right G-action by $P \times G \to P$, $(p, g) \mapsto p \cdot g = R_g p$, where $R : G \to \mathrm{Diff}(P)$ denotes the structure homomorphism of the action. A connection on the fibre bundle $P \to M$ is given by a **horizontal lift** $h : TM \times_M P \to TP$, i.e. a right splitting of the short exact sequence of vector bundles over P,

$$\begin{tikzcd} 0 \arrow[r] & VP \arrow[r] & TP \arrow[r, "{(T\pi, \mathrm{pr}_P)}"] & TM \times_M P \arrow[r] \arrow[l, bend left, "h"] & 0 \end{tikzcd} \quad . \tag{1.7}$$

The group G acts on $\xi_p \in TP$ by

$$\xi_p \cdot g := TR_g \xi_p \,.$$

Since the bundle projection $\pi : P \to M$ is G-invariant, $\pi(R_g p) = \pi(p)$, its tangent map $T\pi : TP \to TM$ is invariant as well, $T\pi(TR_g \xi_p) = T\pi\, \xi_p$. Since TR_g is a map of vector bundles covering R_g, $\mathrm{pr}_P(TR_g \xi_p) = R_g(\mathrm{pr}_P \xi_p)$, the tangent projection $\mathrm{pr}_P : TP \to P$ is G-equivariant. It follows that, when we equip $TM \times_M P$ with the right G-action defined by

$$(v, p) \cdot g := (v, p \cdot g) \,,$$

then the map $(T\pi, \mathrm{pr}_P)$ is G-equivariant. From the G-invariance of $T\pi$ it follows that $VP = \ker T\pi$ is also G-invariant, so that the inclusion $VP \subset TP$ is G-equivariant. The upshot is that the short exact sequence (1.7) is a sequence of G-equivariant maps of vector bundles over P. Therefore, we should require the splitting h of a connection to be G-equivariant as well.

Definition 1.2.6. A **connection on a principal bundle** P or a **principal connection on** P is an equivariant splitting h of the short exact sequence (1.7) of vector bundles over P.

The affine space of connections The set of connections on P is a subset of the vector space of all maps of vector bundles $TM \times_M P \to TP$. However, since the zero map is not a section of $(T\pi, \mathrm{pr}_P)$, connections are not a vector subspace. The difference of two connections h and h' satisfies

$$T\pi\big(h'(v, p) - h(v, p)\big) = 0 \,,$$

that is, $\mu := h' - h$ takes its values in $\ker T\pi = VP$. Conversely, let $\mu : TM \times_M P \to VP$ be a G-equivariant map of vector bundles over P. Then $h + \mu$ satisfies $T\pi\big(h(v, p) + \mu(v, p)\big) = v$, so that h is a G-equivariant splitting of (1.7). The conclusion is that the set of connections on the principal bundle P is an affine space modelled on the vector space of G-equivariant maps $TM \times_M P \to VP$ of vector bundles.

Since the G-actions are free and proper, such a G-equivariant map can be identified with a map on the G-quotients, $(TM \times_M P)/G \to VP/G$. Since G acts trivially on TM, the quotient of the domain is

$$(TM \times_M P)/G \cong TM \times_M (P/G) \cong TM \,.$$

The quotient VP/G of the target has a nice description, too. Every vertical tangent vector of P can be represented by a smooth path $t \mapsto p_t \in P$ with constant base point $\pi(p_t) = \pi(p_0)$. Since the fibre over $\pi(p_0)$ is isomorphic to G, there is a unique smooth path $t \mapsto g_t \in G$ with $g_0 = e$, such that $p_t = p_0 \cdot g_t$. It follows that we can identify the tangent space at every point with $\mathfrak{g}$, which means that we have an isomorphism of vector bundles

$$VP \cong P \times \mathfrak{g}\,.$$

The action of $h \in G$ on the vertical path p_t is given by

$$p_t \cdot h = (p_0 \cdot g_t) \cdot h = (p_0 \cdot h) \cdot h^{-1} g_t h\,.$$

Differentiating with respect to t, we see that the action of G on VP is given on the isomorphic vector bundle $P \times \mathfrak{g}$ by

$$(p, X) \cdot g = (p \cdot g, \mathrm{Ad}_{g^{-1}} X)\,.$$

It follows that the quotient

$$VP/G \cong (P \times \mathfrak{g})/G = P \times_{\mathrm{Ad}} \mathfrak{g}$$

is the vector bundle associated to the principal bundle by the adjoint representation $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$, which is called the **adjoint bundle**. We summarize our findings in the following proposition.

Proposition 1.2.7. *Let P be a principal G-bundle and $P \times_{\mathrm{Ad}} \mathfrak{g}$ the associated adjoint bundle. The set of principal connections on P is an affine space modelled on the vector space*

$$\Gamma^\infty\big(M, T^*M \otimes (P \times_{\mathrm{Ad}} \mathfrak{g})\big) \cong \Omega^1(M) \otimes_{C^\infty(M)} \Gamma^\infty(M, P \times_{\mathrm{Ad}} \mathfrak{g})\,.$$

Corollary 1.2.8. *When the adjoint bundle of P is trivial, $P \times_{\mathrm{Ad}} \mathfrak{g} \cong M \times \mathfrak{g}$, then the affine space of connections is modelled on the vector space of $\mathfrak{g}$-valued 1-forms $\Omega^1(M) \otimes \mathfrak{g}$.*

There are two basic cases, in which the adjoint bundle is trivial, so that Cor. 1.2.8 applies. In the first case P is a trivial bundle. An important example for this is when $M = \mathbb{R}^4$ is Minkowski space. Another example is, when we restrict M to a coordinate ball $U \subset M$. This implies that locally, connections are modelled on the space of $\mathfrak{g}$-valued 1-forms. These forms are called **local connection 1-forms**.

The second case is that G is abelian, so that the adjoint representation is trivial. For example when $G = \mathrm{U}(1)$, so that $\mathfrak{g} = \mathrm{u}(1) = \mathbb{R}$, principal connections are modelled on the vector space of 1-forms on M. This is the case we have in Maxwell theory.

Curvature Taking the quotient by G of the sequence (1.7), we obtain a short sequence of vector bundles over M,

$$0 \longrightarrow P \times_{\mathrm{Ad}} \mathfrak{g} \longrightarrow TP/G \xrightarrow{T\pi} TM \longrightarrow 0 \ , \tag{1.8}$$

which is called the **Atiyah sequence** of the principal bundle P. This sequence of vector bundles induces a sequence of the vector spaces of sections,

$$0 \longrightarrow \Gamma^\infty(M, P \times_{\mathrm{Ad}} \mathfrak{g}) \longrightarrow \mathfrak{X}(P)^G \xrightarrow{\pi_*} \mathfrak{X}(M) \longrightarrow 0 \ , \tag{1.9}$$

where $\mathfrak{X}(P)^G$ denotes the space of G-invariant vector fields on P.

Remark 1.2.9. The right G-action on P induces a left G-action on vector fields by pullback, $g \cdot \xi = R_g^* \xi$. A vector field ξ is G-invariant if it is a fixed point under this action. Observe, that the map $\xi : P \to TP$ of a G-invariant vector field is G-equivariant.

A splitting of (1.8) induces a splitting $h : \mathfrak{X}(M) \to \mathfrak{X}(P)^G$ of (1.9). The **curvature** of the connection is given by

$$F(v,w) := [h(v), h(w)] - h\big([v,w]\big) \, , \tag{1.10}$$

for all $v, w \in \mathfrak{X}(M)$. The curvature vanishes if and only if h is a homomorphism of Lie algebras. If this is the case, the connection is called **flat**.

Remark 1.2.10. Sequence (1.9) can be viewed as an extension of Lie algebras. Then F is the 2-cocycle in the Lie algebroid cohomology that classifies extensions up to isomorphism.

The horizontal lift of every $v \in \mathfrak{X}(M)$ projects to v, $\pi_* h(v) = v$. In other words, $h(v)$ is **π-related** to v. Since the Lie brackets of π-related vector fields are π-related (see e.g. Prop. 8.30 in [Lee13]), the curvature satisfies

$$\begin{aligned} \pi_* F(v,w) &= \pi_*[h(v), h(w)] - \pi_* h\big([v,w]\big) \\ &= [\pi_* h(v), \pi_* h(w)] - [v,w] \\ &= 0 \, . \end{aligned}$$

Moreover, using that $h(fv) = (\pi^* f) h(v)$ and $h(v) \cdot \pi^* f = \pi^*(v \cdot f)$ for every $f \in C^\infty(M)$, a similar calculation shows that F is $C^\infty(M)$-linear in both arguments, which implies that F is a bundle map on $\wedge^2 T^* M$. We conclude that the curvature is a 2-form with values in the adjoint bundle,

$$\begin{aligned} F &\in \Gamma^\infty\big(M, \wedge^2 T^* M \otimes (P \times_{\mathrm{Ad}} \mathfrak{g})\big) \\ &\cong \Omega^2(M) \otimes_{C^\infty(M)} \Gamma^\infty(M, P \times_{\mathrm{Ad}} \mathfrak{g}) \, . \end{aligned}$$

Remark 1.2.11. According to Prop. 1.2.7, the set of connections is an *affine space* modelled on the vector space of 1-forms on M with values in the adjoint bundle. The curvature, however, takes values in the *vector space* of 2-forms on M with values in the adjoint bundle. The reason is that the curvature is defined as difference of two terms in an affine space.

Connection and curvature as invariant forms As it is the case for short exact sequences in any abelian category, a right splitting of the sequence (1.7) induces a left splitting and vice versa. In fact, given a horizontal lift $h : TM \times_M P \to TP$, we obtain a map

$$\begin{aligned} \theta : TP &\longrightarrow VP \\ \xi_p &\longmapsto \theta(\xi_p) := \xi_p - h\big(T\pi\, \xi_p, p\big)\,, \end{aligned}$$

which maps $\xi_p \in VP$ to ξ_p, so it is a retract of the inclusion $VP \to TP$. Moreover, if h is G-equivariant, then so is θ. Using the natural trivialization $VP \cong P \times \mathfrak{g}$, this retract can be viewed as a linear map $\theta : TP \to \mathfrak{g}$, which is equivariant with respect to the action $\xi_p \cdot g = TR_g\xi_p$ on TP and $X \cdot g = \mathrm{Ad}_{g^{-1}}X$ on $\mathfrak{g}$. These observations are summarized in the following proposition.

Proposition 1.2.12. *Let $\Omega^\bullet(P) \otimes \mathfrak{g}$ be equipped with the left G-action defined by*

$$g \cdot (\alpha \otimes X) := R_g^*\alpha \otimes \mathrm{Ad}_g X\,,$$

for all $\alpha \otimes X \in \Omega^\bullet(P) \otimes \mathfrak{g}$. Then a principal connection on P is given by a unique G-invariant form $\theta \in \Omega^1(P) \otimes \mathfrak{g}$ that acts as the identity $\theta(\xi_p) = \xi_p$ on all vertical vectors $\xi_p \in V_pP \cong \mathfrak{g}$, $p \in P$.

Terminology 1.2.13. In view of Prop. 1.2.12, an invariant $\mathfrak{g}$-valued 1-form on P that restricts to the identity on vertical vectors is called a **connection 1-form**.

Given a connection 1-form θ or, equivalently, a horizontal lift h, the **horizontal tangent space** at $p \in P$ is defined as

$$H_p := \ker \theta_p = h(TM \times_M \{p\}) \subset T_pP\,.$$

The **horizontal distribution** $H = \ker \theta \subset TP$ is the Ehresmann connection given by θ. Since θ is G-invariant, so is H. In fact, a connection on a principal bundle can be identified with a G-invariant Ehresmann connection.

Terminology 1.2.14. A form in $\Omega^\bullet(P) \otimes \mathfrak{g}$ is called **horizontal** if it annihilates the vertical tangent bundle VP. A form that is horizontal and G-invariant is called **basic**.

The vector space of all G-invariant forms will be denoted by $(\Omega^\bullet(P) \otimes \mathfrak{g})^G$ and the space of horizontal forms by $\Omega^\bullet(P)_{\mathrm{hor}}$. So the space of basic forms is denoted by $(\Omega^\bullet(P)_{\mathrm{hor}} \otimes \mathfrak{g})^G$.

Proposition 1.2.15. *The set of connection 1-forms is an affine space modelled on the vector space of basic 1-forms.*

Proof. Let θ be a connection 1-form. If θ' another connection 1-form, then $\mu := \theta' - \theta$ is a G-invariant 1-form, such that for all $\xi_p \in VP$ we have $\mu(\xi_p) = \theta'(\xi_p) - \theta(\xi_p) = \xi_p - \xi_p = 0$, so that μ is horizontal. Conversely, if μ is a basic 1-form on P, then $\theta' := \theta + \mu$ is a G-invariant 1-form on P, such that for all $\xi_p \in VP$ we have $\theta'(\xi_p) = \theta(\xi_p) + \mu(\xi_p) = \xi_p + 0 = \xi_p$, so that θ' is a connection 1-form. □

Both Prop. 1.2.15 and Prop. 1.2.7 establish that the set of connections has the natural structure of an affine space, which implies that the affine spaces of the two propositions must be isomorphic. The following lemma makes this explicit.

Lemma 1.2.16. *A connection on the principal bundle P induces an isomorphism of $C^\infty(M)$-modules*

$$\Gamma^\infty\big(M, \wedge^\bullet T^*M \otimes (P \times_{\mathrm{Ad}} \mathfrak{g})\big) \cong (\Omega^\bullet(P)_{\mathrm{hor}} \otimes \mathfrak{g})^G\,. \tag{1.11}$$

Proof. A section σ of $\wedge^k T^*M \otimes (P \times_{\mathrm{Ad}} \mathfrak{g}) \to M$ can be identified with a map

$$\wedge^k TM \longrightarrow P \times_{\mathrm{Ad}} \mathfrak{g}\,,$$

of vector bundles over M, which in turn can be identified with a G-equivariant map

$$\wedge^k TM \times_M P \longrightarrow P \times \mathfrak{g}$$

of vector bundles over P. The horizontal lift induces a G-equivariant isomorphism

$$h : TM \times_M P \xrightarrow{\cong} H$$

of vector bundles over P. This shows that σ can be identified with a G-equivariant linear map

$$\wedge^k H \longrightarrow \mathfrak{g}\,,$$

which can be identified with a G-invariant section of $\wedge^k H^* \otimes \mathfrak{g} \to P$, which in turn can be identified with a basic form

$$\mu_\sigma \in (\Omega^k(P)_{\mathrm{hor}} \otimes \mathfrak{g})^G \cong \Gamma^\infty(M, \wedge^k H \otimes \mathfrak{g})^G\,.$$

From μ_σ we retrieve σ by

$$\sigma(v_1 \wedge \ldots \wedge v_k, p) = \big[p, \mu\big(h(v_1, p) \wedge \ldots \wedge h(v_k, p)\big)\big]\,,$$

for all $v_1, \ldots, v_k \in T_mM$, all $m \in M$, and all p in the fibre over m, where $[p, X]$ for $p \in P$, $X \in \mathfrak{g}$ denotes an equivalence class in $P \times_{\mathrm{Ad}} \mathfrak{g} = (P \times \mathfrak{g})/G$. ☐

Remark 1.2.17. A local trivialization $P|_U \cong U \times G$ induces an isomorphism of each side of Eq. (1.11) with $\Omega^\bullet(U) \otimes \mathfrak{g}$.

A G-invariant vector field ξ is vertical if and only if it projects to the zero vector field, $\pi_*\xi = 0$. If ξ is vertical and χ an arbitrary G-invariant vector field, then

$$\begin{aligned}\pi_*[\xi, \chi] &= [\pi_*\xi, \pi_*\chi] = [0, \pi_*\chi] \\ &= 0\,,\end{aligned}$$

that is, the Lie bracket of a vertical G-invariant vector field with any other G-invariant vector is again vertical.

A connection is flat if the horizontal distribution H is integrable, which by the Frobenius theorem is the case if and only if the space of horizontal vector fields is closed under the Lie bracket. Every vector field $\xi \in \mathfrak{X}(P)^G$ can be decomposed as $\xi = \xi_V + \xi_H$ into its vertical and horizontal parts,

$$\xi_V = \theta(\xi)\,, \quad \xi_H = \xi - \theta(\xi)\,.$$

Since a vector field is horizontal if and only if it is annihilated by θ, the distribution H is involutive if and only if

$$\tilde{F}(\xi, \chi) := \theta\big([\xi_H, \chi_H]\big) \tag{1.12}$$

vanishes for all $\xi, \chi \in \mathfrak{X}(P)^G$. It is straightforward to check that $\tilde{F}(\xi, \chi)$ is $C^\infty(P)$-linear in both arguments, so it is a two form on P. Moreover, $\tilde{F}$ is vertical and annihilates horizontal vector fields, so it can be viewed as a basic 2-form,

$$\tilde{F} \in (\Omega^2(P)_{\mathrm{hor}} \otimes \mathfrak{g})^G\,.$$

Proposition 1.2.18. *The basic 2-form $\tilde{F}$ is identified by the isomorphism of Lem. 1.2.16 with the curvature form F defined in Eq.* (1.10)*.*

Proof. For every $v \in \mathfrak{X}(M)$, the horizontal lift $h(v) \in \mathfrak{X}(P)^G$ is the unique horizontal G-invariant vector field that projects to $\pi_* h(v) = v$. When we evaluate $\tilde{F}$ on the horizontal lifts of two vector fields $v, w \in \mathfrak{X}(M)$, we obtain

$$\begin{aligned}
\tilde{F}\big(h(v), h(w)\big) &= \theta\big([h(v), h(w)]\big) \\
&= \theta\big([h(v), h(w)] - h([v, w])\big) \\
&= [h(v), h(w)] - h([v, w]) \\
&= F(v, w)\,,
\end{aligned}$$

which proves the proposition. □

Notation 1.2.19. In view of Prop. 1.2.18, we will from now on denote the 2-form $\tilde{F}$ defined in Eq. (1.12) also by $F \equiv \tilde{F}$.

The DGLA of invariant forms The de Rham differential on $\Omega^\bullet(P)$ and the Lie bracket on $\mathfrak{g}$ can be extended to the graded vector space $\Omega^\bullet(P) \otimes \mathfrak{g}$, by

$$\begin{gathered}
d(\alpha \otimes X) := d\alpha \otimes X \\
[\alpha \otimes X, \beta \otimes Y] := (\alpha \wedge \beta) \otimes [X, Y]\,,
\end{gathered} \tag{1.13}$$

for all $\alpha \otimes X, \beta \otimes Y \in \Omega^\bullet(P) \otimes \mathfrak{g}$. The following proposition is straightforward to prove.

Proposition 1.2.20. *The differential and bracket* (1.13) *equip the graded vector space $\Omega^\bullet(P) \otimes \mathfrak{g}$ with the structure of a differential graded Lie algebra (DGLA).*

Proposition 1.2.21. *The graded subspace $(\Omega^\bullet(P) \otimes \mathfrak{g})^G \subset \Omega^\bullet(P) \otimes \mathfrak{g}$ of G-invariant forms is a sub-DGLA, i.e. it is closed under the differential and the Lie bracket.*

Proof. Every pullback commutes with the differential, $R_g^* d\alpha = d(R_g^*\alpha)$, and with the product, $R_g^*(\alpha \wedge \beta) = R_g^*\alpha \wedge R_g^*\beta$, for all $\alpha, \beta \in \Omega^\bullet(P)$. The adjoint action commutes with the Lie bracket $\mathrm{Ad}_g[X,Y] = [\mathrm{Ad}_g X, \mathrm{Ad}_g Y]$. With these relations it is easy to show that the bracket of invariant forms $\alpha \otimes X, \beta \otimes Y \in (\Omega^\bullet(P) \otimes \mathfrak{g})^G$ satisfies

$$\begin{aligned} g \cdot [\alpha \otimes X, \beta \otimes Y] &= g \cdot \big((\alpha \wedge \beta) \otimes [X,Y]\big) \\ &= R_g^*(\alpha \wedge \beta) \otimes \mathrm{Ad}_g[X,Y] \\ &= (R_g^*\alpha \wedge R_g^*\beta) \otimes [\mathrm{Ad}_g X, \mathrm{Ad}_g Y] \\ &= [R_g^*\alpha \otimes \mathrm{Ad}_g X, R_g^*\beta \otimes \mathrm{Ad}_g Y] \\ &= [g \cdot (\alpha \otimes X), g \cdot (\beta \otimes Y)] \\ &= [\alpha \otimes X, \beta \otimes Y]\,, \end{aligned}$$

so it is G-invariant. Similarly, we obtain for the differential of a G-invariant form

$$\begin{aligned} g \cdot d(\alpha \otimes X) &= g \cdot (d\alpha \otimes X) \\ &= R_g^* d\alpha \otimes \mathrm{Ad}_g X \\ &= d(R_g^*\alpha) \otimes \mathrm{Ad}_g X \\ &= d(R_g^*\alpha \otimes \mathrm{Ad}_g X) \\ &= d\big(g \cdot (\alpha \otimes X)\big) \\ &= d(\alpha \otimes X)\,, \end{aligned}$$

so it is G-invariant, as well. □

Proposition 1.2.22. *The curvature of a connection 1-form* $\theta \in (\Omega^1(P) \otimes \mathfrak{g})^G$ *is given by*

$$F = -d\theta + \frac{1}{2}[\theta, \theta]\,.$$

Proof. The curvature can be written as

$$\begin{aligned} F(\xi, \chi) &= \theta\big([\xi - \theta(\xi), \chi - \theta(\chi)]\big) \\ &= \theta\big([\xi, \chi] - [\xi, \theta(\chi)] - [\theta(\xi), \chi] + [\theta(\xi), \theta(\chi)]\big) \\ &= \theta([\xi, \chi]) - [\xi, \theta(\chi)] + [\chi, \theta(\xi)] + [\theta(\xi), \theta(\chi)] \end{aligned} \tag{1.14}$$

for all G-invariant vector fields ξ, χ.

By the identification $VP = P \times \mathfrak{g}$, the elements of $\mathfrak{g}$ are the fundamental vector fields of the G-action on P. So if a vector field $\xi \in \mathfrak{X}(P)$ is G-invariant, $R_g^*\xi = \xi$, then the Lie derivative of ξ with respect to all $X \in \mathfrak{g}$ must vanish,

$$[\xi, X] = 0\,. \tag{1.15}$$

Let $\{X_\alpha\} \subset \mathfrak{g}$ be a basis. Then the connection 1-form can be written as $\theta = \theta^\alpha \otimes X_\alpha$. It follows from (1.15), that for G-invariant vector fields $\xi, \chi \in \mathfrak{X}(P)^G$ we have

$$[\xi, \theta(\chi)] = [\xi, \theta^\alpha(\chi) X_\alpha] = \big(\xi \cdot \theta^\alpha(\chi)\big) X_\alpha\,.$$

With this relation we obtain

$$\begin{aligned}(d\theta)(\xi,\chi) &= (d\theta^\alpha)(\xi,\chi)\,X_\alpha \\ &= \big(\xi\cdot\theta^\alpha(\chi) - \chi\cdot\theta^\alpha(\xi) - \theta^\alpha([\xi,\chi])\big)\,X_\alpha \\ &= [\xi,\theta(\chi)] - [\chi,\theta(\xi)] - \theta([\xi,\chi])\,,\end{aligned}$$

which is minus the first three terms of the right hand side of Eq. (1.14). For the last term, we have

$$\begin{aligned}[\theta,\theta](\xi,\chi) &= \iota_\chi\iota_\xi[\theta,\theta] \\ &= \iota_\chi\iota_\xi[\theta^\alpha\otimes X_\alpha, \theta^\beta\otimes X_\beta] \\ &= \iota_\chi\iota_\xi(\theta^\alpha\wedge\theta^\beta)\otimes[X_\alpha,X_\beta] \\ &= \iota_\chi\big(\theta^\alpha(\xi)\,\theta^\beta - \theta^\alpha\,\theta^\beta(\xi)\big)\otimes[X_\alpha,X_\alpha] \\ &= \big(\theta^\alpha(\xi)\,\theta^\beta(\chi) - \theta^\alpha\alpha(\chi)\,\theta^\beta(\xi)\big)\otimes[X_\alpha,X_\beta] \\ &= 2[\theta^\alpha(\xi)X_\alpha, \theta^\beta(\chi)X_\beta] \\ &= 2[\theta(\xi),\theta(\chi)]\,,\end{aligned}$$

from which is follows that $\frac{1}{2}[\theta,\theta](\xi,\chi) = [\theta(\xi),\theta(\chi)]$. We conclude that the sum of $(-d\theta)(\xi,\chi)$ and $\frac{1}{2}[\theta,\theta](\xi,\chi)$ is the right hand side of (1.14). □

Terminology 1.2.23. An element A of a DGLA is called **Maurer-Cartan element** if $dA + \frac{1}{2}[A,A] = 0$. In this terminology, a connection 1-form defines a flat connection if $A = -\theta$ is a Maurer-Cartan element of the DGLA $(\Omega^\bullet(P)\otimes\mathfrak{g})^G$.

Given a connection 1-form θ, we define a linear map by

$$\begin{aligned}d_\theta : \Omega^\bullet(P)\otimes\mathfrak{g} &\longrightarrow \Omega^{\bullet+1}(P)\otimes\mathfrak{g} \\ d_\theta\eta &:= d\eta - [\theta,\eta]\end{aligned} \tag{1.16}$$

for all $\eta\in\Omega^\bullet(P)\otimes\mathfrak{g}$. The map d_θ is a derivation, i.e. it satisfies

$$d_\theta[\eta,\zeta] = [d_\theta\eta,\zeta] + (-1)^{|\eta|}[\eta,d_\theta\zeta]$$

for all $\eta,\zeta\in\Omega^\bullet(P)\otimes\mathfrak{g}$, which can be checked by a straightforward calculation. Since θ is G-invariant, d_θ maps G-invariant forms to G-invariant forms, so it induces a degree 1 derivation on G-invariant forms. Moreover, d_θ is a differential, $d_\theta^2 = 0$, if and only if θ defines a flat connection.

Proposition 1.2.24. *Let θ be a connection 1-form and F its curvature 2-form. Then $d_\theta F = 0$.*

Proof. We have

$$\begin{aligned}d_\theta F &= d\big(-d\theta + \tfrac{1}{2}[\theta,\theta]\big) - \big[\theta, -d\theta + \tfrac{1}{2}[\theta,\theta]\big] \\ &= -d^2\theta + \tfrac{1}{2}\big([d\theta,\theta] - [\theta,d\theta]\big) + [\theta,d\theta] - \tfrac{1}{2}\big[\theta,[\theta,\theta]\big] \\ &= -\tfrac{1}{2}\big[\theta,[\theta,\theta]\big]\,,\end{aligned}$$

where we have used $d^2 = 0$ and $[d\theta,\theta] = -[\theta,d\theta]$. For the remaining term we get from the graded Jacobi identity

$$\begin{aligned}[\theta,[\theta,\theta]] &= [[\theta,\theta],\theta]] - [\theta,[\theta,\theta]] \\ &= -2[\theta,[\theta,\theta]]\,,\end{aligned}$$

which implies that $[\theta,[\theta,\theta]] = 0$. □

1.2.4 Yang-Mills gauge theory

In Yang-Mills gauge theory, the fields are the connections on a principal G-bundle P over a lorentzian 4-manifold M. As we have seen in Prop. 1.2.7, connections are sections of an affine bundle, so they are really fields in the sense of Def. 1.1.1. We define the **gauge field**

$$A := -\theta$$

to be the negative of the connection 1-form. The curvature is given in terms of A by

$$F(A) = dA + \tfrac{1}{2}[A, A]\,. \tag{1.17}$$

The product of fields So far, we have equipped $(\Omega^\bullet(P) \otimes \mathfrak{g})^G$ with the structure of a DGLA. In order to make sense of terms like $F(A) \wedge \star F(A)$, which appear in the lagrangian (1.4) of Maxwell theory, we have to be able to multiply elements of $(\Omega^\bullet(P) \otimes \mathfrak{g})^G$. This is achieved by embedding $\mathfrak{g}$ into its universal enveloping algebra $\mathcal{U}(\mathfrak{g})$, which is the free associative algebra generated by $\mathfrak{g}$ modulo the relations $XY - YX = [X, Y]$ for all $X, Y \in \mathfrak{g}$. The adjoint action of G on $\mathfrak{g}$ extends uniquely to the adjoint action on $\mathcal{U}(\mathfrak{g})$, so we obtain a map

$$(\Omega^\bullet(P) \otimes \mathfrak{g})^G \hookrightarrow \big(\Omega^\bullet(P) \otimes \mathcal{U}(\mathfrak{g})\big)^G\,.$$

The right hand side is a differential graded algebra (DGA). The associative product of $\alpha \otimes a$ and $\beta \otimes b$ in $\Omega^\bullet(P) \otimes \mathcal{U}(\mathfrak{g})$ is denoted by

$$(\alpha \otimes a) \wedge (\beta \otimes b) := (\alpha \wedge \beta) \otimes ab\,.$$

Warning 1.2.25. The product in $\Omega^\bullet(P) \otimes \mathcal{U}(\mathfrak{g})$ is denoted by $\wedge$, even though it is not graded anti-commutative.

The trace Let $\Phi : G \to \mathrm{GL}(k, \mathbb{R})$ be a finite-dimensional representation of G and $\rho : \mathcal{U}(\mathfrak{g}) \to \mathrm{Mat}(k, \mathbb{R})$ the corresponding representation of the universal enveloping algebra. Let $\mathrm{Tr} : \mathrm{Mat}(k, \mathbb{R}) \to \mathbb{R}$ denote the trace. We define

$$\mathrm{Tr}_\rho : \mathcal{U}(\mathfrak{g}) \xrightarrow{\rho} \mathrm{Mat}(k, \mathbb{R}) \xrightarrow{\mathrm{Tr}} \mathbb{R}\,.$$

Note that Tr_ρ inherits the trace property $\mathrm{Tr}_\rho(XY) = \mathrm{Tr}_\rho(YX)$ from the trace of matrices. The action of G induces an action of G on $\mathrm{Mat}(k, \mathbb{R})$ given by $g \cdot B := \Phi(g)B\Phi(g)^{-1}$. The map ρ is G-equivariant with respect to this action and the adjoint action on G. The matrix trace is invariant with respect to the adjoint action, so that Tr_ρ is G-invariant. It follows that the map

$$\Omega^\bullet(P) \otimes \mathcal{U}(\mathfrak{g}) \xrightarrow{\mathrm{id}\otimes\mathrm{Tr}_\rho} \Omega^\bullet(P) \otimes \mathbb{R}$$

is G-invariant, so that it descends to a map on equivariant forms,

$$\begin{aligned} \mathrm{Tr}_\rho : \big(\Omega^\bullet(P)_{\mathrm{hor}} \otimes \mathcal{U}(\mathfrak{g})\big)^G &\longrightarrow \Omega^\bullet(M) \\ \eta \otimes a &\longmapsto \mathrm{Tr}_\rho(a)\,\eta\,, \end{aligned} \tag{1.18}$$

where we have used the isomorphism

$$(\Omega(P)_{\mathrm{hor}})^G \cong \Omega(M)\,.$$

From (1.18) we can deduce that the trace is graded cyclic,

$$\mathrm{Tr}_\rho(\eta_1 \wedge \ldots \wedge \eta_k) = (-1)^{|\eta_1|(|\eta_2|+\ldots+|\eta_k|)}\mathrm{Tr}_\rho(\eta_2 \wedge \ldots \wedge \eta_k \wedge \eta_1)\,. \tag{1.19}$$

Remark 1.2.26. For every Lie algebra there is the adjoint representation on the vector space $\mathfrak{g}$, given by $\mathrm{ad}(X)\,Y = [X, Y]$. The bilinear form $\mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$, $(X, Y) \mapsto \mathrm{Tr}_{\mathrm{ad}}(X, Y)$ is called the **Killing form**. A real Lie algebra is semi-simple if the Killing form is non-degenerate, and it is the Lie algebra of a compact Lie group if the Killing form is negative definite. So when G is semi-simple compact, like SU(2) and SU(3), the trace is taken with respect to the adjoint action. For $G = \mathrm{U}(1)$, however, the adjoint action is trivial, so that the trace has to be taken with respect to a non-zero character of u(1).

Lagrangian and field equations We now have all the technical ingredients to write down the Yang-Mills lagrangian (without current), which is given by

$$L(A) = \mathrm{Tr}_\rho\big(\tfrac{1}{2}\,F(A) \wedge \star F(A)\big)\,, \tag{1.20}$$

where ρ is the adjoint representation for G semisimple and a non-zero element of $\mathfrak{g}^*$ for $G = \mathrm{U}(1)$. The Euler-Lagrange equation is

$$d_A \star F(A) = 0\,, \tag{1.21}$$

where $d_A = d_\theta$ is the gauge equivariant extension of d, which was introduced in Eq. (1.16).

If we view Eq. (1.21) as equation for the field F, we have to add the equation

$$d_A F = 0 \tag{1.22}$$

to the field equations. In Prop. 1.2.24 we have seen that this equation is satisfied for the field $F = F(A)$ that arises as curvature of A. Eqs. (1.21) and (1.22) together are called the the **Yang-Mills equations**.

Example 1.2.27. Let $G = \mathrm{U}(1)$, so that $\mathfrak{g} = \mathbb{R}$. Since U(1) is abelian, the adjoint action is trivial, which implies the isomorphism

$$\big(\Omega^\bullet(P)_{\mathrm{hor}} \otimes \mathbb{R}\big)^{\mathrm{U}(1)} \cong \Omega^\bullet(M) \otimes \mathfrak{g}\,.$$

It follows that if we choose some connection as the origin in the affine space of connections, then connections can be identified with 1-forms on M. The trace can be taken with respect to the representation $\rho = \mathrm{id}_{\mathbb{R}}$, so that $\mathrm{Tr}_\rho = \mathrm{id}$. We thus retrieve the Maxwell lagrangian (1.4) with $j = 0$ from the Yang-Mills lagrangian (1.20).

Remark 1.2.28. At first sight, the Yang-Mills equations look no more complicated than the Maxwell equations. Note, however, that the expression (1.17) for the curvature $F(A)$ contains a quadratic term, so that the field equations contain cubic terms in A. This makes solving the Yang-Mills equation a very difficult non-linear problem. In fact, one of the Millennium prize problems in mathematics is about the solutions of the Yang-Mills equations.

1.2.5 Abelian Chern-Simons theory

Chern-Simons form There are other interesting lagrangians on the space of connections on a principal G-bundle $P \to M$.

Definition 1.2.29. Let A be a gauge field, i.e. $A = -\theta$ for a connection 1-form θ. The 3-form on M given by

$$\omega_{\mathrm{CS}}(A) := \mathrm{Tr}_{\mathrm{ad}}\big(F(A) \wedge A - \tfrac{1}{3} A \wedge A \wedge A\big)$$

is called the **Chern-Simons 3-form** for A.

Proposition 1.2.30. *The Chern-Simons 3-form satisfies*

$$d\omega_{\mathrm{CS}}(A) = \mathrm{Tr}_{\mathrm{ad}}\big(F(A) \wedge F(A)\big) .$$

Proof. Let $\{X_\alpha\} \subset \mathfrak{g}$ be a basis. The square of $A = A^\alpha \otimes X_\alpha$ is given by

$$\begin{aligned} A \wedge A &= (A^\alpha \otimes X_\alpha) \wedge (A^\beta \otimes X_\beta) \\ &= (A^\alpha \wedge A^\beta) \otimes X_\alpha X_\beta \\ &= \tfrac{1}{2}(A^\alpha \wedge A^\beta - A^\beta \wedge A^\alpha) \otimes X_\alpha X_\beta \\ &= \tfrac{1}{2} A^\alpha \wedge A^\beta \otimes (X_\alpha X_\beta - X_\beta X_\alpha) \\ &= \tfrac{1}{2} A^\alpha \wedge A^\beta \otimes [X_\alpha, X_\beta] \\ &= \tfrac{1}{2}[A, A] . \end{aligned}$$

It follows that the curvature form of A can be written as

$$F(A) = dA + A \wedge A . \tag{1.23}$$

Inserting this expression for $F(A)$ into the definition of the Chern-Simons 3-form, we obtain

$$\omega_{\mathrm{CS}}(A) = \mathrm{Tr}_{\mathrm{ad}}\big(dA \wedge A + \tfrac{2}{3} A \wedge A \wedge A\big) .$$

We have to compute the differential of $\omega_{\mathrm{CS}}(A)$. First, we observe that since the trace satisfies Eq. (1.19), we have the relations

$$\mathrm{Tr}(dA \wedge A \wedge A) = \mathrm{Tr}(A \wedge A \wedge dA) = -\mathrm{Tr}(A \wedge dA \wedge A) ,$$

where from now on we write Tr for the trace. In fact, the computations do not depend on the representation with respect to which we take the trace. Eq. (1.19) also yields

$$\mathrm{Tr}(A \wedge A \wedge A \wedge A) = -\mathrm{Tr}(A \wedge A \wedge A \wedge A) ,$$

which implies that

$$\mathrm{Tr}(A \wedge A \wedge A \wedge A) = 0 .$$

With these relations we obtain

$$\begin{aligned} d\omega_{\mathrm{CS}}(A) &= \mathrm{Tr}\big\{d(dA \wedge A) + \tfrac{2}{3} d(A \wedge A \wedge A)\big\} \\ &= \mathrm{Tr}\big\{dA \wedge dA + \tfrac{2}{3}(dA \wedge A \wedge A - A \wedge dA \wedge A + A \wedge A \wedge dA)\big\} \\ &= \mathrm{Tr}\big\{dA \wedge dA + 2\, dA \wedge A \wedge A + A \wedge A \wedge A \wedge A\big\} \\ &= \mathrm{Tr}\big\{F(A) \wedge F(A)\big\} , \end{aligned}$$

where we have used that Tr and d commute, and that d is a derivation. □

Lagrangian and field equation Let M be 3-manifold. The Chern-Simons lagrangian is given by the Chern-Simons 3-form,

$$L(A) := \omega_{\mathrm{CS}}(A)\,.$$

The Euler-Lagrange equation is

$$F(A) = 0\,.$$

In other words, Chern-Simons theory is the theory of flat connections on principal fibre bundles. In fact, it is closely related to secondary characteristic classes in Chern-Weil theory. ***

1.2.6 Poisson sigma models

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1.2.7 General relativity

In general relativity a field is a lorentzian metric on a smooth oriented manifold of dimension n. The vacuum Hilbert-Einstein lagrangian is

$$L(g) := R(g)\,\mathrm{vol}_g\,,$$

where $R(g)$ is the scalar curvature and $\mathrm{vol}_g = \star 1$ the canonical volume form of g. The Euler-Lagrange equation is the vacuum **Einstein equation**

$$G := \mathrm{Ric}(g) - \frac{1}{2}R(g)\,g = 0\,,$$

where $\mathrm{Ric}(g)$ is the Ricci curvature and where the symmetric 2-form G is called the **Einstein tensor**. Pairing the Einstein tensor with the inverse metric, we obtain

$$g^{ij}G_{ij} = R(g) - \frac{n}{2}R(g) = -\frac{n-2}{2}R(g)\,.$$

If $n > 2$ it follows, that every metric that satisfies the Einstein equations has vanishing scalar curvature. This in turn implies that the vacuum Einstein equations are equivalent to

$$\mathrm{Ric}(g) = 0\,.$$

In other words, a metric satisfies the Euler-Lagrange equations of general relativity if it is Ricci flat.

Exercises

Exercise 1.1 (Quotient diffeology of the folded line)**.** Let $X = \mathbb{R}$ with the natural diffeology of the smooth manifold $\mathbb{R}$. Let $Y = \mathbb{R}/\mathbb{Z}_2 \cong [0,\infty)$ be the quotient of the action of $\mathbb{Z}_2 \cong \{1,-1\}$ by multiplication. Determine the quotient diffeology on Y, that is, the finest diffeology, such that the quotient projection $X \to Y$, $x \mapsto |x|$ is a morphisms of diffeological spaces. What are the smooth paths $\gamma : \mathbb{R} \to Y$ that pass through $\gamma(0) = 0$?

Exercise 1.2 (Subspace diffeology at the boundary)**.** Let $X = \mathbb{R}^2$ with the natural diffeology of the smooth manifold $\mathbb{R}^2$. Let $Y = \bar{D}^2 = \{x \in \mathbb{R}^2 \mid \|x\| \leq 1\}$ be the closed disk. Determine the subspace diffeology of Y, that is, the coarsest diffeology, such that the inclusion $X \hookrightarrow Y$ is a morphisms of diffeological spaces. What are the smooth paths $\gamma : \mathbb{R} \to Y$ that pass through $\gamma(0) = (1, 0)$?

Exercise 1.3 (Horizontal exterior differential)**.** Let $P \to M$ be a principal bundle with a connection given by a connection 1-form θ. The horizontal exterior differential of a differential form $\alpha \in \Omega^k(P)$ is defined by

$$(D\alpha)(\xi_0, \dots, \xi_k) := (d\alpha)(\xi_{0H}, \dots, \xi_{kH}),$$

where $\xi_H = \xi - \theta(\xi)$ is the horizontal component of ξ. Show that $D^2 = 0$ if and only if θ defines a flat connection.

Exercise 1.4 (Chern-Simons 5-form)**.** Let $P \to M$ be a principal bundle. Let $F(A)$ denote the curvature 2-form of a gauge field A. Compute the Chern-Simons 5-form, which is the 5-form $\omega(A)$ on M that satisfies

$$d\big(\omega(A)\big) = \mathrm{Tr}_{\mathrm{ad}}\{F(A) \wedge F(A) \wedge F(A)\}$$

and depends polynomially on A and dA.

Chapter 2

Diffeological spaces of fields

The first attempt to view the set $\mathcal{F} = \Gamma^\infty(M, F)$ as a mathematical space is as a topological space, by equipping $\mathcal{F} \subset \mathrm{Hom}(M, F)$ with the subspace topology of the compact-open topology of $\mathrm{Hom}(M, F)$. Recall, that the compact-open topology is the topology that is generated by the open sets $U_{C,V}$ defined for every compact set $C \subset M$ and open set $V \subset F$ by

$$U_{C,V} := \{\varphi \in \mathcal{F} \mid \varphi(C) \subset V\} .$$

However, many if not most of the functions on $\mathcal{F}$, that are relevant in classical field theory will not be continuous with respect to this topology. Consider $F := \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, so that $\mathcal{F} = C^\infty(\mathbb{R})$. Consider the sequence $n \mapsto \varphi_n$ of fields given by

$$\varphi_n(x) := e^{-\frac{1}{2}(x-n)^2} .$$

We can picture this as a travelling gaussian wave. Any translation invariant physical quantity, like the energy $E(\varphi_n) = \frac{1}{2}\int_{\mathbb{R}} |\frac{\partial \varphi_n}{\partial x}|^2 dx = \int_{\mathbb{R}} (x-n)^2 e^{-(x-n)^2} = \frac{1}{4}\sqrt{\pi}$, will be constant for the sequence φ_n. However, in the compact-open topology φ_n converges to the zero function $\varphi_n \to 0$, which can be verified by restricting φ_n to any compact interval. The conclusion is that the compact-open topology on $\mathcal{F}$ will not be particularly useful in field theory.

In many situations the notion of smooth paths is sufficient to get from global to infinitesimal structures. For example, we only need to use the notion of smooth paths of diffeomorphisms given by local flows in order to show that the Lie algebra of the diffeomorphism group $\mathrm{Diff}(M)$ is the Lie algebra of vector fields $\mathfrak{X}(M)$. For this we need the concept of tangent vectors and tangent maps of diffeological spaces.

On smooth manifolds there are two basic definitions of tangent vectors. In the first definition a tangent vector at $m \in M$ is an equivalence class of smooth paths $t \mapsto \gamma_t \in M$ that are tangent at m, i.e. they have the same value m and the same first derivative at $t = 0$ in a chart. (One then has to check that the definition is independent of the chart.) We call this the geometric definition of tangent vectors. In the second definition a tangent vector at m is a derivation at m of the algebra of smooth functions on M, i.e. a linear map $v_m : C^\infty(M) \to \mathbb{R}$, such that $v_m(fg) = (v_m f)\, g(m) + f(m)\, (v_m g)$. We call this the algebraic definition of tangent vectors. There is an obvious map from paths to derivations that sends a path γ to the directional derivative $f \mapsto \frac{d}{dt} f(\gamma_t)|_{t=0}$. Getting back from derivations to paths is a bit more tricky, involving Hadamard's lemma in a chart. This shows that the two definitions are equivalent.

For diffeological spaces the situation is more complicated. There is a variety of different notions of tangent vectors and tangent spaces [Hec95, KM97, Vin08, Sta11, IZ13, CW16]. The definitions of tangent vectors of a diffeological space X fall essentially into two groups: the geometric definitions in terms of equivalence classes of smooth paths in X and the algebraic definitions in terms of derivations of the algebra $C^\infty(X) := \mathrm{Hom}_{\mathcal{D}\mathrm{iffl g}}(X, \mathbb{R})$ of diffeological functions on X. For a comparison of the various approaches see [CW16]. Which of those definitions is the "right" one will depend on the application.

In field theory we want both, that the tangent vectors of the space of fields $\mathcal{F}$ are represented by diffeological paths and that these paths induce the same derivation on the algebra of diffeological function on $\mathcal{F}$. Therefore, we choose a definition, which combines the geometric and algebraic approaches. ***

The price we have to pay for the great flexibility of diffeological spaces is that there cannot be particularly strong geometric statements that hold for all diffeological spaces, since they would have to be valid for all topological spaces. But if we stay close enough to the diffeologies that come from actual smooth maps of manifolds, diffeological spaces are a useful concept in field theory (see e.g. [BFW13]).

2.1 Diffeology

2.1.1 The category of diffeological spaces

Definition 2.1.1 (e.g. Def. 1.5 in [IZ13]). Let X be a set. A **diffeology** on X is a map D that assigns to every open subset $U \subset \mathbb{R}^n$ for every $n \geq 0$ a set $D(U) \subset \mathrm{Hom}_{\mathcal{S}\mathrm{et}}(U, X)$ of maps called **plots**, such that the following conditions are satisfied:

(D1) Every constant map $p : U \to X$ is a plot.

(D2) Let $p : U \to X$ be a map on an open subset $U \subset \mathbb{R}^n$ and $\{U_i\}_{i \in I}$ an open cover of U. If $p|_{U_i} : U_i \to U$ is a plot for every $i \in I$, then p is a plot.

(D3) If $p : U \to X$ is a plot and $f : V \to U$ a smooth map from an open subset $V \subset \mathbb{R}^m$, then $p \circ f$ is a plot.

A set with a diffeology is called a **diffeological space**. A **morphism of diffeological spaces** $f : X \to Y$ is a map of sets such that for every plot $p : U \to X$ the map $f \circ p : U \to Y$ is a plot. The category of diffeological spaces will be denoted by $\mathcal{D}\mathrm{iffl g}$.

Terminology 2.1.2. In the terminology of diffeological spaces, the open subsets of $\mathbb{R}^n$ for all $n \geq 0$ are called **parameter spaces**. Plots are also called **smooth parametrizations** or **smooth families**. A plot $\mathbb{R} \to X$ is called a **smooth path**. A morphism of diffeological spaces is also called a **diffeological map** or a **smooth map** when it is clear from the context that "smooth" refers to the diffeology.

Definition 2.1.3. Let $\mathcal{C}$ be a concrete category, i.e. a category with a faithful functor $|\_|:\mathcal{C}\to\mathcal{S}\mathrm{et}$. Let C be an object of $\mathcal{C}$. A diffeology on the set $|C|$ is called a **diffeology on** C.

Example 2.1.4. Here are a few examples for diffeologies. Let C be an object of a concrete category $\mathcal{C}$.

(a) The **discrete diffeology** or **fine diffeology** on C is the diffeology for which the plots to $|C|$ are the locally constant maps.

(b) The **coarse diffeology** on C is given by $D(U)=\mathrm{Hom}_{\mathcal{S}\mathrm{et}}\big(U,|C|\big)$, i.e. all maps are plots.

(c) Every topological space C is equipped with the **continuous diffeology** given by $D(U)=\mathrm{Hom}_{\mathcal{T}\mathrm{op}}(U,C)$, i.e. maps to $|C|$ are plots if they are continuous.

(d) Every smooth finite-dimensional manifold C is equipped with the **natural diffeology** given by $D(U)=\mathrm{Hom}_{\mathcal{M}\mathrm{fld}}(U,C)$, i.e. maps to $|C|$ are plots if they are infinitely often differentiable.

Def. 2.1.1 is a good working definition of diffeological spaces, that can be easily applied to concrete situations. For general considerations, however, it is useful to rephrase the definition in the language of sheaves: Let $\mathcal{E}\mathrm{ucl}$ denote the category which has as objects all open submanifolds of euclidean spaces $\mathbb{R}^n$, $n\geq 0$ and as morphisms smooth maps between them. $\mathbb{R}^0=*$ is the terminal object in $\mathcal{E}\mathrm{ucl}$. The functor of points,

$$\begin{aligned}|\_|:\mathcal{E}\mathrm{ucl}&\longrightarrow\mathcal{S}\mathrm{et}\\ U&\longmapsto\mathrm{Hom}_{\mathcal{E}\mathrm{ucl}}(*,U)\,,\end{aligned}$$

is faithful, so it equips $\mathcal{E}\mathrm{ucl}$ with the structure of a concrete category. $\mathcal{E}\mathrm{ucl}$ together with open covers is a site, called the **site of euclidean spaces**. Let X be a set. Then the functor

$$\begin{aligned}\mathcal{E}\mathrm{ucl}^{\mathrm{op}}&\longrightarrow\mathcal{S}\mathrm{et}\\ U&\longmapsto\mathrm{Hom}(|U|,X)\,,\end{aligned}\tag{2.1}$$

is a sheaf on $\mathcal{E}\mathrm{ucl}$.

Definition 2.1.5. A subsheaf $D(\_)\subset\mathrm{Hom}(|\_|,X)$ that satisfies

$$D(*)=\mathrm{Hom}(|*|,X)$$

is called a **concrete** sheaf on $\mathcal{E}\mathrm{ucl}$.

Notation 2.1.6. It is customary to omit the forgetful functor $|\_|$ of a concrete category whenever it is clear what the underlying set of an object is and that morphisms are maps of sets. For example, in Def. 2.1.1 we have considered maps $p:U\to X$ rather than maps $p:|U|\to X$. In the same vein, we will write

$$\mathrm{Hom}(|U|,X)\equiv\mathrm{Hom}_{\mathcal{S}\mathrm{et}}(U,X)$$

for $U\in\mathcal{E}\mathrm{ucl}$ and $X\in\mathcal{S}\mathrm{et}$. Moreover, since $|*|=|\mathbb{R}^0|=\{0\}$ is a singleton, i.e. a terminal object in $\mathcal{S}\mathrm{et}$, we write $|*|\equiv *$.

Proposition 2.1.7. *A diffeological space is a concrete sheaf on the site of euclidean spaces. A morphism of diffeological spaces is a morphism of sheaves.*

Proof. Axiom (D3) is equivalent to D being a subpresheaf of (2.1). (D2) is the sheaf axiom for D.

Let us assume that D is a concrete sheaf on $\mathcal{E}\mathrm{ucl}$. Every constant map $p : U \to X$ factors as $U \to * \to X$. Since $D(*) = \mathrm{Hom}_{\mathcal{S}\mathrm{et}}(*, X)$ and since D is a presheaf it follows that $p \in D(U)$, so that (D1) is satisfied. Conversely, assume (D1). Since every map $* \to X$ is constant it follows that $D(*) = \mathrm{Hom}_{\mathcal{S}\mathrm{et}}(*, X)$. We conclude that the condition $D(*) = \mathrm{Hom}_{\mathcal{S}\mathrm{et}}(*, X)$ is equivalent to (D1).

Let (X, D) and (Y, D') be diffeological spaces. Let $\tau : D \to D'$ be a morphism of sheaves, i.e. a natural transformation of the functors. Let

$$f : X \cong D(*) \xrightarrow{\tau_*} D'(*) \cong Y\,,$$

where τ_* is the natural transformation evaluated at the terminal object $* \in \mathcal{E}\mathrm{ucl}$. Let $u \in U$, which can be viewed as a map $u : * \to U$. Due to the naturality of τ we have the following commutative diagram:

$$\begin{tikzcd}
D(U) \arrow[r, "\tau_U"] \arrow[d, "\mathrm{ev}_u"'] & D'(U) \arrow[d, "\mathrm{ev}_u"] \\
D(*) \arrow[r, "f"] & D'(*)
\end{tikzcd}$$

This means that for every plot $p : U \to X$ we have $\big(\tau_U(p)\big)(u) = f\big(p(u)\big)$. Since this holds for all $u \in U$, if follows that $\tau_U(p) = f \circ p \in D'(U)$, so that f is a smooth map. Conversely, a smooth map $f : X \to Y$ defines a natural transformation by $\tau_U : D(U) \to D'(U)$, $p \mapsto f \circ p$. We conclude that we can identify a morphism of diffeological spaces $f : X \to Y$ with the morphism of concrete sheaves given by the composition of plots with f. □

Proposition 2.1.8. *Mapping every smooth manifold M to the set M with the natural diffeology defines a fully faithful injective functor $I : \mathcal{M}\mathrm{fld} \to \mathcal{D}\mathrm{iffl g}$.*

Proof. The natural diffeology of a smooth manifold M is given by the restriction of the representable presheaf $N \to \mathrm{Hom}(N, M)$ from $\mathcal{M}\mathrm{fld}$ to $\mathcal{E}\mathrm{ucl}$. This induces an injective map $I : \mathcal{M}\mathrm{fld} \to \mathcal{D}\mathrm{iffl g}$. Since $\mathcal{E}\mathrm{ucl}$ is dense in $\mathcal{M}\mathrm{fld}$, i.e. every manifold is a colimit of a diagram in $\mathcal{E}\mathrm{ucl}$ (see Sec. 2.1.4), it follows from the Yoneda lemma that this embedding is fully faithful. □

Notation 2.1.9. When it is clear from the context that we are working in the category of diffeological spaces, we will identify $\mathcal{M}\mathrm{fld}$ with its image under the embedding $I : \mathcal{M}\mathrm{fld} \to \mathcal{D}\mathrm{iffl g}$ and write $I(M) \equiv M$ for $M \in \mathcal{M}\mathrm{fld}$.

Corollary 2.1.10. *The plots of a diffeological space X are given by*

$$D(U) = \mathrm{Hom}_{\mathcal{D}\mathrm{iffl g}}(U, X)$$

for all $U \subset \mathbb{R}^n$, $n \geq 0$.

Terminology 2.1.11. Since manifolds are a full subcategory of diffeological spaces, the usage of "smooth" for diffeological spaces is consistent with the meaning "infinitely often differentiable" for manifolds.

Proposition 2.1.12. *The category of diffeological spaces has all small limits, all small colimits, and all exponential objects.*

Proof. The proof is a straightforward exercise in basic category theory: Limits of the sheaves in $\mathcal{D}\mathrm{iffl}\mathrm{g}$ can be taken point-wise. Colimits are taken in presheaves, i.e. point-wise, and then sheafified. Exponential objects are given by the universal property. For a fully elaborated proof see [BH11]. □

Terminology 2.1.13. A category such as $\mathcal{D}\mathrm{iffl}\mathrm{g}$ that contains $\mathcal{M}\mathrm{fld}$ as full subcategory and satisfies the properties of Prop. 2.1.12 is often called a **convenient category** or a **convenient setting** for differential geometry [BH11, KM97, Sta11].

We will denote the exponential objects in $\mathcal{D}\mathrm{iffl}\mathrm{g}$ by

$$\underline{\mathrm{Hom}}(X, Y) = Y^X$$

and call them the **diffeological mapping spaces**. It follows from the universal property of exponential objects and Cor. 2.1.10 that the mapping space diffeology is given by

$$\begin{aligned} D(U) &= \mathrm{Hom}_{\mathcal{D}\mathrm{iffl}\mathrm{g}}\big(U, \underline{\mathrm{Hom}}(X, Y))\big) \\ &:= \mathrm{Hom}_{\mathcal{D}\mathrm{iffl}\mathrm{g}}(U \times X, Y)\,, \end{aligned}$$

which is also called the **functional diffeology**.

Notation 2.1.14. Let X be a diffeological space. Then $C^\infty(X) \equiv \underline{\mathrm{Hom}}(X, \mathbb{R})$ denotes the mapping space of real-valued functions on X.

The functor of points $\mathcal{D}\mathrm{iffl}\mathrm{g} \to \mathcal{S}\mathrm{et}$, $X \mapsto \mathrm{Hom}_{\mathcal{D}\mathrm{iffl}\mathrm{g}}(*, X)$ is faithful, so it equips $\mathcal{D}\mathrm{iffl}\mathrm{g}$ with the structure of a concrete category. This functor has both a left adjoint given by the fine diffeology on X and a right adjoint given by the coarse diffeology. This has the following consequence.

Proposition 2.1.15. *The functor of points* $\mathcal{D}\mathrm{iffl}\mathrm{g} \to \mathcal{S}\mathrm{et}$ *preserves all limits and colimits.*

Proof. Every functor with a left adjoint preserves all limits and every functor with a right adjoint preserves all colimits [ML98, Sec. V.5]. □

2.1.2 Inductions and subductions

Let X be a set with two topologies T and T'. Then T is finer than T', i.e. $T \supset T'$, if there are fewer T-continuous maps than T'-continuous maps to X. This suggests the following definition.

Definition 2.1.16. Let D and D' be two diffeologies on X. If $D(U) \subset D'(U)$ for all U, then D is called **finer** than D' and D' **coarser** than D.

With the notion of finer and coarser diffeologies we can give constructions of diffeologies that are analogous to topological spaces. Let $f : X \to Y$ be a map. When Y is a diffeological space, the **pullback diffeology** of f is the coarsest diffeology on X such that for every plot $p : U \to X$ the map $f \circ p$ is a plot. It is given by

$$(f^* D_Y)(U) = (f_*)^{-1}\bigl(D_Y(U)\bigr)\,,$$

for all U, where D_Y is the diffeology on X [IZ13, Sec. 1.26]. This means that a map $p : U \to X$ is a plot if and only if $f \circ p : U \to Y$ is a plot. When f is injective, the pullback diffeology is also called the **subspace diffeology**.

When X is a diffeological space, the **pushforward diffeology** is the finest diffeology on Y such that for every plot $p : U \to X$ the map $f \circ p$ is a plot. A map $p : U \to Y$, $U \subset \mathbb{R}^n$ is a plot of the pushforward diffeology, if every $u \in U$ has a neighborhood $V \subset U$ such that $p|_V = f \circ \tilde{p}$ for some plot $\tilde{p} : V \to X$ [IZ13, Sec. 1.43]. When f is surjective, the pushforward diffeology is also called the **quotient diffeology**.

Definition 2.1.17 (Secs. 1.29 and 1.46 in [IZ13])**.** Let $f : X \to Y$ be a map of diffeological spaces. If f is injective and X has the pullback diffeology, it is called an **induction**. If f is surjective and Y has the pushforward diffeology, it is called a **subduction**.

Proposition 2.1.18. *Let $\pi : X \to Y$ be a morphism of diffeological spaces and $\sigma : Y \to X$ a section. Then π is a subduction and σ an induction.*

Proof. Let $p : U \to Y$ be a plot. Then $\tilde{p} := \sigma \circ p : U \to X$ is a plot, because σ is a morphism of diffeological spaces. Since σ is a section of π, we have $\pi \circ \tilde{p} = \pi \circ \sigma \circ p = p$. We conclude that Y is equipped with the pushforward diffeology of π. Since π has a section it is surjective, so that it is a subduction.

Let $p : U \to Y$ be some map on $U \subset \mathbb{R}^n$. Since $\pi \circ \sigma = \mathrm{id}_Y$, p is a plot iff $(\pi \circ \sigma) \circ p = \pi \circ (\sigma \circ p)$ is a plot. Since the diffeology on Y is the pushforward diffeology of π, this is the case iff $\sigma \circ p$ is a plot. We conclude that the diffeology on Y is the pullback diffeology of σ. Since σ is a section it is injective, so that it is an induction. □

Corollary 2.1.19. *Injective subductions and surjective inductions are isomorphisms in $\mathcal{D}$iffl$\mathrm{g}$.*

Proposition 2.1.20. *Let*

$$\begin{tikzcd}
X \times_Y Z \arrow[r, "\mathrm{pr}_Z"] \arrow[d, "\mathrm{pr}_X"'] & Z \arrow[d, "g"] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}$$

be a pullback diagram of diffeological spaces.

(i) If the diffeology on Z is the pullback diffeology of g, then the diffeology on the pullback $X \times_Y Z$ is the pullback diffeology of pr_X.

(ii) If the diffeology on Y is the pushforward diffeology of g, then the diffeology on X is the pushforward diffeology of the projection pr_X.

Proof. Consider the following diagram:

$$\begin{tikzcd}
U \arrow[dr, "r"] \arrow[drr, bend left, "q"] \arrow[ddr, bend right, "p"'] & & \\
& X \times_Y Z \arrow[r, "\mathrm{pr}_Z"'] \arrow[d, "\mathrm{pr}_X"] & Z \arrow[d, "g"] \\
& X \arrow[r, "f"'] & Y
\end{tikzcd} \tag{2.2}$$

(i) Let $r : U \to X \times_Y Z$ be a map such that $p := \mathrm{pr}_X \circ r : U \to X$ is a plot. Since f is a morphism of diffeological spaces, $f \circ p$ is smooth. Since the diagram is commutative, $f \circ p = g \circ q$. Assume that the diffeology on Z is the pullback diffeology of g. This implies that $q : U \to Z$ is a plot. Since both p and q are plots, it follows from the universal property of the pullback that the unique induced map r is a plot, as well. We conclude that the diffeology on $X \times_Y Z$ is the pullback diffeology of pr_X.

(ii) Let $\tilde{p} : V \to X$ be a plot. Since f is a morphism of diffeological spaces, $f \circ \tilde{p} : V \to Y$ is a plot. Assume that Y is equipped with the pushforward diffeology of g. Then every point $v \in V$ has a neighborhood $U \subset V$, such that there is a $q : U \to Z$ satisfying $g \circ q = f \circ p$, where p is the restriction of $\tilde{p}$ to U. It follows from the universal property of the pullback that there is a plot $r : U \to X \times_Y Z$ such that $\mathrm{pr}_X \circ r = p$. We conclude that the diffeology on X is the pushforward diffeology of pr_X. □

Corollary 2.1.21. *If g in diagram (2.2) is an induction, then pr_X is an induction. If g is a subduction, then pr_X is a subduction.*

Proof. Injections are stable under pullback. The first statement then follows from Prop. 2.1.20 (i). Similarly, surjections are stable under pullback, so that the second statement follows from Prop. 2.1.20 (ii). □

Proposition 2.1.22. *If a smooth map of diffeological spaces $f : X \to Z$ factors as $f = h \circ g$ through a subduction $g : X \to Y$, then $h : Y \to Z$ is smooth.*

Proof. Let $p : U \to Y$ be a plot. We have to show that $q := h \circ p$ is a plot. By assumption, g is a subduction. This means that every $u \in U$ has a neighborhood V_u such that $p|_{V_u} = g \circ \tilde{p}$ for some plot $\tilde{p} : V_u \to X$. Therefore, $q|_{V_u} = h \circ p|_{V_u} = h \circ g \circ \tilde{p} = f \circ \tilde{p}$. Since f is smooth, $q|_{V_u}$ is a plot. Covering U with such neighborhoods, we obtain an open cover $\{V_u\}_{u \in U}$ such that the restriction of q to every open V_u is a plot. It follows from the sheaf property of the diffeology that q is a plot. □

Terminology 2.1.23. A diffeological space F together with a subduction $F \to X$ will be called a **bundle** of diffeological spaces over X.

Let $F \to X$ be a bundle of diffeological spaces. Then the diffeological space

$$F_x := \{x\} \times_X F$$

is the fibre over $x \in X$. Since $\{x\} \hookrightarrow X$ is trivially an induction and since by Cor. 2.1.21 inductions are stable under pullback, the inclusion $F_x \hookrightarrow F$ is an induction. It follows, that the diffeology of F_x is the subspace diffeology.

Definition 2.1.24. Let $F \to X$ be a bundle of diffeological spaces. Then

$$\Gamma(X,F) := \{\mathrm{id}_X\} \times_{\underline{\mathrm{Hom}}(X,X)} \underline{\mathrm{Hom}}(X,F)$$

is the **diffeological space of sections** of F.

Remark 2.1.25. Since the inclusion $\{\mathrm{id}_X\} \hookrightarrow \underline{\mathrm{Hom}}(X,X)$ is trivially an induction, it follows from Cor. 2.1.21 that $\Gamma(X,F) \subset \underline{\mathrm{Hom}}(X,F)$ is equipped with the subspace topology of the mapping space topology.

2.1.3 Diffeological vector spaces

Definition 2.1.26 (Sec. 3.7 in [IZ13])**.** A diffeological vector space is a vector space X with a diffeology, such that addition and scalar multiplication are morphisms of diffeological spaces.

Let X be a vector space with a diffeology D on the underlying set. In general, (X,D) is not a diffeological vector space, but there is a finest diffeology $D' \supseteq D$ such that (X,D') is a diffeological vector space. This will be called the **free vector space diffeology generated by** D.

Remark 2.1.27. There is a functor from diffeological vector spaces to the category of vector spaces V with a diffeology on the underlying set D, which forgets the compatibility of the diffeological and vector space structure. Mapping (V,D) to the (V,D') is the left adjoint, which is why D' is called the *free* vector space diffeology. In [Vin08, Def. 2.2.1] D' is called the weak vector space diffeology.

Let us describe the free vector space diffeology D' explicitly. Let $p_i \in D(U)$ be plots and $\lambda_i \in C^\infty(U)$ be smooth functions for $1 \leq i \leq k$. Since scalar multiplication and addition are D'-smooth, the map $p: U \to X$ given by

$$p(u) = \lambda^1(u)\,p_1(u) + \ldots + \lambda^k(u)\,p_k(u) \tag{2.3}$$

is a plot in $D'(U)$. It is straightforward to verify that maps that are locally of this form define a diffeology on X, which is the free vector space diffeology D'.

Definition 2.1.28. Let X be a vector space. The free vector space diffeology generated by the fine diffeology on the underlying set of X (see Ex. 2.1.4 (a)) is called the **fine vector space diffeology**.

Remark 2.1.29. Since every diffeology contains the fine diffeology, the fine vector space diffeology is contained in every other vector space diffeology.

Proposition 2.1.30. *Let X be a vector space. The fine vector space diffeology on X is the finest diffeology on the underlying set X such that the restriction of every linear map $\mathbb{R}^n \to X$ to an open subset $U \subset \mathbb{R}^n$, $n \geq 0$ is a plot.*

Proof. Let D denote the diffeology generated by linear plots and D' the fine vector space diffeology. Every linear map $q: \mathbb{R}^k \to X$ is of the form

$$q(\alpha^1, \ldots, \alpha^k) = \alpha^1 x_1 + \ldots + \alpha^k x_k\,,$$

where $x_i \in X$ is the image of the canonical basis vector e_i of $\mathbb{R}^n$. This is of the form (2.3), which shows that $D(U) \subset D'(U)$.

Let $U \subset \mathbb{R}^n$ be an open subset and $\lambda : U \to \mathbb{R}^k$, $u \mapsto (\lambda^1(u), \ldots, \lambda^k(u))$ a smooth map. Since the composition of a smooth function with a plot is a plot, the map $p := q \circ \lambda$ is in $D(U)$, which is of the form

$$p(u) = \lambda^1(u)\, x_1 + \ldots + \lambda^k(u)\, x_k\,, \tag{2.4}$$

Looking at Eq. (2.3), where the plots $p_i(u) = x_i$ are constant maps, we see that every plot in $D'(U)$ is locally of the from (2.4). This shows that $D'(U) \subseteq D(U)$. We conclude that $D = D'$. □

Proposition 2.1.31. *The fine vector space diffeology on a finite dimensional vector space $\mathbb{R}^n$ is the natural diffeology of $\mathbb{R}^n$ viewed as manifold.*

Proof. A map $U \to \mathbb{R}^n$, $u \mapsto \big(\lambda^1(u), \ldots, \lambda^n(u)\big)$ for $U \subset \mathbb{R}^m$ is smooth if and only if every λ^i is smooth. □

Proposition 2.1.32. *Let X and Y be diffeological vector spaces. If X has the fine vector space diffeology, then every linear map $X \to Y$ is smooth.*

Proof. Let $f : X \to Y$ be linear. A plot in the fine vector space diffeology on X is locally of the form (2.4). Since f is linear, $(f \circ p)(u) = \lambda^1(u)\, f(x_1) + \ldots + \lambda^n(u)\, f(x_n)$, which is a plot in the fine vector space diffeology on Y and, therefore, a plot in any vector space diffeology on Y. □

Proposition 2.1.33 (Prop. 3.4 in [CW]). *Let X be a diffeological vector space. The diffeology on X is the fine vector space diffeology if and only if every linear map $X \to \mathbb{R}$ is smooth.*

Warning 2.1.34. The last statement shows that if a diffeological vector space X is not equipped with the fine vector space diffeology, then there are linear functions $X \to \mathbb{R}$ that are not smooth.

Lemma 2.1.35. *Let X be a diffeological space and Y a diffeological vector space. Then $\underline{\mathrm{Hom}}(X, Y)$ with point-wise addition and scalar multiplication is a diffeological vector space.*

Proof. Let $f, g : U \to \underline{\mathrm{Hom}}(X, Y)$ be plots, which are given by morphisms of diffeological spaces $f, g : U \times X \to Y$. Since the addition on Y is smooth, the map $f + g : U \times X \to Y$, $(f + g)(u, x) = f(u, x) + g(u, x)$ is a morphism of diffeological spaces. Similarly for every $\alpha \in \mathbb{R}$, the map $\alpha f : U \times X \to Y$, $(\alpha f)(u, x) = \alpha\, f(u, x)$ is smooth because the scalar multiplication on Y is smooth. □

Definition 2.1.36. Let X and Y be diffeological vector spaces. Then $\underline{\mathrm{Lin}}(X, Y)$ denotes the diffeological vector subspace of all linear maps in $\underline{\mathrm{Hom}}(X, Y)$.

$$X' := \underline{\mathrm{Lin}}(X, \mathbb{R})$$

is the **diffeological dual vector space** of X.

Notation 2.1.37. Many authors use the notation X^* for the diffeological dual of a diffeological vector space. We will follow the convention of topology and functional analysis, reserving X^* for the algebraic dual and using $X' \subset X^*$ for the smooth or continuous dual.

Proposition 2.1.38. *The smooth dual of a fine diffeological vector space is the algebraic dual vector space with the fine vector space diffeology.*

Proof. This follows from Prop. 2.1.33. □

2.1.4 Extensions of functors from manifolds to diffeological spaces

Definition 2.1.39. The **category of plots** of a diffeological space X has as objects plots $p : U \to X$ and as morphisms triangles

$$\begin{tikzcd} U \arrow[rr, "f"] \arrow[dr, "p"'] & & V \arrow[dl, "q"] \\ & X & \end{tikzcd}$$

where $f : U \to V$ is a smooth map.

Notation 2.1.40. The category of plots will be denoted by $\mathcal{E}\mathrm{ucl}{\downarrow}X$, where we identify $\mathcal{E}\mathrm{ucl} \equiv y(\mathcal{E}\mathrm{ucl})$.

Proposition 2.1.41. *Every diffeological space (X, D) is the colimit of its category of plots, i.e. the colimit of the functor $\mathcal{E}\mathrm{ucl}{\downarrow}X \to \mathcal{D}\mathrm{iffl g}$, $(U \to X) \mapsto y(U)$, which we write as*

$$X \cong \operatorname*{colim}_{U \to X} y(U)\,.$$

Proof. The proof follows from Thm. 1 in Sec. III.7 of [ML98] applied to the functor $D : \mathcal{E}\mathrm{ucl}^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$. □

The proposition shows that $y : \mathcal{E}\mathrm{ucl} \to \mathcal{D}\mathrm{iffl g}$, $U \mapsto \mathrm{Hom}_{\mathcal{M}\mathrm{fld}}(-, U)$ is **dense** in $\mathcal{D}\mathrm{iffl g}$, that is, every object in $\mathcal{D}\mathrm{iffl g}$ is the colimit of a diagram in $y(\mathcal{E}\mathrm{ucl})$. This leads to the following observation.

Proposition 2.1.42. *Let $\Phi : \mathcal{M}\mathrm{fld} \to \mathcal{C}$ be a functor to a cocomplete category. Then the left Kan extension of Φ along the embedding $I : \mathcal{M}\mathrm{fld} \to \mathcal{D}\mathrm{iffl g}$ exists and we have the commutative diagram of functors*

$$\begin{tikzcd} \mathcal{E}\mathrm{ucl} \arrow[r, hook, "J"] \arrow[dr, "y"'] & \mathcal{M}\mathrm{fld} \arrow[r, "\Phi"] \arrow[d, "I"'] & \mathcal{C} \\ & \mathcal{D}\mathrm{iffl g} \arrow[ur, "\mathrm{Lan}_I \Phi"'] & \end{tikzcd}$$

*Moreover, **** $\mathrm{Lan}_I \Phi = \mathrm{Lan}_y(\Phi \circ J)$.

Proof. *** □

Notation 2.1.43. Since I is injective and fully faithful, we can identify every manifold M with the diffeological space $M \equiv I(M)$. Prop. 2.1.42 then shows that $(\mathrm{Lan}_I \Phi)(M) = \Phi(M)$. Therefore, we will use the notation

$$\Phi(X) \equiv (\mathrm{Lan}_y \Phi)(X)$$

for all $X \in \mathcal{D}\mathrm{iffl g}$.

Example 2.1.44. Let $\Omega : \mathcal{M}\mathrm{fld} \to \mathrm{dg}\mathcal{V}\mathrm{ec}^{\mathrm{op}}$ be the functor that maps a manifold to its de Rham complex. Then $\Omega(X) := (\mathrm{Lan}_I \Omega)(X)$ is the **de Rham complex** of the diffeological space X.

The left Kan extension of Prop. 2.1.42 is given object-wise as the colimit over the category of plots,

$$\Phi(X) := \operatorname*{colim}_{U \to X} \Phi(U)\,,$$

where we use notation 2.1.43. This colimit can be computed as coequalizer

$$\coprod_{U \xrightarrow{f} V \xrightarrow{q} X} \Phi(U)_{f,q} \rightrightarrows \coprod_{U \xrightarrow{p} X} \Phi(U)_p \longrightarrow \Phi(X)\,,$$

where the arrows on the left map the object $\Phi(U)$ indexed by the morphism $U \xrightarrow{f} V \xrightarrow{q} X$ in $\mathcal{E}\mathrm{ucl}\!\downarrow\! X$ identically to the object $\Phi(U)$ indexed by the domain $q \circ f$ and codomain q, respectively.

If Φ takes values in $\mathcal{S}\mathrm{et}$, the coequalizer is obtained as a quotient

$$\Phi(X) \cong \coprod_{p : U \to X} \Phi(U)_p \;/ \sim\,, \tag{2.5}$$

where $\sim$ is the equivalence relation generated by the following relations: An element $x \in \Phi(U)_p$ is Φ-related to $y \in \Phi(V)_q$ if there is a smooth map $f : U \to V$ such that $q \circ f = p$ and $(\Phi(f))(x) = y$. Two elements in the coproduct are then related by $\sim$ if and only if they are connected by a zigzag of Φ-relations.

Remark 2.1.45. The construction (2.5) still works if Φ is a functor to a concrete category for which the forgetful functor preserves colimits. By Prop. 2.1.15, $\mathcal{D}\mathrm{iffl g}$ is such a category.

2.2 Tangent bundle

2.2.1 Kan extension of the tangent functor

When we want to define the tangent bundle of a diffeological space by the left Kan extension we have to observe that the tangent functor $T : \mathcal{M}\mathrm{fld} \to \mathcal{M}\mathrm{fld}$ does not take values in a cocomplete category. To solve this issue we embed the target $\mathcal{M}\mathrm{fld}$ into $\mathcal{D}\mathrm{iffl g}$.

Definition 2.2.1. Let $T : \mathcal{M}\mathrm{fld} \to \mathcal{M}\mathrm{fld}$ be the tangent functor of manifolds and let $I : \mathcal{M}\mathrm{fld} \to \mathcal{D}\mathrm{iffl g}$ be the natural embedding of Prop. 2.1.8. The left Kan extension

$$T := \mathrm{Lan}_I(I \circ T) : \mathcal{D}\mathrm{iffl g} \longrightarrow \mathcal{D}\mathrm{iffl g}$$

will be called the **tangent functor** of diffeological spaces.

The base point projections $\mathrm{pr}_M : TM \to M$ for all $M \in \mathcal{M}\mathrm{fld}$ define a natural transformation $\mathrm{pr}_{\mathcal{M}\mathrm{fld}} : T \Rightarrow \mathrm{id}_{\mathcal{M}\mathrm{fld}}$. By composition with id_I we get a natural transformation

$$\mathrm{id}_I \circ \mathrm{pr}_{\mathcal{M}\mathrm{fld}} : I \circ T \Longrightarrow I \circ \mathrm{id}_{\mathcal{M}\mathrm{fld}} .$$

By the functoriality of the left Kan extension we obtain a natural transformation

$$\mathrm{pr} := \mathrm{Lan}_I(\mathrm{id}_I \circ \mathrm{pr}_{\mathcal{M}\mathrm{fld}}) : \mathrm{Lan}_I(I \circ T) \Longrightarrow \mathrm{Lan}_I I . \tag{2.6}$$

Using $T = \mathrm{Lan}_I(I \circ T)$ and the natural isomorphism $\mathrm{Lan}_I I \cong \mathrm{id}_{\mathcal{D}\mathrm{iffl g}}$, we obtain a natural transformation $\mathrm{pr} : T \Rightarrow \mathrm{id}_{\mathcal{D}\mathrm{iffl g}}$. The naturality of pr means that we have a commutative diagram

$$\begin{tikzcd} TX \arrow[r, "Tf"] \arrow[d, "\mathrm{pr}_X"'] & TY \arrow[d, "\mathrm{pr}_Y"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

for every morphism of diffeological spaces f.

Definition 2.2.2. Let X be a diffeological space. The diffeological space TX given by Def. 2.2.1 together with the morphism $\mathrm{pr}_X : TX \to X$ given by the natural transformation (2.6) is called the **tangent bundle** of X. The morphism of bundles $Tf : TX \to TY$ is called the **tangent map** or **tangent morphism** of $f \in \mathrm{Hom}_{\mathcal{D}\mathrm{iffl g}}(X, Y)$.

Definition 2.2.3. A **vector field** on a diffeological space X is a section of the tangent bundle $TX \to X$.

Proposition 2.2.4. *Let X and Y be diffeological spaces. Then $T(X \times Y) \cong TX \times TY$.*

Proof. The categories of plots $\mathcal{E}\mathrm{ucl}{\downarrow}X$ and $\mathcal{E}\mathrm{ucl}{\downarrow}Y$ are sifted, so that colimits over them commute with finite products [GU71, ARV10]. Since $T : \mathcal{M}\mathrm{fld} \to \mathcal{M}\mathrm{fld}$ and $I : \mathcal{M}\mathrm{fld} \to \mathcal{D}\mathrm{iffl g}$ preserve products, so does their composition $I \circ T$. □

For a more explicit description of TX we can use that the left Kan extension is given point-wise by the colimit

$$TX := \operatorname*{colim}_{p:U \to X} TU ,$$

where p ranges over all plots to X and where the manifold TU is viewed as diffeological space. By Eq. (2.5) and remark 2.1.45, the colimit is given by the diffeological quotient space

$$TX = \coprod_{p:U \to X} (TU)_p \,/ \sim , \tag{2.7}$$

where the index p distinguishes the different summands of TU. The equivalence relation is given as follows: A vector $v_u \in (TU)_p$ is T-related to a vector $w_v \in (TV)_q$ if there is a smooth map $f : U \to V$ such that $q \circ f = p$ and $Tf\, v_u = w_v$. Two vectors are related by $\sim$ iff they are connected by a zigzag of T-relations.

Notation 2.2.5. For every $v_u \in (TU)_p$ in the disjoint union on the right hand side of (2.7), we denote by $p_* v_u$ the equivalence class it represents.

Let $f : X \to Y$ be a morphism of diffeological spaces. By the naturality of the quotient (2.7), the tangent map Tf maps a tangent vector in TX represented by the $v_u \in (TU)_p$ to the tangent vector in TY that is represented by $v_u \in (TU)_{f \circ p}$, that is

$$Tf\big(p_*(v_u)\big) = (f \circ p)_*(v_u)\,. \tag{2.8}$$

for all plots $p : U \to X$ and all $v_u \in TU$.

2.2.2 Representing tangent vectors by smooth paths

Proposition 2.2.6. *Let $\partial_t = \frac{\partial}{\partial t}$ denote the standard coordinate vector field on $\mathbb{R}$ and $\partial_{t=0} \in T_0\mathbb{R}$ its value at 0. The map*

$$\begin{aligned} \rho_X : \underline{\mathrm{Hom}}(\mathbb{R}, X) &\longrightarrow TX \\ \gamma &\longmapsto \gamma_*(\partial_{t=0})\,, \end{aligned} \tag{2.9}$$

is a subduction which is natural in X.

Proof. TX is equipped with the quotient diffeology of the coproduct diffeology, that is, with the pushforward diffeology of the quotient map $\coprod_p (TU)_p \to TX$. This means that a smooth family $v : W \to TX$, $W \subset \mathbb{R}^n$ of tangent vectors is smooth if it can be lifted locally to a smooth family in one of the summands $(TU)_p$ of the coproduct, i.e. for every point $w \in W$ there is a neighborhood $N \subset W$ of w and a smooth family $\xi : N \to (TU)_p$, $n \mapsto \xi(n)$ such that $p_*\big(\xi(n)\big) = v(n)$. Every family ξ of tangent vectors on the smooth manifold U is represented by a smooth family of paths $\nu : N \to \underline{\mathrm{Hom}}(\mathbb{R}, U)$, that satisfies $\xi(n) = \dot{\nu}(n)_0$. Consider the smooth family of paths $\gamma : N \to \underline{\mathrm{Hom}}(\mathbb{R}, X)$, $\gamma(n) := p \circ \nu(n)$. For every $n \in N$ we have

$$\big(T\nu(n)\big)(\partial_{t=0}) = \xi(n)\,,$$

which shows that $\partial_{t=0} \in (T\mathbb{R})_\gamma(n)$ and $\xi(n) \in (TU)_p$ are T-related. It follows that

$$\begin{aligned} v(n) = p_*\big(\xi(n)\big) &= \gamma(n)_*(\partial_{t=0}) \\ &= \rho_X\big(\gamma(n)\big)\,, \end{aligned}$$

for all $n \in N$. This shows that ρ_X is surjective and that every plot in TX lifts locally to a plot in $\underline{\mathrm{Hom}}(\mathbb{R}, X)$. In other words, ρ_X is a subduction.

Let $f : X \to Y$ be a morphism of diffeological spaces. By equation (2.8) we have for every smooth path $\gamma : \mathbb{R} \to X$ the relation

$$\begin{aligned} (Tf \circ \rho_X)(\gamma) = Tf\big(\gamma_*(\partial_{t=0})\big) = (f \circ \gamma)_*(\partial_{t=0}) &= \rho_Y\big(f \circ \gamma\big) \\ = (\rho_Y \circ f_*)\big(\gamma\big)\,. \end{aligned}$$

Since every tangent vector is represented by a path, it follows that $Tf \circ \rho_X = \rho_Y \circ f_*$. In other terms, $X \mapsto \rho_X$ is a natural transformation $\underline{\mathrm{Hom}}(\mathbb{R}, \_\,) \Rightarrow T$. □

Proposition 2.2.6 shows that every tangent vector in TX is represented by a smooth path in X and that plots of tangent vectors are represented by homotopies of paths. More precisely, a family $v : U \to TX$ of tangent vectors is a plot if every point in U has a neighborhood on which v is represented by a smooth family of paths in X. The naturality of the map means that there is a commutative diagram

$$\begin{tikzcd}
\underline{\mathrm{Hom}}(\mathbb{R}, X) \arrow[r, "f_*"] \arrow[d, "\rho_X"'] & \underline{\mathrm{Hom}}(\mathbb{R}, Y) \arrow[d, "\rho_Y"] \\
TX \arrow[r, "Tf"] & TY
\end{tikzcd}$$

for every morphism of diffeological spaces f. This shows that the tangent map is induced by the pushforward of smooth paths, as in the case of manifolds. The following proposition, too, is completely analogous to the situation for smooth manifolds.

Proposition 2.2.7. *Let X be a diffeological space. The tangent vector on the diffeological space X that is represented by a smooth path $\gamma : \mathbb{R} \to X$ depends only on the germ of γ at 0.*

Proof. Let $i : (-\varepsilon, \varepsilon) \hookrightarrow \mathbb{R}$ be the embedding of a small interval containing 0. The differential of i at 0 is the identity, $Ti(\partial_{t=0}) = \partial_{t=0}$. It follows from the construction of the quotient (2.7) that $\gamma_*(\partial_{t=0}) = (\gamma \circ i)_*(\partial_{t=0})$. Since $\gamma \circ i$ is the restriction of γ to $(-\varepsilon, \varepsilon)$, it follows that restricting the path γ to an open neighborhood of 0 does not change the tangent vector it represents. We conclude that $\rho_X(\gamma)$ depends only on the germ of γ at 0. □

Proposition 2.2.8. *The projection $\mathrm{pr}_X : TX \to X$ is a subduction.*

Proof. As it is the case for exponential objects in any category, the evaluation map

$$\mathrm{ev}_0 : \underline{\mathrm{Hom}}(\mathbb{R}, X) \longrightarrow X$$
$$\gamma \longmapsto \gamma_0 \,,$$

is a morphism, i.e. it is smooth. We have the following commutative diagram of diffeological spaces:

$$\begin{tikzcd}
\underline{\mathrm{Hom}}(\mathbb{R}, X) \arrow[r, "\rho_X"] \arrow[dr, "\mathrm{ev}_0"'] & TX \arrow[d, "\mathrm{pr}_X"] \\
& X
\end{tikzcd}$$

The map $c : X \to \underline{\mathrm{Hom}}(\mathbb{R}, X)$ that maps x to the constant path $c(x)_t = x$ is a smooth section of ev_0. The composition of c with $\rho_X : \underline{\mathrm{Hom}}(\mathbb{R}, X) \to TX$ is smooth. We have $\mathrm{pr}_X \circ (\rho \circ c) = \mathrm{ev}_0 \circ c = \mathrm{id}_X$, that is, $\rho_X \circ c$ is a section of pr_X. It now follows from Prop. 2.1.18 that pr_X is a subduction. □

Definition 2.2.9. The tangent vector in T_xX represented by the constant path $\gamma_t = x$ is called the **zero vector** at x and denoted by 0_x. The map $X \to TX$, $x \mapsto 0_x$ is the **zero section** of TX.

Corollary 2.2.10. *The zero section of $TX \to X$ is an induction.*

Proof. We have seen in the proof of Prop. 2.2.8 that the zero section is a smooth section of pr_X. It follows from Prop. 2.1.18 that the zero section is an induction. □

Corollary 2.2.11. *The evaluation map $\mathrm{ev}_0 : \underline{\mathrm{Hom}}(\mathbb{R}, X) \to X$ is a subduction.*

2.2.3 The derivation of a tangent vector

Definition 2.2.12. Let $C^\infty(X)$ be the algebra of smooth functions on the diffeological space X. A **derivation** of $C^\infty(X)$ **at** $x \in X$ is a linear function $\partial : C^\infty(X) \to \mathbb{R}$ such that

$$\partial(fg) = (\partial f)\, g(x) + f(x)\, (\partial g)$$

for all $f, g \in C^\infty(X)$. The vector space of derivations at x will be denoted by $\mathrm{Der}_x\big(C^\infty(X)\big)$.

Remark 2.2.13. A linear map on a diffeological vector space is not necessarily smooth, so that a priori there may be derivations that are not smooth. ***

By definition, a function $f : X \to \mathbb{R}$ is smooth if for every plot $p : U \to X$ the map $f \circ p$ is a map of smooth manifolds. In particular, if $\gamma : \mathbb{R} \to X$ is a smooth path, then $f \circ \gamma : \mathbb{R} \to \mathbb{R}$ is a smooth function, so that we can define the directional derivative,

$$\partial_\gamma f := \frac{d}{dt} f(\gamma_t)\Big|_{t=0} .$$

It follows from the Leibniz rule that ∂_γ is a derivation of $C^\infty(X)$ at γ_0. Let

$$\underline{\mathrm{Hom}}(\mathbb{R}, X)_x := \{x\} \times_X \underline{\mathrm{Hom}}(\mathbb{R}, X)$$

be the fibre over x of the subduction $\mathrm{ev}_0 : \underline{\mathrm{Hom}}(\mathbb{R}, X) \to X$, i.e. the diffeological space of paths $\gamma : \mathbb{R} \to X$ that start at $\gamma_0 = x$. The directional derivative defines a map of sets

$$\begin{aligned} \underline{\mathrm{Hom}}(\mathbb{R}, X)_x &\longrightarrow \mathrm{Der}_x\big(C^\infty(X)\big) \\ \gamma &\longmapsto \partial_\gamma \,, \end{aligned}$$

The following proposition states that this maps factors through $T_x X$.

Proposition 2.2.14. *There is a unique map* $\tilde{\partial} : T_x X \to \mathrm{Der}_x\big(C^\infty(X)\big)$*, such that*

$$\begin{tikzcd} \underline{\mathrm{Hom}}(\mathbb{R}, X)_x \arrow[d, "\rho_X"'] \arrow[dr, "\partial"] & \\ T_x X \arrow[r, "\tilde{\partial}"] & \mathrm{Der}_x\big(C^\infty(X)\big) \end{tikzcd}$$

commutes.

Proof. Let $p : U \to X$ be a plot and $x = p(u)$ for some $u \in U$. Let $\xi_u \in T_u U$ and $v_x = p_*(\xi_u)$ be the tangent vector on X that is represented by ξ_u. Then

$$\tilde{\partial}_{v_x} f := \langle d(f \circ p), \xi_u \rangle$$

for all $f \in C^\infty(X)$ is a derivation at x. We have to show that $\tilde{\partial}_{v_x}$ is well-defined, i.e. that it does not depend on the representative ξ_u. Let $q : W \to X$ be a plot and $\eta_w \in T_w W$ a tangent vector that is T-related to ξ_u. This means that there is a smooth map $h : U \to W$ such that $p = q \circ h$ and $(T_u h)\xi_u = \eta_w$. Then

$$\begin{aligned} \langle d(f \circ q), \eta_w \rangle &= \langle d(f \circ q), (T_u h)\xi_u \rangle \\ &= \langle d(f \circ q) \circ T_u h, \xi_u \rangle \\ &= \langle d(f \circ q \circ h), \xi_u \rangle \\ &= \langle d(f \circ p), \xi_u \rangle \,. \end{aligned}$$

This shows that if two tangent vectors on the domains of plots are T-related, then they define the same derivation. By transitivity of the equivalence relation of the quotient (2.7) it follows that $\tilde{\partial}_{v_x}$ is well-defined, so that we obtain a map $\tilde{\partial} : T_xX \to \mathrm{Der}_x\big(C^\infty(X)\big)$.

If $v_x = \gamma_*(\partial_{t=0}) = \rho_X(\gamma)$ for a path $\gamma : \mathbb{R} \to X$, then

$$\tilde{\partial}_{\rho_X(\gamma)} f = \langle d(f \circ \gamma), \partial_{t=0} \rangle = \partial_\gamma f \,,$$

which shows that $\tilde{\partial} \circ \rho_X = \partial$, that is, the diagram of the proposition commutes. Moreover, since ρ_X is surjective, $\tilde{\partial}$ is unique. □

Remark 2.2.15. The map $\tilde{\partial}$ is generally neither surjective nor injective [CW16], so it cannot be used to identify T_xX with a subset of $\mathrm{Der}_x(C^\infty(X))$.

***

In general, T_xX is not a vector space. While we cannot add paths, we can rescale the time parameter of a path,

$$\begin{aligned} L^* : \mathbb{R} \times \underline{\mathrm{Hom}}(\mathbb{R}, X) &\longrightarrow \underline{\mathrm{Hom}}(\mathbb{R}, X) \\ (\alpha, \gamma) &\longmapsto (L_\alpha^* \gamma : t \mapsto \gamma_{\alpha t}) \,, \end{aligned}$$

where $L_\alpha : \mathbb{R} \to \mathbb{R}$, $L_\alpha t = \alpha t$ is the multiplication with α. This morphism of diffeological spaces descends to a morphism on the tangent bundle.

Proposition 2.2.16. *There is a unique morphism of diffeological spaces*

$$\begin{aligned} \mathbb{R} \times TX &\longrightarrow TX \\ (\alpha, v_x) &\longmapsto \alpha \cdot v_x \,, \end{aligned}$$

such that the diagram

$$\begin{tikzcd}
\mathbb{R} \times \underline{\mathrm{Hom}}(\mathbb{R}, X) \arrow[r, "L^*"] \arrow[d, "\mathrm{id}_{\mathbb{R}} \times \rho_X"'] & \underline{\mathrm{Hom}}(\mathbb{R}, X) \arrow[d, "\rho_X"] \\
\mathbb{R} \times TX \arrow[r] & TX
\end{tikzcd}$$

commutes. It satisfies

$$\alpha \cdot (\beta \cdot v_x) = (\alpha\beta) \cdot v_x \,, \quad 1 \cdot v_x = v_x \,, \quad 0 \cdot v_x = 0_x$$

for all $\alpha, \beta \in \mathbb{R}$, $x \in X$, and $v_x \in T_xX$.

Lemma 2.2.17. *Two smooth paths $\gamma, \tilde{\gamma} : \mathbb{R} \to X$ represent the same tangent vector on the diffeological space X if and only if there is a commutative diagram*

$$\begin{tikzcd}
\mathbb{R} \arrow[r, "f"] \arrow[dr, "\gamma"'] & \mathbb{R}^2 \arrow[d, "p"] & \mathbb{R} \arrow[l, "\tilde{f}"'] \arrow[dl, "\tilde{\gamma}"] \\
& X &
\end{tikzcd} \tag{2.10}$$

where p is a plot and $f, \tilde{f}$ smooth functions such that $Tf\,(\partial_{t=0}) = T\tilde{f}\,(\partial_{t=0})$.

Proof. This follows from the proof of Prop. 3.4 in [CW16]. □

Proof of Prop. 2.2.16. Let $v_x \in T_x X$ be represented by a path $\gamma : \mathbb{R} \to X$, i.e. $v_x = \rho_X(\gamma) = \gamma_*(\partial_{t=0})$. For the diagram to commute, the scalar multiplication of v_x by $\alpha \in \mathbb{R}$ must be defined by

$$\alpha \cdot v_x = \rho_X(L_\alpha^* \gamma) = (\alpha \cdot \gamma)_*(\partial_{t=0}) \, .$$

We have to show that this map is well-defined.

Let $\tilde{\gamma} : \mathbb{R} \to X$ be another path that represents the same tangent vector as γ. By Lem. 2.2.17, there is a diagram like (2.10), which we can extend to a diagram

$$\begin{tikzcd}
\mathbb{R} \arrow[r, "\alpha f"] \arrow[d, "L_\alpha"'] & \mathbb{R}^2 \arrow[d, "\mathrm{id}"] & \mathbb{R} \arrow[l, "\alpha \tilde{f}"'] \arrow[d, "L_\alpha"] \\
\mathbb{R} \arrow[r, "f"] \arrow[dr, "\gamma"'] & \mathbb{R}^2 \arrow[d, "p"] & \mathbb{R} \arrow[l, "\tilde{f}"'] \arrow[dl, "\tilde{\gamma}"] \\
 & \mathcal{A} &
\end{tikzcd}$$

This shows that the paths $\gamma \circ L_\alpha$ and $\tilde{\gamma} \circ L_\alpha$ represent the same tangent vector.

Let v_x be represented by the path γ. Since $L_\alpha^*(L_\beta^*)\gamma = L_{\alpha\beta}\gamma$ it follows that $\alpha \cdot (\beta \cdot v_x) = (\alpha\beta) \cdot v_x$. Since $L_1^* \gamma = \gamma$ it follows that $1 \cdot v_x = v_x$. Since $L_0^* \gamma$ is the constant path at x, it follows that $0 \cdot v_x = 0_x$. □

Definition 2.2.18. An **$\mathbb{R}$-cone** is a set C together with a map

$$\begin{aligned} \mathbb{R} \times C &\longrightarrow C \\ (\alpha, c) &\longmapsto \alpha \cdot c \end{aligned}$$

such that $\alpha \cdot (\beta \cdot c) = (\alpha\beta) \cdot c$, $1 \cdot c = c$, and $0 \cdot c = 0 \cdot c'$ for all $\alpha, \beta \in \mathbb{R}$, $c, c' \in C$.

With this terminology, Prop. 2.2.16 states that the fibres of the tangent bundle of a diffeological space have the natural structure of $\mathbb{R}$-cones.

2.2.4 Tangent space of a diffeological vector space

We will now raise but not answer the following question: Under what conditions can the tangent bundle of a diffeological vector space X be identified with $X \times X$? In a first step we construct a map $X \times X \to TX$.

Lemma 2.2.19. *Let X be a diffeological vector space. Then the map*

$$\begin{aligned} \lambda : X \times X &\longrightarrow \underline{\mathrm{Hom}}(\mathbb{R}, X) \\ (x, y) &\longmapsto (t \mapsto x + ty) \end{aligned}$$

is a bilinear induction.

Proof. Let $p : U \to X \times X$, $u \mapsto \big(x(u), y(u)\big)$ be a map on $U \subset \mathbb{R}^n$. Assume that $\lambda \circ p$ is a plot. This means that $l_p : U \times \mathbb{R} \to X$, $(u,t) \mapsto x(u) + ty(u)$ is a plot. Therefore, the composition $U \cong U \times \{0\} \to U \times \mathbb{R} \to X$, which is the map $x : U \to X$, is a plot. Similarly, the map $y : U \to X$ is given by the composition of smooth maps

$$U \xrightarrow{(\Delta,1)} U \times U \times \mathbb{R} \xrightarrow{(-x)\times f_p} X \times X \xrightarrow{+} X\,,$$

where $\Delta : U \to U \times U$, $u \mapsto (u,u)$ is the diagonal map, so y is a plot. We conclude that $p = (x,y)$ is a plot, so that the diffeology on $X \times X$ is the pullback diffeology of λ. Since λ is injective, it is an induction. By definition, λ is bilinear. □

Corollary 2.2.20. *The map*

$$X \times X \xrightarrow{\lambda} \underline{\mathrm{Hom}}(\mathbb{R}, X) \xrightarrow{\rho_X} TX$$

is a morphism of diffeological spaces.

Proposition 2.2.21. *Let X be a vector space equipped with the fine vector space diffeology. Then the map $X \times X \to TX$ of Cor. 2.2.20 is an isomorphism of diffeological spaces.*

Proof. First, we show that $\nu : X \times X \to TX$ is injective. If $x \neq x'$, then $\nu(x,y) \neq \nu(x',y')$ since the base points of the tangent vectors are different. It remains to show that $y \neq y'$ implies that $\nu(x,y) \neq \nu(x,y')$. Let $\tilde{\partial}$ be the map of Prop. 2.2.14. Let $\alpha \in X'$ be a smooth linear function. Then

$$\begin{aligned}\tilde{\partial}_{\nu(x,y)}\alpha &= \frac{d}{dt}\alpha(x+ty)|_0 = \frac{d}{dt}(\alpha(x) + t\alpha(y))_0\\ &= \alpha(y)\end{aligned}$$

It follows from Prop. 2.1.32 that every linear map $\alpha : X \to \mathbb{R}$ is a morphism of diffeological spaces, so that we can always find an α such that $\alpha(y) \neq \alpha(y')$. This shows that $\tilde{\partial}$ is injective.

Let $\gamma : \mathbb{R} \to X$ be a smooth path with $\gamma_0 = x$. In a neighborhood $U = (-\varepsilon, \varepsilon)$ of the origin γ is of the form (2.4), i.e. $\gamma_t = \gamma_t^1\, x_1 + \ldots + \gamma_t^k\, x_k$ for smooth functions $\gamma^i \in C^\infty(U)$ and vectors $x_i \in X$. The path γ lies entirely in the finite-dimensional subspace $Y = \mathrm{Span}\{x_1, \ldots, x_k\}$. This means that the directional derivative of any function $f \in C^\infty(X)$ depends only on the restriction of f to Y. Prop. 2.1.31 implies that the subspace diffeology on Y is the natural diffeology on the manifold $Y \cong \mathbb{R}^k$. It follows that the linear path $\bar{\gamma}_t = x + ty$ with $y = \dot{\gamma}_0^1\, x_1 + \ldots + \dot{\gamma}_0^k\, x_k$ represents the same tangent vector as γ. We conclude that ν is surjective.

We have shown that the inverse of ν maps the tangent vector represented by the path $\gamma_t = \gamma_t^1\, x_1 + \ldots + \gamma_t^k\, x_k$ to the pair

$$(x,y) = (\gamma_0^1\, x_1 + \ldots + \gamma_0^k\, x_k, \dot{\gamma}_0^1\, x_1 + \ldots + \dot{\gamma}_0^k\, x_k)\,.$$

This shows that plots of such paths are mapped to plots in $X \times X$, so that ν is an isomorphism of diffeological spaces. □

Prop. 2.2.21 shows that under the strong assumption that X is a fine diffeological vector space, we can identify the set of tangent vectors at a point of X with X. Under what general conditions this is true is, to our best knowledge, an open question.

2.2.5 Fibre-wise linear bundles

Definition 2.2.22. Let $A \to X$ be a diffeological bundle, i.e. a subduction. A **fibre-wise linear structure** on the bundle A consists of two maps of diffeological bundles,

$$\begin{aligned} + : A \times_X A &\longrightarrow A \\ \cdot : \mathbb{R} \times A &\longrightarrow A \end{aligned}$$

called fibre-wise addition and scalar multiplication, that equip every fibre of A with the structure of diffeological vector space, such that the zero section $X \to A$, $x \mapsto 0_x$ is smooth. A diffeological bundle together with a fibre-wise linear structure will be called a **fibre-wise linear diffeological bundle**.

Terminology 2.2.23. The notion of linear diffeological bundles of Def. 2.2.22 is very natural and has appeared under the name **regular vector bundle** in [Vin08], **diffeological vector space over** X in [CW16], and **diffeological vector pseudo-bundle** in [Per16]. I apologize to the reader for following my own idiosyncratic linguistic preference.

Proposition 2.2.24. *A fibre-wise linear structure on a diffeological bundle $A \to X$ induces the structure of a $C^\infty(X)$-module on the diffeological space of sections $\Gamma(X, A)$.*

Proof. Let $f : X \to \mathbb{R}$ be a smooth function and $a \in \Gamma(X, A)$ be a smooth section. Then we have a smooth section

$$fa : X \xrightarrow{\Delta} X \times X \xrightarrow{f \times a} \mathbb{R} \times A \xrightarrow{\cdot} A\,,$$

where $\Delta(x) = (x, x)$ is the diagonal map. This defines a smooth map

$$\begin{aligned} C^\infty(X) \times \Gamma(X, A) &\longrightarrow \Gamma(X, A) \\ (f, a) &\longmapsto fa\,. \end{aligned}$$

If $b \in \Gamma(X, A)$ is another smooth section, then we have a smooth section

$$a + b : X \xrightarrow{\Delta} X \times X \xrightarrow{a \times b} A \times A \xrightarrow{+} A\,.$$

This defines a smooth map

$$\begin{aligned} \Gamma(X, A) \times \Gamma(X, A) &\longrightarrow \Gamma(X, A) \\ (a, b) &\longmapsto a + b\,. \end{aligned}$$

It follows from the defining property of a fibre-wise linear structure that $f(ga) = (fg)a$ and $f(a+b) = fa+fb$, so that we obtain the structure of a $C^\infty(X)$-module. □

2.3 The space of fields

Definition 2.3.1. Let $F \to M$ be a smooth fibre bundle. The diffeological space of sections $\mathcal{F} := \Gamma(M, F)$ (Def. 2.1.24) is called the **space of fields**.

The space of fields is equipped with the subspace diffeology of the diffeological mapping space $\underline{\mathrm{Hom}}(M, F)$. This means that a map $\varphi : U \to \mathcal{F}$, $u \mapsto \varphi_u$ defined on the open subset $U \subset \mathbb{R}^n$ is a plot if the map

$$U \times M \longrightarrow F, \qquad (u, m) \longmapsto \varphi_u(m)$$

is smooth, i.e. an infinitely often differentiable map of finite-dimensional manifolds.

2.3.1 Tangent bundle

Let $\varphi : \mathbb{R} \to \mathcal{F}$, $t \mapsto \varphi_t$ be a smooth path of fields. We define

$$\begin{aligned} \dot{\varphi}_0 : M &\longrightarrow TF \\ m &\longmapsto \frac{d}{dt}\varphi_t(m)\Big|_{t=0} , \end{aligned}$$

where the right hand side is a suggestive notation for the tangent vector in TF represented by the smooth path $t \mapsto \varphi_t(m)$. Since φ_t is a section of M we have that $\pi\big(\varphi_t(m)\big) = m$. It follows that

$$T\pi\big(\dot{\varphi}_0(m)\big) = \frac{d}{dt}\pi\big(\varphi_t(m)\big)\Big|_{t=0} = \frac{d}{dt}m\Big|_{t=0} = 0_m .$$

This means that $\dot{\varphi}_0(m)$ lies in the **vertical tangent bundle** of F,

$$VF := \ker T\pi \,, \tag{2.11}$$

which is a vector bundle over F. We have the following commutative diagram of manifolds:

$$\begin{tikzcd}
M \arrow[rrd, bend left, "\dot{\varphi}_0"] \arrow[rd, dotted] \arrow[rdd, bend right, "\mathrm{id}"'] & & \\
& M \times_F VF \arrow[r] \arrow[d] & VF \arrow[d, "\mathrm{pr}_F"] \\
& M \arrow[r, "\varphi_0"] \arrow[rd, "\mathrm{id}"'] & F \arrow[d, "\pi"] \\
& & M
\end{tikzcd} \tag{2.12}$$

This shows that $\dot{\varphi}_0$ is a section of the bundle $VF \to M$, which covers the section $\varphi_0 = \mathrm{pr}_F \circ \dot{\varphi}_0$. The map

$$\mathrm{pr}_{\mathcal{F}} := (\mathrm{pr}_F)_* : \Gamma(M, VF) \longrightarrow \Gamma(M, F) = \mathcal{F}$$

is a subduction since zero section is a smooth section of $\mathrm{pr}_{\mathcal{F}}$. The fiber over $\varphi \in \mathcal{F}$ is given by

$$\Gamma(M, VF)_\varphi = \Gamma^\infty(M, \varphi^* VF) \,, \tag{2.13}$$

where $\varphi^* VF = M \times_F^{\varphi, \mathrm{pr}_F} VF$ is the pullback fibre bundle. The map

$$\begin{aligned} \tau_{\mathcal{F}} : \underline{\mathrm{Hom}}(\mathbb{R}, \mathcal{F}) &\longrightarrow \Gamma(M, VF) \\ \varphi &\longmapsto \dot{\varphi}_0 \end{aligned}$$

is a morphism of diffeological bundles over $\mathcal{F}$. The following result is one of the reasons for using diffeological spaces in field theory.

Theorem 2.3.2. *Let $F \to M$ be a smooth fibre bundle. Then there is a unique isomorphism $T\mathcal{F} \to \Gamma(M, VF)$ of diffeological bundles over $\mathcal{F}$, such that*

$$\begin{tikzcd}
\underline{\mathrm{Hom}}(\mathbb{R}, \mathcal{F}) \arrow[d, "\rho_{\mathcal{F}}"'] \arrow[dr, "\tau_{\mathcal{F}}"] & \\
T\mathcal{F} \arrow[r, "\cong"] & \Gamma(M, VF)
\end{tikzcd}$$

commutes.

Lemma 2.3.3. *Let $F \to M$ be a smooth fibre bundle and $\varphi, \psi : \mathbb{R} \to \mathcal{F}$ smooth paths in the space of fields. If φ and ψ represent the same tangent vector, then $\dot{\varphi}_0 = \dot{\psi}_0$.*

Proof. By Prop. 2.2.14 φ and ψ induce the same directional derivative, $\tilde{\partial}_{\rho_{\mathcal{F}}(\varphi)} = \tilde{\partial}_{\rho_{\mathcal{F}}(\psi)}$. Let (x^i, u^α) be local bundle coordinates on a neighborhood of $m \in M$. The map $u^\alpha_m : \mathcal{F} \to \mathbb{R}$, $\chi^\alpha(m) = u^\alpha(\chi(m))$ is a smooth function on $\mathcal{F}$. Its directional derivative with respect to φ_t is

$$\begin{aligned}
\tilde{\partial}_{\rho_{\mathcal{F}}(\varphi)} u^\alpha_m &= \frac{d}{dt} u^\alpha_m(\varphi_t)\Big|_{t=0} = \frac{d}{dt} \varphi^\alpha_t(m)\Big|_{t=0} \\
&= \dot{\varphi}^\alpha_0(m)\,.
\end{aligned}$$

Since $\tilde{\partial}_{\rho_{\mathcal{F}}(\varphi)} u^\alpha_m = \tilde{\partial}_{\rho_{\mathcal{F}}(\psi)} u^\alpha_m$ for all $m \in M$ and α, it follows that $\dot{\varphi}_0(m) = \dot{\psi}_0(m)$ for all $m \in M$. □

Lemma 2.3.4. *The map $\tau_{\mathcal{F}} : \underline{\mathrm{Hom}}(\mathbb{R}, \mathcal{F}) \to \Gamma(M, VF)$, $\varphi \mapsto \dot{\varphi}_0$ is a subduction.*

Proof. A section $\eta : M \to VF$ is a vertical field supported on $N = (\mathrm{pr}_F \circ \eta)(M) \subset M$. Since N is an embedded submanifold, we can extend η to a complete vertical vector field $\bar{\eta}$ on F, supported on a tubular neighborhood of N. Let $\Phi : \mathbb{R} \times F \to F$ be the flow integrating $\bar{\eta}$. Then the smooth path $\varphi : \mathbb{R} \to \mathcal{F}$ defined by $\varphi_t(m) := \Phi(t, (\mathrm{pr}_F \circ \eta)(m))$ satisfies $\dot{\varphi}_0 = \eta$. This shows that every section in $\Gamma(M, VF)$ is the time derivative at 0 of a path in $\mathcal{F}$, so that the map $\tau_{\mathcal{F}}$ is surjective.

Let $p : U \times \mathbb{R} \times M \to F$ be a smooth homotopy of sections of F. Then the maps $\dot{p} : U \times M \to F$, $\dot{p}(u, m) = \frac{\partial p}{\partial t}(u, 0, m)$ is smooth. This shows that $\tau_{\mathcal{F}}$ maps plots to plots, so it is smooth.

Let now $q : U \times M \to VF$ define a smooth family of sections of $VF \to M$. It can be extended trivially to a section $\tilde{q} : U \times M \to U \times VF$, $(u, m) \mapsto (u, q(u, m))$ of the vertical tangent bundle of the fibre bundle $\mathrm{id}_U \times \pi : U \times F \to U \times M$. By the same argument as above, we can find a smooth path $\tilde{p} : \mathbb{R} \to \Gamma(U \times M, U \times VF)$, such that $\dot{\tilde{p}}_0 = \tilde{q}$. The path $\tilde{p}$ is of the form $\tilde{p}_t(u, m) = (u, p(t, u, m))$ for a smooth map $(t, u, m) \mapsto p(t, u, m)$, so that $q(u, m) = \frac{\partial p}{\partial t}(u, 0, m)$. It follows that $q = \tau_{\mathcal{F}} \circ p$ for the plot $p : U \to \underline{\mathrm{Hom}}(\mathbb{R}, \mathcal{F})$. We conclude that $\Gamma(M, VF)$ has the pushforward diffeology of $\tau_{\mathcal{F}}$. □

Lemma 2.3.5. *Let $A \to M$ be a smooth vector bundle. Let $a, b : \mathbb{R} \to \mathcal{A}$ be smooth paths of fields. If $\dot{a}_0 = \dot{b}_0$, then a and b represent the same tangent vector on $\mathcal{A}$.*

Proof. In local fibre coordinates $(x^1, \dots, x^n, u^1, \dots, u^k)$ over a neighborhood $V \subset M$, the sections are given by the coordinate functions, which we denote by

$$\begin{aligned} a^\alpha(t,x) &= a^\alpha_t(x^1, \dots, x^n) \\ b^\alpha(t,x) &= b^\alpha_t(x^1, \dots, x^n)\,. \end{aligned}$$

Since $\dot{a}_0 = \dot{b}_0$, the difference $b^\alpha - a^\alpha$ is a function that has vanishing value and vanishing partial derivative with respect to t at $t = 0$. It follows from Hadamard's lemma that there is a smooth function $h^\alpha = h^\alpha(x,t)$ on the local coordinate chart, such that

$$b^\alpha(t,x) - a^\alpha(t,x) = h^\alpha(t,x)t^2\,.$$

Now we define smooth functions $p^\alpha : \mathbb{R}^2 \times V \to \mathbb{R}$, $f : \mathbb{R} \to \mathbb{R}^2$, and $g : \mathbb{R} \to \mathbb{R}^2$ by

$$\begin{aligned} p^\alpha(r,s,x) &:= a^\alpha(r,x) + h^\alpha(r,x)s^2 \\ f(t) &:= (t,0) \\ g(t) &:= (t,t^2)\,. \end{aligned}$$

It is easy to check that

$$a^\alpha = p^\alpha \circ f\,, \quad b^\alpha = p^\alpha \circ g\,,$$

and that

$$(T_0 f)\partial_t = \frac{\partial}{\partial r}\Big|_{(0,0)} = (T_0 g)\partial_t\,.$$

The maps p^α for $1 \leq \alpha \leq k$ define a smooth homotopy of local sections $p_V : \mathbb{R}^2 \to \mathcal{A}(V)$. Since p_V depends linearly on h we can use a standard partition of unity argument to sum the local homotopies to obtain a smooth family of sections $p : \mathbb{R}^2 \to \mathcal{A}$ that makes the following diagram commute:

$$\begin{tikzcd} \mathbb{R} \arrow[r, "f"] \arrow[dr, "a"'] & \mathbb{R}^2 \arrow[d, "p"] & \mathbb{R} \arrow[l, "g"'] \arrow[dl, "b"] \\ & \mathcal{A} & \end{tikzcd} \tag{2.14}$$

We conclude that the smooth paths a_t and b_t represent the same tangent vector on $\mathcal{A}$. □

Terminology 2.3.6. Let $F \to M$ be a fibre bundle and $S \subset F$ a subset. We say that a plot $p : U \to \mathcal{F}$ is **contained in** S if the image of the map $U \times M \to F$, $(u,m) \mapsto p_u(m)$ is contained in S.

Lemma 2.3.7. *Let $F \to M$ be a smooth fibre bundle, let $\varphi : \mathbb{R} \to \mathcal{F}$ be a smooth path of fields, and let $S \subset F$ be a tubular neighborhood of $\varphi(M)$. Then there is a smooth path contained in S that represents the same tangent vector as φ.*

Proof. We can view the smooth path of fields φ as a smooth map $\varphi : \mathbb{R} \times M \to F$. Let $S \subset F$ be a tubular neighborhood of $\varphi_0(M) = \varphi(\{0\} \times M)$. Let $U_i \subset M$ be an open set with compact closure. Then we can find an $\varepsilon_i > 0$ sufficiently small, such that $\varphi\big((-\varepsilon_i, \varepsilon_i) \times U_i\big) \subset S$. Let $\chi_i : \mathbb{R} \to \mathbb{R}$ be a smooth function with the following properties:

(i) $|\chi_i(t)| < \varepsilon_i$ for all t.

(ii) $\chi_i(t) = t$ for $|t| \leq \frac{\varepsilon_i}{2}$.

From these properties it follows that $\varphi\big(\chi_i(\mathbb{R})\times U_i\big) \subset S$ and that $\varphi(\chi_i(t), u) = \varphi(t, u)$ for $|t| < \varepsilon$, $u \in U_i$. Using a standard partition of unity argument, we obtain functions $\varepsilon : M \to \mathbb{R}^+$ and $\chi : \mathbb{R} \times M \to \mathbb{R}$, such that $\psi := \varphi \circ (\chi \times \mathrm{id}_M) : \mathbb{R} \times M \to F$ satisfies the following properties.

(i) $\psi(t, m) \subset S$ for all t and m.

(ii) $\psi(t, m) = \varphi(t, m)$ for $|t| \leq \frac{\varepsilon(m)}{2}$.

Property (i) means that ψ is a smooth path of fields contained in S. Property (ii) means that the restrictions of ψ and φ to the open set $D := \{(\frac{\varepsilon(m)}{2}, m) \mid m \in M\}$ are equal.

It remains to show that φ and ψ represent the same tangent vector. Let the map $p : \mathbb{R}^2 \times M \to F$ be defined as

$$p(r, s, m) := \begin{cases} \varphi\big((1 - \frac{s}{r^2})r + \frac{s}{r^2}\chi(r, m), m\big)\,; & (r, s, m) \notin \mathbb{R} \times D \\ \varphi(r, m)\,; & (r, s, m) \in \mathbb{R} \times D \end{cases}$$

For all $(r, s, m) \in \mathbb{R} \times D$, $r \neq 0$ we have

$$\begin{aligned} \varphi\Big(\Big(1 - \frac{s}{r^2}\Big)r + \frac{s}{r^2}\chi(r, m), m\Big) &= \varphi\Big(\Big(1 - \frac{s}{r^2}\Big)r + \frac{s}{r^2}\, r, m\Big) \\ &= \varphi(r, m)\,. \end{aligned}$$

which shows that p is smooth. Moreover, $p(t, 0, m) = \varphi(m)$ and $p(t, t^2, m) = \psi(t, m)$ for all $t \in \mathbb{R}$. Let $f : \mathbb{R} \to \mathbb{R}^2$ and $g : \mathbb{R} \to \mathbb{R}^2$ be defined as in diagram (2.14). Then $p \circ f = \varphi$ and $p \circ g = \psi$. By the same reasoning as for diagram (2.14), it follows that φ and ψ represent the same tangent vector. □

Proof of Thm. 2.3.2. Lem. 2.3.3 shows that the map $\tau_{\mathcal{F}} : \underline{\mathrm{Hom}}(\mathbb{R}, \mathcal{F}) \to \Gamma(M, VF)$, $\varphi \mapsto \dot{\varphi}_0$ descends to a well-defined map

$$\nu : T\mathcal{F} \longrightarrow \Gamma(M, VF)\,.$$

Since by Prop. 2.2.6 the map $\rho_{\mathcal{F}} : \underline{\mathrm{Hom}}(\mathbb{R}, \mathcal{F}) \to T\mathcal{F}$ is surjective, ν is unique. It is clear from the construction of ν that $(\mathrm{pr}_F)^* \circ \nu = \mathrm{pr}_{\mathcal{F}}$. In other words, ν is a map of bundles (in sets) over $\mathcal{F}$.

By Lem. 2.3.4, $\tau_{\mathcal{F}}$ is a subduction. In particular $\tau_{\mathcal{F}}$ is surjective. This implies that ν is surjective. We now show that ν is injective, as well. Let $\varphi, \psi : \mathbb{R} \to \mathcal{F}$ be smooth paths such that $\dot{\varphi}_0 = \dot{\psi}_0$. We must show that φ and ψ represent the same tangent vector on $\mathcal{F}$ in the quotient (2.7). Let $S \subset F$ be a tubular neighborhood of $\varphi_0(M) = \psi_0(M)$. Since the normal bundle of S is isomorphic to the pullback $\varphi_0^* VF$ of the vertical bundle, there is a smooth map

$$\sigma : \varphi_0^* VF \longrightarrow F$$

and a tubular neighborhood $S' \subset \varphi_0^* VF$ of the zero section, such that the restriction $\bar{\sigma} : S' \to S$ is a diffeomorphism. By Lem. 2.3.7 there are paths φ' and ψ' that are contained in S, so that they can be identified with the paths $a := \bar{\sigma}^{-1} \circ \varphi'$ and $b := \bar{\sigma}^{-1} \circ \psi'$ in the vector bundle $A = \varphi_0^* VF$, which satisfy $\dot{a}_0 = \dot{b}_0$. Now we can apply Lem. 2.3.5, which shows that there is a diagram like (2.14). Applying the pushforward by σ to this diagram we obtain the commutative diagram

$$\begin{tikzcd}
\mathbb{R} \arrow[r, "\sigma\circ f"] \arrow[dr, "\varphi'"'] & \mathbb{R}^2 \arrow[d, "p"] & \mathbb{R} \arrow[l, "\sigma\circ g"'] \arrow[dl, "\psi'"] \\
& \mathcal{F} &
\end{tikzcd}$$

Moreover, since $(T_0 f)\partial_t = (T_0 g)\partial_t$, it follows that $\big(T_0(\sigma \circ f)\big)\partial_t = \big(T_0(\sigma \circ g)\big)\partial_t$. This shows that φ' and ψ' represent the same tangent vector in the quotient (2.7), which implies that φ and ψ represent the same tangent vector $\rho_{\mathcal{F}}(\varphi) = \rho_{\mathcal{F}}(\psi)$. We conclude that ν is injective.

It remains to show that ν and its inverse are smooth. Since $\tau_{\mathcal{F}} = \nu \circ \rho_{\mathcal{F}}$ and since by Prop. 2.2.6 $\rho_{\mathcal{F}}$ is a subduction, it follows from Prop. 2.1.22 that ν is smooth. Similarly, since $\rho_{\mathcal{F}} = \nu^{-1} \circ \tau_{\mathcal{F}}$ and since by Lem. 2.3.4 $\tau_{\mathcal{F}}$ is a subduction, it follows from Prop. 2.1.22 that ν^{-1} is smooth. We conclude that ν is an isomorphism of diffeological spaces. □

Corollary 2.3.8. *The fibre of the diffeological tangent bundle of $T\mathcal{F} \to \mathcal{F}$ over $\varphi \in \mathcal{F}$ is*

$$T_\varphi \mathcal{F} \cong \Gamma(M, \varphi^* VF) \,. \tag{2.15}$$

Terminology 2.3.9. In the language of variational calculus, an element of $T_\varphi \mathcal{F}$ is called an **infinitesimal variation** of φ.

Example 2.3.10. Let $F := M \times N \xrightarrow{\mathrm{pr}_1} M$ be the trivial bundle. Then $\mathcal{F} = \mathrm{Hom}_{\mathcal{M}\mathrm{fld}}(M, N)$ is the set of smooth maps from M to N, equipped with the functional diffeology. The vertical tangent bundle is given by $VF = 0_M \times TN \cong M \times TN \to M \times N$. A tangent vector at $\varphi : M \to N$ is given by a section of

$$M \times_{M\times N} (M \times TN) \cong M \times_N^{\varphi, \mathrm{pr}_N} TN \to M \,.$$

By the universal property of the pullback, such a section is given by a commutative diagram

$$\begin{tikzcd}
M \arrow[r, "\xi"] \arrow[dr, "\varphi"'] & TN \arrow[d, "\mathrm{pr}_N"] \\
& N
\end{tikzcd}$$

If $N = M$, the tangent space at the identity is given by $T_{\mathrm{id}}\mathcal{F} = \mathcal{X}(M)$, the space of vector fields on M.

Corollary 2.3.11. *The tangent bundle of a diffeological space of fields is fibre-wise linear.*

Proof. Let $\mathcal{F} = \Gamma(M, F)$ be a space of fields. By Thm. 2.3.2 we have

$$\begin{aligned} T\mathcal{F} \times_{\mathcal{F}} T\mathcal{F} &\cong \Gamma(M, VF) \times_{\Gamma(M,F)} \Gamma(M, VF) \\ &\cong \{(\xi, \chi) \in \Gamma(M, VF \times_M VF) \mid \mathrm{pr}_F \circ \xi = \mathrm{pr}_F \circ \chi\} \\ &\cong \Gamma(M, VF \times_F VF)\,. \end{aligned}$$

Since $VF \to F$ is a vector bundle we have the structure maps of addition $VF \times_F VF \to VF$ and scalar multiplication $\mathbb{R} \times VF \to VF$. The pushforward of these maps defines a fibre-wise linear structure on the diffeological bundle $\Gamma(M, VF) \to \Gamma(M, F)$. □

Proposition 2.3.12. *Let $A \to M$ be a vector bundle. Then we have an isomorphism*

$$T\mathcal{A} \cong \mathcal{A} \times \mathcal{A}$$

of fibre-wise linear diffeological bundles over $\mathcal{A}$, where $\mathrm{pr}_1 : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ is the trivial bundle.

Proof. We have an isomorphism

$$\begin{aligned} A \times_M A &\longrightarrow VA \\ (a_m, b_m) &\longmapsto \frac{d}{dt}(a_m + t b_m)\big|_{t=0} \end{aligned}$$

of smooth fibre bundles over M, which induces an isomorphism of the spaces of sections

$$\mathcal{A} \times \mathcal{A} \cong \Gamma(M, A \times_M A) \xrightarrow{\cong} \Gamma(M, VA) \cong T\mathcal{A}\,.$$

For every section $a \in \mathcal{A}$, the restriction $\mathcal{A} \cong \{a\} \times \mathcal{A} \to T_a\mathcal{A}$ is a smooth map of diffeological vector spaces. □

Remark 2.3.13. The isomorphism of Prop. 2.3.12 is the smooth map of Cor. 2.2.20.

Example 2.3.14. Let $E \to M$ and $F \to M$ be smooth fibre bundles. Then the product of the space of sections of E and F is itself a space of sections,

$$\mathcal{E} \times \mathcal{F} \cong \Gamma(M, E \times_M F)\,.$$

This shows that the tangent bundle of $\mathcal{E} \times \mathcal{F}$ is given by

$$\begin{aligned} T(\mathcal{E} \times \mathcal{F}) &\cong \Gamma\big(M, V(E \times_M F)\big) \\ &\cong \Gamma(M, VE \times_M VF) \\ &\cong T\mathcal{E} \times T\mathcal{F}\,, \end{aligned}$$

which is a special case of Prop. 2.2.4.

2.3.2 Differential forms

For our purposes, there is no need to consider the general theory of differential forms on diffeological spaces [IZ13, Sec. ***]. We will only be concerned with the diffeological space of fields for which every fibre of the tangent bundle $T\mathcal{F} \to \mathcal{F}$ is the diffeological vector space of sections of a vector bundle.

Definition 2.3.15. Let X be a diffeological vector space. A p-form on X is a morphism of diffeological spaces $X^p \to \mathbb{R}$ that is multilinear and antisymmetric.

Let us denote the p-fold product of a fibre bundle $X \to Y$ in the category $\mathcal{D}\mathrm{iffg}_{/Y}$ of objects over Y by $(X_{/Y})^p$. Then

$$(T\mathcal{F}_{/\mathcal{F}})^p = \underbrace{T\mathcal{F} \times_{\mathcal{F}} \ldots \times_{\mathcal{F}} T\mathcal{F}}_{p\text{-factors}} \tag{2.16}$$

is the p-fold fibre product of $T\mathcal{F} \to \mathcal{F}$. The empty product in the category of objects over $\mathcal{F}$ is the identity of $\mathcal{F}$, so that

$$(T\mathcal{F}_{/\mathcal{F}})^0 = \mathcal{F}\,.$$

The tangent bundle of a space of fields is a fibre-wise linear bundle of diffeological spaces, so that we can extend Def. 2.3.15 to all fibres of the bundle in an obvious way.

Definition 2.3.16. A **differential p-form** on the space of fields $\mathcal{F}$ is a morphism of diffeological spaces

$$\nu : (T\mathcal{F}_{/\mathcal{F}})^p \longrightarrow \mathbb{R}\,,$$

such that the restriction of ν to every fibre is multilinear and antisymmetric.

The fibre product is itself a space of fields,

$$(T\mathcal{F}_{/\mathcal{F}})^p \cong \Gamma\big(M, (VF_{/M})^p\big)\,.$$

This shows that the fibre over $\varphi \in \mathcal{F}$ is given by

$$\begin{aligned}(T\mathcal{F}_{/\mathcal{F}})^p_\varphi &\cong \Gamma\big(M, \varphi^*(VF \times_F \ldots \times_F VF)\big)\\ &\cong \Gamma\big(M, (\varphi^*VF) \times_M \ldots \times_M (\varphi^*VF)\big)\,.\end{aligned}$$

Remark 2.3.17. The diffeological tensor product $(T_\varphi\mathcal{F})^{\otimes p}$ is defined by the universal property that there is a multilinear map $i : (T_\varphi\mathcal{F})^p \to (T_\varphi\mathcal{F})^{\otimes p}$ such that every smooth multilinear map on $(T_\varphi\mathcal{F})^p$ extends to a unique smooth linear map on $(T\mathcal{F})^{\otimes p}$. But since the vector space $T_\varphi\mathcal{F}$ is rather large, this tensor product is hard to describe explicitly. We point out preventively that it is *not* given by

$$\Gamma\big(M, (\varphi^*VF)^{\otimes p}\big) \cong T_\varphi\mathcal{F} \otimes_{C^\infty(M)} \ldots \otimes_{C^\infty(M)} T_\varphi\mathcal{F}\,,$$

which is a quotient of the much larger vector space $(T_\varphi\mathcal{F})^{\otimes p}$.

The set of all differential p-forms is a diffeological vector subspace

$$\Omega^p(\mathcal{F}) \subset \underline{\mathrm{Hom}}\big((T\mathcal{F}_{/\mathcal{F}})^p, \mathbb{R}\big)\,,$$

which is endowed with the structure of a $C^\infty(\mathcal{F})$-module in the usual way. The collection of all diffeological spaces $\Omega^p(\mathcal{F})$, $p \geq 0$ is a graded commutative algebra $\Omega(\mathcal{F})$, with the wedge product of a p-form ν and a q-form ν' defined in the usual way by the antisymmetrized point-wise product

$$\begin{aligned}&(\nu \wedge \nu')(\xi^1_\varphi, \dots, \xi^{p+q}_\varphi)\\ &\qquad := \frac{1}{p!q!} \sum_{\sigma \in S_{p+q}} \mathrm{sgn}(\sigma)\, \nu(\xi^{\sigma(1)}_\varphi, \dots, \xi^{\sigma(p)}_\varphi)\, \nu'(\xi^{\sigma(p+1)}_\varphi, \dots, \xi^{\sigma(p+q)}_\varphi)\end{aligned}$$

for all $\varphi \in \mathcal{F}$ and all $\xi^1_\varphi, \dots, \xi^{p+q}_\varphi \in T_\varphi \mathcal{F}$. The inner derivative with respect to a vector field $\chi \in \mathfrak{X}(\mathcal{F})$ is also defined in the usual manner by

$$(\iota_\chi \nu)(\xi^1_\varphi, \dots, \xi^{p-1}_\varphi) := \nu(\chi_\varphi, \xi^1_\varphi, \dots, \xi^{p-1}_\varphi)\,,$$

for all $\varphi \in \mathcal{F}$.

Let $E \to N$ be another smooth fibre bundle. In Prop. 2.2.4 we have shown that the tangent functor on diffeological spaces commutes with products, so that

$$\begin{aligned}T(\mathcal{F} \times \mathcal{E}) &\cong T\mathcal{F} \times T\mathcal{E}\\ &\cong (T\mathcal{F} \times \mathcal{E}) \times_{\mathcal{F}\times\mathcal{E}} (\mathcal{F} \times T\mathcal{E})\,,\end{aligned}$$

which is the fibre product of fibre-wise linear diffeological bundles over $\mathcal{F} \times \mathcal{E}$. It follows that the fibre at $(\varphi, \psi) \in \mathcal{F} \times \mathcal{E}$ is the direct sum of diffeological vector spaces

$$T_{(\varphi,\psi)}(\mathcal{F} \times \mathcal{E}) \cong T_\varphi \mathcal{F} \oplus T_\psi \mathcal{E}\,.$$

This induces a decomposition of the n-fold fibre product of $T(\mathcal{F} \times \mathcal{E})$ as follows. Let

$$\begin{aligned}T^{(p,q)}(\mathcal{F} \times \mathcal{E}) &:= \big((T\mathcal{F} \times \mathcal{E})_{/(\mathcal{F}\times\mathcal{E})}\big)^p \times_{\mathcal{F}\times\mathcal{E}} \big((\mathcal{F} \times T\mathcal{E})_{/(\mathcal{F}\times\mathcal{E})}\big)^q\\ &\cong \big((T\mathcal{F}_{/\mathcal{F}})^p \times \mathcal{E}\big) \times_{\mathcal{F}\times\mathcal{E}} \big(\mathcal{F} \times (T\mathcal{E}_{/\mathcal{E}})^q\big)\\ &\cong (T\mathcal{F}_{/\mathcal{F}})^p \times (T\mathcal{E}_{/\mathcal{E}})^q\,.\end{aligned} \tag{2.17}$$

The fibre over $(\varphi, \psi) \in \mathcal{F} \times \mathcal{E}$ is

$$T^{(p,q)}_{(\varphi,\psi)}(\mathcal{F} \times \mathcal{E}) \cong (T_\varphi \mathcal{F})^p \oplus (T_\psi \mathcal{E})^q\,.$$

With this notation we have the decomposition

$$\begin{aligned}\big(T(\mathcal{F} \times \mathcal{E})_{/(\mathcal{F}\times\mathcal{E})}\big)^n \cong T^{(n,0)}(\mathcal{F} \times \mathcal{E}) \times_{(\mathcal{F}\times\mathcal{E})} T^{(n-1,1)}(\mathcal{F} \times \mathcal{E}) \times_{(\mathcal{F}\times\mathcal{E})} \cdots\\ \cdots \times_{(\mathcal{F}\times\mathcal{E})} T^{(0,n)}(\mathcal{F} \times \mathcal{E})\,.\end{aligned} \tag{2.18}$$

Definition 2.3.18. A **differential (p,q)-form on** $\mathcal{F} \times \mathcal{E}$ is a fibre-wise linear and antisymmetric map of diffeological spaces

$$\nu : (T\mathcal{F}_{/\mathcal{F}})^p \times (T\mathcal{E}_{/\mathcal{E}})^q \longrightarrow \mathbb{R}\,.$$

The space of all (p,q)-forms will be denoted by $\Omega^{p,q}(\mathcal{F} \times \mathcal{E})$.

It follows from (2.18) that the space of differential n-forms on $\mathcal{F} \times \mathcal{E}$ decomposes as

$$\Omega^n(\mathcal{F} \times \mathcal{E}) \cong \Omega^{n,0}(\mathcal{F} \times \mathcal{E}) \oplus \Omega^{n-1,1}(\mathcal{F} \times \mathcal{E}) \oplus \ldots \oplus \Omega^{0,n}(\mathcal{F} \times \mathcal{E})\,,$$

in complete analogy to smooth manifolds.

Let $A \to M$ be a vector bundle. We have shown in Prop. 2.3.12 that the tangent bundle of the space of fields is $T\mathcal{A} \cong \mathcal{A} \times \mathcal{A} \xrightarrow{\mathrm{pr}_1} \mathcal{A}$. It follows that the n-fold fibre product is given by

$$(T\mathcal{A}_{/\mathcal{A}})^n \cong \mathcal{A} \times \mathcal{A}^n \xrightarrow{\mathrm{pr}_1} \mathcal{A}\,.$$

This shows that a differential n-form on $\mathcal{A}$ is given by a smooth map

$$\nu : \mathcal{A} \times \mathcal{A}^n \longrightarrow \mathbb{R}\,, \tag{2.19}$$

that is multilinear and antisymmetric in $\mathcal{A}^n$.

Since $T\mathcal{A}$ is naturally a trivial bundle, we have the notion of constant vector fields and forms on $\mathcal{A}$. A vector field $\xi : \mathcal{A} \to T\mathcal{A} \cong \mathcal{A} \times \mathcal{A}$ is **constant** if it is of the form $\xi_a = (a, b)$ for some $b \in \mathcal{A}$. This shows that the space of constant vector fields is given by $\mathcal{A}$. A differential n-form ν is **constant** if for any family of constant vector fields $\xi^1, \ldots, \xi^n$ the map $a \mapsto \nu(\xi^1_a, \ldots, \xi^n_a)$ is constant. Viewed as a map of the form (2.19), ν is constant if and only if it does not depend on the first factor $\mathcal{A}$. We thus arrive at the following statement, which is in complete analogy to the case of finite dimensional vector spaces.

Proposition 2.3.19. *Let $A \to M$ be a vector bundle. Constant differential n-forms on the diffeological space $\mathcal{A} = \Gamma(M, A)$ can be identified with n-forms on the vector space $\mathcal{A}$, that is, with smooth multilinear and antisymmetric maps $\mathcal{A}^n \to \mathbb{R}$.*

2.3.3 Fréchet manifold structure

If the diffeological structure is not enough, we can equip $\mathcal{F}$ with the structure of a smooth Fréchet manifold modelled on the tangent spaces $T_\varphi\mathcal{F} = \Gamma^\infty(M, \varphi^*VF)$ with the usual semi-norms of infinitely often differentiable functions on a non-compact manifold [Ham82]. We will not make much use of such Fréchet manifold structures, so we will not go into any more detail here.

Chapter 3

Locality

For a general action $\mathcal{F} \to \mathbb{R}$ there is no mathematical reason why the critical points should be the solution of a PDE, as is the case for most LFTs that come to mind. The condition that guarantees that the Euler-Lagrange equation is a PDE is locality.

Definition 3.0.1. A lagrangian $L : \mathcal{F} \to \Omega^{\mathrm{top}}(M)$ is called **local** if there is a natural number $k \geq 0$, such that for all fields $\varphi \in \mathcal{F}$ and all points $m \in M$ the value of $L(\varphi)$ at m depends only on the partial derivatives of φ at m up to order k.

***

3.1 Jets

3.1.1 Jet bundles

Definition 3.1.1. Two local sections φ and φ' of a smooth fiber bundle $F \to M$ defined on a neighborhood of m have the same **k-jet at** m, denoted by $j^k_m \varphi = j^k_m \varphi'$, if they have the same value and partial derivatives up to k-th order at m.

It is not immediately clear that this is a good definition, since the partial derivatives of a section generally depend on the choice of coordinates. For example, the section of a line bundle is given in local coordinates by an $\mathbb{R}$-valued function. In one coordinate system this function can be constant so that its first derivatives vanish, while in another coordinate system it will have non-zero derivatives. But when the value and all partial derivatives of two sections φ and φ' are equal up to order k for one choice of coordinates, they will be equal in all charts.

Exercise 3.1.2. Let $f, g : M \to \mathbb{R}$ be functions on a smooth n-dimensional manifold. Let $x = (x^1, \ldots, x^n) : U \to \mathbb{R}^n$ be local coordinates on a neighborhood U of m. Let k be a natural number. Show that if

$$\frac{\partial^l f}{\partial x^{i_1} \cdots \partial x^{i_l}}\bigg|_{x(m)} = \frac{\partial^l g}{\partial x^{i_1} \cdots \partial x^{i_l}}\bigg|_{x(m)}$$

for all $l \leq k$ and all indices $1 \leq i_1, \ldots, i_l \leq n$, then these equalities hold in any other coordinate system.

Figure 3.1: Caption

Exercise 3.1.2 shows that having the same partial derivatives at a point m up to a given degree k is an equivalence relation on the space of all local sections on a neighborhood of m. The k-jets are the equivalence classes of this relation.

Definition 3.1.3. Two maps $f, g : M \to N$ of smooth manifolds have the same **k-jet** at $m \in M$ if the sections $m \mapsto (m, f(m))$ and $m \mapsto (m, g(m))$ of the trivial bundle $M \times N \to M$ have the same k-jet at m in the sense of Def. 3.1.1.

Remark 3.1.4. Two sections of $F \to M$ have the same k-jet at m in the sense of Def. 3.1.1 if and only if, when viewed as functions $M \to F$, they have the same k-jet at m in the sense of Def. 3.1.3. In this sense, the two definitions of jets are equivalent.

Terminology 3.1.5. The natural number k in Defs. 3.1.1 and 3.1.3 is called the **order** of the jet.

Example 3.1.6. Two smooth paths $f, g : \mathbb{R} \to M$ have the same 1-jet at 0 if and only if they represent the same tangent vector at the point $f(0)$.

The last example shows that the concept of jets can be viewed as a generalization of tangent vectors in two ways. First, the domain is generalized from a line $\mathbb{R}$ to a higher dimensional manifold, so that tangent vectors are generalized to tangent planes. Second, tangent planes are generalized to surfaces given by higher order polynomials. The geometric meaning of jets is then that two sections have the same jet at m if they have the same value (0-jet), the same tangent plane (1-jet), the same osculating ellipsoid or hyperboloid (2-jet), etc. at m. This is sometimes expressed by saying that, when two sections φ and φ' have the same k-jet at m, they are tangent to k-th order at $\varphi(m)$ (Fig. 3.1).

The analogy with tangent vectors can be taken further by also generalizing the concept of tangent spaces and tangent bundles. The set of all k-jets at m is denoted by

$$J^k_m F = \{ j^k_m \varphi \mid \text{for all open } U \ni m \text{ and all } \varphi \in \Gamma(U, F) \} .$$

The union of all jets at all m will be denoted by

$$J^k F := \bigcup_{m \in M} J^k_m F .$$

On the set of k-jets we have the natural projection

$$\pi_k : J^k F \longrightarrow M , \quad j^k_m \varphi \longmapsto m ,$$

to the base-point of every jet. The fibre of π_k over m is $J^k_m F$.

Example 3.1.7. Let $F = \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be the trivial line bundle over $\mathbb{R}$, so that $\mathcal{F} = C^\infty(\mathbb{R})$. The k-jet of a function $\varphi \in C^\infty(\mathbb{R})$ at $m \in \mathbb{R}$ can be identified with the k-th Taylor polynomial of φ at m. This induces an isomorphism

$$J^k_m(\mathbb{R} \times \mathbb{R}) \cong \mathbb{R}[\varepsilon]/(\varepsilon^{k+1}) .$$

In the language of algebraic geometry this is the ring of functions on the k-th infinitesimal neighborhood of m.

Exercise 3.1.8. Let $F = \mathbb{R} \times Q \to \mathbb{R}$ be a trivial bundle over $\mathbb{R}$. Show that $J^1(\mathbb{R} \times Q) \cong \mathbb{R} \times TQ$.

Exercise 3.1.8 shows that $J^1(\mathbb{R} \times Q)$ has natural structure of a smooth fiber bundle. In fact, this is the case for every J^kF. The way to show this is analogous to showing that the tangent bundle of a smooth manifold is itself a smooth manifold: We choose local bundle coordinates on F and show that these induce local coordinates on J^kF.

Let $(x^1, \dots, x^n, u^1, \dots, u^r)$ be a system of local bundle coordinates of F, that is, (x^i) are the base coordinates and (u^α) the fiber coordinates of some local trivialization. This induces coordinates $(x^i, u^\alpha, u^\alpha_{i_1}, u^\alpha_{i_1,i_2}, \dots, u^\alpha_{i_1,\dots,i_k})$ on J^kF given by

$$\begin{aligned} x^i, u^\alpha_{i_1,i_2,\dots,i_l} &: J^kF \longrightarrow \mathbb{R}\,, \\ x^i\big(j^k_m\varphi\big) &:= x^i(m)\,, \\ u^\alpha_{i_1,i_2,\dots,i_l}\big(j^k_m\varphi\big) &:= \frac{\partial^l(u^\alpha \circ \varphi)}{\partial x^{i_1}\partial x^{i_2}\cdots\partial x^{i_l}}\bigg|_m\,, \end{aligned} \tag{3.1}$$

for all $l \leq k$ and all sequences $i_1, \dots, i_l$ of indices. In order to handle the indices efficiently we will use multi-index notation.

Notation 3.1.9. Let $(x^1, \dots, x^n) = (x^i)$ be local coordinates indexed by $1 \leq i \leq n$. A **multi-index** is an n-tuple $I = (I_1, \dots, I_n) \in \mathbb{N}_0^n$. Multi-indices are used to define compact notation for products such as

$$x^I := (x^1)^{I_1}(x^2)^{I_2}\cdots(x^n)^{I_n}\,.$$

The number

$$|I| := I_1 + I_2 + \ldots + I_n$$

is called the **length** or **order** of I. Our main use of multi-indices is for higher partial derivatives,

$$\begin{aligned} \frac{\partial^{|I|}}{\partial x^I} &:= \frac{\partial^{|I|}}{(\partial x^1)^{I_1}(\partial x^2)^{I_2}\cdots(\partial x^n)^{I_n}} \\ &= \Big(\frac{\partial}{\partial x^1}\Big)^{I_1}\Big(\frac{\partial}{\partial x^2}\Big)^{I_2}\cdots\Big(\frac{\partial}{\partial x^n}\Big)^{I_n} =: \Big(\frac{\partial}{\partial x}\Big)^I\,. \end{aligned}$$

This suggests the following notation for jet bundle coordinates,

$$u^\alpha_I(j^k_m\varphi) := \frac{\partial^{|I|}\varphi^\alpha}{\partial x^I}\bigg|_m\,. \tag{3.2}$$

For every number $1 \leq i \leq n$, we define the **concatenation** of I with i by

$$I, i := (I_1, \dots, I_{i-1}, I_i + 1, I_{i+1}, \dots, I_n)\,.$$

The concatenation of the multi-index $0 = (0, \dots, 0)$ will be denoted by $0, i = i$. This makes the multi-index notation (3.2) consistent with that of Eq. (3.1). That is, if $I = i_1, i_2, \dots, i_l$ is the concatenated multi-index, then $u^\alpha_I = u^\alpha_{i_1,\dots,i_l}$. While multi-indices label the coordinates u^α_I uniquely, the concatenation $i_1, \dots, i_k$ of different sequences can represent the same multi-index. In fact, let I be a multi-index of order k. Then

$$\#\{(i_1,\dots,i_k)\in\{1,\dots,n\}^k \mid I=i_1,\dots,i_k\}=\frac{k!}{I!}\,,$$

where the multi-index factorial is defined by

$$I! := I_1! I_2! \cdots I_n!\,.$$

This combinatorial factor has two be taken into account when changing between the summation over multi-indices I and sequences $i_1,\dots,i_k$. Let C_I be some finite sequence labelled by the multi-index I, then

$$\sum_I C_I = \sum_k \frac{[i_1,\dots,i_k]!}{k!} \sum_{1\le i_1,\dots,i_k\le n} C_{i_1,\dots,i_k}\,, \tag{3.3}$$

where $[i_1,\dots,i_k]!$ denotes the multi-index factorial of the multi-index $I=i_1,\dots,i_k$. The concatenation of two multi-indices is given by the sum

$$I+J=(I_1+J_1,\dots,I_n+J_n)\,.$$

Splitting the sum over a multi-index into the sum over two concatenated multi-indices we again have to take into account combinatorial factors,

$$\sum_I C_I = \sum_J \sum_K \frac{J!K!}{(J+K)!} C_{J+K}\,. \tag{3.4}$$

As special case, we have

$$\sum_I C_I = \sum_J \sum_k \frac{1}{(J_k+1)} C_{J,k}\,. \tag{3.5}$$

Further usages of multi-indices will be explained as they occur.

Remark 3.1.10. The Taylor expansion at the point x_0 of an analytic function $(\varphi^1,\dots,\varphi^r):\mathbb{R}^n\to\mathbb{R}^r$ can be written in multi-index notation as

$$\varphi^\alpha(x)=\sum_{|I|=0}^{\infty}\frac{1}{I!}\frac{\partial^{|I|}\varphi^\alpha}{\partial x^I}\bigg|_{x_0}(x-x_0)^I\,,$$

which shows that the jet bundle coordinates of $j^k_m\varphi$ can be identified with the k-th Taylor polynomial of φ^α at $x_0=(x^1(m),\dots x^n(m))$. In this sense, a k-jet can be viewed as the coordinate independent version of the k-th Taylor polynomial.

It is straight-forward to show that the transition functions from one set of jet bundle coordinates to another are smooth (cf. exercise 3.1.2). The conclusion is the following proposition.

Proposition 3.1.11. *Let $F\to M$ be a smooth fibre bundle. Then J^kF has the natural structure of a smooth manifold and $J^kF\to M$ is a smooth fibre bundle.*

For every $k > l \geq 0$ there is a **forgetful map**

$$\mathrm{pr}_{k,l} : J^k F \longrightarrow J^l F\,, \quad j^k_m \varphi \longmapsto j^l_m \varphi\,,$$

which forgets the partial derivatives of order higher than l. In local jet coordinates it is the projection

$$(x^i, u^\alpha, u^\alpha_{i_1}, \ldots, u^\alpha_{i_1,\ldots,i_k}) \longmapsto (x^i, u^\alpha, u^\alpha_{i_1}, \ldots, u^\alpha_{i_1,\ldots,i_l})\,, \tag{3.6}$$

which shows that $\mathrm{pr}_{k,l}$ is a surjective submersion and a map of fibre bundles over M.

3.1.2 Jet evaluation and prolongation

Definition 3.1.12. The map

$$\begin{aligned} j^k : \mathcal{F} \times M &\longrightarrow J^k F \\ (\varphi, m) &\longmapsto j^k_m \varphi \end{aligned}$$

is called the k-th **jet evaluation**.

In general, the jet evaluations are not surjective. For example, when $F \to M$ is a non-trivial principal bundle then F has no global sections at all, so the image of j^k is empty. Another important example is the bundle of lorentzian metrics in general relativity, which does not have a global section if the base manifold is closed with non-vanishing Euler characteristic. This is the reason why jets are defined to be represented by local sections. Here is a criterion for the surjectivity of the jet evaluations.

Lemma 3.1.13. *Let $F \to M$ be a smooth fibre bundle. The jet evaluations j^k, $k \geq 0$ are all surjective if and only if the evaluation j^0 is surjective, i.e. if for every point of F there is a global section through that point.*

Proof. Assume that j^0 is surjective. Then for any k-jet $j^k_m \varphi$ represented by a local section φ, there is a global section $\psi : M \to F$ such that $\psi(m) = \varphi(m)$. We can choose local bundle coordinates (x^i, u^α) on an open neighborhood $U \times V \subset F$ such that φ is defined on U and such that $\varphi(U)$, $\psi(U)$ are both contained in $U \times V$. Furthermore, we can choose the coordinates such that $\psi^\alpha = 0$ on U. Let f be a smooth bump function on U with support contained in U and locally constant value 1 on a small neighborhood of m. Then there is a smooth global section χ defined by $\chi(x) = \psi(x)$ for $x \notin U$ and $\chi^\alpha(x) = f(x)\varphi^\alpha(x)$ for $x \in U$, which satisfies $j^k_m \chi = j^k_m \varphi$. This shows that every k-jet has a preimage under j^k. □

Proposition 3.1.14. *Let $F \to M$ be a smooth fibre bundle with connected fibres. Then the jet evaluation j^k is surjective for all $k \geq 0$ if and only if F has a global section.*

Proof. Assume that $j^k : \mathcal{F} \times M \to J^k F$ is surjective for all $k \geq 0$. Then the image of j^k is non-empty, so that $\mathcal{F}$ must be non-empty.

Conversely, assume that $\varphi \in \mathcal{F}$. Let $p \in F_m$. Since by assumption F_m is path-connected, there is a smooth path $\gamma : [0,1] \mapsto F_m$ with $\gamma(0) = \varphi(m)$ and $\gamma(1) = p$. Let $U \subset M$ be an open neighborhood of m and $F|_U \cong U \times F_m$ a trivialization in which the section φ is constant, i.e. $\varphi(u) = (u, \varphi(m))$ for all $u \in U$. Let $V \subset U$ be an open ball containing m such that the closure of V is contained in U. Then there is a smooth bump function $f : U \to [0,1]$ such that $f(m) = 1$ and $f(u) = 0$ for all $u \in U \setminus V$. Now we can define a local section $\psi : U \to F$ which is given in the trivialization by $\psi(u) = \bigl(u, \gamma(f(u))\bigr)$. By construction, $\psi(m) = p$ and $\psi(u) = \varphi(u)$ for all $u \in U \setminus V$. The section defined by ψ on U and by φ on $M \setminus U$ is a global smooth section of F through p. This shows that j^0 is surjective. It now follows from Lem. 3.1.13 that j^k is surjective for all $k \geq 0$. □

Proposition 3.1.15. *The jet evaluations $\mathcal{F} \times M \to J^k F$ are smooth maps of diffeological spaces.*

Proof. A path $t \mapsto (\varphi_t, m_t) \in \mathcal{F} \times M$ is smooth in the diffeology if $t \mapsto \varphi_t$ is a smooth homotopy of sections given by a smooth map of manifolds $\varphi : \mathbb{R} \times M \to F$ and if $m : \mathbb{R} \to M$ is a smooth map of manifolds.

Let (x^i, u^α) be local bundle coordinates on F. Then $t \mapsto \varphi^\alpha_t = u^\alpha \circ \varphi_t$ and $t \mapsto m^i_t = x^i(m_t)$ are the paths in local coordinates. Let (x^i, u^α_I) be the induced coordinates on $J^k F$, so that

$$\begin{aligned} x^i\bigl(j^k(\varphi_t, m_t)\bigr) &= m^i_t \\ u^\alpha_I\bigl(j^k(\varphi_t, m_t)\bigr) &= \frac{\partial^{|I|}\varphi}{\partial x^I}(t, m_t)\,. \end{aligned} \tag{3.7}$$

By assumption m^i_t is a smooth function of t. Since all partial derivatives of the smooth map of manifolds φ are smooth, the maps $t \mapsto u^\alpha_I\bigl(j^k(\varphi_t, m_t)\bigr)$ are all smooth. We conclude that $\mathbb{R} \to J^k F$, $t \mapsto j^k(\varphi_t, m_t)$ is a smooth map of manifolds. This argument generalizes from paths to smooth families in $\mathcal{F} \times M$ that are parametrized by open subsets of $\mathbb{R}^n$. □

Proposition 3.1.16. *Let φ be a smooth section of the fibre bundle $F \to M$. The map*

$$j^k\varphi : M \longrightarrow J^k F\,, \quad m \longmapsto j^k_m\varphi\,,$$

*is a smooth section of the k-th jet bundle, called the **k-th jet prolongation** of φ.*

Proof. This is easily checked in local jet coordinates in which $j^k\varphi$ is given by

$$u^\alpha_{i_1,\dots,i_k}(j^k\varphi) = \frac{\partial^k\varphi^\alpha}{\partial x^{i_1}\dots\partial x^{i_k}}\,, \tag{3.8}$$

which is a smooth function of the local base coordinates $(x^1, \dots, x^n)$. □

Notation 3.1.17. In the physics literature, the right hand side of Eq. (3.8) often denotes both, the jet bundle coordinates of the prolongation of a single field φ and the coordinates functions $u^\alpha_{i_1,\dots,i_k}$ themselves. This is analogous to the coordinates $(x^1, \dots, x^n)$ of a manifold, which can denote both, the coordinates of a single point x and the coordinate functions of a chart. For example, consider the action in classical mechanics, $S(q) = \int_{\mathbb{R}} L\big(q^\alpha, \dot{q}^\alpha\big)dt$. On the one hand, $S(q)$ can be viewed as the action of a single path $q^\alpha \in C^\infty(\mathbb{R}, Q)$. In this case, the integrand is a closed 1-form on $\mathbb{R}$, which is always exact. On the other hand, during the derivation of the Euler-Lagrange equation, we discard exact terms under the integral. So for the step "discarding exact terms" to be meaningful, we need to view the arguments of $L(q^\alpha, \dot{q}^\alpha)$ as jet coordinate functions rather than as the coordinates of the first prolongation of a single path q^α.

Terminology 3.1.18. A section of a jet bundle of F that is the prolongation of a section of F is also called **holonomic**, and a section that is not a prolongation **non-holonomic**. This language originated historically from the theory of constrained mechanical systems.

Remark 3.1.19. Prop. 3.1.16 allows us to view the k-th jet evaluation equivalently as map

$$j^k : \mathcal{F} \longrightarrow \Gamma^\infty(M, J^kF)\,, \quad \varphi \longmapsto j^k\varphi\,.$$

Proposition 3.1.20. *Let $f : E \to F$ be a map of smooth fiber bundles over M covering the identity on M. Then*

$$j^kf : J^kE \longrightarrow J^kF\,, \quad j^k_m\varphi \longmapsto j^k_m(f \circ \varphi)\,,$$

*is a well-defined smooth map of fiber bundles called the k-**th prolongation** of f.*

Proof. It follows from the chain rule for partial derivatives that $j^k_m(f \circ \varphi)$ depends only on $j^k_m\varphi$, so that j^kf is well-defined. The chain rule also shows that j^kf is smooth. □

Remark 3.1.21. If $E = M$ is the rank 0 fiber bundle over M, a smooth map $E \to F$ covering the identity is a section of F. Its k-th prolongation in the sense of Prop. 3.1.20 is the prolongation in the sense of Prop. 3.1.16.

Let $f : F \to F'$ and $g : F' \to F''$ be maps of smooth fibre bundles over M that cover the identity on M. Let φ be a section of F. Then

$$\begin{aligned} \big(j^k(g \circ f)\big)(j^k_m\varphi) &= j^k_m\big((g \circ f) \circ \varphi\big) = j^k_m\big(g \circ (f \circ \varphi)\big) \\ &= j^kg\big(j^k_m(f \circ \varphi)\big) = j^kg\big(j^kf(j^k_m\varphi)\big) \\ &= (j^kg \circ j^kf)(j^k_m\varphi)\,, \end{aligned}$$

which shows that the jet prolongation is functorial. This can be stated as follows.

Proposition 3.1.22. *Let $\mathcal{F}\mathrm{ib}_M$ denote the category that has smooth fibre bundles over M as objects and smooth bundle maps covering the identity of M as morphisms. The k-th prolongation is a functor $J^k : \mathcal{F}\mathrm{ib}_M \to \mathcal{F}\mathrm{ib}_M$.*

Example 3.1.23. Let $E = \mathbb{R} \times X$ and $F = \mathbb{R} \times Y$ be trivial bundles over $\mathbb{R}$. A smooth map $f : X \to Y$ of the fibres can be viewed as a bundle map $\tilde{f} : (t, x) \mapsto (t, f(x))$. Its first jet prolongation is given by

$$\begin{aligned} j^1\tilde{f} : J^1(\mathbb{R} \times X) \cong \mathbb{R} \times TX &\longrightarrow \mathbb{R} \times TY \cong J^1(\mathbb{R} \times Y) \\ (t, v) &\longmapsto \big(t, Tf(v)\big)\,, \end{aligned}$$

where we have used exercise 3.1.8. This shows that the first jet prolongation of f at a fixed time is the tangent map of f.

3.1.3 The affine structure of jet bundles

Two local sections φ and φ' of $\pi : F \to M$ have the same 1-jet at m if they have the same value $\varphi(m) = \varphi'(m)$ and the same derivative $T_m\varphi = T_m\varphi' : T_mM \to T_{\varphi(m)}F$. Since φ is a section of π, $T_m\varphi$ is a section of $T_{\varphi(m)}\pi : T_{\varphi(m)}F \to T_mM$. It follows that a 1-jet of F is given by a subspace of a tangent space T_pF which $T\pi$ projects bijectively to the tangent space $T_{\pi(p)}M$. By definition, an Ehresmann connection is given by the choice of such a subspace of the tangent space, called the horizontal tangent space, at every point of the bundle. We thus arrive at the following observation.

Observation 3.1.24. An Ehresmann connection of $F \to M$ can be identified with a section of the bundle $J^1F \to F$.

Observation 3.1.24 can be used to express the bundle $J^1F \to F$ in terms of other definitions of connections. A connection can be given by a horizontal lift,

$$h : TM \times_M F \longrightarrow TF\,,$$

i.e. a section of the map $(T\pi, \mathrm{pr}_F) : TF \to TM \times_M F$, where $\pi : F \to M$ and $\mathrm{pr}_F : TF \to F$ are the bundle projections. Let h' be another horizontal lift. Then

$$T\pi\big(h'(v_m, f) - h(v_m, f)\big) = 0\,.$$

It follows that two horizontal lifts differ at each point $p \in F$ by a linear map $T_{\pi(p)}M \to V_pF$, where $VF := \ker T\pi$ is the vertical tangent bundle of F. The vector space of such linear maps can be identified with

$$\mathrm{Hom}(T_{\pi(p)}M, V_pF) \cong T^*_{\pi(p)}M \otimes V_pF\,.$$

We infer that the difference between two horizontal lifts is given by a section of the vector bundle

$$\pi^*(T^*M) \otimes VF \longrightarrow F\,,$$

where $\pi^*(T^*M) := F \times_M T^*M$ denotes the pullback bundle. Returning to observation 3.1.24, we see that the choice of a horizontal lift h, which can be identified with a section of $J^1F \to F$, induces the following isomorphism of bundles over F,

$$\begin{aligned} J^1F &\longrightarrow \pi^*(T^*M) \otimes VF \\ j^1_m\varphi &\longmapsto \big[v_m \mapsto (T_m\varphi)v_m - h\big(v_m, \varphi(m)\big)\big]\,. \end{aligned}$$

We can summarize this in the following proposition.

Proposition 3.1.25. *Let $\pi : F \to M$ be a smooth fibre bundle. The fiber bundle $J^1F \to F$ is an affine bundle modelled on the vector bundle $\pi^*(T^*M) \otimes VF$.*

From Prop. 3.1.25 we recover the well-known fact that the set of connections, which can be identified with the set of sections of $J^1F \to F$, forms an affine space, as we have seen for connections on principal bundles in Prop. 1.2.7 and Prop. 1.2.15. Another consequence is that the sheaf of sections of $J^1F \to F$ is soft. Prop. 3.1.25 can be generalized to the following statement.

Proposition 3.1.26. *Let $F \to M$ be a smooth fibre bundle. For every $k > 0$, the forgetful map $\operatorname{pr}_{k,k-1} : J^kF \to J^{k-1}F$ is an affine bundle modelled on the vector bundle $\pi^*_{k-1}(S^kT^*M)) \otimes \operatorname{pr}^*_{k-1,0}(VF)$, where $\pi_{k-1} : J^{k-1}F \to M$ is the bundle map and $\operatorname{pr}_{k-1,0} : J^{k-1}F \to F$ the forgetful map.*

Prop. 3.1.26 can be proved using jet coordinates, which is somewhat tedious (see e.g. Thm. 5.1.7 and Thm. 6.2.9 in [Sau89]). We will use that $\operatorname{pr}_k : J^kF \to J^{k-1}F$ is naturally embedded as subbundle into the affine bundle $J^1(J^{k-1}F) \to J^{k-1}F$. The embedding is given by the following lemma.

Lemma 3.1.27. *For all $k, l \geq 0$ there is a natural embedding*

$$\iota_{k,l} : J^{k+l}F \longrightarrow J^k(J^lF)\,, \quad j^{k+l}_m\varphi \longmapsto j^k_m(j^l\varphi)\,, \tag{3.9}$$

for all local sections φ.

Proof. The k-th order partial derivatives of the l-th prolongation of a local section φ of $F \to M$ are the $(k+l)$-th oder partial derivatives of φ. This implies that the k-jet of $j^l\varphi$ at m depends only on the $(k+l)$-jet of φ at m, which shows that $\iota_{k,l}$ is well-defined. It is easily checked in local jet coordinate that $\iota_{k,l}$ is an embedding. □

It is instructive to spell out the embedding of Lem. 3.1.27 in local coordinates. Let (x^i, u^α) be local fibre bundle coordinates on $F|_U$ for some open $U \subset M$. These induce jet bundle coordinates as in Eq. (3.1). A local section $\eta : U \to J^lF$ of the l-th jet bundle is given in local coordinates by

$$\eta = (\eta^\alpha, \eta^\alpha_{i_1}, \ldots, \eta^\alpha_{i_1,\ldots,i_l})\,,$$

where $\eta^\alpha_{i_1,\ldots,i_l} = u^\alpha_{i_1,\ldots,i_l} \circ \eta$. Its k-th jet at m is given in coordinates by

$$j^k_m\eta = \begin{pmatrix} \eta^\alpha, & \eta^\alpha_{i_1}, & \ldots, & \eta^\alpha_{i_1,\ldots,i_l} \\ \frac{\partial \eta^\alpha}{\partial x^{j_1}}, & \frac{\partial \eta^\alpha_{i_1}}{\partial x^{j_1}}, & \ldots, & \frac{\partial \eta^\alpha_{i_1,\ldots,i_l}}{\partial x^{j_1}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^k \eta^\alpha}{\partial x^{j_1} \cdots \partial x^{j_k}}, & \frac{\partial^k \eta^\alpha_{i_1}}{\partial x^{j_1} \cdots \partial x^{j_k}}, & \ldots, & \frac{\partial^k \eta^\alpha_{i_1,\ldots,i_l}}{\partial x^{j_1} \cdots \partial x^{j_k}} \end{pmatrix}_m$$

The embedding $\iota_{k,l}$ maps a $(k+l)$-jet $j^{k+l}_m\varphi$ to

$$\iota_{k,l}(j^{k+l}_m\varphi) = \begin{pmatrix} \varphi^\alpha, & \frac{\partial \varphi^\alpha}{\partial x^{i_1}}, & \ldots, & \frac{\partial^l \varphi^\alpha}{\partial x^{i_1} \cdots \partial x^{i_l}} \\ \frac{\partial \varphi^\alpha}{\partial x^{j_1}}, & \frac{\partial^2 \varphi^\alpha}{\partial x^{i_1} \partial x^{j_1}}, & \ldots, & \frac{\partial^{1+l} \varphi^\alpha}{\partial x^{j_1} \partial x^{i_1} \cdots \partial x^{i_l}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^k \varphi^\alpha}{\partial x^{j_1} \cdots \partial x^{j_k}}, & \frac{\partial^{k+1} \varphi^\alpha}{\partial x^{j_1} \cdots \partial x^{j_k} \partial x^{i_1}}, & \ldots, & \frac{\partial^{k+l} \varphi^\alpha}{\partial x^{j_1} \cdots \partial x^{j_k} \partial x^{i_1} \cdots \partial x^{i_l}} \end{pmatrix}_m$$

The prolongation of the forgetful map $j^k\operatorname{pr}_{l,n} : J^k(J^lF) \to J^k(J^nF)$ drops the last $l - n$ columns of the coordinate matrix.

Proof of Prop. 3.1.26. The map

$$\begin{tikzcd}
J^kF \arrow[rr, "\iota_{1,k-1}"] \arrow[dr] & & J^1(J^{k-1}F) \arrow[dl] \\
& J^{k-1}F &
\end{tikzcd}$$

embeds the fibre bundle $E := J^kF \to J^{k-1}F$ into the bundle $J^1(J^{k-1}F)$, which by Prop. 3.1.25 is an affine bundle modelled on the vector bundle $A = \pi^*_{k-1}T^*M \otimes VJ^{k-1}F$. This means that each fibre of E is equipped with a free and transitive action of the additive group of the corresponding fibre of A.

An element $j^1_m\eta \in J^1(J^{k-1}F)$ represented by a local section $\eta : U \to J^{k-1}F$ is in the image of $\iota_{1,k-1}$ iff there is a local section $\varphi : U \to F$ such that

$$\begin{aligned}
&\begin{pmatrix} \eta^\alpha, & \eta^\alpha_{i_1}, & \dots, & \eta^\alpha_{i_1,\dots,i_{k-1}} \\ \frac{\partial\eta^\alpha}{\partial x^{j_1}}, & \frac{\partial\eta^\alpha_{i_1}}{\partial x^{j_1}}, & \dots, & \frac{\partial\eta^\alpha_{i_1,\dots,i_{k-1}}}{\partial x^{j_1}} \end{pmatrix}_m \\
&= \begin{pmatrix} \varphi^\alpha, & \frac{\partial\varphi^\alpha}{\partial x^{i_1}}, & \dots, & \frac{\partial^l\varphi^\alpha}{\partial x^{i_1}\cdots\partial x^{i_{k-1}}} \\ \frac{\partial\varphi^\alpha}{\partial x^{j_1}}, & \frac{\partial^2\varphi^\alpha}{\partial x^{i_1}\partial x^{j_1}}, & \dots, & \frac{\partial^k\varphi^\alpha}{\partial x^{j_1}\partial x^{i_1}\cdots\partial x^{i_{k-1}}} \end{pmatrix}_m .
\end{aligned} \tag{3.10}$$

We have to show that there is a fibre-wise free and transitive action of additive group the vector bundle $B := \pi^*_{k-1}(S^kT^*M)) \otimes \mathrm{pr}^*_{k-1,0}(VF)$ on $\iota_{l,k-1}(J^kF) \subset J^1(J^{k-1}F)$. An element of B is given by a jet $j^{k-1}_m\varphi \in J^{k-1}F$ together with a linear map

$$\theta : S^kTM \longrightarrow V_{\varphi(m)}F .$$

Given such a θ, there is a local section $\psi : U \to F$, such that $j^{k-1}_m\psi = j^{k-1}_m\varphi$ and

$$\frac{\partial^k\psi^\alpha}{\partial x^{j_1}\partial x^{i_1}\cdots\partial x^{i_k}}\bigg|_m = \frac{\partial^k\varphi^\alpha}{\partial x^{j_1}\partial x^{i_1}\cdots\partial x^{i_k}}\bigg|_m + \theta^\alpha_{i_1,\dots,i_k} .$$

This defines a fibre-wise free and transitive action of $\pi^*_{k-1}(S^kT^*M)) \otimes \mathrm{pr}^*_{k-1,0}(VF)$ on J^kF. *** □

3.2 Local maps

3.2.1 Local maps and differential operators

Definition 3.2.1. Let $\mathcal{F} = \Gamma^\infty(M,F)$ and $\mathcal{F}' = \Gamma^\infty(M,F')$ be the sets of sections of smooth fiber bundles $F \to M$ and $F' \to M$. A map $f : \mathcal{F} \to \mathcal{F}'$ is called **local** of jet order k if there is a smooth map $f_0 : J^kF \to F'$, such that the following diagram commutes:

$$\begin{tikzcd}
\mathcal{F} \times M \arrow[r, "f\times\mathrm{id}_M"] \arrow[d, "j^k"'] & \mathcal{F}' \times M \arrow[d, "j^0"] \\
J^kF \arrow[r, "f_0"] & F'
\end{tikzcd} \tag{3.11}$$

Terminology 3.2.2. A local map in the sense of Def. 3.2.1 is also called a **differential operator**, although this terminology is more commonly used when F and F' are trivial vector bundles, so that $\mathcal{F}$ and $\mathcal{F}'$ are function spaces.

Example 3.2.3. The Laplace operator $f = \Delta : C^\infty(\mathbb{R}^3) \to C^\infty(\mathbb{R}^3)$ of example 1.1.5 descends to the map $f_0 : J^2(\mathbb{R}^3 \times \mathbb{R}) \to \mathbb{R}^3 \times \mathbb{R}$ given by

$$f_0 = \big((x^1, x^2, x^3), u_{11} + u_{22} + u_{33}\big)$$

in terms of jet bundle coordinates.

Example 3.2.4. Let $F' = TM \to M$, so that $\mathcal{F}' = \mathfrak{X}(M)$ is the space of vector fields. The product of the space of vector fields is the space of sections

$$\mathfrak{X}(M) \times \mathfrak{X}(M) \cong \Gamma^\infty(M, TM \times_M TM)\,,$$

of the vector bundle $F := TM \times_M TM$. The Lie bracket of vector fields $\mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ is a local map, which descends to J^1F.

Example 3.2.5. A special case for a fibre bundle over M is the trivial bundle $F' = M \overset{\mathrm{id}}{\to} M$, which is the terminal object in fibre bundles over M. The space of fields is given by a point $* = \{\mathrm{id}_M\}$. The terminal map

$$\mathcal{F} \longrightarrow *$$

descends to the bundle map $J^0F = F \to M$, so it is local of jet order 0. Similarly, every point

$$\iota_\varphi : * \hookrightarrow \mathcal{F}$$

mapping $*$ to a field $\varphi \in \mathcal{F}$ descends to the map $\varphi : J^0M = M \to F$, so it is also local of jet order 0.

Example 3.2.6. The map $f : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$ given by

$$f(\varphi) := \sum_{k=0}^{\infty} 2^{-k}\Big(\arctan \circ \frac{\partial^k\varphi}{\partial x^k}\Big)$$

is not local, since the value of $f(\varphi)$ at x depends on derivatives of arbitrarily large order.

Example 3.2.7. A lagrangian $L : \mathcal{F} \to \Omega^{\mathrm{top}}(M)$ is local in the sense of Def. 3.0.1 if it is local in the sense of Def. 3.2.1.

The composition of differential operators on functions on some domain of $\mathbb{R}^n$ is again a differential operator. This suggests that the composition of local maps $f : \mathcal{F} \to \mathcal{F}'$ and $g : \mathcal{F}' \to \mathcal{F}''$ should be local as well. The map f_0 determines the map f of sections uniquely by

$$f(\varphi) = f_0 \circ j^k\varphi\,, \tag{3.12}$$

for all $\varphi \in \mathcal{F}$. Therefore, we can express the composition of f and g in terms of f_0 and g_0. The maps $f_0 : J^kF \to F'$ and $g_0 : J^lF' \to F''$, to which f and g descend by Def. 3.2.1, cannot be composed directly, since the target of f_0 and the source of g_0 do not match. Instead we have to use Eq. (3.12), which yields

$$\begin{aligned}(g \circ f)(\varphi)\big|_m &= g\big(f(\varphi)\big)\big|_m = g_0\big(j^l_m(f(\varphi)\big) = g_0\big(j^l_m(f_0 \circ j^k\varphi)\big) \\ &= (g_0 \circ j^l f_0)\big(j^l_m(j^k\varphi)\big)\,,\end{aligned}$$

where we have used Prop. 3.1.20. The right hand side is not yet a function on some jet bundle of F. This issue is resolved by Lem. 3.1.27, which leads to the following proposition.

Proposition 3.2.8. *The composition of two local maps is a local map.*

Proof. Let $f : \mathcal{F} \to \mathcal{F}'$ and $g : \mathcal{F}' \to \mathcal{F}''$ be local maps, which descend to $f_0 : J^kF \to F'$ and $g_0 : J^lF' \to F''$, respectively. Let $\iota_{l,k} : J^{k+l}F \to J^l(J^kF)$ be the injective immersion of Lem. 3.1.27 and $j^lf_0 : J^l(J^kF) \to J^lF'$ the l-th jet prolongation of f_0. Then we have the following commutative diagram,

$$\begin{tikzcd}
\mathcal{F} \times M \arrow[rr, "f\times \mathrm{id}_M"] \arrow[d, swap, "j^{k+l}"] & & \mathcal{F}' \times M \arrow[r, "g\times \mathrm{id}_M"] \arrow[d, "j^l"] & \mathcal{F}'' \times M \arrow[d, "j^0"] \\
J^{k+l}F \arrow[r, "\iota_{l,k}"] \arrow[d] & J^l(J^kF) \arrow[r, "j^lf_0"] \arrow[dl] & J^lF' \arrow[r, "g_0"] \arrow[d] & F'' \\
J^kF \arrow[rr, "f_0"] & & F' &
\end{tikzcd}$$

where $J^{k+l}F \to J^kF$, $J^l(J^kF) \to J^kF$, and $J^lF' \to F'$ are the obvious forgetful maps. Defining $f_l := j^lf_0 \circ \iota_{l,k}$, we see that $(g \circ f) \times \mathrm{id}_M$ descends to $g_0 \circ f_l$. We conclude that $g \circ f$ is local. □

Remark 3.2.9. Proposition 3.2.8 is a generalized version of the fact that the composition of a k-th order differential operator with an l-th order differential operator is a differential operator of order $k + l$.

Corollary 3.2.10. *Local maps of smooth sections of fibre bundles over a fixed manifold M form a category.*

Let $F \to M$ a fibre bundle and $F' \to M$ a vector bundle. Let $f_0 : J^kF \to F'$ the map to which a differential operator $f : \mathcal{F} \to \mathcal{F}'$ descends. For example, $f_0 = u_{11} + u_{22} + u_{22}$ for the Laplace operator on $\mathbb{R}^3$. A field $\varphi \in \mathcal{F}$ is a solution of the equation

$$f(\varphi) = 0 \tag{3.13}$$

if and only if

$$M \xrightarrow{j^k\varphi} J^kF \xrightarrow{f_0} F'$$

is the zero map. This shows that Eq. (3.13) is a partial differential equation (PDE).

Remark 3.2.11. Finding solutions of a PDE is generally very difficult. It may be easier to first try to find sections $\psi : M \to J^kF$ of the jet bundle such that $f_0 \circ \psi = 0$. Such sections are called **formal solutions** or **non-holonomic solutions** of the PDE. In a second step, we can determine those formal solutions for which $\psi = j^k\varphi$ is the k-th prolongation of a field $\varphi \in \mathcal{F}$, which are, therefore, sometimes called **holonomic solutions**. The images of the tangent maps of the jet prolongations $Tj^k\varphi : TM \to TJ^kF$ of all fields φ define a distribution on J^kF, called the **Cartan distribution**. If we want to extend a point $x \in f^{-1}(0)$ to a holonomic solution on a neighborhood of m, the tangent space $T_xf^{-1}(0) \subset T_xJ^k_mF$ must, therefore, be a subspace of the Cartan distribution. Pursuing this approach leads to Cartan-Kähler theory [BCG+91].

Remark 3.2.12. For some PDEs it can be proved that every formal solution is connected by a homotopy to an actual solution. To show that the PDE has a solution it then suffices to solve it formally, which is generally much easier. This approach is called the homotopy principle, or h-principle [EM02].

Proposition 3.2.13. *The tangent map of a local map is local of the same jet order.*

Proof. Let $f : \mathcal{F} \to \mathcal{F}'$ be a smooth map of fields. Let $t \mapsto \psi_t \in \mathcal{F}$ be a smooth path with $\psi_0 = \varphi$ that represents the tangent vector $\xi_\varphi := \dot{\psi}_0 \in T\mathcal{F} = \Gamma(M, VF)$. Then the smooth path $t \mapsto f(\psi_t)$ represents the tangent vector $(Tf)\xi_\varphi \in T\mathcal{F}' = \Gamma(M, VF')$.

Assume now that f descends to $f_0 : J^kF \to F'$, so that $f(\psi_t) = f_0 \circ j^k\psi_t$. In local coordinates we obtain

$$\begin{aligned}\big((T_\varphi f)\xi_\varphi\big)^\beta(x) &= \frac{d}{dt}\big(f^\beta(\psi_t)\big)(x)\big|_{t=0} \\ &= \frac{d}{dt} f_0^\beta\big(j^k_x\psi_t\big)\big|_{t=0} \\ &= \frac{\partial f_0^\beta}{\partial u^\alpha_I}(j^k_x\varphi)\,\frac{d}{dt}u^\alpha_I\big(j^k_x\psi_t\big)\big|_{t=0} \\ &= \frac{\partial f_0^\beta}{\partial u^\alpha_I}(j^k_x\varphi)\,\frac{\partial^{|I|}\xi^\alpha_\varphi}{\partial x^I}\,.\end{aligned}$$

The right hand side depends only on derivatives of φ^α and ξ^α at x up to k-th order, i.e. only on $j^k_x\xi_\varphi$. □

Corollary 3.2.14. *Let $f : \mathcal{F} \to \mathcal{F}'$ be a local map of jet order k. Let $\varphi \in \mathcal{F}$. Then the linear map $T_\varphi f : T_\varphi\mathcal{F} \to T_{f(\varphi)}\mathcal{F}'$ is local of jet order k.*

Terminology 3.2.15. The linear differential operator $T_\varphi f$ is called the **linearization at** φ of the differential operator f.

3.2.2 Local maps of products

Let $E \to M$ and $F \to M$ be smooth fibre bundles. In example 2.3.14 we have already noted that the product of the spaces of fields is itself a space of fields,

$$\mathcal{E} \times \mathcal{F} \cong \Gamma(M, E \times_M F)\,.$$

The k-th jet bundle of $E \times_M F$ is given by

$$J^k(E \times_M F) \cong J^kE \times_M J^kF\,.$$

Lemma 3.2.16. *Let $E \to M$ and $F \to M$ be smooth fibre bundles. Then the projection $\mathcal{E} \times \mathcal{F} \to \mathcal{E}$, the diagonal $\mathcal{E} \to \mathcal{E} \times \mathcal{E}$, and the flip $\mathcal{E} \times \mathcal{F} \to \mathcal{F} \times \mathcal{E}$ descend to smooth maps of the fibre bundles over M, i.e. they are local of jet order* 0.

Proof. The projection is induced by the fibre-wise projection $E \times_M F \to E$, the diagonal by the fibre-wise diagonal $E \to E \times_M E$ and the flip is by the fibre-wise flip $E \times_M F \to F \times_M E$. □

Lemma 3.2.17. *Let $E \to M$, $F \to M$, $E' \to M$, and $F' \to M$ be smooth fibre bundles. Let $f : \mathcal{E} \to \mathcal{E}'$ and $g : \mathcal{F} \to \mathcal{F}'$ be a maps of the spaces of fields. If f and g are local, then the product map*

$$f \times g : \mathcal{E} \times \mathcal{F} \longrightarrow \mathcal{E}' \times \mathcal{F}'$$

is local.

Proof. By assumption, f descends to $f_0 : J^k E \to E'$ and g descends to a map $g_0 : J^l F \to F'$. Without loss of generality let $k \geq l$. Then g also descends to the map $g_0' = g_0 \circ \mathrm{pr}_{k,l} : J^k F \to F'$. It follows that $f \times g$ descends to the map $h_0 : J^k(E \times_M F) \to E \times_M F$ defined by

$$h_0\big(j^k_m(\psi, \varphi)\big) = \big(f_0(j^k_m \psi), g_0'(j^k_m \varphi)\big)\,,$$

which shows that $f \times g$ is local. □

Lemma 3.2.18. *Let $E \to M$, $F \to M$, and $F' \to M$ be smooth fibre bundles. Let $f : \mathcal{E} \times \mathcal{F} \to \mathcal{F}'$ be a map of spaces of fields. If f is local then there is a $k < \infty$, such that the maps*

$$f(\_\,, \varphi) : \mathcal{E} \longrightarrow \mathcal{F}'$$
$$f(\psi, \_\,) : \mathcal{F} \longrightarrow \mathcal{F}'$$

are local of jet order k for all $\varphi \in \mathcal{F}$ and $\psi \in \mathcal{F}$.

Proof. The map $f(\_\,, \varphi)$ is given by the composition

$$\mathcal{E} \cong \mathcal{E} \times * \xrightarrow{\mathrm{id}_{\mathcal{E}} \times \iota_\varphi} \mathcal{E} \times \mathcal{F} \xrightarrow{f \times g} \mathcal{F}'\,,$$

where ι_φ is the inclusion of φ of example 3.2.5. Since $\mathrm{id}_{\mathcal{E}}$ and ι_φ are local, their product is local by Lem. 3.2.17. Since $\mathrm{id}_{\mathcal{E}} \times \iota_\varphi$ and f are local, their composition $f(\_\,, \varphi)$ is local by Prop. 3.2.8. An analogous argument shows that $f(\psi, \_\,)$ is local, too. □

3.2.3 Linear local maps of jet order 0 and 1

Assume that $A \to M$ and $B \to M$ are vector bundles. Let $D : \mathcal{A} \to \mathcal{B}$ be a k-th order local map, so it descends to a map $D_0 : J^k A \to B$ for some $k \geq 0$. D is linear if and only if D_0 is in local jet coordinates of the general form

$$D_0^\beta = \sum_{|I|=0}^{k} D_\alpha^{\beta I}(x)\, u_I^\alpha\,,$$

where (x^i, u^α) are local vector bundle coordinates on $A|_U$ for some $U \subset M$, where (x^i, v^β) are coordinates on $B|_U$, and where the $D_\alpha^{\beta I}$ are smooth functions on U. The linear map D is given in terms of these functions by

$$(Da)^\beta = \sum_{|I|=0}^{k} D_\alpha^{\beta I} \frac{\partial^{|I|} a^\alpha}{\partial x^I}\,. \tag{3.14}$$

Proposition 3.2.19. *A linear map $D : \mathcal{A} \to \mathcal{B}$ of sections of vector bundles is induced by a map $D_0 : A \to B$ of vector bundles if and only if it is $C^\infty(M)$-linear, i.e.*

$$D(fa) = f\,Da$$

for all $a \in \mathcal{A}$ and $f \in C^\infty(M)$.

Proof. This can be deduced directly from Eq. (3.14). □

Proposition 3.2.20. *A linear map $D : \mathcal{A} \to \mathcal{B}$ of sections of vector bundles is a first order differential operator if and only if there is a vector bundle map $P : A \to B \otimes TM$, such that*

$$D(fa) = f\,Da + \langle P(a), df \rangle \tag{3.15}$$

for all $a \in \mathcal{A}$ and $f \in C^\infty(M)$.

Proof. Assume that D is a linear first order local map. By Eq. (3.14) it is given in local coordinates by

$$(Da)^\beta = D^\beta_\alpha a^\alpha + D^{\beta i}_\alpha \frac{\partial a^\alpha}{\partial x^i} \,. \tag{3.16}$$

It follows that

$$\big(D(fa)\big)^\beta = D^\beta_\alpha f a^\alpha + f\, D^{\beta i}_\alpha \frac{\partial a^\alpha}{\partial x^i} + a^\alpha D^{\beta i}_\alpha \frac{\partial f}{\partial x^i} \,.$$

So if we define P in local coordinates by

$$P(a)^\beta := a^\alpha D^{\beta i}_\alpha \frac{\partial}{\partial x^i} \,, \tag{3.17}$$

then Eq. (3.15) follows.

Conversely, assume that Eq. (3.15) holds. Let σ_α be the basis of local sections of A such that $u^\alpha(\sigma_{\alpha'}) = \delta^\alpha_{\alpha'}$ and let τ_β be the basis of local sections of B such that $v^\beta(\tau_{\beta'}) = \delta^\beta_{\beta'}$. Let D^α_β be the unique local functions, such that

$$D(\sigma_\alpha) = D^\beta_\alpha \tau_\beta \,.$$

P be given in local coordinates by (3.17) for some local functions $D^{\beta i}_\alpha$. A general local section is of the form $a = a^\alpha \sigma_\alpha$. Using Eq. (3.15), we get

$$\begin{aligned} D(a) = D(a^\alpha \sigma_\alpha) &= a^\alpha D(\sigma_\alpha) + \langle P(\sigma_\alpha), a^\alpha \rangle \\ &= a^\alpha D^\beta_\alpha + D^{\beta i}_\alpha \frac{\partial a^\alpha}{\partial x^i} \,, \end{aligned}$$

which has the form of a linear first order local map. □

3.2.4 Generalized local maps

The class of local maps defined in Def. 3.2.1 is not general enough for our purposes. In a first step, we can relax the condition that f_0 is a bundle map that covers the identity on M, by replacing id_M in Def. 3.2.1 with some diffeomorphism f_M on M. But this is still not general enough to cover diffeomorphism symmetries, such as the action of diffeomorphisms on differential forms by pullback.

Let $f : \mathcal{F} \to \mathcal{F}'$ be a map of fields. For f to be local in a generalized sense, the first requirement is that it can be extended by some map $f_{M'} : \mathcal{F} \times M \to M'$ to a map

$$\begin{aligned} \tilde{f} : \mathcal{F} \times M &\longrightarrow \mathcal{F}' \times M' \\ (\varphi, m) &\longmapsto \big(f(\varphi), f_{M'}(\varphi, m)\big) , \end{aligned}$$

that descends to a map $f_0 : J^kF \to F'$ on a jet bundle. For every $\varphi \in \mathcal{F}$ we have the following commutative diagram

$$\begin{tikzcd}
\mathrm{M} \cong \{\varphi\} \times M \arrow[r] \arrow[dr, "j^k\varphi"] \arrow[ddrr, bend right, "f_{M'}(\varphi)"'] & \mathcal{F} \times M \arrow[r, "\tilde{f}"] \arrow[d] & \mathcal{F}' \times M' \arrow[d, "j^0"] \\
& J^kF \arrow[r, "f_0"] \arrow[dr] & F' \arrow[d, "\pi'"] \\
& & M'
\end{tikzcd}$$

which shows how the map $f_{M'}$ can be reconstructed from f_0.

Definition 3.2.21. Let $\pi : F \to M$ and $\pi' : F' \to M'$ be smooth fibre bundles. Let $f_0 : J^kF \to F'$ be a smooth map of jet manifolds, not necessarily a bundle map. The map

$$\begin{aligned} f_{M'} : \mathcal{F} &\longrightarrow C^\infty(M, M') \\ \varphi &\longmapsto \pi' \circ f_0 \circ j^k\varphi \end{aligned}$$

will be called the **base map** induced by f_0. The set of fields that are mapped by $f_{M'}$ to a diffeomorphism on the base will be denoted by

$$\mathcal{F}_{\text{diff}} := f_{M'}^{-1}\big(\mathrm{Diff}(M, M')\big) .$$

Lemma 3.2.22. *Let $f_0 : J^kF \to F'$ be a smooth map of jet manifolds. Then*

$$\begin{aligned} f_{\mathcal{F}'} : \mathcal{F}_{\text{diff}} &\longrightarrow \mathcal{F}' \\ \varphi &\longmapsto f_0 \circ j^k\varphi \circ f_{M'}(\varphi)^{-1} , \end{aligned} \tag{3.18}$$

is the unique map that makes the diagram

$$\begin{tikzcd}
\mathcal{F}_{\text{diff}} \times M \arrow[r, "(f_{\mathcal{F}'}, f_{M'})"] \arrow[d, "j^k"'] & \mathcal{F}' \times M' \arrow[d, "j^0"] \\
J^kF \arrow[r, "f_0"] & F'
\end{tikzcd}$$

commute.

Proof. The commutativity of the diagram means that for every $\varphi \in \mathcal{F}_{\text{diff}}$

$$f_0(j^k_m\varphi) = j^0\big(f_{\mathcal{F}'}(\varphi), f_{M'}(\varphi, m)\big) ,$$

which can be written as equality of smooth maps from M to F',

$$f_0 \circ j^k\varphi = f_{\mathcal{F}'}(\varphi) \circ f_{M'}(\varphi) .$$

Since $\varphi \in \mathcal{F}_{\text{diff}}$, the map $f_{M'}(\varphi)$ is a diffeomorphism, so we can compose with its inverse on the right. This yields (3.18). □

Terminology 3.2.23. The map $f_{\mathcal{F}'}$ of lemma 3.2.22 will be called the **lift of** f_0.

Example 3.2.24. Let $F = M \times M' \to M$ and $F' = M' \times M \to M'$ be trivial bundles. The sets of sections are $\mathcal{F} \cong C^\infty(M, M')$ and $\mathcal{F}' \cong C^\infty(M', M)$. Let now $f_0 : j^0 F \to F'$ be the flip $f_0(m, m') = (m', m)$. The base map induced by f_0 is the identity $f_{M'}(\Phi) = \Phi$, so that $\mathcal{F}_{\mathrm{diff}} = \mathrm{Diff}(M, M')$. It follows that the lift of f_0 is $f_{\mathcal{F}}(\Phi) = \Phi^{-1}$.

The map f_0 only lifts to a map on the subset $\mathcal{F}_{\mathrm{diff}}$ or, more generally, on any subset $\mathcal{D} \subset \mathcal{F}_{\mathrm{diff}}$. When we restrict the domain of the map of fields to such $\mathcal{D}$, we can also restrict the domain of f_0 to a submanifold of $D^k \subset J^k F$ as long as it contains $j^k(\mathcal{D} \times M)$.

Proposition 3.2.25. *Let $F \to M$ and $F' \to M'$ be smooth fibre bundles and $\mathcal{D} \subset \mathcal{F}$ a subset. Let $f_0 : D^k \to F'$ be a smooth map defined on an embedded submanifold $D^k \subset J^k F$, such that*

(i) $j^k(\mathcal{D} \times M) \subset D^k$;

(ii) For every $\varphi \in \mathcal{D}$, the map $f_{M'}(\varphi) = \pi' \circ f_0 \circ j^k \varphi : M \to M'$ is a diffeomorphism.

Then there is a unique map $f_{\mathcal{F}'} : \mathcal{D} \to \mathcal{F}'$, such that the diagram

$$\begin{tikzcd}
\mathcal{D} \times M \arrow[r, "(f_{\mathcal{F}'}, f_{M'})"] \arrow[d, "j^k"'] & \mathcal{F}' \times M' \arrow[d, "j^0"] \\
D^k \arrow[r, "f_0"] & F'
\end{tikzcd}$$

commutes.

Proof. The maps $f_{M'}$ and $f_{\mathcal{F}'}$ are defined exactly as the base map of Def. 3.2.21 and the lift of f_0 of Lem. 3.2.22. The proof of the commutativity of the diagram is as in Lem. 3.2.22. □

Terminology 3.2.26. The map $f_{\mathcal{F}'}$ of Prop. 3.2.25 is called the **lift of** f_0.

Let $f_0 : J^k F \to F'$ be a smooth map of manifolds. For every local section $\varphi : U \to F$ we have a map

$$f_{M'}(\varphi) := \pi' \circ f_0 \circ j^k \varphi : U \longrightarrow M' .$$

By the inverse function theorem, this map is a local diffeomorphism at $m \in U$ if and only if its tangent map at m is a bijection. By the chain rule, this is a condition on $j^{k+1}_m \varphi$. We denote,

$$(J^{k+1} F)_{\mathrm{diff}} := \{ j^{k+1}_m \varphi \in J^{k+1} F \mid \varphi \text{ is a local diffeomorphism at } m \} .$$

For $l > 1$ we define

$$(J^{k+l} F)_{\mathrm{diff}} := \mathrm{pr}^{-1}_{k+l,k+1}\big((J^{k+1} F)_{\mathrm{diff}} \big) .$$

Definition 3.2.27. Let $f_0 : J^k F \to F'$ be a smooth map. The map

$$\begin{aligned} f_l : (J^{k+l}F)_{\text{diff}} &\longrightarrow J^l F' \\ j^{k+l}_m \varphi &\longmapsto j^l_m \big(f_0 \circ j^k \varphi \circ f_{M'}(\varphi)^{-1} \big) \end{aligned}$$

is called the **l-th prolongation** of f_0.

The prolongations form a commutative diagram

$$\begin{tikzcd} J^k F \arrow[d, "f_0"] & (J^{k+1}F)_{\text{diff}} \arrow[l] \arrow[d, "f_1"] & (J^{k+2}F)_{\text{diff}} \arrow[l] \arrow[d, "f_2"] & \cdots \arrow[l] \\ F' & J^1 F' \arrow[l] & J^2 F' \arrow[l] & \cdots \arrow[l] \end{tikzcd}$$

Furthermore, we observe that

$$j^{k+l}(\mathcal{F}_{\text{diff}} \times M) \subset (J^{k+l}F)_{\text{diff}} \, .$$

Definition 3.2.28. Let $\pi : F \to M$ and $\pi' : F' \to M'$ be smooth fibre bundles. Let $\mathcal{D} \subset \mathcal{F}$ and $\mathcal{D} \subset \mathcal{F}'$ be subsets. A map $f : \mathcal{D} \to \mathcal{D}'$ is called **generalized local** if

(i) f is the lift of a smooth map $f_0 : D^k \to F'$ defined on a submanifold $D^k \subset J^k F$;

(ii) $D^{k+l} := j^{k+l}(\mathcal{D} \times M) \subset J^{k+l}F$ is a submanifold for all $l > 0$.

Proposition 3.2.29. *The composition of local maps in the sense of Def. 3.2.28 is local.*

Proof. Let $f : \mathcal{D} \to \mathcal{D}'$ and $g : \mathcal{D}' \to \mathcal{D}''$ be generalized local maps that are lifts of $f_0 : D^k \to F'$ and $g_0 : D'^l \to F''$, respectively. Since $\mathcal{D}$ is by assumption a subset of $\mathcal{F}_{\text{diff}}$ it follows that $D^{k+l} \subset (J^{k+l}F)_{\text{diff}}$, so that the l-th prolongation of f_0 is defined on D^{k+l}. Since f is the lift of f_0, we have

$$f_l(j^{k+l}_m \varphi) = j^l_{f_{M'}(\varphi,m)}\big(f(\varphi)\big) \, .$$

This implies that the image of f_l lies in $j^l(\mathcal{D}' \times M') \subset D'^l$ and that the following diagram commutes

$$\begin{tikzcd} \mathcal{D} \times M \arrow[r, "{(f, f_{M'})}"] \arrow[d, "j^{k+l}"'] & \mathcal{D}' \times M' \arrow[d, "j^l"] \\ D^{k+l} \arrow[r, "f_l"] & D'^l \end{tikzcd}$$

We conclude that $g \circ f$ is the lift of $g_0 \circ f_l$. □

***

3.3 The theorems of Peetre and Slovák

3.3.1 Locality in topology

In topology, "local" roughly means "compatible with the restriction to open subsets". In this sense, a map $f : \mathcal{F} \to \mathcal{F}'$ of sections of fibre bundles is considered to be local if the restriction of $f(\varphi)$ to any open subset $U \in M$ depends only on the restriction of φ to U. Let $\hat{\mathcal{F}}$ denote the sheaf of sections, given by

$$\hat{\mathcal{F}}(U) := \Gamma^\infty(U, F|_U)\,,$$

for every open $U \subset M$. The set of global sections is $\mathcal{F} = \hat{\mathcal{F}}(M)$. A morphism of sheaves is given by a map $\hat{f}_U : \hat{\mathcal{F}}(U) \to \hat{\mathcal{F}}'(U)$ for every open subset $U \in M$ that commutes with the restrictions to every open subset $V \subset U$, i.e. the following diagram commutes.

$$\begin{tikzcd}
\hat{\mathcal{F}}(U) \arrow[r, "\hat{f}_U"] \arrow[d, "\mathrm{res}_{U,V}"'] & \hat{\mathcal{F}}'(U) \arrow[d, "\mathrm{res}'_{U,V}"] \\
\hat{\mathcal{F}}(V) \arrow[r, "\hat{f}_V"] & \hat{\mathcal{F}}'(V)
\end{tikzcd}$$

A map $f : \mathcal{F} \to \mathcal{F}'$ ought to be considered to be local in the sense of topology if there is a morphism of sheaves $\hat{f} : \hat{\mathcal{F}} \to \hat{\mathcal{F}}'$ such that $f = \hat{f}_M$.

Proposition 3.3.1. *If $f : \mathcal{F} \to \mathcal{F}'$ is local (in the sense of Def. 3.2.1), then it is induced by a morphisms of sheaves.*

Proof. Let $f_0 : J^k F \to F'$ be the map f descends to. Let

$$\hat{f}_U(\varphi) := f_0 \circ j^k \varphi$$

for all $\varphi \in \Gamma^\infty(U, F|_U)$. The restrictions of the jet prolongation $j^k|_U : \Gamma^\infty(U, F|_U) \to \Gamma^\infty(U, J^k F|_U)$ define a morphism of sheaves; and the morphism of fibre bundles f_0 induces a morphism of the sheaves of sections. Therefore, the composition is a morphism of sheaves. □

Let $f : \mathcal{F} \to \mathcal{F}'$ be induced by a morphism of sheaves. Then for every $m \in M$, the restriction of $f(\varphi)$ to a neighborhood U of m depends only on the restriction of f to U. Since the neighborhood U is arbitrarily small, it follows that the value of $f(\varphi)$ at m depends only on the germ of f at m.

Recall that the germ of a function φ at m is the equivalence class of functions ψ that have a the same restriction $\psi|_U = \varphi|_U$ to some neighborhood U of m. If two functions have the same germ, then they have the same partial derivatives to all orders. The converse is clearly not true. For example, the derivatives of the function $\varphi(x) = \exp(-1/x^2)$ on the real line are all zero at $x = 0$, so it has the same jets as $\psi(x) = 0$, but φ and and ψ do not have the same germ at 0. The germ of a section φ of a fibre bundle at some point m contains more information about the function than the jet $j^k_m \varphi$. Therefore, the condition that $f(\varphi)_m$ depends only on the germ of φ at m is weaker than the condition that it depends on a finite jet, as required by the definition 3.2.1 of locality.

3.3.2 Peetre's theorem

Surprisingly, with rather mild additional assumptions a map $f : \mathcal{F} \to \mathcal{F}'$ that is induced by a morphism of sheaves is local (Def. 3.2.1). We first consider the linear case.

Theorem 3.3.2 (Peetre). *Let $A \to M$ and $B \to M$ be vector bundles over a compact base. Let $D : \mathcal{A} \to \mathcal{B}$ be a linear map. If D is induced by a map of sheaves in vector spaces, then it is local.*

Lem. 3.1.13 implies that all jet evaluations $j^k : \mathcal{A} \times M \to J^k A$ are surjective. It follows, that if the map $D : \mathcal{A} \to \mathcal{B}$ descends to a map $J^r A \to B$, then this map must be given by

$$\begin{aligned} D_0 : J^k A &\longrightarrow B \\ j^k_m \varphi &\longmapsto (D\varphi)(m) \,. \end{aligned} \tag{3.19}$$

In the first step, we have to show that the map (3.19) is well defined. For this we will use the following lemma.

Lemma 3.3.3. *Let $D : C^\infty(\mathbb{R}^n, \mathbb{R}^p) \to C^\infty(\mathbb{R}^n, \mathbb{R}^q)$ be a support non-increasing linear map. Then for every point $x \in \mathbb{R}^n$ and every real constant $c > 0$ there is a neighborhood U of x and a natural number $r \geq 0$, such that for all $y \in U \setminus \{x\}$ and $\varphi \in C^\infty(\mathbb{R}^n, \mathbb{R}^p)$ the condition $j^r_y \varphi = 0$ implies $\|(D\varphi)(y)\| \leq c$.*

Proof. Assume that the statement is false. This means that there is a point $x \in \mathbb{R}^n$ and a constant $c > 0$, such that for every neighborhood U of x and every $r \geq 0$ there is a $y \in U$, $y \neq x$ and a $\varphi \in C^\infty(\mathbb{R}^n, \mathbb{R}^p)$, such that $j^k_y \varphi = 0$ and $\|(D\varphi)(x)\| > c$. By choosing a sequence of shrinking neighborhoods $U_0 \supset U_1 \supset \ldots$ with $\bigcap_k U_k = \{x\}$, we can find a sequence $y_k \to x$ and a sequence $\varphi_k \in \mathcal{A}$, such that $j^k_{y_k} \varphi_k = 0$ and $\|(D\varphi_k)(y_k)\| > c$.

By selecting a suitable subsequence, the relations $\|y_k - x\| \leq 4\|y_k - x_j\|$ can be satisfied for all $k > j$. Let us choose smooth maps $\psi_k \in C^\infty(\mathbb{R}^n, \mathbb{R}^p)$ that have the same germ as φ_k at y_k and are zero outside of the ball of radius $\frac{1}{2}$ around y_k. Since the germs are the same, so are the jets $j^k_{y_k} \psi_k = j^k_{y_k} \varphi_k = 0$. Because the jets at y_k are zero, the functions ψ_k can be chosen such that their partial derivatives are bounded in the supremum norm by

$$\left\| \frac{\partial^{|I|} \psi_k}{\partial x^I} \right\|_{\sup} \leq 2^{-k} \,,$$

for all multi-indices I of order $|I| \leq k$. Due to this condition, the map defined point-wise by

$$\psi(y) := \sum_{l=0}^{\infty} \psi_{2l}(y)$$

for all $y \in \mathbb{R}^n$ is smooth. By construction, the points y_{2l+1} lie outside of the support of ψ. By assumption, D is support non-increasing so that y_{2l+1} also lies outside of the support of $D\psi$,

$$(D\psi)(y_{2l+1}) = 0 \,.$$

Since D is support non-increasing, $(D\psi)(y_{2l})$ only depends on the germ of ψ_{2l} at y_{2l} which is equal to the germ of φ_{2l} at y_{2l}, so that

$$(D\psi)(y_{2l}) = (D\varphi)(y_{2l}) .$$

It follows that $y_k \to x$ is a convergent sequence, such that

$$\|(D\psi)(y_{2l})\| > c , \qquad \|(D\psi)(y_{2l+1})\| = 0 ,$$

which shows that $D\psi$ is not continuous at x. This is a contradiction to the assumption that the lemma does not hold. □

In order to show that the D_0 is smooth, we will use Boman's theorem.

Theorem 3.3.4. *Let $f : \mathbb{R}^m \to \mathbb{R}^n$ be a map, such that such that for every smooth path $\gamma : \mathbb{R} \to \mathbb{R}^m$ the path $f \circ \gamma : \mathbb{R} \to \mathbb{R}^n$ is smooth. Then f is smooth.*

Proof. The original proof is in [Bom67]. A more pedagogic proof is found in Thm. 3.4 in [KM97]. □

Proof of Thm. 3.3.2. Choose $c = 1$ and apply Lem. 3.3.3 in a coordinate neighborhood of every point $m \in M$. This yields a cover of neighborhoods U_i with jet orders r_i as in the lemma. Since M is compact, we can choose a finite subcover. Let $r < \infty$ be the maximum of the r_i. Then condition $j^r_m\varphi = 0$ implies $\|(Df)(m)\| < 1$ for all $m \in M$.

Let $j^k_m\varphi = 0$ and assume that $\|(D\varphi)(m)\| = \varepsilon > 0$. Then $j^k_m(\frac{\varepsilon}{2}\varphi) = 0$, but $\|(D\frac{2}{\varepsilon}\varphi)(m)\| = 2 > 1$, which is a contradiction, so that $(D\varphi)(m) = 0$. It follows, that (3.19) is a well defined fibre-wise linear map.

It remains to show that D_0 is smooth. As can be easily seen in local coordinates, every smooth path in J^rA can be written as $t \mapsto j^r_{m_t}\varphi$, where $t \mapsto \varphi_t$ is a smooth family of sections of A and $t \mapsto m_t$ a smooth path in M. Since D is linear, $D\varphi_t$ is a smooth family of smooth maps. It follows that $t \mapsto (D\varphi_t)(m_t)$ is a smooth path. This shows that every smooth path $j^r_{m(t)}\varphi_t$ in J^rA is mapped by D_0 to a smooth path in B. It now follows from Boman's theorem 3.3.4 that D_0 is smooth. □

3.3.3 The nonlinear case

Theorem 3.3.5 (Slovák)**.** *Let $F \to M$, $F' \to M$ be smooth fiber bundles. Let $f : \mathcal{F} \to \mathcal{F}'$ be induced by a morphism of sheaves of diffeological spaces. Then for every $\varphi \in \mathcal{F}$ and every $m \in M$ there is an open neighborhood $U \ni m$ and an open subbundle $E \subset F|_U$ containing $\varphi(U)$, such that the restricted map $f|_{\mathcal{E}}$ is local in the sense of Def. 3.2.1.*

The original proof, which is quite involved, can can be found in [Slo88]. A more pedagogic presentation is in [KMS93]. There is a somewhat modernized formulation of the theorem in [NS]. For a recent discussion of the Peetre-Slovák theorem in relation to field theory, we refer the reader to Appendix A in [KM16, Appendix A].

The original statement of Slovák is somewhat more general. It allows for the basis of the target bundle F' to be a different manifold $M' \neq M$ and assumes that there is a map $\eta : M' \to M$ such that $f(\varphi)|_{m'}$ depends only on the germ of φ at $\eta(m')$ for all $m' \in M'$. But this is the same as saying that there is a morphism of sheaves from the pullback sheaf $\eta^*\hat{\mathcal{F}}$ to $\hat{\mathcal{F}}'$. ***

The condition that f maps smooth families of sections to smooth families of sections is called "regularity" in [Slo88, KMS93]. Here, we just restated regularity in terms of the natural diffeological structure on $\mathcal{F}$.

Corollary 3.3.6. *Let $F \to M$, $F' \to M$ be smooth fiber bundles. Let F be compact. Then a map $f : \mathcal{F} \to \mathcal{F}'$ is local if and only if it is induced by a morphism of sheaves in diffeological spaces.*

A casual way of rephrasing Cor. 3.3.6 is by saying that for sections of compact fibre bundles smooth sheaf-locality is the same as jet-locality. In the non-compact case the jet order may be only locally but not globally finite, so that Def. 3.2.1 is a stronger version of locality. It is debatable, whether global or local finiteness of the jet order is the more appropriate condition in field theory ***. Ultimately, this will depend on and be justified by the application.

We will not give a proof of Thm. 3.3.5. But we will state an important technical step, which is interesting in its own right: The Whitney extension theorem gives the exact conditions for a collection of functions on a closed subset of $\mathbb{R}^n$ to be the partial derivatives of a smooth function on $\mathbb{R}^n$.

Theorem 3.3.7. *Let $K \subset \mathbb{R}^n$ be a closed set. Let $\varphi_I : K \to \mathbb{R}$ be continuous functions defined for all multi-indices $I \in \mathbb{N}_0^n$. The following are equivalent:*

(i) For every $r \geq 0$

$$\varphi_I(b) = \sum_{|J| \leq r} \frac{1}{J!} \varphi_{I+J}(a)(b-a)^J + o(|b-a|^r) \tag{3.20}$$

holds uniformly for $|b-a| \to 0$, $a, b \in K$.

(ii) There is a smooth function $\varphi \in C^\infty(\mathbb{R}^n)$ such that

$$\varphi_I = \frac{\partial^{|I|}\varphi}{\partial x^I}\Big|_K .$$

Proof. The original proof where K was assumed to be compact is in [Whi34]. It was first observed in [Bie80] that K being closed is sufficient. For a more pedagogic proof see [HÖ3]. □

The condition (3.20) for the functions φ_I imply that $\varphi_I = \frac{\partial^{|I|}\varphi}{\partial x^I}$ in the interior of K. Conversely, if φ is a some smooth function and $\varphi_I = \frac{\partial^{|I|}\varphi}{\partial x^I}$, then (3.20) follows from Taylor's theorem. This shows that Eq. (3.20) is always satisfied in the interior of K.

When $K = *$ is a point, condition (3.20) is always satisfied, which implies that *any* collection of real numbers c_I for all multi-indices I can be realized as partial derivatives of a smooth function. This is the content of the Borel lemma. In its simplest form it can be stated as follows.

Lemma 3.3.8. *For any infinite sequence of real numbers $c_0, c_1, c_2, \ldots$ there is a smooth function $\varphi \in C^\infty(\mathbb{R})$, such that $c_n = \frac{d^n\varphi}{dx^n}\big|_{x=0}$.*

3.4 Infinite jets

A local map of fields descends to a map on the manifold of jets of a finite but arbitrarily large order. When two local maps are composed, their jet orders are added. So even though we can describe a single local map in terms of a map on a finite jet manifolds, we need the jet manifolds of all orders to deal with the category of all local maps. This suggests the following definition.

Definition 3.4.1. Two local sections φ and φ' of a smooth fiber bundle $F \to M$ defined on a neighborhood of m have the same **infinite jet** or **∞-jet at** m, denoted by $j^\infty_m \varphi = j^\infty_m \varphi'$, if they have the same k-jet at m for all $k \geq 0$.

Since having the same k-jet at m is an equivalence relation on the set of local sections, having the same ∞-jet is an equivalence relation as well. An ∞-jet is an equivalence class for this relation. The set of all ∞-jets will be denoted by $J^\infty F$.

Given local bundle coordinates (x^i, u^α), $j^\infty_m \varphi$ is uniquely determined by the coordinates $x^i(m)$ of the base point and the jet coordinates

$$u^\alpha_I(j^\infty_m \varphi) = \frac{\partial^{|I|} \varphi^\alpha}{\partial x^I}\bigg|_m$$

for all α and all multi-indices I. Conversely, the Borel lemma 3.3.8 tells us that given numbers c^α_I for all α and I, there is a local section such that $u^\alpha_I(j^\infty_m \varphi) = c^\alpha_I$. In this sense, the infinite collection $\{x^i, u^\alpha, u^\alpha_{i_1}, \ldots\}$ of real valued functions on $J^\infty F$ can be viewed as a set of coordinates.

For every $k \geq 0$, there are natural forgetful maps of sets $\mathrm{pr}_{\infty,k} : J^\infty F \to J^k F$, $j^\infty_m \varphi \mapsto j^k_m \varphi$. The forgetful maps satisfy $\mathrm{pr}_{k,k-1} \circ \mathrm{pr}_{\infty,k} = \mathrm{pr}_{\infty,k-1}$, so they fit in to the commutative cone

$$\begin{tikzcd}
 & & J^\infty F \arrow[dll] \arrow[dl] \arrow[d] \arrow[dr] & \\
F & J^1 F \arrow[l] & J^2 F \arrow[l] & \ldots \arrow[l]
\end{tikzcd}$$

As can be easily seen in jet coordinates, any other cone over the diagram $F \leftarrow J^1 F \leftarrow J^2 F \leftarrow \ldots$ induces a unique map to $J^\infty F$, which shows that $J^\infty F$ is the categorical limit of the sequence of the sets of finite jets.

How do we equip $J^\infty F$ with a differentiable structure? Since the dimension of the jet manifolds $J^k F$ increases with k, the limit of the sequence of the jet manifolds $J^k F$ cannot exist in the category of finite dimensional manifolds. In order to make sense of this limit we, therefore, have to embed $\mathcal{M}\mathrm{fld}$ as subcategory into an ambient category $\mathcal{C}$ in which such limits exist. Let us write down a wish list of some of the properties this category should have.

Wish list 3.4.2. A good category $\mathcal{C}$ for $J^\infty F$ should have the following properties:

(i) There is an injective and fully faithful functor $I : \mathcal{M}\mathrm{fld} \to \mathcal{C}$.

(ii) For every infinite inverse sequence of manifolds $X_0 \leftarrow X_1 \leftarrow \ldots$ the limit $\check{X} := \lim(I(X_0) \leftarrow I(X_1) \leftarrow \ldots)$ exists in $\mathcal{C}$.

(iii) Given a limit $\check{X}$ as in (ii), every morphism $\check{X} \to I(Y)$ to a manifold Y factors as $\check{X} \to I(X_k) \xrightarrow{I(f)} I(Y)$ through a smooth map $f : X_k \to Y$.

(iv) There is a faithful functor $\check{U} : \mathcal{C} \to \mathcal{S}\mathrm{et}$, such that for every limit $\check{X}$ as in (ii) there is a natural isomorphism $\check{U}(\check{X}) \cong \lim_{i \in \mathcal{I}} \mathrm{Hom}_{\mathcal{M}\mathrm{fld}}(*, X_i)$ of sets.

Let us motivate this wish list. Property (i) states that $\mathcal{M}$fld can be embedded as full subcategory into $\mathcal{C}$. Property (ii) ensures that the limit $J^\infty F := \lim(I(J^0 F) \leftarrow I(J^1 F) \leftarrow \ldots)$ exists as a limit of smooth manifolds in $\mathcal{C}$. Property (iii) means that a morphism out of the limit object $J^\infty F$ in $\mathcal{C}$ is given by a smooth map on a finite jet manifold, so that the maps out of $J^\infty F$ are precisely the local maps. Finally, property (iv) requires $\mathcal{C}$ to have the structure of a concrete category that is compatible with the concrete structure on $\mathcal{M}$fld. This will ensure that the limit object $J^\infty F$ in $\mathcal{C}$ has as underlying set the set of infinite jets as defined in Def. 3.4.1. Constructing a category that satisfies these conditions is the goal of the next chapter.

Exercises

Exercise 3.1 (Dimension of jet manifolds)**.** Let $F \to M$ be a smooth fiber bundle with $\dim F = p + q$ and $\dim M = p$. Compute the dimension of $J^k F$.

Exercise 3.2 (Jet bundles of vector bundles)**.** Let $A \to M$ and $B \to M$ be smooth vector bundles. Show the following:

(a) $J^k A \to M$ and $J^k B \to M$ are vector bundles.

(b) $J^k(A \oplus B) \cong J^k A \oplus J^k B$

Exercise 3.3 (Cartan distribution)**.** Let $F \to M$ be a smooth fibre bundle. The **Cartan distribution** $C^k \subset T(J^k F)$ is spanned at every point $j^k_m \varphi \in J^k F$ by the tangent vectors of the form $\xi = T_m(j^k \psi)\, v_m$ for all $v_m \in T_m M$ and all local sections ψ with $j^k_m \psi = j^k_m \varphi$.

(a) Show that C^k is regular.

(b) Compute the rank of C^k.

(c) Show that C^k is not integrable.

Exercise 3.4 (Diffeomorphisms and locality)**.** Let M be a manifold and $F := M \times M \to M$ the projection to the first factor. Let $E = F \times_M F$. Let $f_0 : E \to F$ be the smooth map of manifolds defined by

$$f_0\big((m, m_1), (m, m_2)\big) := (m_1, m_2)\,.$$

(a) Compute the induced base map $f_M : \mathcal{E} \to C^\infty(M, M)$.

(b) Compute $\mathcal{E}_{\mathrm{diff}} = f_M^{-1}\big(\mathrm{Diff}(M, M)\big)$.

(c) Compute the induced map $f_{\mathcal{F}} : \mathcal{E}_{\mathrm{diff}} \to \mathcal{F}$.

Chapter 4

Pro-manifolds

4.1 Ind-categories and pro-categories

Let $\mathcal{C}$ be a category which is not cocomplete, that is, in which not all colimits exist. A natural way of cocompleting the category by adding colimits is to embed it into its category of presheaves by the Yoneda embedding

$$y : \mathcal{C} \longrightarrow \mathcal{S}\mathrm{et}^{\mathcal{C}^{\mathrm{op}}}, \quad y(C) := \mathrm{Hom}(\_, C) .$$

$\mathcal{S}\mathrm{et}$ is cocomplete, so the category of presheaves $\mathcal{S}\mathrm{et}^{\mathcal{C}^{\mathrm{op}}}$ is also cocomplete, since colimits in functor categories can be computed object-wise. If we add *all* colimits to the image of the Yoneda embedding we obtain all of $\mathcal{S}\mathrm{et}^{\mathcal{C}^{\mathrm{op}}}$ since every presheaf is a colimit of representable presheaves. However, the category of presheaves will generally be too big for our purposes. For example, the category of presheaves on smooth manifolds contains the category of topological spaces as subcategory, so it is clear that none of the structures and theorems of differential geometry that make essential use of the smooth structure will carry over to $\mathcal{S}\mathrm{et}^{\mathcal{M}\mathrm{fld}^{\mathrm{op}}}$.

4.1.1 Filtered and cofiltered categories

The colimits we will now consider are those of infinite sequences like

$$C_0 \longrightarrow C_1 \longrightarrow C_2 \longrightarrow \ldots ,$$

that is, a diagram $\omega \to \mathcal{C}$ indexed by the smallest transfinite ordinal

$$\omega = (0 \to 1 \to 2 \to \ldots) .$$

Exercise 4.1.1. Let $\mathcal{C}$ be the partially ordered set $(\mathbb{R}, \leq)$, viewed as category. A functor $x : \omega \to \mathcal{C}$ is an increasing sequence $x_0 \leq x_1 \leq x_2 \leq \ldots$ of real numbers. Show that the functor x has a colimit $y \in \mathbb{R}$ if and only if the sequence of numbers converges to y.

Even if we are primarily interested in diagrams indexed by ω, studying only diagrams of type ω and their colimits is not very natural. Many categorical constructions involving ω-diagrams will produce diagrams of different types. Exercise 4.1.1 also suggests that we may have to consider more general index categories. While every continuous map preserves limits of convergent sequences, the converse is true only if the domain of the map is a first countable topological space. In spaces that are not first countable we have to consider the convergence of filters instead of sequences. The concept of filtered categories is a generalization of the concept of filters.

Definition 4.1.2. A category $\mathcal{I}$ is **filtered** if the following three properties are satisfied:

(i) $\mathcal{I}$ is not empty.

(ii) For any two objects $i_1, i_2 \in \mathcal{I}$, there is a diagram,

$$\begin{tikzcd} i_1 \arrow[dr] & \\ & i \\ i_2 \arrow[ur] & \end{tikzcd}$$

(iii) For any two parallel morphisms $f : i_1 \to i_2$ and $g : i_1 \to i_2$, there is a diagram

$$\begin{tikzcd} i_1 \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & i_2 \arrow[r, "h"] & i \end{tikzcd}$$

such that $hf = hg$.

Example 4.1.3. Let $\mathcal{U}$ be a filter of a topological space X, that is, a non-empty collection of open subsets such that for every pair $U, V \in \mathcal{U}$, $U \cap V$ is also contained in $\mathcal{U}$. We can view $\mathcal{U}$ as a full subcategory of $\mathcal{O}\mathrm{pen}(X)^{\mathrm{op}}$. By definition, $\mathcal{U}$ is non-empty, and any two elements $U_1, U_2 \in \mathcal{U}$ contain $U_1 \cap U_2$, so that (i) and (ii) of Def. 4.1.2 are satisfied. Since the morphism between any two U_1 and U_2, i.e. the inclusion $U_1 \subset U_2$ is unique, two parallel morphisms are equal, so that we can always choose the morphism h of (iii) to be the identity. We conclude that $\mathcal{U}$ is a filtered category.

Proposition 4.1.4. *A category $\mathcal{I}$ is filtered if and only if every finite diagram $D : \mathcal{J} \to \mathcal{I}$ has a cocone.*

Proof. Recall that a cocone over a diagram D is an object $i \in \mathcal{I}$ and a natural transformation $\tau : D \to \Delta^i$, where $\Delta^i : \mathcal{J} \to \mathcal{I}$, $j \mapsto i$ denotes the constant functor with value i. This means that for every $j \in \mathcal{J}$ there is a morphism $\tau_j : D_j \to i$ such that for every $f : j \to j'$ in $\mathcal{J}$ we have $\tau_{j'} \circ Df = \tau_j$. There are three basic examples for cocones:

When $\mathcal{J} = \emptyset$, then a cocone is an object i in $\mathcal{I}$, so that $\mathcal{I}$ is non-empty. When $\mathcal{J}$ has two objects with no arrows between them, then a $\mathcal{J}$-diagram consists of a diagram of type (ii) in Def. 4.1.2. When $\mathcal{J}$ consists of two parallel morphisms from j_1 to j_2, then a cocone is a diagram of type (iii) in Def. 4.1.2. We conclude that if $\mathcal{I}$ has cocones on all finite diagrams, then $\mathcal{I}$ is filtered.

Conversely, assume that $\mathcal{I}$ is filtered and let $D : \mathcal{J} \to \mathcal{I}$ be a finite diagram. If $\mathcal{J} = \emptyset$, then D has a cocone since $\mathcal{I}$ is not empty by property (i) in Def. 4.1.2. Now, assume that $\mathscr{J}$ is not empty and let $\{j_1, \dots, j_n\}$ be its set of objects. Then, for every j_k, j_l in $\mathscr{J}$, there is a diagram

$$\begin{tikzcd}
D(j_k) \arrow[dr] & & D(j_l) \arrow[dl] \\
& i_{kl} &
\end{tikzcd}$$

in $\mathcal{I}$ by property (ii) in Def. 4.1.2. Furthermore, for every $r \leftarrow i \rightarrow s$ in $\mathcal{I}$, there exists an element $t \in \mathcal{I}$ and morphisms $r \rightarrow t$ and $s \rightarrow t$ such that the diagram

$$\begin{tikzcd}
& i \arrow[dl] \arrow[dr] & \\
r \arrow[dr] & & s \arrow[dl] \\
& t &
\end{tikzcd}$$

commutes by properties (ii) and (iii) of Def. 4.1.2. All in all, we get the following commutative diagram

$$\begin{tikzcd}
D(j_1) \arrow[dr] & & D(j_2) \arrow[dl] \arrow[dr] & & \cdots \arrow[dl] & & \cdots \arrow[dr] & & D(j_n) \arrow[dl] \\
& i_{12} \arrow[dr] & & i_{23} \arrow[dl] \arrow[dr] & & \cdots & & i_{(n-1)n} \arrow[dl] & \\
& & i_{13} \arrow[dr] & & \cdots \arrow[dl] \arrow[dr] & & \cdots \arrow[dl] & & \\
& & & i_{14} \arrow[dr] & & \cdots \arrow[dl] & & & \\
& & & & i & & & &
\end{tikzcd}$$

Lastly, for all $f : j_k \rightarrow j_l$ in $\mathscr{J}$, one can choose the element i_{kl} such that the diagram

$$\begin{tikzcd}
D(j_k) \arrow[rr, "D(f)"] \arrow[dr] & & D(j_l) \arrow[dl] \\
& i_{kl} &
\end{tikzcd}$$

commutes again by the properties of a filtered category. As a conclusion $i \in \mathcal{I}$ is a cocone for the finite diagram D.

*** □

Definition 4.1.5. A category $\mathcal{I}$ is **cofiltered** if $\mathcal{I}^{\mathrm{op}}$ is filtered.

Definition 4.1.6. The colimit (limit) of a diagram $D : \mathcal{I} \rightarrow \mathcal{C}$ is called **filtered** (**cofiltered**), when $\mathcal{I}$ is.

Example 4.1.7. The sequence

$$\mathbb{R}^0 \longrightarrow \mathbb{R}^1 \longrightarrow \mathbb{R}^2 \longrightarrow \dots$$

of the inclusions $\mathbb{R}^n \hookrightarrow \mathbb{R}^n \oplus \mathbb{R} \cong \mathbb{R}^{n+1}$ is a filtered diagram. Its colimit is $\coprod_{n=0}^{\infty} \mathbb{R}$, the countably infinite coproduct of $\mathbb{R}$, the elements of which are finite but arbitrarily long sequences of real numbers.

Example 4.1.8. The sequence

$$\mathbb{R}^0 \longleftarrow \mathbb{R}^1 \longleftarrow \mathbb{R}^2 \longleftarrow \dots$$

of the projections $\mathbb{R}^{n+1} \cong \mathbb{R}^n \times \mathbb{R} \twoheadrightarrow \mathbb{R}^n$ is a cofiltered diagram. Its limit is $\prod_{n=0}^{\infty} \mathbb{R}$, the countably infinite product of $\mathbb{R}$, the elements of which are infinite sequences of real numbers.

Example 4.1.9. Let $\mathcal{F} : \mathcal{O}\mathrm{pen}(M)^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$ be a presheaf on the topological space M. Let $\mathcal{O}\mathrm{pen}(M, m) \subset \mathcal{O}\mathrm{pen}(M)$ be the subcategory of open sets containing the point $m \in M$. (This is called the neighborhood filter of m.) The colimit of the functor $\mathcal{O}\mathrm{pen}(M, m)^{\mathrm{op}} \hookrightarrow \mathcal{O}\mathrm{pen}(M)^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$,

$$\mathcal{F}_m := \operatorname*{colim}_{U \ni m} \mathcal{F}(U)\,,$$

is the **stalk** at m, that is, the set of germs at m. (Recall that two elements $\varphi \in \mathcal{F}(U)$, $\varphi' \in \mathcal{F}(U')$ have the same germ at m if they have the same restriction to some open neighborhood of m.)

Given a functor $\Phi : \mathcal{I} \to \mathcal{J}$ and an object $j \in \mathcal{J}$, the **comma category** $j \downarrow \Phi$ has as objects pairs $(i, j \to \Phi(i))$ and as morphisms commutative triangles $j \to \Phi(i) \to \Phi(i')$. A category is called **connected** if every two objects are connected by a finite zigzag of arrows.

Definition 4.1.10. A functor $\Phi : \mathcal{I} \to \mathcal{J}$ is **final** if for every object $j \in \mathcal{J}$ the comma category $j \downarrow \Phi$ is non-empty and connected.

Let $\Phi : \mathcal{I} \to \mathcal{J}$ and $X : \mathcal{J} \to \mathcal{C}$ be functors. If the colimit of X exists, the maps to $(X \circ \Phi)_i = X_{\Phi(i)} \to \operatorname{colim} X$ are a cocone under the diagram $X \circ \Phi$. So if the colimit of $X \circ \Phi$ exists as well, the cocone induces by the universal property of the colimit a unique morphism

$$\operatorname{colim}(X \circ \Phi) \longrightarrow \operatorname{colim} X\,. \tag{4.1}$$

Proposition 4.1.11. *A functor $\Phi : \mathcal{I} \to \mathcal{J}$ is final if and only if for every functor $X : \mathcal{J} \to \mathcal{C}$ for which* $\operatorname{colim}(X \circ \Phi)$ *exists,* $\operatorname{colim} X$ *also exists and the morphism* (4.1) *is an isomorphism.*

Proof. See Thm. 1 and exercise 5 in Sec. IX.3 of [ML98]. □

Example 4.1.12. Let $\mathcal{I} = \omega = \mathcal{J}$ and $\Phi : \omega \to \omega$ be a functor such that the sequence $(\Phi(0), \Phi(1), \dots)$ is unbounded. Then for every j in the target, there is some i such that $j \le \Phi(i)$, which shows that $j \downarrow \Phi$ is non-empty. Moreover, if $j \le \Phi(i')$ then either $\Phi(i) \le \Phi(i')$ or $\Phi(i') \le \Phi(i)$, so that $j \downarrow \Phi$ is connected. We conclude that Φ is final in the sense of Def. 4.1.10.

Example 4.1.13. Let $\mathcal{I} = \omega$ and $\mathcal{J} = \omega \times \omega$. The diagonal functor $\Phi : \omega \to \omega \times \omega$, $i \to (i, i)$ is final. In order to see this, observe that there is a morphism in ω from (i, j) to (i', j') iff $i \le i'$ and $j \le j'$. We can then argue as in the last example to show that Φ is final.

Definition 4.1.14. A functor $\Phi : \mathcal{I} \to \mathcal{J}$ is **initial** if for every object $j \in \mathcal{J}$ the comma category $\Phi \downarrow j$ is non-empty and connected.

Proposition 4.1.15. *A functor $\Phi : \mathcal{I} \to \mathcal{J}$ is initial if and only if for every functor $X : \mathcal{J} \to \mathcal{C}$ for which $\lim(X \circ \Phi)$ exists, $\lim X$ also exists, and the natural morphism*

$$\lim X \longrightarrow \lim(X \circ \Phi)$$

is an isomorphism.

Proof. The proposition is dual to Prop. 4.1.11. □

Terminology 4.1.16. Final functors are sometimes called "cofinal" and initial functors are sometimes called "co-cofinal", e.g. in [KS06]. This can be quite confusing, since "cofinal" is sometimes also used as synonym for "initial" in the sense used here. We will generally adhere to the terminology of [ML98]. And besides, in category theory "coco-x" should always mean the same as "x", which is why there is no category theoretical difference between a coconut and a nut.

Let $\mathcal{I}$ and $\mathcal{J}$ be index categories and $X : \mathcal{I} \times \mathcal{J} \to \mathcal{C}$ a functor to a complete and cocomplete category. The morphisms of the limit cone

$$\lim_{j \in \mathcal{J}} X(i, j) \longrightarrow X(i, j)$$

are natural in i, so they induce a morphism of the colimits over i,

$$\operatorname*{colim}_{i \in \mathcal{I}} \lim_{j \in \mathcal{J}} X(i, j) \longrightarrow \operatorname*{colim}_{i \in \mathcal{I}} X(i, j)\,.$$

These morphisms form a cone over the diagram $j \mapsto X(i, j)$, so by the universal property of the limit this induces a unique morphism

$$\operatorname*{colim}_{i \in \mathcal{I}} \lim_{j \in \mathcal{J}} X(i, j) \longrightarrow \lim_{j \in \mathcal{J}} \operatorname*{colim}_{i \in \mathcal{I}} X(i, j)\,. \tag{4.2}$$

Definition 4.1.17. Let $X : \mathcal{I} \times \mathcal{J} \to \mathcal{C}$ be a functor to a complete and cocomplete category. If the morphism (4.2) is an isomorphism then the limit and colimit are said to **commute**.

Proposition 4.1.18. *Let $\mathcal{I}$ be a small category. The following are equivalent:*

(i) $\mathcal{I}$ is filtered.

(ii) For any finite category $\mathcal{J}$ and any functor $X : \mathcal{I} \times \mathcal{J} \to \mathcal{S}\mathrm{et}$ the colimit over $\mathcal{I}$ and the limit over $\mathcal{J}$ commute.

Proof. See Theorem 3.1.6 in [KS06]. Cf. also Theorem 1 in Sec. IX.2 of [ML98]. □

Corollary 4.1.19. *Filtered colimits and small limits preserve monomorphisms. Dually, small colimits and cofiltered limits preserve epimorphisms.*

Proof. In general, a morphisms $f : S \to T$ is a monomorphism if and only

$$\begin{tikzcd}
S \arrow[r, "\mathrm{id}"] \arrow[d, "\mathrm{id}"'] & S \arrow[d, "f"] \\
S \arrow[r, "f"] & T
\end{tikzcd}$$

is a pullback diagram, which is a finite limit diagram. Since by Prop. 4.1.20 filtered colimits commute with finite limits, filtered colimits preserve monomorphisms. Since limits commute with limits, limits preserve monomorphisms, as well. ☐

Prop. 4.1.18 can be viewed as the most important feature of filtered categories. For more on commuting classes of limits and colimits see [BJLS15]. For completeness and later reference we state the dual of Prop. 4.1.18.

Proposition 4.1.20. *Let $\mathcal{I}$ be a small category. The following are equivalent:*

(i) $\mathcal{I}$ is cofiltered.

(ii) For any finite category $\mathcal{J}$ and any functor $X : \mathcal{I} \times \mathcal{J} \to$ $\mathcal{S}\mathrm{et}$ the limit over $\mathcal{I}$ and the colimit over $\mathcal{J}$ commute.

4.1.2 Definition of ind/pro-categories

Definition 4.1.21. A presheaf is called **ind-representable** if it is isomorphic to the filtered colimit of representable presheaves.

Let us spell out this definition. A presheaf $\hat{X} \in \mathcal{S}\mathrm{et}^{\mathcal{C}^{\mathrm{op}}}$ is ind-representable if $\hat{X} \cong \mathrm{colim}_{\mathcal{I}}\, y(X_i)$ for some functor $X : \mathcal{I} \to \mathcal{C}$ from a small filtered category $\mathcal{I}$. We then say that $\hat{X}$ is **ind-represented** by X.

Definition 4.1.22 (I.8.2 in [Art72])**.** Let $\mathcal{C}$ be a category. The **ind-category** $\mathrm{Ind}(\mathcal{C})$ is the full subcategory of $\mathcal{S}\mathrm{et}^{\mathcal{C}^{\mathrm{op}}}$ of ind-representable presheaves.

The concept dual to ind-categories is that of pro-categories. For the pro-category, we want to enlarge $\mathcal{C}$ by cofiltered limits. Let $X : \mathcal{I} \to \mathcal{C}$ be a cofiltered diagram. Then $X^{\mathrm{op}} : \mathcal{I}^{\mathrm{op}} \to \mathcal{C}^{\mathrm{op}}$ is a filtered diagram. The limit of X is the colimit of X^{op}. So in order to add the limit of X to $\mathcal{C}$ we first embed $\mathcal{C}^{\mathrm{op}}$ in its presheaf category by the Yoneda embedding,

$$y_{\mathcal{C}^{\mathrm{op}}} : \mathcal{C}^{\mathrm{op}} \longrightarrow \mathcal{S}\mathrm{et}^{(\mathcal{C}^{\mathrm{op}})^{\mathrm{op}}} \cong \mathcal{S}\mathrm{et}^{\mathcal{C}}\,.$$

An object in $\mathcal{S}\mathrm{et}^{\mathcal{C}}$ is called a **copresheaf**. The Yoneda embedding of $C \in \mathcal{C}^{\mathrm{op}}$ is given explicitly by

$$\big(y_{\mathcal{C}^{\mathrm{op}}}(C)\big)(A) = \mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(A, C) = \mathrm{Hom}_{\mathcal{C}}(C, A)$$

for all $A \in \mathcal{C}$. The functor $\mathrm{Hom}(C, \_) : \mathcal{C} \to \mathcal{S}\mathrm{et}$ is called a **representable** copresheaf or the copresheaf **represented by** C. Now we can take the colimit of $y_{\mathcal{C}^{\mathrm{op}}} \circ X^{\mathrm{op}}$ inside $\mathcal{S}\mathrm{et}^{\mathcal{C}}$.

Definition 4.1.23. A copresheaf $\check{X} \in \mathcal{S}\mathrm{et}^{\mathcal{C}}$ is **pro-representable** if there is a cofiltered diagram $X : \mathcal{I} \to \mathcal{C}$ such that $\check{X}$ is isomorphic to the colimit of the filtered diagram $y_{\mathcal{C}^{\mathrm{op}}} \circ X^{\mathrm{op}} : \mathcal{I}^{\mathrm{op}} \to \mathcal{S}\mathrm{et}^{\mathcal{C}}$.

Definition 4.1.24. Let $\mathcal{C}$ be a category. The **pro-category** $\mathrm{Pro}(\mathcal{C})$ is the full subcategory of pro-representable copresheaves in $(\mathcal{S}\mathrm{et}^{\mathcal{C}})^{\mathrm{op}}$.

Proposition 4.1.25. *There is an isomorphism of categories* $\mathrm{Pro}(\mathcal{C}) \cong \big(\mathrm{Ind}(\mathcal{C}^{\mathrm{op}})\big)^{\mathrm{op}}$.

Proof. This isomorphism follows directly from the definition. □

Remark 4.1.26. Prop. 4.1.25 is sometimes taken as definition of pro-categories, e.g. In I.8.10 of [Art72].

Terminology 4.1.27. The prefixes "ind" and "pro" derive from the historic names "inductive limit" for colimit and "projective limit" for limit. By abuse of language, an object $\hat{X} \in \mathrm{Ind}(\mathcal{C})$ is called an **ind-object in** $\mathcal{C}$, even though it is not an object of $\mathcal{C}$. Analogously, $\check{X} \in \mathrm{Pro}(\mathcal{C})$ is called a **pro-object in** $\mathcal{C}$. When the objects in the category are named, ind and pro are added as prefixes. For example, a pro-object in the category of finite groups is called a pro-finite group, a pro-object in manifolds a pro-manifold, etc.

Lemma 4.1.28. $\hat{X} := \mathrm{colim}_{i\in\mathcal{I}}\, y(X_i)$ *and* $\hat{Y} := \mathrm{colim}_{j\in\mathcal{J}}\, y(Y_i)$ *presheaves on* $\mathcal{C}$ *represented by the diagrams* $X : \mathcal{I} \to \mathcal{C}$ *and* $Y : \mathcal{J} \to \mathcal{C}$. *Then there is a natural isomorphism*

$$\mathrm{Hom}_{\mathcal{S}\mathrm{et}^{\mathcal{C}^{\mathrm{op}}}}(\hat{X}, \hat{Y}) \cong \lim_{i\in\mathcal{I}} \operatorname*{colim}_{j\in\mathcal{J}} \mathrm{Hom}_{\mathcal{C}}(X_i, Y_j)\,.$$

Proof. We have the natural isomorphisms

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{S}\mathrm{et}^{\mathcal{C}^{\mathrm{op}}}}(\hat{X}, \hat{Y}) &\cong \mathrm{Hom}_{\mathcal{S}\mathrm{et}^{\mathcal{C}^{\mathrm{op}}}}(\operatorname*{colim}_{i\in\mathcal{I}} y(X_i), \hat{Y}) \\
&\cong \lim_{i\in\mathcal{I}} \mathrm{Hom}_{\mathcal{S}\mathrm{et}^{\mathcal{C}^{\mathrm{op}}}}(y(X_i), \hat{Y}) \\
&\cong \lim_{i\in\mathcal{I}} \hat{Y}(X_i) \\
&= \lim_{i\in\mathcal{I}} \big(\operatorname*{colim}_{j\in\mathcal{J}} y(Y_j)\big)(X_i) \\
&= \lim_{i\in\mathcal{I}} \operatorname*{colim}_{j\in\mathcal{J}} \big(y(Y_j)(X_i)\big) \\
&= \lim_{i\in\mathcal{I}} \operatorname*{colim}_{j\in\mathcal{J}} \mathrm{Hom}_{\mathcal{C}}(X_i, Y_j)\,.
\end{aligned}$$

In the first step we have used the colimit representation of $\hat{X}$, in the second step the universal property of colimits, in the third step the Yoneda lemma, in the fourth step the colimit representation of $\hat{Y}$, in the fifth step that colimits of presheaves are computed object-wise, and in the last step the Yoneda lemma again. □

Proposition 4.1.29. *Let* $\mathcal{C}$ *be a category. Let* $\hat{X}, \hat{Y} \in \mathrm{Ind}(\mathcal{C})$ *be ind-represented by the diagrams* $X : \mathcal{I} \to \mathcal{C}$ *and* $Y : \mathcal{J} \to \mathcal{C}$. *Then there is a natural isomorphism*

$$\mathrm{Hom}_{\mathrm{Ind}(\mathcal{C})}(\hat{X}, \hat{Y}) \cong \lim_{i\in\mathcal{I}} \operatorname*{colim}_{j\in\mathcal{J}} \mathrm{Hom}_{\mathcal{C}}(X_i, Y_j)\,. \tag{4.3}$$

Proof. Ind($\mathcal{C}$) is defined to be a *full* subcategory of $\mathcal{S}\mathrm{et}^{\mathcal{C}^{\mathrm{op}}}$, which means that

$$\mathrm{Hom}_{\mathrm{Ind}(\mathcal{C})}(\hat{X},\hat{Y}) = \mathrm{Hom}_{\mathcal{S}\mathrm{et}^{\mathcal{C}^{\mathrm{op}}}}(\hat{X},\hat{Y})\,.$$

The proposition now follows from Lem. 4.1.28. □

Corollary 4.1.30. *Let $\mathcal{C}$ be a category. Let $\check{X},\check{Y} \in \mathrm{Pro}(\mathcal{C})$ be pro-represented by the diagrams $X : \mathcal{I} \to \mathcal{C}$ and $Y : \mathcal{J} \to \mathcal{C}$. There is a natural isomorphism*

$$\mathrm{Hom}_{\mathrm{Pro}(\mathcal{C})}(\check{X},\check{Y}) \cong \lim_{j\in\mathcal{J}} \operatorname*{colim}_{i\in\mathcal{I}} \mathrm{Hom}_{\mathcal{C}}(X_i,Y_j)\,. \tag{4.4}$$

Proof. Using Props. 4.1.25 and 4.1.29, we can express the hom-set in Pro($\mathcal{C}$) as

$$\begin{aligned}\mathrm{Hom}_{\mathrm{Pro}(\mathcal{C})}(\check{X},\check{Y}) &\cong \mathrm{Hom}_{\mathrm{Ind}(\mathcal{C}^{\mathrm{op}})^{\mathrm{op}}}(\check{X},\check{Y})\\ &\cong \mathrm{Hom}_{\mathrm{Ind}(\mathcal{C}^{\mathrm{op}})}(\check{Y},\check{X})\\ &\cong \lim_{j\in\mathcal{J}} \operatorname*{colim}_{i\in\mathcal{I}} \mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(Y_j,X_i)\\ &\cong \lim_{j\in\mathcal{J}} \operatorname*{colim}_{i\in\mathcal{I}} \mathrm{Hom}_{\mathcal{C}}(X_i,Y_j)\,,\end{aligned}$$

which proves the corollary. □

The Yoneda embedding maps $\mathcal{C}$ as full subcategory into the category Ind($\mathcal{C}$). If $\mathcal{C}$ is a category in which all filtered colimits exist, then the Yoneda embedding $y : \mathcal{C} \to \mathrm{Ind}(\mathcal{C})$ has a retract, which is the **colimit functor**

$$\begin{aligned}\bar{U} : \mathrm{Ind}(\mathcal{C}) &\longrightarrow \mathcal{C}\\ \hat{X} &\longmapsto \operatorname*{colim}_{i\in\mathcal{I}} X_i\end{aligned} \tag{4.5}$$

for $\hat{X}$ represented by the diagram $X : \mathcal{I} \to \mathcal{C}$.

Definition 4.1.31. An ind-object (pro-object) in $\mathcal{C}$ is called **strict** if it is represented by a diagram in which every arrow is a monomorphism (epimorphism).

Proposition 4.1.32. *Let $\mathcal{C}$ be a category in which all filtered colimits exist. Let $\hat{X},\hat{Y} \in \mathrm{Ind}(\mathcal{C})$. If $\hat{Y}$ is strict, then the map*

$$\mathrm{Hom}_{\mathrm{Ind}(\mathcal{C})}(\hat{X},\hat{Y}) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(\bar{U}\hat{X},\bar{U}\hat{Y})$$

is injective.

Proof. Let $\hat{X}$ and $\hat{Y}$ be represented by diagrams $X : \mathcal{I} \to \mathcal{C}$ and $Y : \mathcal{J} \to \mathcal{C}$, where all morphisms of Y are monomorphisms. By Cor. 4.1.19, monomorphisms commute with filtered colimits. Therefore, the morphisms of the colimit cone

$$Y_j \longrightarrow \operatorname*{colim}_{j'\in\mathcal{J}} Y_{j'}$$

*** are all monomorphisms. It follows that the induced morphisms

$$\mathrm{Hom}(X,Y_j) \longrightarrow \mathrm{Hom}(X,\operatorname*{colim}_{j'\in\mathcal{J}} Y_{j'})$$

are monomorphisms for any $X \in \mathcal{C}$. Using again that monomorphisms commute with filtered colimits, we infer that

$$\operatorname*{colim}_{j \in \mathcal{J}} \operatorname{Hom}(X, Y_j) \longrightarrow \operatorname{Hom}(X, \operatorname*{colim}_{j \in \mathcal{J}} Y_j)$$

is a monomorphism. Similarly, monomorphisms commute with limits. Therefore,

$$\lim_{i \in \mathcal{I}} \operatorname*{colim}_{j \in \mathcal{J}} \operatorname{Hom}(X_i, Y_j) \longrightarrow \lim_{i \in \mathcal{I}} \operatorname{Hom}(X_i, \operatorname*{colim}_{j \in \mathcal{J}} Y_j) \cong \operatorname{Hom}(\operatorname*{colim}_{i \in \mathcal{I}} X_i, \operatorname*{colim}_{j \in \mathcal{J}} Y_j)$$

is a monomorphism. Using Eq. (4.3) and the definition (4.5) of $\hat{U}$, we conclude that

$$\operatorname{Hom}_{\operatorname{Ind}(\mathcal{C})}(\hat{X}, \hat{Y}) \longrightarrow \operatorname{Hom}_{\mathcal{C}}(\bar{U}\hat{X}, \bar{U}\hat{Y})$$

is an injective map. □

Remark 4.1.33. Prop. 4.1.32 implies that the colimit functor $\bar{U}$, if it exists, is faithful on strict ind-objects. However, it is generally not full. This means that there may be morphisms $\bar{U}(\hat{X}) \to \bar{U}(\hat{Y})$ that do not come from a morphism of the ind-objects $\hat{X} \to \hat{Y}$. Moreover, $\bar{U}$ is generally not essentially injective. This means that non-isomorphic ind-objects $\hat{X} \ncong \hat{Y}$ may have isomorphic colimit-objects $\bar{U}(\hat{X}) \cong \bar{U}(\hat{Y})$. The upshot is that even if all cofiltered limits in $\mathcal{C}$ exist, the objects in $\operatorname{Ind}(\mathcal{C})$ have a richer structure with fewer morphisms between them than those in $\mathcal{C}$.

4.1.3 Functoriality

Let $\Phi : \mathcal{C} \to \mathcal{D}$ be a functor. Since $\mathcal{S}\mathrm{et}^{\mathcal{D}^{\mathrm{op}}}$ is cocomplete, the functor $y_{\mathcal{D}} \circ \Phi$ has a left Kan extension along the Yoneda embedding of $\mathcal{C}$ into $\mathcal{S}\mathrm{et}^{\mathcal{C}^{\mathrm{op}}}$ ***,

$$\hat{\Phi} := \operatorname{Lan}_{y_{\mathcal{C}}}(y_{\mathcal{D}} \circ \Phi) : \mathcal{S}\mathrm{et}^{\mathcal{C}^{\mathrm{op}}} \longrightarrow \mathcal{S}\mathrm{et}^{\mathcal{D}^{\mathrm{op}}} ,$$

which we will call the **Yoneda extension** of Φ. The evaluation of $\hat{\Phi}$ on $\hat{X} = \operatorname{colim}(y_{\mathcal{C}} \circ X) = \operatorname{colim}_{i \in \mathcal{I}} y_{\mathcal{C}}(X_i)$ for some diagram $X : \mathcal{I} \to \mathcal{C}$ is given explicitly by

$$\hat{\Phi}(\hat{X}) = \operatorname*{colim}_{i \in \mathcal{I}} y_{\mathcal{D}}\big(\Phi(X_i)\big) .$$

By the Yoneda lemma, $\mathcal{C}$ is dense in $\mathcal{S}\mathrm{et}^{\mathcal{C}^{\mathrm{op}}}$, i.e. every presheaf is the colimit of representable presheaves. It follows that the following diagram commutes:

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "\Phi"] \arrow[d, "y_{\mathcal{C}}"'] & \mathcal{D} \arrow[d, "y_{\mathcal{D}}"] \\
\mathcal{S}\mathrm{et}^{\mathcal{C}^{\mathrm{op}}} \arrow[r, "\hat{\Phi}"] & \mathcal{S}\mathrm{et}^{\mathcal{D}^{\mathrm{op}}}
\end{tikzcd}$$

Moreover, if $\mathcal{I}$ is filtered, so that $\hat{X}$ is ind-represented by X, then $\hat{\Phi}(\hat{X})$ is ind-represented by $\Phi \circ X : \mathcal{I} \to \mathcal{D}$. We can draw the following conclusion.

Proposition 4.1.34. *The Yoneda extension of a functor $\Phi : \mathcal{C} \to \mathcal{D}$ restricts to a functor of the ind-categories*

$$\operatorname{Ind}(\Phi) : \operatorname{Ind}(\mathcal{C}) \longrightarrow \operatorname{Ind}(\mathcal{D}) .$$

Corollary 4.1.35. *The Yoneda extensions of functors* $\Phi : \mathcal{C} \to \mathcal{D}$ *and* $\Psi : \mathcal{C} \to \mathcal{D}^{\mathrm{op}}$ *restrict to functors*

$$\begin{aligned} \mathrm{Pro}(\Phi) : \mathrm{Pro}(\mathcal{C}) &\longrightarrow \mathrm{Pro}(\mathcal{D}) \\ \mathrm{Ind}(\Psi) : \mathrm{Ind}(\mathcal{C}) &\longrightarrow \mathrm{Pro}(\mathcal{D})^{\mathrm{op}} \\ \mathrm{Pro}(\Psi) : \mathrm{Pro}(\mathcal{C}) &\longrightarrow \mathrm{Ind}(\mathcal{D})^{\mathrm{op}} . \end{aligned}$$

where $\mathrm{Pro}(\Phi) := \mathrm{Ind}(\Phi^{\mathrm{op}})^{\mathrm{op}}$ *and* $\mathrm{Pro}(\Psi) := \mathrm{Ind}(\Psi^{\mathrm{op}})^{\mathrm{op}}$.

Remark 4.1.36. Prop. 4.1.34 and Cor. 4.1.35 tell us that mapping a category to its ind-category or its pro-category is functorial, i.e. there are functors $\mathrm{Ind} : \mathcal{C}\mathrm{at} \to \mathcal{C}\mathrm{at}$ and $\mathrm{Pro} : \mathcal{C}\mathrm{at} \to \mathcal{C}\mathrm{at}$.

Example 4.1.37. Consider the sequence of euclidean spaces $\mathbb{R}^0 \to \mathbb{R}^1 \to \dots$ from example 4.1.7, which represents an ind-object in the category of finite-dimensional vector spaces. The composition with the dual yields that sequence $(\mathbb{R}^0)^* \leftarrow (\mathbb{R}^1)^* \leftarrow \dots$, which represents a pro-object in finite dimensional vector spaces. Taking the dual again, we get back the ind-object we started with.

The reflexivity of ind/pro-finite dimensional vector spaces is one of the advantages of working in ind- and pro-categories. Taking the algebraic dual of an infinite dimensional vector space always raises the cardinality of the dimension. For example, the dual of the colimit of the sequence $\mathbb{R}^0 \to \mathbb{R}^1 \to \dots$ is $(\coprod_{n=0}^{\infty} \mathbb{R})^* \cong \prod_{n=0}^{\infty} \mathbb{R}^*$, which is the limit of the sequence $(\mathbb{R}^0)^* \leftarrow (\mathbb{R}^1)^* \leftarrow \dots$. But taking the dual again, yields a vector space of the unwieldy dimension $2^{(2^{\aleph_0})}$. Adding a Banach structure and taking bounded duals can make an infinite dimensional vector space reflexive. But when we only have a Fréchet structure, as in the example of smooth sections of a vector bundle, we are out of luck: The dual of a Fréchet space is again a Fréchet space if and only if it was a Banach space to begin with.

Proposition 4.1.38. *For any two categories* $\mathcal{C}$ *and* $\mathcal{D}$*, there are natural equivalences*

$$\begin{aligned} \mathrm{Ind}(\mathcal{C} \times \mathcal{D}) &\simeq \mathrm{Ind}(\mathcal{C}) \times \mathrm{Ind}(\mathcal{D}) \\ \mathrm{Pro}(\mathcal{C} \times \mathcal{D}) &\simeq \mathrm{Pro}(\mathcal{C}) \times \mathrm{Pro}(\mathcal{D}) . \end{aligned}$$

Proof. Let $(\hat{X}, \hat{Y}) \in \mathrm{Ind}(\mathcal{C}) \times \mathrm{Ind}(\mathcal{D})$ a pair of ind-objects represented by diagrams $X : \mathcal{I} \to \mathcal{C}$ and $Y : \mathcal{J} \to \mathcal{D}$. It is straight-forward to show that the product of two filtered categories is filtered (Prop. 3.2.1 (iii) in [KS06]). Therefore, the product functor $X \times Y : \mathcal{I} \times \mathcal{J} \to \mathcal{C} \times \mathcal{D}$ represents an ind-object in $\mathcal{C} \times \mathcal{D}$. We thus obtain a map

$$\mathrm{Ind}(\mathcal{C}) \times \mathrm{Ind}(\mathcal{D}) \longrightarrow \mathrm{Ind}(\mathcal{C} \times \mathcal{D}) . \tag{4.6}$$

Because the product of functors $X \times Y$ is natural in both the domain and the target, the map (4.6) is a functor. And since the Yoneda embedding commutes with products, this functor is fully faithful.

Consider an object $\hat{Z}$ in $\mathrm{Ind}(\mathcal{C} \times \mathcal{D})$ represented by a functor $Z : \mathcal{I} \to \mathcal{C} \times \mathcal{D}$, $i \mapsto X_i \times Y_i$, where $X : \mathcal{I} \to \mathcal{C}$ and $Y : \mathcal{I} \to \mathcal{D}$ are the two components of Z. Since the diagonal functor $\Delta : \mathcal{I} \to \mathcal{I} \times \mathcal{I}$ is final (exercise 4.2), $X \times Y : \mathcal{I} \times \mathcal{I} \to \mathcal{C} \times \mathcal{D}$ and Z represent isomorphic ind-objects. This shows that the fully faithful functor (4.6) is essentially surjective, so it is an equivalence of categories.

There is an isomorphism $(\mathcal{C} \times \mathcal{D})^{\mathrm{op}} \cong \mathcal{C}^{\mathrm{op}} \times \mathcal{D}^{\mathrm{op}}$ for any pair of categories. We thus obtain

$$\begin{aligned}\mathrm{Pro}(\mathcal{C} \times \mathcal{D}) &\cong \big(\mathrm{Ind}\big((\mathcal{C} \times \mathcal{D})^{\mathrm{op}}\big)\big)^{\mathrm{op}} \\ &\cong \big(\mathrm{Ind}(\mathcal{C}^{\mathrm{op}} \times \mathcal{D}^{\mathrm{op}})\big)^{\mathrm{op}} \\ &\simeq \big(\mathrm{Ind}(\mathcal{C}^{\mathrm{op}}) \times \mathrm{Ind}(\mathcal{D}^{\mathrm{op}})\big)^{\mathrm{op}} \\ &\cong \mathrm{Ind}(\mathcal{C}^{\mathrm{op}})^{\mathrm{op}} \times \mathrm{Ind}(\mathcal{D}^{\mathrm{op}})^{\mathrm{op}} \\ &\cong \mathrm{Pro}(\mathcal{C}) \times \mathrm{Pro}(\mathcal{D})\,,\end{aligned}$$

which finishes the proof. □

4.1.4 Finite limits and colimits in ind/pro-categories

Even finite limits and colimits in ind/pro-categories can be difficult to compute. Matters become easier if for a diagram $\hat{D} : \mathcal{A} \to \mathrm{Ind}(\mathcal{C})$ the objects $\hat{D}(a) \in \mathrm{Ind}(\mathcal{C})$ can be ind-represented by diagrams $D(a) : \mathcal{I} \to \mathcal{C}$ indexed by the same filtered category $\mathcal{I}$ and the morphisms of the diagram are all represented by natural transformations $D(a) \to D(b)$. Such a D is called a **level-representation** of $\hat{D}$. If a level-representation of $\hat{D}$ exists, then its limit and colimit can be computed level-wise, which is the statement of the following result, first proved in [AM69].

Proposition 4.1.39. *Let $\mathcal{I}$ be a small filtered category. Then the functor*

$$\begin{aligned}\mathcal{C}^{\mathcal{I}} &\longrightarrow \mathrm{Ind}(\mathcal{C}) \\ X &\longmapsto \operatorname*{colim}_{i \in \mathcal{I}} y(X_i)\end{aligned} \tag{4.7}$$

commutes with finite limits and finite colimits.

Proof. Let $D : \mathcal{A} \to \mathcal{C}^{\mathcal{I}}$, $a \mapsto D(a)$ be a diagram indexed by a finite category $\mathcal{A}$. Assume that the colimit of D exists. Since colimits in functor categories are computed object-wise this means that the colimit of the functor $\mathcal{A} \to \mathcal{C}$, $a \mapsto D(a)_i$ exists for all $i \in \mathcal{I}$.

Let us denote the functor (4.7) by F. The image of this diagram under F is

$$\begin{aligned}FD : \mathcal{A} &\longrightarrow \mathrm{Ind}(\mathcal{C}) \\ a &\longmapsto \operatorname*{colim}_{i \in \mathcal{I}} y\big(D(a)_i\big)\,.\end{aligned}$$

Let $\hat{Y} \in \mathrm{Ind}(\mathcal{C})$ be ind-represented by the filtered diagram $Y : \mathcal{J} \to \mathcal{C}$. We have the natural bijections

$$\begin{aligned}\mathrm{Hom}_{\mathrm{Ind}(\mathcal{C})}(\operatorname*{colim}_{a \in \mathcal{A}} FD(a), \hat{Y}) &\cong \lim_{a \in \mathcal{A}} \mathrm{Hom}_{\mathrm{Ind}(\mathcal{C})}(FD(a), \hat{Y}) \\ &\cong \lim_{a \in \mathcal{A}} \lim_{i \in \mathcal{I}} \operatorname*{colim}_{j \in \mathcal{J}} \mathrm{Hom}_{\mathcal{C}}(D(a)_i, Y_j) \\ &\cong \lim_{i \in \mathcal{I}} \lim_{a \in \mathcal{A}} \operatorname*{colim}_{j \in \mathcal{J}} \mathrm{Hom}_{\mathcal{C}}(D(a)_i, Y_j) \\ &\cong \lim_{i \in \mathcal{I}} \operatorname*{colim}_{j \in \mathcal{J}} \lim_{a \in \mathcal{A}} \mathrm{Hom}_{\mathcal{C}}(D(a)_i, Y_j) \\ &\cong \lim_{i \in \mathcal{I}} \operatorname*{colim}_{j \in \mathcal{J}} \mathrm{Hom}_{\mathcal{C}}(\operatorname*{colim}_{a \in \mathcal{A}} D(a)_i, Y_j) \\ &\cong \mathrm{Hom}_{\mathrm{Ind}(\mathcal{C})}\big(F(\operatorname*{colim}_{a \in \mathcal{A}} D(a)), \hat{Y}\big)\,,\end{aligned}$$

where we have used the universal property of the colimit of FD, the commutativity of limits, formula (4.3) for the morphisms in an ind-category, the commutativity of finite limits with filtered colimits stated in Prop. 4.1.18, the universal property of the colimit of the functor $D(\_)_i : \mathcal{A} \to \mathcal{C}$, and formula (4.3) again. Since this bijection holds for all $\hat{Y}$, we conclude that F commutes with the colimits over $\mathcal{A}$.

Assume now that the limit of D exists. Then we have the natural bijections

$$\begin{aligned}
\mathrm{Hom}_{\mathrm{Ind}(\mathcal{C})}(\hat{Y}, \lim_{a\in\mathcal{A}} FD(a)) &\cong \lim_{a\in\mathcal{A}} \mathrm{Hom}_{\mathrm{Ind}(\mathcal{C})}(\hat{Y}, FD(a)) \\
&\cong \lim_{a\in\mathcal{A}} \lim_{j\in\mathcal{J}} \operatorname*{colim}_{i\in\mathcal{I}} \mathrm{Hom}_{\mathcal{C}}(Y_j, D(a)_i) \\
&\cong \lim_{j\in\mathcal{J}} \lim_{a\in\mathcal{A}} \operatorname*{colim}_{i\in\mathcal{I}} \mathrm{Hom}_{\mathcal{C}}(Y_j, D(a)_i) \\
&\cong \lim_{j\in\mathcal{J}} \operatorname*{colim}_{i\in\mathcal{I}} \lim_{a\in\mathcal{A}} \mathrm{Hom}_{\mathcal{C}}(Y_j, D(a)_i) \\
&\cong \lim_{j\in\mathcal{J}} \operatorname*{colim}_{i\in\mathcal{I}} \mathrm{Hom}_{\mathcal{C}}(Y_j, \lim_{a\in\mathcal{A}} D(a)_i) \\
&\cong \mathrm{Hom}_{\mathrm{Ind}(\mathcal{C})}\big(\hat{Y}, F(\lim_{a\in\mathcal{A}} D(a))\big)\,,
\end{aligned}$$

which shows that F commutes with the limits over $\mathcal{A}$. □

Example 4.1.40. Let $\hat{X}$, $\hat{Y}$ be ind-objects in a category $\mathcal{C}$ with finite products, that are represented by the filtered diagrams $X, Y : \mathcal{I} \to \mathcal{C}$. Then the product $\hat{X} \times \hat{Y}$ exists and is represented by $\mathcal{I} \to \mathcal{C}$, $i \mapsto X_i \times Y_i$.

Remark 4.1.41. The map $\mathcal{C}^{\mathcal{I}} \to \mathrm{Ind}(\mathcal{C})$ does in general not commute with infinite limits or colimits. In fact, it does not even commute with filtered colimits, even though $\mathrm{Ind}(\mathcal{C})$ is a cocompletion of $\mathcal{C}$ by filtered colimits. In example 4.2.5 we will give an example for this phenomenon.

A finite diagram $\hat{D}$ can fail to have a level-representation only if $\mathcal{A}$ has "loops", i.e. no non-trivial endomorphisms [Isa02]. For example, a level-representation exists for every diagram consisting of a finite number of ind-objects without morphisms between them or for every diagram consisting of a pair of parallel morphism between a pair of ind-objects [KS06, Cor. 6.3.15] since the (co)limits of such diagrams are (co)products and (co)equalizers, Prop. 4.1.39 implies that all finite (co)products and (co)equalizers exist in $\mathrm{Ind}(\mathcal{C})$ if they exist in $\mathcal{C}$. Since every finite (co)limit can be obtained by a (co)equalizer of a finite (co)product we arrive at the following corollary.

Corollary 4.1.42. *If $\mathcal{C}$ has all finite coproducts, coequalizers, colimits, products, equalizers, or limits then so does* $\mathrm{Ind}(\mathcal{C})$.

This result can be slightly improved. In Prop. 6.1.18 of [KS06] it is shown that having finite coproducts in $\mathcal{C}$ implies that $\mathrm{Ind}(\mathcal{C})$ has small coproducts. As a consequence, if $\mathcal{C}$ has finite colimits, then $\mathrm{Ind}(\mathcal{C})$ has small colimits. For later reference, we state the dual of Prop. 4.1.39 for pro-categories, which follows immediately from Prop. 4.1.39.

Proposition 4.1.43 (Prop. 4.1 in App. A of [AM69])**.** *Let $\mathcal{I}$ be a small cofiltered category. Then the natural functor $\mathcal{C}^{\mathcal{I}} \to \mathrm{Pro}(\mathcal{C})$ commutes with finite limits and finite colimits.*

4.1.5 Concrete categories

Most categories we will deal with are concrete, that is, the objects can be viewed as sets with additional structure.

Definition 4.1.44 (***)**.** A category $\mathcal{C}$ together with a faithful forgetful functor $U : \mathcal{C} \to \mathcal{S}\mathrm{et}$ is called **concrete**.

A category can be concrete for different choices of the forgetful functor, so being concrete is a structure and not a property. In many categories the objects are by definition sets with additional structure, such as groups, rings, algebras, vector spaces, topological spaces, manifolds, etc. In that case, there is the obvious forgetful functor that discards the additional structure.

Proposition 4.1.45. *Let* $(\mathcal{C}, U)$ *be a concrete category. Let*

$$\hat{U} := \mathrm{Lan}_{\mathcal{C}\to\mathrm{Ind}(\mathcal{C})} U : \mathrm{Ind}(\mathcal{C}) \longrightarrow \mathcal{S}\mathrm{et}$$

be the left Kan extension of U *to* $\mathrm{Ind}(\mathcal{C})$*. Then* $(\mathrm{Ind}(\mathcal{C}), \hat{U})$ *is a concrete category.*

Proof. Let $\hat{X}, \hat{Y} \in \mathrm{Ind}(\mathcal{C})$ be ind-represented by $X : \mathcal{I} \to \mathcal{C}$ and $Y : \mathcal{J} \to \mathcal{C}$, respectively. First, we observe that the Yoneda extension of the forgetful functor is given by $\hat{U}\hat{X} = \mathrm{colim}_{i\in I} UX_i$. It follows that

$$\mathrm{Hom}_{\mathcal{S}\mathrm{et}}(\hat{U}\hat{X}, \hat{U}\hat{Y}) \cong \lim_{i\in\mathcal{I}} \operatorname*{colim}_{j\in\mathcal{J}} \mathrm{Hom}_{\mathcal{S}\mathrm{et}}(UX_i, UY_j)\,. \tag{4.8}$$

Since U is faithful, the forgetful map $\mathrm{Hom}_{\mathcal{C}}(X_i, Y_j) \to \mathrm{Hom}_{\mathcal{S}\mathrm{et}}(UX_i, UY_j)$ is injective for all $i \in \mathcal{I}$, $j \in \mathcal{J}$. By Cor. 4.1.19 filtered colimits preserve monomorphisms. It follows that the forgetful map

$$\operatorname*{colim}_{j\in\mathcal{J}} \mathrm{Hom}_{\mathcal{C}}(X_i, Y_j) \longrightarrow \operatorname*{colim}_{j\in\mathcal{J}} \mathrm{Hom}_{\mathcal{S}\mathrm{et}}(UX_i, UY_j) \tag{4.9}$$

is a monomorphism. By Cor. 4.1.19 small limits preserve monomorphisms. It follows that the map

$$\lim_{i\in\mathcal{I}} \operatorname*{colim}_{j\in\mathcal{J}} \mathrm{Hom}_{\mathcal{C}}(X_i, Y_j) \longrightarrow \lim_{i\in\mathcal{I}} \operatorname*{colim}_{j\in\mathcal{J}} \mathrm{Hom}_{\mathcal{S}\mathrm{et}}(UX_i, UY_j) \tag{4.10}$$

is a monomorphism. Using the isomorphisms (4.3) and (4.8), we conclude that the map

$$\mathrm{Hom}_{\mathrm{Ind}(\mathcal{C})}(\hat{X}, \hat{Y}) \longrightarrow \mathrm{Hom}_{\mathcal{S}\mathrm{et}}(\hat{U}\hat{X}, \hat{U}\hat{Y})$$

is a monomorphism as well, i.e. $\hat{U}$ is faithful. □

Corollary 4.1.46. *Let* $(\mathcal{C}, U)$ *be a concrete category. Let*

$$\check{U} := \mathrm{Ran}_{\mathcal{C}\to\mathrm{Pro}(\mathcal{C})} U : \mathrm{Pro}(\mathcal{C}) \longrightarrow \mathcal{S}\mathrm{et}$$

be the right Kan extension of U *to* $\mathrm{Pro}(\mathcal{C})$*. Then* $(\mathrm{Pro}(\mathcal{C}), \check{U})$ *is a concrete category.*

Proof. The proof follows from Prop. 4.1.25. □

Remark 4.1.47. The category of presheaves on any category $\mathcal{C}$ is concrete with the forgetful functor $\hat{X} \mapsto \coprod_{C \in \mathcal{C}} \hat{X}(C)$. But this functor is quite different from the one of Prop. 4.1.45.

Cor. 4.1.46 states that if $\mathcal{C}$ is a concrete category then there is a faithful functor $\check{U}$ on $\mathrm{Pro}(\mathcal{C})$ such that for every $\check{X} \in \mathrm{Pro}(\mathcal{C})$ pro-represented by $X : \mathcal{I} \to \mathcal{C}$ we have

$$\check{U}\check{X} = \lim_{i \in \mathcal{I}} UX_i \,.$$

In many categories the forgetful functor is the functor of morphisms

$$U(C) = \mathrm{Hom}_{\mathcal{C}}(I, C)$$

out of a test object I. Such a U is called the **functor of I-points**. The Kan extension of U is now given by

$$\check{U}\check{X} \cong \mathrm{Hom}_{\mathrm{Pro}(\mathcal{C})}\big(y(I), \check{X}\big) \,,$$

where we have used formula (4.4) for the hom-sets in $\mathrm{Pro}(\mathcal{C})$. This shows that $\check{U}$ is also the functor of I-points, where we identify I with the presheaf it represents. In geometric categories, such as topological spaces and smooth manifolds, the test object is typically the terminal object $I = *$. Since the Yoneda embedding commutes with limits $y(*)$ is the terminal object in $\mathrm{Ind}(\mathcal{C})$. In the category of vector spaces, the test object is $I = \mathbb{R}$.

4.1.6 Tensor products, algebras, derivations

The tensor product of vector spaces is an example for a closed symmetric monoidal structure. We recall that a **monoidal structure** on a category $\mathcal{C}$ consists of a functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, called the **tensor product** and an object $1 \in \mathcal{C}$, called the **tensor unit**, that equip $\mathcal{C}$ with a weakly associative and unital multiplication. That means that there are natural isomorphisms $a_{A,B,C} : (A \otimes B) \otimes C \to A \otimes (B \otimes C)$, $l_A : 1 \otimes A \to A$ and $r_A : A \otimes 1 \to A$ satisfying certain coherence axioms. The tensor product is called **symmetric** if there is a natural isomorphism $\tau_{A,B} : A \otimes B \to B \otimes A$ with $\tau_{A,B} \circ \tau_{B,A} = \mathrm{id}_{A \otimes B}$, satisfying additional coherence axioms involving a, l, and r. A monoidal category is called **closed** if for every $B \in \mathcal{C}$ the functor $\_ \otimes B : A \mapsto A \otimes B$ has a right adjoint $C \mapsto \underline{\mathrm{Hom}}(B, C)$, i.e. there is a natural isomorphism

$$\mathrm{Hom}_{\mathcal{C}}(A \otimes B, C) \cong \mathrm{Hom}_{\mathcal{C}}\big(A, \underline{\mathrm{Hom}}(B, C)\big) \,.$$

For the full definition of closed symmetric monoidal categories see for example Ch. VII in [ML98] or Sec. 1 in [Kel05].

Terminology 4.1.48. The object $\underline{\mathrm{Hom}}(A, B)$ is called the **internal** or **inner hom-object**. It is also denoted by $[A, B]$ or A^B.

Example 4.1.49. The category $\mathcal{V} = \mathcal{V}\mathrm{ec}$ with the tensor product $\otimes$ of vector spaces, the tensor unit $1 = \mathbb{R}$, and the usual vector space of linear maps $\underline{\mathrm{Hom}}(V, W)$ is a closed symmetric monoidal category.

By Prop. 4.1.34 the functor $\otimes$ induces a functor $\mathrm{Ind}(\otimes)$ on $\mathrm{Ind}(\mathcal{C}\times\mathcal{C})$. Composing this functor with the equivalence of Prop. 4.1.38, we obtain a functor

$$\hat{\otimes} : \mathrm{Ind}(\mathcal{C}) \times \mathrm{Ind}(\mathcal{C}) \xrightarrow{\cong} \mathrm{Ind}(\mathcal{C} \times \mathcal{C}) \xrightarrow{\mathrm{Ind}(\otimes)} \mathrm{Ind}(\mathcal{C})\,, \tag{4.11}$$

which maps ind-objects $\hat{A}$, $\hat{B}$ represented by diagrams $A : \mathcal{I} \to \mathcal{C}$ and $B : \mathcal{J} \to \mathcal{C}$ to the ind-object represented by $\mathcal{I} \times \mathcal{J} \to \mathcal{C}$, $(i, j) \mapsto A_i \otimes B_j$.

Proposition 4.1.50. *Let* $(\mathcal{C}, \otimes, 1)$ *be a monoidal category. Then the functor* $\hat{\otimes}$ *of Eq.* (4.11) *and the object* $\hat{1} := y(1) \in \mathrm{Ind}(\mathcal{C})$ *are a monoidal structure on* $\mathrm{Ind}(\mathcal{C})$*.*

Proof. The associativity of $\hat{\otimes}$ follows from the associativity of $\otimes$ and of the product in categories. That $\hat{1}$ is the unit of $\hat{\otimes}$ follows immediately from 1 being the unit of $\otimes$. *** ☐

Remark 4.1.51. Eq. (4.11) is an example for the **Day convolution product** of functors on a monoidal category [Day70].

A special case for a monoidal structure is the biproduct $\oplus$ of an additive category such as $\mathcal{V}\mathrm{ec}$. In fact, it can be shown that not only the biproduct, but the entire structure of an abelian category extends to the ind-category.

Proposition 4.1.52 (Thm. 8.6.5 in [KS06])**.** *Let* $\mathcal{C}$ *be an abelian category, then* $\mathrm{Ind}(\mathcal{C})$ *is an abelian category, such that the embedding* $\mathcal{C} \to \mathrm{Ind}(\mathcal{C})$ *is exact.*

When we have a tensor product on a category, we can define many algebraic structures internal to this category. In fact, any algebraic structure that is given by an operad or a prop can be generalized to any monoidal category. For example a monoid internal to $(\mathcal{C}, \otimes, 1)$ consists of an object $A \in \mathcal{C}$, a multiplication morphism $\mu : A \otimes A \to A$, and a unit morphism $e : 1 \to A$, such that the following diagrams commute:

$$\begin{tikzcd}
A \otimes A \otimes A \arrow[r, "\mathrm{id}\otimes\mu"] \arrow[d, "\mu\otimes\mathrm{id}"'] & A \arrow[d, "\mu"] \\
A \arrow[r, "\mu"] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
1 \otimes A \arrow[d, "e\otimes\mathrm{id}"'] & A \arrow[l, "l^{-1}"'] \arrow[r, "r^{-1}"] \arrow[d, "\mathrm{id}"] & A \otimes 1 \arrow[d, "\mathrm{id}\otimes e"] \\
A \otimes A \arrow[r, "\mu"] & A & A \otimes A \arrow[l, "\mu"']
\end{tikzcd}$$

Terminology 4.1.53. A monoid in $(\mathcal{S}\mathrm{et}, \times, *)$ is a monoid in the usual sense, which motivates the terminology. A monoid in $(\mathcal{V}\mathrm{ec}, \otimes, \mathbb{R})$ is an algebra. So when $\mathcal{V}\mathrm{ec}$ or, more generally, the category of modules over a ring is the basic category, a monoid internal to $(\mathcal{C}, \otimes, 1)$ is also called an algebra in $\mathcal{C}$.

Definition 4.1.54. A monoid internal to a monoidal category $(\mathcal{C}, \otimes, 1)$ will be called an **algebra in** $\mathcal{C}$. The category of algebras in $\mathcal{C}$ is denoted by $\mathcal{A}\mathrm{lg}(\mathcal{C})$. When $\mathcal{C} = \mathcal{V}\mathrm{ec}$, we abbreviate $\mathcal{A}\mathrm{lg} \equiv \mathcal{A}\mathrm{lg}(\mathcal{V}\mathrm{ec})$.

Let us spell out the structure of an algebra on an ind-object $\hat{A}$ represented by the diagram $A : \mathcal{I} \to \mathcal{V}\mathrm{ec}$. The tensor square $\hat{A} \,\hat{\otimes}\, \hat{A}$ is represented by the diagram $\mathcal{I} \times \mathcal{I} \to \mathcal{V}\mathrm{ec}$, $(i, j) \mapsto A_i \otimes A_j$. A map $\mu : \hat{A} \,\hat{\otimes}\, \hat{A} \to \hat{A}$ is represented by a family of morphisms

$$\mu_{i,j} : A_i \otimes A_j \longrightarrow A_{k(i,j)}\,. \tag{4.12}$$

This map is an associative multiplication if the families of morphisms

$$\begin{aligned}\mu_{i_1 i_2, i_3} &:= \mu_{k(i_1,i_2),i_3} \circ (\mu_{i_1,i_2} \otimes \mathrm{id}) : A_{i_1} \otimes A_{i_2} \otimes A_{i_3} \longrightarrow A_{k(k(i_1,i_2),i_3)}\\ \mu_{i_1, i_2 i_3} &:= \mu_{i_1,k(i_2,i_3)} \circ (\mathrm{id} \otimes \mu_{i_2,i_3}) : A_{i_1} \otimes A_{i_2} \otimes A_{i_3} \longrightarrow A_{k(i_1,k(i_2,i_3))}\end{aligned}$$

for all $i_1, i_2, i_3 \in \mathcal{I}$ represent the same morphism in $\mathrm{Ind}(\mathcal{V}\mathrm{ec})$. This is the case if there are commutative diagrams

$$\begin{tikzcd} & A_{i_1} \otimes A_{i_2} \otimes A_{i_3} \arrow[dl, "\mu_{i_1 i_2, i_3}"'] \arrow[dr, "\mu_{i_1, i_2 i_3}"] & \\ A_{k(k(i_1,i_2),i_3)} \arrow[dr] & & A_{k(i_1,k(i_2,i_3))} \arrow[dl] \\ & A_l & \end{tikzcd} \tag{4.13}$$

where the unmarked arrows are morphisms of the diagram $A : \mathcal{I} \to \mathcal{V}\mathrm{ec}$. Similarly, the unit of the algebra is given by a map $e : \mathbb{R} \to A_i$, such that there are commutative diagrams

$$\begin{tikzcd} \mathbb{R} \otimes A_j \arrow[d, "e \otimes \mathrm{id}"'] & A_j \arrow[l, "l^{-1}"'] \arrow[r, "r^{-1}"] \arrow[dd] & A_j \otimes \mathbb{R} \arrow[d, "\mathrm{id} \otimes e"] \\ A_i \otimes A_j \arrow[d, "\mu_{i,j}"'] & & A_j \otimes A_i \arrow[d, "\mu_{j,i}"] \\ A_{k(i,j)} \arrow[r] & A_l & A_{k(j,i)} \arrow[l] \end{tikzcd} \tag{4.14}$$

where again the unmarked arrows are some morphisms of the diagram $A : \mathcal{I} \to \mathcal{V}\mathrm{ec}$.

Example 4.1.55. Let $\bar{A}$ be a vector space with a filtration $A_0 \subset A_1 \subset A_2 \subset \ldots \subset \bar{A}$, which can be viewed as a sequence $A : \omega \to \mathcal{V}\mathrm{ec}$ of monomorphisms with colimit $\bar{A}$. An associative multiplication $\bar{\mu} : \bar{A} \otimes \bar{A} \to \bar{A}$ is **filtered** if $\mu(A_i \otimes A_j) \subset A_{i+j}$. Then the restrictions

$$\mu_{i,j} := \bar{\mu}|_{A_i \otimes A_j} : A_i \otimes A_j \longmapsto A_{k(i,j)}$$

for all $i, j \in \omega$ and $k(i,j) = i + j$ represent an associative multiplication on the ind-vector space $\hat{A}$ represented by the diagram A. The unit $e \in A_0$ of $\bar{\mu}$ is also a unit of the multiplication on $\hat{A}$.

Proposition 4.1.56. *Let $(\mathcal{C}, \otimes, 1)$ be a monoidal category. Let $F_{\mathcal{C}} : \mathcal{A}\mathrm{lg}(\mathcal{C}) \to \mathcal{C}$ denote the natural functor that forgets the structure morphisms of an algebra object. Then there is an injective and faithful functor $I : \mathrm{Ind}\big(\mathcal{A}\mathrm{lg}(\mathcal{C})\big) \to \mathcal{A}\mathrm{lg}\big(\mathrm{Ind}(\mathcal{C})\big)$, such that the diagram*

$$\begin{tikzcd} \mathrm{Ind}\big(\mathcal{A}\mathrm{lg}(\mathcal{C})\big) \arrow[rr, "I"] \arrow[dr, "\mathrm{Ind}(F_{\mathcal{C}})"'] & & \mathcal{A}\mathrm{lg}\big(\mathrm{Ind}(\mathcal{C})\big) \arrow[dl, "F_{\mathrm{Ind}(\mathcal{C})}"] \\ & \mathrm{Ind}(\mathcal{C}) & \end{tikzcd} \tag{4.15}$$

commutes.

Proof. The diagonal functor $\mathcal{I} \to \mathcal{I} \times \mathcal{I}$, $i \to (i,i)$ is final (exercise 4.2). This implies that the diagram $\mathcal{I} \times \mathcal{I} \to \mathcal{V}\mathrm{ec}$, $(i,j) \mapsto A_i \otimes A_j$ and the diagram $\mathcal{I} \to \mathcal{V}\mathrm{ec}$, $i \mapsto A_i \otimes A_i$ represent the same ind-vector space $\hat{A} \otimes \hat{A}$. More precisely, the family of maps $\mathrm{id} : A_i \otimes A_i \to A_i \otimes A_i$ induces an isomorphism of presheaves

$$\operatorname*{colim}_{i \in \mathcal{I}} y(A_i \otimes A_i) \xrightarrow{\cong} \operatorname*{colim}_{(i,j) \in \mathcal{I} \times \mathcal{I}} y(A_i \otimes A_j)\,. \tag{4.16}$$

For every pair $i, j \in \mathcal{I} \times \mathcal{J}$, let $m(i,j)$ be in $\mathcal{I}$ such that there are maps $i \to m(i,j)$ and $j \to m(i,j)$. Then there are morphisms $A_i \to A_{m(i,j)}$ and $A_j \to A_{m(i,j)}$ in the filtered diagram $A : \mathcal{I} \to \mathcal{V}\mathrm{ec}$. Their tensor product yields a family of morphisms

$$\Delta_{i,j} : A_i \otimes A_j \longrightarrow A_{m(i,j)} \otimes A_{m(i,j)}\,,$$

which represents the inverse of (4.16).

Let $\hat{A}_{\mathrm{alg}} \in \mathrm{Ind}(\mathcal{A}\mathrm{lg}(\mathcal{C}))$ be represented by $\mathcal{I} \to \mathcal{A}\mathrm{lg}(\mathcal{C})$, $i \mapsto (A_i, \mu_i, e_i)$. The family of morphisms $\mu_i : A_i \otimes A_i \to A_i$ defines a morphism $\mu : \hat{A} \otimes \hat{A} \to \hat{A}$ of ind-objects in $\mathcal{C}$. Composing the morphisms with $\Delta_{i,j}$ yields the family of morphisms

$$\mu_{i,j} : A_i \otimes A_j \xrightarrow{\Delta_{i,j}} A_{m(i,j)} \otimes A_{m(i,j)} \xrightarrow{\mu_{m(i,j)}} A_{m(i,j)}\,.$$

which represents μ on the diagram $(i,j) \mapsto A_i \otimes A_j$. From the associativity of μ_i and the fact that all maps in the diagram $A : \mathcal{I} \to \mathcal{C}$ are homomorphisms of algebras, it follows that there is a commutative diagram (4.13) for all $i_1, i_2, i_3 \in \mathcal{I}$. We conclude that μ is an associative multiplication on $\hat{A}$.

Since any arrow $\sigma : A_0 \to A_i$ of the diagram A is a homomorphism of unital algebras, we have $e_i = \sigma(e_0)$. This implies that the morphisms $e : y(1) \to \hat{A}$ of ind-objects in $\mathcal{C}$ that is represented by $e_0 : 1 \to A_0$ makes the diagrams (4.14) commutative, so that e is the unit of μ.

So far we have shown that the structure morphisms μ_i, e_i of any $\hat{A}_{\mathrm{alg}} \in \mathrm{Ind}(\mathcal{A}\mathrm{lg}(\mathcal{C}))$ represent the morphisms of an algebra structure on the underlying ind-object $\hat{A} \in \mathrm{Ind}(\mathcal{C})$. A morphism $f : \hat{A}_{\mathrm{alg}} \to \hat{B}_{\mathrm{alg}}$ of ind-algebras is a represented by a family $f_i : A_i \to B_i$ of morphisms of algebra objects in $\mathcal{C}$. The morphisms f_i induce a morphism $f : \hat{A} \to \hat{B}$ of the underlying ind-objects in $\mathcal{C}$. It is straight-forward to check that f is compatible with the induced algebra structures on $\hat{A}$ and $\hat{B}$, i.e. f is a morphisms of algebras in $\mathrm{Ind}(\mathcal{C})$. We conclude that we have a functor $I : \mathrm{Ind}\big(\mathcal{A}\mathrm{lg}(\mathcal{C})\big) \to \mathcal{A}\mathrm{lg}\big(\mathrm{Ind}(\mathcal{C})\big)$.

By definition, $\hat{A}_{\mathrm{alg}}$ and $I(\hat{A}_{\mathrm{alg}})$ have the same underlying $\hat{A} \in \mathrm{Ind}(\mathcal{C})$, which means that the diagram (4.15) commutes. A morphism in $\mathrm{Ind}(\mathcal{A}\mathrm{lg}(\mathcal{C}))$ is given by a morphism in $\mathrm{Ind}(\mathcal{C})$ that satisfies compatibility conditions with the algebra structures. This implies that the forgetful morphism $\mathrm{Ind}(\mathcal{A}\mathrm{lg}(\mathcal{C})) \to \mathrm{Ind}(\mathcal{C})$ is faithful. Since diagram (4.15) commutes, I must be faithful as well. Finally, if the morphisms $\mu_i, \mu'_i : A_i \otimes A_i \to A_i$ and $e_i, e'_i : 1 \to A_i$ represent the same ind-algebra $\hat{A}_{\mathrm{alg}}$, then the induced morphisms $\mu, \mu' : \hat{A} \otimes \hat{A} \to \hat{A}$, $e, e' : y(1) \to \hat{A}$ of ind-objects in $\mathcal{C}$ are equal. We conclude that I is injective on objects. □

Proposition 4.1.57. $(\mathcal{V}, \otimes, 1)$ *be a closed symmetric monoidal category that has all filtered colimits. Then the colimit functor* $\bar{U} : \mathrm{Ind}(\mathcal{V}) \to \mathcal{V}$ *defined in Eq.* (4.5) *preserves tensor products, i.e. there is a natural isomorphisms*

$$\bar{U}(\hat{A} \mathbin{\hat{\otimes}} \hat{B}) \cong \bar{U}(\hat{A}) \otimes \bar{U}(\hat{B}) \,,$$

for all $\hat{A}, \hat{B} \in \operatorname{Ind}(\mathcal{V}\mathrm{ec})$.

Lemma 4.1.58. *The tensor product of a closed symmetric monoidal category commutes with colimits in each factor.*

Proof. Let $\mathcal{V}$ be a closed symmetric monoidal category. Let $A : \mathcal{I} \to \mathcal{V}$ be a diagram that has a colimit. We have natural isomorphisms

$$\begin{aligned} \operatorname{Hom}\big((\operatorname*{colim}_{i\in I} A_i) \otimes B, C\big) &\cong \operatorname{Hom}\big(\operatorname*{colim}_{i\in I} A_i, \underline{\operatorname{Hom}}(B,C)\big) \\ &\cong \lim_{i\in I} \operatorname{Hom}\big(A_i, \underline{\operatorname{Hom}}(B,C)\big) \\ &\cong \operatorname{Hom}\big(\operatorname*{colim}_{i\in I}(A_i \otimes B), C\big) \,, \end{aligned}$$

for all $B, C \in \mathcal{V}$. It follows that

$$(\operatorname*{colim}_{i\in I} A_i) \otimes B \cong \operatorname*{colim}_{i\in I}(A_i \otimes B) \,.$$

By the symmetry of $\otimes$ it follows that the tensor product commutes with colimits in the second factor, as well. □

Proof of Prop 4.1.57. Let $\hat{A}, \hat{B} \in \operatorname{Ind}(\mathcal{V})$ be represented by diagrams $A : \mathcal{I} \to \mathcal{C}$ and $B : \mathcal{J} \to \mathcal{C}$. The tensor product $\hat{A} \mathbin{\hat{\otimes}} \hat{B}$ is represented by $\mathcal{I} \times \mathcal{J} \to \mathcal{V}$, $(i,j) \to A_i \otimes B_j$. Therefore,

$$\begin{aligned} \bar{U}(\hat{A} \mathbin{\hat{\otimes}} \hat{B}) &= \operatorname*{colim}_{(i,j)\in\mathcal{I}\times\mathcal{J}} A_i \otimes B_j \\ &\cong \operatorname*{colim}_{i\in\mathcal{I}} \operatorname*{colim}_{j\in\mathcal{J}}(A_i \otimes B_j) \\ &\cong \operatorname*{colim}_{i\in\mathcal{I}}\big(A_i \otimes (\operatorname*{colim}_{j\in\mathcal{J}} B_j)\big) \\ &\cong (\operatorname*{colim}_{i\in\mathcal{I}} A_i) \otimes (\operatorname*{colim}_{j\in\mathcal{J}} B_j) \\ &\cong \bar{U}(\hat{A}) \otimes \bar{U}(\hat{B}) \,, \end{aligned}$$

where we have used Lem. 4.1.58 twice. □

Corollary 4.1.59. *The colimit functor* $\bar{U} : \operatorname{Ind}(\mathcal{V}) \to \mathcal{V}$ *induces a functor*

$$\mathcal{A}\mathrm{lg}(\operatorname{Ind}(\mathcal{V})) \longrightarrow \mathcal{A}\mathrm{lg}(\mathcal{V}) \,.$$

Example 4.1.60. It follows from Cor. 4.1.32 that the colimit functor $\bar{U} : \operatorname{Ind}(\mathcal{V}\mathrm{ec}) \to \mathcal{V}\mathrm{ec}$ is faithful on strict ind-objects. Cor. 4.1.59 then implies that an algebra structure on the strict ind-vector space $\hat{A}$ can be identified with an algebra structure on the underlying vector space $\bar{A} := \bar{U}(\hat{A})$. Note, however, that $\bar{U}$ is neither essentially injective nor full (Rmk. 4.1.33). This means that non-isomorphic ind-vector spaces $\hat{A} \ncong \hat{B}$ can have isomorphic underlying vector spaces $\bar{A} \cong \bar{B}$, and that there may be algebra structures on $\bar{A}$ that do not arise from an algebra structure on $\hat{A}$.

Example 4.1.61. The category $\mathcal{V} = \mathrm{gr}\mathcal{V}\mathrm{ec}$ of $\mathbb{Z}$-graded vector spaces is closed symmetric monoidal. The tensor product of two graded vector spaces $V_\bullet$ and $W_\bullet$ is given by

$$(V \otimes W)_n = \bigoplus_{p+q=n} V_p \otimes W_q \,.$$

The tensor unit is $\mathbb{R}$ viewed as graded vector space concentrated in degree 0. The symmetric structure is $\tau(v \otimes w) = (-1)^{|v|\,|w|} w \otimes v$. The inner hom-object is the graded vector space

$$\underline{\mathrm{Hom}}_{\mathrm{gr}\mathcal{V}\mathrm{ec}}(V, W)_n = \prod_{p \in \mathbb{Z}} \underline{\mathrm{Hom}}_{\mathcal{V}\mathrm{ec}}(V_p, W_{p+n}) \,.$$

By Cor. 4.1.32 the colimit functor $\bar{U} : \mathrm{Ind}(\mathrm{gr}\mathcal{V}\mathrm{ec}) \to \mathrm{gr}\mathcal{V}\mathrm{ec}$ on ind-vector spaces is faithful on strict ind-objects. Cor. 4.1.59 then shows, that an algebra structure on a strict ind-graded vector space $\hat{A}$ can be identified with an algebra structure on the graded vector space $\bar{U}(A)$. An algebra in graded vector spaces is the same thing as a graded algebra.

Definition 4.1.62. Let $(\mathcal{C}, \otimes, 1)$ be an additive monoidal category. Let (A, μ, e) be an algebra object in $\mathcal{C}$. A **derivation** of A is a morphism $\delta : A \to A$ such that the diagram

$$\begin{tikzcd}
A \otimes A \arrow[r, "\mu"] \arrow[d, "\delta\otimes\mathrm{id}+\mathrm{id}\otimes\delta"'] & A \arrow[d, "\delta"] \\
A \otimes A \arrow[r, "\mu"] & A
\end{tikzcd} \tag{4.17}$$

commutes.

Proposition 4.1.63. *Let A be an algebra in an additive monoidal category $\mathcal{C}$. Then* $\mathrm{Der}(A)$ *is closed under the commutator of composition.*

Proof. This is shown by a direct calculation, which is analogous to the case of algebras is $\mathcal{V}$ec. □

4.1.7 Enrichment

In the definition of an **enriched category** $\mathcal{C}$ the hom-*sets* are replaced by hom-*objects* in a monoidal category $(\mathcal{V}, \otimes, 1)$. More precisely, a **category enriched over** $\mathcal{V}$ consists of a class of objects $\mathcal{C}$, a hom-object $\underline{\mathrm{Hom}}(A, B) \in \mathcal{V}$ for every pair $A, B \in \mathcal{C}$, and morphisms of the composition and the identities

$$\begin{aligned} \circ : \underline{\mathrm{Hom}}(B, C) \otimes \underline{\mathrm{Hom}}(A, B) &\longrightarrow \underline{\mathrm{Hom}}(A, C) \\ \mathrm{id}_A : 1 &\longrightarrow \underline{\mathrm{Hom}}(A, A) \,, \end{aligned}$$

for all $A, B, C \in \mathcal{C}$, that satisfy axioms that generalize the axioms of a category up to isomorphisms, together with coherence axioms involving the morphisms a, l, and r of the monoidal structure. For the full definition see e.g. [Kel05].

Example 4.1.64. Every additive category is enriched over the category of abelian groups.

To every enriched category we can associate an ordinary category by applying the functor of 1-points,

$$\begin{aligned} U : \mathcal{V} &\longrightarrow \mathcal{S}\mathrm{et} \\ A &\longmapsto \mathrm{Hom}_{\mathcal{V}}(1, A) \end{aligned}$$

to the hom-objects. This functor is monoidal, i.e. there is a natural map $U(A) \times U(B) \to U(A \otimes B)$ for all $A, B \in \mathcal{V}$. It follows that the sets

$$\mathrm{Hom}(A, B) := \mathrm{Hom}_{\mathcal{V}}\big(1, \underline{\mathrm{Hom}}(A, B)\big)$$

satisfy the axioms of a (non-enriched) category. When U is faithful, the enrichment can be viewed as structure on the hom-sets of the underlying category that is compatible with the composition of morphisms. For example, this is the case for $\mathcal{V} = \mathcal{V}\mathrm{ec}$, where $\mathrm{Hom}_{\mathcal{V}\mathrm{ec}}(V, W)$ has the structure of a vector space and the composition $f \circ g$ is linear in f and g. We have all been using this long before we even knew what a category, let alone an enriched category is. This is the low-brow point of view we want to adopt here.

Proposition 4.1.65. *Every closed symmetric monoidal category is enriched over itself by the internal hom-objects.*

Proof. *** □

Proposition 4.1.66. *Let $\mathcal{V}$ be a closed symmetric monoidal category that has all filtered colimits and all cofiltered limits. Then $\mathrm{Ind}(\mathcal{V})$ is enriched over $\mathcal{V}$ with the hom-objects*

$$\underline{\mathrm{Hom}}_{\mathrm{Ind}(\mathcal{V})}(\hat{A}, \hat{B}) \cong \lim_{i \in \mathcal{I}} \operatorname*{colim}_{j \in \mathcal{J}} \underline{\mathrm{Hom}}_{\mathcal{V}}(A_i, B_j)\,,$$

for $\hat{A}, \hat{B} \in \mathrm{Ind}(\mathcal{C})$ represented by diagrams $A : \mathcal{I} \to \mathcal{C}$ and $B : \mathcal{J} \to \mathcal{C}$.

Proof. ***. □

Let $\mathcal{C}$ be a monoidal category enriched over an additive closed symmetric monoidal category $\mathcal{V}$. For every object $V \in \mathcal{V}$ with a point $e : 1 \to V$ we have a linear map $\Delta : V \to V \otimes V$ given by

$$\begin{tikzcd}
V \arrow[d, "\mathrm{diag}"] \\
V \oplus V \arrow[d, "r^{-1} \oplus l^{-1}"] \\
(V \otimes 1) \oplus (1 \otimes V) \arrow[d, "(\mathrm{id}_V \otimes e) \oplus (e \otimes \mathrm{id}_V)"] \\
(V \otimes V) \oplus (V \otimes V) \arrow[d, "+"] \\
V \otimes V
\end{tikzcd}$$

In a concrete category this map is given by

$$\Delta(v) = v \otimes e + e \otimes v\,.$$

Consider now the object $\underline{\mathrm{End}}(A) := \underline{\mathrm{Hom}}(A, A)$ in $\mathcal{V}$ for some $A \in \mathcal{C}$ with the point $e = \mathrm{id}_A$, so that the linear map Δ is defined. By definition of a monoidal enriched category, the tensor product of $\mathcal{C}$ is an enriched functor, i.e. there are morphisms

$$\underline{\mathrm{Hom}}(A, B) \otimes \underline{\mathrm{Hom}}(A', B') \xrightarrow{\otimes} \underline{\mathrm{Hom}}(A \otimes A', B \otimes B')\,,$$

For all $A, A', B, B' \in \mathcal{C}$. In particular, we have a morphism

$$\underline{\mathrm{End}}(A) \otimes \underline{\mathrm{End}}(A) \xrightarrow{\otimes} \underline{\mathrm{End}}(A \otimes A)\,.$$

Let now (A, μ, e) be an algebra in $\mathcal{C}$. The multiplication $\mu : A \otimes A \to A$ induces composition morphisms

$$\begin{aligned} \mu_* &: \underline{\mathrm{End}}(A \otimes A) \longrightarrow \underline{\mathrm{Hom}}(A \otimes A, A) \\ \mu^* &: \underline{\mathrm{End}}(A) \longrightarrow \underline{\mathrm{Hom}}(A \otimes A, A)\,. \end{aligned}$$

Now we can define the enriched derivation object of the algebra A as the equalizer

$$\begin{tikzcd} \underline{\mathrm{Der}}(A) \arrow[r] & \underline{\mathrm{End}}(A) \arrow[r, shift left, "\mu^*"] \arrow[r, shift right, "\mu_* \circ \otimes \circ \Delta"'] & \underline{\mathrm{Hom}}(A \otimes A, A) \end{tikzcd}\,. \tag{4.18}$$

Proposition 4.1.67. *The enriched derivation object* $\underline{\mathrm{Der}}(A)$ *as defined in Eq.* (4.18) *is a Lie algebra object in* $\mathcal{V}$.

4.2 Sequential ind/pro-objects

Definition 4.2.1. An ind-object (pro-object) is called **sequential** if it is represented by a diagram indexed by ω (ω^{op}).

Spelling out this definition, we see that a strict sequential ind-object in $\mathcal{C}$ is represented by a sequence

$$X_0 \xrightarrow{\sigma_0} X_1 \xrightarrow{\sigma_1} X_2 \xrightarrow{\sigma_2} \ldots\,,$$

such that every σ_i is a monomorphism. Dually, a strict sequential pro-object in $\mathcal{C}$ is represented by a sequence

$$X_0 \xleftarrow{\sigma_0} X_1 \xleftarrow{\sigma_1} X_2 \xleftarrow{\sigma_2} \ldots\,,$$

such that every σ_i is an epimorphism. Many of the ind-objects and pro-objects we are interested in arise from such diagrams, so we will study them in more detail.

4.2.1 Representation of morphisms

There is an explicit description of the set of morphisms between sequential ind-objects.

Proposition 4.2.2. *Let $\hat{X}$ and $\hat{Y}$ be sequential ind-objects in $\mathcal{C}$ represented by the sequences $X_0 \overset{\sigma_0}{\to} X_1 \overset{\sigma_1}{\to} \dots$ and $Y_0 \overset{\tau_0}{\to} Y_1 \overset{\tau_1}{\to} \dots$. A morphism in $\mathrm{Hom}_{\mathrm{Ind}(\mathcal{C})}(\hat{X}, \hat{Y})$ is represented by a diagram*

$$\begin{tikzcd}
X_0 \arrow[r] \arrow[d, "f_0"] & X_1 \arrow[r] \arrow[d, "f_1"] & X_2 \arrow[r] \arrow[d, "f_2"] & \dots \\
Y_{j(0)} \arrow[r] & Y_{j(1)} \arrow[r] & Y_{j(2)} \arrow[r] & \dots
\end{tikzcd} \tag{4.19}$$

where $j(i) \leq j(i+1)$ for all $i \in \omega$.

Moreover, if all target indices $j(i)$ are chosen to be minimal in the sense that no f_i factors like

$$\begin{tikzcd}
 & X_i \arrow[dl, "f_i'"'] \arrow[d, "f_i"] \\
Y_{j(i)-1} \arrow[r, "\tau_{j(i)-1}"'] & Y_{j(i)}
\end{tikzcd}$$

and if $\hat{Y}$ is strict, then every f_i is unique.

Proof. In the first step we calculate the inner colimit of Eq. (4.3). The set $\mathrm{colim}_j \mathrm{Hom}(X_i, Y_j)$ is the quotient of the disjoint union of all $\mathrm{Hom}(X_i, Y_j)$, $j \geq 0$ modulo the equivalence relation that is generated by $f \sim \tau_j \circ f$ for all $f \in \mathrm{Hom}(X_i, Y_j)$, $j \geq 0$,

$$\operatorname*{colim}_j \mathrm{Hom}(X_i, Y_j) = \coprod_j \mathrm{Hom}(X_i, Y_j) / \sim \ . \tag{4.20}$$

Since the index category ω is ordered and bounded from below every element of the quotient has a representative $f_i : X_i \to Y_{j(i)}$ for which $j(i)$ is minimal. From the minimality it follows that $j(i) \leq j(i+1)$.

In the second step we construct the limit of Eq. (4.3). The diagram of which we have to compute the limit is

$$C_0 \xleftarrow{\sigma_0^*} C_1 \xleftarrow{\sigma_1^*} C_2 \xleftarrow{\sigma_2^*} \dots ,$$

where $C_i := \mathrm{colim}_j \mathrm{Hom}(X_i, Y_j)$ and

$$\begin{aligned}
\sigma_i^* : \operatorname*{colim}_j \mathrm{Hom}(X_{i+1}, Y_j) &\longrightarrow \operatorname*{colim}_j \mathrm{Hom}(X_i, Y_j) \\
[f_{i+1}] &\longmapsto [f_{i+1} \circ \sigma_i] .
\end{aligned}$$

Every equivalence class in C_i has a representative $f_i : X_i \to Y_{j(i)}$ for which $j(i)$ is minimal. An element in the limit is given by a sequence

$$([f_0], [f_1], [f_2], \dots) \in \prod_i \operatorname*{colim}_j \mathrm{Hom}(X_i, Y_j)$$

with the property that $\sigma_i^*[f_{i+1}] = [f_i]$ for all i. This means that for every $f_i : X_i \to Y_{j(i)}$ and $f_{i+1} : X_{i+1} \to Y_{j(i+1)}$ we have a commutative square

$$\begin{tikzcd}
X_i \arrow[r, "\sigma_i"] \arrow[d, "f_i"] & X_{i+1} \arrow[d, "f_{i+1}"] \\
Y_{j(i)} \arrow[r, "\tau"] & Y_{j(i+1)}
\end{tikzcd}$$

where

$$\tau : Y_{j(i)} \xrightarrow{\tau_{j(i)}} Y_{j(i)+1} \longrightarrow \ldots \xrightarrow{\tau_{j(i+1)-1}} Y_{j(i+1)}\,.$$

The commutativity of the infinite diagram of the proposition is equivalent to the commutativity of these squares for all i.

We have already seen that the target indices $j(i)$ can be chosen to be minimal. Assume now that $\hat{Y}$ is strict, i.e. all morphisms τ_j are monomorphisms. This implies that if two morphisms $f, f' : X_i \to X_j$ with the same domain and target represent the same equivalence class $[f : X_i \to Y_j] = [f' : X_i \to Y_j]$ in the colimit (4.20), then they are equal $f = f'$. In particular, the morphism $f_i : X_i \to Y_{j(i)}$ that represents $[f_i]$ is unique. □

The composition of an ind-morphism $\hat{X} \to \hat{Y}$ as in Prop. 4.2.2 with another ind-morphism $\hat{g} : \hat{Y} \to \hat{Z}$ of sequential ind-objects represented by a family $g : Y_j \to Y_{k(j)}$ is represented by the family of morphisms obtained by stacking the two diagrams of type (4.19).

$$\begin{tikzcd}
X_0 \arrow[r] \arrow[d, "f_0"] & X_1 \arrow[r] \arrow[d, "f_1"] & X_2 \arrow[r] \arrow[d, "f_2"] & \ldots \\
Y_{j(0)} \arrow[r] \arrow[d, "g_{j(0)}"] & X_{j(1)} \arrow[r] \arrow[d, "g_{j(1)}"] & X_{j(2)} \arrow[r] \arrow[d, "g_{j(2)}"] & \ldots \\
Z_{k(j(0))} \arrow[r] & Z_{k(j(1))} \arrow[r] & Z_{k(j(2))} \arrow[r] & \ldots
\end{tikzcd} \tag{4.21}$$

Note that, even if $i \mapsto j(i)$ and $j \mapsto k(j)$ are minimal in the sense of Prop. 4.2.2, the numbers $i \mapsto k(j(i))$ may not.

Corollary 4.2.3. *Let $\hat{X}$ be a sequential ind-object in $\mathcal{C}$ represented by the sequence $X_0 \overset{\sigma_0}{\to} X_1 \overset{\sigma_1}{\to} \ldots$ and let C be an object in $\mathcal{C}$.*

(i) A morphism in $\mathrm{Hom}_{\mathrm{Ind}(\mathcal{C})}(\hat{X}, C)$ is represented by a unique family of morphisms $\{f_i : X_i \to C\}_{i\in\omega}$ satisfying $f_{i+1} \circ \sigma_i = f_i$.

(ii) A morphism in $\mathrm{Hom}_{\mathrm{Ind}(\mathcal{C})}(C, \hat{X})$ is represented by a morphism $f : C \to X_i$. Moreover, if i is minimal and $\hat{X}$ is strict, then f is unique.

Warning 4.2.4. The Yoneda embedding commutes with limits but does not commute with all colimits. This means that even if a diagram $X = (X_0 \to X_1 \to \ldots)$ does have a colimit $\mathrm{colim}_i\, X_i$ in $\mathcal{C}$ it is generally not true that $\mathrm{colim}_i\, X_i$ viewed as constant ind-object is isomorphic to the ind-object represented by X. The next example illustrates this phenomenon.

Example 4.2.5 (Exhaustion of the real line)**.** Let $\mathcal{C} = \mathcal{M}\mathrm{fld}$ be the category of smooth finite-dimensional manifolds. Consider the sequence of embeddings of open intervals,

$$X := \big((-1,1) \hookrightarrow (-2,2) \hookrightarrow (-3,3) \hookrightarrow \ldots\big)\,.$$

On the one hand, a morphism of ind-manifolds from the constant ind-object $\mathbb{R}$ to the ind-manifold $\hat{X}$ represented by this sequence is, according to Prop. 4.2.2, given by a smooth map from $\mathbb{R}$ to one of the intervals $(-n, n)$, in other words, by a bounded function on the real line. On the other hand, the colimit of X is given by the real line $\mathbb{R}$, so that a morphism from $\mathbb{R}$ to the colimit of X is, therefore, a smooth, not necessarily bounded function.

Corollary 4.2.6. *Let $\check{X}$ and $\check{Y}$ be sequential pro-objects in $\mathcal{C}$ represented by the sequences $X_0 \xleftarrow{\sigma_0} X_1 \xleftarrow{\sigma_1} \ldots$ and $Y_0 \xleftarrow{\tau_0} X_1 \xleftarrow{\tau_1} \ldots$. A morphism in $\mathrm{Hom}_{\mathrm{Pro}(\mathcal{C})}(\check{X}, \check{Y})$ is given by a diagram*

$$\begin{tikzcd}
X_{i(0)} \arrow[d, "f_0"] & X_{i(1)} \arrow[l] \arrow[d, "f_1"] & X_{i(2)} \arrow[l] \arrow[d, "f_2"] & \ldots \arrow[l] \\
Y_0 & Y_1 \arrow[l] & Y_2 \arrow[l] & \ldots \arrow[l]
\end{tikzcd}$$

where all $i(j) \le i(j+1)$ for all $j \in \omega$.

Moreover, if all source indices $i(j)$ are chosen to be minimal and if $\check{X}$ is strict, then every f_i is unique.

Proof. The corollary is obtained from Prop. 4.2.2 by using the isomorphism of Prop. 4.1.25. □

Corollary 4.2.7. *Let $\check{X}$ be as in Cor. 4.2.6 and let C be an object in $\mathcal{C}$.*

(i) A morphism in $\mathrm{Hom}_{\mathrm{Pro}(\mathcal{C})}(C, \check{X})$ is uniquely given by a family of morphisms $\{f_i : C \to X_i\}_{i \in \omega}$ satisfying $\sigma_i \circ f_{i+1} = f_i$.

(ii) A morphism in $\mathrm{Hom}_{\mathrm{Pro}(\mathcal{C})}(\check{X}, C)$ is represented by a morphism $f : X_i \to C$. Moreover, if i is minimal and $\check{X}$ is strict, then f is unique.

4.2.2 Sections, retractions, isomorphisms, derivations

Choosing the target indices $j(i)$ to be minimal makes the family of morphisms representing an ind-morphism unique, the minimal choice may be difficult or not natural. For example, the identity morphism of a sequential ind-object $\hat{X}$, is naturally represented by the family $\mathrm{id} : X_i \to X_i$, even though $j(i) = i$ is not the minimal choice when $\sigma_{i-1} : X_{i-1} \to X_i$ is an isomorphism. The price we have to pay is that different families of morphisms may represent the same ind-morphism. The next proposition gives a criterion to decide when this is the case.

Proposition 4.2.8. *Let $\hat{X}$ and $\hat{Y}$ be sequential ind-objects as in Prop. 4.2.2. Two families of morphisms $f_i : X_i \to Y_{j(i)}$ and $f'_i : X_i \to Y_{j'(i)}$, with $j(i)$ and $j'(i)$ not* *necessarily minimal, represent the same morphism of ind-objects if and only if for every $i \in \omega$ one of the following two diagrams commutes,*

$$\begin{tikzcd} X_i \arrow[d, "f_i"'] \arrow[dr, "f'_i"] & \\ Y_{j(i)} \arrow[r] & Y_{j'(i)} \end{tikzcd} \quad \text{or} \quad \begin{tikzcd} X_i \arrow[d, "f'_i"'] \arrow[dr, "f_i"] & \\ Y_{j'(i)} \arrow[r] & Y_{j(i)} \end{tikzcd}$$

depending on whether $j(i) \leq j'(i)$ or $j(i) \geq j'(i)$.

Proof. *** □

Corollary 4.2.9. *Let $\hat{X}$ be a strict sequential ind-object in $\mathcal{C}$ represented by the sequence $X_0 \xrightarrow{\sigma_0} X_1 \xrightarrow{\sigma_1} \dots$. A family of morphisms $f_i : X_i \to X_{j(i)}$ represents the identity morphism of $\hat{X}$ if and only if for every $i \in \omega$ one of the following two conditions is satisfied.*

(i) If $i \leq j(i)$, then f_i is equal to $X_i \xrightarrow{\sigma} X_{j(i)}$.

(ii) If $i > j(i)$, then $X_{j(i)} \xrightarrow{\sigma} X_i$ is an isomorphism and f_i its inverse.

Proof. We apply Prop. 4.2.8 to the case $\hat{Y} = \hat{X}$ and $f'_i := \mathrm{id}_{X_i}$. When $i \leq j(i)$, the second diagram of Prop. 4.2.8 must commute, which is equivalent to condition (i).

When $i > j(i)$, the first diagram of diagram of Prop. 4.2.8 must commute, that is, $\sigma \circ f_i = \mathrm{id}_{X_i}$. Composing on the right with σ yields $\sigma \circ f_i \circ \sigma = \sigma$. By the assumption of strictness of $\hat{X}$, the morphism $\sigma : X_{j(i)} \to X_i$ is a monomorphism, so it follows that $f_i \circ \sigma = \mathrm{id}_{X_{j(i)}}$, i.e. f_i is the left and right inverse of σ, which is condition (ii). □

Corollary 4.2.10. *Let $\hat{X}$ be a strict sequential ind-object in $\mathcal{C}$ represented by the sequence $X_0 \xrightarrow{\sigma_0} X_1 \xrightarrow{\sigma_1} \dots$ in which none of the arrows is an isomorphism. Then the family of morphisms $\mathrm{id}_{X_i} : X_i \to X_i$ is the unique representative of the identity morphism with minimal target indices.*

With Cor. 4.2.9 and the composition of ind-morphisms in terms of the representing families by diagram (4.21), we can easily determine the conditions for families of morphisms to represent sections, retractions, or isomorphisms in the ind-category. Spelling these conditions out would be highly redundant, though.

Example 4.2.11. Let $\hat{X}$ be the strict sequential ind-object in $\mathcal{C}$ represented by the diagram $X : \omega \to \mathcal{C}$. In example 4.1.12 we have seen that every unbounded order preserving map $\Phi : \omega \to \omega$ is final, which implies that the ind-object $\hat{X}'$ represented by $X \circ \Phi$ is isomorphic to $\hat{X}$. The isomorphism $f : \hat{X}' \to \hat{X}$ is represented by the family of morphisms $X'_i = X_{\Phi(i)} \xrightarrow{\mathrm{id}} X_{\Phi(i)}$.

Exercise 4.2.12. Find a family of morphisms representing the inverse of the isomorphism f of example 4.2.11.

As before, we can use the isomorphism of ind- and pro-categories of Prop. 4.1.25 to obtain the dual propositions for pro-objects. We give just one example, because we will need it later for the description of vector fields on pro-manifolds as sections on the pro-tangent bundle.

Proposition 4.2.13. *Let $\check{X}$ and $\check{Y}$ be sequential pro-objects in $\mathcal{C}$ represented by $X_0 \xleftarrow{\sigma_0} X_1 \xleftarrow{\sigma_1} \dots$ and $Y_0 \xleftarrow{\tau_0} X_1 \xleftarrow{\tau_1} \dots$. Let $\check{f} : \check{X} \to \check{Y}$ be a morphism which is represented by the family $(f_i : X_i \to Y_i)_{i \in \omega}$.*

A morphism $\check{g} : \check{Y} \to \check{X}$ represented by a family $(g_i : Y_{j(i)} \to X_i)_{i\in\omega}$ is a section of $\check{f}$ if and only if for every $i \in \omega$ one of the following two conditions is satisfied.

(i) If $i \leq j(i)$, then $f_i \circ g_i$ is equal to $Y_{j(i)} \xrightarrow{\tau} Y_i$.

(ii) If $i > j(i)$, then $Y_i \xrightarrow{\tau} Y_{j(i)}$ is an isomorphism and $f_i \circ g_i$ its inverse.

Remark 4.2.14. When in the sequence $X_0 \xleftarrow{\tau_0} X_1 \xleftarrow{\tau_1} \dots$ a morphism τ_i is an isomorphism, we can skip X_{i+1} and replace τ_i with $\tau_i \circ \tau_{i+1} : X_{i+2} \to X_i$ without changing the pro-object. Unless the sequence is stably constant, i.e. τ_i is an isomorphism for all $i \gg 0$, we obtain by reiterating this procedure a **reduced sequence** for which none of the connecting isomorphisms τ_i is an isomorphisms. If we assume further that the sequence is strict, i.e. all τ_i are epimorphisms, it follows that no composition of connecting morphisms is an isomorphism. In that case, condition (ii) of Prop. 4.2.13 cannot occur.

Example 4.2.15. Let $X_0 \xleftarrow{\sigma_0} X_1 \xleftarrow{\sigma_1} \dots$ be a sequence representing the pro-object $\check{X}$. By condition (i) of Prop. 4.2.13, the morphism $\check{\sigma} : \check{X} \to \check{X}$ represented by the family $\sigma_k : X_{k+1} \to X_k$ is a section of the identity morphism, which is represented by $\mathrm{id}_{X_k} : X_k \to X_k$. We conclude that $\check{\sigma}$ represents the identity morphism of $\check{X}$.

Proposition 4.2.16. *Let $A_0 \xrightarrow{\sigma_0} A_1 \xrightarrow{\sigma_1} \dots$ be a sequence of algebras. Then a derivation of the algebra in ind-vector spaces we obtain from Prop. 4.1.56 is represented by a family of linear maps $\delta_i : A_i \to A_{j(i)}$, $i \in \omega$, such that for all i and all $a, b \in A_i$,*

$$\delta_i(ab) = (\delta_i a)\,\sigma(b) + \sigma(a)\,(\delta_i b)\,,$$

where $\sigma : A_i \to A_{j(i)}$ is the linear map of the diagram A.

Proof. By Prop. 4.2.2 a morphism $\delta : \hat{A} \to \hat{A}$ is represented by a family of morphisms $\delta_i : A_i \mapsto A_{j(i)}$. Let $a, b \in A_i$ and let $\sigma : A_i \to A_{j(i)}$ denote the map of the diagram $A : \omega \to \mathcal{V}\mathrm{ec}$. If the diagram (4.17) commutes, then

$$\begin{aligned} \delta_i(ab) &= (\delta_i \circ \mu_i)(a \otimes b) \\ &= \big(\mu \circ (\delta_i \otimes \mathrm{id} + \mathrm{id} \otimes \delta_i \circ \mu)\big)(a \otimes b) \\ &= (\delta_i a)\,\sigma(b) + \sigma(a)\,(\delta_i b)\,. \end{aligned}$$

Let $a \in A_i$ and $b \in A_j$ be elements that live in different levels of the ind-algebra. The product of a and b in the algebra $\hat{A}$ is given by first mapping them to a higher level A_k, $k \geq i, j$ by the maps $A_i \to A_k$ and $A_j \to A_k$ in the diagram $A : \omega \to \mathcal{V}\mathrm{ec}$ and multiplying them there. □

4.3 Differential geometry on pro-manifolds

A **pro-manifold** is a pro-object in the category $\mathcal{M}\mathrm{fld}$ of smooth finite-dimensional manifolds. In our wish list 3.4.2, we have given conditions for a category to be a good setting for the differential geometry of infinite jets. Our wishes have been granted.

Proposition 4.3.1. *The category* Pro($\mathcal{M}$fld) *satisfies the conditions of the wish list 3.4.2.*

Proof. (i) The Yoneda embedding $y : \mathcal{M}\text{fld} \to \text{Pro}(\mathcal{M}\text{fld})$ is injective and fully faithful. (ii) An infinite inverse sequence $X_0 \leftarrow X_1 \leftarrow \dots$ of manifolds is a diagram $X : \omega^{\text{op}} \to \mathcal{M}\text{fld}$ indexed by the cofiltered category ω^{op}. The limit of $y \circ X$ exists, because it is the copresheaf $\check{X}$ represented by X. (iii) was shown in Cor. 4.2.7. (iv) The functor $\check{U} : \text{Pro}(\mathcal{M}\text{fld}) \to \mathcal{S}\text{et}$ of Cor. 4.1.46 has the required properties. □

As shown in Sec. 4.1.5, the forgetful functor $\check{U}$ on Pro($\mathcal{M}$fld) is given by the functor of points, $\check{U}\check{X} = \text{Hom}(*, \check{X})$, where $* = y(\mathbb{R}^0)$ is the terminal object in Pro($\mathcal{M}$fld). So a **point** of a pro-manifold $\check{X}$ is a morphism $x : * \to \check{X}$.

Proposition 4.3.2. *Let $\check{X}$ be a strict sequential pro-manifold represented by $X_0 \xleftarrow{\sigma_0} X_1 \xleftarrow{\sigma_1} \dots$. Then every point $x : * \to \check{X}$ is given by a unique sequence $x_0, x_1, x_2, \dots$ of points $x_i \in X_i$ such that $x_i = \sigma_i(x_{i+1})$ for all $i \geq 0$.*

Proof. The proposition is a special case of Cor. 4.2.6. □

4.3.1 Tangent bundle and vector fields

Prop. 4.1.34 and Cor. 4.1.35 state that covariant and contravariant functors extend to functors between the ind/pro-categories. Therefore, all *functorial* constructions on smooth manifolds generalize to the pro-manifolds in a straight-forward way. Since pro-manifolds typically arise via cofiltered diagrams of manifolds that fail to have a limit in $\mathcal{M}$fld, we will describe the generalized geometric structures in terms of these diagrams.

The first case we will consider is the tangent functor T from finite-dimensional smooth manifolds to vector bundles, which assigns to every $M \in \mathcal{M}\text{fld}$ the tangent bundle $TM \to M$ and to every smooth map $f : M \to N$ the tangent map $Tf : TM \to TN$. According to Cor. 4.1.35, T induces a functor from pro-manifolds to pro-vector bundles.

Definition 4.3.3. Let $\check{X}$ be a pro-manifold represented by $X : \mathcal{I} \to \mathcal{M}\text{fld}$. The **tangent bundle** of $\check{X}$ is the pro-vector bundle represented by $T \circ X$, which will be denoted by $T\check{X}$.

The tangent bundle of a sequential pro-manifold is represented by the diagram

$$\begin{tikzcd}
TX_0 \arrow[d, "\text{pr}_0"] & TX_1 \arrow[l, "T\sigma_0"'] \arrow[d, "\text{pr}_1"] & TX_2 \arrow[l, "T\sigma_1"'] \arrow[d, "\text{pr}_2"] & \dots \arrow[l] \\
X_0 & X_1 \arrow[l, "\sigma_0"'] & X_2 \arrow[l, "\sigma_1"'] & \dots \arrow[l]
\end{tikzcd}$$

This diagram can be viewed as morphism $\text{pr} : T\check{X} \to \check{X}$ of the total pro-manifold of the tangent bundle to the pro-manifold $\check{X}$. Just as for ordinary bundles, it will usually be clear from the context whether $T\check{X}$ denotes the total pro-manifold of the tangent bundle, or the tangent bundle as pro-object in vector bundles.

A single **tangent vector** of $\check{X}$ is a point $v : * \to T\check{X}$ of the total pro-manifold of the tangent bundle. Every tangent vector v projects to its base point $\text{pr}(v) :=$

$\mathrm{pr} \circ v : * \to \check{X}$. The **tangent space** $T_x\check{X}$ at a point $x : * \to \check{X}$ is defined as the pull-back

$$\begin{tikzcd}
T_x\check{X} \arrow[r] \arrow[d] & T\check{X} \arrow[d, "\mathrm{pr}"] \\
* \arrow[r, "x"] & \check{X}
\end{tikzcd}$$

which exists by Prop. 4.1.43 because the pullback $(T_x\check{X})_i = * \times_{X_i}^{x_i,\mathrm{pr}} TX_i \cong T_{x_i}X_i$ exists for every $i \in \mathcal{I}$ of the index category of a representing diagram $X : \mathcal{I} \to \mathcal{M}\mathrm{fld}$. This also shows that $T_x\check{X}$ is a pro-finite-dimensional vector space represented by the diagram $i \mapsto T_{x_i}X_i$. Let $\check{Y}$ be a pro-manifold represented by $Y : \mathcal{J} \to \mathcal{M}\mathrm{fld}$ and $f : \check{X} \to \check{Y}$ a morphism of pro-manifolds represented by the family $f_j : X_{k(j)} \to Y_j$. Then the **tangent morphism** $Tf : T\check{X} \to T\check{Y}$ is the morphism of pro-manifolds (or pro-vector bundles) represented by the family $Tf_j : TX_{k(j)} \to TY_j$. It maps a tangent vector $v : * \to T\check{X}$ to the tangent vector $Tf\, v := Tf \circ v : * \to T\check{Y}$.

A **vector field** on $\check{X}$ is a section of $\mathrm{pr} : T\check{X} \to \check{X}$. The **value** of a vector field $v : \check{X} \to T\check{X}$ at the point $x : * \to \check{X}$ is the tangent vector $v_x := v \circ x : * \to T\check{X}$. The following proposition describes vector fields on a sequential pro-manifold in terms of the representing sequences.

Proposition 4.3.4. *A vector field v on the sequential pro-manifold represented by $X_0 \overset{\sigma_0}{\leftarrow} X_1 \overset{\sigma_1}{\leftarrow} \dots$ is represented by a family of smooth maps $(v_i : X_{k(i)} \to TX_i)_{i\in\omega}$ such that the diagram*

$$\begin{tikzcd}
TX_0 & TX_1 \arrow[l, "T\sigma_0"'] & TX_2 \arrow[l, "T\sigma_1"'] & \dots \arrow[l] \\
X_{k(0)} \arrow[u, "v_0"'] & X_{k(1)} \arrow[l, "\sigma"'] \arrow[u, "v_1"'] & X_{k(2)} \arrow[l, "\sigma"'] \arrow[u, "v_2"'] & \dots \arrow[l]
\end{tikzcd}$$

commutes and for all $i \geq 0$ we have:

(i) If $i \leq k(i)$, then $\mathrm{pr}_i \circ v_i$ is equal to $X_{k(i)} \overset{\sigma}{\to} X_i$.

(ii) If $i > k(i)$, then $\sigma : X_i \overset{\sigma}{\to} X_{k(i)}$ is an isomorphism and $\mathrm{pr}_i \circ v_i$ its inverse.

Proof. The proposition follows immediately from Cor. 4.2.6 and Prop. 4.2.13. □

All functors on vector bundles, such the functors mapping a vector bundle E to the sum $E \oplus E$, the tensor square $E \otimes E$, exterior powers $\wedge^k E$, etc. extend by Cor. 4.1.35 to pro-vector bundles. Composing them with the tangent functor extends these constructions to the tangent bundle of a pro-manifolds. For example, $\wedge^k T\check{X}$ is the pro-vector bundle represented by the sequence

$$\wedge^k TX_0 \xleftarrow{\wedge^k T\sigma_0} \wedge^k TX_1 \xleftarrow{\wedge^k T\sigma_1} \wedge^k TX_2 \longleftarrow \dots$$

A section of $\wedge T\check{X}$ is a k**-vector field** on the pro-manifold $\check{X}$.

Remark 4.3.5. Constructions that are not functorial, do generally not extend to pro-vector objects by applying them to every object of a representing diagram. For example, mapping a vector bundle to its dual or to its space of sections is not functorial.

A vector field v on a manifold M can be identified with its action on smooth functions, which is a derivation of the $\mathbb{R}$-algebra of smooth functions $C^\infty(M)$, i.e. a linear map

$$\begin{aligned} C^\infty(M) &\longrightarrow C^\infty(M) \\ f &\longmapsto v \cdot f\,, \end{aligned}$$

that satisfies the Leibniz rule

$$v \cdot (fg) = (v \cdot f)\, g + f\, (v \cdot g)\,.$$

The algebraic description of vector fields is typically the best for working with algebraic structures in differential geometry. For example, it is easy to check that the commutator of two derivations is a derivation, which shows that the space of vector fields is equipped with a Lie bracket. Therefore, we would like to generalize this point of view to the pro-manifold setting.

Mapping a smooth manifold to its algebra of smooth functions is a functor $C^\infty : \mathcal{M}\mathrm{fld} \to \mathcal{A}\mathrm{lg}^{\mathrm{op}}$, which by Cor. 4.1.35 induces a functor

$$C^\infty : \mathrm{Pro}(\mathcal{M}\mathrm{fld}) \longrightarrow \mathrm{Ind}(\mathcal{A}\mathrm{lg})^{\mathrm{op}}\,,$$

which maps the pro-manifold represented by $X : \mathcal{I} \to \mathcal{M}\mathrm{fld}$ to the ind-algebra represented by $(C^\infty \circ X)^{\mathrm{op}} : \mathcal{I}^{\mathrm{op}} \to \mathcal{A}\mathrm{lg}$. Since mapping an algebra to its vector space of derivations is *not* functorial, there is no derivation functor that we could extend by Prop. 4.1.34 to a functor on ind-algebras. Instead, we will show that an ind-algebra can be viewed as an algebra object (i.e. a monoid) internal to ind-vector spaces.

4.3.2 Vector fields as derivations

Proposition 4.3.6. *Let $\check{X}$ be the pro-manifold represented by the cofiltered diagram $X : \mathcal{I} \to \mathcal{M}\mathrm{fld}$. Then there is a natural bijection between sections of the tangent bundle $T\check{X} \to \check{X}$ in pro-manifolds and the derivations of the algebra of smooth functions $C^\infty(\check{X})$ in ind-vector spaces.*

For ordinary manifolds, the map from vector fields to derivations is obvious, mapping the tangent vector at every point to its directional derivative. The difficult part is to show that this map has an inverse, for which Hadamard's lemma is used. For pro-manifolds the situation is similar. The map from vector fields to derivations is straight-forward, while for the inverse map we need the following lemma.

Lemma 4.3.7. *Let $\tau : Y \to X$ be a smooth map of manifolds. Let $\delta : C^\infty(X) \to C^\infty(Y)$ be a linear map such that $\delta(fg) = (\delta f)\,(\tau^* g) + (\tau^* f)\,(\delta g)$ for all $f, g \in C^\infty(X)$. Then there is a unique map $v : Y \to TX$ making the diagram*

$$\begin{tikzcd} Y \arrow[r, "v"] \arrow[dr, "\tau"'] & TX \arrow[d, "\mathrm{pr}"] \\ & X \end{tikzcd}$$

commutative, such that $(\delta f)(y) = v_y \cdot f$ for all $f \in C^\infty(X)$ and $y \in Y$.

Proof. Let $f \in C^\infty(X)$ and $y \in Y$. Let $(x^1, \dots, x^n)$ be local coordinates centered at $\big(\tau(y)\big)^i = 0$. By Hadamard's lemma $f(x) = f(0) + h_i(x) x^i$, for some functions $h_i \in C^\infty(X)$. At $x = 0$ we have $h_i(0) = \frac{\partial f}{\partial x^i}(0)$. We thus obtain

$$\begin{aligned}(\delta f)(y) &= \big\{(\delta h_i)(\tau^* x^i) + (\tau^* h_i)(\delta x^i)\big\}_y = (\delta x^i)(y)\,\frac{\partial f}{\partial x^i}(0) \\ &= v_y \cdot f\,,\end{aligned}$$

where $v_y = (\delta x^i)(y) \frac{\partial}{\partial x^i}$. □

Proof of Prop. 4.3.6. We give the proof for a sequential pro-manifold $\check{X}$ represented by the diagram $X_0 \xleftarrow{\tau_0} X_1 \xleftarrow{\tau_1} \dots$. Furthermore, we will assume for simplicity that the sequence is strict and reduced, every morphisms τ_i is an epimorphism but not an isomorphism. This is the case we will need later. The proof for a general pro-manifold is analogous.

Let $v : \check{X} \to T\check{X}$ be a vector field on $\check{X}$. By Prop. 4.3.4 is represented by a family of smooth maps $v_i : X_{k(i)} \to X_i$, $i \in \omega$ such that

$$\begin{tikzcd} X_{k(i)} \arrow[r, "v_i"] \arrow[dr, "\tau_{i\leftarrow k(i)}"'] & TX_i \arrow[d, "\mathrm{pr}"] \\ & X_i \end{tikzcd} \tag{4.22}$$

commutes. This defines a map

$$\begin{aligned}\delta_i : C^\infty(X_i) &\longrightarrow C^\infty(X_{k(i)}) \\ f &\longmapsto (y \mapsto v_y \cdot f)\,.\end{aligned}$$

for every $i \in \omega$. Since by Prop. 4.3.4 the maps v_i satisfy $T\tau_i \circ v_i = v_{i-1} \circ \tau_{k(i-1)\leftarrow k(i)}$, the maps δ_i satisfy $\delta_i \circ \tau_i^* = \tau^*_{k(i-1)\leftarrow k(i)} \circ \delta_{i-1}$. This shows that the family δ_i represents an endomorphism of the ind-vector space $C^\infty(\check{X})$, which is represented by the diagram

$$C^\infty(X_0) \xrightarrow{\tau_0^*} C^\infty(X_1) \xrightarrow{\tau_1^*} C^\infty(X_2) \xrightarrow{\tau_0^*} \dots$$

The Leibniz rule for the directional derivative states that

$$v_y \cdot fg = (v_y \cdot f)\, g(\tau(y)) + f(\tau(y))\, (v_y \cdot g)\,,$$

were $\tau = \tau_{i\leftarrow k(i)}$. This shows that $(\delta_i)_{i\in\omega}$ represents a derivation of $C^\infty(\check{X})$.

Conversely, let δ be a derivation of $C^\infty(\check{X})$ represented by maps $\delta_i : C^\infty(X_i) \to C^\infty(X_{k(i)})$. Then lemma 4.3.7 tells us that every δ_i is the directional derivative given by a unique smooth map $v_i : X_{k(i)} \to TX_i$. Since the family δ_i represents a morphism of ind-vector spaces, the family v_i represents a morphism $v : \check{X} \to T\check{X}$ of pro-manifolds. Moreover, since diagram (4.22) commutes, Prop. 4.3.4 implies that v is a section of the bundle projection $T\check{X} \to \check{X}$. □

Corollary 4.3.8. *The set of vector fields on a pro-manifold is a Lie algebra object in* Ind($\mathcal{V}$ec).

Proof. This follows directly from Props. 4.3.6 and Prop. 4.1.63. □

To get a better intuition for vector fields on graded manifolds we will spell out in local coordinates the structures we have on the pro-manifold represented by the diagram

$$\mathbb{R}^0 \longleftarrow \mathbb{R}^1 \longleftarrow \mathbb{R}^2 \longleftarrow \dots$$

where $\mathbb{R}^{i+1} \to \mathbb{R}^i$ is the projection to the first i-factors (cf. example 4.1.8). Let us denote this pro-manifold by $\check{\mathbb{R}}^\infty$. In local coordinates every submersion is a composition of such projections, so that $\check{\mathbb{R}}^\infty$ is the local model for a large class of pro-manifolds. ***

Let $(x^1, \dots, x^i)$ be the canonical local coordinates of $\mathbb{R}^i$. Then a point $p : * \to \check{\mathbb{R}}^\infty$ can be identified with the infinite sequence $(x^1(p), x^2(p), \dots)$. In fact, the underlying set is

$$\check{U}(\check{\mathbb{R}}^\infty) = \prod_{i=1}^{\infty} \mathbb{R}\,.$$

A function $f : * \to C^\infty(\check{\mathbb{R}}^\infty)$ is a smooth function $f \in C^\infty(\mathbb{R}^i)$ for some i, that is, a function $f = f(x^1, \dots, x^i)$ that depends smoothly on a finite number of coordinates. A tangent vector is an element of the set

$$\check{U}(T\check{\mathbb{R}}^\infty) = \prod_{i=1}^{\infty} T\mathbb{R}\,.$$

Let $(\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^i})$ be the coordinate vector fields on $\mathbb{R}^i$. Then a tangent vector $v_p : * \mapsto T\check{\mathbb{R}}^\infty$ at the point $p = (p^1, p^2, \dots)$ is given by an infinite sequence

$$\prod_{i=1}^{\infty} T\mathbb{R} \ni \Big(v_p^1 \frac{\partial}{\partial x^1}\Big|_{p^1}, v_p^2 \frac{\partial}{\partial x^2}\Big|_{p^1}, \dots\Big) \equiv v_p^1 \frac{\partial}{\partial x^1}\Big|_{p^1} + v_p^2 \frac{\partial}{\partial x^2}\Big|_{p^2} + \dots$$

for $v_p^i \in \mathbb{R}$, where the infinite sum on the right hand side is a somewhat abusive but more suggestive notation. A vector field $v \in \mathfrak{X}(\check{\mathbb{R}}^\infty)$ represented by the maps $v_i : \mathbb{R}^{k(i)} \to T\mathbb{R}^i$, where we recall that $k(i) \leq k(i+1)$, is given by the infinite sum

$$v = v^1 \frac{\partial}{\partial x^1} + v^2 \frac{\partial}{\partial x^2} + \dots = v^i \frac{\partial}{\partial x^i}\,,$$

where $v^i \in C^\infty(\mathbb{R}^{k(i)})$ are the component functions of v. Note that the v^i are different from the maps v_i representing the morphism of pro-manifolds, which are given by

$$v_i(x^1, \dots, x^{k(i)}) = v^1(x^1, \dots, x^{k(1)}) \frac{\partial}{\partial x^1} + \dots + v^i(x^1, \dots, x^{k(i)}) \frac{\partial}{\partial x^i}\,.$$

The action of v on $f \in C^\infty(\mathbb{R}^i)$ is given by

$$v \cdot f = v^1 \frac{\partial f}{\partial x^1} + \dots + v^i \frac{\partial f}{\partial x^i}\,,$$

which is a function in $C^\infty(\mathbb{R}^{k(i)})$. Let w be a vector field represented by the maps $w_i : \mathbb{R}^{l(i)} \to T\mathbb{R}^i$. The Lie bracket of v and w is given by

$$[v, w] = \Big(v^j \frac{\partial w^i}{\partial x^j} - w^j \frac{\partial v^i}{\partial x^j}\Big) \frac{\partial}{\partial x^i}\,.$$

The difference to the usual formula is that the sum over i is infinite. While the index j runs from 1 to ∞ as well, the condition that all component functions v^i and w^i are smooth functions on a finite-dimensional manifold ensures that the sum over j is finite.

4.3.3 Cartan calculus

Assigning to a manifold the complex of differential forms is a functor $\Omega : \mathcal{M}\mathrm{fld} \to \mathrm{dg}\mathcal{A}\mathrm{lg}^{\mathrm{op}}$ to differential graded algebras. By Cor. 4.1.35 this induces a functor

$$\Omega \equiv \mathrm{Pro}(\Omega) : \mathrm{Pro}(\mathcal{M}\mathrm{fld}) \longrightarrow \mathrm{Ind}(\mathrm{dg}\mathcal{A}\mathrm{lg})^{\mathrm{op}} .$$

When $\check{X} \in \mathrm{Pro}(\mathcal{M}\mathrm{fld})$ is represented by the cofiltered diagram $X : \mathcal{I} \to \mathcal{M}\mathrm{fld}$, then $\Omega(\check{X})$ is represented by the filtered diagram $\mathcal{I}^{\mathrm{op}} \to \mathrm{dg}\mathcal{A}\mathrm{lg}$, $i \mapsto \Omega(X_i)$.

The category of graded vector spaces is a closed symmetric monoidal category (see example 4.1.61). So by Prop. 4.1.66, $\mathrm{Ind}(\mathrm{gr}\mathcal{V}\mathrm{ec})$ is enriched over $\mathrm{gr}\mathcal{V}\mathrm{ec}$. A **differential form on** $\check{X}$ is then given by a graded linear map $\alpha : \mathbb{R} \to \Omega(\check{X})$. The graded vector space of all differential forms is given by

$$\underline{\mathrm{Hom}}_{\mathrm{Ind}(\mathrm{gr}\mathcal{V}\mathrm{ec})}\big(\mathbb{R}, \Omega(\check{X})\big) \cong \operatorname*{colim}_{i \in \mathcal{I}} \Omega(\check{X}) = \bar{U}\big(\Omega(\check{X})\big) \in \mathrm{gr}\mathcal{V}\mathrm{ec} ,$$

which is the underlying graded vector space defined in Eq. (4.5). Every differential form α is represented by an element $\alpha \in \Omega^p(X_i)$, where p is the degree of α.

$\Omega(\check{X})$ is a ind-graded algebra. By Prop. 4.1.56 we can view $\Omega(\check{X})$ as an algebra in the category $\mathrm{Ind}(\mathrm{gr}\mathcal{V}\mathrm{ec})$ of ind-$\mathbb{Z}$ graded vector spaces. The product on $\Omega(\check{X})$ will be denoted as usual by $\wedge$. Let $\alpha, \beta \in \Omega(\check{X})$ be represented by $\alpha \in \Omega^p(X_i)$ and $\beta \in \Omega^q(X_j)$. Since the index category $\mathcal{I}$ is cofiltered, there is are morphisms $i \leftarrow k \to j$ in $\mathcal{I}$. They are mapped by the functor X to morphisms

$$X_i \xleftarrow{\tau_{i\leftarrow k}} X_k \xrightarrow{\tau_{j\leftarrow k}} X_j .$$

The product $\alpha \wedge \beta$ is then represented by

$$\tau^*_{i\leftarrow k}\alpha \wedge \tau^*_{j\leftarrow k}\beta \in \Omega^{p+q}(X_k) . \tag{4.23}$$

This shows that $\wedge$ is graded commutative.

Proposition 4.3.9. *The graded vector space* $\underline{\mathrm{Der}}(\Omega(\check{X}))$ *of enriched derivations defined in Eq.* (4.18) *is a Lie algebra in* $\mathrm{gr}\mathcal{V}\mathrm{ec}$*, i.e. a graded Lie algebra.*

Proof. This follows from Prop. 4.1.67. □

Proposition 4.3.10. *The family of de Rham differentials* $d_i : \Omega^\bullet(X_i) \to \Omega^{\bullet+1}(X_i)$ *represents a degree 1 derivation* d *in* $\underline{\mathrm{Der}}\big(\Omega(\check{X})\big)$.

Proof. Each de Rham differential d_i represents a degree 1 element in $\underline{\mathrm{End}}(\Omega(X_i))$, so the family $\{d_i\}$ represents an degree 1 element d in $\underline{\mathrm{End}}(\Omega(\check{X}))$. Let $\alpha, \beta \in \Omega(\check{X})$ be represented by $\alpha \in \Omega^p(X_i)$ and $\beta \in \Omega^q(X_j)$. Their $\wedge$-product is represented by 4.23, so $d(\alpha \wedge \beta)$ is represented by

$$\begin{aligned} d_k(\tau^*_{i\leftarrow k}\alpha \wedge \tau^*_{j\leftarrow k}\beta) &= d_k\tau^*_{i\leftarrow k}\alpha \wedge \tau^*_{j\leftarrow k}\beta + (-1)^p \tau^*_{i\leftarrow k}\alpha \wedge d_k\tau^*_{j\leftarrow k}\beta \\ &= \tau^*_{i\leftarrow k}d_i\alpha \wedge \tau^*_{j\leftarrow k}\beta + (-1)^p \tau^*_{i\leftarrow k}\alpha \wedge \tau^*_{j\leftarrow k}d_j\beta , \end{aligned} \tag{4.24}$$

where we have used that the de Rham differentials commute with pullbacks. The right hand side of Eq. (4.24) represents $d\alpha \wedge \beta + (-1)^p \alpha \wedge d\beta$, which shows that d is a derivation. $\square$

For every tangent vector v_m on a manifold M, let $\iota_v : \Omega^1(M) \to \mathbb{R}$, $i_{v_m}\alpha = \langle \alpha, v_m \rangle$ denote the evaluation of 1-forms on v_m. Let $f : M \to N$ be a smooth map. Recall that the pullback $f^*\alpha$ of a 1-form $\alpha \in \Omega^1(N)$ is defined by $\iota_{v_m} f^*\alpha = \iota_{Tf\,v_m}\alpha$. This means that for a tangent vector on the pro-manifold $\check{X}$ represented by $v_{x,i} : * \to TX_i$, we have commutative diagrams

$$\begin{tikzcd}
\Omega^1(X_i) \arrow[r, "\tau^*"] \arrow[d, "\iota_{v_{x,i}}"'] & \Omega^1(X_j) \arrow[dl, "\iota_{v_{x,j}}"] \\
\mathbb{R} &
\end{tikzcd}$$

where $\tau : X_j \to X_i$ is a morphism of the diagram $X : \mathcal{I} \to \mathcal{M}\mathrm{fld}$, so that $v_{x,i} = T\tau\, v_{x,j}$. This shows that the family of maps $\iota_{v,i} : \Omega^1(X_i) \to \mathbb{R}$ represents a morphism of ind-vector spaces

$$\iota_{v_x} : \Omega^1(\check{X}) \longrightarrow \mathbb{R}\,,$$

which is the evaluation of 1-forms on $\check{X}$ on the tangent vector v_x. Let now $v : \check{X} \to T\check{X}$ be a vector field represented by the smooth maps $v_i : X_{k(i)} \to TX_i$. For every $\alpha \in \Omega^1(\check{X})$ we have the family of smooth maps

$$\begin{aligned}
(i_v\alpha)_i : X_{k(i)} &\longrightarrow \mathbb{R} \\
x &\longmapsto \iota_{v_x}\alpha
\end{aligned}$$

which defines a morphism of ind-manifolds $\iota_v\alpha : \check{X} \to \mathbb{R}$. If α is represented by $\alpha \in \Omega^1(X_i)$, then $\iota_v\alpha$ is represented by $(\iota_v\alpha)_i \in C^\infty(X_{k(i)})$, which is given explicitly by

$$(\iota_v\alpha)_i(x) = \langle \alpha_{\tau(x)}, v_{i,x} \rangle\,.$$

where $\tau : X_{k(i)} \to X_i$ is a smooth map of the diagram X. This map depends linearly on α, so we obtain a morphism of ind-vector spaces

$$\iota_v : \Omega^1(\check{X}) \longrightarrow C^\infty(\check{X})\,,$$

which is the pairing of 1-forms with the vector field v in the setting of pro-manifolds.

In order to extend the pairing to the inner derivative on higher degree differential forms we use that $\Omega(\check{X})$ is generated as graded commutative algebra by functions and 1-forms. For every function $f \in C^\infty(\check{X})$ we set

$$\iota_v f := 0\,.$$

For $\alpha, \beta \in \Omega^1(\check{X})$ we define

$$\iota_v(\alpha \wedge \beta) := \iota_v\alpha \wedge \beta - \alpha \wedge \iota_v\beta\,. \tag{4.25}$$

Note that in order to represent the right hand side by a 1-form on X_l we have to first pull-back all factors along some smooth maps

$$\begin{tikzcd}
 & & X_l \arrow[dll] \arrow[dl] \arrow[d] \arrow[dr] & \\
X_{k(i)} & X_j & X_i & X_{k(j)}
\end{tikzcd}$$

in the diagram X and then multiply and add them in $\Omega(X_l)$. Iterating (4.25), we obtain a derivation of $\Omega(\check{X})$. Let us summarize the result.

Proposition 4.3.11. *Let $v \in \mathfrak{X}(\check{X})$ be a vector field on the pro-manifold $\check{X}$. Then the pairing extends to a unique degree -1 derivation in $\underline{\mathrm{Der}}(\Omega(\check{X}))$.*

Proposition 4.3.12. *In the graded Lie algebra $\underline{\mathrm{Der}}\big(\Omega(\check{X})\big)$ let*

$$\mathcal{L}_v := [\iota_v, d] \,.$$

*denote the **Lie derivative** with respect to the vector field $v \in \mathfrak{X}(\check{X})$. Then*

$$\begin{aligned} [\mathcal{L}_v, \iota_w] = \iota_{[v,w]} \,, \quad [\mathcal{L}_v, \mathcal{L}_w] = \mathcal{L}_{[v,w]} \,, \\ [d,d] = [\iota_v, \iota_w] = [\mathcal{L}_v, d] = 0 \,, \end{aligned}$$

Proof. The proof is completely analogous to the proof for ordinary manifolds. The relations only have to be checked on the generators of the algebra $\Omega(\check{X})$, which are functions f and exact 1-forms $\alpha = df$. Since d is a differential, $[d,d] = 2d^2 = 0$. Since $\iota_v \iota_w f = 0$ and $\iota_v \iota_w \alpha = 0$ for degree reasons, $[\iota_v, \iota_w] = 0$. Using the graded Jacobi identity, we obtain

$$\begin{aligned} [\mathcal{L}_v, d] &= [[\iota_v, d], d] = [\iota_v, [d,d]] - [[\iota_v, d], d] \\ &= -[\mathcal{L}_v, d] \,, \end{aligned}$$

which implies $[d, \mathcal{L}_v] = 0$. On functions, we have $\mathcal{L}_v f = \iota_v df = v \cdot f$. It follows that

$$\begin{aligned} [\mathcal{L}_v, \iota_w] df &= v \cdot (w \cdot f) - w \cdot (v \cdot f) = [v,w] \cdot f \\ &= \iota_{[v,w]} df \end{aligned}$$

Moreover, for degree reasons we have $[\mathcal{L}_v, \iota_w] f = 0 = \iota_{[v,w]} f$. Together this implies the relation $[\mathcal{L}_v, \iota_w] = \iota_{[v,w]}$. Finally, we compute

$$\begin{aligned} [\mathcal{L}_v, \mathcal{L}_w] &= [\mathcal{L}_v, [\iota_w, d]] = [[\mathcal{L}_v, \iota_w], d] - [\iota_v, [\mathcal{L}_v, d]] = [\iota_{[v,w]}, d] \\ &= \mathcal{L}_{[v,w]} \,, \end{aligned}$$

which finishes the proof. □

Terminology 4.3.13. The graded Lie subalgebra of $\underline{\mathrm{Der}}\big(\Omega(\check{X})\big)$ generated by d, ι_v, $\mathcal{L}_v$ for all $v \in \mathfrak{X}(\check{X})$ is called the **Cartan calculus** on the pro-manifold $\check{X}$.

Let us spell out the Cartan calculus on the pro-manifold represented by $\mathbb{R}^0 \leftarrow \mathbb{R}^1 \leftarrow \ldots$ in terms of local coordinates $(x^1, x^2, \ldots)$ as at the end of 4.3.2. Let dx^i denote the coordinate 1-forms. They are dual to the coordinate vector fields $\iota_{\frac{\partial}{\partial x^i}} dx^j = \delta_i^j$. Every 1-form α is given by a finite sum

$$\alpha = \alpha_1 dx^1 + \ldots + \alpha_n dx^n = \alpha_n dx^n ,$$

where $\alpha_i \in C^\infty(\mathbb{R}^{k(i)})$. Let l be the maximum of all indices $\{n, k(1), \ldots, k(i)\}$. Then we can view all functions as functions on $C^\infty(\mathbb{R}^l)$ and therefore view α as a 1-form on $\mathbb{R}^l$. Similarly, a general p-form is given by a finite sum

$$\omega = \sum_{0 < i_1 < \ldots < i_p \leq n} \omega_{i_1, \ldots, i_p} dx^{i_1} \wedge \ldots \wedge dx^{i_p} ,$$

where $\alpha_{i_1, \ldots, i_p} \in C^\infty(\mathbb{R}^k)$ for some k. The de Rham differential of a function f on $\mathbb{R}^n$ is given by by the finite sum

$$df = \frac{\partial f}{\partial x^1} dx^1 + \ldots + \frac{\partial f}{\partial x^n} dx^n .$$

Since the sums are finite, the inner derivative with respect to a vector field, which is given by an infinite sum $v = v^i \frac{\partial}{\partial x^i}$ is well-defined. For example, the pairing of v with the 1-form α is given by the finite sum

$$\iota_v \alpha = v^1 \alpha_1 + \ldots + v^n \alpha_n .$$

The upshot is that in local coordinates the de Rham calculus is given by the usual formulas. The difference is that a vector field is generally given by an infinite sums of partial derivatives. But since, functions depend only on a finite number of coordinates and forms are given by finite sums over products of coordinate 1-forms, all operations are well-defined.

4.3.4 Relation with Fréchet manifolds

Exercises

Exercise 4.1. Show that every category with a terminal object is filtered.

Exercise 4.2. Show that for every filtered category the diagonal functor $\mathcal{I} \to \mathcal{I} \times \mathcal{I}$, $i \mapsto (i, i)$ is final.

Exercise 4.3. Let $\Phi : \mathcal{I} \to \mathfrak{J}$ be a final functor. Show that if $\mathcal{I}$ is filtered, then $\mathfrak{J}$ is filtered.

Exercise 4.4. Let $D : \mathcal{M}\mathrm{fld} \to \mathcal{D}\mathrm{iffl}\mathrm{g}$ denote the functor that maps a manifold to its natural diffeology. Consider the functor $\bar{D} : \mathrm{Pro}(\mathcal{M}\mathrm{fld}) \to \mathcal{D}\mathrm{iffl}\mathrm{g}$ that maps a pro-manifold $\check{X}$ represented by $X : \mathcal{I} \to \mathcal{M}\mathrm{fld}$ to

$$\bar{D}(\check{X}) := \lim_{i \in \mathcal{I}} D(X_i)$$

Show that there is a natural isomorphism

$$\mathrm{Hom}_{\mathrm{Pro}(\mathcal{M}\mathrm{fld})}(M, \check{X}) \cong \mathrm{Hom}_{\mathcal{D}\mathrm{iffl}\mathrm{g}}(DM, \bar{D}\check{X}) ,$$

for all $M \in \mathcal{M}\mathrm{fld}$ and $\hat{X} \in \mathrm{Pro}(\mathcal{M}\mathrm{fld})$. Is there an similar relation between statement for $\mathrm{Hom}_{\mathrm{Pro}(\mathcal{M}\mathrm{fld})}(\check{X}, M)$ and $\mathrm{Hom}_{\mathcal{D}\mathrm{iffl}\mathrm{g}}(\bar{D}\check{Y}, DM)$?

Chapter 5

The variational bicomplex

Heuristically, let us assume that we have a good functorial notion of a complex of differential forms on the diffeological space $\mathcal{F} \times M$. Assume further that F has connected fibres and $\mathcal{F}$ is non-empty, so that j^∞ is surjective by Prop. 3.1.14. Then we can identify the de Rham complex of $J^\infty F$ with the subcomplex

$$\Omega_{\mathrm{loc}}(\mathcal{F} \times M) := (j^\infty)^* \Omega(J^\infty F) \subset \Omega(\mathcal{F} \times M)\,.$$

Even though we will not describe this point of view in a mathematically rigorous way, it provides us with a useful geometric intuition. Since the de Rham complex on a product manifold is a bicomplex, viewing $\Omega(J^\infty F)$ as the subset of local forms on $\mathcal{F} \times M$ suggests that $\Omega(J^\infty F)$ ought to be a bicomplex as well.

5.1 The de Rham complex of the infinite jet bundle

Definition 5.1.1. Let $F \to M$ be a smooth finite-dimensional fibre bundle. The **infinite jet manifold** $J^\infty F$ is the pro-object in the category of smooth fibre bundles over the fixed base manifold M that is represented by the sequence

$$J^0 F \xleftarrow{\mathrm{pr}_{1,0}} J^1 F \xleftarrow{\mathrm{pr}_{2,1}} J^2 F \longleftarrow \ldots$$

The morphism of pro-manifolds $\pi : J^\infty F \to M$ represented by the bundle projections $\pi_i : J^i F \to M$ is called the **infinite jet bundle**.

Remark 5.1.2. Since the functor of points $U : \mathcal{M}\mathrm{fld} \to \mathcal{S}\mathrm{et}$, $M \mapsto \mathrm{Hom}_{\mathcal{M}\mathrm{fld}}(*, M)$ is faithful, so is its right Kan extension to pro-manifolds $\check{U} : \mathrm{Pro}(\mathcal{M}\mathrm{fld}) \to \mathcal{S}\mathrm{et}$, $\check{U}(\check{X}) = \mathrm{Hom}_{\mathrm{Pro}(\mathcal{M}\mathrm{fld})}(*, \check{X})$, as we have shown in Cor. 4.1.46. The underlying set of the infinite jet manifold

$$\mathrm{Hom}_{\mathrm{Pro}(\mathcal{M}\mathrm{fld})}(*, J^\infty F) \cong \lim_{i \in \omega} \mathrm{Hom}_{\mathcal{M}\mathrm{fld}}(*, J^i F)$$

is the *set* of infinite jets of F defined in Def. 3.4.1.

5.1.1 Vertical and horizontal tangent vectors

The diagram of all jet evaluations

$$\begin{tikzcd}
\mathcal{F} \times M \arrow[d, "j^0"] \arrow[dr, "j^1"] \arrow[drr, "j^2"] & & & \\
J^0F & J^1F \arrow[l] & J^2F \arrow[l] & \dots \arrow[l]
\end{tikzcd}$$

represents a morphism of pro-objects $j^\infty : \mathcal{F} \times M \longrightarrow J^\infty F$, where $\mathcal{F} \times M$ is the constant pro-object. But in what category do the objects and morphisms live? In a first step, we could view $\mathcal{F}$ as a discrete manifold, so that j^∞ is a morphism of pro-manifolds. But this completely ignores the geometry of $\mathcal{F}$. We will show that the functional diffeology of $\mathcal{F}$ is the smooth structure that will be the most useful for our purposes.

Recall that the functor $\mathcal{M}\mathrm{fld} \to \mathcal{D}\mathrm{iffl g}$ that maps a smooth manifold to its natural diffeology is injective, faithful, and full. In other words, the category of smooth manifolds is a full subcategory of the category of diffeological spaces. This functor induces an embedding $\mathrm{Pro}(\mathcal{M}\mathrm{fld}) \to \mathrm{Pro}(\mathcal{D}\mathrm{iffl g})$ that is also injective, full, and faithful. *** By this embedding, we can view a pro-manifold as a pro-diffeological space.

Definition 5.1.3. The morphism of pro-diffeological spaces

$$j^\infty : \mathcal{F} \times M \longrightarrow J^\infty F$$

represented by the jet evaluations $j^k : \mathcal{F} \times M \to J^kF$ (which are smooth by Prop. 3.1.15) will be called the diffeological **infinite jet evaluation**.

In Sec. 2.3.2 we have seen that the tangent bundle of a product of two spaces of fields decomposes into the fibre product of fibre-wise linear diffeological bundles. It follows that the tangent bundle of $\mathcal{F} \times M$ splits as

$$\begin{aligned} T(\mathcal{F} \times M) &\cong (T\mathcal{F} \times M) \times_{\mathcal{F}\times M} (\mathcal{F} \times TM) \\ &=: (T\mathcal{F} \times M) \oplus (\mathcal{F} \times TM)\,, \end{aligned} \tag{5.1}$$

where we use the Whitney sum notation for the fibre product, as it is standard for ordinary vector bundles. By Thm. 2.3.2, the fibres of $T\mathcal{F} \to \mathcal{F}$ are vector spaces, so that the isomorphism (5.1) splits every fibre of $T(\mathcal{F} \times M) \to \mathcal{F} \times M$ into the direct sum of vector spaces,

$$T_{(\varphi,m)}(\mathcal{F} \times M) \cong T_\varphi\mathcal{F} \oplus T_mM\,. \tag{5.2}$$

We will call $T_\varphi\mathcal{F}$ the **vertical** tangent space and T_mM the **horizontal** tangent space.

Since the infinite jet evaluation is a morphism of pro-diffeological spaces it has a tangent map

$$\begin{tikzcd}
T(\mathcal{F} \times M) \arrow[r, "Tj^\infty"] \arrow[d] & TJ^\infty F \arrow[d] \\
\mathcal{F} \times M \arrow[r, "j^\infty"] & J^\infty F
\end{tikzcd}$$

which is a morphism of pro-diffeological vector bundles. The main result of this section is that the splitting (5.1) descends along Tj^∞ to a splitting of $TJ^\infty F$. This can be stated as follows.

Theorem 5.1.4. *There is a commutative diagram of pro-objects in the category of fibre-wise linear diffeological bundles*

$$\begin{tikzcd}
(T\mathcal{F} \times M) \oplus (\mathcal{F} \times TM) \arrow[r, "\cong"] \arrow[d, "\alpha\oplus\beta"'] & T(\mathcal{F} \times M) \arrow[d, "Tj^\infty"] \\
J^\infty(VF) \oplus (J^\infty F \times_M TM) \arrow[r, "\cong"] & TJ^\infty F
\end{tikzcd}$$

where α is the infinite jet evaluation of $T\mathcal{F}$ and where β maps $(\varphi, v_m) \mapsto (j^\infty_m \varphi, v_m)$.

Terminology 5.1.5. $J^\infty(VF) \hookrightarrow TJ^\infty F$ is called the **vertical tangent bundle** and $J^\infty F \times_M TM \hookrightarrow TJ^\infty F$ the **horizontal tangent bundle** of $J^\infty F$.

The proof of Thm. 5.1.4 is constructive and the basis for the cohomological formulation of the calculus of variations. First, we recall from Thm. 2.3.2 that the tangent bundle of $\mathcal{F}$ is given by $T\mathcal{F} \cong \Gamma^\infty(M, VF)$, so that a tangent vector $\xi_\varphi \in T\mathcal{F}$ consists of a field $\varphi \in \mathcal{F}$ together with a section ξ of $\varphi^* VF$. In local coordinates $\xi(m) = \xi^\alpha(m) \frac{\partial}{\partial u^\alpha}\big|_{\varphi(m)}$, where ξ^α are local functions on M. There are induced jet coordinates $(x^i, u^\alpha_I, \dot{u}^\alpha_I)$ on $J^k VF$, where

$$\dot{u}^\alpha_I(j^k_m \xi) := \frac{\partial^{|I|}\xi^\alpha}{\partial x^I}\Big|_m ,$$

for $|I| \leq k$. This notation is motivated by the jet coordinates of a tangent vector represented by a path $t \mapsto \varphi_t$, which are given by

$$\dot{u}^\alpha_I(j^k_m \dot{\varphi}_0) = \frac{d}{dt}\Big(u^\alpha_I(j^k_m \varphi_t)\Big)_{t=0} .$$

In terms of these jet coordinates we can compute the tangent map of the jet evaluations explicitly.

Proposition 5.1.6. *The tangent map $T_{(\varphi,m)} j^k : T_\varphi \mathcal{F} \times T_m M \to T_{j^k_m \varphi} J^k F$ of the k-th jet evaluation at $(\varphi, m) \in \mathcal{F} \times M$ is given in local coordinates by*

$$(T_{(\varphi,m)} j^k)(\xi_\varphi, v_m) = \sum_{|I|=0}^{k} \dot{u}^\alpha_I(j^k_m \xi_\varphi) \frac{\partial}{\partial u^\alpha_I} + v^i \Big(\frac{\partial}{\partial x^i} + \sum_{|I|=0}^{k} u^\alpha_{I,i}(j^{k+1}_m \varphi) \frac{\partial}{\partial u^\alpha_I} \Big) . \tag{5.3}$$

Proof. In Eq. (3.7) we have expressed the k-jet of a smooth path $t \mapsto (\varphi_t, m_t)$ in terms of local jet coordinates. To compute the tangent map in terms of coordinates, we have to compute the time derivative of the coordinates of these paths. This yields

$$\frac{d}{dt} x^i\big(j^k(\varphi_t, m_t)\big)\Big|_{t=0} = \dot{m}^i_0 , \tag{5.4}$$

for the coordinates of M. For the fibre coordinates of $J^\infty F \to M$ we obtain

$$\begin{aligned}\frac{d}{dt} u_I^\alpha\big(j^k(\varphi_t, m_t)\big)\Big|_{t=0} &= \frac{d}{dt}\Big(\frac{\partial^{|I|}\varphi^\alpha}{\partial x^I}(t, m_t)\Big)_{t=0} \\ &= \Big(\frac{\partial\partial^{|I|}\varphi^\alpha}{\partial t \partial x^I}(t, m_t) + \frac{\partial\partial^{|I|}\varphi^\alpha}{\partial x^i \partial x^I}(t, m_t)\, \dot{m}_t^i\Big)_{t=0} \\ &= \frac{\partial^{|I|}\dot{\varphi}_0^\alpha}{\partial x^I}(m_0) + \frac{\partial^{|I|+1}\varphi_0^\alpha}{\partial x^{I,i}}(m_0)\, \dot{m}_0^i\,,\end{aligned} \tag{5.5}$$

where we have used the chain rule and that partial derivatives commute. Eqs. (5.4) and (5.5) show that the tangent map is given by

$$\begin{aligned}(T_{(\varphi_0,m_0)}j^k)(\dot{\varphi}_0, \dot{m}_0) &= \dot{m}_0^i \frac{\partial}{\partial x^i} + \sum_{|I|=0}^{k} \big(u_I^\alpha(j_{m_0}^k \dot{\varphi}_0) + \dot{m}_0^i\, u_{I,i}^\alpha(j_{m_0}^{k+1}\varphi_0)\big)\frac{\partial}{\partial u_I^\alpha} \\ &= u_I^\alpha(j_{m_0}^k \dot{\varphi}_0)\frac{\partial}{\partial u_I^\alpha} + \dot{m}_0^i\Big(\frac{\partial}{\partial x^i} + \sum_{|I|=0}^{k} u_{I,i}^\alpha(j_{m_0}^{k+1}\varphi_0)\frac{\partial}{\partial u_I^\alpha}\Big)\,,\end{aligned}$$

where first summand depends linearly on $\dot{\varphi}_0$ and the second linearly on $\dot{m}_0$. Using the notation $\xi_\varphi := (\varphi_0, \dot{\varphi}_0)$ and $v_m = (m_0, \dot{m}_0)$ for the tangent vectors represented by the paths, we obtain Eq. (5.3). □

Notation 5.1.7. Writing an element $(\xi_\varphi, v_m) \in T_\varphi\mathcal{F} \times T_m M$ as sum $(\xi_\varphi, v_m) = \xi_\varphi + v_m$, the tangent map on each summand can be written as

$$\begin{aligned}(T_{(\varphi,m)}j^k)\xi_\varphi &= (T_{\varphi,m}j^k)(\xi_\varphi, 0_m) \\ (T_{(\varphi,m)}j^k)v_m &= (T_{\varphi,m}j^k)(0_\varphi, v_m)\,.\end{aligned}$$

When it is clear what the domain is, we will, therefore, denote the restriction of Tj^∞ to the vertical and horizontal tangent spaces and tangent bundles simply by Tj^∞, for example in the following corollaries 5.1.8 and 5.1.9.

Corollary 5.1.8. *The restriction of Tj^k to the vertical tangent bundle factors as*

$$\begin{tikzcd}
T\mathcal{F} \times M \arrow[d, "\alpha_k"'] \arrow[dr, "Tj^k"] & \\
J^k(VF) \arrow[r, "\tau_k"] \arrow[d, "j^k\mathrm{pr}_F"'] & TJ^kF \arrow[d] \\
J^kF \arrow[r, "\mathrm{id}"] & J^kF
\end{tikzcd}$$

where $\alpha_k := j_{T\mathcal{F}}^k$ is the k-th jet evaluation of $T\mathcal{F} \cong \Gamma^\infty(M, VF)$ and where τ_k is a morphism of fibre-wise linear diffeological bundles given by

$$\tau_k\big(j_m^k \dot{\varphi}_0\big) = \frac{d}{dt}\big(j_m^k \varphi_t\big)_{t=0}\,,$$

for every smooth path $t \mapsto \varphi_t$ of local sections of F.

Corollary 5.1.9. *The restriction of Tj^k to the horizontal tangent bundle factors as*

$$\begin{tikzcd}
\mathcal{F} \times TM \arrow[d, "\beta_{k+1}"'] \arrow[dr, "Tj^k"] & \\
J^{k+1}F \times_M TM \arrow[r, "\sigma_k"] \arrow[d] & TJ^kF \arrow[d] \\
J^{k+1}F \arrow[r, "\mathrm{pr}_{k+1,k}"] & J^kF
\end{tikzcd}$$

where β_{k+1} sends $(\varphi, v_m) \mapsto (j^{k+1}_m\varphi, v_m)$ and where σ_k is a morphism of fibre-wise linear diffeological bundles given by

$$\sigma_k\big(j^{k+1}_m\varphi, \dot{m}_0\big) = \frac{d}{dt}\big(j^k_{m_t}\varphi\big)_{t=0},$$

for all smooth paths $t \mapsto m_t \in M$.

Proof of Thm. 5.1.4. First, we recall that for any sequential pro-object represented by the diagram $X_0 \overset{\nu_0}{\leftarrow} X_1 \overset{\nu_1}{\leftarrow} \ldots$, the family of morphisms $\nu_k : X_{k+1} \to X_k$ represents the identity morphism of $\check{X}$ (see example 4.2.15). In particular, the family of forgetful maps $\mathrm{pr}_{k+1,k} : J^{k+1}F \to J^kF$ represent the identity of $J^\infty F$ and the family $\mathrm{pr}_{k+1,k} : J^{k+1}(VF) \to J^k(VF)$ represents the identity of $J^\infty(VF)$. This shows that the family $\tau_k : J^kVF \to TJ^kF$ and the family

$$\begin{tikzcd}
J^{k+1}(VF) \arrow[r, "\tau_k \circ \mathrm{pr}_{k+1,k}"] \arrow[d] & TJ^kF \arrow[d] \\
J^{k+1}F \arrow[r, "\mathrm{pr}_{k+1,k}"] & J^kF
\end{tikzcd}$$

represent the same morphism of pro-vector bundles covering the identity of $J^\infty F$.

The Whitney sum of the pro-vector bundles $J^\infty(VF) \to J^\infty F$ and $J^kF \times_M TM \to J^kF$ is given by the pull-back over $J^\infty F$. By Prop. 4.1.43 the pullback can be computed level-wise. That is, the pro-vector bundle

$$J^\infty(VF) \oplus (J^\infty F \times_M TM) := J^\infty(VF) \times_{J^\infty F} (J^\infty F \times_M TM)$$

is represented by the sequence of the pullbacks of vector bundles indexed by $k \in \omega$,

$$J^k(VF) \oplus (J^kF \times_M TM) = J^k(VF) \times_{J^kF} (J^kF \times_M TM),$$

with the obvious forgetful maps from level $(k+1)$ to level k. The map $f_k := (\tau_k \circ \mathrm{pr}_{k+1,k}) \oplus \sigma_k$ is a morphism of vector bundles

$$\begin{tikzcd}
J^{k+1}(VF) \oplus (J^{k+1}F \times_M TM) \arrow[r, "f_k"] \arrow[d] & TJ^kF \arrow[d] \\
J^{k+1}F \arrow[r, "\mathrm{pr}_{k+1,k}"] & J^kF
\end{tikzcd}$$

The family $(f_k)_{k\in\omega}$ represents a morphism of pro-objects in the category of fibre-wise linear diffeological bundles,

$$\begin{tikzcd}
(T\mathcal{F} \times M) \oplus (\mathcal{F} \times TM) \arrow[r, "\cong"] \arrow[d] & T(\mathcal{F} \times M) \arrow[d, "Tj^\infty"] \\
J^\infty VF \oplus (J^\infty F \times_M TM) \arrow[r, "f"] & TJ^\infty F
\end{tikzcd}$$

where the left vertical map is given according to Cor. 5.1.8 and Cor. 5.1.9 by $\alpha \oplus \beta$.

It remains to show that f is an isomorphism, which we will do by constructing an inverse. First, we note that

$$J^k(VF) \times_{J^kF} (J^kF \times_M TM) \cong J^k(VF) \times_M TM\,.$$

Let $g_k : TJ^{k+1}F \to J^k(VF) \times_M TM$ be defined by

$$g_k\bigg(\Big(\sum_{|I|=0}^{k+1} \xi^\alpha_I \frac{\partial}{\partial u^\alpha_I} + v^i_m \frac{\partial}{\partial x^i}\Big)_{j^{k+1}_m\varphi}\bigg) = \Big(\sum_{|I|=0}^{k} \big(\xi^\alpha_I - v^i_m u^\alpha_{I,i}(j^{k+1}_m\varphi)\big)\frac{\partial}{\partial u^\alpha_I}, v^i_m \frac{\partial}{\partial x^i}\Big)_{j^k_m\varphi}\,,$$

The family g_k represents a morphism of pro-vector bundles

$$g : TJ^\infty F \longrightarrow J^\infty(VF) \oplus (J^\infty F \times_M TM)\,.$$

The composition $g \circ f$ is represented by the family $(g \circ f)_k = g_k \circ f_{k+1}$. In local coordinates this map is given by

$$(g_k \circ f_{k+1})\bigg(\Big(\sum_{|I|=0}^{k+2} \xi^\alpha_I \frac{\partial}{\partial u^\alpha_I}, v^i_m \frac{\partial}{\partial x^i}\Big)_{j^{k+2}_m\varphi}\bigg) = \Big(\sum_{|I|=0}^{k} \xi^\alpha_I \frac{\partial}{\partial u^\alpha_I}, v^i_m \frac{\partial}{\partial x^i}\Big)_{j^k_m\varphi}\,,$$

which shows that $(g \circ f)_k$ is a morphism of the diagram representing $J^\infty(VF) \oplus (J^\infty \times_M TM)$. It follows that $g \circ f$ is the identity morphism. Similarly, we can show using local coordinates that $f_k \circ g_{k+1}$ is a morphism of the diagram representing $TJ^\infty F$, so that $f \circ g$ is the identity morphism as well. We conclude that f is an isomorphism. □

Warning 5.1.10. The morphisms f_k that represent the splitting f of the pro-vector bundle $TJ^\infty F \to J^\infty F$ are surjective but not injective, so that f_k does not induce a splitting of $TJ^kF \to J^kF$ for any $k < \infty$. This is one of the main reasons why we have to work with the infinite jet bundle.

Remark 5.1.11. A vector $v \in T_{j^k_m\varphi}J^kF$ is in the image $f_k(J^{k+1}F \times_M TM)$ of the $(k+1)$-level of the horizontal tangent bundle if and only if there is a local section ψ such that $v = (T_m j^k\psi)X_m$ for some $X_m \in TM$. (This implies that $j^k_m\psi = j^k_m\varphi$, but v will generally depend on the $(k+1)$-jet of ψ.) The span at every fibre of TJ^kF of all vectors in the image of f_k is called the **Cartan distribution** on J^kF.

A tangent vector $v : * \to TJ^\infty F$ is called **vertical**, if it factors as $* \to J^\infty(VF) \to TJ^\infty F$ through the vertical tangent bundle. Analogously, v is called **horizontal** if it factors as $* \to J^\infty F \times_M TM \to TJ^\infty F$ through the horizontal tangent bundle. A vector field is called vertical (horizontal) if all its values are. As corollary to Thm. 5.1.4 we obtain the following statement.

Corollary 5.1.12. *The vector space of vector fields on $J^\infty F$ decomposes as*

$$\mathfrak{X}(J^\infty F) \cong \mathfrak{X}_{\mathrm{vert}}(J^\infty F) \oplus \mathfrak{X}_{\mathrm{hor}}(J^\infty F) \tag{5.6}$$

into the spaces of vertical and horizontal vector fields. Moreover, we have the natural isomorphisms of vector spaces

$$\begin{aligned} \mathfrak{X}_{\mathrm{vert}}(J^\infty F) &\cong \Gamma\bigl(J^\infty F, J^\infty(VF)\bigr) \\ \mathfrak{X}_{\mathrm{hor}}(J^\infty F) &\cong \mathrm{Hom}(J^\infty F, TM)\,. \end{aligned}$$

Cor. 5.1.12 means that every vector field $v \in \mathfrak{X}(J^\infty F)$ has a unique decomposition $v = v_{\mathrm{vert}} + v_{\mathrm{hor}}$ into a vertical and a horizontal vector field. Let us compute this decomposition in local jet coordinates, in which a vector field $v \in \mathfrak{X}(J^\infty F)$ has the general form

$$v = v^i \frac{\partial}{\partial x^i} + \sum_{|I|=0}^{\infty} v_I^\alpha \frac{\partial}{\partial u_I^\alpha}\,, \tag{5.7}$$

where the components v^i and v_I^α are functions on $J^\infty F$, that is, each is a smooth function on some finite jet manifold. From Eq. (5.3) we deduce that the tangent map of the infinite jet evaluation is given by

$$(Tj^\infty)(\xi_\varphi, v_m) = \sum_{|I|=0}^{\infty} \dot{u}_I^\alpha(j_m^\infty \xi) \frac{\partial}{\partial u_I^\alpha} + v^i \Bigl(\frac{\partial}{\partial x^i} + \sum_{|I|=0}^{\infty} u_{I,i}^\alpha (j_m^\infty \varphi) \frac{\partial}{\partial u_I^\alpha} \Bigr)\,.$$

From this equation we can read off an explicit formula for the decomposition of Cor. 5.1.12. The horizontal component is given by

$$v_{\mathrm{hor}} = v^i D_i\,,$$

where

$$D_i := \frac{\partial}{\partial x^i} + \sum_{|I|=0}^{\infty} u_{I,i}^\alpha \frac{\partial}{\partial u_I^\alpha}\,. \tag{5.8}$$

For the vertical component $v_{\mathrm{vert}} = v - v_{\mathrm{hor}}$ we obtain

$$v_{\mathrm{vert}} = \sum_{|I|=0}^{\infty} (v_I^\alpha - v^i u_{I,i}^\alpha)\, \frac{\partial}{\partial u_I^\alpha}\,.$$

Since v_I^α and v^i are arbitrary, a vertical vector field is of the general form $\sum_{|I|=0}^{\infty} \xi_I^\alpha \frac{\partial}{\partial u_I^\alpha}$ with arbitrary coefficient functions $\xi_I^\alpha \in C^\infty(J^\infty F)$.

Remark 5.1.13. Let $f \in C^\infty(J^k F)$ be a local function. Then $D_i f$ is a function defined on a local coordinate neighborhood of $J^{k+1}F$. When we evaluate it at a jet represented by a local section φ, we obtain

$$\begin{aligned} (D_i f)(j_x^{k+1}\varphi) &= \frac{\partial f}{\partial x^i}(j_x^k \varphi) + \sum_{|I|=0}^{k} \Bigl(\frac{\partial}{\partial x^i} \frac{\partial^{|I|}\varphi^\alpha}{\partial x^I} \Bigr) \frac{\partial f}{\partial u_I^\alpha}(j_x^k \varphi) \\ &= \frac{\partial}{\partial x^i}\,(f \circ j^k \varphi)\big|_x\,. \end{aligned} \tag{5.9}$$

In other words, D_i acts on holonomic sections of the jet bundle as the partial derivative with respect to x^i.

Remark 5.1.14. The space of vertical vector fields is involutive, i.e. closed under the Lie bracket. A straightforward calculation shows that $[D_i, D_j] = 0$, which implies that the space of horizontal vector fields is involutive, as well. The horizontal distribution is called the **Cartan distribution** on $J^\infty F$.

Remark 5.1.15. The map $\sigma : J^\infty F \times_M TM \to TJ^\infty F$ can be viewed as the horizontal lift of a connection on $TJ^\infty F$, which is called the **Cartan connection**.

5.1.2 The variational bicomplex

The splitting of pro-vector bundles of Thm. 5.1.4 induces a splitting of the ind-vector space of 1-forms. More precisely, the statement is the following.

Corollary 5.1.16. *Let $g_k : TJ^{k+1}F \to J^k(VF) \oplus (J^kF \times_M TM)$ be the morphisms of vector bundles defined in the proof of Thm. 5.1.4 that represent an isomorphism of pro-vector bundles. Let $J^k(V^*F) \to J^kF$ denote the dual vector bundle of $J^k(VF) \to J^kF$. Then the family of linear maps*

$$g_k^* : \Gamma\big(J^kF, J^k(V^*F)\big) \oplus \Gamma\big(J^kF, J^kF \times_M T^*M\big) \longrightarrow \Omega^1(J^{k+1}F)$$

represents an isomorphism of ind-vector spaces.

Proof. Every isomorphism of pro-vector bundles induces an isomorphism of sections of the dual bundles. Therefore, the corollary follows from Thm. 5.1.4. □

The maps g_k are surjective but not injective. Therefore, g_k^* is injective but not surjective, so that g_k^* does not induce a splitting of $\Omega^1(J^kF)$ for any $k \geq 0$. This is the dual statement to what we have pointed out in warning 5.1.10 for the tangent bundles. But since g_k^* is injective, we can identify the two summands of the domain of g_k^* with their images under g_k^* in $\Omega^1(J^kF)$.

Definition 5.1.17. The vector spaces

$$\begin{aligned} \Omega^{1,0}(J^{k+1}F) &:= g_k^* \,\Gamma\big(J^kF, J^k(V^*F)\big) \\ \Omega^{0,1}(J^{k+1}F) &:= g_k^* \,\Gamma(J^kF, J^kF \times_M T^*M) \,. \end{aligned}$$

for all $k \geq 0$ are the vector spaces of **vertical** and **horizontal** 1-forms.

From Def. 5.1.17 we obtain for the subspace of (p, q)-forms

$$\begin{aligned} \Omega^{p,q}(J^kF) &= g_k^* \,\Gamma\big(J^kF, \wedge^p J^k(V^*F) \times_{J^kF} (J^kF \times_M \wedge^q T^*M)\big) \\ &= g_k^* \,\Gamma\big(J^kF, \wedge^p J^k(V^*F) \times_M \wedge^q T^*M)\big) \,. \end{aligned} \tag{5.10}$$

We point out once more that $\Omega^{1,0}(J^kF) \oplus \Omega^{0,1}(J^kF)$ is a proper subspace of $\Omega^1(J^kF)$ for every $k > 0$, so that

$$\bigoplus_{n=p+q} \Omega^{p,q}(J^kF) \subsetneq \Omega^n(J^kF)$$

is a proper subspace as well. In other words, there is no natural splitting of the space of 1-forms and no natural bigrading of the space of forms on any of the finite jet manifolds J^kF.

Let $\Omega^{p,q}(J^\infty)$ denote the ind-vector space represented by the sequence

$$\Omega^{p,q}(F) \subset \Omega^{p,q}(J^1 F) \subset \Omega^{p,q}(J^2 F) \subset \dots .$$

Then Cor. 5.1.16 implies that we have a decomposition of ind-vector spaces

$$\Omega^n(J^\infty F) \cong \bigoplus_{n=p+q} \Omega^{p,q}(J^\infty F) . \tag{5.11}$$

For calculations we need to determine the local coordinate form of vertical and horizontal forms. We begin with the following observation.

Lemma 5.1.18. *A 1-form* $\mu \in \Omega^1(J^\infty F)$ *is vertical if and only if* $\iota_v \mu = 0$ *for all* $v \in \mathfrak{X}_{\mathrm{hor}}(J^\infty F)$. *It is horizontal if and only if* $\iota_v \mu = 0$ *for all* $v \in \mathfrak{X}_{\mathrm{vert}}(J^\infty F)$.

Proof. This follows from the non-degeneracy of the pairing of vector fields and 1-forms on $J^\infty F$. □

Lem. 5.1.18 can be used to compute the local form of vertical and horizontal 1-forms in jet coordinates. Let $\mathbf{d}$ denote the de Rham differential of $\Omega(J^{k+1}F)$. A 1-form $\mu \in \Omega(J^{k+1}F)$ is given locally by

$$\mu = \mu_i \mathbf{d}x^i + \sum_{|I|=0}^{k+1} \mu_\alpha^I \mathbf{d}u_I^\alpha , \tag{5.12}$$

where we have written out the sum to emphasize that it is finite. As $C^\infty(J^\infty F)$-module, $\mathfrak{X}_{\mathrm{hor}}(J^\infty F)$ is locally spanned by the basis of local vector fields $\{D_i\}$ defined in Eq. (5.8). The condition for μ to be vertical is therefore

$$0 = \iota_{D_i} \mu = \mu_i + \sum_{|I|=0}^{k+1} u_{I,i}^\alpha \mu_I^\alpha .$$

We can write this condition as

$$\mu_i + \sum_{|I|=0}^{k} u_{I,i}^\alpha \mu_I^\alpha = \sum_{|I|=k+1} u_{I,i}^\alpha \mu_I^\alpha .$$

The left hand side does only depend on jet coordinates up to order $k+1$, whereas the right hand side also depends linearly on the jet coordinates of order $k+2$. Since the equation must hold for all values of jet coordinates of order $k+2$, it follows that both sides must vanish independently. The right hand side vanishes if $\mu_I^\alpha = 0$ for $|I| = k+1$. The vanishing of the left hand side yields an expression for μ_i by the μ_I^α. We conclude that μ is vertical if and only if it is of the local form

$$\mu = \sum_{|I|=0}^{k} \mu_I^\alpha (\mathbf{d}u_I^\alpha - u_{I,i}^\alpha \mathbf{d}x^i) = \mu_I^\alpha \theta_\alpha^I ,$$

where

$$\theta_I^\alpha := \mathbf{d}u_I^\alpha - u_{I,i}^\alpha \mathbf{d}x^i .$$

The 1-forms $\theta_\alpha^I \in \Omega^1(J^{|I|+1}F)$ are linearly independent at every point, so that they are a local basis of the $C^\infty(J^{|I|+1}F)$-module $\Omega^{1,0}(J^{|I|+1}F)$.

Terminology 5.1.19. In the language of variational calculus the 1-forms θ^I_α are called **contact forms**.

As $C^\infty(J^\infty F)$-module, $\mathfrak{X}_{\mathrm{vert}}(J^\infty F)$ is locally spanned by the infinite sums of the vertical coordinate vector fields $\{\frac{\partial}{\partial u^\alpha_I}\}$. This shows that the conditions

$$0 = \iota_{\frac{\partial}{\partial u^\alpha_I}} \mu = \mu^I_\alpha$$

for μ to be horizontal are satisfied if and only if μ is of the form $\mu = \mu_i \mathbf{d}x^i$. We have shown the following.

Lemma 5.1.20. *A local 1-form $\mu \in \Omega^1(J^\infty F)$ given in local coordinates by Eq.* (5.12) *decomposes as $\mu = \mu_{\mathrm{vert}} + \mu_{\mathrm{hor}}$ into its vertical and horizontal components*

$$\mu_{\mathrm{vert}} = \mu^I_\alpha \theta^\alpha_I \,, \quad \mu_{\mathrm{hor}} = (\mu_i + \mu^I_\alpha u^\alpha_{I,i}) \mathbf{d}x^i \,. \tag{5.13}$$

A form $\omega \in \Omega^{p,q}(J^\infty F)$ is given in local coordinates by a finite sum

$$\omega = \omega^{I_1,\dots,I_p}_{\alpha_1,\dots\alpha_p,j_1,\dots,j_p} \theta^{\alpha_1}_{I_1} \wedge \ldots \wedge \theta^{\alpha_p}_{I_p} \wedge \mathbf{d}x^{j_1} \wedge \ldots \wedge \mathbf{d}x^{j_q} \,,$$

where the coefficients $\omega^{I_1,\dots,I_p}_{\alpha_1,\dots\alpha_p,j_1,\dots,j_q}$ are functions in $C^\infty(J^\infty F)$.

Let $\mathrm{pr}_{\Omega^{p,q}} : \Omega(J^\infty F) \to \Omega^{p,q}(J^\infty F)$ denote the projection onto the vector space of degree (p,q)-forms. The vertical component δ and the horizontal component d of the differential $\mathbf{d}$ are given by the linear maps

$$\begin{aligned} \delta^{p,q} &: \Omega^{p,q}(J^\infty F) \longrightarrow \Omega^{p+1,q}(J^\infty F) \,, \quad \delta^{p,q} := \mathrm{pr}_{\Omega^{p,q+1}} \circ \mathbf{d}|_{\Omega^{p,q}} \,, \\ d^{p,q} &: \Omega^{p,q}(J^\infty F) \longrightarrow \Omega^{p,q+1}(J^\infty F) \,, \quad d^{p,q} := \mathrm{pr}_{\Omega^{p,q+1}} \circ \mathbf{d}|_{\Omega^{p,q}} \,. \end{aligned}$$

Proposition 5.1.21. *The bigraded vector space with the vertical differential δ and the horizontal differential d is a differential bicomplex.*

Proof. This is a standard argument. We must show that $\mathbf{d} = \delta + d$ which implies that $\delta^2 = 0$, $d^2 = 0$, and $\delta d = -d\delta$. For $\mathbf{d}$ acting on functions this is clear by definition. For $\mathbf{d}|_{\Omega^{0,1}}$ we have

$$\begin{aligned} \mathbf{d}|_{\Omega^{0,1}} &= (\mathrm{pr}_{\Omega^{2,0}} + \mathrm{pr}_{\Omega^{1,1}} + \mathrm{pr}_{\Omega^{0,2}}) \circ \mathbf{d}|_{\Omega^{0,1}} \\ &= \mathrm{pr}_{\Omega^{2,0}} \circ \mathbf{d}|_{\Omega^{0,1}} + \delta + d \,, \end{aligned}$$

so we have to show that $\mathrm{pr}_{\Omega^{2,0}} \circ \mathbf{d}|_{\Omega^{0,1}} = 0$. Let $\mu \in \Omega^{0,1}(J^\infty F)$. Evaluated on two vertical vector fields $v, w \in \mathfrak{X}(J^\infty F)_{\mathrm{vert}}$ the differential can be written as

$$\begin{aligned} (\mathbf{d}\mu)(v,w) &= v \cdot \mu(w) - w \cdot \mu(v) - \mu([v,w]) \\ &= -\mu([v,w]) \,, \end{aligned}$$

where we have used that $\mu(v) = 0 = \mu(w)$ because μ is horizontal and v, w vertical. We see that $\mathrm{pr}_{\Omega^{2,0}} \circ \mathbf{d}|_{\Omega^{0,1}} = 0$ iff $\mathfrak{X}(J^\infty F)_{\mathrm{vert}}$ is involutive. Analogously, $\mathrm{pr}_{\Omega^{0,2}} \circ \mathbf{d}|_{\Omega^{0,1}} = 0$ iff $\mathfrak{X}(J^\infty F)_{\mathrm{hor}}$ is involutive. The spaces of vertical and horizontal vector fields are both involutive (Rmk. 5.1.14), so that $\mathbf{d}\omega = \delta\omega + d\omega$ for an arbitrary 1-form ω. Since functions and 1-forms generate the graded algebra $\Omega(J^\infty F)$, it follows that $\mathbf{d} = \delta + d$. ☐

We can depict the variational bicomplex by the diagram

$$\begin{tikzcd}
\vdots & \vdots & & \vdots \\
\Omega^{1,0}(J^\infty F) \arrow[r, "d"] \arrow[u, "\delta"] & \Omega^{1,1}(J^\infty F) \arrow[r, "d"] \arrow[u, "\delta"] & \cdots \arrow[r, "d"] & \Omega^{1,\mathrm{top}}(J^\infty F) \arrow[u, "\delta"] \\
\Omega^{0,0}(J^\infty F) \arrow[r, "d"] \arrow[u, "\delta"] & \Omega^{0,1}(J^\infty F) \arrow[r, "d"] \arrow[u, "\delta"] & \cdots \arrow[r, "d"] & \Omega^{0,\mathrm{top}}(J^\infty F) \arrow[u, "\delta"]
\end{tikzcd} \tag{5.14}$$

where top $= \dim M$.

Terminology 5.1.22. The vertical differential δ is also called the **variation**. The horizontal differential d is also called the **spacetime differential**.

Let us compute the differentials in local coordinates. From Eq. (5.13) we obtain

$$\begin{aligned}
\delta x^i &= (\mathbf{d}x^i)_{\mathrm{vert}} = 0 \\
dx^i &= (\mathbf{d}x^i)_{\mathrm{hor}} = \mathbf{d}x^i \\
\delta u^\alpha_I &= (\mathbf{d}u^\alpha_I)_{\mathrm{vert}} = \theta^\alpha_I \\
du^\alpha_I &= (\mathbf{d}u^\alpha_I)_{\mathrm{hor}} = u^\alpha_{I,i} dx^i \,.
\end{aligned}$$

For a function $f \in \Omega^{0,0}(J^\infty F)$ we thus obtain

$$\delta f = \Big(\frac{\partial f}{\partial x^i} \mathbf{d}x^i + \frac{\partial f}{\partial u^\alpha_I} \mathbf{d}u^\alpha_I \Big)_{\mathrm{vert}} = \frac{\partial f}{\partial u^\alpha_I} \delta u^\alpha_I \,, \tag{5.15a}$$

$$df = \Big(\frac{\partial f}{\partial x^i} \mathbf{d}x^i + \frac{\partial f}{\partial u^\alpha_I} \mathbf{d}u^\alpha_I \Big)_{\mathrm{hor}} = \frac{\partial f}{\partial x^i} dx^i + u^\alpha_{I,i} \frac{\partial f}{\partial u^\alpha_I} dx^i = (D_i f)\, dx^i \,. \tag{5.15b}$$

Using the relations $\delta^2 = 0$, $d^2 = 0$, and $\delta d = -d\delta$, we can easily compute the differentials of the coordinate 1-forms,

$$\begin{aligned}
\delta(dx^i) &= -d\delta x^i = 0 \\
d(dx^i) &= 0 \\
\delta(\delta u^\alpha_I) &= 0 \\
d(\delta u^\alpha_I) &= -\delta(du^\alpha_I) = -\delta(u^\alpha_{I,i} dx^i) = -\delta u^\alpha_{I,i} \wedge dx^i \,.
\end{aligned}$$

Using the formulas for the differentials of functions and coordinate 1-forms, as well as the fact that δ and d are derivations, we can compute the differentials of an arbitrary form $\omega \in \Omega^{p,q}(J^\infty F)$ which can be expressed in local coordinates as

$$\omega = \omega^{I_1 \dots I_p}_{\alpha_1 \dots \alpha_p i_1 \dots i_q} \delta u^{\alpha_1}_{I_1} \wedge \ldots \wedge \delta u^{\alpha_p}_{I_p} \wedge dx^{i_1} \wedge \ldots \wedge dx^{i_q} \,. \tag{5.16}$$

Here the coefficients $\omega^{I_1 \dots I_p}_{\alpha_1 \dots \alpha_p i_1 \dots i_q}$ are functions on $J^\infty F$. Note that the sum is finite, i.e. there is a k such that the terms vanish for $|I| > k$.

The inner derivatives of the differentials with respect to the coordinate vector fields are

$$\begin{aligned}\iota_{\frac{\partial}{\partial x^j}} dx^i &= \delta^i_j \\ \iota_{\frac{\partial}{\partial u^\beta_J}} dx^i &= 0 \\ \iota_{\frac{\partial}{\partial x^j}} \delta u^\alpha_I &= -u^\alpha_{I,j} \\ \iota_{\frac{\partial}{\partial u^\beta_J}} \delta u^\alpha_I &= \delta^\alpha_\beta \delta^J_I \,.\end{aligned}$$

5.1.3 Strictly vertical and horizontal vector fields

So far we have seen that the product structure of $\mathcal{F} \times M$ induces a splitting of the tangent bundle of $J^\infty F$ into a horizontal and vertical subspace. The product structure $\mathcal{F} \times M$ also enables us to lift vector fields on $\mathcal{F}$ and vector fields on M to vector fields on $\mathcal{F} \times M$, using the trivial connection of the bundles $\mathcal{F} \times M \to \mathcal{F}$ and $\mathcal{F} \times M \to M$, respectively. These lifts can be characterized infinitesimally as follows.

Proposition 5.1.23. *Let $X \times Y$ be a product of manifolds. Let d_X and d_Y be the differentials of the bicomplex $\Omega(X \times Y)$. A vector field $v \in \mathfrak{X}(X \times Y)$ is the lift of a vector field on X if and only if $[\iota_v, d_Y] = 0$.*

Proof. In local coordinates $(x^1, \ldots, x^p, y^1, \ldots, y^q)$ a vector field v is of the form

$$v = a^i(x,y)\frac{\partial}{\partial x^i} + b^i(x,y)\frac{\partial}{\partial y^i}\,,$$

which is the lift of a vector field on X iff the functions $\frac{\partial a^i}{\partial y^k} = 0$ and $b^i = 0$. For any function $f \in C^\infty(X \times Y)$ we have

$$[\iota_v, d_Y]f = \iota_v d_Y f = b^i \frac{\partial f}{\partial y^i}\,.$$

This shows that $[\iota_v, d_Y]f = 0$ for all functions f iff $b^i = 0$. For a 1-form $\mu = \alpha_i(x,y)dx^i + \beta_i(x,y)dy^i$ we have

$$\begin{aligned}[\iota_v, d_Y]\mu &= (\iota_v d_Y + d_Y \iota_v)\mu \\ &= \iota_v\Big(\frac{\partial \alpha_i}{\partial y^j} dy^j \wedge dx^i + \frac{\partial \beta_i}{\partial y^j} dy^j \wedge dy^i\Big) + d_Y(a^i\alpha_i + b^i\beta_i) \\ &= \Big(\frac{\partial \alpha_i}{\partial y^j}(b^j dx^i - a^i dy^j) + \frac{\partial \beta_i}{\partial y^j}(b^j dy^i - b^i dy^j)\Big) \\ &\quad + \Big(\frac{\partial a^i}{\partial y^j}\alpha_i + a^i \frac{\partial \alpha_i}{\partial y^j} + \frac{\partial b^i}{\partial y^j}\beta_i + b^i \frac{\partial \beta_i}{\partial y^j}\Big) dy^j \\ &= \frac{\partial a^i}{\partial y^j}\alpha_i dy^j + \Big(\frac{\partial \alpha_i}{\partial y^j} b^j dx^i - \frac{\partial \beta_j}{\partial y^i} b^i dy^j + \frac{\partial b^i}{\partial y^j}\beta_i dy^j\Big)\,.\end{aligned}$$

The first term vanishes for all 1-forms μ iff a^i does not depend on the y^i. The second term vanishes iff $b^i = 0$.

We conclude that v is a lift of a vector field on X iff $[i_v, d_Y]$ annihilates all functions and 1-forms. Since functions and 1-forms generate $\Omega(X \times Y)$ as $\mathbb{R}$-algebra and since $[\iota_v, d_Y]$ is a derivation, this is the case iff $[\iota_v, d_Y] = 0$. □

Definition 5.1.24. A vector field $v \in \mathfrak{X}(J^\infty F)$ will be called **strictly vertical** if $[\iota_v, d] = 0$ and **strictly horizontal** if $[\iota_v, \delta] = 0$.

Remark 5.1.25. For a strictly vector field v as in the proof Prop. 5.1.23 we obtain $0 = [\iota_v, d]x^\alpha = \iota_v dx^\alpha$, which shows that it is vertical. Analogously, a strictly horizontal vector field v satisfies $0 = [\iota_v, \delta]u^\alpha_I = \iota_v \delta u^\alpha_I$, which shows that it is horizontal.

Proposition 5.1.26. *We have the following graded Lie brackets:*

$$\begin{gathered}[\iota_\xi, \delta] = \mathcal{L}_\xi\,, \quad [\mathcal{L}_\xi, \iota_{\xi'}] = \iota_{[\xi,\xi']}\,, \quad [\mathcal{L}_\xi, \mathcal{L}_{\xi'}] = \mathcal{L}_{[\xi,\xi']}\,, \\ [\delta, \delta] = [\iota_\xi, \iota_{\xi'}] = [\mathcal{L}_\xi, \delta] = 0\,,\end{gathered}$$

for all strictly vertical vector fields ξ, ξ',

$$\begin{gathered}[\iota_X, d] = \mathcal{L}_X\,, \quad [\mathcal{L}_X, \iota_{X'}] = \iota_{[X,X']}\,, \quad [\mathcal{L}_X, \mathcal{L}_{X'}] = \mathcal{L}_{[X,X']}\,, \\ [d, d] = [\iota_X, \iota_{X'}] = [\mathcal{L}_X, d] = 0\,,\end{gathered}$$

for all strictly horizontal vector fields X, X', and

$$\begin{gathered}[\delta, d] = [\delta, \iota_X] = [\delta, \mathcal{L}_X] = 0 \\ [\iota_\xi, d] = [\iota_\xi, \iota_X] = [\iota_\xi, \mathcal{L}_X] = 0 \\ [\mathcal{L}_\xi, d] = [\mathcal{L}_\xi, \iota_X] = [\mathcal{L}_\xi, \mathcal{L}_X] = 0\,.\end{gathered}$$

In other words, we have two commuting Cartan calculi, the vertical and the horizontal Cartan calculus on $\Omega(J^\infty F)$, each satisfying the relations of Prop. 4.3.12.

Proof. The relations follow directly from the relations of Prop. 4.3.12, from the fact that we have a bicomplex (Prop. 5.1.21), and from the definition 5.1.24 of strictly vertical and horizontal vector fields. □

Lemma 5.1.27. *A vector field $v \in \mathfrak{X}(J^\infty F)$ is strictly horizontal if and only if it is of the local form*

$$v = v^i(x) D_i\,,$$

for smooth functions $v^i \in C^\infty(M)$.

Proof. Since $[\iota_v, \delta]$ is a derivation, it is zero if it vanishes on functions f and the coordinate 1-forms dx^i and δu^α_i, which generate the algebra $\Omega(J^\infty F)$ locally. In local coordinates v is given by Eq. (5.7), so we obtain

$$\begin{aligned}[\iota_v, \delta] f &= \iota_v \frac{\partial f}{\partial u^\alpha_I} \delta u^\alpha_I \\ &= \frac{\partial f}{\partial u^\alpha_I} (v^\alpha_I - u^\alpha_{I,i} v^i)\,,\end{aligned}$$

where we have used that $\delta u_I^\alpha = \theta_I^\alpha = \mathbf{d}u_I^\alpha - u_{I,i}^\alpha \mathbf{d}x^i$. This vanishes for all functions iff $v_I^\alpha = u_{I,i}^\alpha v^i$, i.e. iff v is of the form

$$v = v^i \frac{\partial}{\partial x^i} + u_{I,i}^\alpha v^i \frac{\partial}{\partial u_I^\alpha} = v^i D_i \,,$$

which means that v is horizontal. Next, we obtain

$$\begin{aligned}[\iota_v, \delta] dx^i &= \iota_v \delta dx^i + \delta \iota_v dx^i \\ &= \frac{\partial v^i}{\partial u_I^\alpha} \delta u_I^\alpha \,,\end{aligned}$$

which vanishes iff v^i does not depend on the fibre coordinates u_I^α. Finally, we get

$$[\iota_v, \delta] \delta u_I^\alpha = \delta \iota_v u_I^\alpha + \delta(\iota_v \delta u_I^\alpha) \,,$$

which vanishes when v is horizontal such that the expression in parentheses vanishes. This shows that the last equation does not yield an additional condition. We conclude that v is strictly horizontal if it is horizontal with the coefficient functions v^i depending only on the base coordinates x^i. □

Conceptually, strictly horizontal vector fields in $\mathfrak{X}(J^\infty F)$ play the role of the lifts of vector fields on M to vector fields on $\mathcal{F} \times M$. This interpretation can be made rigorous by observing that the Cartan distribution can be viewed as Ehresmann connection on $J^\infty F \to M$. The corresponding lift of vector fields is given in local coordinates by

$$\begin{aligned}\mathfrak{X}(M) &\longrightarrow \mathfrak{X}(J^\infty F) \\ v^i(x) \frac{\partial}{\partial x^i} &\longmapsto v^i(x) D_i \,.\end{aligned}$$

The analogous interpretation of strictly vertical vector fields as lifts of vector fields on $\mathcal{F}$ is more subtle, since $J^\infty F$ is not a bundle over $\mathcal{F}$.

5.1.4 Equivalence of strictly vertical and local vector fields

Definition 5.1.28. A vector field $\xi : \mathcal{F} \to T\mathcal{F}$ **projects** to a vector field on $J^\infty F$ if there is a diagram of pro-diffeological spaces

$$\begin{tikzcd} \mathcal{F} \times M \arrow[r, "\xi \times \mathrm{id}_M"] \arrow[d, "j^\infty"'] & T\mathcal{F} \times M \arrow[d, "Tj^\infty"] \\ J^\infty F \arrow[r, "v"] & TJ^\infty F \end{tikzcd} \tag{5.17}$$

where $T\mathcal{F} \times M \subset T\mathcal{F} \times TM$ is the subspace embedded by the zero section of TM.

The diagram (5.17) is similar to the condition for ξ to be a local map. In fact, in Thm. 2.3.2 we have shown that $T\mathcal{F} \cong \Gamma^\infty(M, VF)$, so that a vector field on $\mathcal{F}$ is given by a map

$$\xi : \Gamma^\infty(M, F) \longrightarrow \Gamma^\infty(M, VF) \,, \tag{5.18}$$

such that $(\mathrm{pr}_F)_* \xi = \mathrm{id}_\mathcal{F}$, where $\mathrm{pr}_F : VF \to F$ is the bundle projection.

Definition 5.1.29. A vector field on $\mathcal{F}$ is called **local** if the map (5.18) is local in the sense of Def. 3.2.1.

Terminology 5.1.30. ***

Remark 5.1.31. Lem. 6.1.18 shows that there is a good supply of local vector fields.

Remark 5.1.32. By definition, a local vector field $\xi : \mathcal{F} \to T\mathcal{F}$ descends to a smooth map $v_0 : J^k F \to VF$ covering the identity on M. Since $(\mathrm{pr}_F)_* \xi = \mathrm{id}_F$, the map v_0 covers the identity on F.

Terminology 5.1.33 (***)**.** A smooth map $v_0 : J^k F \to VF$ covering the identity of F is called an **evolutionary "vector field"**.

Remark 5.1.34. An evolutionary "vector field" $v_0 : J^k F \to VF$ is not a vector field on $J^\infty F$, which is why we put quotes around it. But it induces a vector field ξ on $\mathcal{F}$ given by $\xi_\varphi := v_0 \circ j^k \varphi$ for all $\varphi \in \mathcal{F}$.

In order to view a local vector field on $\mathcal{F}$ as a vector field on $J^\infty F$, we have to prolong the corresponding evolutionary "vector field" $v_0 : J^k F \to VF$. In Prop. 3.2.8 we have used the maps

$$J^{k+l} F \xrightarrow{\iota_{l,k}} J^l(J^k F) \xrightarrow{j^l v_0} J^l VF\,, \tag{5.19}$$

where $\iota_{l,k} : J^{k+l} F \to J^l(J^k F)$ is the embedding (3.9), that maps $j^{l+k}_m \varphi$ to $j^l_m(j^k \varphi)$, and $j^l v_0 : J^l(J^k) \to J^l VF$ is the l-th prolongation of v_0 defined in Prop. 3.1.20. The maps (5.19) represent a morphism of pro-manifolds $J^\infty F \to J^\infty VF$. In order to obtain a map to $TJ^\infty F$, we need to use the map of Cor. 5.1.8.

Definition 5.1.35. Let $v_0 : J^k F \to VF$ be a map of bundles over F, i.e. an evolutionary "vector field". The the smooth map

$$v_l : J^{k+l} F \xrightarrow{\iota_{l,k}} J^l(J^k F) \xrightarrow{j^l \eta} J^l(VF) \xrightarrow{\tau_l} TJ^l F$$

for $l \geq 0$, is called the **l-th prolongation** of v_0.

Proposition 5.1.36. *Let $v_0 : J^k F \to VF$ be a smooth map covering the identity of F, i.e. an evolutionary "vector field". Then the family $v_l : J^{k+l} F \to TJ^l F$ of smooth maps represents a vector field $v : J^\infty F \to TJ^\infty F$, which is called the* ***infinite prolongation*** *of v_0.*

Proof. We have the following row of commutative squares

$$\begin{tikzcd}
J^{k+l+1} F \arrow[r, "\iota_{l+1,k}"] \arrow[d] & J^{l+1}(J^k F) \arrow[r, "j^{l+1} v_0"] \arrow[d] & J^{l+1} VF \arrow[r, "\tau_{l+1}"] \arrow[d] & TJ^{l+1} F \arrow[d, "T\mathrm{pr}_{l+1,l}"] \\
J^{k+l} F \arrow[r, "\iota_{l,k}"] & J^l(J^k F) \arrow[r, "j^l v_0"] & J^l VF \arrow[r, "\tau_l"] & TJ^l F
\end{tikzcd}$$

where the unmarked vertical arrows are the obvious forgetful maps. The commutativity of the outer rectangle shows that the prolongations v_l represent a morphism $v : J^\infty F \to TJ^\infty F$ of pro-manifolds.

In order to show that v is a section of $TJ^\infty F \to J^\infty F$ we consider the following diagram:

$$\begin{tikzcd}
J^{k+l}F \arrow[r, "\iota_{l,k}"] \arrow[d, "\mathrm{pr}_{k+l,l}"] & J^l(J^kF) \arrow[r, "j^lv_0"] \arrow[d, "j^l\mathrm{pr}_{k,0}"] & J^lVF \arrow[r, "\tau_l"] \arrow[d, "j^l\mathrm{pr}_F"] & TJ^lF \arrow[d, "\mathrm{pr}_{J^lF}"] \\
J^lF \arrow[r, "\mathrm{id}"] & J^lF \arrow[r, "\mathrm{id}"] & J^lF \arrow[r, "\mathrm{id}"] & J^lF
\end{tikzcd}$$

It follows from the definition of $\iota_{l,k}$ of Lem. 3.1.27 that the first square commutes. By assumption, v_0 covers the identity, i.e. $\mathrm{pr}_F \circ v_0 = \mathrm{pr}_{k,0}$. By applying the l-th prolongation functor we obtain $j^l\mathrm{pr}_F \circ j^lv_0 = j^l\mathrm{pr}_{k,0}$, which is the commutativity of the second square. The commutativity of the third square follows from the definition of τ_l of Cor. 5.1.8. From the commutativity of all squares follows the commutativity of the outer rectangle, which is the condition of Prop. 4.3.4 for the maps v_l to represent a section of $TJ^\infty F \to J^\infty F$. □

Theorem 5.1.37. *Let $F \to M$ be a smooth fibre bundle. Let $v : J^\infty F \to TJ^\infty F$ be a vector field on the pro-manifold $J^\infty F$. The following are equivalent:*

(i) v is strictly vertical.

(ii) v is the infinite prolongation of an evolutionary "vector field".

(iii) There is a local vector field on $\mathcal{F}$ that projects to v.

Moreover, the local vector field projecting to v is unique.

The situation of 5.1.37 can be summarized in the following diagram of pro-diffeological spaces:

$$\begin{tikzcd}
\mathcal{F} \times M \arrow[r, "\xi \times \mathrm{id}_M"] \arrow[d, "j^\infty"'] & T\mathcal{F} \times M \arrow[d, "\alpha"] \\
J^\infty F \arrow[r, "v"] \arrow[d] & J^\infty(VF) \arrow[d] \\
J^kF \arrow[r, "v_0"] & VF
\end{tikzcd}$$

Here, we have used that a vertical vector field $v : J^\infty F \to TJ^\infty F$ takes its values in the horizontal tangent space $V(J^\infty F) \hookrightarrow TJ^\infty F$ as defined in Thm. 5.1.4. Thm. 5.1.37 states that given a strictly vertical vector field v, there is a unique ξ that makes this diagram commutative. The map v_0 is not determined uniquely by v. It is unique only if we require the jet order k to be minimal. In general, a local vector field ξ does not determine v or v_0 uniquely. In fact, if $\mathcal{F} = \emptyset$, then *any* v_0 and its prolongation v will make the diagram commutative. If we assume the jet evaluations to be surjective (see Lem. 3.1.13), then v is uniquely determined by ξ and v_0 if we require k to be minimal. The proof of Thm. 5.1.37 relies on the following technical lemmas.

Notation 5.1.38. For every multi-index $I = (I_1, \dots, I_n)$ and $n = \dim M$, we denote

$$D_I := D_1^{I_1} D_2^{I_2} \cdots D_n^{I_n} .$$

In particular, $D_{i_1,\dots,i_k} = D_{i_1} \cdots D_{i_k}$.

Lemma 5.1.39. *A vector field $v \in \mathfrak{X}(J^\infty F)$ is strictly vertical if and only if it is of the form*

$$v = (D_I v^\alpha) \frac{\partial}{\partial u_I^\alpha}\,,$$

for some functions $v^\alpha \in C^\infty(J^\infty F)$.

Proof. Let $v = v_I^\alpha \frac{\partial}{\partial u_I^\alpha}$ be an arbitrary vector field on $J^\infty F$. Locally, the variational bicomplex is generated by the coordinate functions x^i, u_I^α and the coordinate 1-forms dx^i, δu_I^α. The operator $[\iota_v, d]$ is a derivation, so that it suffices to check the relation $[\iota_v, d] = 0$ on the generators. On x^i we obtain the condition

$$[\iota_v, d] x^i = \iota_v dx^i = v^i = 0\,,$$

so that v must be vertical, as already noted. On u_I^α we obtain $[\iota_v, d] u_I^\alpha = \iota_v du_I^\alpha = \iota_v u_{I,i}^\alpha dx^i = u_{Ii}^\alpha v^i = 0$, which follows from the first condition. On the horizontal coordinate one forms we have $[\iota_v, d] dx^i = d\iota_v dx^i = dv^i = 0$ which also follows from the first equation. On the vertical coordinate 1-forms we get

$$\begin{aligned}[\iota_v, d]\delta u_I^\alpha &= \iota_v d\delta u_I^\alpha + d(\iota_v \delta u_I^\alpha) \\ &= \iota_v(-\delta u_{I,i}^\alpha \wedge dx^i) + dv_I^\alpha \\ &= -v_{I,i}^\alpha\, dx^i + v^i \delta u_{I,i}^\alpha + (D_i v_I^\alpha)\, dx^i\,.\end{aligned}$$

Assuming that $v^i = 0$ we obtain the condition

$$v_{I,i}^\alpha = D_i v_I^\alpha\,.$$

By induction, this implies that $v_{i_1,\ldots,i_n}^\alpha = D_{i_1} \cdots D_{i_n} v^\alpha = D_{i_1,\ldots i_k} v^\alpha$. This proves the lemma. □

Lemma 5.1.40. *Let $f : F \to \tilde{F}$ be a map of smooth fibre bundles over M covering the identity of M. Let x^i be local coordinates on a neighborhood U of m, u^α fibre coordinates of F, and $\tilde{u}^\beta$ fibre coordinates of $\tilde{F}$, both over U. Then the k-th prolongation $j^k f : J^k F \to J^k \tilde{F}$ is given in the induced jet bundle coordinates by*

$$f_I^\beta = D_I f^\beta\,,$$

for all multi-indices I with $|I| \le k$, where $f_I^\beta = \tilde{u}_I^\beta \circ j^k f$.

Proof. In Prop. 3.1.20 the k-th prolongation $j^k f$ was defined as the map that sends $j_m^k \varphi$ to $j_m^k (f \circ \varphi)$. In local coordinates we have

$$\begin{aligned}(\tilde{u}_{i_1,\ldots,i_l}^\beta \circ j^k f)(j_x^k \varphi) &= \tilde{u}_{i_1,\ldots,i_l}^\beta \big((j^k f)(j_x^k \varphi)\big) \\ &= \tilde{u}_{i_1,\ldots,i_l}^\beta \big(j_x^k (f \circ \varphi)\big) \\ &= \frac{\partial^l (f^\beta \circ \varphi)}{\partial x^{i_1} \cdots \partial x^{i_l}} \\ &= \frac{\partial^{l-1}}{\partial x^{i_1} \cdots \partial x^{i_{l-1}}} \frac{\partial (f^\beta \circ \varphi)}{\partial x^{i_l}} \\ &= \frac{\partial^{l-1}}{\partial x^{i_1} \cdots \partial x^{i_{l-1}}} \big[(D_{i_l} f^\beta) \circ j^1 \varphi\big] \\ &= \frac{\partial^{l-2}}{\partial x^{i_1} \cdots \partial x^{i_{l-2}}} \big[(D_{i_{l-1}} D_{i_l} f^\beta) \circ j^2 \varphi\big] \\ &= (D_{i_1} \cdots D_{i_l} f^\beta)(j_x^l \varphi)\,,\end{aligned}$$

where we in the last step we have repeatedly applied Eq. (5.9). Note, that while the right hand side depends only on the l-jet of φ, it can be viewed as function on the k-jet. □

Lemma 5.1.41. *Let $\xi : \mathcal{F} \to T\mathcal{F}$ be a local vector field that descends to a smooth map $v_0 : J^kF \to VF$. Then ξ projects to the infinite prolongation $v : J^\infty F \to TJ^\infty F$ of v_0.*

Proof. As we have already noted in Rmk. 5.1.32, v_0 is an evolutionary "vector field", i.e. it covers the identity of F. Moreover, as we have noted in Rmk. 5.1.32, ξ is given in terms of v_0 by the relation

$$\xi_\varphi(m) = v_0(j^k_m\varphi)\,, \tag{5.20}$$

for all $(\varphi, m) \in \mathcal{F} \times M$. Let $\xi_\varphi \in T_\varphi\mathcal{F}$ be represented by the path $t \mapsto \varphi_t$ in $\mathcal{F}$, i.e. $\xi_\varphi = \dot\varphi_0$. Then the tangent map of $j^l : \Gamma^\infty(M, VF) \to VF$ is given by

$$\begin{aligned}
(Tj^l)(\xi_\varphi, m) = (Tj^l)(\dot\varphi_0, m) &= \frac{d}{dt}(j^l_m\varphi_t)\Big|_{t=0} \\
&= \tau_l(j^l_m\dot\varphi_0) = \tau_l(j^l_m\xi_\varphi) \\
&= \tau_l\big(j^l_m(v_0 \circ j^k\varphi)\big) \\
&= \big(\tau_l \circ j^lv_0 \circ j^l(j^k\varphi)\big)(m) \\
&= \big(\tau_l \circ j^lv_0 \circ \iota_{l,k} \circ j^{k+l}\big)(\varphi, m) \\
&= v_l(j^{k+l}_m\varphi)\,,
\end{aligned}$$

where we have used the definition of τ_l from Cor. 5.1.8 and the definition of $\iota_{l,k}$ from Lem. 3.1.27. This shows that the diagram

$$\begin{tikzcd}
\mathcal{F} \times M \arrow[r, "\xi \times \mathrm{id}_M"] \arrow[d, "j^{k+l}"'] & T\mathcal{F} \times M \arrow[d, "Tj^l"] \\
J^{k+l}F \arrow[r, "v_l"] & TJ^lF
\end{tikzcd}$$

commutes for all $l \geq 0$. We conclude that ξ descends to the vector field on $J^\infty F$ that is represented by the prolongations v_l. □

Proof of Thm. 5.1.37. Let $v_0 : J^kF \to VF$ be an evolutionary "vector field" given in local bundle coordinates by $v_0 = v_0^\alpha \frac{\partial}{\partial u^\alpha}$. It follows from Lem. 5.1.40 that the infinite prolongation $v = v_I^\alpha \frac{\partial}{\partial u_I^\alpha}$ of v_0 is given by $v_I^\alpha = D_I v_0^\alpha$. Lem. 5.1.39 now implies that (i) and (ii) are equivalent.

Let $\xi : \mathcal{F} \to T\mathcal{F}$, $\xi \mapsto \xi_\varphi$ be a local vector field that descends to the smooth map $v_0 : J^kF \to VF$. In Rmk. 5.1.32 we have already noted that v_0 is an evolutionary "vector field". Conversely, we have noted in Rmk. 5.1.34 that for every evolutionary "vector field" v_0, there is a unique vector field ξ on $\mathcal{F}$ that descends to v_0. Moreover, we have shown in Lem. 5.1.41 that ξ projects to the infinite prolongation of v_0. We conclude that (ii) and (iii) are equivalent. □

5.1.5 Basic forms

Definition 5.1.42. A differential form $\omega \in \Omega(J^\infty F)$ is called **vertically invariant** if $\mathcal{L}_\xi \omega = 0$ for all vertical vector fields $\xi \in \mathfrak{X}(J^\infty F)$. A horizontal form that is vertically invariant is called **basic**.

Proposition 5.1.43. *A differential form $\omega \in \Omega(J^\infty F)$ is basic if and only if it is the pullback of a form on the base manifold M by the projection $J^\infty F \to M$.*

Proof. Let $\omega \in \Omega^{0,q}(J^\infty)$ be a horizontal form. In local coordinates we have $\omega = \omega_{i_1,\dots,i_q}\, dx^{i_1} \wedge \dots \wedge dx^{i_q}$, where the $\omega_{i_1,\dots,i_q}$ are functions on $J^\infty F$. For the action of the Lie derivative with respect to a vertical coordinate vector field we get

$$\begin{aligned}\mathcal{L}_{\frac{\partial}{\partial u_I^\alpha}}\omega &= \frac{\partial}{\partial u_I^\alpha} \lrcorner (d+\delta)\omega \\ &= \frac{\partial}{\partial u_I^\alpha} \lrcorner \Big((D_j\omega_{i_1,\dots,i_q}) dx^j \wedge dx^{i_1} \wedge \dots dx^{i_q} \\ &\quad + \sum_{|J|=0}^{\infty} \frac{\partial^{|J|}\omega_{i_1,\dots,i_q}}{\partial u_J^\beta}\, \delta u_J^\beta \wedge dx^{i_1} \wedge \dots dx^{i_q}\Big) \\ &= \frac{\partial^{|I|}\omega_{i_1,\dots,i_q}}{\partial u_I^\alpha}\, dx^{i_1} \wedge \dots \wedge dx^{i_q}\,.\end{aligned}$$

We conclude that, in local coordinates, $\omega = \omega_{i_1,\dots,i_q}(x)\, dx^{i_1} \wedge \dots \wedge dx^{i_q}$, that is, ω is the pullback of a form on M. For a general vertical vector field $\xi = \xi_I^\alpha \frac{\partial}{\partial u_I^\alpha}$, we have $\mathcal{L}_\xi \omega = \iota_\xi \mathbf{d}\omega = \xi_I^\alpha(\frac{\partial}{\partial u_I^\alpha} \lrcorner \mathbf{d}\omega) = 0$, so we do not obtain an additional condition on ω. □

Remark 5.1.44. We can define a form $\omega \in \Omega(J^\infty F)$ to be horizontally basic if $\mathcal{L}_X \omega = 0$ for all horizontal vector fields $X \in \mathfrak{X}(J^\infty F)$. However, it turns out that this condition is only satisfied by constant functions, so that it is not a useful concept.

5.2 Cohomology of the variational bicomplex

In our setup, the variational bicomplex consists of a bigraded commutative ind-algebra $\Omega(J^\infty F)$ with the vertical and horizontal derivations δ, which are elements of the graded Lie algebra of internal derivations $\underline{\mathrm{Der}}(\Omega(J^\infty F))$. In cohomology it is more common to view the ind-bigraded algebra, which is represented by the sequence $\Omega(J^0 F) \to \Omega(J^1 F) \to \Omega(J^2 F) \to \dots$, as filtration

$$\Omega(J^0 F) \subset \Omega(J^1 F) \subset \Omega(J^2 F) \subset \dots \subset \bar{\Omega}(J^\infty F)\,,$$

of bigraded algebras, where

$$\bar{\Omega}(J^\infty F) := \operatorname*{colim}_{k \in \omega} \Omega(J^k F)$$

is the colimit in bigraded algebras. The multiplication of the algebra satisfies

$$\Omega(J^k F)\, \Omega(J^l F) \subset \Omega^{\max(j,l)}(J^0 F)\,,$$

and the differentials satisfy

$$\delta\Omega^{p,q}(J^kF) \subset \Omega^{p+1,q}(J^kF)\,, \quad d\Omega^{p,q}(J^kF) \subset \Omega^{p,q+1}(J^{k+1}F)\,,$$

as can be deduced from the local coordinate expressions for δ and d. Viewing the variational ind-bicomplex as filtered bicomplex makes allows us to apply the method of spectral sequences without modification, although we will need only a very simple version of it.

5.2.1 Cohomological partial integration

Let $\alpha, \beta \in \Omega(M)$ be compactly supported differential forms, such that $d\alpha \wedge \beta \in \Omega^{\mathrm{top}}(M)$ is a form of degree top $= \dim M$, so that it can be integrated over M. Then $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^{|\alpha|}\alpha \wedge d\beta$, so that by Stokes' theorem

$$\int_M d\alpha \wedge \beta = -\int_M (-1)^{|\alpha|}\alpha \wedge d\beta + \int_{\partial M} \alpha \wedge \beta\,.$$

If $\partial M = 0$, then the second term on the right hand side vanishes, so that we obtain the coordinate free version of partial integration. The procedure does not depend on taking the integrals and can be stated in terms of the integrands as

$$[d\alpha \wedge \beta] = -[(-1)^{|\alpha|}\alpha \wedge d\beta]\,,$$

where the brackets denote the cohomology classes. This formula, which holds for forms with arbitrary support and in all degrees, can be viewed as cohomological version of partial integration. It generalizes to the d-cohomology classes of the variational bicomplex and is an important step in the computation of its horizontal cohomology classes.

Using the local coordinate formulas for d, we get

$$\begin{aligned}\mathcal{L}_{D_i}\delta u_I^\alpha &= (\iota_{D_i}d + d\iota_{D_i})\delta u_I^\alpha = \iota_{D_i}(-\delta u_{I,i}^\alpha \wedge dx^i)\\ &= \delta u_{I,i}^\alpha\,.\end{aligned} \tag{5.21}$$

Notation 5.2.1. For every multi-index $I = (I_1, \dots, I_n)$ and $n = \dim M$ we denote

$$\mathcal{L}_{D_I} = (\mathcal{L}_{D_1})^{I_1}(\mathcal{L}_{D_2})^{I_2}\cdots(\mathcal{L}_{D_n})^{I_n}\,.$$

In particular, $\mathcal{L}_{D_{i_1,\dots,i_k}} = \mathcal{L}_{D_{i_1}}\cdots\mathcal{L}_{D_{i_k}}$.

From Eq. (5.21) we deduce the formula

$$\delta u_I^\alpha = \mathcal{L}_{D_I}\delta u^\alpha\,.$$

A form $\omega \in \Omega^{p,\mathrm{top}}(J^\infty F)$ for $p > 0$ can be written locally as

$$\omega = \delta u_I^\alpha \wedge \tau_\alpha^I\,,$$

where the $(p-1, \mathrm{top})$-forms τ_α^I are given by

$$\tau_\alpha^I = \frac{1}{p}\Big(\frac{\partial}{\partial u_I^\alpha} \lrcorner\, \omega\Big)\,, \tag{5.22}$$

Using the derivation property of the Lie derivative we get

$$\begin{aligned}\delta u^\alpha_{i_1,\dots,i_k} \wedge \tau^{i_1,\dots,i_k}_\alpha &= (\mathcal{L}_{D_{i_k}} \delta u^\alpha_{i_1,\dots,i_{k-1}}) \wedge \tau^{i_1,\dots,i_k}_\alpha \\ &= -\delta u^\alpha_{i_1,\dots,i_{k-1}} \wedge \mathcal{L}_{D_{i_k}} \tau^{i_1,\dots,i_k}_\alpha + \mathcal{L}_{D_{i_k}} (\delta u^\alpha_{i_1,\dots,i_{k-1}} \wedge \tau^{i_1,\dots,i_k}_\alpha)\,,\end{aligned} \tag{5.23}$$

where there is no summation over repeated indices. Since τ^I_α is of top horizontal degree, the second term on the right hand side is exact, so that Eq. (5.23) can be viewed as a cohomological version of partial integration. Applying Eq. (5.23) recursively to the first term on the right hand side, we obtain

$$\begin{aligned}\delta u^\alpha_{i_1,\dots,i_k} \wedge \tau^{i_1,\dots,i_k}_\alpha &= \delta u^\alpha \wedge (-1)^k (\mathcal{L}_{D_{i_1}} \cdots \mathcal{L}_{D_{i_k}} \tau^{i_1,\dots,i_k}_\alpha) \\ &\quad + \sum_{l=1}^{k} (-1)^{k-l} \mathcal{L}_{D_{i_l}} \big(\delta u^\alpha_{i_1,\dots,i_{l-1}} \wedge (\mathcal{L}_{D_{i_{l+1}}} \cdots \mathcal{L}_{D_{i_k}} \tau^{i_1,\dots,i_k}_\alpha)\big)\,.\end{aligned} \tag{5.24}$$

We will now rewrite this equation in multi-index notation. Using Eq. (3.3), we get

$$\sum_k \sum_{i_1,\dots,i_k} \frac{[i_1,\dots,i_k]!}{k!}\, \delta u^\alpha_{i_1,\dots,i_k} \wedge \tau^{i_1,\dots,i_k}_\alpha = \omega\,.$$

The sum of the first term on the right hand side of Eq. (5.24) is given by

$$\begin{aligned}P\omega :&= \sum_k \sum_{i_1,\dots,i_k} \frac{[i_1,\dots,i_k]!}{k!} (-1)^k \delta u^\alpha \wedge (\mathcal{L}_{D_{i_1}} \cdots \mathcal{L}_{D_{i_k}} \tau^{i_1,\dots,i_k}_\alpha) \\ &= \delta u^\alpha \wedge \sum_I (-1)^{|I|} \mathcal{L}_{D_I} \tau^I_\alpha\,.\end{aligned}$$

Using Eq. (5.22), we can write this as

$$P\omega := \delta u^\alpha \wedge \frac{1}{p} \sum_I (-1)^{|I|} \mathcal{L}_{D_I} \Big(\frac{\partial}{\partial u^\alpha_I} \lrcorner\, \omega \Big)\,. \tag{5.25}$$

Since the second term of the right hand side of Eq. (5.23) is exact, the sum is also exact. We conclude that in local coordinates every form $\omega \in \Omega^{p,\mathrm{top}}(J^\infty F)$, $p > 0$, can be written as

$$\omega = P\omega + d\eta\,,$$

for some $\eta \in \Omega^{p,\mathrm{top}-1}(J^\infty F)$.

Theorem 5.2.2 (Thm. 2.12 in [And89])**.** *There is a unique family of linear operators $P : \Omega^{p,\mathrm{top}}(J^\infty F) \to \Omega^{p,\mathrm{top}}(J^\infty F)$, $p > 0$, which is defined in local coordinates by Eq.* (5.25)*. It has the following properties:*

(i) $\omega - P\omega$ is locally d-exact for all $\omega \in \Omega^{p,\mathrm{top}}(J^\infty F)$, $p > 0$.

(ii) P is a projection, $P^2 = P$.

(iii) $Pd = 0$.

(iv) $(P\delta)^2 = 0$.

Definition 5.2.3. The operator $\Omega^{p,\mathrm{top}}(J^\infty F) \to \Omega^{p+1,\mathrm{top}}(J^\infty F)$, $\omega \mapsto P\delta\omega$ is called the **Euler operator** and denoted by $E := P\delta$.

5.2.2 The acyclicity theorem

Theorem 5.2.4 (Thm. 5.1 in [And89]). *For $p > 0$, the augmented horizontal complex*

$$0 \to \Omega^{p,0}(J^\infty F) \xrightarrow{d} \Omega^{p,1}(J^\infty F) \xrightarrow{d} \dots \xrightarrow{d} \Omega^{p,\mathrm{top}}(J^\infty F) \xrightarrow{P} \Omega^{p,\mathrm{top}}_{\mathrm{fun}}(J^\infty F) \to 0$$

is exact.

Corollary 5.2.5. *Let P be the partial integration operator of Thm. 5.2.2; let $\omega \in \Omega^{p,\mathrm{top}}(J^\infty F)$ for $p > 0$. Then $\omega - P\omega$ is d-exact.*

The rest of this section is devoted to the proof of this theorem. We first prove local exactness by the construction of explicit homotopy operators. In a second step we use a partition of unity and the generalized Mayer-Vietoris sequence to deduce global exactness.

Proposition 5.2.6. *Let $F = \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n = M$ a trivial vector bundle. Then the complex of Thm. 5.2.4 is exact.*

5.2.3 The cohomology of the Euler-Lagrange complex

Theorem 5.2.7. *The cohomology of the Euler-Lagrange complex*

$$\begin{aligned} 0 \longrightarrow \Omega^{0,0}(J^\infty F) &\xrightarrow{d} \Omega^{0,1}(J^\infty F) \xrightarrow{d} \dots \\ \dots \xrightarrow{d} \Omega^{0,n-1}(J^\infty F) &\xrightarrow{d} \Omega^{0,n}(J^\infty F) \xrightarrow{P\delta} \Omega^{1,n}_{\mathrm{fun}}(J^\infty F) \xrightarrow{P\delta} \Omega^{2,n}_{\mathrm{fun}}(J^\infty F) \longrightarrow \dots \end{aligned}$$

where $n = \dim M$, is isomorphic to the de Rham cohomology of the manifold F, that is,

$$H^q\big(\Omega^{0,\bullet}(J^\infty F), d\big) \cong H^q(F)\,, \qquad 0 \le q \le n-1 \tag{5.26a}$$

$$\frac{\ker\big(P\delta : \Omega^{0,n}(J^\infty F) \to \Omega^{1,n}_{\mathrm{fun}}(J^\infty F)\big)}{d\big(\Omega^{0,n-1}(J^\infty F)\big)} \cong H^n(F) \tag{5.26b}$$

$$H^p\big(\Omega^{\bullet,n}_{\mathrm{fun}}(J^\infty F), P\delta\big) \cong H^{n+p}(F)\,, \qquad p \ge 1\,. \tag{5.26c}$$

Warning 5.2.8. In equation (5.26a) of Thm. 5.9 of [And89] it is is erroneously claimed that (5.26a) holds for n. (This would imply that the horizontal cohomology of closed forms in $\Omega^{0,n}(J^\infty F)$ for a vector bundle F over a non-compact manifold M vanishes.) The correct statement is Eq. (5.26b).

Exercises

Exercise 5.2.9. Let $E \to M$ and $F \to M$ be smooth fibre bundles. Show that there is a natural isomorphism of pro-manifolds

$$J^\infty(E \times_M F) \cong J^\infty E \times_M J^\infty F\,.$$

Chapter 6

Local diffeological forms

6.1 Local forms on $\mathcal{F} \times M$

6.1.1 Local forms as $\Omega(M)$-valued forms on $\mathcal{F}$

In Sec. 2.3.2 we have defined differential forms on a diffeological space of fields $\mathcal{F}$. This definition extends to a product of fields $\mathcal{F} \times \mathcal{E}$. In this section, we will consider the case that $E = M \to *$, so that $\mathcal{E} \cong M$ is the base manifold. A differential n-form $\nu \in \Omega(\mathcal{F} \times M)$ is a fibre-wise multilinear and antisymmetric morphism of diffeological spaces

$$\nu : \big(T(\mathcal{F} \times M)_{/\mathcal{F}\times M}\big)^n \longrightarrow \mathbb{R}\,.$$

Analogously, an n-form on $J^\infty F$ can be viewed as a fibre-wise multilinear morphism of pro-manifolds

$$\omega : \big((TJ^\infty F)_{/J^\infty F}\big)^n \longrightarrow \mathbb{R}\,.$$

The precomposition of ω with the tangent map $Tj^\infty : T(\mathcal{F} \times M) \to TJ^\infty F$ on every factor of the fibre product,

$$\begin{tikzcd}
\big(T(\mathcal{F} \times M)_{/\mathcal{F}\times M}\big)^n \arrow[r, "(j^\infty)^*\omega"] \arrow[d, "(Tj^\infty)^n"'] & \mathbb{R} \\
\big((TJ^\infty F)_{/J^\infty F}\big)^n \arrow[ru, "\omega"'] &
\end{tikzcd}$$

is an n-form $(j^\infty)^*\omega \in \Omega^n(\mathcal{F} \times M)$ called the **pullback** of ω by the infinite jet evaluation.

Definition 6.1.1. A differential form on $\mathcal{F} \times M$ is called **local** if it is the pullback of a form on $J^\infty F$ by the infinite jet evaluation. The bigraded vector space of local forms is denoted by $\Omega_{\mathrm{loc}}(\mathcal{F} \times M)$.

Theorem 6.1.2. *Let $\bar\Omega(J^\infty F) := \mathrm{colim}_k\, \Omega(J^k F)$ denote the colimit of bigraded algebras. The pullback of forms on $J^\infty F$ by the infinite jet evaluation,*

$$(j^\infty)^* : \bar\Omega(J^\infty F) \longrightarrow \Omega_{\mathrm{loc}}(\mathcal{F} \times M)\,,$$

is a surjective morphism of bigraded algebras. Moreover, if $j^0 : \mathcal{F} \times M \to F$ is surjective then $(j^\infty)^*$ is an isomorphism.

Proof. Let $\omega \in \Omega^{p,q}(J^\infty F)$. This means that ω is is a $(p+q)$-form on some finite jet bundle $J^{k-1}F$, that factors like

$$\begin{tikzcd}
(J^k(VF)_{/J^kF})^p \times_{J^kF} \big((J^kF \times_M TM)_{/J^kF}\big)^q \arrow[r, "i_{p,q}"] \arrow[dr] & (TJ^{k-1}F_{/J^{k-1}F})^{p+q} \arrow[d, "\omega"] \\
 & \mathbb{R}
\end{tikzcd}$$

through $i_{p,q} = f^p_{k-1} \times f^q_{k-1}$, where f_{k-1} is defined as in the proof of Thm. 5.1.4. We also have the commutative diagram

$$\begin{tikzcd}
\big((T\mathcal{F} \times M)_{/(\mathcal{F}\times M)}\big)^p \times_{\mathcal{F}\times M} \big((\mathcal{F} \times TM)_{/(\mathcal{F}\times M)}\big)^q \arrow[r] \arrow[d, "\alpha^p_k \times \beta^q_k"'] & (T(\mathcal{F} \times M)_{/(\mathcal{F}\times M)})^{p+q} \arrow[d, "(Tj^{k-1})^{p+q}"] \\
(J^k(VF)_{/J^kF})^p \times_{J^kF} \big((J^kF \times_M TM)_{/J^kF}\big)^q \arrow[r] & (TJ^{k-1}F_{/J^{k-1}F})^{p+q}
\end{tikzcd}$$

where α_k is defined in Cor. 5.1.8 and β_k in Cor. 5.1.9. This shows that $\alpha^p_k \times \beta^q_k$ is the restriction of Tj^{k-1} applied to every factor of the fibre-product. Combining the two diagrams, we see that the pullback of a (p,q)-form on $J^\infty F$ is a (p,q)-form on $\mathcal{F} \times M$. In other words, $(j^\infty)^*$ is a morphism of bigraded ind-vector spaces.

The wedge product on both $\bar{\Omega}(J^\infty F)$ and $\Omega(\mathcal{F}\times M)$ is defined as antisymmetrization of the point-wise multiplication. This shows that $(j^\infty)^*$ is an homomorphism of rings.

Assume that j^0 is surjective. This implies by Lem. 3.1.13 that all jet evaluations j^k are surjective. Moreover, we can see from the local coordinate expression of Tj^k given in Prop. 5.1.6 that j^k is a submersion. It follows that the precomposition with Tj^k and, hence, with $(Tj^k)^p$ is injective. Since $(j^\infty)^*$ is a surjection onto its image $\Omega_{\mathrm{loc}}(\mathcal{F} \times M)$, it is follows that it is an isomorphism. □

By definition, a (p,q)-form on $\mathcal{F}\times M$ is a fibre-wise multilinear and antisymmetric map of diffeological spaces

$$\nu : (T\mathcal{F}_{/\mathcal{F}})^p \times (TM_{/M})^q \longrightarrow \mathbb{R}\,.$$

where we recall the notation (2.16) for the fibre product. The domain

$$(T\mathcal{F}_{/\mathcal{F}})^p \times (TM_{/M})^q \cong \Gamma(M, VF \times_M \ldots \times_M VF) \times (TM \times_M \ldots \times_M TM)$$

is is a fibre-wise linear diffeological bundle over $\mathcal{F} \times M$. As we have shown in Thm. 2.3.2, the fibre over $(\varphi, m) \in \mathcal{F} \times M$ is the diffeological vector space

$$\big((T\mathcal{F}_{/\mathcal{F}})^p \times (TM_{/M})^q\big)_{(\varphi,m)} \cong \Gamma(M, \varphi^*VF)^p \oplus (T_mM)^q\,.$$

Every (p,q)-form can be equivalently viewed as a $\Omega^q(M)$-valued p-form on $\mathcal{F}$, i.e. as fibre-wise multilinear and antisymmetric map of diffeological spaces defined by

$$\begin{aligned} \tilde{\nu} : (T\mathcal{F}_{/\mathcal{F}})^p &\longrightarrow \Omega^q(M) \\ \big(\tilde{\nu}(\xi^1_\varphi, \ldots, \xi^p_\varphi)\big)(v^1_m, \ldots, v^q_m) &:= \nu(\xi^1_\varphi, \ldots, \xi^p_\varphi, v^1_m, \ldots, v^q_m)\,, \end{aligned} \tag{6.1}$$

for all $(\varphi, m) \in \mathcal{F} \times M$, all $\xi^1_\varphi, \ldots, \xi^p_\varphi \in T_\varphi\mathcal{F}$ and all $v^1_m, \ldots, v^q_m \in T_mM$. The advantage of this point of view is that (6.1) is a map of sections of fibre bundles, so that we can impose the usual condition of locality.

Proposition 6.1.3. *A form $\nu \in \Omega^{p,q}(\mathcal{F} \times M)$ is local in the sense of Def. 6.1.1 if and only if the associated map $\tilde{\nu}$ defined in* (6.1) *is local in the sense of Def. 3.2.1.*

Lemma 6.1.4. *A (p,q)-form on $\mathcal{F} \times M$ is local if and only if it is the pullback by the infinite jet evaluation of a (p,q)-form on $J^\infty F$.*

Proof. The map $\tilde{\nu}$ of (6.1) is local in the sense of Def. 3.2.1 if and only if there is a commutative diagram

$$\begin{tikzcd}
(T\mathcal{F}_{/\mathcal{F}})^p \times M \arrow[r, "\tilde{\nu}\times \mathrm{id}_M"] \arrow[d] & \Omega^q(M) \times M \arrow[d] \\
(J^k(VF)_{/J^kF})^p \arrow[r, "\tilde{\omega}"] & \wedge^q T^*M
\end{tikzcd} \tag{6.2}$$

where $\tilde{\omega}$ covers the identity on M and where the vertical arrows are the jet evaluations. Since $\tilde{\nu}$ is fibre-wise multilinear and antisymmetric, $\tilde{\omega}$ can be chosen to be fibre-wise multilinear and antisymmetric as well. The map $\tilde{\omega}$ gives rise to a fibre-wise multilinear and antisymmetric map

$$\omega : (J^k(VF)_{/J^\infty F})^p \times_M (TM_{/M})^q \longrightarrow \mathbb{R}\,,$$

which is defined by

$$\omega(\eta^1_{j^k_m\varphi}, \dots, \eta^p_{j^k_m\varphi}, v^1_m, \dots, v^q_m) := \big(\tilde{\omega}(\eta^1_{j^k_m\varphi}, \dots, \eta^p_{j^k_m\varphi})\big)(v^1_m, \dots, v^q_m)\,.$$

It follows that

$$\nu(\xi^1_\varphi, \dots, \xi^p_\varphi, v^1_m, \dots, v^q_m) = \omega(j^k_m\xi^1_\varphi, \dots, j^k_m\xi^p_\varphi, v^1_m, \dots, v^q_m)\,.$$

where $j^k_m : T\mathcal{F} \cong \Gamma(M, VF) \to J^k_m(VF)$ denotes the jet evaluation at $m \in M$. We have the commutative diagram

$$\begin{tikzcd}
\big((T\mathcal{F} \times M)_{/(\mathcal{F}\times M)}\big)^p \times_{\mathcal{F}\times M} \big((\mathcal{F} \times TM)_{/(\mathcal{F}\times M)}\big)^q \arrow[r, "\cong"] \arrow[d, "\alpha^p_k\times\beta^q_k"'] & (T\mathcal{F}_{/\mathcal{F}})^p \times (TM_{/M})^q \arrow[d, "(j^k_{T\mathcal{F}})^p\times \mathrm{id}^q_{TM}"] \\
(J^k(VF)_{/J^kF})^p \times_{J^kF} \big((J^kF \times_M TM)_{/J^kF}\big)^q \arrow[r, "\cong"] & (J^k(VF)_{/J^kF})^p \times_M (TM_{/M})^q
\end{tikzcd}$$

where α_k is defined in Cor. 5.1.8 and β_k in Cor. 5.1.9. Identifying the isomorphic bundles in this diagram, we obtain the following commutative diagram

$$\begin{tikzcd}
\big((T\mathcal{F} \times M)_{/(\mathcal{F}\times M)}\big)^p \times_{\mathcal{F}\times M} \big((\mathcal{F} \times TM)_{/(\mathcal{F}\times M)}\big)^q \arrow[r, "\nu"] \arrow[d, "\alpha^p_k\times\beta^q_k"'] & \mathbb{R} \\
(J^k(VF)_{/J^kF})^p \times_{J^kF} \big((J^kF \times_M TM)_{/J^kF}\big)^q \arrow[ur, "\omega"'] &
\end{tikzcd}$$

Moreover, a straightforward generalization of the proof of Thm. 5.1.4 yields the commutative diagram

$$\begin{tikzcd}
\big((T\mathcal{F} \times M)_{/(\mathcal{F}\times M)}\big)^p \times_{\mathcal{F}\times M} \big((\mathcal{F} \times TM)_{/(\mathcal{F}\times M)}\big)^q \arrow[r] \arrow[d, "\alpha^p_k\times\beta^q_k"'] & (T(\mathcal{F} \times M)_{/(\mathcal{F}\times M)})^{p+q} \arrow[d, "(Tj^{k-1})^{p+q}"] \\
(J^k(VF)_{/J^kF})^p \times_{J^kF} \big((J^kF \times_M TM)_{/J^kF}\big)^q \arrow[r] & (TJ^{k-1}F_{/J^{k-1}F})^{p+q}
\end{tikzcd}$$

which shows that $\alpha_k^p \times \beta_k^q$ is the restriction of Tj^{k-1} applied to every factor of the fibre-product. We conclude that $\nu = (j^\infty)^*\omega$.

Conversely, if $\omega \in \Omega^{p,q}(J^kF)$, then $\nu := \omega \circ (\alpha_k^p \times \beta_k^q) = (j^\infty)^*\omega$ is a local (p, q)-form on $\mathcal{F} \times M$. This concludes the proof. $\square$

6.1.2 Evaluation of forms at fields

The bundle $T\mathcal{F}$ can be restricted to any subset $X \subset \mathcal{F}$. When we equip X with the subspace diffeology we can form the pullback in diffeological spaces

$$\begin{tikzcd}
X \times_{\mathcal{F}} T\mathcal{F} \arrow[r, "i'"] \arrow[d, "\mathrm{pr}_X"'] & T\mathcal{F} \arrow[d, "\mathrm{pr}_{\mathcal{F}}"] \\
X \arrow[r, "i"'] & \mathcal{F}
\end{tikzcd}$$

In Prop. 2.2.8 we have shown that $\mathrm{pr}_{\mathcal{F}}$ is a subduction. It then follows from Cor. 2.1.21 that pr_X is an subduction. In other words, $X \times_{\mathcal{F}} T\mathcal{F} \to X$ is a diffeological bundle. By definition of the subspace diffeology, i is an induction. Cor. 2.1.21 then implies that i' is an induction. Moreover, the subbundle $X \times_{\mathcal{F}} T\mathcal{F} \to X$ inherits a fibre-wise linear structure from $T\mathcal{F}$.

The restriction of a form $\nu \in \Omega^{p,q}(\mathcal{F} \times M)$ to the subbundle over $X \times M$,

$$\nu|_X : \big(X \times_{\mathcal{F}} (T\mathcal{F}_{/\mathcal{F}})^p\big) \times (TM_{/M})^q \hookrightarrow (T\mathcal{F}_{/\mathcal{F}})^p \times (TM_{/M})^q \xrightarrow{\nu} \mathbb{R}\,,$$

is a smooth fibre-wise multilinear and antisymmetric map. When $X = \{\varphi\}$, then $\{\varphi\} \times_{\mathcal{F}} T\mathcal{F} \cong T_\varphi\mathcal{F}$ is the tangent space at $\varphi \in \mathcal{F}$.

Definition 6.1.5. Let $\nu \in \Omega^{p,q}(\mathcal{F} \times M)$ and $\varphi \in \mathcal{F}$. The restriction of ν to the subbundle over $\{\varphi\} \times M$ will be denoted by

$$\nu_\varphi : (T_\varphi\mathcal{F})^p \times (TM_{/M})^q \longrightarrow \mathbb{R}$$

and called the **evaluation of ν at φ**.

Terminology 6.1.6. When $\nu_\varphi = 0$ is the zero map, ν is said to be **zero at** φ or to **vanish at** φ. When $\nu|_X = 0$ is the zero map, ν is said to **vanish on** X.

Remark 6.1.7. The evaluation of ν at φ can be equivalently viewed as the smooth multilinear and antisymmetric map

$$\tilde{\nu}_\varphi : (T_\varphi\mathcal{F})^p \hookrightarrow (T\mathcal{F}_{/\mathcal{F}})^p \xrightarrow{\tilde{\nu}} \Omega^q(M)\,.$$

When $\nu \in \Omega^{0,q}(\mathcal{F} \times M)$, then the evaluation at φ is given by $\tilde{\nu}_\varphi = \tilde{\nu}(\varphi)$.

An n-form on $J^\infty F$ is given by a fibre-wise linear and antisymmetric map

$$\omega : \big((TJ^\infty F)_{/J^\infty F}\big)^n \longrightarrow \mathbb{R}\,.$$

For a local version of the evaluation of ω at $\varphi \in \mathcal{F}$ we restrict the domain of ω to the subbundle over the image of the infinite jet prolongation $(j^\infty\varphi)(M) \cong M$, which is isomorphic to the pullback bundle

$$\begin{tikzcd}
M \times_{J^\infty F} \big((TJ^\infty F)_{/J^\infty F}\big)^n \arrow[r, "i"] \arrow[d] & \big((TJ^\infty F)_{/J^\infty F}\big)^n \arrow[d, "\mathrm{pr}"] \\
M \arrow[r, "j^\infty\varphi"] & J^\infty F
\end{tikzcd}$$

Definition 6.1.8. Let $\omega \in \Omega^n(J^\infty F)$ and $\eta \in \Gamma(M, J^\infty F)$. The restriction of ω to the pullback bundle along η will be denoted by

$$\omega_\eta : M \times^{\eta;\mathrm{pr}}_{J^\infty F} \big((TJ^\infty F)_{/J^\infty F}\big)^n \hookrightarrow \big((TJ^\infty F)_{/J^\infty F}\big)^n \xrightarrow{\omega} \mathbb{R}$$

and called the **evaluation of ω at η.**

Terminology 6.1.9. Let $\varphi \in \mathcal{F}$. The evaluation of $\omega \in \Omega(J^\infty F)$ at $j^\infty\varphi$ is called the **evaluation of ω at φ.** A form $\omega \in \Omega(J^\infty F)$ is said to be **zero at φ** or to **vanish at φ** when $\omega_{j^\infty\varphi} = 0$.

The following lemma shows that the notions of evaluation of forms on $\mathcal{F} \times M$ and forms on $J^\infty F$ at fields are compatible.

Lemma 6.1.10. *Let $\omega \in \Omega^n(J^\infty F)$ and let $v^1, \dots, v^n \in T_\varphi\mathcal{F} \times TM$. Then*

$$\big((j^\infty)^*\omega\big)_\varphi(v^1, \dots, v^n) = \omega_{j^\infty\varphi}(Tj^\infty v^1, \dots, Tj^\infty v^n)\,.$$

Proof. We have the following commutative diagram:

$$\begin{tikzcd}
(\{\varphi\} \times M) \times_{\mathcal{F}\times M} \big(T(\mathcal{F} \times M)_{/\mathcal{F}\times M}\big)^n \arrow[r, "i'"] \arrow[d, "\mathrm{id}_M \times (Tj^\infty)^n"'] & \big(T(\mathcal{F} \times M)_{/\mathcal{F}\times M}\big)^n \arrow[d, "(Tj^\infty)^n"'] \arrow[dr, "(j^\infty)^*\omega"] & \\
M \times^{j^\infty\varphi,\mathrm{pr}}_{J^\infty F} \big((TJ^\infty F)_{/J^\infty F}\big)^n \arrow[r, "i"'] & \big((TJ^\infty F)_{/J^\infty F}\big)^n \arrow[r, "\omega"'] & \mathbb{R}
\end{tikzcd}$$

The evaluations of the forms at φ are given by

$$\omega_{j^\infty\varphi} = \omega \circ i\,, \qquad \big((j^\infty)^*\omega\big)_\varphi = \big((j^\infty)^*\omega\big) \circ i'\,,$$

so that the lemma follows from the commutativity of the diagram. □

Lemma 6.1.11. *Let $\omega \in \Omega^{0,q}(J^\infty F)$ and $\varphi \in \mathcal{F}$. Then*

$$\big(\widetilde{(j^\infty)^*\omega}\big)(\varphi) = (j^\infty\varphi)^*\omega\,,$$

holds in $\Omega^q(M)$.

Proof. This is Lem. 6.1.10 for $p = 0$. □

Lemma 6.1.12. *Let $\omega \in \Omega(J^\infty F)$ and $\varphi \in \mathcal{F}$. Then ω vanishes at φ if and only $(j^\infty)^*\omega$ does.*

Proof. Let $\hat{v} \in T_{j^k_m\varphi}J^kF$. By working in a tubular neighborhood of $\varphi(M) \subset F$ we can find a path $t \mapsto (\psi_t, m_t) \in \mathcal{F} \times M$ such that $\psi_0 = \varphi$ and $\frac{d}{dt}j^k_{m(t)}\psi_t = v$. This shows that $v := (\dot{\psi}_0, \dot{m}_0) \in T_\varphi \times M$ is mapped by Tj^k to $\hat{v}$. It follows that for all $\hat{v}^1, \dots, \hat{v}^n \in T_{j^k_m\varphi}J^kF$ there are $v^1, \dots, v^n \in T_\varphi\mathcal{F} \times TM$ such that

$$\omega_{j^k\varphi}(\hat{v}^1, \dots, \hat{v}^n) = \omega_{j^k\varphi}(Tj^kv^1, \dots, Tj^kv^n)\,.$$

The lemma now follows from Lem. 6.1.10. □

Remark 6.1.13. Note that Lem. 6.1.12 holds even when $(j^\infty)^*$ is not injective.

6.1.3 The PDE of a local form

Let us now view $\omega \in \Omega^n(J^kF)$ as a section $\omega : J^kF \to \wedge^n T^* J^k F$. Let us denote precomposition of the form with a section $\eta \in \Gamma(M, J^kF)$ of the k-th jet bundle by

$$\omega_\eta : M \xrightarrow{\eta} J^kF \xrightarrow{\omega} \wedge^n T^* J^k F\,. \tag{6.3}$$

We have the following commutative diagram,

$$\begin{tikzcd}
M \arrow[drr, bend left, "\omega_\eta"] \arrow[dr, dotted] \arrow[ddr, bend right, "\mathrm{id}"'] & & \\
& M \times_{J^kF} (\wedge^n T^* J^k F) \arrow[r] \arrow[d] & \wedge^n T^* J^k F \arrow[d, shift left, "\mathrm{pr}_{J^kF}"] \\
& M \arrow[r, "\eta"] \arrow[dr, "\mathrm{id}"'] & J^kF \arrow[u, shift left, "\omega"] \arrow[d, "\mathrm{pr}_M"] \\
& & M
\end{tikzcd}$$

which is analogous to the diagram (2.12). It shows that ω_η is a section of the bundle $\wedge^n T^* J^k F \to M$ and that $\mathrm{pr}_{J^\infty F} \circ \omega_\eta = \eta$. When $\eta = j^\infty \varphi$ is the infinite jet prolongation of a field $\varphi \in \mathcal{F}$, we obtain the section

$$\omega_{j^k\varphi} \in \Gamma(M, \wedge^n T^* J^k F)$$

which can be identified with the evaluation of ω at φ. The map

$$\begin{aligned} \Gamma(M, \wedge^n T^* J^k F) &\longrightarrow \Gamma(M, J^kF) \\ \sigma &\longmapsto \mathrm{pr}_{J^kF} \circ \sigma \end{aligned} \tag{6.4}$$

is a fibre-wise linear diffeological bundle and the map

$$\begin{aligned} \Gamma(M, J^kF) &\longrightarrow \Gamma(M, \wedge^n T^* J^k F) \\ \eta &\longmapsto \omega_\eta \end{aligned}$$

is a section of this bundle. When we restrict this section to prolongations of fields by precomposition of with the jet prolongation $j^k : \mathcal{F} \to \Gamma(M, J^kF)$, we obtain a differential operator on $\mathcal{F}$.

Definition 6.1.14. Let $\omega \in \Omega(J^kF)$. The local map

$$\begin{aligned} D_\omega : \mathcal{F} &\longrightarrow \Gamma(M, \wedge^n T^* J^k F) \\ \varphi &\longmapsto \omega_{j^k\varphi} \end{aligned} \tag{6.5}$$

is called the **differential operator associated to** ω. The equation

$$\omega_{j^k\varphi} = 0 \tag{6.6}$$

is called the k-th order **PDE associated to** ω.

Warning 6.1.15. If $F \to M$ is a vector bundle, the bundle $\wedge^n T^* J^k F \to M$ is a vector bundle, so that the target $\Gamma(M, \wedge^n T^* J^k F)$ of the differential operator $\varphi \mapsto \omega_{j^k\varphi}$ is a vector space. But the 0 on the right hand side of the PDE (6.6) must not be viewed as the zero in this vector space. It is to be viewed as the evaluation $0 = 0_{j^k\varphi}$ of the zero section of the fibre-wise linear diffeological bundle (6.4). Eq. (6.6) is then properly understood as the equality $\omega_{j^k\varphi} = 0_{j^k\varphi}$ in the vector space $\Gamma(M, (j^k\varphi)^* \wedge^n T^* J^k F)$.

In local coordinates a form $\omega \in \Omega^{p,q}(J^k F)$ is given by (5.16), so that the PDE is given by a system of equations

$$\omega^{I_1 \dots I_p}_{\alpha_1 \dots \alpha_p i_1 \dots i_q}\Big(\varphi^\alpha, \frac{\partial \varphi^\alpha}{\partial x^{i_1}}, \dots, \frac{\partial^k \varphi^\alpha}{\partial x^{i_1} \cdots \partial x^{i_k}}\Big) = 0\,.$$

Definition 6.1.16. Let $\omega \in \Omega^n(J^k F)$ and $\varphi \in \mathcal{F}$. The PDE

$$(T_\varphi D_\omega)\xi_\varphi = 0$$

for $\xi_\varphi \in \Gamma(M, \varphi^* VF)$ is called the **linearization at** φ of the PDE $\omega_{j^k\varphi} = 0$.

6.1.4 Extension of tangent vectors and forms

In the case of finite-dimensional manifolds, it is easy to show in local coordinates that every tangent vector can be extended to a vector field. Dually, every linear form at a single tangent space can be extended to a differential form on the entire manifold. It is not clear, whether this property carries over to the diffeological space of fields. Moreover, we can ask, whether every the extension can be chosen to be local. We will now show that both questions have an affirmative answer. The main technical lemma used for the proofs is the following:

Lemma 6.1.17. *Let $E \to M$ be a fibre bundle and $A \to E$ a vector bundle. Let $\sigma \in \Gamma(M, E)$ be an arbitrary section and $\sigma^* A = M \times_E^{\sigma, \mathrm{pr}_E} A$ the pullback. Let $i : \sigma^* A \to A$ be the natural inclusion, which induces a map $i_* : \Gamma(M, \sigma^* A) \to \Gamma(M, A)$. Then for every $\tau \in \Gamma(M, \sigma^* A)$ there is a $\bar\tau \in \Gamma(E, A)$ such that $\bar\tau \circ \sigma = i \circ \tau$.*

Proof. Consider the following commutative diagram:

$$\begin{tikzcd}
\sigma^* A \arrow[r, "\cong"] \arrow[d, shift right] \arrow[rr, bend left, "i"] & A\big|_{\sigma(M)} \arrow[r, hook] \arrow[d, shift right] & A \arrow[d, shift right] \\
M \arrow[r, "\cong"] \arrow[u, shift right, "\tau"'] \arrow[rr, bend right, "\sigma"'] & \sigma(M) \arrow[r, hook] \arrow[u, shift right, "\tau'"'] & E \arrow[u, shift right, dotted, "\exists\bar\tau"']
\end{tikzcd}$$

Using the isomorphisms in the left square, a section τ of $\sigma^* A \to M$ can be identified with a section τ' of the restriction of $A \to E$ to the image of σ. As is the case for any section of a fibre bundle, $\sigma(M)$ is a closed embedded submanifold of E. And as is the case for any vector bundle, the sheaf of sections of $A \to E$ is soft, i.e. a section supported on a closed subset extends to a global section. In particular, the section τ' of $A|_{\sigma(M)} \to \sigma(M)$ extends to some section $\bar\tau$ of $A \to E$. From the commutativity of the outer rectangle of the diagram we obtain the relation $\bar\tau \circ \sigma = i \circ \tau$. □

Lemma 6.1.18. *Every* $\xi_\varphi \in T\mathcal{F}$ *can be extended to a local vector field* $\xi : \mathcal{F} \to T\mathcal{F}$.

Proof. Recall that $T\mathcal{F} \cong \Gamma^\infty(M, VF)$ and that $T_\varphi\mathcal{F} = \Gamma(M, \varphi^* VF)$. We can now apply Lem. 6.1.17 for $E := F \to M$, $A := VF \to F$, $\sigma := \varphi \in \mathcal{F}$, $\tau := \xi_\varphi \in \Gamma(M, VF)$. This shows that there is a $\bar{\tau} \in \Gamma(F, VF)$ extending ξ_φ. We can now define the vector field $\xi : \mathcal{F} \to T\mathcal{F}$ by $\xi(\psi) = \bar{\tau} \circ \psi$. By construction, $\xi(\varphi) = \xi_\varphi$ and ξ descends to $\bar{\tau}$ so it is local. □

Lemma 6.1.19. *Let* $\eta \in \Gamma(M, J^k F)$ *and* $\tau \in \Gamma(M, \eta^* \wedge^n T^* J^k F)$. *Then there is an* n*-form* $\omega \in \Omega^n(J^k F)$, *such that* $\tau = \omega_\eta$.

Proof. We apply Lem. 6.1.17 to $E := J^k F \to M$, $A := \wedge^n T^* J^k F \to J^k F$, $\sigma := \eta$, and τ as it is. This shows that there is a section $\omega \in \Gamma(J^k F, \wedge^n T^* J^k F) = \Omega^n(J^k F)$, such that $\tau = \omega \circ \eta = \omega_\eta$. □

Proposition 6.1.20. *Let*

$$\lambda : (T_\varphi \mathcal{F})^p \longrightarrow \Omega^q(M)$$

be a local multilinear map. Then there is a local form $\nu \in \Omega^{p,q}(\mathcal{F} \times M)$, *such that* τ *is the evaluation of* ν *at* φ.

Proof. Since τ is local, it descends to a linear map

$$\lambda_0 : \wedge^p J^k(\varphi^* VF) \longrightarrow \wedge^q T^* M \,.$$

This map can be viewed as section of the bundle

$$\begin{aligned} \wedge^p J^k(\varphi^* V^* F) \times_M \wedge^q T^* M &\cong \wedge^p J^k(M \times_F^{\varphi,\mathrm{pr}} V^* F) \times_M \wedge^q T^* M \\ &\cong \big(M \times_{J^k F}^{j^k\varphi,\mathrm{pr}} \wedge^p J^k(V^* F)\big) \times_M \wedge^q T^* M \\ &\cong (j^k\varphi)^* \big(\wedge^p J^k(V^* F) \times_M \wedge^q T^* M\big) \,. \end{aligned}$$

We now apply Lem. 6.1.17 to $E := J^k F \to M$, $A := \wedge^p J^k(V^* F) \times_M \wedge^q T^* M$, $\sigma := j^k\varphi$, and $\tau = \lambda_0$. This shows that there is a section

$$\omega \in \Gamma\big(J^k F, \wedge^p J^k(V^* F) \times_M \wedge^q T^* M\big) \,,$$

such that $\omega \circ j^k\varphi = \lambda_0$. By Eq. (5.10), we can view ω as a form in $\Omega^{p,q}(J^k F)$. Let $\nu := (j^\infty)^*\omega$. It follows from Lem. 6.1.10 that for all $v^1, \dots, v^{p+q} \in T_\varphi\mathcal{F} \times TM$ we have

$$\begin{aligned} \nu(v^1, \dots, v^{p+q}) &= \big((j^\infty)^*\omega\big)(v^1, \dots, v^{p+q}) \\ &= \omega\big((Tj^\infty)v^1, \dots, (Tj^\infty)v^{p+q}\big) \\ &= \lambda_0\big((Tj^\infty)v^1, \dots, (Tj^\infty)v^{p+q}\big) \\ &= \lambda(v^1, \dots, v^{p+q}) \,. \end{aligned}$$

We conclude that $\nu_\varphi = \lambda$. □

6.1.5 Evaluation of vector fields

The evaluation of a vector field $\xi : \mathcal{F} \to T\mathcal{F}$ at $\varphi \in \mathcal{F}$ has a counterpart for a vector field $v : J^\infty F \to TJ^\infty F$ on the infinite jet bundle. Let us denote the precomposition of v with a section $\eta \in \Gamma(M, J^\infty F)$ by

$$v_\eta : M \xrightarrow{\eta} J^\infty F \xrightarrow{v} TJ^\infty F\,. \tag{6.7}$$

We have the following commutative diagram,

$$\begin{tikzcd}
M \arrow[rrd, bend left, "v_\eta"] \arrow[rd, dotted] \arrow[rdd, bend right, "\mathrm{id}"'] & & \\
& M \times_{J^\infty F} (TJ^\infty F) \arrow[r] \arrow[d] & TJ^\infty F \arrow[d, shift left, "\mathrm{pr}_{J^\infty F}"] \\
& M \arrow[r, "\eta"] \arrow[rd, "\mathrm{id}"'] & J^\infty F \arrow[u, shift left, "v"] \arrow[d, "\mathrm{pr}_M"] \\
& & M
\end{tikzcd}$$

It shows that v_η is a section of the bundle $TJ^\infty F \to M$ and that $\mathrm{pr}_{J^\infty F} \circ \omega_\eta = \eta$.

Definition 6.1.21. Let $v : J^\infty F \to TJ^\infty F$ be a vector field on $J^\infty F$ and $\varphi \in \mathcal{F}$ a field. The section $v_{j^\infty \varphi} \in \Gamma(M, TJ^\infty F)$ will be called the **evaluation of v at φ**.

Definition 6.1.22. Let $\xi_\varphi \in T_\varphi \mathcal{F}$. The map

$$\hat{\xi}_{j^\infty \varphi} : M \xrightarrow{\cong} \{\xi_\varphi\} \times M \xrightarrow{Tj^\infty} TJ^\infty F \tag{6.8}$$

will be called the **infinite prolongation** of ξ_φ.

In local coordinates $\xi_\varphi \in \Gamma(M, VF)$ is given by $\xi_\varphi = \xi^\alpha_\varphi \frac{\partial}{\partial u^\alpha}$, where $\xi^\alpha_\varphi \in C^\infty(M)$. From the local coordinate formula (5.3) for Tj^∞ we deduce that

$$\hat{\xi}_{j^\infty \varphi} = \frac{\partial^{|I|} \xi^\alpha_\varphi}{\partial x^I} \frac{\partial}{\partial u^\alpha_I}\,. \tag{6.9}$$

Proposition 6.1.23. *Let $\omega \in \Omega^{p,q}(J^\infty F)$. Then*

$$\iota_{\xi^p_\varphi} \cdots \iota_{\xi^1_\varphi} (j^\infty)^* \omega = \iota_{\xi^p_{j^\infty \varphi}} \cdots \iota_{\xi^1_{j^\infty \varphi}} \omega \in \Omega^q(M)$$

for all $\varphi \in \mathcal{F}$ and all $\xi^1_\varphi, \ldots, \xi^p_\varphi \in T_\varphi \mathcal{F}$.

Proof. This follows from Def. 6.1.22 and Lem. 6.1.10. □

Proposition 6.1.24. *Let $\zeta : M \to TJ^\infty F$ be a smooth map. The following are equivalent:*

(i) ζ is the infinite prolongation $\hat{\xi}_{j^\infty \varphi}$ of a tangent vector $\xi_\varphi \in T\mathcal{F}$ at $\varphi \in \mathcal{F}$.

(ii) ζ is the evaluation $v_{j^\infty \varphi}$ of a strictly vertical vector field $v \in \mathfrak{X}(J^\infty F)$ at φ.

Proof. Assume (ii). By Thm. 5.1.37, v is the infinite prolongation of an evolutionary "vector field" $\eta : J^k F \to VF$. Define $\xi_\varphi := \eta \circ j^k\varphi \in \Gamma(M, \varphi^* VF) = T_\varphi \mathcal{F}$. In local coordinates $v = (D_I \eta^\alpha)\frac{\partial}{\partial u_I^\alpha}$ and $\xi_\varphi = (\eta^\alpha \circ j^k\varphi)\frac{\partial}{\partial u^\alpha}$. It follows that

$$\begin{aligned} v_{j^\infty\varphi} &= (D_I \eta^\alpha \circ j^k\varphi)\frac{\partial}{\partial u_I^\alpha} = \frac{\partial^{|I|}(\eta^\alpha \circ j^k\varphi)}{\partial x^I}\frac{\partial}{\partial u_I^\alpha} = \frac{\partial^{|I|}\xi^\alpha}{\partial x^I}\frac{\partial}{\partial u_I^\alpha} \\ &= \hat{\xi}_{j^\infty\varphi}\,, \end{aligned} \tag{6.10}$$

where we have used Eq. (6.9). This shows that $\hat{\xi}_{j^\infty\varphi} = \zeta$. We conclude that (ii) implies (i)

Assume (i). By Lem. 6.1.18, ξ_φ can be extended to a local vector field $\xi \in \mathfrak{X}(\mathcal{F})$. By locality, ξ descends to an evolutionary vector field $\eta : J^k F \to VF$. And since ξ extends ξ_φ, we have $\xi_\varphi = \eta \circ j^k\varphi$. The infinite prolongation of η is a strictly vertical vector field on $J^\infty F$, which we denote by v. In local coordinates we have $\xi_\varphi = (\eta^\alpha \circ j^k\varphi)\frac{\partial}{\partial u^\alpha}$ and $v = (D_I\eta^\alpha)\frac{\partial}{\partial u_I^\alpha}$. It follows by Eq. (6.10) that $v_{j^\infty\varphi} = \zeta$. We conclude that (i) implies (ii). ☐

Theorem 6.1.25. *Let ξ be a vector field on $\mathcal{F}$ and v a vector field on $J^\infty F$. The following are equivalent:*

(i) The evaluation $v_{j^\infty\varphi}$ is the infinite prolongation of ξ_φ for all $\varphi \in \mathcal{F}$.

(ii) v is strictly vertical and ξ the unique local vector field that projects to v by Thm. 5.1.37.

Proof. *** ☐

6.2 Cartan calculus

6.2.1 Inner derivative

We now turn to the Cartan calculus on local forms. Let $\nu \in \Omega^{p,q}(\mathcal{F} \times M)$ and $\chi \in \mathfrak{X}(\mathcal{F})$. The inner derivative $\iota_\chi \nu$ is $(p-1,q)$-form, which is given by

$$(\iota_\chi \nu)(\xi_\varphi^1, \dots, \xi_\varphi^{p-1}, v_m^1, \dots, v_m^q) = \nu(\chi_\varphi, \xi_\varphi^1, \dots, \xi_\varphi^{p-1}, v_m^1, \dots, v_m^q)\,.$$

Similarly, for $w \in \mathfrak{X}(M)$ the inner derivative $\iota_w \nu$ is a $(p, q-1)$-form, which is given by

$$(\iota_w \nu)(\xi_\varphi^1, \dots, \xi_\varphi^p, v_m^1, \dots, v_m^{q-1}) = (-1)^p \nu(\xi_\varphi^1, \dots, \xi_\varphi^p, w, v_m^1, \dots, v_m^{q-1})\,.$$

Proposition 6.2.1. *Let $\omega \in \Omega^{p,q}(J^\infty F)$. Let $\hat{\xi}$ be a strictly vertical vector field on $J^\infty F$ and ξ the unique local vector field on $\mathcal{F}$ that projects to $\hat{\xi}$ by Thm. 5.1.37. Then*

$$\iota_\xi (j^\infty)^* \omega = (j^\infty)^* \iota_{\hat{\xi}} \omega\,.$$

Proof. The vector field ξ projects to $\hat{\xi} : J^\infty F \to TJ^\infty F$, which means that

$$(T_{(\varphi,m)} j^\infty)\xi_\varphi = \hat{\xi}(j_m^\infty \varphi)\,.$$

Let us denote $T := T_{(\varphi,m)}j^\infty$ for compact notation. Then

$$\begin{aligned}
&\big(\iota_{\xi_\varphi}(j^\infty)^*\omega\big)(\chi^1_\varphi,\dots,\chi^{p-1}_\varphi,v^1_m,\dots,v^q_m)\\
&=\omega\big(T\xi_\varphi,T\chi^1_\varphi,\dots,T\chi^{p-1}_\varphi,Tv^1_m,\dots,Tv^q_m\big)\\
&=\omega\big(\hat{\xi}(j^\infty_m\varphi),T\chi^1_\varphi,\dots,T\chi^{p-1}_\varphi,Tv^1_m,\dots,Tv^q_m\big)\\
&=(\iota_{\hat{\xi}(j^\infty_m\varphi)}\omega)\big(T\chi^1_\varphi,\dots,T\chi^{p-1}_\varphi,Tv^1_m,\dots,Tv^q_m\big)\\
&=\big((j^\infty)^*\iota_{\hat{\xi}}\omega\big)(\chi^1_\varphi,\dots,\chi^{p-1}_\varphi,v^1_m,\dots,v^q_m)\,,
\end{aligned}$$

for all $\chi^1_\varphi,\dots,\chi^{p-1}_\varphi\in T_\varphi\mathcal{F}$ and all $v^1_m,\dots,v^q_m\in T_mM$. □

Corollary 6.2.2. *If ν is a local form on $\mathcal{F}\times M$ and ξ a local vector field on $\mathcal{F}$, then $\iota_\xi\nu$ is a local form.*

Proposition 6.2.3. *Let $\omega\in\Omega^{p,q}(J^\infty F)$. Let v be a vector field on M and $\hat{v}$ the strictly horizontal vector field on $J^\infty F$ to which v lifts by the Cartan connection. Then*

$$\iota_v(j^\infty)^*\omega=(j^\infty)^*\iota_{\hat{v}}\omega\,.$$

Proof. By Rmk. 5.1.15, the horizontal lift $\hat{v}\in\mathfrak{X}(J^\infty F)$ of a vector field $v\in\mathfrak{X}(M)$ is defined by

$$(T_{(\varphi,m)}j^\infty)v_m=\hat{v}(j^\infty_m\varphi)\,.$$

The rest of the proof is analogous to the proof of Prop. 6.2.1. □

6.2.2 Horizontal differential

Definition 6.2.4. Let ν be a (p,q)-form on $\mathcal{F}\times M$. By $d\nu$ we denote the $(p,q+1)$-form $\mathcal{F}\times M$ that is given by the map

$$\begin{aligned}
\widetilde{d\nu}:(T\mathcal{F}_{/\mathcal{F}})^p&\longrightarrow\Omega^{q+1}(M)\\
(\xi^1_\varphi,\dots,\xi^p_\varphi)&\longmapsto(-1)^pd_M\big(\tilde{\nu}(\xi^1_\varphi,\dots,\xi^p_\varphi)\big)\,,
\end{aligned}$$

where $\tilde{\nu}:(T\mathcal{F}_{/\mathcal{F}})^p\to\Omega^q(M)$ is the map defined in (6.1) and where d_M is the de Rham differential on $\Omega(M)$.

Lemma 6.2.5. *Let $\nu\in\Omega(\mathcal{F}\times M)$ and $\xi\in\mathfrak{X}(\mathcal{F})$. Then $\iota_\xi d\nu=-d\iota_\xi\nu$.*

Proof. Let $\nu\in\Omega^{p,q}(\mathcal{F}\times M)$. Then

$$\begin{aligned}
(\widetilde{\iota_\xi d\nu})(\chi^1_\varphi,\dots,\chi^{p-1}_\varphi)&=(\widetilde{d\nu})(\xi_\varphi,\chi^1_\varphi,\dots,\chi^{p-1}_\varphi)\\
&=(-1)^pd_M\big(\tilde{\nu}(\xi_\varphi,\chi^1_\varphi,\dots,\chi^{p-1}_\varphi)\big)\\
&=(-1)^pd_M\big((\widetilde{\iota_\xi\nu})(\chi^1_\varphi,\dots,\chi^{p-1}_\varphi)\big)\\
&=-(\widetilde{d\iota_\xi\nu})(\chi^1_\varphi,\dots,\chi^{p-1}_\varphi)\big)\,,
\end{aligned}$$

for all $\chi^1_\varphi,\dots,\chi^p_\varphi\in T_\varphi\mathcal{F}$ and $\varphi\in\mathcal{F}$. □

Proposition 6.2.6. *Let $\omega\in\Omega^{p,q}(J^\infty F)$. Then*

$$(j^\infty)^*d\omega=d\big((j^\infty)^*\omega\big)\,. \tag{6.11}$$

Proof. Let $f \in \Omega^{0,0}(J^\infty F)$ be a function, $v \in \mathfrak{X}(M)$ a vector field, $\hat{v} \in \mathfrak{X}(J^\infty F)$ its horizontal lift, and $\varphi \in \mathcal{F}$ a field. Then

$$\big(\widetilde{(j^\infty)^* f}\big)(\varphi) = (j^\infty\varphi)^* f\,, \tag{6.12}$$

for every $\varphi \in \mathcal{F}$, where $j^\infty\varphi : M \to J^\infty F$ is the infinite jet prolongation of φ. It follows from Eq. (5.9) that

$$v \cdot \big((j^\infty\varphi)^* f\big) = (j^\infty\varphi)^*(\hat{v} \cdot f)\,. \tag{6.13}$$

We then get

$$\begin{aligned}
\big[\iota_v d\big((j^\infty)^* f\big)\big]_\varphi &= \iota_v d_M\big[\big(\widetilde{(j^\infty)^* f}\big)(\varphi)\big] \\
&= v \cdot \big[(j^\infty\varphi)^* f\big] \\
&= (j^\infty\varphi)^*(\hat{v} \cdot f) \\
&= \big[(j^\infty)^*(\iota_{\hat{v}} df)\big]_\varphi \\
&= \big[\iota_v\big((j^\infty)^* df\big)\big]_\varphi\,,
\end{aligned}$$

where we have used first Def. 6.2.4, then Eq. (6.12), then Eq. (6.13), and in the last step Prop. 6.2.3. Since this relation holds for all $v \in \mathfrak{X}(M)$ and all $\varphi \in \mathcal{F}$, it follows that Eq. (6.11) holds for all functions $f \in \Omega^{0,0}(J^\infty F)$.

Let now $\omega \in \Omega^{0,1}(J^\infty F)$, let $v, w \in \mathfrak{X}(M)$ be vector fields, and $\hat{v}, \hat{w} \in \mathfrak{X}(J^\infty F)$ their horizontal lifts. Then

$$\begin{aligned}
\big[\iota_w\iota_v d\big((j^\infty)^*\omega\big)\big]_\varphi &= \iota_w\iota_v d_M\big((j^\infty\varphi)^*\omega\big) \\
&= (\iota_v d\iota_w - \iota_w d\iota_v - \iota_{[v,w]})\big((j^\infty\varphi)^*\omega\big) \\
&= \iota_v d\big((j^\infty\varphi)^*\iota_{\hat{w}}\omega\big) - \iota_w d\big((j^\infty\varphi)^*\iota_{\hat{v}}\omega\big) - \big((j^\infty\varphi)^*\iota_{[\hat{v},\hat{w}]}\omega\big) \\
&= \iota_v\big((j^\infty\varphi)^* d\iota_{\hat{w}}\omega\big) - \iota_w\big((j^\infty\varphi)^* d\iota_{\hat{v}}\omega\big) - \big((j^\infty\varphi)^*\iota_{[\hat{v},\hat{w}]}\omega\big) \\
&= (j^\infty\varphi)^*(\iota_{\hat{v}} d\iota_{\hat{w}} - \iota_{\hat{w}} d\iota_{\hat{v}} - \iota_{[\hat{v},\hat{w}]})\omega \\
&= (j^\infty\varphi)^*\iota_{\hat{w}}\iota_{\hat{v}} d\omega \\
&= \iota_w\iota_v(j^\infty\varphi)^* d\omega \\
&= [\iota_w\iota_v(j^\infty)^* d\omega]_\varphi\,,
\end{aligned}$$

where we have used the Chevalley-Eilenberg formula for the differential of a 2-form, Prop. 6.2.1, and that d commutes with $(j^\infty)^*$ on functions. We conclude that Eq. (6.11) holds for all $\omega \in \Omega^{0,1}(J^\infty F)$.

Let $\omega \in \Omega^{1,0}(J^\infty F)$ and let $\xi_\varphi \in T_\varphi\mathcal{F}$. By Lem. 6.1.18, we can extend ξ_φ to a local vector field $\xi \in \mathfrak{X}(\mathcal{F})$. Let $\hat{\xi} \in \mathfrak{X}(J^\infty F)$ be a strictly vertical vector field to which ξ projects. Then

$$\begin{aligned}\iota_{\xi_\varphi}(j^\infty)^* d\omega &= \big[\iota_\xi (j^\infty)^* d\omega\big]_\varphi \\ &= \big[(j^\infty)^* \iota_{\hat{\xi}} d\omega\big]_\varphi \\ &= \big[-(j^\infty)^* d\iota_{\hat{\xi}}\omega\big]_\varphi \\ &= \big[-d\big((j^\infty)^*\iota_{\hat{\xi}}\omega\big)\big]_\varphi \\ &= \big[-d\big(\iota_\xi (j^\infty)^*\omega\big)\big]_\varphi \\ &= \big[\iota_\xi d\big((j^\infty)^*\omega\big)\big]_\varphi \\ &= \iota_{\xi_\varphi} d\big((j^\infty)^*\omega\big)\,,\end{aligned}$$

where we have used Prop. 6.2.1, that $\hat{\xi}$ is strictly vertical, that d commutes with $(j^\infty)^*$ on the function $\iota_{\hat{\xi}}\omega$, and Lem. 6.2.5. Since this relation holds for all $\xi_\varphi \in T\mathcal{F}$, it follows that Eq. (6.11) holds for all $\omega \in \Omega^{1,0}(J^\infty F)$.

The algebra $\Omega^{p,q}(J^\infty M)$ is generated by functions $\Omega^{0,0}(J^\infty F)$, horizontal 1-forms $\Omega^{0,1}(J^\infty F)$, and vertical 1-forms $\Omega^{1,0}(J^\infty F)$. We have shown, that Eq. (6.11) holds on this set of generators. Since the differential d on $\Omega(J^\infty F)$ is a derivation and since by Thm. 6.1.2 the pullback $(j^\infty)^*$ is a morphism of algebras, we conclude that Eq. (6.11) holds for all $\omega \in \Omega(J^\infty F)$. □

Corollary 6.2.7. *The map* $d : \Omega^{p,q}(\mathcal{F} \times M) \to \Omega^{p,q+1}(\mathcal{F} \times M)$ *restricts to a degree* $(0,1)$ *differential on* $\Omega_{\mathrm{loc}}(\mathcal{F} \times M)$.

Corollary 6.2.8. *Let* $\omega \in \Omega^{0,q}(J^\infty F)$ *and* $\varphi \in \mathcal{F}$. *Then* $d_M (j^\infty\varphi)^*\omega = (j^\infty\varphi)^* d\omega$.

Proof. The map $\tilde{\omega} : \mathcal{F} \to \Omega^q(M)$ is given by $\tilde{\omega}(\varphi) = (j^\infty\varphi)^*\omega$. From Prop. 6.2.6 we obtain $d_M(j^\infty\varphi)^*\omega = [d(j^\infty)^*\omega]_\varphi = [(j^\infty)^* d\omega]_\varphi = (j^\infty\varphi)^* d\omega$. □

6.2.3 Vertical differential

Definition 6.2.9. Let $\nu \in \Omega^{0,q}(\mathcal{F} \times M)$. The **diffeological differential** $\delta\nu \in \Omega^{1,q}(\mathcal{F} \times M)$ is the form given by the linear map

$$\widetilde{\delta\nu} : T\mathcal{F} \xrightarrow{T\tilde{\nu}} T\Omega^q(M) \cong \Omega^q(M) \times \Omega^q(M) \xrightarrow{\mathrm{pr}_2} \Omega^q(M)\,,$$

where $T\tilde{\nu}$ is the diffeological tangent map of $\tilde{\nu} : \mathcal{F} \to \Omega^q(M)$.

Proposition 6.2.10. *Let* $\omega \in \Omega^{0,q}(J^\infty F)$. *Then*

$$(j^\infty)^*\delta\omega = \delta\big((j^\infty)^*\omega\big)\,. \tag{6.14}$$

Proof. The diffeological tangent map is given by

$$(T_{\varphi_0}\tilde{\nu})\dot{\varphi}_0 = \frac{d}{dt}\tilde{\nu}(\varphi_t)\big|_{t=0} \in T_{\tilde{\nu}(\varphi_0)}\Omega^q(M)\,,$$

for every smooth path $t \mapsto \varphi_t \in \mathcal{F}$. First, we consider a function $f \in \Omega^{0,0}(J^k F)$. In local coordinates we have

$$\begin{aligned}
\frac{d}{dt}(j^k\varphi_t)^* f \,\big|_{t=0} &= \frac{d}{dt}\big(f \circ j^k\varphi_t\big)_{t=0} \\
&= \Big\{ \frac{\partial f}{\partial x^i}(j^k\varphi_t)\frac{\partial x^i}{\partial t} + \sum_{|I|=0}^{k} \frac{\partial f}{\partial u_I^\alpha}(j^k\varphi_t)\frac{\partial}{\partial t}\Big(\frac{\partial^{|I|}\varphi^\alpha}{\partial x^I}\Big)\Big\}_{t=0} \\
&= \sum_{|I|=0}^{k} \frac{\partial f}{\partial u_I^\alpha}(j^k\varphi_0)\, \dot{u}_I^\alpha(\dot\varphi_0) \\
&= \sum_{|I|=0}^{k} \frac{\partial f}{\partial u_I^\alpha}(j^k\varphi_0)\, \iota_{Tj^k\dot\varphi_0}\delta u_I^\alpha \\
&= \iota_{Tj^k\dot\varphi_0}\delta f\,.
\end{aligned}$$

Since this holds for all $k \geq 0$, we obtain

$$\begin{aligned}
(T_{\varphi_0}\widetilde{(j^\infty)^* f})\dot\varphi_0 &= \frac{d}{dt}\big(\widetilde{(j^\infty)^* f}\big)(\varphi_t)\big|_{t=0} \\
&= \frac{d}{dt}(j^\infty\varphi_t)^* f \,\big|_{t=0} \\
&= \iota_{Tj^\infty\dot\varphi_0}\delta f\,.
\end{aligned}$$

This shows that Eq. (6.14) holds for functions. For a d-exact $(0,1)$-form df we have

$$\begin{aligned}
\frac{d}{dt}(j^\infty\varphi_t)^* df \,\big|_{t=0} &= \frac{d}{dt}d_M(j^\infty\varphi_t)^* f \,\big|_{t=0} \\
&= d_M\frac{d}{dt}(j^\infty\varphi_t)^* f \,\big|_{t=0} \\
&= d_M(\iota_{Tj^\infty\dot\varphi_0}\delta f) \\
&= -\iota_{Tj^\infty\dot\varphi_0}d(\delta f) \\
&= \iota_{Tj^\infty\dot\varphi_0}\delta(df)\,.
\end{aligned}$$

This shows that Eq. (6.14) holds for all exact $(0,1)$-forms df.

Eq. (6.14), which we want to prove, is local, so it suffices to check it locally, i.e. when restricting the infinite jet bundle to an open subset $U \subset M$ of the base. $\Omega^{0,\bullet}(J^\infty F)$ is generated locally by functions and exact 1-forms, for which we have shown that Eq. (6.14) holds. Since δ is a derivation and since by Thm. 6.1.2 $(j^\infty)^*$ is a homomorphism of ind-algebras, it follows that Eq. (6.14) holds for all forms in $\Omega^{0,\bullet}(J^\infty F)$. □

Proposition 6.2.11. *Let* $\nu \in \Omega^{0,q}(\mathcal{F} \times M)$. *Then*

$$d\delta\nu = -\delta d\nu\,.$$

Proof. Since $T\Omega^q(M) \cong \Omega^q(M) \times \Omega^q(M)$ by Prop. 2.3.12, every tangent vector (α, β) of $\Omega^q(M)$ is represented by an affine path $t \mapsto \alpha + t\beta$. Since the de Rham differential is linear, $d_M(\alpha + t\beta) = d_M\alpha + t d_M\beta$, the tangent map of d_M is

$$Td_M = d_M \times d_M : \Omega^q(M) \times \Omega^q(M) \longrightarrow \Omega^{q+1}(M) \times \Omega^{q+1}(M)\,.$$

With this we obtain the following commutative diagram

$$\begin{tikzcd}
T\mathcal{F} \arrow[r, "T\tilde{\nu}"] \arrow[rr, bend left, "T(d_M \circ \tilde{\nu})"] \arrow[dr, "\widetilde{\delta\nu}"'] & \Omega^q(M) \times \Omega^q(M) \arrow[r, "Td_M"] \arrow[d, "\mathrm{pr}_2"] & \Omega^{q+1}(M) \times \Omega^{q+1}(M) \arrow[d, "\mathrm{pr}_2"] \\
& \Omega^q(M) \arrow[r, "d_M"'] & \Omega^{q+1}(M)
\end{tikzcd}$$

The commutativity of the outer quadrilateral diagram means that

$$\widetilde{\delta(d\nu)} = d_M \circ \widetilde{\delta\nu} = -\widetilde{d(\delta\nu)}\,,$$

where we have used the definition 6.2.4 of d. Removing the tilde on both sides finishes the proof. □

6.3 Cohomology of local forms

6.3.1 Local families of forms and vector fields

Let $E \to M$ be another smooth fibre bundle. In Sec. 2.3.2 we have seen that the product of spaces of fields is itself a space of field,

$$\mathcal{E} \times \mathcal{F} = \Gamma(M, E \times_M F)\,,$$

so that all statements we have proved about local forms on $\mathcal{F}$ apply to local forms on $\mathcal{E} \times \mathcal{F}$. The additional structure we obtain is that the space of forms has $\mathbb{Z}^2$-grading. A form $\nu \in \Omega^{p,q,r}(\mathcal{E} \times \mathcal{F} \times M)$ can be equivalently viewed as a p-form on $\mathcal{E}$ with values in $\Omega^{q,r}(\mathcal{F} \times M)$, i.e. as a fibre-wise multilinear and antisymmetric map

$$\begin{aligned} \nu_1 : (T\mathcal{E}_{/\mathcal{E}})^p &\longrightarrow \Omega^{q,r}(\mathcal{E} \times M) \\ (\chi^1_\psi, \dots, \chi^p_\psi) &\longmapsto \iota_{\chi^p_\psi} \cdots \iota_{\chi^1_\psi} \nu\,. \end{aligned} \tag{6.15}$$

Since this definition is graded symmetric in $\mathcal{E}$ and $\mathcal{F}$ we can exchange the roles of $\mathcal{E}$ and $\mathcal{F}$, so that the form ν can be equivalently viewed as a q-form on $\mathcal{F}$ with values in $\Omega^{p,r}(\mathcal{E} \times M)$, i.e. as a fibre-wise multilinear and antisymmetric map

$$\begin{aligned} \nu_2 : (T\mathcal{F}_{/\mathcal{F}})^q &\longrightarrow \Omega^{p,r}(\mathcal{F} \times M) \\ (\xi^1_\varphi, \dots, \xi^q_\varphi) &\longmapsto (-1)^{pq} \iota_{\xi^q_\varphi} \cdots \iota_{\xi^1_\varphi} \nu\,. \end{aligned} \tag{6.16}$$

By definition, we have

$$\begin{aligned} \iota_{\xi^q_\varphi} \cdots \iota_{\xi^1_\varphi} \big(\nu_1(\chi^1_\psi, \dots \chi^q_\psi)\big) &= \iota_{\xi^q_\varphi} \cdots \iota_{\xi^1_\varphi} \iota_{\chi^p_\psi} \cdots \iota_{\chi^1_\psi} \nu \\ &= (-1)^{pq} \iota_{\chi^p_\psi} \cdots \iota_{\chi^1_\psi} \iota_{\xi^q_\varphi} \cdots \iota_{\xi^1_\varphi} \nu \\ &= \iota_{\chi^p_\psi} \cdots \iota_{\chi^1_\psi} \big(\nu_2(\xi^1_\varphi, \cdots, \xi^q_\varphi)\big)\,. \end{aligned}$$

which shows that the maps (6.15) and (6.16) amount to a change of notation.

Proposition 6.3.1. *Let $\nu \in \Omega^{p,q,r}(\mathcal{E} \times \mathcal{F} \times M)$. If ν is local, then there is a natural number $k < \infty$, such that the forms*

$$\nu_1(\chi^1_\psi, \dots \chi^q_\psi) \in \Omega^{q,r}(\mathcal{F} \times M)$$
$$\nu_2(\xi^1_\varphi, \cdots, \xi^q_\varphi) \in \Omega^{p,r}(\mathcal{E} \times M)$$

are local of jet order bounded by k for all $\psi \in \mathcal{E}$, $\chi^1_\varphi, \dots, \chi^1_p \in T_\psi\mathcal{E}$ and all $\varphi \in \mathcal{F}$, $\xi^1_\varphi, \dots, \xi^1_q \in T_\varphi\mathcal{F}$.

Proof. The proof follows directly from Lem. 3.2.18 replacing E with $(E_{/M})^p$, F with $(F_{/M})^q$, and F' with $\wedge^r T^*M$. □

Definition 6.3.2. A family of differential forms

$$\mathcal{E} \longrightarrow \Omega^{q,r}(\mathcal{F} \times M)$$

will be called **local**, if it is local when viewed as differential form in $\Omega^{0,q,r}(\mathcal{E}\times\mathcal{F}\times M)$.

A family of forms $\nu : \mathcal{E} \to \Omega^{q,r}(\mathcal{F} \times M)$ is local if there is a form

$$\omega \in \Omega^{p+q,r}\big(J^\infty(E \times_M F)\big),$$

such that $\nu = (j^\infty)^*\omega$. By Lem. 6.1.4 locality of ν implies that there is a $k < \infty$, such that the following two properties hold:

(a) ν takes values in $(j^k)^*\Omega^{q,r}(J^kF) \subset \Omega^{q,r}(\mathcal{F} \times M)$.

(b) The map $\iota_{\xi^q_\varphi} \cdots \iota_{\xi^1_\varphi}\nu : \mathcal{E} \to \Omega^p(M)$ descends to J^kE for all fields $\varphi \in \mathcal{F}$ and all tangent vectors $\xi^i_\varphi \in T_\varphi\mathcal{F}$.

Remark 6.3.3. Properties (a) and (b) are generally not sufficient for the locality of ν.

In the spirit of Def. 6.3.2, we can also define a local family of vector fields. For this we recall from Thm. 2.3.2 that $T\mathcal{F} = \Gamma(M, VF)$ is a space of sections over M.

Definition 6.3.4. A family $\xi : \mathcal{E} \to \mathfrak{X}(\mathcal{F})$, $\psi \mapsto \xi_\psi$ of vector fields is called **local** if the map

$$\begin{aligned}\tilde{\xi} : \mathcal{E} \times \mathcal{F} &\longrightarrow T\mathcal{F}\\ (\psi, \varphi) &\longmapsto \xi_\psi(\varphi)\end{aligned}$$

is a local map.

Proposition 6.3.5. *Let $\xi : \mathcal{E} \to \mathfrak{X}(\mathcal{F})$ be a family of vector fields. If ξ is local, then there is a $k < \infty$, such that the following two properties hold:*

(i) For every $\psi \in \mathcal{E}$ the vector field $\xi_\psi \in \mathfrak{X}(\mathcal{F})$ is a local vector field that descends to an evolutionary "vector field" $\eta_\psi : J^kF \to VF$.

(ii) For every $\varphi \in \mathcal{F}$ the map

$$\begin{aligned}\xi(\varphi) : \mathcal{E} &\longrightarrow T_\varphi\mathcal{F}\\ \psi &\longmapsto \xi_\psi(\varphi)\end{aligned}$$

descends to J^kE.

Proof. The proof follows directly from Lem. 3.2.18 for $F' = VF$. □

Proposition 6.3.6. *Let $\nu : \mathcal{E} \to \Omega^{q,r}(\mathcal{F} \times M)$ be a family of differential forms and $\xi : \mathcal{E} \to \mathfrak{X}(\mathcal{F})$ a family of vector fields. If ν and ξ are both local, then*

$$\begin{aligned} \iota_\xi \nu : \mathcal{E} &\longrightarrow \Omega^{q-1,r}(\mathcal{F} \times M) \\ \psi &\longmapsto \iota_{\xi_\psi} \nu_\psi \end{aligned}$$

is a local family of differential forms.

Proof. The family of differential forms ν can be viewed as map $\tilde{\nu} : \mathcal{E} \times (T\mathcal{F}_{/\mathcal{F}})^q \to \Omega^r(M)$. Let $\Delta_\mathcal{E} : \mathcal{E} \to \mathcal{E} \times \mathcal{E}$ denote the diagonal map and $\mathrm{pr}_\mathcal{F} : (T\mathcal{F}_{/\mathcal{F}})^{q-1} \to \mathcal{F}$ the bundle projection. The family of differential forms $\iota_\xi \nu$ can be viewed as a map $\mathcal{E} \times (T\mathcal{F}_{/\mathcal{F}})^{q-1} \to \Omega^r(M)$, which is given by the composition

$$\begin{aligned} \mathcal{E} \times (T\mathcal{F}_{/\mathcal{F}})^{q-1} &\xrightarrow{\Delta_\mathcal{E} \times (\mathrm{pr}_\mathcal{F}, \mathrm{id}^{q-1}_{T\mathcal{F}})} \mathcal{E} \times \mathcal{E} \times \mathcal{F} \times (T\mathcal{F}_{/\mathcal{F}})^{q-1} \\ &\xrightarrow{\mathrm{id}_\mathcal{E} \times \tilde{\xi} \times \mathrm{id}^{q-1}_{T\mathcal{F}}} \mathcal{E} \times (T\mathcal{F}_{/\mathcal{F}})^q \\ &\xrightarrow{\tilde{\nu}} \Omega^r(M)\,, \end{aligned} \tag{6.17}$$

Assume that $\tilde{\nu}$ and $\tilde{\xi}$ are local. In Lem. 3.2.16 we have shown that Δ is local, in Lem. 3.2.17 that the product of local maps is local, and in Prop. 3.2.8 that the composition of local maps is local. The map (6.17) is defined by products and compositions of local maps, so it is local. □

Definition 6.3.7. A family of differential forms $\nu : \mathcal{E} \to \Omega^{q,r}(\mathcal{F} \times M)$ will be called **d-closed** if $d \circ \nu = 0$. It is called **d-exact** if there is a family of differential forms $\mu : \mathcal{E} \to \Omega^{q,r-1}(\mathcal{F} \times M)$, such that $\nu = d \circ \mu$.

Remark 6.3.8. It follows from the definition 6.2.4 of the differential d that a family of differential forms $\nu : \mathcal{E} \to \Omega^{q,r}(\mathcal{F} \times M)$ is d-closed (d-exact) if and only if it is d-closed (d-exact) when viewed as form in $\Omega^{0,q,r}(\mathcal{E} \times \mathcal{F} \times M)$.

Proposition 6.3.9. *Let $\nu : \mathcal{E} \to \Omega^{q,r}(\mathcal{F} \times M)$ be a family of differential forms, where $q > 0$ and $r < \dim M$. If ν is d-closed and local, then ν is d-exact.*

Proof. We can view ν as form $\bar{\nu} \in \Omega^{q,r}\big((\mathcal{E} \times \mathcal{F}) \times M\big)$. If ν is local, then by Lemma. 6.1.4 $\nu = (j^\infty)^* \omega$ for some $\omega \in \Omega^{q,r}\big(J^\infty(E \times_M F)\big)$. If ν is d-closed, then by Prop. 6.2.6 ω is d-closed. It now follows from the acyclicity theorem 5.2.4, that $\omega = d\tau$. The pullback $\bar{\mu} = (j^\infty)^* \tau$ defines a family $\mu : \mathcal{E} \to \Omega^{q,r-1}(\mathcal{F} \times M)$ that satisfies $d \circ \mu = \nu$. □

6.3.2 Linear local families of forms

Let $A \to M$ be a vector bundle, so that the space of sections $\mathcal{A}$ is a diffeological vector space. Then we can require smooth families of vector fields and forms to be linear. When $\nu : \mathcal{A} \to \Omega^{q,r}(\mathcal{F} \times M)$ is a linear family of forms and $\xi : \mathcal{A} \to \mathfrak{X}(\mathcal{F})$ a family of vector fields, then $\iota_\xi \nu$ as defined in Prop. 6.3.6 is linear as well.

Proposition 6.3.10. *Let $\nu : \mathcal{A} \to \Omega^{q,r}(\mathcal{F} \times M)$ be a linear family of differential forms and $\xi : \mathcal{A} \to \mathfrak{X}(\mathcal{F})$ a linear family of vector fields. If both ν and ξ are local, then the linear family of local forms $\iota_\xi \nu : \mathcal{A} \to \Omega^{q-1,r}(\mathcal{F} \times M)$ is local.*

Proof. This follows from Prop. 6.3.6. □

In Sec. 2.3.2 we have seen that multilinear and antisymmetric smooth maps on $\mathcal{A}^p$ can be viewed as constant p-forms on $\mathcal{A}$. In particular, a linear family of differential forms $\nu : \mathcal{A} \to \Omega^{q,r}(\mathcal{F} \times M)$ can be viewed as constant 1-form on $\mathcal{A}$ with values in $\Omega^{q,r}(\mathcal{F} \times M)$ which is given by the map

$$\nu \circ \mathrm{pr}_2 : T\mathcal{A} \cong \mathcal{A} \times \mathcal{A} \xrightarrow{\mathrm{pr}_2} \mathcal{A} \xrightarrow{\nu} \Omega^{q,r}(\mathcal{F} \times M)\,.$$

Conversely, a 1-form $\mu : T\mathcal{A} \to \Omega^{q,r}(\mathcal{F} \times M)$ is local if it factors as $\mu = \nu \circ \mathrm{pr}_2$ through a map $\nu : \mathcal{A} \to \Omega^{q,r}(\mathcal{F} \times M)$, which is linear since a form is by definition fibre-wise linear. We thus obtain a bijection between linear families and constant 1-forms.

Lemma 6.3.11. *Let $\nu : \mathcal{A} \to \Omega^{q,r}(\mathcal{F} \times M)$ be a linear family of differential forms. If ν is local, then the form in $\Omega^{1,q,r}(\mathcal{A} \times \mathcal{F} \times M)$ that is given by $\nu \circ \mathrm{pr}_2$ is local as well.*

Proof. The family of differential forms ν can be viewed as map $\tilde{\nu} : \mathcal{A} \times (T\mathcal{F}_{/\mathcal{F}})^q \to \Omega^r(M)$. The $(1, q, r)$-form defined by $\nu \circ \mathrm{pr}_2$ is given by the composition

$$\begin{aligned} \widetilde{\nu \circ \mathrm{pr}_2} : T\mathcal{A} \times (T\mathcal{F}_{/\mathcal{F}})^q &\xrightarrow{\cong} \mathcal{A} \times \mathcal{A} \times (T\mathcal{F}_{/\mathcal{F}})^q \\ &\xrightarrow{\mathrm{pr}_2 \times \mathrm{id}^q_{T\mathcal{F}}} \mathcal{A} \times (T\mathcal{F}_{/\mathcal{F}})^q \\ &\xrightarrow{\tilde{\nu}} \Omega^r(M)\,. \end{aligned} \tag{6.18}$$

Assume that $\tilde{\nu}$ is local. The projection $\mathrm{pr}_2 : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ is induced by the fibre-wise projection $A \times_M A \to A$ onto the second factor, so that pr_2 is local. Since $\widetilde{\nu \circ \mathrm{pr}_2}$ is given by products and compositions of local maps, it is local. □

Lemma 6.3.12. *Let $\nu : \mathcal{A} \to \Omega^{q,r}(\mathcal{F} \times M)$ be a linear family of differential forms. Then $d_M \circ \nu = 0$ if and only if the form in $\Omega^{1,q,r}(\mathcal{A} \times \mathcal{F} \times M)$ that is given by $\nu \circ \mathrm{pr}_2$ is d-closed.*

Proof. By Def. 6.3.7 and Def. 6.2.4, the form given by family $\nu \circ \mathrm{pr}_2$ is d-closed if and only if the map

$$0 = d_M \circ (\widetilde{\nu \circ \mathrm{pr}_2}) = (d_M \circ \tilde{\nu}) \circ (\mathrm{pr}_2 \times \mathrm{id}^q_{T\mathcal{F}})\,,$$

where we have used Eq. (6.18). Since $\mathrm{pr}_2 \times \mathrm{id}^q_{T\mathcal{F}}$ is surjective, this is the case if and only if $d_M \circ \tilde{\nu} = 0$, which by Rmk. 6.3.8 is the case if and only if the form in $\Omega^{1,q,r}(\mathcal{A} \times \mathcal{F} \times M)$ that is given by $\tilde{\nu} \circ \mathrm{pr}_2$ is d-closed. □

Proposition 6.3.13. *Let $A \to M$ be a smooth vector bundle of non-zero rank. Let $\nu : \mathcal{A} \to \Omega^r(M)$ be a linear family of differential forms. Assume that the following three conditions are satisfied:*

(i) ν *is local,*

(ii) $d_M \circ \nu = 0$,

(iii) $r < \dim M$.

Then there is a local linear family of differential forms $\mu : \mathcal{A} \to \Omega^{r-1}(M)$, *such that* $\nu = d_M \circ \mu$.

Proof. The linear family ν gives rise to the $(1, r)$-form $\sigma := \nu \circ \mathrm{pr}_2 : T\mathcal{A} \to \Omega^r(M)$ on $\mathcal{A} \times M$. Assume that ν is a local family of forms, then Lem. 6.3.11 shows that σ is a local form, i.e. the pullback $\sigma = (j^\infty)^*\omega$ of a form $\omega \in \Omega^{1,r}(J^\infty F)$. Assume furthermore that $d_M \circ \nu = 0$. Then Lem. 6.3.12 shows that σ is d-closed. It follows from Prop. 6.2.6 that $d(j^\infty)^*\omega = (j^\infty)^* d\omega = 0$. Since for a vector bundle $A \to M$ the evaluation map $j^0 : \mathcal{A} \times M \to A$ is surjective, Thm. 6.1.2 shows that $(j^\infty)^*$ is injective. This implies that $d\omega = 0$. Finally, assume that $r < \dim M$. Then the acyclicity theorem 5.2.4 implies that there is a form $\alpha \in \Omega^{0,r-1}(J^\infty F)$ such that $\omega = d\alpha$. The pullback $\tau := (j^\infty)^*\alpha \in \Omega^{1,r-1}(\mathcal{F} \times M)$ then satisfies $d\tau = d(j^\infty)^*\alpha = (j^\infty)^* d\alpha = (j^\infty)^*\omega = \nu'$. Spelling out the definition 6.2.4 of d, we conclude that $d_M \circ \tau = \sigma = \nu \circ \mathrm{pr}_2$.

Let the embedding of the fibre of $T\mathcal{A} \to \mathcal{A}$ over $b \in \mathcal{A}$ be denoted by

$$\begin{aligned} i_b : \mathcal{A} &\longrightarrow \mathcal{A} \times \mathcal{A} \cong T\mathcal{A} \\ a &\longmapsto (b, a)\,, \end{aligned}$$

which is a section of pr_2. Since i_b descends to $A \to A \times_M A$, $a_m \mapsto (b(m), a_m)$, it is local. Let $\mu := \tau \circ \iota_b$. Since τ and ι_b is local, μ is local. And since τ is fibre-wise linear, μ is linear. We now have the following commutative diagram:

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "i_b"] \arrow[dr, "\mathrm{id}_{\mathcal{A}}"'] & \mathcal{A} \times \mathcal{A} \arrow[r, "\tau"] \arrow[d, "\mathrm{pr}_2"] & \Omega^{r-1}(M) \arrow[d, "d_M"] \\
 & \mathcal{A} \arrow[r, "\nu"] & \Omega^r(M)
\end{tikzcd}$$

This shows that $\nu = d_M \circ \mu$, which finishes the proof. □

Definition 6.3.14. Let $A \to M$ be a smooth vector bundle. A form $\omega \in \Omega^{0,r}(J^\infty A)$ will be called **vertically linear** if the map

$$\widetilde{(j^\infty)^*\omega} : \mathcal{A} \longrightarrow \Omega^r(M)$$

is linear.

Proposition 6.3.15. *Let* $\omega \in \Omega^{0,r}(J^\infty A)$ *be vertically linear and* $r < \dim M$. *If* ω *is d-closed, then it is d-exact.*

Proof. Let $\nu := \widetilde{(j^\infty)^*\omega}$. By assumption ω is vertically linear so that ν is a d-closed local linear family of forms. Prop. 6.3.13 implies that there is a a local linear $\mu : \mathcal{A} \to \Omega^{r-1}(M)$ such that $\nu = d_M \circ \mu$. By definition of locality, this means that there is an $\alpha \in \Omega^{0,r-1}(\mathcal{A} \times M)$, such that $\mu = \widetilde{(j^\infty)^*\alpha}$. It follows that $(j^\infty)^*\omega = d(j^\infty)^*\alpha = (j^\infty)^* d\alpha$. Since $A \to M$ is a vector bundle, $j^0 : \mathcal{A} \times M \to A$ is surjective, so that by Thm. 6.1.2 $(j^\infty)^*$ is injective. We conclude that $\omega = d\alpha$. □

6.3.3 Closed and exact forms at fields

Definition 6.3.16. A form $\nu \in \Omega^{p,q}(\mathcal{F} \times M)$ will be called **d-closed at** $\varphi \in \mathcal{F}$ if $(d\nu)_\varphi = 0$. It will be called **d-exact at** φ if there is a $\lambda \in \Omega^{p,q}(\mathcal{F} \times M)$ such that $\nu_\varphi = (d\lambda)_\varphi$.

Remark 6.3.17. A form $\nu \in \Omega^{p,q}(\mathcal{F} \times M)$ is d-closed if and only if it is d-closed at all fields $\varphi \in \mathcal{F}$. If ν is d-exact, then it is d-exact at all $\varphi \in \mathcal{F}$. The converse of the last statement, however, is not true. For example, consider the case that M is non-compact, so that $H^{\text{top}}(M) = 0$. Then every lagrangian form $\mathcal{L} \in \Omega^{0,\text{top}}(\mathcal{F} \times M)$ is exact at every $\varphi \in \mathcal{F}$, which of course does not imply that $\mathcal{L}$ is d-exact.

Proposition 6.3.18. *If $\nu \in \Omega^{p,q}(\mathcal{F} \times M)$ is d-exact at $\varphi \in \mathcal{F}$, then ν is d-closed at φ.*

Proof. It follows from Rmk. 6.1.7 and the definition 6.2.4 of the differential d on $\Omega(\mathcal{F} \times M)$ that ω is exact at φ if there is a form λ such that

$$\begin{aligned}\tilde{\omega}_\varphi(\xi^1_\varphi, \dots, \xi^p_\varphi) &= \widetilde{d\lambda}_\varphi(\xi^1_\varphi, \dots, \xi^p_\varphi)\\ &= d_M\big(\lambda_\varphi(\xi^1_\varphi, \dots, \xi^p_\varphi)\big)\end{aligned}$$

for all $\xi^1_\varphi, \dots, \xi^p_\varphi \in T_\varphi\mathcal{F}$. It follows that

$$\begin{aligned}\widetilde{d\omega}_\varphi(\xi^1_\varphi, \dots, \xi^p_\varphi) &= d_M\big(\tilde{\omega}_\varphi(\xi^1_\varphi, \dots, \xi^p_\varphi)\big)\\ &= d_M\big(\widetilde{d\lambda}_\varphi(\xi^1_\varphi, \dots, \xi^p_\varphi)\big)\\ &= d_M^2\big(\lambda_\varphi(\xi^1_\varphi, \dots, \xi^p_\varphi)\big)\\ &= 0\end{aligned}$$

for all $\xi^1_\varphi, \dots, \xi^p_\varphi \in T_\varphi\mathcal{F}$. This shows that $(d\omega)_\varphi = 0$. □

Definition 6.3.19. A form $\omega \in \Omega(J^\infty F)$ is said to be **d-closed at** $\varphi \in \mathcal{F}$ if $d\omega$ vanishes at φ, $(d\omega)_{j^\infty\varphi} = 0$. It will be called **$d$-exact at** φ if there is an $\alpha \in \Omega(J^\infty F)$ such that $\omega_{j^\infty\varphi} = (d\alpha)_{j^\infty\varphi}$.

Proposition 6.3.20. *Let $\omega \in \Omega^{p,q}(J^\infty F)$ and $\varphi \in \mathcal{F}$.*

(i) ω is d-closed at φ if and only if $(j^\infty)^*\omega \in \Omega^{p,q}(\mathcal{F} \times M)$ is d-closed at φ.

(ii) If ω is d-exact at φ, then $(j^\infty)^*\omega$ is d-exact at φ.

Proof. Let $\omega \in \Omega^n(J^\infty F)$, where $n = p + q$, and let $\nu := (j^\infty)^*\omega \in \Omega^n(\mathcal{F} \times M)$. By Prop. 6.2.6 we have $d\nu = (j^\infty)^*\omega = (j^\infty)^*d\omega$. Lem. 6.1.12 implies that $(j^\infty)^*d\omega$ vanishes at φ if and only if $d\omega$ vanishes at φ. We conclude that $d\nu$ vanishes at φ if and only if $d\omega$ vanishes at φ, which proves (i).

Assume that there is a form $\alpha \in \Omega^{p,q-1}(J^\infty F)$, such that $\omega - d\alpha$ vanishes at φ. Let $\lambda := (j^\infty)^*\alpha$. Then $\nu - d\lambda = (j^\infty)^*\omega - d(j^\infty)^*\alpha = (j^\infty)^*(\omega - d\alpha)$ vanishes at φ by Lem. 6.1.12, which proves (ii). □

Proposition 6.3.21. *Let $\varphi \in \mathcal{F}$. The following are equivalent:*

(i) $\omega \in \Omega^{0,q}(J^\infty F)$ *is d-exact at* φ.

(ii) $(j^\infty)^*\omega \in \Omega^{0,q}(\mathcal{F} \times M)$ *is d-exact at* φ.

(iii) $(j^\infty\varphi)^*\omega \in \Omega^q(M)$ *is exact.*

Proof. Assume (i). Then Prop. 6.3.20 implies (ii).

Assume (ii), i.e. $\nu := (j^\infty)^*\omega$ is d-exact at φ. This means that there is a $\lambda \in \Omega^{0,q-1}(\mathcal{F} \times M)$, such that $\nu_\varphi = (d\lambda)_\varphi$. We have

$$(j^\infty\varphi)^*\omega = \nu_\varphi = (d\lambda)_\varphi = d_M\big(\tilde{\lambda}(\varphi)\big)\,,$$

where in the last step we have used Def. 6.2.4. We conclude that (iii) holds.

Assume (iii). By Lem. 6.1.11 this means that $(j^\infty\varphi)^*\omega = d_M\tau$ for some $\tau \in \Omega^{q-1}(M)$. Define a form $\alpha \in \Omega^{0,q-1}(J^\infty F)$ by $\alpha_{j^\infty_m\psi}(v^1_m, \ldots, v^q_m) := \tau(v^1_m, \ldots, v^q_m)$ for all $\psi \in \mathcal{F}$. By construction, the form α satisfies $(j^\infty\varphi)^*\alpha = \tau$, which implies that

$$(j^\infty\varphi)^*\omega = d_M\tau = d_M\big((j^\infty\varphi)^*\alpha\big) = (j^\infty\varphi)^*d\alpha\,,$$

where we have used Prop. 6.2.6. This shows that (iii) implies (i), which concludes the proof. □

Proposition 6.3.22. *If* $\omega \in \Omega^{p,q}(J^\infty F)$ *is d-exact at* $\varphi \in \mathcal{F}$*, then* ω *is d-closed at* φ.

Proof. Assume that $\omega_{j^\infty\varphi} = (d\alpha)_{j^\infty\varphi}$. It follows from Prop. 6.3.20 that $(j^\infty)^*\omega$ is d-exact at φ. Prop. 6.3.18 then shows that $(j^\infty)^*\omega$ is d-closed at φ, i.e. $d(j^\infty)^*\omega = (j^\infty)^*\omega$ vanishes at φ. With Lem. 6.1.12 we conclude that $d\omega$ vanishes at φ, i.e. ω is d-closed at φ. □

Proposition 6.3.23. *Let* $\omega \in \Omega^{p,q}(J^\infty F)$ *for* $p > 0$ *and* $q < \dim M$. *If* ω *is d-closed at* $\varphi \in \mathcal{F}$*, then* ω *is d-exact at* φ.

Proof. As observed in Rmk 6.1.7, the evaluation of $(j^\infty)^*\omega$ at φ can be viewed as a map

$$\nu : (T_\varphi\mathcal{F})^p \longhookrightarrow (T\mathcal{F}_{/\mathcal{F}})^p \xrightarrow{\widetilde{(j^\infty)^*\omega}} \Omega^q(M)\,.$$

The domain of ν is the space of sections of the vector bundle $A := \big((\varphi^*VF)_{/M}\big)^p$. This map has the following properties:

- Since $p > 0$, the rank of A is non-zero.
- The map ν is linear.
- The inclusion $(T_\varphi\mathcal{F})^p \hookrightarrow (T\mathcal{F}_{/\mathcal{F}})^p$ is local, since it is induced by the inclusion of fibre bundles $\varphi^*VF \hookrightarrow VF$ of every factor of A. The map $\widetilde{(j^\infty)^*\omega}$ is local by Prop. 6.1.3. Since the composition of local maps is local, ν is local.
- By assumption ω is closed at φ. By Prop 6.3.20, $(j^\infty)^*\omega$ is closed at φ, which means that $d_M \circ \nu = 0$.

- By assumption, $q < \dim M$.

This shows that all conditions of Prop. 6.3.13 are satisfied. It follows that there is a local multilinear map $\mu : \mathcal{A} = (T_\varphi \mathcal{F})^p \to \Omega^{q-1}(M)$ such that $\nu = d_M \circ \mu$.

By Prop. 6.1.20, there is a form $\alpha \in \Omega^{p,q}(J^\infty F)$, such that $\big((j^\infty)^* \alpha\big)_\varphi = \mu$. This implies

$$\begin{aligned}\big((j^\infty)^*\omega\big)_\varphi &= \nu \\ &= d_M \circ \mu \\ &= d_M \circ \big((j^\infty)^*\alpha\big)_\varphi \\ &= \big(d(j^\infty)^*\alpha\big)_\varphi \\ &= \big((j^\infty)^* d\alpha\big)_\varphi .\end{aligned}$$

In other words, $(j^\infty)^*(\omega - d\alpha)$ vanishes at φ. By Lem. 6.1.12 this implies that $\omega - d\alpha$ vanishes at φ, which concludes the proof. □

Proposition 6.3.24. *Let $\omega \in \Omega^{p,\mathrm{top}}(J^\infty F)$ where $p > 0$, let $\varphi \in \mathcal{F}$, and let P be the interior Euler operator. The following are equivalent:*

(i) ω is d-exact at φ

(ii) $P\omega$ vanishes at φ.

For the proof of Prop. 6.3.24 follows from the following two technical lemmas, which we will also need for the theory of generalized Jacobi fields.

Lemma 6.3.25. *Let $\omega \in \Omega(J^\infty F)$ and let $v \in \mathfrak{X}(J^\infty F)$ be a horizontal vector field. If ω vanishes at $\varphi \in \mathcal{F}$, then $\mathcal{L}_v\omega$ vanishes at φ.*

Proof. The condition $(\mathcal{L}_v\omega)_{j^\infty\varphi} = 0$ is local, so it can be checked in local coordinates in which the vector field is of the form $v = v^i D_i$ for some functions $v^i \in C^\infty(J^\infty F)$. First, consider the case that $f \in \Omega^0(J^\infty F)$ a function. Then

$$(\mathcal{L}_v f)_{j^\infty\varphi} = \big(v^i(D_i f)\big)_{j^\infty\varphi} = (v^i \circ j^\infty\varphi)\frac{\partial}{\partial x^i}(f \circ j^\infty\varphi) .$$

If $f \circ j^\infty\varphi \in C^\infty(M)$ is zero, then the right hand side is zero, which proves the statement for 0-forms. Let $\omega \in \Omega^{p,q}(J^\infty F)$. In local coordinates

$$\begin{aligned}\omega &= \omega^{I_1\dots I_p}_{\alpha_1\dots\alpha_p i_1\dots i_q}\delta u^{\alpha_1}_{I_1} \wedge \dots \wedge \delta u^{\alpha_p}_{I_p} \wedge dx^{i_1} \wedge \dots \wedge dx^{i_q} \\ &= \omega^{I_1\dots I_p}_{\alpha_1\dots\alpha_p i_1\dots i_q}\tau^{\alpha_1\dots\alpha_p i_1\dots i_q}_{I_1\dots I_p} ,\end{aligned}$$

where

$$\tau^{\alpha_1\dots\alpha_p i_1\dots i_q}_{I_1\dots I_p} := \delta u^{\alpha_1}_{I_1} \wedge \dots \wedge \delta u^{\alpha_p}_{I_p} \wedge dx^{i_1} \wedge \dots \wedge dx^{i_q} .$$

The form ω vanishes at φ if and only if the functions $\omega^{I_1\dots I_p}_{\alpha_1\dots\alpha_p i_1\dots i_q}$ vanish at φ. For the Lie derivative with respect to v we obtain

$$\mathcal{L}_v\omega = (\mathcal{L}_v\omega^{I_1\dots I_p}_{\alpha_1\dots\alpha_p i_1\dots i_q})\tau^{\alpha_1\dots\alpha_p i_1\dots i_q}_{I_1\dots I_p} + \omega^{I_1\dots I_p}_{\alpha_1\dots\alpha_p i_1\dots i_q}(\mathcal{L}_v\tau^{\alpha_1\dots\alpha_p i_1\dots i_q}_{I_1\dots I_p}) .$$

Assume that the functions $\omega^{I_1\dots I_p}_{\alpha_1\dots\alpha_p i_1\dots i_q}$ vanish at φ. We have already shown that their Lie derivatives with respect to v vanish at φ, so that both terms on the right hand side vanish at φ. □

Lemma 6.3.26. *Let* $\omega \in \Omega^{p,\mathrm{top}}(J^\infty F)$ *where* $p > 0$*, let* $\varphi \in \mathcal{F}$*, and let* P *be the interior Euler operator. If* ω *vanishes at* φ*, then* $P\omega$ *vanishes at* φ*.*

Proof. The condition $(P\omega)_{j^\infty\varphi} = 0$ is local, so it can be checked in local coordinates, in which $P\omega$ is given by Eq. (5.25), that is

$$P\omega = \delta u^\alpha \wedge \frac{1}{p} \sum_I (-1)^{|I|} \mathcal{L}_{D_I}\Big(\frac{\partial}{\partial u^\alpha_I} \lrcorner\, \omega\Big)\,. \tag{6.19}$$

Assume that ω vanishes at φ. Then $\frac{\partial}{\partial u^\alpha_I} \lrcorner\, \omega$ vanishes at φ. It follows from Lem. 6.3.25 that

$$\mathcal{L}_{D_I}\Big(\frac{\partial}{\partial u^\alpha_I} \lrcorner\, \omega\Big) = (\mathcal{L}_{D_1})^{I_1} \cdots (\mathcal{L}_{D_n})^{I_n}\Big(\frac{\partial}{\partial u^\alpha_I} \lrcorner\, \omega\Big)$$

vanishes at φ. Since each summand on the right hand side of Eq. (6.19) vanishes at φ, so does the sum $P\omega$. □

Proof of Prop. 6.3.24. Assume (i). Then there is a form $\alpha \in \Omega^{p,q-1}(J^\infty F)$, so that $\omega - d\alpha$ vanishes at φ. By Lem. 6.3.26, it follows that $P(\omega - d\alpha) = P\omega$ vanishes at φ.

Conversely, assume (ii). By Cor. 5.2.5, $\omega - P\omega = d\alpha$ for some form $\alpha \in \Omega^{p,q-1}(J^\infty F)$. Then

$$\omega_{j^\infty\varphi} = (P\omega + d\alpha)_{j^\infty\varphi} = (d\alpha)_{j^\infty\varphi}\,,$$

which shows that ω is exact at φ. □

Lemma 6.3.27. *Let* $\omega \in \Omega^{p,\mathrm{top}}(J^\infty F)$*. Let* $\nu := (j^\infty)^*\omega \in \Omega^{p,\mathrm{top}}(\mathcal{F} \times M)$*. Let* $\varphi \in \mathcal{F}$*. If the base* M *of the fibre bundle* $F \to M$ *is closed, then the following are equivalent:*

(i) $\int_M \tilde{\nu}(\xi^1_\varphi, \ldots, \xi^p_\varphi) = 0$ *for all* $\xi^1_\varphi, \ldots, \xi^p_\varphi \in T_\varphi\mathcal{F}$*.*

(ii) $P\omega_{j^\infty} = 0$ *is exact at* φ*.*

Proof. Since $\omega - P\omega = d\alpha$ by Cor. 5.2.5, we obtain for the integral

$$\begin{aligned}
\int_M \tilde{\nu}(\xi^1_\varphi, \ldots, \xi^p_\varphi) &= \int_M \iota_{\xi^p_\varphi} \cdots \iota_{\xi^1_\varphi} (j^\infty)^*\omega \\
&= \int_M \iota_{\hat{\xi}^p_{j^\infty\varphi}} \cdots \iota_{\hat{\xi}^1_{j^\infty\varphi}} \omega \\
&= \int_M \iota_{\hat{\xi}^p_{j^\infty\varphi}} \cdots \iota_{\hat{\xi}^1_{j^\infty\varphi}} P\omega + \int_M d_M \iota_{\hat{\xi}^p_{j^\infty\varphi}} \cdots \iota_{\hat{\xi}^1_{j^\infty\varphi}} \alpha \\
&= \int_M \iota_{\hat{\xi}^p_{j^\infty\varphi}} \cdots \iota_{\hat{\xi}^1_{j^\infty\varphi}} P\omega
\end{aligned}$$

for all tangent vectors $\xi^1_\varphi, \ldots, \xi^p_\varphi \in T_\varphi\mathcal{F}$ with infinite prolongations $\hat{\xi}^1_{j^\infty\varphi}, \ldots, \hat{\xi}^p_{j^\infty\varphi}$.

We first consider the case $p = 1$. Then

$$\begin{aligned}
\int_M \iota_{\hat{\xi}_{j^\infty\varphi}} P\omega = \int_M \iota_{\xi_\varphi} (j^\infty)^* P\omega = \int_M \iota_{\xi_\varphi} d\mu = \int_M d_M(\iota_{\xi_\varphi}\mu) \\
= 0\,.
\end{aligned}$$

for all $\xi_\varphi \in T_\varphi \mathcal{F}$, where $\hat{\xi}_{j^\infty\varphi}$ is the infinite prolongation of ξ_φ. In local coordinates $P\omega = \delta u^\alpha \wedge P_\alpha \, dx^1 \wedge \ldots \wedge dx^n$ where $P_\alpha \in C^\infty(J^\infty F)$, $\xi_\varphi = \xi^\alpha_\varphi \frac{\partial}{\partial u^\alpha}$ where $\xi^\alpha_\varphi \in C^\infty(M)$, and $\hat{\xi}_{j^\infty\varphi}$ is given by Eq. (6.9). The integral now takes the form

$$\int_M \iota_{\hat{\xi}_{j^\infty\varphi}} P\omega = \int_M \xi^\alpha_\varphi(x) P_\alpha(j^\infty_x \varphi) \, dx^1 \wedge \ldots \wedge dx^n \,,$$

which vanishes for all functions $\xi^\alpha_\varphi \in C^\infty(M)$ if and only if $P_\alpha(j^\infty_x\varphi) = 0$ for all x. This is the case if and only if $P\omega_{j^\infty\varphi} = 0$. *** □

Proposition 6.3.28. *Let $\omega \in \Omega^{p,\mathrm{top}}(J^\infty F)$ where $p > 0$ and let $\varphi \in \mathcal{F}$. If the base M of the fibre bundle $F \to M$ is a closed manifold, then the following are equivalent:*

- *(i) $(j^\infty)^*\omega$ is d-exact at φ.*
- *(ii) ω is d-exact at φ.*

Proof. Let $\nu := (j^\infty)^*\omega$. Assume (i), which means that there is a form $\lambda \in \Omega^{1,\mathrm{top}-1}(\mathcal{F} \times M)$, such that $\nu_\varphi = (d\lambda)_\varphi$. It follows that

$$\begin{aligned} \int_M \tilde{\nu}(\xi^1_\varphi, \ldots, \xi^p_\varphi) &= \int_M d_M \tilde{\lambda}(\xi^1_\varphi, \ldots, \xi^p_\varphi) \\ &= 0 \,, \end{aligned}$$

for all $\xi^1_\varphi, \ldots, \xi^p_\varphi \in T_\varphi \mathcal{F}$. Lem. 6.3.27 then implies that $P\omega$ vanishes at φ. By Prop. 6.3.24 it follows that ω is exact at φ. We conclude that (i) implies (ii). In Prop. 6.3.20 it was already shown that (ii) implies (i), which finishes the proof □

6.3.4 Relative horizontal cohomology

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Chapter 7

The action principle

Recall from Sec. 1.1 that a lagrangian is a smooth map $\mathcal{L} : \mathcal{F} \to \Omega^{\mathrm{top}}(M)$. When M is closed we can define the action integral by

$$S(\varphi) := \int_M \mathcal{L}(\varphi)\,, \tag{7.1}$$

The action principle states that the critical points of S are the solutions of the equations of motion. If $\mathcal{L}$ is a local map, then the critical points of the action are the solutions of a PDE, the Euler-Lagrange equation. We will give a proof of this statement in Thm. 7.1.6.

When M is not compact, the action integral will generally not be defined for all fields. In order to obtain a mathematically rigorous action principle for this case, the notions of lagrangian, action, critical points, etc. have to be rephrased in terms of the cohomology of forms on $\mathcal{F} \times M$ and the variational bicomplex. In a first attempt to sidestep integration over M by homological methods, we could look at the map

$$\begin{aligned} \mathcal{F} &\longrightarrow H^{\mathrm{top}}(M) \\ \varphi &\longmapsto [\mathcal{L}(\varphi)]\,, \end{aligned} \tag{7.2}$$

where the bracket denotes the de Rham cohomology class in $H^{\mathrm{top}}(M)$. When M is a closed connected and orientable manifold, then $H^{\mathrm{top}}(M) \cong \mathbb{R}$. In this case, (7.2) can be thought of the usual action divided by the total volume of M. When M is non-compact, however, $H^{\mathrm{top}}(M) = 0$ so that (7.2) is the zero map. A better approach is to formulate the action principle in terms of the variational bicomplex, using the relation between local $\Omega(M)$-valued forms on $\mathcal{F}$ and forms on $J^\infty F$ that we have established in Sec. 6.1.1.

7.1 The action principle

7.1.1 Lagrangian form and Euler-Lagrange form

As we have seen in Sec. 2.3.2, a smooth map $\mathcal{L} : \mathcal{F} \to \Omega^{\mathrm{top}}(M)$ can be identified with a differential form $\mathcal{L} \in \Omega^{0,\mathrm{top}}(\mathcal{F} \times M)$. In Prop. 6.1.3 we have shown that the map $\mathcal{L}$ is local if, when viewed as a differential form on $\mathcal{F} \times M$, it is the pullback of a form $L \in \Omega^{0,\mathrm{top}}(J^\infty F)$. This means that $\mathcal{L}$ is given by

$$\mathcal{L}(\varphi) = (j^\infty \varphi)^* L$$

for all $\varphi \in \mathcal{F}$. The form L will be the primary object by which we study a local lagrangian field theory. In Thm. 6.1.2 we have shown that L is generally not uniquely determined by $\mathcal{L}$, so that it is part of the data of a field theory.

Definition 7.1.1. A **local lagrangian field theory** is given by a manifold M, a smooth fibre bundle $F \to M$, and a form $L \in \Omega^{0,\mathrm{top}}(J^\infty F)$ called the **lagrangian form**.

A lagrangian form is given in local coordinates by

$$L = L(x^i, u^\alpha, \ldots, u^\alpha_{i_1,\ldots,i_k})\, dx^1 \wedge \ldots \wedge dx^n \,,$$

where n is the dimension of M and k the jet order of the form L. When we evaluate the lagrangian $\mathcal{L} : \mathcal{F} \to \Omega^n(M)$ at $\varphi \in \mathcal{F}$, we obtain

$$\mathcal{L}(\varphi) = L\Big(x^i, \varphi^\alpha, \frac{\partial \varphi^\alpha}{\partial x^{i_1}}, \ldots, \frac{\partial^k \varphi^\alpha}{\partial x^{i_1} \cdots \partial x^{i_k}}\Big)\, dx^1 \wedge \ldots \wedge dx^n \,,$$

which is the usual expression for the integrand of the action integral found in physics textbooks.

Definition 7.1.2. Let L be a lagrangian form. The form

$$EL \in \Omega^{1,\mathrm{top}}(J^\infty F) \,,$$

where $E = P\delta$ is the Euler operator (Def. 5.2.3), is called the **Euler-Lagrange form** of L.

Let EL be represented by a form in $\Omega^{\mathrm{top}+1}(J^k F) = \Gamma(J^k F, \wedge^{\mathrm{top}+1} T^* J^k F)$. We can evaluate EL at every point of $J^k F$, in particular at every point of the prolongation $j^k\varphi : M \to J^k F$ of a field $\varphi \in \mathcal{F}$. This yields a map

$$\begin{aligned} \mathcal{F} &\longrightarrow \Gamma(M, \wedge^{\mathrm{top}+1} T^* J^k F) \\ \varphi &\longmapsto (EL_{j^k\varphi} : m \mapsto EL_{j^k_m \varphi}) \,. \end{aligned}$$

If we do not want to specify the jet-order k, we can denote the map on the right hand side also by $EL_{j^\infty \varphi}$.

Definition 7.1.3. The equation

$$EL_{j^\infty \varphi} = 0 \tag{7.3}$$

for $\varphi \in \mathcal{F}$ is called the Euler-Lagrange equation.

In local coordinates, the Euler-Lagrange form is given by

$$EL = E_\alpha \delta u^\alpha \wedge dx^1 \wedge \ldots \wedge dx^n \,, \tag{7.4}$$

where $E^\alpha = E^\alpha(x^i, u^\beta, u^\beta_{i_1}, \ldots, u^\beta_{i_1,\ldots,i_k})$ are functions on some finite jet manifold $J^k F$. The Euler-Lagrange equation is the k-th order PDE given in local coordinates by

$$E_\alpha\Big(x^i, \varphi^\beta, \frac{\partial \varphi^\beta}{\partial x^{i_1}}, \ldots, \frac{\partial^k \varphi^\beta}{\partial x^{i_1} \cdots \partial x^{i_k}}\Big) = 0 \,.$$

Using the local coordinate formula (5.15a) for the vertical differential δ and the formula (5.25) for the interior Euler operator P, we see that E_α is given in terms of L by

$$E_\alpha = \sum_{|I|\geq k} (-1)^{|I|} D_I \Big(\frac{\partial L}{\partial u_I^\alpha} \Big) .$$

The Euler-Lagrange equation then takes the local coordinate form

$$\sum_{|I|\geq k} (-1)^{|I|} \frac{\partial^{|I|}}{\partial x^I} \Big(\frac{\partial L}{\partial u_I^\alpha} \circ j^k\varphi \Big) = 0 .$$

Notation 7.1.4. In the physics literature it is customary to use the same notation for the coordinate functions u_I^α and their evaluation at a field, i.e. $u_I^\alpha \equiv \frac{\partial^{|I|}\varphi^\alpha}{\partial x^I}$. With this notation, the Euler-Lagrange equation is written as

$$\sum_{|I|\geq k} (-1)^{|I|} \frac{\partial^{|I|}}{\partial x^I} \left(\frac{\partial L}{\partial (\frac{\partial^{|I|}\varphi^\alpha}{\partial x^I})} \right) = 0 .$$

Definition 7.1.5. Let (M, F, L) be a local LFT. The **diffeological space of solutions** of the Euler-Lagrange equation will be denoted by $\mathfrak{F}_{\text{shell}}$. That is, $\mathfrak{F}_{\text{shell}} = \{\varphi \in \mathfrak{F} \mid EL_{j^\infty\varphi} = 0\} \subset \mathfrak{F}$ equipped with the subspace diffeology.

7.1.2 The cohomological action principle

Theorem 7.1.6. *Let (M, F, L) be a local lagrangian field theory over a closed manifold M. Then $\varphi \in \mathfrak{F}$ is a diffeological critical point of the action if and only if φ is a solution of the Euler-Lagrange equation.*

Proof. The action $S : \mathfrak{F} \to \mathbb{R}$ is the composition of the lagrangian $\mathfrak{L} : \mathfrak{F} \to \Omega^{\text{top}}(M)$, $\varphi \mapsto (j^\infty\varphi)^*L$ with the integration $\int : \Omega^{\text{top}}(M) \to \mathbb{R}$. It follows from Prop. 6.2.10 that the diffeological tangent map of $\mathfrak{L}$ is given by

$$(T_\varphi \mathfrak{L})\xi_\varphi = \iota_{\xi_\varphi} (j^\infty)^* \delta L .$$

Since M is compact, the integration is a smooth map. As we have shown in Prop. 2.3.12, the diffeological tangent space of $\Omega^{\text{top}}(M)$ at a form ω is isomorphic to $\Omega^{\text{top}}(M)$ itself. (Note, that this is not true for arbitrary diffeological vector spaces.) With this identification, the tangent map of the smooth linear map $\int_M : \Omega^{\text{top}}(M) \to \mathbb{R}$ at ω is isomorphic to $\int_M$ itself, by way of the following commutative diagram:

$$\begin{tikzcd}
T_\omega \Omega^{\text{top}}(M) \arrow[r, "T_\omega \int_M"] \arrow[d, "\cong"'] & T_{\int_M \omega}\mathbb{R} \arrow[d, "\cong"] \\
\Omega^{\text{top}}(M) \arrow[r, "\int_M"] & \mathbb{R}
\end{tikzcd}$$

Identifying the top and bottom rows of this diagram, we obtain for the tangent map of the action

$$(T_\varphi S)(\xi_\varphi) = \int_M \iota_{\xi_\varphi} (j^\infty)^* \delta L . \tag{7.5}$$

Lem. 6.3.27 states that the right hand side vanishes for all $\xi_\varphi \in T_\varphi \mathcal{F}$ if and only if $(P\delta L)_{j^\infty\varphi} = EL_{j^\infty\varphi} = 0$. We conclude that $T_\varphi S = 0$ if and only if φ is a solution of the Euler-Lagrange equation. □

In order to recast the action in cohomological terms, we observe that when M is compact, the integration of $\mathcal{L}(\varphi)$ over M can be viewed as the duality pairing of the the fundamental class $[M] \in H_{\text{top}}(M)$ with the de Rham cohomology class $[\mathcal{L}(\varphi)] \in H^{\text{top}}(M)$. When M is not compact, there is a pairing of $[M]$ with the compactly supported de Rham cohomology in top degree (see e.g. [BT82]). However, for many important lagrangians $\mathcal{L}(\varphi)$ is generally not compactly supported. The idea is now that we view the lagrangian $\mathcal{L} : \mathcal{F} \to \Omega^{\text{top}}(M)$ as a $(0, \text{top})$-form on $\mathcal{F} \times M$ and replace the integration over M with the d-cohomology class of the integrand,

$$\int_M \mathcal{L} \quad \rightsquigarrow \quad [\mathcal{L}]_d \,,$$

where the differential d was defined in Def. 6.2.4. The class $[\mathcal{L}]_d$ can then be viewed as a cohomological replacement for the action.

For the action principle, we have to determine the zeros of the differential of the action. For closed M we have seen in Thm. 7.1.6 that the diffeological derivative of the action at $\varphi \in \mathcal{F}$ is given by $T_\varphi \mathcal{F} \to \mathbb{R}$, $\xi_\varphi \mapsto \int_M \iota_{\xi_\varphi} \delta\mathcal{L}$, where $\delta\mathcal{L} : T\mathcal{F} \to \Omega^{\text{top}}(M)$ is the diffeological differential of $\mathcal{L}$, as defined in Def. 6.2.9. In the cohomological setting the condition for the integral to be zero at φ has to be replaced by the condition for the integrand to be exact at φ. In a first attempt, we could require $\iota_{\xi_\varphi} \delta\mathcal{L} \in \Omega^{\text{top}}(M)$ to be exact for all $\xi_\varphi \in T_\varphi \mathcal{F}$. However, as we have already pointed out, $H^{\text{top}}(M) = 0$ when M is not compact in which case this condition is vacuous.

The right notion of exactness of a form at a field was given in Def. 6.3.16: $\delta\mathcal{L}$ is d-exact at φ if there is a $\lambda \in \Omega^{0,\text{top}-1}(\mathcal{F} \times M)$ such that $\iota_{\xi_\varphi} \delta\mathcal{L} = \iota_{\xi_\varphi} d\lambda$ for all $\xi_\varphi \in T_\varphi \mathcal{F}$. It follows from the definition 6.2.4 of d that $\iota_{\xi_\varphi} d\lambda = d_M(\iota_{\xi_\varphi} \lambda)$, so that the exactness of $\delta\mathcal{L}$ at φ implies that $\iota_{\xi_\varphi} \delta\mathcal{L} = d_M(\iota_{\xi_\varphi} \lambda)$ for all ξ_φ. We can summarize the translation of the basic ingredients of the action principle into homological language by the following table:

differential geometry	homology
smooth map $\mathcal{L} : \mathcal{F} \to \Omega^{\text{top}}(M)$	differential form $\mathcal{L} \in \Omega^{0,\text{top}}(\mathcal{J} \times M)$
smooth function $S = \int_M \mathcal{L}$	cohomology class $[\mathcal{L}]_d$
S has critical point at φ	$\delta\mathcal{L}$ is d-exact at φ

In order to establish the relation between the action and the Euler-Lagrange equation, which is a PDE, we have to consider local lagrangians.

Terminology 7.1.7. Let (M, F, L) be a local LFT. The horizontal cohomology class $[L]_d \in H_d^{0,\text{top}}(J^\infty F)$ will be called the **action cohomology class** or, short, the **action class**.

Proposition 7.1.8. *Let $F \to M$ be a smooth fibre bundle. If two lagrangian form $L, L' \in \Omega^{0,\text{top}}(J^\infty F)$ represent the same action class $[L]_d = [L']_d$, then they have the same Euler-Lagrange form $EL = EL'$.*

Proof. By definition, two lagrangian forms L and L' define the cohomological action if and only if they differ by a d-exact form, $L - L' = d\alpha$ for $\alpha \in \Omega^{0,\mathrm{top}}(J^\infty F)$. It follows that

$$EL - EL' = Ed\alpha = P\delta d\alpha = -Pd(\delta\alpha) = 0\,,$$

where in the last step we have used Thm. 5.2.2 (iii). □

Remark 7.1.9. The converse of Prop. 7.1.2 is not true in general. By Thm. 5.2.7, the obstruction lies in $H^{\dim M}(F)$, i.e. the converse holds if and only if this cohomology class is zero. For example, this is the case when $F \to M$ is a vector bundle and M is non-compact.

Theorem 7.1.10 (Cohomological action principle)**.** *Let (M, F, L) be a local LFT. Then δL is exact at $\varphi \in \mathcal{F}$ if and only if φ is a solution of the Euler-Lagrange equation.*

Proof. By Prop. 6.3.24, δL is exact at φ if and only if $P\delta L = EL$ vanishes at φ, that is, if and only if $EL_{j^\infty\varphi} = 0$. □

We emphasize that the proof of Thm. 7.1.10 sidesteps integration altogether. It only uses, via Prop. 6.3.24, very basic local properties of the interior Euler operator P, which is the cohomological replacement for partial integration.

7.1.3 The Helmholtz problem

7.2 Symmetries and Noether's theorems

Noether's first theorem relates symmetries of the action and conserved currents. Before we state the theorem we will define these concepts.

7.2.1 Symmetries of the action class

Assume that M is closed, so that the action function (7.1) is defined. A vector field $\xi \in \mathfrak{X}(\mathcal{F})$ is a **symmetry of the action function** if the diffeological derivative of the function $S : \mathcal{F} \to \mathbb{R}$ with respect to ξ vanishes, $\xi \cdot S = 0$. It follows from the definition 6.2.9 of the diffeological differential and the linearity of the integral that the diffeological derivative in the direction of the tangent vector ξ_φ is given by

$$\iota_{\xi_\varphi}\delta S = \int_M (\iota_{\xi_\varphi}\delta\mathcal{L})(\varphi)\,.$$

Since M is closed, the right hand side vanishes if and only if the de Rham cohomology class of the integrand vanishes. When M is not closed we require that the d-cohomology class of $\iota_\xi\delta\mathcal{L} \in \Omega^{0,\mathrm{top}}(\mathcal{F}\times M)$ vanishes.

Definition 7.2.1. Let $\mathcal{L} : \mathcal{F} \to \Omega^{\mathrm{top}}(M)$ be a lagrangian. A vector field $\xi \in \mathfrak{X}(\mathcal{F})$ is a **symmetry of the action class** $[\mathcal{L}]_d$ if there is a form $\nu \in \Omega^{0,\mathrm{top}-1}(\mathcal{F}\times M)$ such that $\iota_\xi\delta\mathcal{L} = d\nu$.

Remark 7.2.2. Prop. 6.2.11 states that δ and d commute on $(0,q)$-forms, so that we obtain $\iota_\xi\delta(d\lambda) = d(\iota_\xi\delta\lambda)$ for all $\lambda \in \Omega^{0,\mathrm{top}-1}(\mathcal{F}\times M)$. This shows that the condition of Def. 7.2.1 only depends on the d-cohomology class of $\mathcal{L}$ as it is suggested by the terminology.

Terminology 7.2.3. A symmetry of a local lagrangian $\mathcal{L}$ is called **local** if both the vector field ξ and the form ν in Def. 7.2.1 are local.

The notion of symmetry of a local LFT, where the lagrangian form lives in the variational bicomplex, has to be expressed in terms of vector fields and forms on $J^\infty F$. In Prop. 5.1.26 we have shown that the vertical and the horizontal Cartan calculi on the infinite jet bundle commute. As a consequence, the action of the vertical differential, the inner, and the Lie derivatives with respect to strictly vertical vector fields on forms descend to actions on the d-cohomology classes. That is, the actions

$$\delta[\omega]_d := [\delta\omega]_d\,, \quad \iota_\xi[\omega]_d := [\iota_\xi\omega]_d\,, \quad \mathcal{L}_\xi[\omega]_d := [\mathcal{L}_\xi\omega]_d\,,$$

for $\omega \in \Omega(J^\infty F)$ and a strictly vertical vector field $\xi \in \mathfrak{X}(J^\infty F)$ are well-defined. The Lie derivative with respect to a strictly horizontal vector field also commutes with the horizontal differential, so that we have a well defined action

$$\mathcal{L}_X[\omega]_d := [\mathcal{L}_X\omega]_d\,.$$

for every strictly horizontal vector field $X \in \mathfrak{X}(J^\infty F)$.

Definition 7.2.4. Let (M,F,L) be an LFT. A strictly vertical vector field $\xi \in \mathfrak{X}(J^\infty F)$ is called a **symmetry** of the action class if $\mathcal{L}_\xi[L]_d = 0$, i.e. if there is an $\alpha \in \Omega^{0,\mathrm{top}-1}(J^\infty F)$ such that $\mathcal{L}_\xi L = d\alpha$.

Terminology 7.2.5. A symmetry of the action class in the sense of Def. 7.2.4 is often called a generalized symmetry of the lagrangian, where a non-generalized or manifest symmetry is defined by $\mathcal{L}_\xi L = 0$ [Fre06]. Sometimes a symmetry ξ is called generalized if is not the prolongation of a vertical vector field on F. In this terminology, a non-generalized symmetry is the prolongation of an evolutionary "vector field" of the form $\eta = \xi^\alpha(x^i,u^\alpha)\frac{\partial}{\partial u^\alpha}$ on F [Olv93].

Remark 7.2.6. The Lie derivative with respect to a strictly horizontal vector field X on a lagrangian form L is given by

$$\mathcal{L}_X L = [\iota_X, d+\delta]L = [\iota_X, d]L = d(\iota_X L)\,,$$

so that $\mathcal{L}_X[L]_d = 0$. This shows that every strictly horizontal vector field X is trivially a symmetry of the action class.

Proposition 7.2.7. *Let $F \to M$ be a smooth fibre bundle. Let $\mathcal{L} : \mathcal{F} \to \Omega^{\mathrm{top}}(M)$ be a local lagrangian, so that $\mathcal{L} = (j^\infty)^* L$ for some $L \in \Omega^{0,\mathrm{top}}(J^\infty F)$. If $\hat{\xi} \in \mathfrak{X}(J^\infty F)$ is a symmetry of the action class $[L]_d$, then the corresponding local vector field $\xi \in \mathfrak{X}(\mathcal{F})$ given by Thm. 5.1.37 is a symmetry of the action class $[\mathcal{L}]_d$. Moreover, if $j^0 : \mathcal{F}\times M \to F$ is surjective, then the converse statement holds as well.*

Proof. Conversely, assume that $\iota_{\hat{\xi}}\delta L = d\alpha$. Using Prop. 6.2.6 and Prop. 6.2.10, we obtain

$$\begin{aligned}\iota_\xi \delta \mathcal{L} &= \iota_\xi \delta (j^\infty)^* \delta L = \iota_\xi (j^\infty)^* \delta L = (j^\infty)^* \iota_{\hat{\xi}} \delta L \\ &= (j^\infty)^* d\alpha = d(j^\infty)^* \alpha \\ &= d\nu \,,\end{aligned}$$

where $\nu := (j^\infty)^* d\alpha$.

Conversely, assume ξ is a local symmetry, so that $\iota_\xi \delta L = d\nu$, where $\nu = (j^\infty)^*\alpha$ for some $\alpha \in \Omega^{0,\mathrm{top}-1}(J^\infty F)$. An analogous calculation shows that $(j^\infty)^* \iota_{\hat{\xi}} \delta L = (j^\infty)^* d\alpha$. If we assume that j^0 is surjective, Thm. 6.1.2 states that $(j^\infty)^*$ is injective. We conclude that $\iota_{\hat{\xi}}\delta L = d\alpha$. □

Definition 7.2.8. Let $\mathcal{L} \in \Omega^{0,\mathrm{top}}(\mathcal{F} \times M)$ be a lagrangian form and P a set. A **family of symmetries** of $[\mathcal{L}]_d$ is a map $\xi : P \to \mathfrak{X}(\mathcal{F})$, such that there is a map $\nu : P \to \Omega^{0,\mathrm{top}-1}(\mathcal{F} \times M)$ satisfying $\iota_{\xi_p} \delta \mathcal{L} = d\nu_p$ for all $p \in P$.

When we want to impose additional conditions on a family of symmetries, such as smoothness, locality, or linearity, we have to impose them on both ξ and ν. This leads to the following definition.

Definition 7.2.9. A family of symmetries as in Def. 7.2.8 is called

(i) **linear** if P is a vector space and ξ and ν are linear maps;

(ii) **local** if $\mathcal{L}$ is local, P is the space of sections of a smooth fibre bundle, ξ is a local family of vector fields (Def. 6.3.4), and ν is a local family of forms (Def. 6.3.2);

(iii) **linear local** if P is the vector space of sections of a smooth vector bundle, and ξ is a linear and local family of symmetries.

Remark 7.2.10. If P in Def. 7.2.9 is a vector space and ξ a linear family of vector fields such that every ξ_p is a symmetry, we can always chose ν to be linear by choosing the values of ν for a basis of the vector space and extending it linearly. Since the condition $\mathcal{L}_{\xi_p}\mathcal{L} = d\nu_p$ is linear in both ξ_p and ν_p, it holds for all linear combinations. The condition in Def. 7.2.9 (i) that ν be linear was included for clarity.

If $\xi : \mathcal{E} \to \mathfrak{X}(\mathcal{F})$ is a local symmetry, then by Prop. 6.3.5 ξ_e is a local vector field for every $e \in \mathcal{E}$, which can be identified by Thm. 5.1.37 with a strictly vector field $\hat{\xi}_e \in \mathfrak{X}(J^\infty F)$. Similarly, the form ν_e is the pullback of a form $\alpha_e \in \Omega^{0,\mathrm{top}-1}(J^\infty F)$. For a local family of symmetries of $\mathcal{L} = (j^\infty)^* L$ we thus obtain maps

$$\begin{aligned}\hat{\xi} &: \mathcal{E} \longrightarrow \mathfrak{X}(J^\infty F) \\ \alpha &: \mathcal{E} \longrightarrow \Omega^{0,\mathrm{top}-1}(J^\infty F) \,.\end{aligned}$$

From $\iota_{\xi_e}\delta\mathcal{L} = d\nu_e$ it follows that $(j^\infty)^*(\iota_{\hat{\xi}_e}\delta L - d\alpha_e) = 0$. However, when $(j^\infty)^*$ is not injective we cannot conclude that $\hat{\xi}_e$ is a symmetry of $[L]_d$. Therefore, we need a separate definition of a local family of symmetries of the action class of a lagrangian form on the infinite jet bundle.

Definition 7.2.11. Let $L \in \Omega^{0,\mathrm{top}}(J^\infty F)$ be a lagrangian form and P a set. A **family of symmetries** of $[L]_d$ is a map $\xi : P \to \mathfrak{X}(J^\infty F)$ with values in strictly vertical vector fields, such that there is a map $\alpha : P \to \Omega^{0,\mathrm{top}-1}(J^\infty F)$ satisfying $\mathcal{L}_{\xi_p} L = d\alpha_p$ for all $p \in P$.

Definition 7.2.12. A family of symmetries as in Def. 7.2.11 is called **(linear) local** if $P = \mathcal{E}$ is the space of sections of a smooth fibre (vector) bundle $E \to M$, and the induced maps $\mathcal{E} \to \mathfrak{X}(\mathcal{F})$ and $\mathcal{E} \to \Omega^{0,\mathrm{top}-1}(\mathcal{F} \times M)$ are (linear) local families.

Remark 7.2.13. Note that $\mathfrak{X}(J^\infty F)$ is not the space of sections of any smooth fibre bundle, so that we cannot impose the condition of locality on the map $\xi : \mathcal{E} \to \mathfrak{X}(J^\infty F)$. The notion of locality of Def. 3.2.1 makes only sense for the induced map $\mathcal{E} \times \mathcal{F} \to T\mathcal{F}$.

Remark 7.2.14. If we unpack Def. 7.2.9 (ii) and Def. 7.2.12, we find that a family of vector fields on $\mathcal{F}$ or of strictly vertical vector fields on $J^\infty F$ is local if it is induced by a map

$$J^k E \times_M J^k F \longrightarrow VF\,.$$

Similarly, a family of n-forms on $\mathcal{F} \times M$ or on $J^\infty F$ is local if it is induced by

$$J^k E \times_M J^k F \longrightarrow \wedge^n T^* J^k F\,.$$

The only difference between the Def. 7.2.9 (ii) and Def. 7.2.12 of local families of symmetries is that, when $(j^\infty)^*$ is not injective, the condition that $\xi_e \in \mathfrak{X}(\mathcal{F})$ is a symmetry does not imply that $\hat{\xi}_e \in \mathfrak{X}(J^\infty F)$ is a symmetry. When $(j^\infty)^*$ is injective, the two notions are equivalent.

7.2.2 Currents and charges

Definition 7.2.15. A differential form $j \in \Omega^{0,\mathrm{top}-1}(\mathcal{F} \times M)$ is called a **current**.

Integrating a current j over a closed oriented and cooriented codimension 1 submanifold $S \subset M$ yields a smooth map

$$q_S : \mathcal{F} \longrightarrow \mathbb{R}\,, \quad q_S(\varphi) := \int_S j_\varphi\,, \tag{7.6}$$

which is called the corresponding **charge** on S. Here $j_\varphi = \tilde{j}(\varphi)$ is the evaluation of j at $\varphi \in \mathcal{F}$ (see Def. 6.1.5 and Rmk. 6.1.7).

Assume that the spacetime manifold M is locally split into time and space, i.e. there is an embedding

$$\sigma : \mathbb{R} \times \Sigma \hookrightarrow M\,,$$

where Σ is a closed oriented manifold. Then we can integrate over the time slices,

$$q_t(\varphi) := q_{\sigma_t(\Sigma)}(\varphi)\,, \tag{7.7}$$

which can be viewed is the total charge on Σ as a function of time. Let $t, x^1, \ldots, x^{n-1}$ be local coordinates of $\mathbb{R} \times \Sigma$. Then a local current has the local coordinate form

$$j_\varphi = \rho_\varphi(t,x)\,\mathrm{vol}_\Sigma + j^k_\varphi(t,x)\,dt \wedge \Big(\frac{\partial}{\partial x^k} \lrcorner \mathrm{vol}_\Sigma\Big)\,,$$

where $\mathrm{vol}_\Sigma = dx^1 \wedge \ldots \wedge dx^{n-1}$ is the volume form on Σ, and where ρ_φ and j^k_φ are smooth local functions on M.

Terminology 7.2.16. The smooth function $\rho_\varphi \in C^\infty(M)$ is called the **charge density** and the vector field $j^k_\varphi \frac{\partial}{\partial x^k} \in \mathfrak{X}(M)$ the **current density**, e.g. the electric charge density and the electric current density in Maxwell theory, or the mass density and the material flow density in fluid dynamics.

Local currents can be viewed as representing in a coordinate independent way local observables, i.e. locally defined physical quantities like charge densities and their flows. If two currents differ by an exact current, $j - j' = d\beta$, the corresponding charges are the same, $q_S = q'_S$ (assuming that S is closed). In this case j and j' represent the same physical quantity. As before, we also define the notion of current in terms of the infinite jet bundle.

Definition 7.2.17. A form $j \in \Omega^{0,\mathrm{top}-1}(J^\infty F)$ is also called a **current**.

A current $j \in \Omega^{0,\mathrm{top}-1}(J^\infty F)$ can be pulled back along the infinite jet evaluation to a current $\eta := (j^\infty)^* j \in \Omega^{0,\mathrm{top}-1}(\mathcal{F} \times M)$, where we apologize for the double usage of the letter j. The **charge** of j is by definition the charge of η as given by the integral 7.6.

Definition 7.2.18. Let (M, F, L) be a local LFT. A form in $\Omega(\mathcal{F}\times M)$ or in $\Omega(J^\infty F)$ is called **conserved** if it is d-closed at all solutions $\varphi \in \mathcal{F}_{\mathrm{shell}}$ of the Euler-Lagrange equation.

When a current $j \in \Omega^{0,\mathrm{top}-1}(\mathcal{F}\times M)$ is conserved, then the corresponding charge $q_S(\varphi)$ for $\varphi \in \mathcal{F}_{\mathrm{shell}}$ depends only on the homology class of S. In particular, $t \mapsto q_t(\varphi)$ as defined in Eq. (7.7) is constant. More generally, let M be a cobordism and $f : M \to [0,1]$ a Morse function such that $f^{-1}(0) = (\partial M)_{\mathrm{in}}$ and $f^{-1}(1) = (\partial M)_{\mathrm{out}}$. This can be viewed as time parametrization of M where $S_t := f^{-1}(t)$ is the time t slice. As before, $q_t(\varphi) := q_{S_t}(\varphi)$ is constant for $\varphi \in \mathcal{F}_{\mathrm{shell}}$.

Proposition 7.2.19. *Let j in $\Omega^{0,\mathrm{top}-1}(\mathcal{F} \times M)$ be a current and $q_S : \mathcal{F} \to \mathbb{R}$ the corresponding charge on the closed codimension 1 submanifold $S \subset M$. If $dj = 0$, then q_S is constant along any smooth path in $\mathcal{F}$.*

Proof. *** Prop. 6.2.10 implies that the diffeological differential of $\hat{j} : \mathcal{F} \to \Omega^{\mathrm{top}-1}(M)$ is given by

$$\delta\hat{j} = \widetilde{(j^\infty)^*\delta j}\,.$$

Integrating over a closed codimension 1 submanifold $S \subset M$, we see that the diffeological derivative of the charge $q_S : \mathcal{F} \to \mathbb{R}$ is given by

$$(T_\varphi q_S)(\xi_\varphi) = \int_S (j^\infty)^* \iota_{\xi_\varphi} \delta j\,,$$

where we have used Prop. 6.2.1. Assume that $dj = 0$. This implies that $d(\delta j) = -\delta dj = 0$, so that it follows from the acyclicity theorem 5.2.4 that δj is d-exact. *** □

Corollary 7.2.20. *Let $S \subset M$ be a closed oriented codimension 1 submanifold. If a current $j \in \Omega^{1,\mathrm{top}-1}(\mathcal{F} \times M)$ is closed and $\mathcal{F}$ is connected by piece-wise smooth paths, then the charge $q_S : \mathcal{F} \to \mathbb{R}$ is constant.*

From the viewpoint of physics, Prop. 7.2.19 and Cor. 7.2.20 tell us that d-closed currents do not represent particularly interesting observables. This also shows why conserved currents are required to be d-closed on $\mathcal{F}_{\mathrm{shell}}$ only.

7.2.3 Noether's first theorem

Proposition 7.2.21. *Let (M, F, L) be a local LFT. Then there is a $\gamma \in \Omega^{1,\mathrm{top}-1}(J^\infty F)$ such that*

$$\delta L = EL - d\gamma \,. \tag{7.8}$$

Proof. This is Cor. 5.2.5 for $\omega = L$. □

We will call γ a **boundary form** of the LFT. It follows from the acyclicity theorem 5.2.4 that the boundary form is determined by the LFT up to an exact form.

Theorem 7.2.22 (Noether's first theorem)**.** *Let (M, F, L) be a local LFT and γ a boundary form. Let $\xi \in \mathfrak{X}(J^\infty F)$ be a symmetry of the action cohomology class $[L]_d$ such that $\mathcal{L}_\xi L = d\alpha$. Then the current*

$$j := \alpha - \iota_\xi \gamma$$

is conserved.

Proof. Since ξ is strictly vertical, we have $\iota_\xi L = 0$ and $d\iota_\xi \gamma = -\iota_\xi d\gamma$. We obtain

$$dj = d(\alpha - \iota_\xi \gamma) = \mathcal{L}_\xi L + \iota_\xi d\gamma = \iota_\xi(\delta L + d\gamma) = \iota_\xi EL \,,$$

which vanishes on shell. □

Terminology 7.2.23. The current j of Thm. 7.2.22 is called a **Noether current** and the corresponding charge a **Noether charge** of the symmetry ξ.

Remark 7.2.24. The Noether current of Thm. 7.2.22 depends on the choice of both α and γ. Two Noether currents j and j' for the same symmetry ξ differ by a closed current, $d(j_\xi - j'_\xi) = 0$. It then follows from Prop. 7.2.19 that the Noether charge is unique up to a charge that is locally constant, i.e. constant on the connected components of $\mathcal{F}$.

Definition 7.2.25. Let L be a lagrangian form, $j \in \Omega^{0,\mathrm{top}-1}(J^\infty F)$ a current, and $\xi \in \mathfrak{X}(J^\infty F)$ a strictly vertical vector field. If

$$dj = \iota_\xi EL \,,$$

then (j, ξ) is called a **Noether pair**.

Let (j, ξ) be a Noether pair. Then

$$\begin{aligned} \mathcal{L}_\xi L = \iota_\xi \delta L = \iota_\xi(EL - d\gamma) &= dj + d\iota_\xi \gamma \\ &= d\alpha \,, \end{aligned}$$

where $\alpha = j + \iota_\xi \gamma$. This shows that ξ is a symmetry of $[L]_d$ and j its Noether current.

While the proof of Thm. 7.2.22 makes Noether's first theorem look deceptively simple, it is the mathematical implementation of one of the most important principles in physics, the relation between Lie group symmetries and the fundamental physical quantities like momentum, energy, and charge. Here is a table:

external (spacetime) symmetry	conserved quantity
space translations	linear momentum
space rotations	angular momentum
time translation	energy
velocity transf. (Galilei group) $x \mapsto x - vt$	center of mass
Boosts (Lorentz group) $x \mapsto \frac{x-vt}{\sqrt{1-(v/c)^2}}$	center of mass

internal (gauge) symmetry	conserved quantity
U(1) gauge symmetry	electric charge
SU(2) gauge symmetry	hypercharge
SU(3) gauge symmetry	color charge

7.2.4 Noether currents for linear local families of symmetries

We recall from definition 6.3.19 that a current $\eta \in \Omega^{0,\text{top}-1}(J^\infty F)$ is d-exact at $\varphi \in \mathcal{F}$ if there is a form $\beta \in \Omega^{0,\text{top}-1}(J^\infty F)$ such that $(j^\infty\varphi)^*(\eta - d\beta) = 0$. By Prop. 6.3.21, this is the case if and only if $(j^\infty\varphi)^*\eta \in \Omega^{\text{top}-1}(M)$ is exact.

Theorem 7.2.26. *Let (M, F, L) be a local LFT with boundary form γ; let $A \to M$ be a smooth vector bundle of non-zero rank; let $\xi : \mathcal{A} \to \mathfrak{X}(J^\infty F)$ be a linear local family of symmetries, so that there is a linear local family of forms $\alpha : \mathcal{A} \to \Omega^{0,\text{top}-1}(J^\infty F)$ satisfying $\mathcal{L}_{\xi_a} L = d\alpha_a$. Then for every $a \in \mathcal{A}$ the Noether current $j_a = \alpha_a - \iota_{\xi_a}\gamma$ is d-exact at every $\varphi \in \mathcal{F}_{\text{shell}}$.*

Proof. Let $\varphi \in \mathcal{F}_{\text{shell}}$. Since ξ and α are linear local families, and since $(j^\infty)^*\gamma$ is a local form, the map

$$\begin{aligned} \nu_\varphi : \mathcal{A} &\longrightarrow \Omega^{\text{top}-1}(M) \\ a &\longmapsto (j^\infty\varphi)^* j_a \,. \end{aligned}$$

is linear and local, as well. By Noether's first theorem 7.2.22, $\nu_\varphi(a)$ is closed for all $a \in \mathcal{A}$, so that $d_M \circ \nu_\varphi = 0$. This shows that ν_φ satisfies all conditions of Prop. 6.3.13, so that there is a local linear family $\mu : \mathcal{A} \to \Omega^{\text{top}-2}(M)$ satisfying $\nu_\varphi = d_M \circ \mu$. It follows that for every $a \in \mathcal{A}$ the form $(j^\infty)^* j_a \in \Omega^{0,\text{top}-1}(\mathcal{F} \times M)$ is exact at φ. With Prop. 6.3.21 we conclude that j_a is d-exact at φ. □

Remark 7.2.27. Thm. 7.2.26 was stated without proof in Thm. 15 b) of [Zuc87], where it was attributed to E. Noether's original article [Noe18] of 1918. However, while the general idea may be extrapolated from §6 of [Noe18], the proof of Thm. 7.2.26 for general background manifolds M and general fibre bundles F relies on concepts and technical results that were not available at the time. The proof uses diffeology and variational cohomology, in particular Prop. 6.3.21 and Prop. 6.3.13, which in turn relies on Prop. 2.3.12, Thm. 2.3.2, and, crucially, on the acyclicity theorem 5.2.4. The proof given here can be found in [Ber19].

Corollary 7.2.28. *Assume the situation of Thm. 7.2.26. Let $S \subset M$ be a closed oriented codimension 1 submanifold. Then the charge $q_a(\varphi) := \int_S j_a(\varphi)$ vanishes for all $\varphi \in \mathcal{F}_{\text{shell}}$ and all $a \in \mathcal{A}$.*

Remark 7.2.29. By adding to j_a of Thm. 7.2.26 a current η that is d-closed but not d-exact at φ, we obtain $(\xi_a, j_a + \eta)$ a Noether pair. This shows that there may be other Noether currents for ξ_a that are not d-exact at $\varphi \in \mathcal{F}_{\text{shell}}$.

Remark 7.2.30. Thm. 7.2.26 states that, given a $\varphi \in \mathcal{F}_{\text{shell}}$, there is a form $\beta \in \Omega^{0,\text{top}-2}(J^\infty F)$ such that $(j^\infty\varphi)^*(\eta - d\beta) = 0$. The form β generally depends on φ, so that Thm. 7.2.26 does *not* state that the pullback of j_a to $\mathcal{F}_{\text{shell}} \times M$ is d-exact.

7.2.5 Noether's second theorem

Noether's second theorem relates linear local families of symmetries with local linear degeneracies of the Euler-Lagrange equation. ***

7.3 Jacobi fields

7.3.1 Linearization of the Euler-Lagrange equation

In Sec. 6.1.3 we have explained how a local form like EL can be viewed as the differential operator $\varphi \mapsto D_{EL}\varphi = EL_{j^\infty\varphi}$. The associated PDE, $D_{EL}\varphi = 0$, is the Euler-Lagrange equation. In Prop. 3.2.13 we have shown that the tangent map of a local map like D_{EL} is local of the same jet order. The tangent map $T_\varphi D_{EL}$ is called the linearization of D_{EL} at φ (Terminology 3.2.15). The PDE

$$(T_\varphi D_{EL})\xi_\varphi = 0\,, \tag{7.9}$$

is the linearization of the Euler-Lagrange equation at φ (Def. 6.1.16).

Definition 7.3.1. The solution of the linearization (7.9) of the Euler-Lagrange equation at some $\varphi \in \mathcal{F}_{\text{shell}}$ is called a **Jacobi field**.

Before we give a more explicit description of Jacobi fields we recall from Lem. 6.1.18 that every $\xi_\varphi \in T_\varphi\mathcal{F}$ can be extended to a local vector field ξ on $\mathcal{F}$ and that ξ projects to a strictly vertical vector field $\hat{\xi} \in \mathfrak{X}(J^\infty F)$, which by Thm. 5.1.37 is the infinite prolongation of an evolutionary "vector field".

Lemma 7.3.2. *Let $\omega \in \Omega^{p,q}(J^\infty F)$ and $\varphi \in \mathcal{F}$, such that $\omega_{j^\infty\varphi} = 0$. Let $\xi_\varphi \in T_\varphi\mathcal{F}$. Let $\hat{\xi}$ be a strictly vertical vector field on $J^\infty F$ that extends the infinite prolongation $\hat{\xi}_{j^\infty\varphi}$ of ξ_φ. Then the evaluation of $\mathcal{L}_{\hat{\xi}}\omega$ at φ depends only on ξ_φ and not on the extension $\hat{\xi}$.*

Proof. The form ω on the pro-manifold $J^\infty F$ is represented by a form on a finite jet bundle J^kF, $k < \infty$, which we also denote by ω. The sheaf of differential (p,q)-forms on J^kF is locally free, which means that locally $\omega = \omega_l\tau^l$, where $\{\tau_l\}$ is a local frame of (p,q)-forms and $\omega_l \in C^\infty(J^kF)$ the coefficient functions. The explicit form of $\{\tau_l\}$ can be deduced from the local coordinate form (5.16), but does not matter here. The assumption $\omega_{j^\infty\varphi} = 0$ is equivalent to $\omega_l \circ j^k\varphi = 0$ for all l.

The Lie derivative of ω with respect to a vector field $\hat{\xi} \in \mathfrak{X}(J^\infty F)$ evaluated at φ is given locally by

$$\begin{aligned}\big(\mathcal{L}_{\hat{\xi}}(\omega_l \tau^l)\big)_{j^\infty} &= \big((\mathcal{L}_{\hat{\xi}}\omega_l)\tau^l + \omega_l(\mathcal{L}_{\hat{\xi}}\tau^l)\big)_{j^\infty\varphi} \\ &= (\mathcal{L}_{\hat{\xi}}\omega_l)_{j^\infty\varphi}\tau^l_{j^\infty\varphi} + (\omega_l \circ j^\infty\varphi)\,(\mathcal{L}_{\hat{\xi}}\tau^l)_{j^\infty\varphi} \\ &= (\iota_{\hat{\xi}}\delta\omega_l)_{j^\infty\varphi}\,\tau^l_{j^\infty\varphi} \\ &= (\iota_{\hat{\xi}_{j^\infty\varphi}}\delta\omega_l)\,\tau^l_{j^\infty\varphi}\,.\end{aligned} \tag{7.10}$$

This shows that the right hand side only depends on the infinite prolongation $\hat{\xi}_{j^\infty\varphi}$ of ξ_φ. □

Notation 7.3.3. Let ω be a form on $J^\infty F$, let $\varphi \in \mathcal{F}$ such that $\omega_{j^\infty\varphi} = 0$, and let $\hat{\xi}$ be a strictly vertical vector field on $J^\infty F$. Since by Lem. 7.3.2 the evaluation of $\mathcal{L}_{\hat{\xi}}\omega$ at $\varphi \in \mathcal{F}$ depends only on ξ_φ, we will use the notation

$$\mathcal{L}_{\xi_\varphi}\omega := (\mathcal{L}_{\hat{\xi}}\,\omega)_{j^\infty\varphi}\,.$$

Lemma 7.3.4. *Let $\omega \in \Omega^n(J^\infty F)$ and let $D_\omega : \mathcal{F} \to \Gamma(M, \wedge^n T^* J^\infty F)$, $\varphi \mapsto \omega_{j^\infty\varphi}$ the associated differential operator (Def. 6.1.14). Let $\varphi \in \mathcal{F}$ be such that $\omega_{j^\infty\varphi} = 0$. Then*

$$(T_\varphi D_\omega)\xi_\varphi = \mathcal{L}_{\xi_\varphi}\omega\,. \tag{7.11}$$

Proof. In local coordinates we have $\omega = \omega_l \tau^l$, where $\{\tau_l\}$ is a local frame of (p,q)-forms as in the proof of Lem. 7.3.2. Assume that $\varphi \in \mathcal{F}$ satisfies $\omega_{j^\infty\varphi} = 0$. This is the case iff $\omega_l \circ j^\infty\varphi = 0$. Let $t \mapsto \psi_t \in \mathcal{F}$ be a smooth path, such that $\psi_0 = \varphi$. The image of $\xi_\varphi := \dot{\psi}_0 \in T_\varphi\mathcal{F}$ under the diffeological tangent map D_ω is given by

$$\begin{aligned}(T_\varphi D_\omega)\xi_\varphi &= \frac{d}{dt}D_\omega\psi_t\big|_{t=0} \\ &= \frac{d}{dt}\omega_{j^\infty\psi_t}\big|_{t=0} \\ &= \frac{d}{dt}\Big((\omega_l \circ j^\infty\psi_t)\,\tau^l_{j^\infty\psi_t}\Big)_{t=0} \\ &= \Big(\frac{d}{dt}(\omega_l \circ j^\infty\psi_t)\Big)_{t=0}\tau^l_{j^\infty_m\varphi} + (\omega_l \circ j^\infty\varphi)\Big(\frac{d}{dt}\tau^l_{j^\infty\psi_t}\Big)_{t=0} \\ &= \frac{d}{dt}\big(\omega_l \circ j^\infty\psi_t\big)\big|_{t=0}\,\tau^l_{j^\infty_m\varphi}\,,\end{aligned}$$

where we have used that $\omega \circ j^\infty\varphi = 0$. In local coordinates, the first factor of the right hand side can be written as

$$\begin{aligned}\frac{d}{dt}\big(\omega_l \circ j^\infty\psi_t\big)\big|_{t=0} &= \frac{d}{dt}(u^\alpha_I \circ \psi_t)\big|_{t=0}\Big(\frac{\partial\omega_l}{\partial u^\alpha_I} \circ j^\infty\psi_0\Big) \\ &= \frac{d}{dt}\frac{\partial^I\psi^\alpha_t}{\partial u^\alpha_I}\big|_{t=0}\Big(\frac{\partial\omega_l}{\partial u^\alpha_I} \circ j^\infty\psi_0\Big) \\ &= \frac{\partial^{|I|}\xi^\alpha_\varphi}{\partial x^I}\Big(\frac{\partial\omega_l}{\partial u^\alpha_I} \circ j^\infty\varphi\Big) \\ &= \iota_{\hat{\xi}_{j^\infty\varphi}}\delta\omega_l\,,\end{aligned}$$

where $\hat{\xi}_{j^\infty\varphi}$ is the infinite prolongation of ξ_φ as introduced in Def. 6.1.22. Let $\hat{\xi} \in \mathfrak{X}(J^\infty F)$ be a local vector field that extends $\hat{\xi}_{j^\infty\varphi}$. We conclude that locally we have

$$(T_\varphi D_\omega)\xi_\varphi = (\iota_{\hat{\xi}_\varphi} \delta\omega_l)\,\tau^l\,. \tag{7.12}$$

The right hand sides of Eq. (7.12) and Eq. (7.10) are equal, which implies Eq. (7.11). □

Proposition 7.3.5. *Let (M, F, L) be a local LFT and $\varphi \in \mathcal{F}_{\text{shell}}$. Then $\xi_\varphi \in T_\varphi\mathcal{F}$ is a Jacobi field if and only if*

$$\mathcal{L}_{\xi_\varphi} EL = 0\,.$$

Proof. Let $\varphi \in \mathcal{F}_{\text{shell}}$. By definition, $\xi_\varphi \in T_\varphi\mathcal{F}$ is a Jacobi field if it lies in the kernel of $T_\varphi D_{EL}$. Lem. 7.3.4 for $\omega = EL$ states that $(T_\varphi D_{EL})\xi_\varphi = \mathcal{L}_{\xi_\varphi} EL$. This shows that ξ_φ is a Jacobi field if and only if $\mathcal{L}_{\xi_\varphi} EL = 0$. □

In local coordinates $EL = E_\alpha \delta u^\alpha \wedge \mathrm{vol}$, where $\mathrm{vol} = dx^1 \wedge \ldots \wedge dx^n$, $n = \dim M$. Let k be the jet order of EL. The Lie derivative with respect to ξ_φ, where $\varphi \in \mathcal{F}_{\text{shell}}$, is given by

$$\begin{aligned}\mathcal{L}_{\xi_\varphi} EL &= (\iota_{\hat{\xi}_\varphi} \delta E_\alpha)\delta u^\alpha \wedge \mathrm{vol} \\ &= \sum_{|J|=0}^{k} \frac{\partial E_\alpha}{\partial u_J^\beta}(j^k\varphi)\,\frac{\partial^{|J|}\xi_\varphi^\beta}{\partial x^J}\,\delta u^\alpha \wedge \mathrm{vol}\,,\end{aligned}$$

which vanishes if and only if the **Jacobi equation**

$$\sum_{|J|=0}^{k} \frac{\partial E_\alpha}{\partial u_J^\beta}(j^k\varphi)\,\frac{\partial^{|J|}\xi_\varphi^\beta}{\partial x^J} = 0 \tag{7.13}$$

is satisfied.

7.3.2 Tangent vectors on shell

Proposition 7.3.6. *Let (M, F, L) be a local LFT. If $\xi_\varphi \in T\mathcal{F}$ is tangent to the diffeological space $\mathcal{F}_{\text{shell}}$ of solutions of the Euler-Lagrange equation, then it is a Jacobi field.*

Proof. By definition of the subspace diffeology of $\mathcal{F}_{\text{shell}} \subset \mathcal{F}$, every tangent vector in $\xi_\varphi \in T_\varphi\mathcal{F}_{\text{shell}}$ is represented by a smooth path $t \mapsto \psi_t \in \mathcal{F}_{\text{shell}}$. This means that $EL_{j^\infty\psi_t} = 0$ for all t. It follows that

$$(T_\varphi D_{EL})\xi_\varphi = (T_\varphi D_{EL})\dot{\psi}_0 = \frac{d}{dt} EL_{j^\infty\psi_t}\big|_{t=0} = 0\,.$$

It follows from Prop. 7.3.5 that ξ_φ is a Jacobi field. □

The converse of Prop. 7.3.6 is not true in general, since not every solution of the Jacobi equation is represented by a path in $\mathcal{F}_{\text{shell}}$. The first obstruction to extending Jacobi fields to paths arises when the Euler-Lagrange equation viewed as function on J^kF is degenerate as in the following example.

Example 7.3.7. Let $M = \mathbb{R}$ and $F = \mathbb{R} \times Q \to \mathbb{R}$ with $Q = \mathbb{R}$, so that the space of fields is $\mathcal{F} \cong C^\infty\big(\mathbb{R}\big)$. Consider the lagrangian $L = \frac{1}{3}\dot{q}^3 dt$. The Euler-Lagrange form is $EL = -2\dot{q}\ddot{q}\delta q \wedge dt$, so the Euler-Lagrange equation is

$$2\dot{q}\ddot{q} = 0\,,$$

that is $\frac{d}{dt}\dot{q}^2 = 0$, which is equivalent to $\dot{q}^2 = v^2$ for some constant velocity $v \in \mathbb{R}$. The solutions are the constant velocity paths,

$$\mathcal{F}_{\text{shell}} = \{q \in C^\infty(\mathbb{R}) \mid q(t) = x + vt\,,\ x, v \in \mathbb{R}\}\,.$$

The tangent space of $\mathcal{F}$ at the constant path $q(t) = 0$ is given by $T_0\mathcal{F} = C^\infty(\mathbb{R})$. The subspace of vectors tangent to $\mathcal{F}_{\text{shell}}$ is given by

$$T_0\mathcal{F}_{\text{shell}} = \{\xi \in C^\infty(\mathbb{R}) \mid \xi(t) = \alpha + \beta t\,,\ \alpha, \beta \in \mathbb{R}\}\,.$$

The Jacobi equation (7.13) for $\xi \in T_{x+vt}\mathcal{F} \cong C^\infty(\mathbb{R})$ is

$$v\ddot{\xi} = 0\,.$$

When $v = 0$, the equation is trivially satisfied for every $\xi \in C^\infty(\mathbb{R})$. We conclude that *every* tangent vector at a constant path $q(t) = x$ is a Jacobi field.

The essential property of example 7.3.7 is that the component functions $E_\alpha : J^kF \to \mathbb{R}$ of the Euler-Lagrange are degenerate in the sense that there are tangent vectors to J^kF that annihilate all E_α but are not tangent to the zero locus of the E_α. As a consequence, the obstruction to extending Jacobi fields to paths in $\mathcal{F}_{\text{shell}}$ is local. More precisely, there are Jacobi fields ξ and points $m \in M$ where the restriction $\xi|_U$ to any neighborhood U of m cannot be represented by a path of local solutions of the Euler-Lagrange equation. There are also global obstructions as the next example shows.

Example 7.3.8. Let $M = \mathbb{R}$ and $F = \mathbb{R} \times Q \to \mathbb{R}$ with $Q = (-1, 1)$, so that the space of fields is $\mathcal{F} \cong C^\infty\big(\mathbb{R}, (-1, 1)\big)$, the space of smooth paths in the open interval $(-1, 1)$. Let $L = \frac{1}{2}\dot{q}^2 dt$ be the lagrangian of the free particle in Q. The Euler-Lagrange form is $EL = -\ddot{q}\delta q \wedge dt$. Since any path of non-zero constant velocity would have to leave $(-1, 1)$ eventually, the space of solutions is the space of constant paths,

$$\mathcal{F}_{\text{shell}} = \{q \in \mathcal{F} \mid q(t) = x\,,\ x \in (-1, 1)\}\,.$$

The tangent space at the constant path $q = 0$ is given by

$$T_0\mathcal{F} = \Gamma\big(\mathbb{R}, \mathbb{R} \times T_0(-1, 1)\big) \cong C^\infty(\mathbb{R})\,.$$

The subspace of vectors tangent to $\mathcal{F}_{\text{shell}}$ is given by constant functions,

$$T_0\mathcal{F}_{\text{shell}} = \{\xi \in C^\infty(\mathbb{R}) \mid \xi(t) = \alpha\,,\ \alpha \in \mathbb{R}\}\,.$$

The Jacobi equation (7.13) for $\xi \in C^\infty(\mathbb{R})$ is $\ddot{\xi} = 0$, the solutions of which are of the form $\xi = \alpha + \beta t$ for $\alpha, \beta \in \mathbb{R}$. We conclude that there are Jacobi fields with $\beta \neq 0$ that are not tangent vectors to $\mathcal{F}_{\text{shell}}$.

Remark 7.3.9. We could *define* the tangent spaces of the variety $\mathcal{F}_{\text{shell}}$ to be given by all Jacobi fields, as it is done in Def. 7 of [Zuc87]. In other words, we could use the Zariski tangent space of algebraic geometry for the variety $\mathcal{F}_{\text{shell}}$ rather than the diffeological tangent space. However, this would be inconsistent with the diffeological description of the spaces of fields and obscure the interesting geometric phenomena exhibited by examples 7.3.7 and 7.3.8.

Definition 7.3.10. Let (M, F, L) be a local LFT. A solution $\varphi \in \mathcal{F}_{\text{shell}}$ of the Euler-Lagrange equation for which $T_\varphi \mathcal{F}_{\text{shell}}$ is equal to the space of Jacobi fields will be called **non-degenerate**.

7.3.3 Symmetries and Jacobi fields

Proposition 7.3.11. *Let (M, F, L) be a local LFT. If the local vector field $\xi \in \mathfrak{X}(\mathcal{F})$ is a symmetry of the action class, then $\xi_\varphi \in T_\varphi \mathcal{F}$ is a Jacobi field for all $\varphi \in \mathcal{F}_{\text{shell}}$.*

Lemma 7.3.12. *Let $\omega \in \Omega^{1,\text{top}}(J^\infty F)$ be a source form and $\chi \in \mathfrak{X}(J^\infty F)$ a strictly vertical vector field. If $\varphi \in \mathcal{F}$ satisfies $\omega_{j^\infty\varphi} = 0$, then*

$$(\mathcal{L}_\chi \omega)_{j^\infty\varphi} = \big(P(\mathcal{L}_\chi \omega)\big)_{j^\infty\varphi} . \tag{7.14}$$

Proof. Eq. (7.14) is local, so it can be checked in local coordinates. By assumption ω is a source form, so we have in local coordinates

$$\omega = \delta u^\alpha \wedge \omega_\alpha \tau ,$$

where $\tau = dx^1 \wedge \ldots \wedge dx^n$ for $n = \dim M$ is the volume form of the local coordinates. For the Lie derivative we get

$$\mathcal{L}_\chi \omega = \delta u^\alpha \wedge (\mathcal{L}_\chi \omega_\alpha)\tau + \delta\chi^\alpha \wedge \omega_\alpha \tau ,$$

where we have used that χ is strictly vertical. Using formula (5.25) for P we see that the first term on the right hand side satisfies

$$P(\delta u^\alpha \wedge (\mathcal{L}_\chi \omega_\alpha)\tau) = \delta u^\alpha \wedge (\mathcal{L}_\chi \omega_\alpha)\tau .$$

For the second term we obtain

$$P(\delta\chi^\alpha \wedge \omega_\alpha \tau) = \delta u^\beta \wedge \tau \, \frac{1}{p} \sum_I (-1)^{|I|} D_I\Big(\frac{\partial \chi^\alpha}{\partial u^\beta_I}\, \omega_\alpha\Big) .$$

For any $\varphi \in \mathcal{F}$ we have

$$\begin{aligned}\bigg(D_I\Big(\frac{\partial \chi^\alpha}{\partial u^\beta_I}\, \omega_\alpha\Big)\bigg)_{j^\infty\varphi} &= \frac{\partial^{|I|}}{\partial x^I}\bigg(\Big(\frac{\partial \chi^\alpha}{\partial u^\beta_I}\, \omega_\alpha\Big) \circ j^\infty\varphi\bigg) \\ &= \frac{\partial^{|I|}}{\partial x^I}\bigg(\Big(\frac{\partial \chi^\alpha}{\partial u^\beta_I} \circ j^\infty\varphi\Big)\, (\omega_\alpha \circ j^\infty\varphi)\bigg) ,\end{aligned} \tag{7.15}$$

where we have used Rmk. 5.1.13. Assume now that the field φ satisfies $\omega_{j^\infty\varphi} = 0$. This is the case iff $\omega_\alpha \circ j^\infty\varphi = 0$, so that the right hand side of Eq. (7.15) vanishes. This implies that

$$\big(P(\delta\chi^\alpha \wedge \omega_\alpha \tau)\big)_{j^\infty\varphi} = 0 .$$

Putting things together, we obtain

$$\begin{aligned}\big(P(\mathcal{L}_\chi\omega)\big)_{j^\infty\varphi} &= \big(P(\delta u^\alpha \wedge (\mathcal{L}_\chi\omega_\alpha)\tau)\big)_{j^\infty\varphi} + \big(P(\delta\chi^\alpha \wedge \omega_\alpha\tau)\big)_{j^\infty\varphi} \\ &= \big(\delta u^\alpha \wedge (\mathcal{L}_\chi\omega_\alpha)\tau\big)_{j^\infty\varphi} \\ &= \big(\delta u^\alpha \wedge (\mathcal{L}_\chi\omega_\alpha)\tau\big)_{j^\infty\varphi} + (\delta\chi^\alpha \wedge \omega_\alpha\tau)_{j^\infty\varphi} \\ &= (\mathcal{L}_\chi\omega)_{j^\infty\varphi}\,,\end{aligned}$$

where we have used that $(\delta\chi^\alpha \wedge \omega_\alpha\tau)_{j^\infty\varphi} = 0$ since $\omega_\alpha \circ j^\infty\varphi = 0$. □

Proof of Prop. 7.3.11. Let $\xi \in \mathfrak{X}(\mathcal{F})$ be a local vector field and $\hat{\xi} \in \mathfrak{X}(J^\infty F)$ the strictly vertical vector field to which ξ descends by Thm. 5.1.37. Assume that ξ is a symmetry of the action class, so that $\mathcal{L}_{\hat{\xi}}L = d\alpha$. Then

$$\begin{aligned}\mathcal{L}_{\hat{\xi}}EL &= \mathcal{L}_{\hat{\xi}}(\delta L + d\gamma) \\ &= \delta\mathcal{L}_{\hat{\xi}}L + d\mathcal{L}_{\hat{\xi}}\gamma \\ &= \delta d\alpha + d\mathcal{L}_{\hat{\xi}}\gamma \\ &= d(-\delta\alpha + \delta\iota_{\hat{\xi}}\gamma + \iota_{\hat{\xi}}\delta\gamma) \\ &= d(-\delta j + \iota_{\hat{\xi}}\delta\gamma)\,,\end{aligned}$$

where $j = \alpha - \iota_{\hat{\xi}}\gamma$ is the Noether current. It then follows from Thm. 5.2.2 that $P(\mathcal{L}_{\hat{\xi}}EL) = 0$. Assume that φ is a solution of the Euler-Lagrange equation $EL_{j^\infty\varphi} = 0$. We now apply Lem. 7.3.12 to $\omega = EL$, which shows that

$$\mathcal{L}_{\xi_\varphi}EL = (\mathcal{L}_{\hat{\xi}}EL)_{j^\infty\varphi} = \big(P(\mathcal{L}_{\hat{\xi}}EL)\big)_{j^\infty\varphi} = 0\,,$$

i.e. ξ_φ is a Jacobi field. □

Remark 7.3.13. Prop. 7.3.11 was stated in Prop. 13 a) of [Zuc87] which refers to a forthcoming paper for the proof. To my best knowledge this announced paper has never appeared. Nonetheless, the statement of Prop. 13 a) has been used subsequently in the literature. For example, it is used as the first step in the proof of Prop. 2.76 of [DF99].

7.3.4 Presymplectic structures

The boundary form γ is determined by the lagrangian only up to a closed form.

Proposition 7.3.14. *Let (M, F, L) be a local LFT. The form $\delta\gamma$, where γ is a boundary form, is unique up to a d-exact form.*

Proof. The boundary form γ is unique up to a closed form. So if γ' is another boundary form $\tau := \gamma' - \gamma$ is d-closed. For $\delta\tau = \delta\gamma' - \delta\gamma$ we obtain $d\delta\tau = -\delta d\tau = 0$. It now follows from the acyclicity theorem 5.2.4 that $\delta\tau$ is exact. □

Terminology 7.3.15. In [Zuc87] $\delta\gamma$ is called the universal current.

Lemma 7.3.16. *Let $\varphi \in \mathcal{F}_{\text{shell}}$. Let $\xi_\varphi, \chi_\varphi \in T_\varphi\mathcal{F}$ be Jacobi fields and $\hat{\xi}_{j^\infty\varphi}$, $\hat{\xi}_{j^\infty\varphi}$ their infinite prolongations. Then*

$$\iota_{\hat{\xi}_{j^\infty\varphi}}\iota_{\hat{\chi}_{j^\infty\varphi}}\delta EL = 0\,.$$

Proof. Let $\hat{\xi}$, $\hat{\chi}$ be strictly vertical vector fields extending $\hat{\xi}_{j^\infty\varphi}$ and $\hat{\xi}_{j^\infty\varphi}$. We have

$$\begin{aligned}\iota_{\hat{\xi}}\iota_{\hat{\chi}}\delta EL &= (\iota_{\hat{\xi}}\mathcal{L}_{\hat{\chi}} - \iota_{\hat{\xi}}\delta\iota_{\hat{\chi}})EL \\ &= (\iota_{\hat{\xi}}\mathcal{L}_{\hat{\chi}} - \mathcal{L}_{\hat{\xi}}\iota_{\hat{\chi}})EL \\ &= (\iota_{\hat{\xi}}\mathcal{L}_{\hat{\chi}} - \iota_{\hat{\chi}}\mathcal{L}_{\hat{\xi}} - \iota_{[\hat{\xi},\hat{\chi}]})EL\,.\end{aligned}$$

When we evaluate the right hand side at φ, the first two terms vanish because $\hat{\xi}$ and $\hat{\chi}$ are infinite prolongations of Jacobi fields. The last term vanishes because φ is a solution of the Euler-Lagrange equation. □

Proposition 7.3.17. *Let $\varphi \in \mathcal{F}$, let $\xi_\varphi, \chi_\varphi \in T_\varphi\mathcal{F}$ Jacobi fields, and let $\hat{\xi}_{j^\infty\varphi}$, $\hat{\chi}_{j^\infty\varphi}$ be their infinite prolongations. Then*

$$\iota_{\hat{\xi}_{j^\infty\varphi}}\iota_{\hat{\chi}_{j^\infty\varphi}}d(\delta\gamma) = 0\,.$$

Proof. We have

$$\begin{aligned}d(\delta\gamma) = -\delta d\gamma &= \delta(EL - \delta L) \\ &= \delta EL\,.\end{aligned}$$

The proposition now follows from Lem. 7.3.16. □

Let $S \subset M$ be closed oriented codimension 1 submanifold. Integrating $\delta\gamma$ over S yields a 2-form ω_S on $\mathcal{F}$, which is defined by

$$\omega_S(\xi_\varphi, \chi_\varphi) := \int_S \iota_{\chi_\varphi}\iota_{\xi_\varphi}(j^\infty)^*\delta\gamma = \int_S \iota_{\hat{\chi}_{j^\infty\varphi}}\iota_{\hat{\xi}_{j^\infty\varphi}}\delta\gamma$$

for all $\xi_\varphi, \chi_\varphi \in T\mathcal{F}$. It follows from Prop. 7.3.14 that ω_S is independent of the choice of the boundary form γ. While ω_S does depend on S, it follows from Prop. 7.3.17 that ω_S is conserved on shell in the following sense: For two Jacobi fields ξ_φ and χ_φ, $\omega_S(\xi_\varphi, \chi_\varphi)$ depends only on the homology class of S.

Proposition 7.3.18. *Let (M, F, L) be a local LFT. Let $(j, \hat{\xi})$ be a Noether pair. Let ξ be the unique local vector field on $\mathcal{F}$ that projects to $\hat{\xi}$ by Thm. 5.1.37. Let $S \subset M$ a closed oriented codimension 1 submanifold and q_S the charge of j on S. Then*

$$(\iota_\xi\omega_S - \delta q_S)_\varphi = 0$$

for all $\varphi \in \mathcal{F}_{\text{shell}}$.

Proof. As shown in the proof of Prop. 7.3.11, we have

$$d(\iota_{\hat{\xi}}\delta\gamma - \delta j) = \mathcal{L}_{\hat{\xi}}EL\,.$$

Prop. 7.3.11 states that the right hand side vanishes at $\varphi \in \mathcal{F}_{\text{shell}}$. In other words, $\iota_{\hat{\xi}}\delta\gamma - \delta j$ is d-closed at φ. Prop. 6.3.23 then implies that $\iota_{\hat{\xi}}\delta\gamma - \delta j$ is d-exact at φ. We conclude that $(\iota_\xi\omega_S - \delta q_S)_\varphi = \int_S(\iota_{\hat{\xi}}\delta\gamma - \delta j)_\varphi = 0$. □

Proposition 7.3.19. *Let (M, F, L) be a local LFT with boundary form γ. Let $A \to M$ be a smooth vector bundle of non-zero rank. Let $\xi : \mathcal{A} \to \mathfrak{X}(F)$ be a linear local family of symmetries. Let $S \subset M$ be a closed codimension 1 submanifold. Then*

$$\omega_S(\xi_{a,\varphi}, \chi_\varphi) = 0$$

for all $a \in \mathcal{A}$, all $\varphi \in \mathcal{F}_{\mathrm{shell}}$, and all Jacobi fields $\chi_\varphi \in T_\varphi \mathcal{F}$.

Proof. Let $\varphi \in \mathcal{F}_{\mathrm{shell}}$. Let χ_φ be a Jacobi field and $\hat{\chi}_{j^\infty\varphi}$ its infinite prolongation. Let $\hat{\xi}_{a,j^\infty\varphi}$ be the infinite prolongation of $\xi_{a,\varphi}$. The map

$$\begin{aligned} \nu : \mathcal{A} &\longrightarrow \Omega^{1,\mathrm{top}-1}(M) \\ a &\longmapsto \iota_{\hat{\chi}_{j^\infty\varphi}} \iota_{\hat{\xi}_{a,j^\infty\varphi}} \delta\gamma \end{aligned}$$

is linear and local. By Prop. (7.3.11), $\xi_{a,\varphi}$ is a Jacobi field. Prop. 7.3.17 implies that

$$\begin{aligned} (d_M \circ \nu)(a) &= d_M(\iota_{\hat{\chi}_{j^\infty\varphi}} \iota_{\hat{\xi}_{a,j^\infty\varphi}} \delta\gamma) = \iota_{\hat{\chi}_{j^\infty\varphi}} \iota_{\hat{\xi}_{a,j^\infty\varphi}} d(\delta\gamma) \\ &= 0\,. \end{aligned}$$

We conclude that ν satisfies the conditions of Prop. 6.3.13, so that $\nu = d_M \circ \nu$ for some local linear map $\mu : \mathcal{A} \to \Omega^{1,\mathrm{top}-1}(M)$. It follows that

$$\omega_S(\xi_{a,\varphi}, \chi_\varphi) = \int_S d_M(\mu(a)) = 0\,,$$

which finishes the proof. □

Remark 7.3.20. Prop. 7.3.19 is stated as Thm. 13 b) in [Zuc87]. The sketch of a proof given there is not correct, however, as it requires the assumption that χ_φ be a diffeological tangent vector and not only a Jacobi field. With this stronger assumption, we can give the following short proof as proposed in [Zuc87]:

By Thm. 7.2.26, the Noether charge $q_{a,S}$ of $\xi(a)$ vanishes on $\mathcal{F}_{\mathrm{shell}}$. It follows, that $\iota_{\chi_\varphi} \delta q_{a,S} = 0$ for any tangent vector $\chi_\varphi \in T\mathcal{F}_{\mathrm{shell}}$. From Prop. 7.3.18 we deduce that $\omega_S(\xi_{a,\varphi}, \chi_\varphi) = \iota_{\chi_\varphi} \iota_{\xi_{a,\varphi}} \omega_S = \iota_{\chi_\varphi} \delta q_{a,S} = 0$.

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