Lagrangian Field Theory

Diffeology, Variational Cohomology, Multisymplectic Geometry

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Chapter 1

What is a Lagrangian Field Theory?

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1.1 Fields

In classical physics, a field describes the state of a system by assigning to every point of a geometric space or object the value of some physical quantity at that point. An example of a field is the function that assigns to every point of a solid the temperature at that point. Another example is the field that assigns the wind velocity to every point on the surface of the Earth. Such assignments are generally assumed to be smooth maps. This is an idealization, of course, as the two examples show, in which the physical systems consist of discrete atoms. But it has led to very accurate descriptions of physical phenomena. In mathematics, the idealization is promoted to a definition.

Definition 1.1.1. A **field** is a smooth section of a smooth fiber bundle $F \to M$. The set of all fields is denoted by $\mathcal{F} := \Gamma^\infty(M, F)$.

Example 1.1.2. In the example of the temperature field, the fiber bundle is $F = M \times [0, \infty) \to M$, where M is the manifold describing the solid. This shows that F is generally not a vector bundle. In the example of the air velocity field, the fiber bundle is the tangent bundle $F = TS^2 \to S^2$ of the sphere, which shows that F is generally not a trivial bundle.

Terminology 1.1.3. In physics, the base manifold of the fiber bundle is called the **background** geometry or the **spacetime**, the latter especially in fundamental theories such as gauge theory or general relativity. F is sometimes called the **configuration bundle** or **configuration space bundle**, and the typical fiber of F the **configuration space** or the **field content**. $\mathcal{F}$ is usually called the **space of fields**, although it often remains unclear or implicit what "space" means mathematically.

Example 1.1.4. Let $M = \mathbb{R}$ and $F := Q \times \mathbb{R}$ be a trivial bundle. Then $\mathcal{F} = C^\infty(\mathbb{R}, Q)$ is the space of smooth paths in Q. If we replace $\mathbb{R}$ with S^1 then $\mathcal{F}$ is the free loop space of Q.

1.2 The action principle in its "mythological" form

In a field theory, the fields are usually subject to a **field equation** $f(\varphi) = 0$, where $f : \mathcal{F} \to V$ is a map to a vector space V. The solutions of the field equation are those fields that are governed by the laws of physics or that possess some desired mathematical properties. Typically, f is a differential operator.

Example 1.2.1. Let $M \subset \mathbb{R}^3$ be a 3-dimensional submanifold with boundary ∂M. Let $F := M \times \mathbb{R} \to M$, so that $\mathcal{F} = C^\infty(M)$. In electrostatics, $\varphi \in C^\infty(M)$ is viewed as the electric potential. The field equation is $\Delta\varphi = 0$, where Δ is the Laplace operator. The solutions of the field equation are harmonic functions subject to boundary conditions on ∂M.

Terminology 1.2.2. In physics, the fields that solve the field equations are often called **on-shell** and those that do not **off-shell**. This terminology comes from the so-called mass-shell (German: Massen*schale*), which is the positive energy *sheet* of the hyperboloid of the 4-momentum $(p_0, p_1, p_2, p_3) \in \mathbb{R}^4$ of a relativistic particle of rest mass $m^2 = (p_0)^2 - (p_1)^2 - (p_2)^2 - (p_3)^2$. In this sense "shell" is a mistranslation of "Schale". In early quantum field theory, where the momenta are represented by partial derivatives on the wave functions, the mass-shell has come to denote the space of solutions of the equation of motion $\Box\varphi = m^2$ of the free relativistic particle.

The set of solutions of the field equation will be denoted by $\mathcal{F}_{\text{shell}} := f^{-1}(0)$. In general, $\mathcal{F}_{\text{shell}} \subset \mathcal{F}$ is not a smooth variety, but has singularities. The field equations are often quite complicated. The main tool to study them is the **action principle**. In its ideal form it is stated as follows.

Action principle 1.2.3. There is a smooth function

$$S : \mathcal{F} \longrightarrow \mathbb{R},$$

called the **action**, such that $\varphi \in \mathcal{F}$ is a solution of the field equation if and only if it is a critical point of S.

The value of this principle is that it is usually much easier to construct and study a field theory via its action than via its field equations. For example, a diffeomorphism $\Phi \in \mathrm{Diff}(\mathcal{F})$ acts naturally on functions on $\mathcal{F}$ by pullback. So Φ is a symmetry of the field theory given by an action S if $\Phi^*S = S$. It follows that Φ maps critical points of S to critical points, i.e. $\Phi(\mathcal{F}_{\text{shell}}) = \mathcal{F}_{\text{shell}}$. Conversely, if the symmetries are known, like the Lorentz transformations of special relativity, the requirement for S to be invariant restricts the possible actions of the theory. For such reasons, the action principle is one of the most important guiding principles in both classical and quantum field theory.

Mathematically, however, the action principle 1.2.3 is often not rigorously true. In his 2011 Felix Klein lectures Graeme Segal called it the "mythological picture" of field theory. One of the main goal of these notes is to explain how the action principle can be restated so that it is rigorously true, sufficiently general to cover the most relevant field theories, such as general relativity, and compatible with the current mathematical tools used in field theory.

1.3 Classical mechanics

1.3.1 The action principle in classical mechanics

What is the action? And how do we get from the action to the field equations? The basic example is a classical mechanical system, where $M = \mathbb{R}$ is time and $F = Q \times \mathbb{R}$, so that a field is a smooth path $q : \mathbb{R} \to Q$. Let us assume for simplicity that $Q = \mathbb{R}^n$. When the system is at rest, it will have to be at a critical point of the potential energy $V : Q \to \mathbb{R}$. When the system moves, the kinetic energy has to be taken into account as well. The action turns out to be given by the difference of kinetic and potential energy,

$$S(q) := \int_{\mathbb{R}} \left\{ \tfrac{1}{2}\dot{q}^i(t)\dot{q}^i(t) - V\big(q(t)\big) \right\} dt\,,$$

where $q^i(t)$ are the components of the path, where repeated indices are being summed over, $\dot{q}^i(t)\dot{q}^i(t) = \sum_{i=1}^n \dot{q}^i(t)\dot{q}^i(t)$, and where we have chosen units in which the mass is $m = 1$.

Problem 1.3.1. The integral over $\mathbb{R}$ that defines the action is generally divergent.

In a first attempt to avoid problem 1.3.1, we could consider only those q that have a finite action, but the solutions of the field equation may not satisfy this condition. For example, consider the case of a free particle where $V(q) = 0$. The solutions of the equations of motion are paths of constant velocity. So only if the velocity is zero the action is finite.

In a second attempt to solve problem 1.3.1, we as mathematicians could assume M to be closed, that is, compact without boundary [Abb01]. In the case of classical mechanics this would mean, however, that time is S^1 so that we would only consider periodic solutions. The assumption that M is closed will also exclude some of the most interesting spacetimes, like Minkowski spacetime or many realistic physical models for the curved spacetime of the universe we live in.

In a third attempt, we can restrict the domain of integration to a compact interval $[a, b]$ for the action to be finite. We will denote this action by $S_{[a,b]}$. Following the action principle 1.2.3, we now have to compute the critical points of $S_{[a,b]}$. Let $q : [a, b] \to Q$ be a smooth path. Since we have assumed for simplicity that Q is a vector space, $T_q\mathcal{F} \cong \mathcal{F}$. Therefore, a tangent vector $\xi \in T_q\mathcal{F}$ can be represented by smooth family of paths $\mathbb{R} \ni \varepsilon \mapsto q_\varepsilon \in C^\infty(\mathbb{R}, Q)$ given by $q_\varepsilon = q + \varepsilon\xi$. The derivative of $S_{[a,b]}$ in the direction of ξ is obtained by inserting $q + \varepsilon\xi$ and expanding the result to first order in ε.

$$\begin{aligned}
&S_{[a,b]}(q + \varepsilon\xi) - S_{[a,b]}(q) \\
&= \varepsilon \int_a^b \Big\{ \dot{q}^i(t)\dot{\xi}^i(t) - \frac{\partial V}{\partial q^i}\big(q(t)\big)\,\xi^i(t) \Big\} dt + \mathcal{O}(\varepsilon^2) \\
&= \varepsilon \int_a^b \Big\{ \frac{d}{dt}\big(\dot{q}^i(t)\xi^i(t)\big) - \ddot{q}^i(t)\xi^i(t) - \frac{\partial V}{\partial q^i}\big(q(t)\big)\,\xi^i(t) \Big\} dt + \mathcal{O}(\varepsilon^2) \\
&= -\,\varepsilon \int_a^b \Big\{ \ddot{q}^i(t) + \frac{\partial V}{\partial q^i}\big(q(t)\big) \Big\} \xi^i(t)\, dt + \varepsilon \int_a^b \frac{d}{dt}\big(\dot{q}^i(t)\xi^i(t)\big)\, dt + \mathcal{O}(\varepsilon^2)\,.
\end{aligned}$$

Let us first consider variations ξ^i that have compact support in $[a,b]$, so that the second integral vanishes. The first integral vanishes for all ξ^i if and only if q^i satisfies the field equation

$$\ddot{q}^i = -\frac{\partial V}{\partial q^i}\,,$$

which is the equation of motion of a point particle in a potential V. The second integral is given by

$$\int_a^b \frac{d}{dt}\big(\dot{q}^i(t)\xi^i(t)\big)\,dt = \dot{q}^i(b)\,\xi^i(b) - \dot{q}^i(a)\,\xi^i(a)\,.$$

Now we consider variations ξ^i that have their support concentrated in small neighborhoods around the boundary points a and b. By keeping $\xi^i(a)$ and $\xi^i(b)$ constant while shrinking the support, we can make the first integral arbitrarily small. The conclusion is that the second integral has to vanish for all ξ^i independently of the first, which is the case if and only if

$$\dot{q}^i(a) = 0 \quad \text{and} \quad \dot{q}^i(b) = 0\,.$$

This is certainly not a condition we want to impose on q.

We can modify the action principle by requiring $\xi^i(a) = 0 = \xi^i(b)$. But then the solutions of the field equation are not the critical points of S but rather points where the derivative of S vanishes on a subset of vectors in $T_q\mathcal{F}$. Moreover, we have to require the conditions for all compact intervals $[a,b]$. In terms of differential topology, we are pairing the de Rham 1-cocycle represented by the integrand with the 1-cycle represented by the interval. In light of the Poincaré duality between cohomology and homology on a manifold, this suggest that the derivation of the field equation might be formulated in the framework of cohomology. We will return to this point of view in Chapter 5 and Chapter 6.

1.3.2 Lagrangians

In the example of classical mechanics we have seen that the action is obtained by integrating for every field q a volume form over the spacetime manifold $\mathbb{R}$.

Definition 1.3.2. A smooth function $L : \mathcal{F} \to \Omega^n(M)$, where $n = \dim M$, is called a **lagrangian**.

Remark 1.3.3. For simplicity, we shall assume that M is oriented. If M is non-orientable, we have to tensor before integration with the determinant bundle of M as it is done in [DF99].

Given a lagrangian L, we tentatively define the action by

$$S(\varphi) := \int_M L(\varphi)\,.$$

But, as we have seen, even for classical mechanics the action is generally not finite, so it is certainly not a smooth map to $\mathbb{R}$. The issues come from the integration over the non-closed manifold $\mathbb{R}$.

When we review the derivation of the equation of motion carefully, we see that we did not need to compute any integrals. All we did is to discard exact terms under the integral. This means that we can just as well study the cohomology class of the integrand without ever pairing it with the fundamental class $[M]$. We will return to this idea in Chapter 6.

Definition 1.3.4. A **Lagrangian Field Theory** (LFT) consists of a smooth fiber bundle $F \to M$ and a lagrangian $L : \mathcal{F} \to \Omega^n(M)$.

For a general action $\mathcal{F} \to \mathbb{R}$ there is no mathematical reason why the critical points should be the solution of a PDE, as is the case for most LFTs that come to mind. The following condition guarantees that the Euler–Lagrange equation is a PDE.

Definition 1.3.5. A lagrangian $L : \mathcal{F} \to \Omega^n(M)$ is called **local** if there is a natural number $k \geq 0$, such that the value of $L(\varphi)$ at m depends smoothly on m and only the partial derivatives of φ at m up to order k.

1.3.3 Presymplectic structure

The vertical tangent bundle of $\pi_Q : TQ \to Q$ is given by

$$VTQ = \ker(T\pi_Q : TTQ \to TQ) \cong TQ \times_Q TQ\,.$$

Let $f : TQ \to \mathbb{R}$ be a smooth function. Restricting the differential $df : TTQ \to \mathbb{R}$ to the vertical tangent bundle yields a map

$$df\big|_{VTQ} : TQ \times_Q TQ \longrightarrow \mathbb{R}\,.$$

Since this map is linear in the second factor (the one that can be identified with the vertical tangent vectors), we can identify it with a smooth map

$$\mathrm{Leg}_f : TQ \longrightarrow T^*Q\,,$$

which is the **Legendre transformation** generated by f.

Let ω_{T^*Q} denote the canonical symplectic form on T^*Q. Its pullback by the Legendre transformation,

$$\omega = \mathrm{Leg}_f^* \omega_{T^*Q}\,,$$

is a presymplectic form on TQ, which is symplectic if and only if Leg_f is a local diffeomorphism.

Definition 1.3.6. The function $f : TQ \to \mathbb{R}$ is said to satisfy the **Legendre condition** if Leg_f is a local diffeomorphism.

The lagrangian function for a particle in a time-dependent potential,

$$\mathcal{L} = \frac{1}{2}\dot{q}^i\dot{q}^i - V(q,t)\,, \tag{1.1}$$

is a function on $\mathbb{R} \times TQ$. (In Example 3.1.7 we will see that $\mathbb{R} \times TQ$ is the first jet manifold of the configuration bundle $\mathbb{R} \times Q \to \mathbb{R}$.) If we choose a time $t_0 \in \mathbb{R}$ and restrict $\mathcal{L}$ to $\{t_0\} \times TQ \cong TQ$, we obtain a function

$$\mathcal{L}_{t_0} : TQ \longrightarrow \mathbb{R}\,,$$

that generates the Legendre transformation $\mathrm{Leg}_{\mathcal{L}_{t_0}} : TQ \to T^*Q$.

Let q^i be local coordinates of Q and $(q^i, \dot{q}^i)$ the induced coordinates on TQ, which are given by

$$q^i\Big(x, v^j \frac{\partial}{\partial q^j}\Big) = q^i(x)\,, \qquad \dot{q}^i\Big(x, v^j \frac{\partial}{\partial q^j}\Big) = v^i$$

for all $(x,v) \in TQ$. Let (q^i, p_i) be the induced coordinates on T^*Q, which are given by

$$q^i(x, \alpha_j dq^j) = q^i(x)\,, \qquad p_i(x, \alpha_j dq^j) = \alpha_i$$

for all $(x, \alpha) \in T^*Q$. The Legendre transformation generated by $\mathcal{L}_{t_0}$ maps a vector $(x,v) \in TQ$ to

$$\mathrm{Leg}_{\mathcal{L}_{t_0}}(x,v) = \Big(x, \frac{\partial \mathcal{L}_{t_0}}{\partial \dot{q}^i}(x,v)\, dq^i\Big)\,.$$

The pullback of the canonical 1-form $p_i dq^i$ on T^*Q by the Legendre transformation is

$$\mathrm{Leg}^*_{\mathcal{L}_{t_0}}(p_i dq^i) = \frac{\partial \mathcal{L}_{t_0}}{\partial \dot{q}^i} dq^i\,.$$

The pullback of the canonical symplectic form $\omega_{T^*Q} = d(p_i dq^i) = dp_i \wedge dq^i$ on T^*Q is

$$\begin{aligned}
\omega &= \mathrm{Leg}^*_{\mathcal{L}_{t_0}} d(p_j dq^j) = d\mathrm{Leg}^*_{\mathcal{L}_{t_0}}(p_j dq^j) \\
&= d\Big(\frac{\partial \mathcal{L}_{t_0}}{\partial \dot{q}^j}\Big) \wedge dq^j \\
&= \frac{\partial^2 \mathcal{L}_{t_0}}{\partial q^i \partial \dot{q}^j}\, dq^i \wedge dq^j + \frac{\partial^2 \mathcal{L}_{t_0}}{\partial \dot{q}^i \partial \dot{q}^j}\, d\dot{q}^i \wedge dq^j\,.
\end{aligned}$$

In local coordinates, the Legendre transformation is given by the coordinate transformation

$$(q^i, \dot{q}^i) \longmapsto \Big(q^i, \frac{\partial \mathcal{L}_{t_0}}{\partial \dot{q}^i}\Big)\,.$$

The Jacobi matrix of this map is of the form

$$J(\mathrm{Leg}_{\mathcal{L}_{t_0}}) = \begin{pmatrix} \delta_{ij} & 0 \\ * & \frac{\partial^2 \mathcal{L}_{t_0}}{\partial \dot{q}^i \partial \dot{q}^j} \end{pmatrix}\,.$$

By the inverse function theorem we conclude that $\mathcal{L}_{t_0}$ satisfies the Legendre condition if and only if $\frac{\partial^2 \mathcal{L}_{t_0}}{\partial \dot{q}^i \partial \dot{q}^j}$ is an invertible matrix at all points of TQ. For more on the Legendre condition in symplectic geometry see e.g. Chapter 20 of [CdS01].

For a lagrangian function of the form (1.1) the Jacobi matrix is given by $\frac{\partial^2 \mathcal{L}_{t_0}}{\partial \dot{q}^i \partial \dot{q}^j} = \delta_{ij}$, which satisfies the Legendre condition. This, however, is not always the case.

Example 1.3.7. Let Q be a manifold with a riemannian metric g. The length of a path $q : [0,1] \to Q$ is given by the integral

$$S(q) = \int_{t=0}^{1} \sqrt{g_{ij}\big(q(t)\big)\, \dot{q}^i(t)\dot{q}^j(t)}\, dt$$

The lagrangian

$$L = \sqrt{g_{ij}\, \dot{q}^i \dot{q}^j}\, dt$$

of this action does not satisfy the Legendre condition.

The presymplectic form ω is an important ingredient of classical mechanics. It is used to study symmetries, it describes the dynamics as hamiltonian flows, and it defines the Poisson bracket, which is the most important algebraic structure for quantization. However, as we have seen, we have to contend with the following issues:

1. The Legendre transformation is defined for lagrangians that depend only on t, q, $\dot{q}$, but not on higher derivatives.
2. The Legendre transformation generally depends on the choice of a time t_0.
3. The Legendre condition must be satisfied for ω to be symplectic.

In classical mechanics, the assumptions we must make to avoid these issues are mild and usually satisfied. Most lagrangians depend only on the first derivatives of q and if not, we can convert the lagrangian into a first order lagrangian on a larger configuration bundle. Most lagrangians do not depend explicitly on time, so the Legendre transformation is the same for all choices of t_0. And for lagrangians with the usual kinetic energy term $\frac{1}{2}\dot{q}^i\dot{q}^i$ the Legendre condition is satisfied. For field theories with spacetime dimension larger than 1, however, the three issues pose major technical and conceptual problems.

1.4 Maxwell theory

1.4.1 Minkowski space

Maxwell theory is the classical theory of electromagnetic fields. Its background geometry is physical spacetime given by a lorentzian 4-manifold M. The most basic choice for M is Minkowski space, that is, $M = \mathbb{R}^4$ equipped with the metric

$$\begin{aligned} \eta &= \tfrac{1}{2}\eta_{ij} dx^i dx^j \\ &= \tfrac{1}{2}\big(-(dx^0)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2\big)\,, \end{aligned}$$

where x^0 is the time-coordinate, and x^1, x^2, x^3 are the space-coordinates.

Remark 1.4.1. We define lorentzian metrics to have the signature $(-1,1,1,1)$, which is sometimes called the "east coast" convention, the signature $(1,-1,-1,-1)$ being called the "west coast" convention. The advantage of the east coast convention is that the metric induces the usual euclidean scalar product on 3-space $\mathrm{Span}\{x^1, x^2, x^3\}$.

Terminology 1.4.2. A tangent vector $v \in TM$ on a lorentzian manifold is called **space-like** if $\eta(v,v) > 0$, **light-like** if $\eta(v,v) = 0$, and **time-like** if $\eta(v,v) < 0$. A submanifold $S \subset M$ is called space-like, light-like, or time-like, if all tangent vectors in TS are.

Recall that every bilinear form $\langle\ ,\ \rangle$ on a vector space V can be extended to a bilinear form $\langle\ ,\ \rangle : \wedge^k V \times \wedge^k V \to \mathbb{R}$ on the k-th exterior power by

$$\langle v_1 \wedge \ldots \wedge v_k, w_1 \wedge \ldots \wedge w_k \rangle := \det\big(\langle v_i, w_j\rangle_{1\le i,j\le k}\big)\,. \tag{1.2}$$

We consider the fiber-wise scalar product given by the inverse of η,

$$\begin{aligned} \langle\ ,\ \rangle : T^*M \times_M T^*M &\longrightarrow \mathbb{R} \\ \langle \alpha_i dx^i, \beta_j dx^j \rangle &:= \eta^{ij}\alpha_i\beta_j\,, \end{aligned}$$

where η^{ij} denotes the inverse matrix of η_{ij}, i.e. $\eta^{ij}\eta_{jk} = \delta^i_k$. By (1.2) this induces a bilinear form on differential k-forms,

$$\langle\ ,\ \rangle : \Omega^k(M) \times \Omega^k(M) \longrightarrow C^\infty(M)\,.$$

Let us equip M with the standard orientation for which (x^0, x^1, x^2, x^3) is an oriented chart. Then there is a unique oriented volume form $\mathrm{vol} \in \Omega^4(M)$ that is normalized, $\langle \mathrm{vol}, \mathrm{vol}\rangle = 1$. In terms of coordinate 1-forms, it is given by

$$\mathrm{vol} = dx^0 \wedge \ldots \wedge dx^3\,.$$

The volume form is used to define a **Hodge structure** (see e.g. Sec. 3.3 of [Jos17]), that is, a $C^\infty(M)$-linear map

$$\star : \Omega^k(M) \longrightarrow \Omega^{\dim M - k}(M)$$

uniquely determined by the defining equation

$$\alpha \wedge \star\beta = \langle \alpha, \beta\rangle\, \mathrm{vol}\,,$$

for all $\alpha, \beta \in \Omega^k(M)$ and all k. Note that $\mathrm{vol} = \star 1$. The Hodge-$\star$ satisfies

$$\star(\star\alpha) = (\det\eta)(-1)^{(\dim M - |\alpha|)|\alpha|}\alpha\,, \tag{1.3}$$

where $\det\eta$ is the determinant of the metric in any orthonormal basis and $|\alpha|$ the degree of the form α. For a metric of signature $(-1,1,1,1)$ we have $\det\eta = -1$.

1.4.2 Charges and currents

Electric charges and currents generate the electromagnetic field. In physics, a time-dependent charge density is a smooth function ρ on Minkowski space and a current density a vector field $v = v^1\frac{\partial}{\partial x^1} + v^2\frac{\partial}{\partial x^2} + v^3\frac{\partial}{\partial x^3}$ on M with components only in the space directions.

The total charge $q_{S,t}$ contained in a submanifold $S \subset \mathbb{R}^3$ of space at time t is given by the integral

$$q_{S,t} = \int_{\{t\}\times S} \rho\, dx^1 \wedge dx^2 \wedge dx^3\,.$$

The flux of the current through the surface ∂S at time t is given by

$$\begin{aligned}\Phi_{S,t} &:= \int_{\{t\}\times\partial S} \iota_v(dx^1 \wedge dx^2 \wedge dx^3)\\ &= \int_{\{t\}\times S} d\iota_v(dx^1 \wedge dx^2 \wedge dx^3)\\ &= \int_{\{t\}\times S} (\operatorname{div} v)\, dx^1 \wedge dx^2 \wedge dx^3\,,\end{aligned}$$

where we have used Stokes' theorem and $\operatorname{div} v = \frac{\partial v^i}{\partial x^i}$.

The current density describes the flow of charge through space, so if the charge is conserved, then the rate of change of the charge in every space-region S must be equal to the negative flux through the surface of S, $\frac{d}{dt} q_{S,t} = -\Phi_{S,t}$. This is the case if and only if

$$\frac{\partial \rho}{\partial t} = -\operatorname{div} v\,. \tag{1.4}$$

We obtain a form of condition (1.4) that does not rely on the splitting of the manifold M into time and space directions by combining the charge density and the current density into the 4-vector field

$$J := \rho \frac{\partial}{\partial x^0} + v^1 \frac{\partial}{\partial x^1} + v^2 \frac{\partial}{\partial x^2} + v^3 \frac{\partial}{\partial x^3}\,.$$

The de Rham differential of $\iota_J \mathrm{vol}$ is

$$d\,\iota_J \mathrm{vol} = \Big(\frac{\partial \rho}{\partial x^0} + \operatorname{div} v \Big)\, \mathrm{vol}\,.$$

The conclusion is that Eq. (1.4) holds if and only if $j := \iota_J \mathrm{vol}$ is closed. This suggests the following definition:

Definition 1.4.3. Let M be an n-dimensional manifold. A form $j \in \Omega^{n-1}(M)$ is called a **current**. A current is **conserved** if it is closed, $dj = 0$.

Terminology 1.4.4. In physics, it is usually the vector field J that is called the 4-current. For our purposes, Def. 1.4.3 is more convenient. Unlike for J, the condition in Def. 1.4.3 for a current to be conserved does not involve the volume form.

1.4.3 Gauge symmetry

The fields for Maxwell theory on Minkowski space are 1-forms. That is, the configuration bundle is $T^*M \to M$ and the space of fields

$$\mathcal{F} = \Omega^1(M)\,.$$

In Maxwell theory it is customary to denote the fields by the letter A. The lagrangian for the electromagnetic field generated by a current $j = \iota_J \mathrm{vol}$ is

$$\begin{aligned} L(A) &= \big(\tfrac{1}{2} \langle dA, dA \rangle + \iota_J A \big)\, \mathrm{vol}\\ &= \tfrac{1}{2}\, dA \wedge \star dA + j \wedge A\,. \end{aligned} \tag{1.5}$$

The Euler–Lagrange equation is

$$d \star dA = j\,. \tag{1.6}$$

The equation $d(dA) = 0$, which is satisfied for any field A, is also part of the Maxwell equations. Note that Eq. (1.6) implies that $dj = 0$, that is, j is conserved.

Terminology 1.4.5. In physics, A is usually called the **gauge field**, in order to distinguish it from the **electromagnetic field** $F := dA$. (Denoting the electromagnetic field with F is so standard in physics, that I could not resolve to use a different letter in order to distinguish it from our notation for the configuration bundle.)

If we view Eq. (1.6) as equation $d \star F = j$ for the electromagnetic field F, not assuming that the field is the differential of a 1-form A, we have to add the equation

$$dF = 0 \tag{1.7}$$

to the field equations. Equation (1.6) and Equation (1.7) together are the **Maxwell equations**.

The Maxwell equations are invariant under the Lorentz group, the group of linear transformations of $\mathbb{R}^4$ that leave the bilinear form η invariant. A careful study of these symmetries led Einstein in 1905 to the development of special relativity [Ein05]. In addition to this **external symmetry** group that acts on the spacetime manifold, there is the **internal symmetry** group $\big(C^\infty(M), +, 0\big)$ that acts on the fields by

$$\begin{aligned} C^\infty(M) \times \Omega^1(M) &\longrightarrow \Omega^1(M) \\ (f, A) &\longmapsto A + df\,. \end{aligned}$$

A careful study of this symmetry, called **local gauge symmetry**, led to the development of more general gauge theories.

1.5 General relativity

1.5.1 Hilbert–Einstein lagrangian and field equations

In general relativity a field is a lorentzian metric on a smooth oriented manifold of dimension n. The vacuum Hilbert–Einstein lagrangian is

$$L(g) := R(g)\,\mathrm{vol}_g\,,$$

where $R(g)$ is the scalar curvature and $\mathrm{vol}_g = \star 1$ the canonical volume form of g. The Euler–Lagrange equation is the vacuum **Einstein equation**

$$G := \mathrm{Ric}(g) - \frac{1}{2} R(g)\, g = 0\,,$$

where $\mathrm{Ric}(g)$ is the Ricci curvature and where the symmetric 2-form G is called the **Einstein tensor**. Pairing the Einstein tensor with the inverse metric, we obtain

$$g^{ij} G_{ij} = R(g) - \frac{n}{2} R(g) = -\frac{n-2}{2} R(g)\,.$$

If $n > 2$ it follows, that every metric that satisfies the Einstein equations has vanishing scalar curvature. This in turn implies that the vacuum Einstein equations are equivalent to

$$\mathrm{Ric}(g) = 0 \,.$$

In other words, a metric satisfies the Euler–Lagrange equations of general relativity if it is Ricci flat.

1.5.2 Mathematical features of general relativity

Here are some of the mathematical features of general relativity that make the theory difficult, but interesting to study:

1. The configuration bundle is not a vector bundle. It is a subbundle of the vector bundle of symmetric 2-forms, but due to the Lorentz signature of the metric, the fibers are not convex. As a consequence, local fields cannot be added by a partition of unity argument.

2. The fibers of the configuration bundle are not connected.

3. Local sections of the fiber bundle can generally not be expanded to global sections. If the spacetime manifold M is closed (compact without boundary) with non-vanishing Euler characteristic, then there are no global sections.

4. The lagrangian depends on the 2nd derivatives of the fields.

5. The field equation is a 2nd order PDE, that is, of the same order as the lagrangian.

6. The lagrangian and the field equation are not polynomial in the fields and its derivatives, since the Ricci and scalar curvature involve the inverse of the metric field g and the volume form the inverse of the square root $\sqrt{|\det g|}$ of its determinant.

7. The lagrangian $L : \mathcal{F} \to \Omega^n(M)$ is $\mathrm{Diff}(M)$-equivariant with respect to the action on metrics and forms by pullback. (If we view L as $(0, n)$-form on $\mathcal{F} \times M$, the form is invariant.) The diffeomorphism symmetry is an external symmetry, which means that it acts not only on the fibers of the configuration bundle but also on the base manifold M.

These properties should serve as preventive medicine against oversimplifying assumptions that exclude general relativity. They also show how the properties of field theories can differ from gauge theories such as Maxwell-Theory, which often inform the development of mathematical theories, generalizations, and approaches to quantization.

Exercises

Exercise 1.1 (Symplectic structure on the cotangent bundle)**.** Let Y be a smooth manifold and $\pi_Y : T^*Y \to Y$ its cotangent bundle. Let $\pi_{T^*Y} : T(T^*Y) \to T^*Y$ denote the projection of the tangent bundle of T^*Y. The **canonical** 1**-form** λ on T^*Y is defined by

$$\lambda(v) = \big\langle \pi_{T^*Y}(v), T\pi_Y(v) \big\rangle$$

for all $v \in T(T^*Y)$, where the pairing denotes the pairing of the tangent space and its dual. Let

$$\omega = -d\lambda\,.$$

Show that ω is a **symplectic form**, which means that ω is closed and non-degenerate. (A 2-form ω is non-degenerate, if the associated map of vector bundles $TX \to T^*X$, $v \mapsto \iota_v\omega$, is an isomorphism.) This ω is called the **canonical symplectic form** on the cotangent bundle.

Exercise 1.2 (Poisson brackets on presymplectic manifolds)**.** Let ω be a closed 2-form on the manifold X (also called a **presymplectic form**). A pair (f, v) of a function $f \in C^\infty(X)$ and a vector field $v \in \mathfrak{X}(X)$ is called **hamiltonian** if

$$\iota_v\omega = -df\,.$$

A function or a vector field is called **hamiltonian** if it belongs to a hamiltonian pair. The subspace of hamiltonian functions will be denoted by $C^\infty_{\mathrm{ham}}(X)$. The **Poisson bracket** of two hamiltonian functions $f, g \in C^\infty_{\mathrm{ham}}(X)$ with hamiltonian vector fields v and w, respectively, is defined by

$$\{f, g\} = \iota_w\iota_v\omega\,,$$

which is a smooth function on X.

(i) Show that the Poisson bracket is well-defined on hamiltonian functions, that is, $\{f, g\}$ does not depend on the choice of hamiltonian vector fields v and w.

(ii) Show that $\{f, g\}$ is hamiltonian. Is the product fg of two hamiltonian functions hamiltonian?

(iii) Show that the Poisson bracket satisfies the Jacobi identity,

$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0$$

for all $f, g, h \in C^\infty_{\mathrm{ham}}(X)$.

(iv) Show that the Poisson bracket is a derivation in each argument, that is

$$\{f, gh\} = \{f, g\}h + g\{f, h\}$$

for all $f, g, h \in C^\infty_{\mathrm{ham}}(X)$.

Exercise 1.3 (Hamiltonian group action)**.** Consider the symplectic structure on the cotangent bundle of $\mathbb{R}^3$ as defined in Exercise 1, that is, the symplectic manifold $(T^*\mathbb{R}^3, \omega)$ with coordinates $(q^1, q^2, q^3, p_1, p_2, p_3)$ on $T^*\mathbb{R}^3 \cong \mathbb{R}^3 \times \mathbb{R}^3 \cong \mathbb{R}^6$. Let

$$\mathrm{SO}(3) = \{A \in \mathrm{GL}(3;\mathbb{R}) \mid A^t A = \mathrm{id} \text{ and } \det(A) = 1\}$$

be the **special orthogonal group** of rotations of $\mathbb{R}^3$. Its Lie algebra

$$\mathrm{so}(3) = \{A \in \mathrm{gl}(3;\mathbb{R}) \mid A^t = -A\}$$

is the space of 3×3 skew-symmetric matrices. It can be identified with $\mathbb{R}^3$ while the Lie bracket on so(3) can be identified with the exterior product on $\mathbb{R}^3$:

$$\begin{aligned}
\mathrm{so}(3) = \{A \in \mathrm{gl}(3;\mathbb{R}) \mid A + A^t = 0\} &\longrightarrow \mathbb{R}^3 \\
A = \begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix} &\longmapsto (a_1, a_2, a_3) = \vec{a} \\
[A, B] = AB - BA &\longmapsto \vec{a} \times \vec{b}\,.
\end{aligned}$$

(i) Show that the SO(3)-action on $\mathbb{R}^3$ lifts to an action on the cotangent bundle

$$\Psi : \mathrm{SO}(3) \times T^*\mathbb{R}^3 \longrightarrow T^*\mathbb{R}^3$$

that preserves the symplectic form, that is, the structure diffeomorphisms $\Psi_A \in \mathrm{Diff}(T^*\mathbb{R}^3)$ satisfy $\Psi_A^*\omega = \omega$ for all $A \in \mathrm{SO}(3)$.

The infinitesimal Lie algebra action is

$$\begin{aligned}
\rho : \mathrm{so}(3) &\longrightarrow \mathfrak{X}(\mathbb{R}^6) \\
\rho(\vec{a})(\vec{q}, \vec{p}) &:= (\vec{a} \times \vec{q}, \vec{a} \times \vec{p})\,.
\end{aligned}$$

(ii) Show that ρ is a morphism of Lie algebras.

(iii) Show that there is a linear map

$$\mu : \mathrm{so}(3) \longrightarrow C^\infty(T^*\mathbb{R}^3)$$

such that $\big(\mu(\vec{a}), \rho(\vec{a})\big)$ is a hamiltonian pair and μ a morphism of Lie algebras.

(The map μ is called the momentum map of the action.)

Exercise 1.4 (Chern-Simons 5-form)**.** Let $P \to M$ be a principal bundle. Let $F(A)$ denote the curvature 2-form of a gauge field A. Compute the Chern-Simons 5-form, which is the 5-form $\omega(A)$ on M that satisfies

$$d\big(\omega(A)\big) = \mathrm{Tr}_{\mathrm{ad}}\{F(A) \wedge F(A) \wedge F(A)\}$$

and depends polynomially on A and dA.

Chapter 2

Diffeological spaces of fields

So far our "space" of fields $\mathcal{F} = \Gamma(M, F)$ is just a set. What is the geometric structure on $\mathcal{F}$ that we need in classical field theory? In order to formulate the action principle we need the notion of "variations" in $\mathcal{F}$, which are families

$$p : U \longrightarrow \mathcal{F}$$

parametrized by open subsets $p : U \subset \mathbb{R}^n$, $n \geq 0$. In order to define the geometric structure of variations we have to decide which families we consider to be smooth. For the set of sections of a smooth fiber bundle, the natural choice is the smooth homotopies of sections. That is, p is called smooth if the map

$$\begin{aligned} U \times M &\longrightarrow F \\ (u, m) &\longmapsto \big(p(u)\big)(m) \end{aligned}$$

is a smooth map of manifolds.

2.1 Diffeology

2.1.1 From plots to concrete sheaves

Definition 2.1.1 (e.g. Def. 1.5 in [IZ13])**.** A **diffeological space** is a set X together with a collection of maps $p : U \to X$, called **plots**, for all open subsets $U \subset \mathbb{R}^n$, $n \geq 0$ that satisfy the following conditions:

(i) Every constant map $p : U \to X$ is a plot.

(ii) Let $U \subset \mathbb{R}^n$ be an open subset and $\{U_i\}_{i \in I}$ an open cover. If $p|_{U_i} : U_i \to X$ is a plot for every $i \in I$, then p is a plot.

(iii) If $p : U \to X$ is a plot and $f : V \to U$ a smooth map from an open subset $V \subset \mathbb{R}^m$, then $p \circ f$ is a plot.

A **morphism of diffeological spaces** $f : X \to Y$ is a map of sets such that for every plot $p : U \to X$ the map $f \circ p : U \to Y$ is a plot. The category of diffeological spaces will be denoted by $\mathcal{D}\mathrm{flg}$.

Terminology 2.1.2. The collection of plots is called a **diffeology** on X. The open subsets of $\mathbb{R}^n$ for all $n \geq 0$ are sometimes called **parameter spaces**. Plots are also called **smooth parametrizations** or **smooth families**. A plot $\mathbb{R} \to X$ is called a **smooth path**. A map of sets with diffeology that is a morphism of diffeological spaces is called **diffeological** or **smooth** when it is clear from the context that "smooth" refers to the diffeology.

Example 2.1.3. Here are some of the most basic examples for diffeologies:

(a) The **fine diffeology**, or **discrete diffeology**, or **smallest diffeology** on a set X is the diffeology for which the plots are the locally constant maps.[1]

(b) The **coarse diffeology**, or **indiscrete diffeology**, or **trivial diffeology**, or **largest diffeology** on a set X is the diffeology for which all maps are plots.

(c) Every topological space X is equipped with the **continuous diffeology** for which the plots are the continuous maps.

(d) Every smooth finite-dimensional manifold M is equipped with the **manifold diffeology** or **smooth diffeology** for which the plots are the smooth, that is, infinitely often differentiable maps.

Definition 2.1.1 is the original definition of diffeological spaces that conveys the geometric idea and can be easily applied to concrete situations. For general considerations, however, it is useful to rephrase the definition in the language of sheaves.

Let $\mathcal{E}\mathrm{ucl}$ denote the category which has all open subsets of euclidean spaces $\mathbb{R}^n$, $n \geq 0$ as objects and all smooth maps as morphisms. Open covers define a Grothendieck pretopology, that is, the following three conditions are satisfied: (i) Isomorphisms are covers. (ii) The cover of a cover is a cover. (iii) The pullback of a cover along a smooth map is a cover.

Definition 2.1.4. The small category $\mathcal{E}\mathrm{ucl}$ together with the Grothendieck topology generated by the pretopology of open covers will be called the **site of euclidean spaces**.

The technicalities of Grothendieck topologies will not be important here, since sheaves on $\mathcal{E}\mathrm{ucl}$ can be defined in the same way as for topological spaces. $\mathcal{E}\mathrm{ucl}$ is **subcanonical**, which means that for every cover $\{U_i \to U\}$, the diagram

$$\textstyle\coprod_{i,j} U_i \times_U U_j \rightrightarrows \coprod_i U_i \longrightarrow U \tag{2.1}$$

is a coequalizer. The pullback $U_i \times_U U_j = U_i \cap U_j$ is the intersection, so that the coequalizer can be interpreted geometrically as glueing the open subsets U_i along their intersections. A sheaf is a contravariant functor that preserves this glueing.

[1] In [BH11, Example (2), p. 5794] it is stated incorrectly that the discrete diffeology is given by the constant maps.

Definition 2.1.5. A functor $F : \mathcal{E}\mathrm{ucl}^{\mathrm{op}} \to \mathcal{C}$ is a **sheaf** if

$$F(U) \longrightarrow \textstyle\prod_i F(U_i) \rightrightarrows \textstyle\prod_{i,j} F(U_i \times_U U_j)$$

is an equalizer for every open cover $\{U_i \to U\}$. A functor $G : \mathcal{E}\mathrm{ucl} \to \mathcal{D}$ is a **cosheaf** if $G^{\mathrm{op}} : \mathcal{E}\mathrm{ucl}^{\mathrm{op}} \to \mathcal{D}^{\mathrm{op}}$ is a sheaf.

Terminology 2.1.6. A faithful functor $|\_| : \mathcal{C} \to \mathcal{S}\mathrm{et}$, $X \to |X|$ is called a **concrete structure**. $|X|$ is called the **underlying set**. A category with a concrete structure is called a **concrete category**.

Practically, the objects of a concrete category are sets with structure and the morphisms are maps of sets that respect this structure. Most of the categories we first learn about are concrete. When $\mathcal{C}$ has a terminal object, the concrete structure is often given by the **functor of points** $X \mapsto \mathcal{C}(*, X)$. When the concrete structure is obvious, the notation $|\_|$ is often omitted by abuse of notation. For example, in the Definition 2.1.1 of diffeological spaces, we wrote $p : U \to X$, where the domain should really be denoted by $|U|$, the set underlying the euclidean space $U \in \mathcal{E}\mathrm{ucl}$.

Proposition 2.1.7. *The site $\mathcal{E}\mathrm{ucl}$ is **concrete**, which means that the following properties hold:*

(i) $\mathcal{E}\mathrm{ucl}$ has a terminal object $*$.

(ii) The functor of points $|\_| : \mathcal{E}\mathrm{ucl} \to \mathcal{S}\mathrm{et}$, $U \mapsto \mathcal{E}\mathrm{ucl}(*, U)$ is faithful.

(iii) For every cover $\{U_i \to U\}$, the induced map of sets $\coprod_i |U_i| \to |U|$ is surjective.

Proof. $\mathbb{R}^0$ is the terminal object. Property (ii) follows from the definition of smooth maps and (iii) from the definition of open covers. □

Proposition 2.1.8. *Let S be a set. Then the presheaf*

$$\begin{aligned} \bar{S} : \mathcal{E}\mathrm{ucl}^{\mathrm{op}} &\longrightarrow \mathcal{S}\mathrm{et} \\ U &\longmapsto \mathcal{S}\mathrm{et}\big(|U|, S\big) . \end{aligned}$$

is a sheaf.

Proof. Let $\{U_i \to U\}$ be an open cover. By Proposition 2.1.7 (iii), the map $\coprod_i |U_i| \to |U|$ is surjective. In the category of sets every epimorphism is effective, so that

$$\textstyle\coprod_i |U_i| \times_{|U|} \coprod_j |U_j| \rightrightarrows \coprod_i |U_i| \longrightarrow |U| \tag{2.2}$$

is a coequalizer. The set on the left can be rewritten as

$$\begin{aligned} \textstyle\coprod_i |U_i| \times_{|U|} \coprod_j |U_j| &\cong \textstyle\coprod_{i,j} |U_i| \times_{|U|} |U_j| \\ &\cong \textstyle\coprod_{i,j} |U_i \times_U U_j| , \end{aligned} \tag{2.3}$$

where we have first used that in $\mathcal{S}\mathrm{et}$ pullbacks commute with coproducts and then that the functor of points preserves limits.

By applying the functor $\mathcal{S}\mathrm{et}(\_\,, S)$ to the coequalizer (2.2) and using (2.3), we obtain the diagram

$$\bar{S}(U) \longrightarrow \prod_i \bar{S}(U_i) \rightrightarrows \prod_{i,j} \bar{S}(U_i \times_U U_j) \ .$$

Since (2.1) is a coequalizer and since the hom-functor $\mathcal{S}\mathrm{et}(\_\,, S)$ maps colimits to limits, this is an equalizer diagram. We conclude that $\bar{S}$ is a sheaf. $\square$

Let $X : \mathcal{E}\mathrm{ucl}^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$ be a presheaf and $U \in \mathcal{E}\mathrm{ucl}$. By applying X to a point $* \xrightarrow{u} U$, we obtain a map $X(u) : X(U) \to X(*)$, which we can evaluate at all $p \in X(U)$. This gives rise to the map

$$\begin{aligned} \alpha_U : X(U) &\longrightarrow \mathcal{S}\mathrm{et}\big(|U|, X(*)\big) \\ p &\longmapsto \big((* \xrightarrow{u} U) \mapsto X(u)(p)\big) \,. \end{aligned} \tag{2.4}$$

For every smooth map $f : U \to V$ in $\mathcal{E}\mathrm{ucl}$ and every $q \in X(V)$ we have

$$\begin{aligned} X(u)\big(X(f)(q)\big) &= \big(X(u) \circ X(f)\big)(q) \\ &= X(f \circ u)(q) \\ &= X\big(|f|(u)\big)(q) \,, \end{aligned}$$

where $|f| : |U| \to |V|$ maps the point $* \xrightarrow{u} U$ to $* \xrightarrow{u} U \xrightarrow{f} V$. This relation can be expressed by the commutative diagram

$$\begin{tikzcd} X(V) \arrow[r, "X(f)"] \arrow[d, "\alpha_V"'] & X(U) \arrow[d, "\alpha_U"] \\ \mathcal{S}\mathrm{et}\big(|V|, X(*)\big) \arrow[r, "|f|^*"'] & \mathcal{S}\mathrm{et}\big(|U|, X(*)\big) \end{tikzcd} \tag{2.5}$$

which shows that α_U is natural in U. In other words, we have a morphism

$$\alpha : X \longrightarrow \overline{X(*)}$$

of presheaves, where $\overline{X(*)}$ is defined as in Proposition 2.1.8.

Definition 2.1.9. A presheaf $X : \mathcal{E}\mathrm{ucl}^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$ is **concrete** if α is a monomorphism, that is, if the maps α_U defined in (2.4) are injective for all $U \in \mathcal{E}\mathrm{ucl}$. A sheaf is concrete if it is concrete as a presheaf. A morphism between concrete sheaves is a morphism of the underlying presheaves.

Example 2.1.10. The sheaf $\bar{S}$ of Proposition 2.1.8 is trivially concrete.

Theorem 2.1.11. *The category of diffeological spaces is equivalent to the category of concrete sheaves on* $\mathcal{E}\mathrm{ucl}$.

Proof. Let $X(U) \subset \mathcal{S}\mathrm{et}\big(|U|, |X|\big)$, $U \in \mathcal{E}\mathrm{ucl}$ be a diffeology on the set $|X|$. In a first step, we extend $U \mapsto X(U)$ to a presheaf. Axiom (iii) of Definition 2.1.1 implies that for every smooth map $f : U \to V$, the pullback $|f|^* : \mathcal{S}\mathrm{et}(|V|, |X|) \to \mathcal{S}\mathrm{et}(|U|, |X|)$

restricts to a map $X(f) : X(V) \to X(U)$. The functoriality of $|f|^*$ implies the functoriality of $X(f)$. It follows that X is a presheaf on $\mathcal{E}\mathrm{ucl}$.

Next we observe that, by Axiom (i) of Definition 2.1.1, all constant maps are plots, which implies that $X(\mathbb{R}^0) = |X|$, where, for clarity, we denote the terminal object in $\mathcal{E}\mathrm{ucl}$ by $\mathbb{R}^0 \equiv *$. It follows that the presheaf is concrete. From Axiom (ii) of Definition 2.1.1 it follows that X is a sheaf.

Let $Y(U) \subset \mathcal{S}\mathrm{et}\big(|U|, |Y|\big)$, $U \in \mathcal{E}\mathrm{ucl}$ be a diffeology on $|Y|$. A map $\varphi : |X| \to |Y|$ is a morphism of diffeological spaces if and only if for all $U \in \mathcal{E}\mathrm{ucl}$ the pushforward $\varphi_* : \mathcal{S}\mathrm{et}\big(|U|, |X|\big) \to \mathcal{S}\mathrm{et}\big(|U|, |Y|\big)$ restricts to a map $\hat{\varphi}_U : X(U) \to Y(U)$. Since the pushforward φ_* commutes with the pullback $|f|^* : |U| \to |V|$ for all smooth maps $f : U \to V$, $\hat{\varphi}_U$ is natural in U. In other words, $\hat{\varphi} : X \to Y$ is a morphism of presheaves. Since the pushforward is a functorial, so is $\varphi \mapsto \hat{\varphi}$. We conclude that we have constructed a functor from diffeological spaces to concrete sheaves.

For every point $u : \mathbb{R}^0 \to U$, we have the commutative diagram

$$\begin{tikzcd}
X(U) \arrow[r, "\hat{\varphi}_U"] \arrow[d, hook] \arrow[dd, bend right=60, "X(u)"'] & Y(U) \arrow[d, hook] \arrow[dd, bend left=60, "Y(u)"] \\
\mathcal{S}\mathrm{et}\big(|U|, |X|\big) \arrow[r, "\varphi_*"] \arrow[d, "\mathrm{ev}_u"'] & \mathcal{S}\mathrm{et}\big(|U|, |Y|\big) \arrow[d, "\mathrm{ev}_u"] \\
X(\mathbb{R}^0) \arrow[r, "\varphi = \hat{\varphi}_{\mathbb{R}^0}"'] & Y(\mathbb{R}^0)
\end{tikzcd}$$

where ev_u is the evaluation of a map $p : |U| \to |X|$ at u. Since this is commutative for every u, it follows that the morphism of presheaves $\hat{\varphi}$ is uniquely determined by φ and that every $\hat{\varphi}$ arises in this way. We conclude that our functor from diffeological spaces to concrete sheaves is full and faithful.

In the last step, we have to show that the functor is essentially surjective. Let X be a concrete sheaf on $\mathcal{E}\mathrm{ucl}$. Consider the collection of maps

$$\alpha_U\big(X(U)\big) \subset \mathcal{S}\mathrm{et}\big(|U|, X(\mathbb{R}^0)\big) \,.$$

for all $U \in \mathcal{E}\mathrm{ucl}$. The commutative diagram (2.5) for the terminal morphism $t : U \to \mathbb{R}^0$ is

$$\begin{tikzcd}
X(\mathbb{R}^0) \arrow[r, "X(t)"] \arrow[d, "\alpha_{\mathbb{R}^0}"'] & X(U) \arrow[d, "\alpha_U"] \\
\mathcal{S}\mathrm{et}\big(\mathbb{R}^0, X(\mathbb{R}^0)\big) \arrow[r, "|t|^*"'] & \mathcal{S}\mathrm{et}\big(|U|, X(\mathbb{R}^0)\big)
\end{tikzcd}$$

Since α_* is an isomorphism, it follows that the image of $|t|^*$ is contained in the image of α_U. Since every constant map $|U| \to X(\mathbb{R}^0)$ factors as $|U| \to \mathbb{R}^0 \to X(\mathbb{R}^0)$ through the terminal map, the constant maps are the image of $|t|^* : \mathcal{S}\mathrm{et}\big(\mathbb{R}^0, X(\mathbb{R}^0)\big) \to \mathcal{S}\mathrm{et}\big(|U|, X(\mathbb{R}^0)\big)$. We conclude that $\alpha_U(X(U))$ contains all constant maps, so that Axiom (i) of Definition 2.1.1 is satisfied. Axiom (ii) of Definition 2.1.1 follows from the sheaf property of X and Axiom (iii) from diagram (2.5). We conclude that $\alpha_U\big(X(U)\big)$, $U \in \mathcal{E}\mathrm{ucl}$ is a diffeology on $X(\mathbb{R}^0)$. Since α_U is a monomorphism for all U, there is a natural bijection $X(U) \cong \alpha_U(U)$. This shows that the concrete sheaves

$U \to X(U)$ and $U \mapsto \alpha_U(U)$ are isomorphic. Since the second sheaf arises from a diffeological space, it follows that every concrete sheaf X is isomorphic to one in the image of the functor. This shows that the functor from diffeological spaces to concrete sheaves is essentially surjective, which concludes the proof. □

Theorem 2.1.11 and its constructive proof enables us to go back and forth between two equivalent descriptions of diffeological spaces that each has its advantages. The geometric definition in terms of plots is best suited for explicit computations, the descriptions of examples, and the relation to the traditional methods of analysis and differential geometry. The categorical definition in terms of concrete sheaves is best suited for abstract structural considerations, the efficient understanding of universal properties, and the relation to more recent developments such as in homotopy theory or higher geometric structures.

2.1.2 Categorical properties of diffeological spaces

As is the case for every category of concrete sheaves, $\mathcal{D}\mathrm{flg}$ is a quasi-topos [BH11, Thm. 5.25]. This implies that it has a number of good categorical properties.

Proposition 2.1.12. *As any category of concrete sheaves on a concrete site,* $\mathcal{D}\mathrm{flg}$ *has the following properties:*

(a) $\mathcal{D}\mathrm{flg}$ *has all small limits and small colimits.*

(b) $\mathcal{D}\mathrm{flg}$ *is locally cartesian closed, that is, for every object X in* $\mathcal{D}\mathrm{flg}$ *the overcategory* $\mathcal{D}\mathrm{flg}\!\downarrow\! X$ *is cartesian closed.*

(c) Strong monomorphisms and strong epimorphisms are effective.

(d) (Strong) monomorphisms and (strong) epimorphisms are stable under pullback.

(e) $\mathcal{D}\mathrm{flg}$ *is quasiadhesive, that is, the pushout of a strong monomorphism is a strong monomorphism and the pushout square is a pullback square.*

(f) The initial object is strict, that is, every morphism $X \to \emptyset$ is an isomorphism.

(g) Coproducts are disjoint, that is, $X \to X \sqcup Y \leftarrow Y$ are monomorphisms and $X \times_{X \sqcup Y} Y \cong \emptyset$.

(h) The functor of points $\mathcal{D}\mathrm{flg} \to \mathcal{S}\mathrm{et}$, $X \to \mathcal{D}\mathrm{flg}(*, X)$ *is faithful. It has a left and a right adjoint, so that it preserves limits and colimits.*

Before we explain the statements of Proposition 2.1.12 in more detail, we state an additional property that is a consequence of the site $\mathcal{E}\mathrm{ucl}$ being subcanonical.

Proposition 2.1.13. *Every representable presheaf on* $\mathcal{E}\mathrm{ucl}$ *is a concrete sheaf.*

Proof. Let V be an object of $\mathcal{E}\mathrm{ucl}$. Since the functor of points is faithful, the map

$$\mathcal{E}\mathrm{ucl}(U,V) \longrightarrow \mathcal{S}\mathrm{et}(|U|,|V|) = \mathcal{S}\mathrm{et}\big(|U|, \mathcal{E}\mathrm{ucl}(*,V)\big)$$

is injective for all U. This shows that the presheaf $U \mapsto \mathcal{E}\mathrm{ucl}(U,V)$ represented by V is concrete. $\mathcal{E}\mathrm{ucl}$ is subcanonical, which means that (2.1) is a coequalizer. Since the hom-functor preserves colimits, applying $\mathcal{E}\mathrm{ucl}(\_,V)$ yields an equalizer, so that $\mathcal{E}\mathrm{ucl}(\_,V)$ is a sheaf. $\square$

Let $I : \mathcal{D}\mathrm{flg} \to \mathcal{S}\mathrm{et}^{\mathcal{E}\mathrm{ucl}^{\mathrm{op}}}$ denote the inclusion of concrete sheaves into the category of presheaves. Being a concrete presheaf is a property of a presheaf, so that I is injective. By definition, a morphism of concrete sheaves is a morphism of presheaves, so that I is full and faithful. Proposition 2.1.13 states that the Yoneda embedding $\mathbb{Y} : \mathcal{E}\mathrm{ucl} \to \mathcal{S}\mathrm{et}^{\mathcal{E}\mathrm{ucl}^{\mathrm{op}}}$, $U \mapsto \mathcal{E}\mathrm{ucl}(\_,U)$ takes its values in the image of I, so that we have a commutative diagram

$$\begin{tikzcd} & \mathcal{S}\mathrm{et}^{\mathcal{E}\mathrm{ucl}^{\mathrm{op}}} \\ \mathcal{E}\mathrm{ucl} \arrow[ur, "\mathbb{Y}"] \arrow[r, "y"'] & \mathcal{D}\mathrm{flg} \arrow[u, "I"'] \end{tikzcd}$$

where y is the Yoneda embedding with restricted codomain. The sheaf yU is given by $(yU)(V) = (\mathbb{Y}U)(V) = \mathcal{E}\mathrm{ucl}(V,U)$.

Proposition 2.1.14. *The functor $y : \mathcal{E}\mathrm{ucl} \to \mathcal{D}\mathrm{flg}$ is injective, full, and faithful.*

Proof. Since the Yoneda embedding is injective, so is y. As already explained, I is full and faithful. By the Yoneda lemma, Y is full and faithful. Since both I and Y are full and faithful and $Iy = Y$, it follows that y is full and faithful. $\square$

Since I is full and faithful, the Yoneda lemma implies that the evaluation of the concrete sheaf $X \in \mathcal{D}\mathrm{flg}$ on $U \in \mathcal{E}\mathrm{ucl}$ is given by

$$\begin{aligned} X(U) &\cong \mathcal{S}\mathrm{et}^{\mathcal{E}\mathrm{ucl}^{\mathrm{op}}}(YU, IX) \cong \mathcal{S}\mathrm{et}^{\mathcal{E}\mathrm{ucl}^{\mathrm{op}}}(IyU, IX) \\ &\cong \mathcal{D}\mathrm{flg}(yU, X)\,. \end{aligned} \tag{2.6}$$

It follows that limits in $\mathcal{D}\mathrm{flg}$ are computed pointwise and that I preserves limits. By the adjoint functor theorem, I has a left adjoint,

$$K : \mathcal{S}\mathrm{et}^{\mathcal{E}\mathrm{ucl}^{\mathrm{op}}} \rightleftarrows \mathcal{D}\mathrm{flg} : I\,, \tag{2.7}$$

which was computed and studied in [BH11, Sec. 5.3]. Explicitly, K is given by a procedure called concretization followed by the Grothendieck plus construction. From this construction it follows that if a presheaf on $\mathcal{E}\mathrm{ucl}$ is already a concrete sheaf, that is, if it is in the image of I, then both constructions do nothing. It follows that the left adjoint K is a retract, $KI \cong \mathrm{id}_{\mathcal{D}\mathrm{flg}}$, which implies that the colimit of a diagram $X : \mathcal{I} \to \mathcal{D}\mathrm{flg}$ can be computed as

$$\begin{aligned} \operatorname*{colim}_{i\in\mathcal{I}} X_i &\cong \operatorname*{colim}_{i\in\mathcal{I}} KIX_i \\ &\cong K \operatorname*{colim}_{i\in\mathcal{I}} IX_i\,, \end{aligned}$$

that is, by first computing the colimit in presheaves and then applying K. As a further consequence, it can be shown that y is dense:

Proposition 2.1.15 (Prop. 51 in [BH11]). *Every $X \in \mathcal{D}\mathrm{flg}$ is the colimit of $y \downarrow X \to \mathcal{E}\mathrm{ucl} \to \mathcal{D}\mathrm{flg}$, which we will write as*

$$X \cong \operatorname*{colim}_{yU \to X} yU \,. \tag{2.8}$$

Proof. As is the case for any presheaf, $IX \cong \mathrm{colim}_{YU \to IX} YU$. Since I is full and faithful, the morphisms $IyU \to IX$ are in bijection with the morphisms $yU \to X$, so that the colimit can be written as $IX \cong \mathrm{colim}_{yU \to X} IyU$. From this, we get

$$\begin{aligned} X \cong KIX \cong K \operatorname*{colim}_{yU \to X} IyU \cong \operatorname*{colim}_{yU \to X} KIyU \\ \cong \operatorname*{colim}_{yU \to X} yU \,, \end{aligned}$$

where we have used that K is a left ajoint, so that it preserves colimits. □

Terminology 2.1.16. The category $y \downarrow X$ is called the **category of plots** of X.

Warning 2.1.17. It is customary and convenient to identify notationally the domain of a plot $U \in \mathcal{E}\mathrm{ucl}$ with the diffeological space $yU \in \mathcal{D}\mathrm{flg}$. In this chapter, we deal with subtleties of Kan extensions along y where this identification would invite wrong proofs by notation (a trap the author has fallen into more than once). Therefore, we will always spell out the embedding y.

2.1.3 Categorical properties in terms of plots

In Proposition 2.1.12, we have seen a long list of good properties of the category $\mathcal{D}\mathrm{flg}$. In this section we will spell out some of the properties explicitly. For this the following fact is useful.

Remark 2.1.18. The diffeologies on a given set X are partially ordered by inclusion $D \subset D'$ if $D(U) \subset D'(U)$ for all $U \in \mathcal{E}\mathrm{ucl}$. The diffeology D is then called **smaller** or **finer** than D' and D' **larger** or **coarser** than D. This is in analogy to topology, where a topology T on X is finer than T', if there are fewer T-continuous maps than T'-continuous maps to X. With the partial order the diffeologies on X form a complete lattice [IZ13, Sec. 1.25]. The infimum of a family $\{D_i\}$ is given by the intersection $D_{\inf}(U) := \bigcap_i D_i(U)$. The supremum is given by the intersection of all diffeologies that contain all D_i.

Functional diffeology Proposition 2.1.12 states that $\mathcal{D}\mathrm{flg}$ is locally cartesian closed. This means that for every object $X \in \mathcal{D}\mathrm{flg}$, the overcategory $\mathcal{D}\mathrm{flg} \downarrow X$ is cartesian closed. This means that $\mathcal{D}\mathrm{flg} \downarrow X$ has all finite products and all exponential objects. As is the case in any overcategory, the product in $\mathcal{D}\mathrm{flg} \downarrow X$ is the pullback over X. That is, the product of $A \to X$ and $B \to X$ is $A \times_X B \to X$. The empty product, which is the terminal object, is $\mathrm{id}_X : X \to X$. The exponential by $A \to X$ is the right adjoint to the functor $A \times_X \_$. Having a right adjoint implies that $A \times_X \_$ preserves colimits.

When $X = *$, then $\mathcal{D}\mathrm{flg} \downarrow * \cong \mathcal{D}\mathrm{flg}$ and the fiber product is the product in $\mathcal{D}\mathrm{flg}$. We denote the exponential objects in $\mathcal{D}\mathrm{flg}$ by

$$\underline{\mathcal{D}\mathrm{flg}}(X, Y) \equiv Y^X$$

and call them the **diffeological mapping spaces**. The diffeology, which is determined by the universal property

$$\mathcal{D}\mathrm{flg}\big(yU, \underline{\mathcal{D}\mathrm{flg}}(X,Y)\big) \cong \mathcal{D}\mathrm{flg}(yU \times X, Y)\,,$$

is called the **functional diffeology**. Its plots are the smooth homotopies of morphisms of diffeological spaces.

Discrete and indiscrete diffeology By Proposition 2.1.12, the forgetful functor $\mathcal{D}\mathrm{flg} \to \mathcal{S}\mathrm{et}$, $X \mapsto \mathcal{D}\mathrm{flg}(*, X)$ has left and right adjoints. The left adjoint equips a set S with the discrete diffeology, for which the plots are the locally constant maps. The right adjoint equips S with the trivial diffeology, for which all maps are plots. In other words, the discrete diffeology on a set is the free diffeology, the indiscrete diffeology is the cofree diffeology.

Notation 2.1.19. Let S be a set. We will denote by $\ddot{S}$ the discrete diffeology on S and by $\bar{S}$ the indiscrete diffeology. (The dots remind us of the discrete points, the bar of its opposite.) Since the indiscrete diffeology on S is the sheaf defined in Proposition 2.1.8, we use the same notation. For the set $|X|$ underlying a diffeological space X, we will, for lighter notation, drop the vertical bars and write $\ddot{X} \equiv |\ddot{X}|$ and $\bar{X} \equiv \overline{|X|}$.

Inductions and subductions Some of the statements of Proposition 2.1.12 involve strong monomorphisms and epimorphisms, which we will explain in more detail.

Proposition 2.1.20. *A smooth map $X \to Y$ of diffeological spaces is a monomorphism (an epimorphism) if and only if it is injective (surjective).*

Proof. Let $f : X \to Y$ be a morphism of diffeological spaces. The forgetful functor $\mathcal{D}\mathrm{flg} \to \mathcal{S}\mathrm{et}$ is faithful, so it reflects monomorphisms and epimorphisms. To "reflect" means that, if the map of sets $|f| : |X| \to |Y|$ is a monomorphism or epimorphism, then so is f. By Proposition 2.1.12, the functor of points preserves limits, so that it preserves monomorphisms, and it preserves colimits, so that it preserves epimorphisms. That is, if f is a monomorphism or an epimorphism, then so is $|f|$. Since the monomorphisms (epimorphisms) in $\mathcal{S}\mathrm{et}$ are the injective (surjective) maps, the proposition follows. □

A morphism $i : X \to Y$ is said to have the **right lifting property** with respect to a morphism $p : A \to B$ or, equivalently, p is said to have the **left lifting property** with respect to i, if every commutative diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "p"'] & X \arrow[d, "i"] \\ B \arrow[r, "g"'] \arrow[ur, dashed, "\exists"] & Y \end{tikzcd} \tag{2.9}$$

has a diagonal lift.

Example 2.1.21. In $\mathcal{S}\mathrm{et}$, all surjective maps have the left lifting property with respect to all injective maps.

We can now define three important properties of monomorphisms and epimorphisms. While this makes sense in any category, it will be of particular relevance in diffeological spaces.

- A monomorphism is called **strong**, if it has the right lifting property with respect to all epimorphisms. Dually, an epimorphism is called **strong** if it has the left lifting property with respect to all monomorphisms.
- A monomorphism $X \to Y$ is called **regular** if it is the equalizer $X \to Y \rightrightarrows Z$ of a pair of parallel morphisms. Dually, an epimorphism $Y \to Z$ is **regular** if it is the coequalizer $X \rightrightarrows Y \to Z$ of a pair of parallel arrows.
- A monomorphism $X \to Y$ is called **effective** if $X \to Y \rightrightarrows Y \sqcup_X Y$ is an equalizer. Dually, an epimorphism $X \to Y$ is called **effective** if $X \times_Y X \rightrightarrows X \to Y$ is a coequalizer.

Proposition 2.1.22. *Let $f : X \to Y$ be a monomorphism or an epimorphism of diffeological spaces. We have the following implications: f is effective $\Rightarrow$ f is regular $\Rightarrow$ f is strong.*

Proof. Since an effective monomorphism is by definition an equalizer, it is regular. For a regular monomorphism, it follows from the universal property of the equalizer that it has the right lifting property with respect to epimorphisms. The proof for epimorphisms is dual. □

Proposition 2.1.23. *Strong monomorphisms and strong epimorphisms have the following properties:*

(i) The composition of strong monomorphisms is a strong monomorphism. If $f \circ g$ is a strong monomorphism, then g is a strong monomorphism.

(ii) The composition of strong epimorphisms is a strong epimorphism. If $f \circ g$ is a strong epimorphism, then f is a strong epimorphism.

(iii) If $f \circ g$ is an isomorphism, then f is a strong epimorphism and g is a strong monomorphism. In other words, split monomorphisms and split epimorphisms are strong.

(iv) If a strong monomorphism is an epimorphism, then it is an isomorphism. Dually, if a strong epimorphism is a monomorphism, then it is an isomorphism.

Proof. The proof is a straightforward exercise in basic category theory. □

Definition 2.1.24. Let Y be a diffeological space and $S \to |Y|$ a map of sets, which we can view as morphism $f : \bar{S} \to \bar{Y}$, where we recall that $\bar{S}$ and $\bar{Y}$ denotes the indiscrete diffeology. Then

$$f^*Y := Y \times_{\bar{Y}} \bar{S}$$

is the **pullback diffeology** on S.

Proposition 2.1.25. *Let Y be a diffeological space and $S \to |Y|$ a map of sets, which we can view as morphism $f : \bar{S} \to \bar{Y}$. The following are equivalent:*

(i) f is a monomorphism, that is, injective.

(ii) $f^*Y \to Y$ is a strong monomorphism.

Proof. Assume that the map $f^*Y \to Y$ is a strong monomorphism. It follows that the underlying map of sets is injective. We conclude that (ii) implies (i).

Assume that f is injective, so that $f : \bar{S} \to \bar{Y}$ is a monomorphism. Consider a commutative diagram

$$\begin{tikzcd}
A \arrow[r] \arrow[d, two heads] & Y \times_{\bar{Y}} \bar{S} \arrow[r] \arrow[d] & \bar{S} \arrow[d, hook, "f"] \\
B \arrow[r] \arrow[ur, dashed, "\exists\psi"] \arrow[urr, dashed, "\exists\varphi"'] & Y \arrow[r] & \bar{Y}
\end{tikzcd}$$

where $A \to B$ is an epimorphism. Since f is a monomorphism and $A \to B$ an epimorphism, a unique lift φ exists in $\mathcal{S}\mathrm{et}$. Since the diffeology on $\bar{S}$ is cofree, φ is smooth. The existence of a unique lift ψ follows from the universal property of the pullback. We conclude that $f^*Y \to Y$ is a strong monomorphism, so that (i) induces (ii). $\square$

Corollary 2.1.26. *A monomorphism $f : X \to Y$ of diffeological spaces is strong if and only if the morphism $X \to f^*Y$ given by the universal property of the pullback is an isomorphism.*

Corollary 2.1.27. *Let $\Omega = \{0, 1\}$ equipped with the indiscrete diffeology. Then there is a natural bijection between the images of strong monomorphisms to X (strong subobjects) and morphisms $X \to \Omega$.*

Proof. Let $t : * \to \Omega$, $* \mapsto 1$ be the “truth” map. The image of a monomorphism $f : A \to X$ can be identified with the characteristic function $\chi_f : X \to \Omega$ that maps x to 1 if it is in the image of f and to 0 otherwise. The pullback of t along χ_f is isomorphic to A. The only difference to an elementary topos is that the monomorphism must be strong. $\square$

Definition 2.1.28. Let X be a diffeological space and $|X| \to S$ a map of sets, which we view as morphism $f : \ddot{X} \to \ddot{S}$, where we recall that $\ddot{S}$ and $\ddot{X}$ denotes the discrete diffeology. Then

$$f_*X := \ddot{S} \sqcup_{\ddot{X}} X$$

is the **pushforward diffeology** on S.

Proposition 2.1.29. *Let X be a diffeological space and $|X| \to S$ a map of sets, which we can view as morphism $f : \ddot{X} \to \ddot{S}$. The following are equivalent:*

(i) f is an epimorphism, that is, surjective.

(ii) $X \to f_*X$ is a strong epimorphism.

Proof. The proof is dual to that of Proposition 2.1.25. $\square$

Corollary 2.1.30. *An epimorphism $f : X \to Y$ of diffeological spaces is strong if and only if the morphism $f_*X \to Y$ given by the universal property of the pushout is an isomorphism.*

Terminology 2.1.31. The pullback diffeology along a monomorphism is also called the **subspace diffeology**, the pushforward diffeology along an epimorphism the **quotient diffeology**.

Proposition 2.1.32. *The pullback diffeology f^*Y is the largest diffeology on the set $|X|$ such that $|f|$ is smooth. The pushforward diffeology f_*X is the smallest diffeology on $|Y|$ such that $|f|$ is smooth.*

Proof. Let X' be a diffeology on the set $|X|$ such that $|f|$ is smooth. By the universal property of the pullback, there is a morphism $X' \to f^*Y$, which is the identity on $|X|$. It follows that $X'(U) \subset (f^*X)(U)$ for all $U \in \mathcal{E}\mathrm{ucl}$.

Let Y' be a diffeology on the set $|Y|$ such that $|f|$ is smooth. By the universal property of the pushforward, there is a morphism $f_*X \to Y'$, which is the identity on $|Y|$. It follows that the $(f_*X)(U) \subset Y'(U)$ for all $U \in \mathcal{E}\mathrm{ucl}$.

A more elaborate proof can be found in in [IZ13, Sec. 1.26] for the pullback diffeology and in [IZ13, Sec. 1.43] for the pushforward diffeology. □

Terminology 2.1.33. A monomorphism $f : X \to Y$ of diffeological spaces such that the diffeology on X is the pullback diffeology is called an **induction** [IZ13, Sec. 1.29]. If f is an epimorphism such that the diffeology on Y is the pushforward diffeology, it is called a **subduction** [IZ13, Sec. 1.46].

Corollary 2.1.26 shows that the inductions are the strong monomorphisms and Corollary 2.1.30 that the subductions are the strong epimorphisms. This was first proved in Prop. 34 and Prop. 37 of [BH11].

Proposition 2.1.34. *The following are equivalent:*

(i) *$f : X \to Y$ is a subduction.*

(ii) *A map $p : |U| \to |Y|$ is a plot if and only if every $u_0 \in U$ has an open neighborhood $U_0 \subset U$ such that the restriction of p to $|U_0|$ lifts to a plot of X, that is, there is a plot $q : |U_0| \to |X|$ such that $p|_{|U_0|} = |f| \circ q$.*

Proof. Assume that f is surjective. Let us denote by $D(U) \subset \mathcal{S}\mathrm{et}(|U|, |Y|)$ for all $U \subset \mathcal{E}\mathrm{ucl}$ the maps that have the local lifting property of (ii). Since f is surjective, $D(U)$ contains all constant maps. Since the lifting property is local, $p : |U| \to |X|$ is in $D(U)$ if and only if all its restrictions $p|_{|U_i|}$ to an open cover $\{U_i \to U\}$ have the lifting property. If p has the lifting property and $\varphi : V \to U$ is a smooth map, a local lift $q : |U_0| \to |X|$ for U_0 an open neighborhood of $u_0 = \varphi(v_0)$ gives rise to a local lift $(q \circ |\varphi|)|_{|\varphi^{-1}(U_0)|}$ on the open neighborhood $V_0 = \varphi^{-1}(U_0)$ of v_0. We conclude that $D(U)$ is a diffeology on $|Y|$.

Let $f : X \to Y$ be a morphism of diffeological spaces. By definition this means that for every plot $q : |U| \to |X|$, the map $p = |f| \circ q : |U| \to |Y|$ is a plot of Y. Since p has the lifting property, $p \in D(U)$. We conclude that $D(U) \subset Y(U)$. By

Proposition 2.1.32 the pushforward diffeology is the smallest diffeology such that $|f|$ is smooth. We conclude that if $|f|$ is surjective, then D is the pushforward diffeology.

Assume (ii). Since every constant map $|U| \to |Y|$ is a plot, it has the local lifting property, which implies that f is surjective. It follows that $D(U)$ is the pushforward diffeology, so that (i) follows.

Assume (i). This means that f is surjective and Y has the pushforward diffeology. We have proved that the pushforward diffeology is D, so that (ii) follows. □

It follows from Equation (2.6) that a plot $p : |U| \to |Y|$ can be identified with a morphism $p : yU \to Y$. The local lifting property of subductions can then be expressed by the commutative diagram of diffeological spaces

$$\begin{tikzcd} & yU_0 \arrow[r, "q"] \arrow[d, hook] & X \arrow[d, "f"] \\ * \arrow[ur] \arrow[r, "u_0"'] & yU \arrow[r, "p"'] & Y \end{tikzcd}$$

for all $u_0 \in U$ and an open neighborhood $U_0 \subset U$ of u_0.

Proposition 2.1.35. *Let $i : Y \to Z$ be an induction. Let X be a diffeological space and $f : |X| \to |Y|$ a map of sets. If the composition $|i| \circ f : |X| \to |Z|$ is smooth, that is, a morphism $i \circ f : X \to Z$ of diffeological spaces, then f is smooth.*

Proof. Recall that $\ddot{X}$ denotes the set X with the discrete diffeology, which is the free diffeology on $|X|$, so that f can be identified with a unique morphism $f : \ddot{X} \to Y$. Moreover, there is a morphism $\ddot{X} \to X$ with id_X as underlying map of sets, which is surjective so that it is an epimorphism. Let $\varphi : X \to Z$ be the morphism of diffeological spaces with underlying map of sets $|\varphi| = |i| \circ f$. We have the commutative diagram

$$\begin{tikzcd} \ddot{X} \arrow[r, "f"] \arrow[d, two heads] & Y \arrow[d, hook, "i"] \\ X \arrow[r, "i\circ f"'] \arrow[ur, dashed, "\exists"] & Z \end{tikzcd}$$

An induction is a strong monomorphism, so that there is a diagonal lift. This shows that f is smooth. □

Proposition 2.1.36. *Let $r : X \to Y$ be a subduction. Let Z be a diffeological space and $f : |Y| \to |Z|$ a map of sets. If the composition $f \circ |r| : |X| \to |Z|$ is smooth, that is, a morphism $f \circ r : X \to Z$ of diffeological spaces, then f is smooth.*

Proof. The proof is dual to the proof of Proposition 2.1.35. □

D-topology

Definition 2.1.37 (Sec. 2.8 in [IZ13])**.** The ***D*-topology** on a diffeological space is the finest topology (on the underlying set) such that every plot is continuous.

Explicitly, a subset $Y \subset X$ is open in the D-topolgy if and only if for every plot $p : yU \to X$, the preimage $p^{-1}(Y) \subset U$ is open. Every morphism of diffeological spaces is continuous with respect to the D-topologies. An open subset $S \subset X$ of a diffeological space is naturally equipped with the subspace diffeology, so that the inclusion $i : S \to X$ is an open induction.

The D-topology is determined by the smooth curves only, so that many different diffeologies induce the same topology [CSW14, Thm. 3.7]. The discrete diffeology induces the discrete topology. *** Mapping a diffeology on X to the induced topology is left adjoint to mapping a topology to the continuous diffeology [CSW14, Prop. 3.3]. In general, neither the unit nor the counit of the adjunction is an isomorphism.

2.1.4 Computing limits and colimits

Computing limits in $\mathcal{D}\mathrm{flg}$ is straightforward, as the following result shows.

Proposition 2.1.38. *The limit of a diagram $X : \mathcal{I} \to \mathcal{D}\mathrm{flg}$, $i \mapsto X_i$ is given by the set $\lim_{i\in\mathcal{I}} |X_i|$ with the diffeology for which a map $p : |U| \to \lim_i |X_i|$ is a plot if and only if the compositions with all maps of the limit cone,*

$$|U| \xrightarrow{p} \lim_{i\in\mathcal{I}} |X_i| \xrightarrow{\mathrm{pr}_i} |X_i| \,,$$

are plots.

Proof. The functor of points $X \mapsto |X| = \mathcal{D}\mathrm{flg}(*, X)$ preserves limits, so that $|\lim_i X_i| \cong \lim_i |X_i|$. Moreover, the sheaf of a diffeological space X is given by $X(U) = \mathcal{D}\mathrm{flg}(yU, X)$. It follows that

$$\begin{aligned}(\lim_i X_i)(U) &\cong \mathcal{D}\mathrm{flg}(yU, \lim_i X_i) \cong \lim_i \mathcal{D}\mathrm{flg}(yU, X_i) \\ &\cong \lim_i X_i(U)\,.\end{aligned}$$

In other words, a plot to the limit is given by a collection of plots to all X_i. □

Example 2.1.39. Let $X_1 \xrightarrow{f} Y \xleftarrow{g} X_2$ be morphisms of diffeological spaces. A map $p : |U| \to |X_1 \times_Y X_2|$, is a plot if and only if $p_1 = |\mathrm{pr}_1| \circ p$, $p_2 = |\mathrm{pr}_2| \circ p$, and $|f| \circ p_1 = |g| \circ p_2$ are plots.

The computation of colimits is more involved. Every colimit can be computed as a coproduct followed by a coequalizer [ML98, Thm. X.5.3], [KS06, Prop. 2.4], so that it suffices to consider these two cases. We begin by a description of coproducts in $\mathcal{D}\mathrm{flg}$.

Proposition 2.1.40. *The coproduct of a family of diffeological spaces $\{X_i\}_{i\in I}$ is given by the coproduct of the underlying sets*

$$\left|\textstyle\coprod_i X_i\right| = \textstyle\coprod_i |X_i|$$

with the following diffeology. Let $U \subset \mathbb{R}^n$, $n \geq 0$ be an open subset. A map $p : |U| \to \left|\coprod_i X_i\right|$ is a plot if and only if every $u \in U$ has a neighborhood $U_i \subset U$

such that the restriction of p to $|U_i|$ factors through a plot $p_i : yU_i \to X_i$. That is, we have the following commutative diagram:

$$\begin{tikzcd} yU_i \arrow[r, "p_i"] \arrow[d, hook] & X_i \arrow[d] \\ yU \arrow[r, "p"'] & \coprod_i X_i \end{tikzcd}$$

Proof. Let $X = \coprod_i X_i$ and let $p : yU \to X$ be a plot. Since equipping X_i with the induced topology is a left adjoint functor, it preserves coproducts. It follows that the induced topology on X is the coproduct topology. By definition of the induced topology, $p : yU \to X$ is continuous, so that $U_i := p^{-1}(X_i)$ is open and closed, so that we can identify U with $\coprod_i U_i$. The restriction of p to U_i factors through a plot $p_i : yU_i \to X_i$. Since a point $u \in U$ is contained in some U_i, we obtain the commutative diagram of the proposition. □

Remark 2.1.41. Since every $U \in \mathcal{E}\mathrm{ucl}$ is second countable, it follows that only countably many of the U_i in Proposition 2.1.40 can be non-empty. In particular, the image of a plot is concentrated in countably many components of the coproduct. Moreover, if U is connected, then the image of p is concentrated in a single component X_i.

Proposition 2.1.42. *The coequalizer of parallel arrows $X \rightrightarrows Y$ of diffeological spaces is given by the coequalizer of sets,*

$$|X| \rightrightarrows |Y| \xrightarrow{h} C\,,$$

with the pushforward diffeology h_*Y on C.

Proof. Since the forgetful functor $X \to |X|$ preserves colimits, the underlying set of the coequalizer is the coequalizer of the underlying sets. The morphism h of the coequalizer is a regular epimorphism, so a subduction. □

The colimit of any functor $X : \mathcal{I} \to \mathcal{D}\mathrm{flg}$, $i \to X_i$ can be computed by a coequalizer and products as [ML98, Thm. X.5.3]

$$\coprod_{f \in \mathrm{Mor}(\mathcal{I})} X_{\mathrm{dom} f} \rightrightarrows \coprod_{i \in \mathcal{I}} X_i \longrightarrow \operatorname{colim} X\,, \tag{2.10}$$

where the two arrows on the left are given by the morphisms

$$\begin{tikzcd} & X_{\mathrm{dom} f} \\ X_{\mathrm{dom} f} \arrow[ur, "\mathrm{id}"] \arrow[dr, "Xf"'] & \\ & X_{\mathrm{codom} f} \end{tikzcd}$$

for all morphisms f of $\mathcal{I}$. We can now combine Propositions 2.1.40 and 2.1.42 which yields the following procedure to compute a colimit in $\mathcal{D}\mathrm{flg}$.

Proposition 2.1.43. *The colimit of a functor $X : \mathcal{I} \to \mathcal{D}\mathrm{flg}$ is the colimit of sets,*

$$|\operatorname*{colim}_i X_i| = \operatorname*{colim}_i |X_i|$$

with the following diffeology. Let $U \subset \mathbb{R}^n$, $n \geq 0$ be an open subset. A map $p : |U| \to |\operatorname{colim} X|$ is a plot if and only if every $u_0 \in U$ has a neighborhood $U_0 \subset U$ such that the restriction of p to U_0 factors through a plot $p_0 : yU_0 \to X_i$ of some X_i. That is, we have a commutative diagram

$$\begin{tikzcd} yU_0 \arrow[r, "p_0"] \arrow[d, hook] & X_i \arrow[d] \\ yU \arrow[r, "p"'] & \operatorname{colim}_i X_i \end{tikzcd}$$

Remark 2.1.44. The colimit in sets in Proposition 2.1.43 is given explicitly by the quotient

$$|\operatorname*{colim}_i X_i| = \left|\textstyle\coprod_{i\in\mathcal{I}} X_i\right| / \sim ,$$

where $\sim$ is the equivalence relation generated by the relations

$$x \sim y :\Leftrightarrow \exists\, f \in \operatorname{Mor}(\mathcal{I}) : (Xf)x = y ,$$

for all $x, y \in \coprod_i |X_i|$.

Example 2.1.45. Let $X \xleftarrow{f} Z \xrightarrow{g} Y$ be morphisms of diffeological spaces. The pushout $X \sqcup_Z Y$ is the coequalizer

$$Z \underset{g}{\overset{f}{\rightrightarrows}} X \sqcup Y \xrightarrow{h} X \sqcup_Z Y \quad .$$

A map $p : |U| \to |X \sqcup_Z Y|$ is a plot if every $u_0 \in U$ has a neighborhood U_0 such that there is a plot $q : yU_0 \to X$ or a plot $q : yU_0 \to Y$ such that $p|_{U_0} = h \circ q$.

2.2 The tangent functor

2.2.1 Differential forms and tangent vectors

The de Rham complex A large part of the structure of differential geometry is local, which means that they are first defined on the open subsets $U \subset \mathbb{R}^n$ of the charts and then glued together on an atlas. For example, the de Rham complex of differential forms is defined on U as the free graded antisymmetric $C^\infty(U)$-algebra generated by the coordinate differentials $\{du^1, \dots, du^n\}$ with the usual differential. With the pullback of forms along smooth maps $f : U \to V$ between charts, this yields a contravariant functor $U \to \Omega(U)$ from $\mathcal{E}\mathrm{ucl}$ to differential graded rings.

A manifold is obtained by glueing together its charts, which can be written as

$$M \cong \operatorname*{colim}_{U \to M} U ,$$

where the colimit in manifolds is taken over the maximal atlas. The glueing of the local de Rham complex model to the de Rham complex on the manifold can be written as

$$\Omega(M) \cong \lim_{U \to M} \Omega(U)\,,$$

where the colimit in manifolds becomes a limit in differential graded algebras because Ω is contravariant.

We have proved in Proposition 2.1.15, that a diffeological space X is given by the colimit

$$X \cong \operatorname*{colim}_{yU \to X} yU\,,$$

where the colimit is taken over the category $y{\downarrow}X$ of plots of X. In other words, a diffeological spaces is obtained by glueing together its plots. This suggest that for the definition of the de Rham complex of X we simply replace the charts by plots,

$$\Omega(X) := \lim_{yU \to X} \Omega(U)\,, \tag{2.11}$$

where the limit in differential graded rings is taken over $y{\downarrow}X$.

Explicitly, an element $\alpha \in \Omega(X)$ is given by a family $\{\alpha_p\}_{p:yU\to X}$ of forms $\alpha_p \in \Omega(U)$ on the domains of all plots $p : yU \to X$ such that for every smooth map $f : V \to U$,

$$f^*\alpha_p = \alpha_{f^*p}\,,$$

where $f^*p = p \circ f : yV \to X$ is the pullback plot. The differential and ring structure on $\Omega(X)$ is given by the differential and ring structure of the forms α_p. This explicit description is the standard definition of the de Rham complex on diffeological spaces [IZ13, Sec. 6.28].

The tangent functor The tangent functor is given on the domains $U \subset \mathbb{R}^n$ of charts by

$$TU := U \times \mathbb{R}^n$$

and on smooth maps $f : U \to V \subset \mathbb{R}^m$ by

$$\begin{aligned} Tf : TU &\longrightarrow TV \\ (u, \dot{u}^i) &\longmapsto \Big(f(u), \frac{\partial f^\alpha}{\partial u^i}\dot{u}^i\Big)\,, \end{aligned}$$

where we use the summation convention for the repeated index i and where $1 \leq \alpha \leq m$. Its extension to a manifold M is given by

$$TM \cong \operatorname*{colim}_{U \to M} TU\,,$$

where the colimit in smooth manifolds is taken over the charts of the maximal atlas of M. Since $\mathcal{M}\mathrm{fld}$ is not cocomplete it has to be shown that this colimit exists.

For a diffeological space X we replace the category of charts by the category of plots and obtain the definition

$$TX := \operatorname*{colim}_{yU \to X} yTU\,, \tag{2.12}$$

where the colimit is taken in diffeological spaces.

Since the colimit (2.12) defining the tangent functor is taken in diffeological spaces, the tangent fibers are generally not vector spaces. The geometric reason is that the tangent fiber at x is an infinitesimal model of the space at x. More precisely, T_xX describes the directions in which smooth paths can leave or enter x with finite velocity. The vector space structure of the tangent fibers of a manifold reflects the fact that the space itself looks locally like a vector space.

Other definitions of the tangent functor In the literature, a number of inequivalent definitions for the tangent functor have been used, that force the fiber to be a vector space. For an overview see [CW16]. For a diffeological space that is not a manifold, forcing the tangent fiber to be a vector space obscures some of the local geometric information. For example, the tangent vector space at 0 of three smooth lines through the origin of $\mathbb{R}^2$ is $\mathbb{R}^3$, whereas the tangent fiber as defined in (2.12) is an accurate local model of the diffeological space.

There is another class of definitions of tangent vectors and tangent bundles that involve the smooth real-valued functions on the diffeological space. An *external* tangent vector is a derivation of the germ of smooth functions at a point. Many definitions require such a tangent vector to be represented by a smooth path and use its action on functions to define the tangency condition of two paths [Vin08, GW21]. This includes the original definition of Souriau [Sou70] and definitions that are used in the context of diffeological groups [Les03, Mag18].

We avoid involving smooth functions on the diffeological spaces in our strictly internal definition of tangent vectors for two reasons. The first reason is that this would be at odds with the conceptually simple categorical approach we use. The second reason is that there are many interesting spaces, such as noncommutative tori [DI85], that have a rich diffeological structure, but a poor ring of functions on them. In fact, non-commutative tori turn out to be elastic***, which shows that there is no reason to exclude them here by using a definition of tangent space that relies on a good supply of smooth functions on the diffeological space.

2.2.2 The tangent functor as left Kan extension

Recall that, if it exists, the pointwise left Kan extension of a functor $F : \mathcal{E}\mathrm{ucl} \to \mathcal{C}$ along the inclusion $y : \mathcal{E}\mathrm{ucl} \to \mathcal{D}\mathrm{flg}$ evaluated at $X \in \mathcal{D}\mathrm{flg}$ is given by the colimit

$$\begin{aligned}(\mathrm{Lan}_y F)X &= \mathrm{colim}(y{\downarrow}X \longrightarrow \mathcal{E}\mathrm{ucl} \xrightarrow{F} \mathcal{C}) \\ &= \operatorname*{colim}_{yU\to X} FU\,,\end{aligned} \tag{2.13}$$

where $y{\downarrow}X \to \mathcal{E}\mathrm{ucl}$ maps a plot $yU \to X$ to its domain U [ML98, Sec. X.5]. We see that $\Omega(X)$ as defined in (2.11) is the pointwise left Kan extension of the de Rham functor on euclidean spaces to diffeological spaces. Similarly, TX as defined in (2.12) is the pointwise left Kan extension

$$T = \mathrm{Lan}_y\, y\hat{T} : \mathcal{D}\mathrm{flg} \longrightarrow \mathcal{D}\mathrm{flg}\,,$$

where we put a hat on the tangent functor of euclidean spaces

$$\hat{T} : \mathcal{E}\mathrm{ucl} \longrightarrow \mathcal{E}\mathrm{ucl}$$

for disambiguation.

The realization that $\Omega(X)$ and TX are given by a pointwise left Kan extension leads to a number of useful observations.

Proposition 2.2.1. *Let $F : \mathcal{E}\mathrm{ucl} \to \mathcal{C}$ be a functor to a cocomplete category. Then the diagram*

$$\begin{tikzcd} \mathcal{E}\mathrm{ucl} \arrow[r, "F"] \arrow[d, "y"'] & \mathcal{C} \\ \mathcal{D}\mathrm{flg} \arrow[ur, "\mathrm{Lan}_y F"'] & \end{tikzcd}$$

commutes, that is,

$$(\mathrm{Lan}_y F)yU = FU \tag{2.14}$$

for all $U \in \mathcal{E}\mathrm{ucl}$.

Proof. Since y is full and faithful, the statement follows from [ML98, Cor. 3, Sec. X.3] or from [Kel05, Prop. 4.23]. □

Warning 2.2.2. It is common to denote the functor and its Kan extension with the same letter, "$FX = (\mathrm{Lan}_y F)X$", assuming that it is clear from the context or the type of argument which one is meant. If in addition, the embedding $y : \mathcal{E}\mathrm{ucl} \to \mathcal{D}\mathrm{flg}$ is omitted from the notation, the last statement looks like a tautology, "$FU = FU$", which would be a wrong "proof" by notation.

A natural transformation $\alpha_U : FU \to GU$ of functors $F, G : \mathcal{E}\mathrm{ucl} \to \mathcal{C}$ induces a morphism of the colimits (2.13) defining the pointwise left Kan extension. This morphism is natural in X. The upshot is that the left Kan extension is a functor

$$\mathrm{Lan}_y(\_) : \mathrm{Fun}(\mathcal{E}\mathrm{ucl}, \mathcal{C}) \longrightarrow \mathrm{Fun}(\mathcal{D}\mathrm{flg}, \mathcal{C}) \,,$$

where $\mathrm{Fun}(\mathcal{D}, \mathcal{C})$ denotes the category that has functors $\mathcal{D} \to \mathcal{C}$ as objects and natural transformations as morphisms. Moreover, it follows from $y : \mathcal{E}\mathrm{ucl} \to \mathcal{D}\mathrm{flg}$ being full and faithful that $\mathrm{Lan}_y(\_)$ is full and faithful. An important case is the extension of endofunctors of euclidean spaces, such as the tangent functor, to diffeological spaces.

Proposition 2.2.3. *The pointwise left Kan extension defines a functor of the categories of endofunctors*

$$\mathbb{L} := \mathrm{Lan}_y\, y(\_) : \mathrm{End}(\mathcal{E}\mathrm{ucl}) \longrightarrow \mathrm{End}(\mathcal{D}\mathrm{flg}) \tag{2.15}$$

that is full and faithful.

By Proposition 2.1.43, the colimit (2.13) of the pointwise left Kan extension is given as follows. As set it is given by the colimit of the underlying sets, which is the set of equivalence classes

$$|(\mathbb{L}F)X| = \coprod_{p : yU \to X} |FU_p| \,/ \sim \,, \tag{2.16}$$

where we have used $|yFU_p| = |FU_p|$ and where the equivalence relation is given as follows. We say that an element $\zeta_u \in |FU_p|$ is F**-related** to $\eta_v \in |FV_q|$ if there is a smooth map $f : U \to V$ such that $q \circ y(f) = p$ and $Ff\,\zeta_u = \eta_v$. Two elements are related by $\sim$ if and only if they are connected by a finite zigzag of F-relations. That is, $\zeta \in |FU|$ and $\eta \in |FV|$ are equivalent if there is a sequence of elements $\theta^i \in |FW_i|$, $1 \leq i \leq n$ and a commutative diagram of plots like

$$\begin{tikzcd} yU \arrow[r, "f_1"] \arrow[drr, "p"'] & yW_1 \arrow[dr] & \cdots \arrow[l, "f_2"'] \arrow[r, "f_n"] & yW_n \arrow[dl] & yV \arrow[l, "f_{n+1}"'] \arrow[dll, "q"] \\ & & X & & \end{tikzcd}$$

such that $Ff_i\,\theta^{i-1} = \theta^i = Ff_{i+1}\,\theta^{i+1}$ for all $1 \leq i \leq n$, where we set $\theta^0 := \zeta$ and $\theta^{n+1} := \eta$. Note that the direction of any of the arrows f_i in the diagram may be reversed.

A map $p : |U| \to |(\mathbb{L}F)X|$ is a plot if and only if for every point $u_0 \in U$ there is a neighborhood $U_0 \subset U$ of u_0, a plot $q : yV \to X$, and a smooth map $p_0 : U_0 \to FV$ such that the diagram

$$\begin{tikzcd} yU_0 \arrow[r, "yp_0"] \arrow[d, hook] & yFV \arrow[d, "(\mathbb{L}F)q"] \\ yU \arrow[r, "p"'] & (\mathbb{L}F)X \end{tikzcd} \tag{2.17}$$

commutes, where we have used that $(\mathbb{L}F)yV = yFV$ by Proposition 2.2.1.

2.2.3 The bundle of cones

There is a number of endofunctors of euclidean spaces that are a natural structure of the tangent functor, such as the bundle projection

$$\begin{aligned} \pi_U : \hat{T}U &\longrightarrow U \\ (u, u_1) &\longmapsto u\,, \end{aligned}$$

the zero section

$$\begin{aligned} \hat{0}_U : U &\longrightarrow \hat{T}U \\ u &\longmapsto (u, 0)\,, \end{aligned}$$

and the scalar multiplication

$$\begin{aligned} \hat{\kappa}_U : \mathbb{R} \times \hat{T}U &\longrightarrow \hat{T}U \\ \big(r, (u, u_1)\big) &\longmapsto (u, ru_1)\,. \end{aligned}$$

Their pointwise left Kan extensions will be denoted by

$$\begin{aligned} \pi_X := (\mathbb{L}\hat{\pi})_X : TX &\longrightarrow X \\ 0_X := (\mathbb{L}\hat{0})_X : X &\longrightarrow TX \\ \kappa_X := (\mathbb{L}\hat{\kappa})_X : \mathbb{R} \times TX &\longrightarrow TX \end{aligned}$$

The functoriality of $\mathbb{L}$ implies that commutative diagrams of $\hat{\pi}$, $\hat{0}$, $\hat{\kappa}$ get mapped to commutative diagrams of π, 0, and κ. It follows that 0 is a section of π, $\pi_X \circ 0_X = \mathrm{id}_X$. The $\mathbb{R}$-multiplication is a morphism of bundles, that is, the diagram

$$\begin{tikzcd} \mathbb{R} \times TX \arrow[rr, "\kappa_X"] \arrow[dr, "\pi_X \circ \mathrm{pr}_2"'] & & TX \arrow[dl, "\pi_X"] \\ & X & \end{tikzcd}$$

commutes. The compatibility of the scalar multiplication with the multiplication $m : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ can be expressed in terms of the following three commutative diagrams:

(i) Associativity:

$$\begin{tikzcd} \mathbb{R} \times \mathbb{R} \times TX \arrow[r, "\mathrm{id}_{\mathbb{R}} \times \kappa_X"] \arrow[d, "m \times \mathrm{id}_{TX}"'] & \mathbb{R} \times TX \arrow[d, "\kappa_X"] \\ \mathbb{R} \times TX \arrow[r, "\kappa_X"'] & TX \end{tikzcd}$$

(ii) Unitality:

$$\begin{tikzcd} \{1\} \times TX \arrow[r, hook] \arrow[dr, "\cong"'] & \mathbb{R} \times TX \arrow[d, "\kappa_X"] \\ & TX \end{tikzcd}$$

(iii) Compatibility of zeros:

$$\begin{tikzcd} \{0\} \times TX \arrow[r, hook] \arrow[d, "\pi_X \circ \mathrm{pr}_2"'] & \mathbb{R} \times TX \arrow[d, "\kappa_X"] \\ X \arrow[r, "0_X"'] & TX \end{tikzcd}$$

What is the geometric interpretation of these commutative diagrams? The map $\pi_X : TX \to X$ can be viewed as a bundle in a very general sense, since there are generally no local trivializations. The fiber over a point $x : * \to X$ is defined by

$$T_x X := * \times_X^{x, \pi_X} TX \, .$$

Since $x : * \to X$ is trivially an induction and since inductions are stable under pullbacks, the natural morphism $T_x X \to TX$ is an induction. In other words, $T_x X$ is a diffeological subspace of TX. Since π_X has the section 0_X, it follows that π_X is a subduction, so that X is the diffeological quotient space obtained by identifying all points of a fiber.

A subset of a real vector space that is invariant under the $\mathbb{R}$-multiplication is called an $\mathbb{R}$-cone. Note that such a cone is generally not convex. If there is no ambient vector space, we can axiomatize the properties of $\mathbb{R}$-invariance as follows.

Definition 2.2.4. An abstract $\mathbb{R}$**-cone** consists of a set V, a map $\kappa : \mathbb{R} \times V \to V$, $(\alpha, v) = \alpha \cdot v$, and a distinguished element $0 \in V$, such that

(i) $\alpha \cdot (\beta \cdot v) = (\alpha\beta) \cdot v$,

(ii) $1 \cdot v = v$,

(iii) $0 \cdot v = 0$,

for all $\alpha, \beta \in \mathbb{R}$ and $v \in V$.

Properties (i), (ii), (iii) state that κ defines an action of the multiplicative monoid of $\mathbb{R}$ on V, such that $1 \in \mathbb{R}$ acts as identity, and $0 \in \mathbb{R}$ maps all points to the tip of the cone $0 \in V$. The induced action of the group of units $\mathbb{R}^\times = \mathbb{R} \setminus \{0\}$ is generally not free.

Remark 2.2.5. Assume that V is a Hausdorff topological space and κ continuous. Let

$$\mathrm{Stab}(v) := \{\alpha \in \mathbb{R} \mid \alpha \cdot v = v \text{ for all } v \in V\}$$

denote the stabilizer of $v \in V$. Assume that $q \in \mathrm{Stab}(v)$ of norm $|q| \neq 1$. If $|q| < 1$, then q^k converges to 0 as k goes to infinity. It follows that the sequence $q^k \cdot v = v$ converges to $0 \cdot v = 0$, which implies that $v = 0$. Similarly, if $|q| > 1$, then q^{-k} converges to 0, so that $q^{-k} \cdot v = v$ converges to $0 \cdot v = 0$, which also implies that $v = 0$. If $v = 0$, then $\mathrm{Stab}(v) = \mathbb{R}$.

We conclude that the stabilizer of any v is either $\mathbb{R}$, $\{1\}$, or $\{1, -1\}$. In the first case, the $\mathbb{R}$-orbit of v is $\{0\}$; in the second case, the orbit is homeomorphic to $\mathbb{R}$; in the third case, the orbit is homeomorphic $[0, \infty)$. V is the union of all orbits. The intersection of any two orbits is $\{0\}$.

Terminology 2.2.6. Let $\mathcal{C}$ be a category and Wibble an algebraic theory. Let $X \in \mathcal{C}$. A Wibble object in $\mathcal{C}{\downarrow}X$ will be called a **bundle of Wibbles over** X.

We will consider the case that $\mathcal{C} = \mathcal{D}\mathrm{flg}$ and that Wibble is a monoid, group, abelian group, module, $\mathbb{R}$-vector space, or $\mathbb{R}$-cone. If $W \to X$ is a bundle of Wibbles, then W_x is a Wibble object in $\mathcal{D}\mathrm{flg}$. In other words, every fiber of a bundle of Wibbles is a Wibble, which justifies the terminology. Note, that the notion of bundle of Wibbles does not make any assumptions on local trivializations, whatsoever. So a bundle of vector spaces over a manifold M is considerably more general than a vector bundle over M.

Remark 2.2.7. The purpose of Terminology 2.2.6 is to unify (for the purpose of this paper) the varied terminology found in the literature and to use a term that is self-explanatory for a category theorist. In [Ros84, p. 1] a bundle of (abelian) groups over an endofunctor $F : \mathcal{C} \to \mathcal{C}$ is called a “natural (abelian) group bundle over F”. A bundle of vector spaces over a diffeological space X is called a “regular vector bundle” in [Vin08], a “diffeological vector space over X” in [CW16], and a “diffeological vector pseudo-bundle” in [Per16].

Proposition 2.2.8. *The natural morphisms $\pi_X : TX \to X$, $0_X : X \to TX$, $\kappa_X : \mathbb{R} \times TX \to TX$ equip the diffeological tangent space with the structure of a bundle of $\mathbb{R}$-cones over X.*

Proof. The statement is, by definition, equivalent to the commutative diagrams of this section. □

2.2.4 Compatibility with products, coproducts, and subductions

Since $y : \mathcal{E}\mathrm{ucl} \to \mathcal{D}\mathrm{flg}$ is full and faithful, a smooth map $f : U \to V$ of euclidean spaces is a strong epimorphism if and only $yf : yU \to yV$ is a strong epimorphism, which is the same thing as a subduction. For this reason we will call a strong epimorphism f a subduction. This is the case if every point $v_0 \in V$ has an open neighborhood $V_0 \subset V$ such that there is a smooth map $g_0 : V_0 \to U$ satisfying $f \circ g_0 = \mathrm{id}_{V_0}$. In short, f is a subduction if it has local sections. In particular, every surjective submersion is a subduction.

Proposition 2.2.9. *Let $\alpha : F \to G$ be a natural transformation of endofunctors of $\mathcal{E}\mathrm{ucl}$. If $\alpha_U : FU \to GU$ is a subduction for all $U \in \mathcal{E}\mathrm{ucl}$, then $(\mathbb{L}\alpha)_X : (\mathbb{L}F)X \to (\mathbb{L}G)X$ is a subduction for all $X \in \mathcal{D}\mathrm{flg}$.*

Proof. Since α_U is a subduction, so is $y\alpha_U$. Subductions in $\mathcal{D}\mathrm{flg}$ are the same as regular epimorphisms, so that $y\alpha_U$ is a regular epimorphism for all $U \in \mathcal{E}\mathrm{ucl}$. The left Kan extension $\alpha_X = (\mathbb{L}\alpha)_X$ is given by the colimit over the category of plots $y \downarrow X$. Since colimits preserve regular epimorphisms, α_X is a regular epimorphism, that is, a subduction. □

Proposition 2.2.10. *If a functor $F : \mathcal{E}\mathrm{ucl} \to \mathcal{E}\mathrm{ucl}$ preserves finite products, then so does $\mathbb{L}F : \mathcal{D}\mathrm{flg} \to \mathcal{D}\mathrm{flg}$.*

Lemma 2.2.11. *Let X_1 and X_2 be diffeological spaces. The functor*

$$\begin{aligned} y\downarrow(X_1 \times X_2) &\longrightarrow (y\downarrow X_1) \times (y\downarrow X_2) \\ \big(yU \xrightarrow{p} X_1 \times X_2\big) &\longmapsto \big(yU \xrightarrow{\mathrm{pr}_1 \circ p} X_1, yU \xrightarrow{\mathrm{pr}_2 \circ p} X_2\big) \end{aligned} \tag{2.18}$$

is final.

Proof. Let the functor (2.18) be denoted by D. We have to show that for every

$$\big(yW_1 \xrightarrow{r_1} X_1, yW_2 \xrightarrow{r_2} X_2\big) \in (y\downarrow X_1) \times (y\downarrow X_2)$$

the category $D\downarrow(r_1, r_2)$ is connected.

Let Z be a diffeological space. Let $p : yU \to Z$ and $q : yV \to Z$ be plots. Let $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$. By using a diffeomorphism $\varphi : \mathbb{R}^n \to \{(x^1, \dots, x^n) \mid x_1 < 0\}$ we obtain a diffeomorphism $U \cong \varphi(U)$, where $\varphi(U)$ is an open subset of the left half space of $\mathbb{R}^m$. Assume without loss of generality that $n \geq m$. Then $\tilde{U} := \varphi(U) \times \mathbb{R}^{n-m}$ is an open subset of the left half space of $\mathbb{R}^n$. $U \cong U \times \{0\} \hookrightarrow \varphi(U) \times \mathbb{R}^{n-m} = \tilde{U}$ is a retract, so that p factors through the plot

$$\tilde{p} : y\tilde{U} \xrightarrow{\cong} yU \times y\mathbb{R}^m \xrightarrow{\mathrm{pr}_1} yU \xrightarrow{p} Z\,.$$

Using a diffeomorphism from $\mathbb{R}^n$ to the right half space of $\mathbb{R}^n$, we obtain a diffeomorphism $\psi : V \to \tilde{V}$, where $\tilde{V}$ is an open subset of the right half space of $\mathbb{R}^n$. Let $\tilde{q} := \psi^{-1} \circ q : y\tilde{V} \to Z$. Since $\tilde{U}$ and $\tilde{V}$ are disjoint we have a commutative diagram of plots:

$$\begin{tikzcd} & y\tilde{U} \cup y\tilde{V} \arrow[d, "(\tilde{p},\tilde{q})"] & \\ yU \arrow[ur] \arrow[r, "p"] & Z & yV \arrow[ul] \arrow[l, "q"'] \end{tikzcd}$$

In other words, every pair of plots is connected by a cospan. (This implies that $y\downarrow Z$ is sifted.)

Let now $f_i : U \to W_i$ and $g_i : V \to W_i$ be smooth maps for $i \in \{1,2\}$. The pairs (f_1, f_2) and (g_1, g_2) can be viewed as elements in $D\downarrow(r_1, r_2)$. The construction of the cospans for $Z = yW_i$ yields the diagrams

$$\begin{tikzcd} & y\tilde{U} \cup y\tilde{V} \arrow[d] & \\ yU \arrow[ur] \arrow[r, "f_i"] \arrow[dr] & yW_i \arrow[d, "r_i"] & yV \arrow[ul] \arrow[l, "g_i"'] \arrow[dl] \\ & X_i & \end{tikzcd}$$

for $i \in \{1,2\}$. By the universal property of the product, we obtain the commutative diagram

$$\begin{tikzcd} & y\tilde{U} \cup y\tilde{V} \arrow[d] & \\ yU \arrow[ur] \arrow[r, "(f_1,f_2)"] \arrow[dr] & yW_1 \times yW_2 \arrow[d, "r_1\times r_2"] & yV \arrow[ul] \arrow[l, "(g_1,g_2)"'] \arrow[dl] \\ & X_1 \times X_2 & \end{tikzcd}$$

This shows that (f_1, f_2) and (g_1, g_2) are connected by a cospan in $D\downarrow(r_1, r_2)$. We conclude that D is final. □

Proof of Prop. 2.2.10. Let X_1 and X_2 be diffeological spaces. We have the isomorphisms

$$\begin{aligned} (\mathbb{L}F)X_1 \times (\mathbb{L}F)X_2 &\cong \big(\operatorname*{colim}_{yU_1\to X_1} yFU_1\big) \times \big(\operatorname*{colim}_{yU_2\to X_2} yFU_2\big) \\ &\cong \operatorname*{colim}_{yU_1\to X_1} \big(yFU_1 \times \operatorname*{colim}_{yU_2\to X_2} yFU_2\big) \\ &\cong \operatorname*{colim}_{yU_1\to X_1} \operatorname*{colim}_{yU_2\to X_2} \big(yFU_1 \times yFU_2\big) \\ &\cong \operatorname*{colim}_{yU_1\to X_1} \operatorname*{colim}_{yU_2\to X_2} yF(U_1 \times U_2) \\ &\cong \operatorname*{colim}_{yU\to X_1\times X_2} yFU \\ &\cong (\mathbb{L}F)(X_1 \times X_2)\,, \end{aligned}$$

where we have used the definition of the pointwise left Kan extension, the fact that the product is cocontinuous in each argument since it is a left adjoint, that y and F preserve finite products, and in the second last step Lemma 2.2.11. By induction, it follows that $\mathbb{L}F$ preserves finite products. □

Let $F : \mathcal{I} \to \mathrm{End}(\mathcal{E}\mathrm{ucl})$, $i \mapsto F_i$ be a functor. Due to the universal properties of colimits and limits, we have for every $X \in \mathcal{D}\mathrm{flg}$ the natural morphism

$$\operatorname*{colim}_{yU\to X} \lim_{i\in\mathcal{I}} yF_iU \longrightarrow \lim_{i\in\mathcal{I}} \operatorname*{colim}_{yU\to X} yF_iU\,.$$

Assuming that the limit $\lim_{i\in\mathcal{J}} F_i$ exists in $\mathrm{End}(\mathcal{E}\mathrm{ucl})$, it can be written as the natural transformation

$$\mathbb{L}\lim_i F_i \longrightarrow \lim_i \mathbb{L}F_i\,, \tag{2.19}$$

where we have used that y preserves limits. This is not an isomorphism unless the colimit and the limit commute.

Let $G : \mathcal{J} \to \mathrm{End}(\mathcal{E}\mathrm{ucl})$ be another diagram, such that $\lim_i G_i$ exists. Any natural transformation $\alpha_i : F_i \to G_i$ induces a commutative diagram

$$\begin{tikzcd} \mathbb{L}\lim_i F_i \arrow[r, "\mathbb{L}(\lim_i \alpha_i)"] \arrow[d] & \mathbb{L}\lim_i G_i \arrow[d] \\ \lim_i \mathbb{L}F_i \arrow[r, "\lim_i \mathbb{L}\alpha_i"'] & \lim_i \mathbb{L}G_i \end{tikzcd} \tag{2.20}$$

Proposition 2.2.12. *Let $F_1, \dots, F_k \in \mathrm{End}(\mathcal{E}\mathrm{ucl})$ be a finite family of endofunctors. Then we have an isomorphism*

$$\mathbb{L}(F_1 \times \ldots \times F_k) \cong \mathbb{L}F_1 \times \ldots \times \mathbb{L}F_k\,.$$

Proof. Since finite products exist in $\mathcal{E}\mathrm{ucl}$, they exist in $\mathrm{End}(\mathcal{E}\mathrm{ucl})$. Let $X \in \mathcal{D}\mathrm{flg}$. As remarked in the proof of Lemma 2.2.11, the index category $y\!\downarrow\! X$ over which the colimits of the Kan extension are taken is sifted. Since sifted colimits commute with finite products, the natural transformation (2.19) is an isomorphism at all $X \in \mathcal{D}\mathrm{flg}$. □

It is well-known, that the left Kan extension of an arbitrary functor along the Yoneda embedding $Y : \mathcal{E}\mathrm{ucl} \to \mathcal{S}\mathrm{et}^{\mathcal{E}\mathrm{ucl}^{\mathrm{op}}}$ preserves all colimits (Proposition B.0.1). This is not true for Kan extensions along $y : \mathcal{E}\mathrm{ucl} \to \mathcal{D}\mathrm{flg}$. Already for the preservation of coproducts we have to make additional assumptions.

Recall that a functor $F : \mathcal{E}\mathrm{ucl} \to \mathcal{C}$ is a **cosheaf** if $F^{\mathrm{op}} : \mathcal{E}\mathrm{ucl}^{\mathrm{op}} \to \mathcal{C}^{\mathrm{op}}$ is a sheaf. Explicitly, this means that for every cover $\{U_i \to U\}$ the diagram

$$\textstyle\coprod_{i,j} F(U_i \cap U_j) \rightrightarrows \coprod_i FU_i \longrightarrow FU \tag{2.21}$$

is a coequalizer. In other words, F maps an open cover of U to an open cover of FU.

Example 2.2.13. The following functors on $\mathcal{E}\mathrm{ucl}$ are cosheaves:

(a) the tangent functor $\hat{T} : \mathcal{E}\mathrm{ucl} \to \mathcal{E}\mathrm{ucl}$;

(b) fiber products of the tangent functor such as $\mathcal{E}\mathrm{ucl} \to \mathcal{E}\mathrm{ucl}$, $U \mapsto TU \times_U TU$;

(c) if $F : \mathcal{E}\mathrm{ucl} \to \mathcal{C}$ and $G : \mathcal{E}\mathrm{ucl} \to \mathcal{E}\mathrm{ucl}$ are cosheaves, then so is their composition FG;

(d) the de Rham functor $\hat{\Omega} : \mathcal{E}\mathrm{ucl} \to \mathrm{dg}\mathcal{A}\mathrm{lg}^{\mathrm{op}}$;

(d) if $F : \mathcal{M}\mathrm{fld}^{\mathrm{op}} \to \mathcal{C}$ is a sheaf on the big site of manifolds and open covers, then the restriction $F : \mathcal{E}\mathrm{ucl} \hookrightarrow \mathcal{M}\mathrm{fld} \to \mathcal{C}^{\mathrm{op}}$ is a cosheaf.

Proposition 2.2.14. *If* $F : \mathcal{E}\mathrm{ucl} \to \mathcal{C}$ *is a cosheaf, then its left Kan extension along* $y : \mathcal{E}\mathrm{ucl} \to \mathcal{D}\mathrm{flg}$ *preserves coproducts.*

Proof. Let $\mathcal{E}\mathrm{ucl}_{\mathrm{con}}$ denote the full subcategory of $\mathcal{E}\mathrm{ucl}$ of connected open subsets of euclidean spaces. Let us denote by $J : \mathcal{E}\mathrm{ucl}_{\mathrm{con}} \to \mathcal{E}\mathrm{ucl}$ the inclusion as full and faithful subcategory. Since $\mathcal{E}\mathrm{ucl}_{\mathrm{con}}$ is small and $\mathcal{C}$ cocomplete, the left Kan extension of any functor $G : \mathcal{E}\mathrm{ucl}_{\mathrm{con}} \to \mathcal{C}$ along J exists and is pointwise. Since every $U \in \mathcal{E}\mathrm{ucl}$ has a cover by connected open subsets, J is dense. It follows*** that the successive Kan extensions first along J and then along y is the left Kan extension along the composition yJ,

$$\mathrm{Lan}_y(\mathrm{Lan}_J\, G) \cong \mathrm{Lan}_{yJ}\, G\,. \tag{2.22}$$

Let us consider the case $G = FJ : \mathcal{E}\mathrm{ucl}_{\mathrm{con}} \to \mathcal{C}$. By assumption F is a cosheaf, which implies that

$$(\mathrm{Lan}_J\, FJ)U = \operatorname*{colim}_{JV \to U} FJV \cong FU\,,$$

for all $U \in \mathcal{E}\mathrm{ucl}$, that is, $F \cong \mathrm{Lan}_J\, FJ$. From (2.22) we conclude that

$$\mathrm{Lan}_y\, F \cong \mathrm{Lan}_{yJ}\, FJ\,.$$

In other words, the left Kan extension of F along y is naturally isomorphic to the left Kan extension of F restricted to connected open subsets of $\mathbb{R}^n$, $n \geq 0$ along the natural embedding $\mathcal{E}\mathrm{ucl}_{\mathrm{con}} \to \mathcal{D}\mathrm{flg}$.

Let $X = \amalg_{i \in \mathcal{I}} X_i$ be a coproduct of diffeological spaces X_i. The functor $\mathcal{D}\mathrm{flg} \to \mathcal{T}\mathrm{op}$ which maps a diffeology to the D-topology has a right adjoint, so that it preserves all limits. In particular, the diffeological subspaces $X_i \subset X$ are open and closed in the D-topology. It follows that every plot $yJU \to X$ for $U \in \mathcal{E}\mathrm{ucl}_{\mathrm{con}}$, which is a continuous map with respect to the underlying D-topologies, factors through a single summand X_i. The conclusion is that the set of all plots from U to X is the union of the plots to X_i for all $i \in \mathcal{I}$,

$$\mathrm{Hom}\big(yJU, \amalg_i X_i\big) \cong \coprod_i \mathrm{Hom}\big(yJU, X_i\big)\,.$$

In other words, the functor $\mathcal{D}\mathrm{flg}(yJU, \_) : \mathcal{E}\mathrm{ucl}_{\mathrm{con}} \to \mathcal{S}\mathrm{et}$ preserves coproducts.

The pointwise left Kan extension can be expressed by the coend

$$(\mathrm{Lan}_{yJ}\, FJ)(X) \cong \int^{U \in \mathcal{E}\mathrm{ucl}_{\mathrm{con}}} \mathcal{D}\mathrm{flg}(yJU, X) \otimes FJU\,,$$

for all $X \in \mathcal{D}\mathrm{flg}$, where the copower functor $\_ \otimes Y : \mathcal{S}\mathrm{et} \to \mathcal{D}\mathrm{flg}$ is the left adjoint of $\mathcal{D}\mathrm{flg}(Y, \_) : \mathcal{D}\mathrm{flg} \to \mathcal{S}\mathrm{et}$. We see that the left Kan extension is the composition of the functor

$$\mathcal{D}\mathrm{flg}(yJU, \_) : \mathcal{D}\mathrm{flg} \longrightarrow \mathcal{S}\mathrm{et}$$

with the functor

$$\_ \otimes FJU : \mathcal{S}\mathrm{et} \longrightarrow \mathcal{D}\mathrm{flg}\,,$$

followed by the coend. We have already seen that the first functor $\mathcal{D}\mathrm{flg}(yJU, \_)$ preserves coproducts. The second functor $\_ \otimes FJU$ preserves coproducts because it is a left adjoint. Finally, the coend is given by a colimit, so that it, too, preserves coproducts. We conclude that the left Kan extension $\mathrm{Lan}_{yJ}\, FJ \cong \mathrm{Lan}_y\, F$ preserves coproducts. $\square$

Corollary 2.2.15. *If $F : \mathcal{E}\mathrm{ucl} \to \mathcal{E}\mathrm{ucl}$ is a cosheaf, then $\mathbb{L}F$ preserves coproducts.*

Proposition 2.2.16. *If $F : \mathcal{E}\mathrm{ucl} \to \mathcal{E}\mathrm{ucl}$ is a cosheaf, then $\mathbb{L}F$ preserves subductions.*

Proof. Let $p : yU \to (\mathbb{L}F)Y$ be a plot. This means that for every point $u_0 \in U$ there is a neighborhood U_0, a plot $q : yV \to Y$, and a smooth map $p_0 : U_0 \to FV$, such that the restriction of p to $yU_0 \hookrightarrow yU$ is equal to $(\mathbb{L}F)q \circ yp_0$, as explained for Diagram (2.17).

Let $f : X \to Y$ be a subduction. Then V has an open cover $\{\varphi_i : V_i \to V\}_{i\in I}$ such that for every i there is a $q_i : yV_i \to Y$ satisfying $f \circ q_i = q|_{V_i}$. Since F is a cosheaf, $\{FV_i\}$ is a cover of FV. This implies that there is a V_i, such that $p_0(u_0)$ is contained in FV_i, so the open subset $p_0^{-1}(V_i) \subset U_0$ contains u_0. The situation can be summarized by the following commutative diagram:

$$\begin{tikzcd}
yp_0^{-1}(FV_i) \arrow[r] \arrow[d, hook] & yFV_i \arrow[r, "="] \arrow[d, "yF\varphi_i"] & (\mathbb{L}F)yV_i \arrow[r, "(\mathbb{L}F)q_i"] \arrow[d, "(\mathbb{L}F)y\varphi_i"] & (\mathbb{L}F)X \arrow[dd, "(\mathbb{L}F)f"] \\
yU_0 \arrow[r, "yp_0"] \arrow[d, hook] & yFV \arrow[r, "="] & (\mathbb{L}F)yV \arrow[dr, "(\mathbb{L}F)q"] & \\
yU \arrow[rrr, "p"'] & & & (\mathbb{L}F)Y
\end{tikzcd}$$

The outer commutative square shows that p has a local lift to $(\mathbb{L}F)X$, defined on the open neighborhood $p_0^{-1}(FV_i)$ of u_0. We conclude that $(\mathbb{L}F)f$ is a subduction. $\square$

Warning 2.2.17. Even though every subduction is a regular epimorphism, that is, given by a coequalizer, Proposition 2.2.16 does not imply that the left Kan extension of a cosheaf preserves coequalizers.

Proposition 2.2.18. *Let $X : \mathcal{I} \to \mathfrak{D}\mathrm{flg}$ be a functor. If $F : \mathcal{E}\mathrm{ucl} \to \mathcal{E}\mathrm{ucl}$ is a cosheaf, then the natural morphism*

$$\operatorname{colim}(\mathbb{L}F)X \longrightarrow (\mathbb{L}F)\operatorname{colim} X$$

is a subduction.

Proof. Applying $\mathbb{L}F$ to diagram (2.10) and using that, by Proposition 2.2.15, $\mathbb{L}F$ preserves coproducts, we obtain the diagram

$$\begin{tikzcd}
\coprod\limits_{f\in \mathrm{Mor}(\mathcal{I})} (\mathbb{L}F)X_{\mathrm{dom} f} \arrow[r, shift left] \arrow[r, shift right] & \coprod\limits_{i\in\mathcal{I}} (\mathbb{L}F)X_i \arrow[r] \arrow[dr, "(\mathbb{L}F)\pi"'] & \operatorname{colim}(\mathbb{L}F)X \arrow[d, "\varphi"] \\
& & (\mathbb{L}F)\operatorname{colim} X
\end{tikzcd}$$

where φ is the morphism of the proposition, which is given by the universal property of the coequalizer. By Proposition 2.2.16 $(\mathbb{L}F)\pi$ is a subduction, so that φ is a subduction. $\square$

Corollary 2.2.19. *The tangent functor of diffeological spaces preserves finite limits, small colimits, and subductions.*

Proof. The tangent functor of euclidean spaces preserves finite products and is a cosheaf. The statements follow from Proposition 2.2.10, Corollary 2.2.15, and Propositions 2.2.16. □

2.2.5 Representing tangent vectors by paths

Every smooth path $\gamma : \mathbb{R} \to U$ in $U \in \mathcal{E}\mathrm{ucl}$, $t \mapsto \gamma_t$ can be mapped to its tangent vector at $t = 0$, which gives rise to the map

$$\begin{aligned} \mathcal{E}\mathrm{ucl}(\mathbb{R}, U) &\longrightarrow \hat{T}U \\ \gamma &\longmapsto \dot{\gamma}_0 := \Big(\gamma_0, \frac{d\gamma_t}{dt}\Big|_{t=0}\Big) . \end{aligned} \tag{2.23}$$

Since every tangent vector in $U \subset \mathbb{R}^n$ is represented by a smooth path, this map is surjective. Every smooth homotopy $h : V \times \mathbb{R} \to U$ of paths parametrized by $V \subset \mathbb{R}^m$ gives naturally rise to a smooth family of tangent vectors

$$\begin{aligned} V &\longrightarrow \hat{T}U \\ v &\longmapsto \Big(h(v,t), \frac{\partial h}{\partial t}(v,0)\Big) . \end{aligned}$$

Conversely, every smooth family $V \to \hat{T}U$, $v \mapsto \big(u(v), \eta(v)\big)$ is obtained locally from the smooth homotopy $h(t,v) := u(v) + t\eta(v)$, where the domain of h has to be restricted to an open subset of $V \times \mathbb{R}$ such that the values $h(v,t)$ remain in U. The conclusion is that if we view $\mathbb{R}$, U, V, and $\hat{T}U$ as diffeological spaces and equip $\mathcal{E}\mathrm{ucl}(\mathbb{R}, U)$ with the functional diffeology, then (2.23) is a subduction, which we denote by

$$\hat{\partial}_U : \underline{\mathcal{D}\mathrm{flg}}(y\mathbb{R}, yU) \longrightarrow y\hat{T}U . \tag{2.24}$$

Moreover, $\hat{\partial}_U$ is natural in U, which means that for all smooth maps $f : U \to V$ the diagram

$$\begin{array}{ccc} \underline{\mathcal{D}\mathrm{flg}}(y\mathbb{R}, yU) & \xrightarrow{f_*} & \underline{\mathcal{D}\mathrm{flg}}(y\mathbb{R}, yV) \\ {\scriptstyle \hat{\partial}_U}\big\downarrow & & \big\downarrow{\scriptstyle \hat{\partial}_V} \\ y\hat{T}U & \xrightarrow[\hat{T}f]{} & y\hat{T}V \end{array}$$

commutes.

Proposition 2.2.20. *The left Kan extension of* (2.24) *to* $\mathcal{D}\mathrm{flg}$*,*

$$\partial_X : \underline{\mathcal{D}\mathrm{flg}}(y\mathbb{R}, X) \longrightarrow TX , \tag{2.25}$$

is a natural subduction.

The difficult technical part of the proof of Proposition 2.2.20 is in the following lemma:

Lemma 2.2.21. *Let* $V \in \mathcal{E}\mathrm{ucl}$ *and* $X \in \mathcal{D}\mathrm{flg}$. *The natural morphism*

$$\operatorname*{colim}_{yU\to X} \underline{\mathcal{D}\mathrm{flg}}(yV, yU) \longrightarrow \underline{\mathcal{D}\mathrm{flg}}(yV, \operatorname*{colim}_{yU\to X} yU)\,, \tag{2.26}$$

is an isomorphism.

Proof. First, we show that (2.26) is a bijection on the underlying sets. We have

$$\begin{aligned} \mathcal{D}\mathrm{flg}(yV, \operatorname*{colim}_{yU\to X} yU) &\cong \mathcal{D}\mathrm{flg}(yV, X) \\ &\cong \mathcal{S}\mathrm{et}^{\mathcal{E}\mathrm{ucl}^{\mathrm{op}}}(IyV, IX) \\ &\cong \mathcal{S}\mathrm{et}^{\mathcal{E}\mathrm{ucl}^{\mathrm{op}}}(YV, \operatorname*{colim}_{YU\to IX} YU) \\ &\cong \big(\operatorname*{colim}_{YU\to IX} YU\big)V \\ &\cong \operatorname*{colim}_{YU\to IX} \big((YU)V\big) \\ &\cong \operatorname*{colim}_{YU\to IX} \mathcal{S}\mathrm{et}^{\mathcal{E}\mathrm{ucl}^{\mathrm{op}}}(YV, YU) \\ &\cong \operatorname*{colim}_{yU\to X} \mathcal{D}\mathrm{flg}(yV, yU)\,, \end{aligned} \tag{2.27}$$

where we have used Proposition 2.1.15, that I is full and faithful, the Yoneda lemma, that colimits of functors are computed pointwise, and, in the last step, that $Y = Iy$.

For every plot $p : yU \to X$, we have the pushforward

$$\underline{\mathcal{D}\mathrm{flg}}(yV, yU) \xrightarrow{p_*} \underline{\mathcal{D}\mathrm{flg}}(yV, X) \cong \underline{\mathcal{D}\mathrm{flg}}(yV, \operatorname*{colim}_{yU\to X} yU)\,,$$

where we have used Proposition 2.1.15. By the universal property of the colimit, the pushforwards induce the morphism (2.26). Using the formula for colimits in terms of a coequalizer of coproducts ***, we obtain the commutative diagram

$$\begin{tikzcd} \coprod\limits_{yU\to X} \underline{\mathcal{D}\mathrm{flg}}(yV, yU) \arrow[r, "\psi"] \arrow[d] & \underline{\mathcal{D}\mathrm{flg}}\big(yV, \coprod\limits_{yU\to X} yU\big) \arrow[d, "\chi"] \\ \operatorname*{colim}\limits_{yU\to X} \underline{\mathcal{D}\mathrm{flg}}(yV, yU) \arrow[r] & \underline{\mathcal{D}\mathrm{flg}}\big(yV, \operatorname*{colim}\limits_{yU\to X} yU\big) \end{tikzcd} \tag{2.28}$$

where ψ is given by the universal property of the coproduct. The left vertical arrow is a coequalizer, so a fortiori a strong epimorphism.

Let $p : yW \to \underline{\mathcal{D}\mathrm{flg}}(yV, X)$ be a plot. By definition of the functional diffeology, this is the case if and only if the associated map $\tilde{p} : yW \times yV \to X$, $(w, v) \mapsto \big(p(w)\big)(v)$ is a plot. Since y preserves products, we have the diagram

$$\begin{tikzcd} & y(W\times V) \arrow[d, "\tilde{p}"] \\ yW \times yV \arrow[ur, "\mathrm{id}"] \arrow[r, "\tilde{p}"'] & X \end{tikzcd}$$

By the universal property of the inner hom, this gives rise to the "tautological" lift of p

$$\begin{tikzcd} & \underline{\mathcal{D}\mathrm{flg}}\big(yV, y(W \times V)\big) \arrow[d, "p_*"] \\ yW \arrow[ur, "q"] \arrow[r, "p"'] & \underline{\mathcal{D}\mathrm{flg}}(yV, X)\,, \end{tikzcd}$$

where

$$\begin{aligned} q : yW &\longrightarrow \underline{\mathcal{D}\mathrm{flg}}\big(yV, y(W \times yV)\big) \\ w &\longmapsto \big(v \mapsto (w, v)\big)\,. \end{aligned}$$

We thus obtain the commutative diagram

$$\begin{tikzcd} \underline{\mathcal{D}\mathrm{flg}}\big(yV, y(W \times V)\big) \arrow[r] & \coprod\limits_{yU \to X} \underline{\mathcal{D}\mathrm{flg}}(yV, yU) \arrow[d, "\chi \circ \psi"] \\ yW \arrow[u, "q"] \arrow[ur, "\hat{p}"] \arrow[r, "p"'] & \underline{\mathcal{D}\mathrm{flg}}(yV, \operatorname*{colim}\limits_{yU \to X} yU) \end{tikzcd}$$

This shows that every plot of $\underline{\mathcal{D}\mathrm{flg}}(yV, X)$ lifts to a plot $\hat{p}$, which implies that $\chi \circ \psi$ is a strong epimorphism. It follows from the commutativity of diagram (2.28) and Proposition 2.1.23 (ii) that φ is a strong epimorphism.

The forgetful functor $\mathcal{D}\mathrm{flg}(*, \_) : \mathcal{D}\mathrm{flg} \to \mathcal{S}\mathrm{et}$ preserves colimits (Proposition 2.1.12), so that we obtain the natural isomorphisms

$$\begin{aligned} \mathcal{D}\mathrm{flg}\big(*, \operatorname*{colim}_{yU \to X} \underline{\mathcal{D}\mathrm{flg}}(yV, yU)\big) &\cong \operatorname*{colim}_{yU \to X} \mathcal{D}\mathrm{flg}(yV, yU) \\ &\cong \mathcal{D}\mathrm{flg}(yV, \operatorname*{colim}_{yU \to X} yU) \\ &\cong \mathcal{D}\mathrm{flg}(yV, X) \\ &\cong \mathcal{D}\mathrm{flg}\big(*, \underline{\mathcal{D}\mathrm{flg}}(yV, X)\big)\,, \end{aligned}$$

where in the second step we have used (2.27). This shows that the morphism (2.26) is a bijection on the underlying sets. In particular, it is a monomorphism.

By Proposition 2.1.23, every strong epimorphism that is a monomorphism is an isomorphism. We conclude that (2.26) is an isomorphism. □

Warning 2.2.22. It is tempting to try to prove Lemma 2.2.21 by simply invoking the enriched Yoneda lemma. However, the colimit of a diagram in $\mathcal{D}\mathrm{flg}$ is generally different from its colimit in the category of presheaves, so that it cannot be computed pointwise. This is why we need to use isomorphism (2.27). For the same reason I believe that the lemma does not hold in general if yV is replaced with a non-representable diffeological space.

Proof of Proposition 2.2.20. The left Kan extension along y of the functor $\mathcal{E}\mathrm{ucl} \to \mathcal{D}\mathrm{flg}$, $U \mapsto \underline{\mathcal{D}\mathrm{flg}}(y\mathbb{R}, yU)$ is given pointwise by

$$\big(\mathrm{Lan}_y\, \underline{\mathcal{D}\mathrm{flg}}(y\mathbb{R}, \_)\big)(X) = \operatorname*{colim}_{yU \to X} \underline{\mathcal{D}\mathrm{flg}}(yV, yU)\,.$$

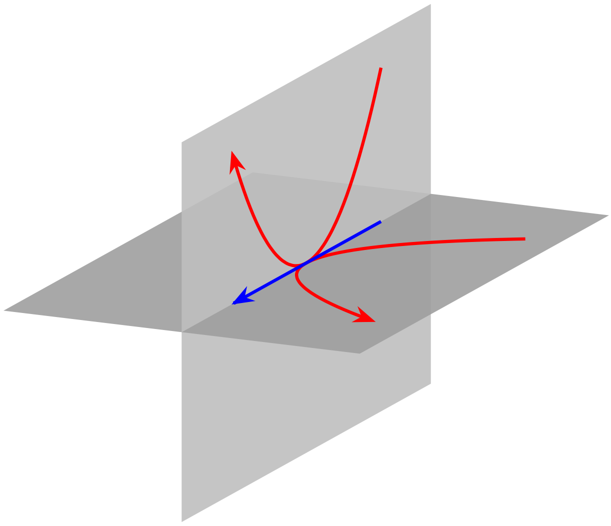


Figure 2.1: Two intersecting planes with the subspace diffeology of $\mathbb{R}^3$. The red paths are not tangent to each other on the same plot. They represent the same tangent vector because each path is tangent to the blue path on a different plot.

Lemma (2.2.21) for $V = \mathbb{R}$ shows that the domain of the left Kan extension is isomorphic to $\underline{\mathfrak{Dflg}}(y\mathbb{R}, X)$. Since (2.24) is a strong epimorphism and since strong epimorphisms are preserved by the left Kan extension, it follows that ∂_X is a strong epimorphism. $\square$

Proposition (2.2.20) can be interpreted as follows. Every tangent vector is represented by a path. Every smooth family of tangent vectors is represented by a smooth family of paths. The naturality of ∂_X means that the pushforward of paths along a morphism $f : X \to Y$ descends to the tangent map, that is, the diagram

$$\begin{tikzcd} \underline{\mathfrak{Dflg}}(y\mathbb{R}, X) \arrow[r, "f_*"] \arrow[d, "\partial_X"'] & \underline{\mathfrak{Dflg}}(y\mathbb{R}, Y) \arrow[d, "\partial_Y"] \\ TX \arrow[r, "Tf"'] & TY \end{tikzcd}$$

commutes. In this sense, the left Kan extension of the tangent functor implements a version of the kinematic definition of tangent vectors as equivalence classes of smooth paths. Note, however, that two paths that represent the same tangent vector in TX need not be tangent to each other on the same plot, as Figure 2.1 illustrates.

The natural morphisms $\pi_X : TX \to X$, $0_X : X \to TX$, and $\kappa_X : \mathbb{R} \times TX \to TX$, which equip TX with the structure of a bundle of $\mathbb{R}$-cones over X, are induced by morphisms on the space of paths as follows. The evaluation of paths at $t = 0$,

$$\begin{aligned} \mathrm{ev}_X : \underline{\mathfrak{Dflg}}(\mathbb{R}, X) &\longrightarrow X \\ \gamma &\longmapsto \gamma(0)\,, \end{aligned}$$

descends to the bundle projection π_X. That is, the diagram

$$\begin{tikzcd} \underline{\mathcal{D}\mathrm{flg}}(y\mathbb{R}, X) \arrow[d, "\partial_X"'] \arrow[dr, "\mathrm{ev}_X"] & \\ TX \arrow[r, "\pi_X"'] & X \end{tikzcd}$$

commutes. The inclusion of X as constant paths,

$$\begin{aligned} c_X : X &\longrightarrow \underline{\mathcal{D}\mathrm{flg}}(y\mathbb{R}, X) \\ x &\longmapsto (t \mapsto x)\,, \end{aligned}$$

descends to the zero section. That is, the diagram

$$\begin{tikzcd} & \underline{\mathcal{D}\mathrm{flg}}(y\mathbb{R}, X) \arrow[d, "\partial_X"] \\ X \arrow[ur, "c_X"] \arrow[r, "0_X"'] & TX \end{tikzcd}$$

commutes. Finally, the linear rescaling of the time parameter of paths,

$$\begin{aligned} \sigma_X : \mathbb{R} \times \underline{\mathcal{D}\mathrm{flg}}(\mathbb{R}, X) &\longrightarrow \underline{\mathcal{D}\mathrm{flg}}(\mathbb{R}, X) \\ (\alpha, \gamma) &\longmapsto (t \mapsto \gamma_{\alpha t})\,, \end{aligned} \tag{2.29}$$

induces the $\mathbb{R}$-multiplication. That is, the diagram

$$\begin{tikzcd} \mathbb{R} \times \underline{\mathcal{D}\mathrm{flg}}(\mathbb{R}, X) \arrow[r, "\sigma_X"] \arrow[d, "\mathrm{id}_{\mathbb{R}} \times \partial_X"'] & \underline{\mathcal{D}\mathrm{flg}}(\mathbb{R}, X) \arrow[d, "\partial_X"] \\ \mathbb{R} \times TX \arrow[r, "\kappa_X"'] & TX \end{tikzcd} \tag{2.30}$$

commutes.

2.3 The diffeological space of fields

2.3.1 The tangent functor of mapping spaces

The natural bijection

$$\mathcal{D}\mathrm{flg}\big(\underline{\mathcal{D}\mathrm{flg}}(X, Y), \underline{\mathcal{D}\mathrm{flg}}(X, Y)\big) \xrightarrow{\cong} \mathcal{D}\mathrm{flg}\big(\underline{\mathcal{D}\mathrm{flg}}(X, Y) \times X, Y\big)$$

maps the identity on $\underline{\mathcal{D}\mathrm{flg}}(X, Y)$ to a morphism

$$\mathrm{ev}_{X,Y} : \underline{\mathcal{D}\mathrm{flg}}(X, Y) \times X \longrightarrow Y\,,$$

which can be viewed as the evaluation of the morphisms in $\underline{\mathcal{D}\mathrm{flg}}(X, Y)$ at the points of X. The domain of its tangent morphism $T\mathrm{ev}_{X,Y}$ is

$$T\big(\underline{\mathcal{D}\mathrm{flg}}(X, Y) \times X\big) \cong T\underline{\mathcal{D}\mathrm{flg}}(X, Y) \times TX\,,$$

where we use that the tangent functor preserves finite products (Corollary 2.2.19). By precomposing $T\mathrm{ev}_{X,Y}$ with the zero section of $TX \to X$, we obtain a morphism

$$T\underline{\mathfrak{Dflg}}(X,Y) \times X \xrightarrow{\mathrm{id}\times 0_X} T\underline{\mathfrak{Dflg}}(X,Y) \times TX \xrightarrow{T\mathrm{ev}_{X,Y}} TY\,.$$

Using the adjunction between products and mapping spaces, this morphism can be viewed as morphism

$$T\underline{\mathfrak{Dflg}}(X,Y) \longrightarrow \underline{\mathfrak{Dflg}}(X,TY)\,, \tag{2.31}$$

which is natural in X and Y.

Due to the naturality of the map from paths to tangent vectors, we have the commutative diagram

$$\begin{tikzcd} \underline{\mathfrak{Dflg}}\big(y\mathbb{R}, \underline{\mathfrak{Dflg}}(X,Y)\times X\big) \arrow[r, "(\mathrm{ev}_{X,Y})_*"] \arrow[d, "\partial_{\underline{\mathfrak{Dflg}}(X,Y)\times X}"'] & \underline{\mathfrak{Dflg}}(y\mathbb{R}, Y) \arrow[d, "\partial_Y"] \\ T\underline{\mathfrak{Dflg}}(X,Y)\times TX \arrow[r, "T\mathrm{ev}_{X,Y}"'] & TY \end{tikzcd}$$

The map $c_X : X \to \underline{\mathfrak{Dflg}}(y\mathbb{R}, X)$ to constant paths descends to the zero section 0_X, so that we have the commutative diagram

$$\begin{tikzcd} \underline{\mathfrak{Dflg}}\big(y\mathbb{R}, \underline{\mathfrak{Dflg}}(X,Y)\big)\times X \arrow[r, "\mathrm{id}\times c_X"] \arrow[d, "\partial_{\underline{\mathfrak{Dflg}}(X,Y)}\times \mathrm{id}_X"'] & \underline{\mathfrak{Dflg}}\big(y\mathbb{R}, \underline{\mathfrak{Dflg}}(X,Y)\times X\big) \arrow[d, "\partial_{\underline{\mathfrak{Dflg}}(X,Y)\times X}"] \\ T\underline{\mathfrak{Dflg}}(X,Y)\times X \arrow[r, "\mathrm{id}\times 0_X"'] & T\underline{\mathfrak{Dflg}}(X,Y)\times TX \end{tikzcd}$$

Juxtaposing the last two commutative squares, we obtain the commutative square

$$\begin{tikzcd} \underline{\mathfrak{Dflg}}\big(y\mathbb{R}, \underline{\mathfrak{Dflg}}(X,Y)\big)\times X \arrow[r] \arrow[d, "\partial_{\underline{\mathfrak{Dflg}}(X,Y)}\times \mathrm{id}_X"'] & \underline{\mathfrak{Dflg}}(y\mathbb{R}, Y) \arrow[d, "\partial_Y"] \\ T\underline{\mathfrak{Dflg}}(X,Y)\times X \arrow[r] & TY \end{tikzcd}$$

By the adjunction between products and mapping spaces, this diagram is mapped to the following commutative diagram:

$$\begin{tikzcd} \underline{\mathfrak{Dflg}}\big(y\mathbb{R}, \underline{\mathfrak{Dflg}}(X,Y)\big) \arrow[r, "\cong"] \arrow[d, "\partial_{\underline{\mathfrak{Dflg}}(X,Y)}"'] & \underline{\mathfrak{Dflg}}\big(X, \underline{\mathfrak{Dflg}}(y\mathbb{R},Y)\big) \arrow[d, "(\partial_Y)_*"] \\ T\underline{\mathfrak{Dflg}}(X,Y) \arrow[r] & \underline{\mathfrak{Dflg}}(X,TY)\,, \end{tikzcd} \tag{2.32}$$

where the bottom horizontal arrow is the morphism (2.31).

Definition 2.3.1. Let $f : Y \to X$ be a morphism of diffeological spaces. The diffeological space

$$\Gamma(X,Y) := * \times^{\mathrm{id}_X, f_*}_{\underline{\mathfrak{Dflg}}(X,X)} \underline{\mathfrak{Dflg}}(X,Y)\,. \tag{2.33}$$

will be called the **space of sections** of f.

Since $* \hookrightarrow \underline{\mathfrak{Dflg}}(X,X)$, $* \mapsto \mathrm{id}_X$ is a strong monomorphism and since strong monomorphisms are stable under pullback, the map $\Gamma(X,Y) \to \underline{\mathfrak{Dflg}}(X,Y)$ is a strong monomorphism. In other words, the space of sections is equipped with the subspace diffeology of the mapping space, which means that a map of sets $p : U \to \Gamma(X,Y)$ is a plot if and only if the map $\tilde{p} : U \times X \to Y$, $\tilde{p}(u,x) = p(u)(x)$ is smooth.

Using the natural morphism (2.31), we obtain a commutative diagram

$$\begin{tikzcd} * \arrow[r, "0_{\mathrm{id}_X}"] \arrow[d] & T\underline{\mathfrak{Dflg}}(X,X) \arrow[d] & T\underline{\mathfrak{Dflg}}(X,Y) \arrow[l, "T(f_*)"'] \arrow[d] \\ * \arrow[r, "0_X"'] & \underline{\mathfrak{Dflg}}(X,TX) & \underline{\mathfrak{Dflg}}(X,TY) \arrow[l, "(Tf)_*"] \end{tikzcd}$$

which induces a morphism of the limits of each row,

$$* \times^{0_{\mathrm{id}_X}, T(f_*)}_{T\underline{\mathfrak{Dflg}}(X,X)} T\underline{\mathfrak{Dflg}}(X,Y) \longrightarrow * \times^{0_X,(Tf)_*}_{\underline{\mathfrak{Dflg}}(X,TX)} \underline{\mathfrak{Dflg}}(X,TY)\,. \tag{2.34}$$

By the universal property of the pullback, we have a natural morphism

$$T\Gamma(X,Y) \longrightarrow * \times_{T\underline{\mathfrak{Dflg}}(X,X)} T\underline{\mathfrak{Dflg}}(X,Y)\,. \tag{2.35}$$

Using that the functor $\underline{\mathfrak{Dflg}}(X, \_)$ preserves limits, we obtain the natural isomorphisms

$$\begin{aligned} & * \times^{0_X,(Tf)_*}_{\underline{\mathfrak{Dflg}}(X,TX)} \underline{\mathfrak{Dflg}}(X,TY) \\ \cong\ & * \times^{\mathrm{id}_X,\mathrm{id}}_{\underline{\mathfrak{Dflg}}(X,X)} \underline{\mathfrak{Dflg}}(X,X) \times^{(0_X)_*,(Tf)_*}_{\underline{\mathfrak{Dflg}}(X,TX)} \underline{\mathfrak{Dflg}}(X,TY) \\ \cong\ & * \times_{\underline{\mathfrak{Dflg}}(X,X)} \underline{\mathfrak{Dflg}}\big(X, X \times^{0_X,Tf}_{TX} TY\big) \\ \cong\ & \Gamma(X,VY)\,, \end{aligned} \tag{2.36}$$

where

$$VY := X \times^{0_X,Tf}_{TX} TY \tag{2.37}$$

is the space of tangent vectors in the kernel of $Tf : TY \to TX$, that is, the vertical tangent space. Note that we have to regard VY as bundle over X rather than Y. Composing (2.34), (2.35), and (2.36), we obtain a natural morphism

$$T\Gamma(X,Y) \longrightarrow \Gamma(X,VY)\,. \tag{2.38}$$

The restriction of Diagram (2.32) to the space of sections yields the commutative diagram

$$\begin{tikzcd} \underline{\mathfrak{Dflg}}\big(y\mathbb{R}, \Gamma(X,Y)\big) \arrow[d, "\partial_{\Gamma(X,Y)}"'] \arrow[dr] & \\ T\Gamma(X,Y) \arrow[r] & \Gamma(X,VY) \end{tikzcd} \tag{2.39}$$

Question 2.3.2. Under what conditions is (2.31) or (2.38) an isomorphism?

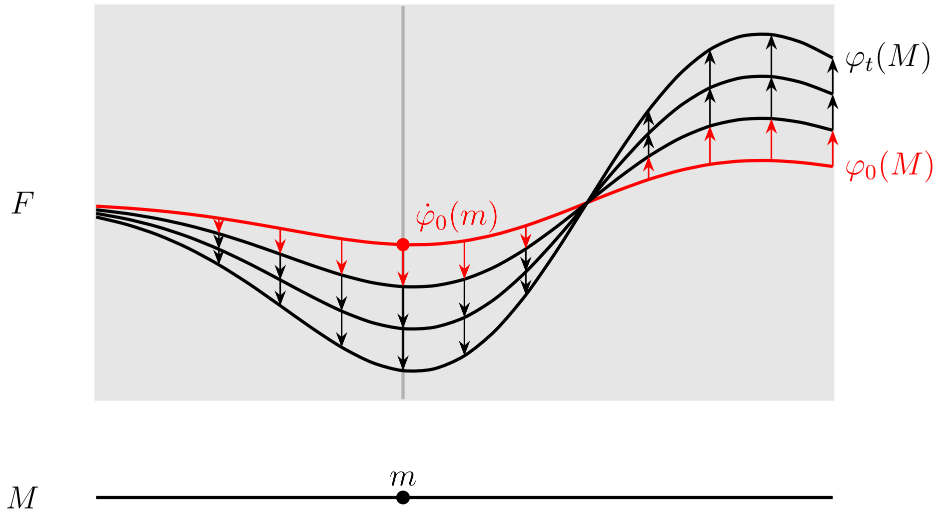


Figure 2.2: A path of sections φ_t of a fiber bundle $F \to M$. The point $\varphi_t(m) \in F$ moves vertically in the fiber above m. The velocity $\dot\varphi_0(m)$ at $t = 0$ for all m defines a vector field supported at $\varphi_0(M)$, which is depicted in red.

2.3.2 The tangent functor of the space of fields

Definition 2.3.3. Let $F \to M$ be the smooth fiber bundle of a field theory. The diffeological space of sections $\mathcal{F} = \Gamma(M, F)$ is called the **space of fields**.

The space of fields is equipped with the subspace diffeology of the functional diffeology. This means that a map $\varphi : U \to \mathcal{F}$, $u \mapsto \varphi_u$ defined on the open subset $U \subset \mathbb{R}^n$ is a plot if and only if

$$U \times M \longrightarrow F, \qquad (u, m) \longmapsto \varphi_u(m)$$

is a smooth map of finite-dimensional manifolds.

The morphism $T\,\underline{\mathrm{Hom}}(M, F) \to \underline{\mathrm{Hom}}(M, TF)$ maps the tangent vector represented by the path $t \mapsto \varphi_t$ to the map in $\Gamma(M, TF)$ that sends m to the tangent vector represented by the path $t \mapsto \varphi_t(m)$. If φ_t is a section of the bundle projection $\rho : F \to M$, then $\rho\big(\varphi_t(m)\big) = m$. It follows that

$$T\rho\big(\dot\varphi_0(m)\big) = \frac{d}{dt}\rho\big(\varphi_t(m)\big)\Big|_{t=0} = \frac{d}{dt} m\Big|_{t=0} = 0_m\,,$$

which shows that the tangent vector $\dot\varphi_0(m)$ lies in the vertical tangent bundle of $F \to M$. This is depicted in Figure 2.2.

We have the following commutative diagram of manifolds:

$$
\begin{tikzcd}
M \arrow[drr, bend left, "\dot{\varphi}_0"] \arrow[dr, dashed] \arrow[ddr, bend right, "\mathrm{id}"'] & & \\
& M \times_F VF \arrow[r] \arrow[d] & VF \arrow[d, "\pi_F"] \\
& M \arrow[r, "\varphi_0"] \arrow[dr, "\mathrm{id}"'] & F \arrow[d] \\
& & M
\end{tikzcd}
\tag{2.40}
$$

This shows that $\dot{\varphi}_0$ is a section of the bundle $VF \to M$, which covers the section $\varphi_0 = \pi_F \circ \dot{\varphi}_0$. The map

$$(\pi_F)_* : \Gamma(M, VF) \longrightarrow \Gamma(M, F) = \mathcal{F}$$

is a subduction since the zero section defines a smooth section of $(\pi_F)_*$. The $(\pi_F)_*$-fiber over $\varphi \in \mathcal{F}$ is given by

$$\Gamma(M, VF)_\varphi = \Gamma^\infty(M, \varphi^* VF)\,,$$

where $\varphi^* VF = M \times_F^{\varphi, \pi_F} VF$ is the pullback bundle.

Theorem 2.3.4. *Let $F \to M$ be a smooth fiber bundle and $\mathcal{F} = \Gamma(M, F)$ the space of fields. Then* (2.38) *is an isomorphism, so that the diffeological tangent bundle of the space of fields is given by*

$$T\mathcal{F} \cong \Gamma(M, VF)\,.$$

Proof. Diagram (2.32) for $\rho : F \to M$ yields the commutative diagram

$$
\begin{tikzcd}
\underline{\mathcal{D}\mathrm{flg}}(y\mathbb{R}, \mathcal{F}) \arrow[d, "\partial_{\mathcal{F}}"'] \arrow[dr, "\mu"] & \\
T\mathcal{F} \arrow[r, "\nu"'] & \Gamma(M, VF)
\end{tikzcd}
\tag{2.41}
$$

The structure of the proof is the following. In the first part, we will show that μ is a subduction. This implies that ν is a subduction. In the second part, which is the hardest, we will show that ν is injective. Since every injective subduction is an isomorphism, we conclude that ν is an isomorphism.

Proof that $\boldsymbol{\mu}$ is a subduction A section $\eta : M \to VF$ is a vertical vector field supported on $S = (\pi_F \circ \eta)(M) \subset F$. Since S is an embedded submanifold, we can extend η to a vertical vector field $\bar{\eta}$ on F, supported on a tubular neighborhood of S, so that η is complete. Let $\Phi : \mathbb{R} \times F \to F$ be the flow integrating $\bar{\eta}$. Then the smooth path $\varphi : \mathbb{R} \to \mathcal{F}$ defined by $\varphi_t(m) := \Phi(t, (\pi_F \circ \eta)(m))$ satisfies $\dot{\varphi}_0 = \eta$. This shows that every section in $\Gamma(M, VF)$ is the time derivative at 0 of a path in $\mathcal{F}$, so that the map μ is surjective.

Let now $p : yU \to \Gamma(M, VF)$ be a plot, which we can view as smooth homotopy $\tilde{p} : U \times M \to VF$. The smooth map $\eta : U \times M \to U \times VF$, $\eta(u,m) := \big(u, \tilde{p}(u,m)\big)$ is a section of the vertical tangent bundle of the fiber bundle $\mathrm{id}_U \times \rho : U \times F \to U \times M$. By the same argument as in the last paragraph, we can find a smooth path $\varphi : \mathbb{R} \to \Gamma(U \times M, U \times F)$, such that $\dot{\varphi}_0 = \eta$. The path φ is of the form $\tilde{\varphi}_t(u,m) = (u, \tilde{q}(t,u,m))$ for a smooth map $\tilde{q} : \mathbb{R} \times U \times M \to F$, which we can view as a plot $q : yU \mapsto \underline{\mathcal{D}\mathrm{flg}}(y\mathbb{R}, \mathcal{F})$. By construction, we have $\tilde{p}(u,m) = \frac{\partial \tilde{q}}{\partial t}(0,u,m)$. In terms of the plots p and q, this means that $p = \mu \circ q$. We conclude that μ is a subduction.

Proof that $\boldsymbol{\nu}$ is injective Let the tangent vector $\eta \in T\mathcal{F}$ be represented by the path $t \mapsto \varphi_t \in \mathcal{F}$. The section $\varphi_0 \in \mathcal{F}$ is the basepoint of η. Let $N \subset F$ be a tubular neighborhood of the embedded submanifold $S := \varphi_0(M) \subset F$. In the first step, we will show that η is represented by a path $t \mapsto \psi_t \in \mathcal{F}$ contained in N.

We can view φ as a smooth map $\tilde{\varphi} : \mathbb{R} \times M \to F$, $(t,m) \mapsto \varphi_t(m)$. In particular, $\tilde{\varphi}(\{0\} \times M) = \varphi_0(M) = S$. Let $U_i \subset M$ be an open set with compact closure, for example an open ball. Then we can find an $\varepsilon_i > 0$ sufficiently small, such that $\tilde{\varphi}\big((-\varepsilon_i, \varepsilon_i) \times U_i\big) \subset N$. Let $\alpha_i : \mathbb{R} \to \mathbb{R}$ be a smooth function with the following properties:

(i) $|\alpha_i(t)| < \varepsilon_i$ for all t.

(ii) $\alpha_i(t) = t$ for $|t| \leq \frac{1}{2}\varepsilon_i$.

From these properties it follows that $\varphi\big(\alpha_i(\mathbb{R}) \times U_i\big) \subset N$ and that $\varphi(\alpha_i(t), u) = \varphi(t,u)$ for $|t| < \varepsilon_i$, $u \in U_i$. Using a standard partition of unity argument, we obtain smooth functions $\varepsilon : M \to \mathbb{R}^+$ and $\alpha : \mathbb{R} \times M \to \mathbb{R}$, such that

(i) $|\alpha(t,m)| < \varepsilon(m)$

(ii) $\alpha(t,m) = t$ for $|t| \leq \frac{1}{2}\varepsilon(m)$

Let

$$\tilde{\psi} := \tilde{\varphi} \circ (\alpha, \mathrm{id}_M) : \mathbb{R} \times M \longrightarrow F$$

which is a homotopy of sections of $F \to M$. Properties (i) and (ii) of ε and α imply

(i') $\tilde{\psi}(t,m) \in S$ for all t and m.

(ii') $\tilde{\psi}(t,m) = \tilde{\varphi}(t,m)$ for $|t| \leq \frac{\varepsilon(m)}{2}$.

The geometric interpretation is that α is a function that squeezes $\tilde{\varphi}$ into N without changing it on the subset

$$D := \big\{(t,m) \in \mathbb{R} \times M \mid |t| < \tfrac{1}{2}\varepsilon(m)\big\},$$

which is a tubular neighborhood of $\{0\} \times M \subset \mathbb{R} \times M$. Next, we will show that ψ represents the same tangent vector as φ.

Since α is smooth, the function

$$\alpha'(t,m) := \frac{\alpha(t,m)-\alpha(0,m)}{t} = \frac{1}{t}\alpha(t,m)$$

extends smoothly to $t=0$, where it has the value $\alpha'(0,m) = (\frac{\partial}{\partial t}\alpha)(0,m) = 1$. (This is often referred to as Hadamard's lemma.) Since α' is smooth, the function

$$\alpha''(t,m) := \frac{\alpha'(t,m)-\alpha'(0,m)}{t} = \frac{1}{t^2}\alpha(t,m) - \frac{1}{t}$$

extends smoothly to $t=0$. Consider the smooth map $\beta : \mathbb{R}^2 \times M \to \mathbb{R}$ defined by

$$\beta(s,t,m) := t - s\alpha''(t,m)\,.$$

It satisfies

$$\begin{aligned}\beta(0,t,m) &= t\\ \beta(t^2,t,m) &= \alpha(t,0)\,.\end{aligned}$$

Consider the smooth map $\tilde{\chi} : \mathbb{R}^2 \times M \to F$ given by

$$\tilde{\chi}(s,t,m) := \tilde{\varphi}\big(\beta(s,t,m),m\big)\,.$$

It satisfies, $\rho\big(\tilde{\chi}(s,t,m)\big) = m$ so that it is a smooth homotopy $(s,t) \mapsto \chi_{(s,t)} \in \mathcal{F}$, $\chi_{(s,t)}(m) := \tilde{\chi}(s,t,m)$ of sections of $F \to M$. Moreover, $\tilde{\chi}(0,t,m) = \tilde{\varphi}(t,m)$ and

$$\begin{aligned}\tilde{\chi}(0,t,m) &= \tilde{\varphi}(t,m)\\ \tilde{\chi}(t^2,t,m) &= \tilde{\varphi}\big(t,\alpha(t,m)\big)\tilde{\psi}(t,m)\,.\end{aligned}$$

This means that $\varphi_t = \chi_{(0,t)}$ and $\psi_t = \chi_{(t^2,t)}$, so that we have the commutative diagram of plots

$$\begin{tikzcd} \mathbb{R} \arrow[r, "f"] \arrow[dr, "\varphi"'] & \mathbb{R}^2 \arrow[d, "\chi"] & \mathbb{R} \arrow[l, "g"'] \arrow[dl, "\psi"] \\ & \mathcal{F} & \end{tikzcd}$$

where $f(t) := (0,t)$ and $g(t) := (t^2,t)$. Since $Tf(0,1) = Tg(0,1)$, we see that the paths φ_t and ψ_t represent the same tangent vector in $\mathcal{F}$.

Let now $\varphi' : \mathbb{R} \to \mathcal{F}$ be a smooth path such that $\dot{\varphi}'_0 = \dot{\psi}_0$, that is $\mu(\varphi') = \mu(\varphi)$. We must show that φ and φ' represent the same tangent vector on $\mathcal{F}$ in the quotient (2.16). The embedded submanifold $S := \varphi_0(M) = \varphi'_0(M)$ has a tubular neighborhood $N \subset F$, that is, N is an open subset such that the inclusion $S \hookrightarrow N$ extends to a diffeomorphism from the normal bundle of S to N. Since the normal bundle of $\varphi_0(M)$ is isomorphic to $\varphi_0^* VF$, we have a diffeomorphism

$$\kappa : \varphi_0^* VF \xrightarrow{\cong} N \tag{2.42}$$

which is an isomorphism of fiber bundles over S. This induces an isomorphism of spaces of sections

$$\kappa_* : \Gamma(S, \varphi_0^* VF) \xrightarrow{\cong} \Gamma(S,N)\,. \tag{2.43}$$

As we have shown, there are paths ψ, ψ' in $\underline{\mathcal{D}\mathrm{flg}}(y\mathbb{R}, \mathcal{F})$ that represent the same tangent vectors as φ, φ' and are contained in N. Since they represent the same tangent vectors, we have $\dot\psi_0 = \dot\varphi_0$ and $\dot\varphi'_0 = \dot\psi'_0$. Since by assumption $\dot\varphi_0 = \dot\varphi'_0$ it follows that $\dot\psi_0 = \dot\psi'_0$. Since ψ and ψ' are contained in S, they are paths in the subspace $\Gamma(S, N) \subset \Gamma(M, F)$, so that they are mapped by the inverse of the isomorphism (2.43) to the paths $a := \kappa^{-1} \circ \varphi$ and $a' := \kappa^{-1}$ in the space of sections of the vector bundle $A = \varphi_0^* VF \to \varphi_0(M) = S$. Moreover, since (2.42) is an isomorphism, we have $\dot a_0 = \dot a'_0$.

In local fiber coordinates $(x^1, \ldots, x^n, u^1, \ldots, u^k)$ over a neighborhood $V \subset M$, the sections are given by the coordinate functions, which we denote by

$$\begin{aligned}
\tilde a^\alpha(t, x) &= a_t^\alpha(x^1, \ldots, x^n) \\
\tilde a'^\alpha(t, x) &= a_t'^\alpha(x^1, \ldots, x^n)\,.
\end{aligned}$$

Since $\dot a_0 = \dot a'_0$, the difference $\tilde a'^\alpha - \tilde a^\alpha$ is a function that has vanishing value and vanishing partial derivative with respect to t at $t = 0$. It follows from Hadamard's lemma that there is a smooth function $h^\alpha = h^\alpha(x, t)$ on the local coordinate chart, such that

$$\tilde a'^\alpha(t, x) - \tilde a^\alpha(t, x) = t^2 h^\alpha(t, x)\,.$$

Now we define smooth functions $\tilde p^\alpha : \mathbb{R}^2 \times V \to \mathbb{R}$ by

$$\tilde p^\alpha(s, t, x) := \tilde a^\alpha(t, x) + s^2 h^\alpha(t, x)\,.$$

It is easy to check that

$$\tilde a^\alpha = \tilde p^\alpha \circ f\,, \quad \tilde a'^\alpha = \tilde p^\alpha \circ g\,,$$

where f and g are given as above.

The maps $\tilde p^\alpha$ for $1 \le \alpha \le k$ define a smooth homotopy of local sections $\tilde p_V : \mathbb{R}^2 \to \Gamma(V, A|_V)$. Since p_V depends linearly on h we can use a standard partition of unity argument to sum the local homotopies h_{V_i} of a cover $V_i \subset S$ which yields a smooth homotopy of global sections $p : \mathbb{R}^2 \to \Gamma(S, A)$ that makes the following diagram commute:

$$\begin{tikzcd}
\mathbb{R} \arrow[r, "f"] \arrow[dr, "a"'] & \mathbb{R}^2 \arrow[d, "p"] & \mathbb{R} \arrow[l, "g"'] \arrow[dl, "a'"] \\
& \Gamma(S, A) &
\end{tikzcd}$$

By composing this diagram with the isomorphism (2.43), we obtain:

$$\begin{tikzcd}
\mathbb{R} \arrow[r, "f"] \arrow[ddr, "\psi"'] & \mathbb{R}^2 \arrow[d, "\kappa \circ p"] & \mathbb{R} \arrow[l, "g"'] \arrow[ddl, "\psi'"] \\
& \Gamma(S, N) \arrow[d, hook] & \\
& \Gamma(M, F) &
\end{tikzcd}$$

This shows that the paths ψ and ψ' represent the same tangent vector in $T\mathcal{F}$, which implies that φ and φ' represent the same tangent vector. In the notation of diagram (2.40), we have shown the following. Assume that $\varphi, \varphi' \in \underline{\mathcal{D}\mathrm{flg}}(y\mathbb{R}, \mathcal{F})$ satisfy

$\mu(\varphi) = \mu(\varphi')$, then $\partial_{\mathcal{F}}(\varphi) = \partial_{\mathcal{F}}(\varphi')$. Since $\mu = \nu \circ \partial_{\mathcal{F}}$ this means that ν is injective on the image of $\partial_{\mathcal{F}}$. Since $\partial_{\mathcal{F}}$ is surjective, we conclude that ν is injective. Since ν is a subduction, as we have already proved, it follows from Proposition 2.1.23 (iv) that it is an isomorphism. This concludes the proof. ☐

Remark 2.3.5. The proof of Theorem 2.3.4 uses in an essential way a number of properties of smooth manifolds: the extension of vector fields, the local triviality of the tangent bundle, the existence of a partition of unity, the integration of vector fields to flows, and the existence of tubular neighborhoods. For this reason, there is no obvious adaptation of the proof to more general diffeological spaces.

Corollary 2.3.6. *Let M and N be smooth manifolds viewed as diffeological spaces with the smooth diffeology. Then*

$$T\underline{\mathfrak{D}\mathrm{flg}}(M,N) \cong \underline{\mathfrak{D}\mathrm{flg}}(M,TN)\,.$$

Proof. The proof follows from Theorem 2.3.4 for the trivial bundle $F := M \times N \to M$. ☐

Corollary 2.3.7. *The fiber of the diffeological tangent bundle $T\mathcal{F} \to \mathcal{F}$ over $\varphi \in \mathcal{F}$ is*

$$T_\varphi \mathcal{F} \cong \Gamma(M, \varphi^* VF)\,. \tag{2.44}$$

Terminology 2.3.8. In the language of variational calculus, an element of $T_\varphi\mathcal{F}$ is called an **infinitesimal variation** of φ.

Corollary 2.3.9. *Let $A \to M$ be a smooth vector bundle and $\mathcal{A} = \Gamma(M, A)$ its space of sections. Then we have an isomorphism*

$$T\mathcal{A} \cong \mathcal{A} \times \mathcal{A}$$

Proof. The smooth map of fiber bundles $A \times_M A \to VA$ that maps (a_m, b_m) to the vertical tangent vector represented by the path $t \mapsto a_m + tb_m$ is an isomorphism. It follows that $\Gamma(M, VA) \cong \Gamma(M, A \times_M A) \cong \mathcal{A} \times \mathcal{A}$. ☐

Remark 2.3.10. The isomorphism of Corollary 2.3.9 is a morphism of bundles, where $\mathrm{pr}_1 : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ is the trivial bundle. This implies that $T_0\mathcal{A} \cong \mathcal{A}$. In other words, the tangent fibers of $\mathcal{A}$ can be identified with $\mathcal{A}$. Diffeological vector spaces with this property are called **tangent-stable** in [Blo].

2.3.3 Elastic diffeological spaces

A considerable part of the infinitesimal differential geometric computations on a smooth manifold M can be carried out in its Cartan calculus, which consists of the tangent bundle $TM \to M$, the Lie bracket of vector fields, the graded algebra of differential forms $\Omega(M)$, together with the de Rham differential d, the inner derivative ι_v and the Lie derivative $\mathcal{L}_v$ for every vector field v, which satisfy the relations

$$[d,d] = 0\,, \quad [\iota_v, \iota_w] = 0\,, \quad [\iota_v, d] = \mathcal{L}_v\,,$$
$$[\mathcal{L}_v, \iota_w] = \iota_{[v,w]}\,, \quad [\mathcal{L}_v, d] = 0\,, \quad [\mathcal{L}_v, \mathcal{L}_w] = \mathcal{L}_{[v,w]}\,,$$

where the bracket is the graded commutator of graded derivations of $\Omega(M)$. For example, local definitions and calculations of symplectic geometry can typically be worked out in the Cartan calculus, such as hamiltonian vector fields, Poisson brackets, hamiltonian actions, Dirac structures, generalized complex geometry, contact structures, the L_∞-algebra of a multisymplectic structure, homotopy momentum maps, infinitesimal models for equivariant cohomology, etc. In Lagrangian Field Theory, the derivation of the Euler–Lagrange equations, local symmetries, Noether's theorems, the theory of Jacobi fields, etc. take place in the Cartan calculus of the infinite jet bundle, also known as the variational bicomplex [DF99].

Question 2.3.11. What are the conditions a diffeological space must satisfy so that it is equipped with a natural Cartan calculus?

Of course, there are always the tautological conditions which promote the desired outcome to axioms, in our case the existence of a Cartan calculus. The task is to identify a set of conditions that is minimal or at least so small that it can be verified in a wide range of cases.

We have already defined the de Rham complex $\Omega(X)$ and the tangent functor TX of diffeological spaces by pointwise left Kan extension. How do we define the Lie bracket of vector fields on a diffeological space? The first guess is to start from the Lie algebras $\mathfrak{X}(U) = \Gamma(U, TU)$ of vector fields on all plots $U \to X$. However, $U \mapsto \mathfrak{X}(U)$ is not a functor, so that the left Kan extension cannot be applied. We could map the vector fields to the space of derivations of $C^\infty(X) = \Omega^0(X)$, which is equipped with the commutator bracket. However, this map is generally not injective, and even if it is, its image may not be closed under the bracket. Worse, the map $X \mapsto \mathrm{Der}(C^\infty(X))$ is still not a functor, so that this does not solve the problem of naturality. The conclusion is that the spaces of vector fields on plots are not a good starting point for the construction of a natural Cartan calculus on diffeological spaces.

Fortunately, the situation has been analyzed carefully by Rosický who has identified the natural structure of the tangent functor that is needed to define the Lie bracket of vector fields [Ros84]. He defines an abstract tangent structure on a category $\mathcal{C}$ to be an endofunctor $T : \mathcal{C} \to \mathcal{C}$ together with the natural transformations of the bundle projection $\pi_X : TX \to X$, zero section $0_X : X \to TX$, fiberwise addition $+_X : TX \times_X TX \to TX$, exchange of order of differentiation $\tau_X : T^2X \to T^2X$, and inclusion of the tangent fibers into the vertical tangent space $\lambda_X : TX \to T^2X$, which have to satisfy a rather long list of axioms. It is instructive to see how all these structures come together to define the Lie bracket of vector fields, avoiding any reference to the commutator bracket of derivations of some structure ring.

The main advantage of Rosický's approach is that all the structure is given by functors and natural transformations, to which we can apply the left Kan extension. However, this does not yield an abstract tangent structure on all diffeological spaces. The main issue is that the pointwise left Kan extension, which is given by a colimit, does not preserve limits, in particular the pullback on which the fiberwise addition of tangent vectors is defined. More precisely, the natural morphism

$$\operatorname*{colim}_{yU \to X} y(\hat{T}U \times_U \hat{T}U) \longrightarrow TX \times_X TX \,, \tag{2.45}$$

is not an isomorphism for all diffeological spaces X. In fact, this map is generally neither surjective nor injective, as the following two examples show.

Example 2.3.12 (Axis cross of the plane)**.** Consider the subset $\{(x,y) \in \mathbb{R}^2 \mid xy = 0\} \subset \mathbb{R}^2$ with the subspace diffeology. The two tangent vectors at the orgin in the direction of the x-axis and the y-axis cannot be represented on the same plot (Figure 2.3). It follows that (2.45) is not surjective.

Example 2.3.13 (Folded line)**.** Consider the diffeological quotient space of the action $\mathbb{Z}_2 \times \mathbb{R} \to \mathbb{R}$, $(k,x) \mapsto kx$, where $\mathbb{Z}_2 = \{1,-1\}$. The quotient map $\mathbb{R} \to \mathbb{R}/\mathbb{Z}_2$ is a plot. The tangent vectors $(0,1)$ and $(0,-1)$ on its domain represent the same tangent vector on $\mathbb{R}/\mathbb{Z}_2$. This implies that the pairs $\zeta = \big((0,1),(0,1)\big)$ and $\eta = \big((0,1),(0,-1)\big)$ in $T\mathbb{R} \times_{\mathbb{R}} T\mathbb{R}$ represent the same pair of tangent vectors in $T(\mathbb{R}/\mathbb{Z}_2) \times_{\mathbb{R}/\mathbb{Z}_2} T(\mathbb{R}/\mathbb{Z}_2)$. Since the tangent morphism of every morphism of plots preserves the sum of a pair of tangent vectors at a point and since the sum of η is zero but that of ζ is not, the two pairs cannot be equivalent in $\operatorname{colim}_{U \to \mathbb{R}/\mathbb{Z}_2} TU \times_U TU$. We conclude that (2.45) is not injective.

The axiom of elasticity Only if (2.45) is an isomorphism, the left Kan extension of the addition $\hat{+}_U$ of tangent vectors on plots is a morphism $+_X : TX \times_X TX \to TX$ that can be viewed as a fiberwise addition of tangent vectors on the diffeological space X. Therefore, requiring (2.45) to be an isomorphism is the first condition we have to impose for a diffeological space to have a natural Cartan calculus.

A k-form in $\Omega(X)$ is a family of k-forms on all plots $U \to X$ that are compatible with the pullbacks along morphisms of plots. A vector field, however, is not represented by a family of vector fields on the plots. For this reason, there is no natural operation of inner derivative on $\Omega(X)$. For the inner derivative, we have to define a k-form as a fiberwise multilinear and antisymmetric morphism

$$\alpha : \underbrace{TX \times_X \ldots \times_X TX}_{=: T_k X} \longrightarrow \mathbb{R}\,.$$

(We avoid defining a tensor product, which would entail the usual technical issues of completion when the fibers are infinite-dimensional.) The notation $T_k X$ for the k-fold fiber product is standard in the literature on abstract tangent structures. The inner derivative of α with respect to a vector field $v : X \to TX$ is then given by precomposition

$$\iota_v \alpha : T_{k-1}X \xrightarrow{\cong} X \times_X T_{k-1}X \xrightarrow{v \times_X \mathrm{id}} T_k X \xrightarrow{\alpha} \mathbb{R}\,.$$

If we define forms as maps $T_k X \to \mathbb{R}$, how can we define the differential? The differential of a function $f : X \to \mathbb{R}$ is given by the tangent map,

$$df : TX \xrightarrow{Tf} T\mathbb{R} \xrightarrow{\cong} \mathbb{R} \times \mathbb{R} \xrightarrow{\mathrm{pr}_2} \mathbb{R}\,.$$

However, the functions and exact 1-forms do not generate the ring of forms, so that this construction cannot be extended to higher forms.

We are now in the following dilemma. Either we define differential forms as families of forms on the plots, in which case we have a differential but no inner derivative. Or we define them as fiberwise multilinear and antisymmetric morphisms $T_k X \to \mathbb{R}$, in which case we have an inner derivative, but no differential. The way out is to require that the two notions of differential forms coincide.

We have already imposed the condition that (2.45) is an isomorphism, which induces an isomorphism

$$\mathcal{D}\mathrm{flg}(TX \times_X TX, \mathbb{R}) \xrightarrow{\cong} \lim_{yU \to X} \mathcal{E}\mathrm{ucl}(\hat{T}U \times_U \hat{T}U, \mathbb{R})\,. \tag{2.46}$$

It is easy to see that this isomorphism is equivariant with respect to the exchange of the two factors for the fiber product. Moreover, the maps are fiberwise multilinear on $T_k X$ if and only if they are on all $T_k U$. This shows that the isomorphism (2.46) induces an isomorphism from fiberwise multilinear and antisymmetric morphisms on $TX \times_X TX$ to $\Omega^2(X)$. Since we need such an isomorphism for forms of arbitrary degree k, we have to impose the following axiom:

Axiom (E1). The natural morphisms

$$\theta_{k,X} : (\mathbb{L}\hat{T}_k)X \longrightarrow T_k X\,,$$

are isomorphisms for all $k > 1$.

This axiom has the following geometric interpretation. Every tangent vector $v_x \in T_x X$ is represented by a path. One can picture this by stretching out x in the direction of v_x to a smooth path $\gamma : (-\varepsilon, \varepsilon) \to X$ of short but non-zero length through $\gamma(0) = x$, such that the coordinate tangent vector $\frac{\partial}{\partial t}$ at the origin of the interval is mapped by $T_0\gamma$ to v_x. In this sense, every point of a diffeological space has some elasticity in a single infinitesimal direction.

However, we generally cannot simultaneously stretch out x in the directions of several tangent vectors $v_x^1, \ldots, v_x^k \in TX_x$. That is, we cannot always find a plot $p : U \to X$ with $p(0) = x$ such that $(T_0 p)\frac{\partial}{\partial t^i} = v_x^i$, where $(t^1, \ldots, t^k)$ are the canonical coordinates of $U \subset \mathbb{R}^k$. And even if we can find such a plot, it may happen that the tangent map Tp is not injective at 0, so that we cannot identify the tangent vectors on X with the coordinate vectors on U. This identification is possible at every point $x \in X$ if and only if the morphism $\theta_{k,X}$ is a bijection. If in addition we want this condition to be compatible with the smooth structure, then we have to make the stronger assumption that $\theta_{k,X}$ is an isomorphism of diffeological spaces. In this sense, Axiom (E1) captures the geometric idea of the "elasticity" of a diffeological space in which any finite set of tangent directions can be streched out to a smooth "membrane" given by the image of a plot.

Example 2.3.14 (Pasta diffeologies)**.** We can equip a smooth manifold M with an alternative diffeology by defining the plots be all smooth maps $p : U \to M$ such that the rank of $Tp : TU \to TM$ is everywhere less than or equal to r. Since (i) the precomposition of p with a smooth function f does not increase the rank, (ii) the rank is a local property, and (iii) the rank of constant maps is zero, this defines a diffeology, which we call the rank-r-restricted diffeology. The rank-r-restricted diffeology is r-dimensional in the sense of Definition 1.12 in [Mag13].

For $r = 0$ we obtain the discrete diffeology. If $r = 1$, then every plot factors through $\mathbb{R}$, so that we obtain the **Spaghetti diffeology** [IZ13, Sec. 1.10, footnote

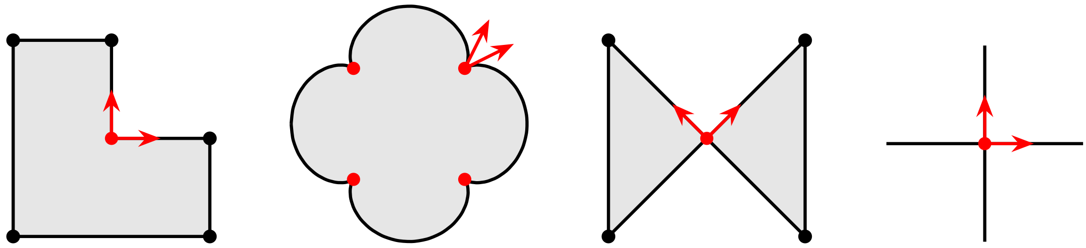


Figure 2.3: Diffeological subspaces of $\mathbb{R}^2$ with non-elastic points marked in red, at which two tangent directions cannot be represented on the same plot.

1]. The case $r = 2$ might then be called the **Fettuccine diffeology**. It was suggested by the participants of the AMS-EMS-SMF meeting 2022 in Grenoble that the case $r = 3$ should be called the **Gnocchi diffeology**. For the rank-r-restricted diffeology the morphism $\theta_{k,M}$ of Axiom (E1) is an isomorphism for all $k \leq r$ but not for $r < k < \dim M$.

The additional axioms So far we have the Axiom (E1) that ensures that we have a fiberwise addtion on TX and an inner derivative on differential forms. For the definition of the Lie bracket we need more structure. In particular, we need a natural morphism $\tau_X : T^2X \to T^2X$ that exchanges the order of differentiation when we apply the tangent functor twice. On a euclidean space $U \subset \mathbb{R}^n$, every tangent vector is represented by a path $\mathbb{R} \to U$ on some plot, so that a tangent vector on the manifold of tangent vectors is represented by a smooth path of smooth paths, which is the same as a smooth map $\mathbb{R}^2 \to U$. Exchanging the order of differentiation is achieved by exchanging the parameters,

$$\begin{aligned} \tau_{1\leftrightarrow 2} : \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ (t_1, t_2) &\longmapsto (t_2, t_1)\,, \end{aligned}$$

which descends by the commutative diagram

$$\begin{array}{ccc} C^\infty(\mathbb{R}^2, U) & \xrightarrow{\tau^*_{1\leftrightarrow 2}} & C^\infty(\mathbb{R}^2, U) \\ \big\downarrow & & \big\downarrow \\ \hat{T}^2U & \xrightarrow[\hat{\tau}_U]{} & \hat{T}^2U \end{array} \tag{2.47}$$

to an endomorphism of $\hat{T}^2U$.

When we extend $\hat{\tau}_U$ to diffeological spaces, the problem arises that the left Kan extension does not preserve the product of endofunctors, that is, the natural morphism

$$\theta^2_X : (\mathbb{L}\hat{T}^2)X \longrightarrow T^2X$$

is generally not an isomorphism. We could impose the condition that θ^2_X is an isomorphism, but this would be unnecessarily strong. It suffices to require the left Kan extension of τ_U to descend to a morphism $\tau_X : T^2X \to T^2X$. It can be shown that θ^2_X is a subduction for all X, so that such a τ_X is unique. This condition can be expressed more intuitively in terms of the smooth families in the same way as for euclidean spaces. We can show that we can represent elements in T^2X by plots $\mathbb{R}^2 \to X$. More precisely, we have a subduction

$$\underline{\mathcal{D}\mathrm{flg}}(\mathbb{R}^2, X) \longrightarrow T^2X \,.$$

The second axiom can now be expressed in a way that is completely analogous to diagram (2.47).

Axiom (E2). There is a natural morphism $\tau_X : T^2X \to T^2X$, such that the diagram

$$\begin{tikzcd} \underline{\mathcal{D}\mathrm{flg}}(\mathbb{R}^2, X) \arrow[r, "\tau_{1\leftrightarrow 2}^*"] \arrow[d] & \underline{\mathcal{D}\mathrm{flg}}(\mathbb{R}^2, X) \arrow[d] \\ T^2X \arrow[r, "\tau_X"'] & T^2X \end{tikzcd}$$

commutes.

Next, consider the natural morphism $\lambda_X : TX \to T^2X$ that maps $v \in TX$ to the vertical tangent vector on TX represented by the path $t \mapsto tv$. On a smooth manifold, this morphism induces an isomorphism between every tangent space and the tangent space of the tangent space. For diffeological vector spaces this can fail, as the following example shows.

Example 2.3.15. Consider $\mathbb{R}^n$ equipped with k-times differentiable maps as plots. This is a diffeological vector space that we denote by $\mathbb{R}^n_{C^k}$. Its tangent diffeological space is given for $k > 0$ by

$$T\mathbb{R}^n_{C^k} \cong \mathbb{R}^n_{C^k} \times \mathbb{R}^n_{C^{k-1}} \,,$$

which shows that the vector space and its tangent fiber are not isomorphic. Assume that $k > 1$, so that we can apply the tangent functor twice. The vertical lift,

$$\begin{aligned} \lambda_{\mathbb{R}^n_{C^k}} : \mathbb{R}^n_{C^k} \times \mathbb{R}^n_{C^{k-1}} &\longrightarrow \mathbb{R}^n_{C^k} \times \mathbb{R}^n_{C^{k-1}} \times \mathbb{R}^n_{C^{k-1}} \times \mathbb{R}^n_{C^{k-2}} \\ (x, v) &\longmapsto (x, 0, 0, v) \,, \end{aligned}$$

is not a subduction.

The definition of the Lie bracket in terms of the tangent structure yields a map from X to the vertical subbundle of T^2X restricted to the zero section of TX. We have to be able to identify this bundle with TX for the bracket to be again a vector field. This condition is not specific to diffeological spaces. A vector field on a C^k-manifold is a C^k-map. The commutator of two such vector fields is a C^{k-1}-map which is, therefore, not a vector field on the C^k-manifold. To exclude such phenomena we have to impose the following axiom:

Axiom (E3). The vertical lift $\lambda_X : TX \to T^2X$ is an induction.

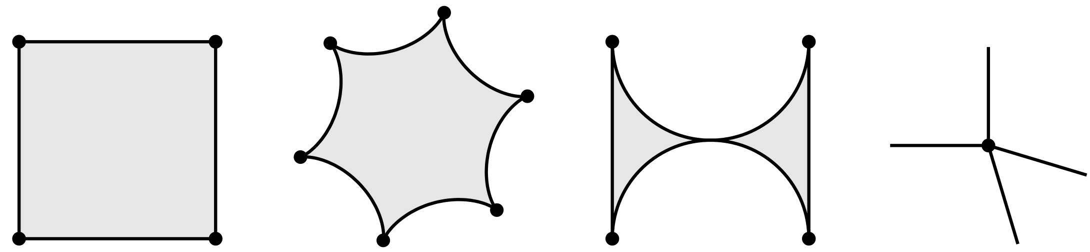


Figure 2.4: Elastic diffeological subspaces of $\mathbb{R}^2$. The tangent spaces are 0 at the marked points, $\mathbb{R}$ at points on the black lines, and $\mathbb{R}^2$ at gray points in the interior.

There are two more axioms. For smooth manifolds the tangent functor commutes with pullbacks over submersions. This follows from the local standard form of submersions, which is proved using the implicit function theorem. Such a genuinely analytic result cannot hold for all diffeological spaces, which is why we need to impose the following axiom:

Axiom (E4). The tangent functor commutes with fiber products of the tangent bundle, $TT_kX \cong T_kTX$.

Finally, we want the diffeological spaces that satisfy our axioms to form a category. This requires the collection of diffeological spaces that satisfy the axioms to be closed under the functors T_k, which leads to the following axiom:

Axiom (E5). For every finite set of positive integers $k_1, \dots, k_n$ the diffeological space $X' := T_{k_1} \cdots T_{k_n} X$ satisfies axioms (E1) through (E4).

Definition 2.3.16. A diffeological space that satisfies Axioms (E1)-(E5) will be called **elastic**.

Theorem 2.3.17. *On elastic diffeological spaces there is a natural Cartan calculus.*

Remark 2.3.18. If we drop Axiom (E5), then we still have a natural Cartan calculus on X. We call a diffeological space that satisfies Axioms (E1)-(E4) **weakly elastic**. The category of weakly elastic spaces is not closed under the functors T_k.

Theorem 2.3.19. *The diffeological space of sections $\Gamma(M, F)$ of a smooth fiber bundle $F \to M$ is elastic.*

Exercises

Exercise 2.1 (Concrete presheaves)**.** Let $X : \mathcal{E}\mathrm{ucl}^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$ be a concrete presheaf. (The elements in the image of the injection $X(U) \hookrightarrow \mathcal{S}\mathrm{et}(|U|, X(*))$ will be called plots.) Show that the following are equivalent:

(i) X is a sheaf.

(ii) A map $p : |U| \to X(*)$ is a plot if for every cover $\{U_i \to U\}$ all restrictions $p_i : |U_i| \to X(*)$ are plots.

Exercise 2.2 (Diffeologies on the half plane)**.** Let $\mathbb{R}^2$ be equipped with the manifold diffeology. Let $\mathbb{H} := \mathbb{R} \times [0, \infty)$ denote the upper half plane. $\mathbb{H}$ equipped with the subspace diffeology will be denoted by $\mathbb{H}_{\text{sub}}$. $\mathbb{H}$ equipped with the quotient diffeology of $\rho : \mathbb{R}^2 \to \mathbb{R}^2/\mathbb{Z}_2 \cong \mathbb{H}$, where $\mathbb{Z} = \{1, -1\}$ acts on $\mathbb{R}^2$ by $k \cdot (x, y) = (x, ky)$, will be denoted by $\mathbb{H}_{\text{quo}}$.

Show that the identity map of $\mathbb{H}$ is a morphism of diffeological spaces $\mathbb{H}_{\text{sub}} \to \mathbb{H}_{\text{quo}}$, but not a morphism from $\mathbb{H}_{\text{quo}}$ to $\mathbb{H}_{\text{sub}}$. Show that π does not have a section.

Exercise 2.3 (Non-standard diffeologies on a manifold)**.** Show that the following collections of plots define diffeologies on a smooth manifold M:

(i) The plots are the k-times differentiable maps $p : U \to M$. (C^k-diffeology)

(ii) Let $\mathcal{S}$ be a foliation of M. The plots are the smooth maps $p : U \to M$ such that the image of $Tp : TU \to TM$ is contained in $T\mathcal{S}$. (foliation diffeology)

(iii) The plots are the smooth maps $p : U \to M$ for which the rank of $Tp : TU \to TM$ is everywhere less than or equal to 1. (Spaghetti diffeology)

Can you come up with a definition for the Fettuccine diffeology?

Exercise 2.4. Show that strong monomorphisms and strong epimorphisms are stable under retracts, that is, if there is a commutative diagram

$$\begin{tikzcd}
X \arrow[r, "i"] \arrow[d, "f"] \arrow[rr, bend left, "\mathrm{id}_X"] & Y \arrow[r, "p"] \arrow[d, "g"] & X \arrow[d, "f"] \\
X' \arrow[r, "i'"] \arrow[rr, bend right, "\mathrm{id}_{X'}"'] & Y' \arrow[r, "p'"] & X'
\end{tikzcd}$$

and g is a strong monomorphism (strong epimorphism), then so is f.

Exercise 2.5. Compute the diffeological tangent spaces of the following diffeological spaces:

(a) the half plane with the subspace diffeology $\mathbb{H}_{\text{sub}}$ from Exercise 6

(b) the folded plane with the quotient diffeology $\mathbb{H}_{\text{quot}}$ from Exercise 6

(c) the 1st quadrant $X = \{(x, y) \in \mathbb{R}^2 \mid x \geq 0 \ \wedge \ y \geq 0\}$ with the subspace diffeology

(d) the 1st and 3rd quadrant $Y = \{(x, y) \in \mathbb{R}^2 \mid xy \geq 0\}$ with the subspace diffeology

Exercise 2.6. Let M be a closed manifold, that is, compact without boundary. Let vol be a volume form on M, let $m_1, m_2 \in M$ be two points, and let $K : M \to$

$\underline{\mathcal{D}\mathrm{flg}}(M,\mathbb{R})$ be a smooth map of diffeological spaces. Consider the following maps of sets:

$$\begin{aligned} f : \underline{\mathcal{D}\mathrm{flg}}(M,\mathbb{R}) &\longrightarrow \mathbb{R} \\ f(\varphi) &= \varphi(m_1) + \varphi(m_2) \end{aligned}$$

$$\begin{aligned} g : \underline{\mathcal{D}\mathrm{flg}}(M,\mathbb{R}) &\longrightarrow \mathbb{R} \\ g(\varphi) &= \int_M \varphi^2 \,\mathrm{vol} \end{aligned}$$

$$\begin{aligned} h : \underline{\mathcal{D}\mathrm{flg}}(M,\mathbb{R}) &\longrightarrow \underline{\mathcal{D}\mathrm{flg}}(M,\mathbb{R}) \\ \big(h(\varphi)\big)(m) &= \int_M K(m)\varphi \,\mathrm{vol}\,. \end{aligned}$$

Show that all three maps are morphisms of diffeological spaces and compute their tangent maps.

Exercise 2.7. Let $C^\infty_{L^1}(\mathbb{R})$ denote the set of integrable smooth functions on $\mathbb{R}$, so that the map

$$\begin{aligned} S : C^\infty_{L^1}(\mathbb{R}) &\longrightarrow \mathbb{R} \\ \varphi &\longmapsto \int_{x=-\infty}^{\infty} \varphi(x)\, dx \end{aligned}$$

is defined. Show that S is not smooth with respect to the subspace diffeology of $\underline{\mathcal{D}\mathrm{flg}}(\mathbb{R},\mathbb{R})$. (Hint: Find a smooth path of integrable functions $t \mapsto h_t \in C^\infty(\mathbb{R})$, such that the integral of h_t for $t \neq 0$ is constant and non-zero, but $h_0 = 0$.) Is the map $\varphi \mapsto \int_{x=-\infty}^{\infty} \varphi^2(x)\, dx$ smooth with respect to the subspace diffeology of $C^\infty_{L^2}(\mathbb{R})$?

Chapter 3

Locality and jets

3.1 Jets

3.1.1 Jet bundles

Definition 3.1.1. Two local sections φ and φ' of a smooth fiber bundle $F \to M$ defined on a neighborhood of m have the same **k-jet at** m, denoted by $j^k_m\varphi = j^k_m\varphi'$, if they have the same value and partial derivatives up to k-th order at m.

It is not immediately clear that this is a good definition, since the partial derivatives of a section generally depend on the choice of coordinates. For example, the section of a line bundle is given in local coordinates by an $\mathbb{R}$-valued function $\varphi \in C^\infty(M)$. While this function may be constant in one set of coordinates, so that its derivatives vanish, it will generally have non-zero derivatives in other coordinates. But if two functions φ and φ' have the same value and first derivatives at m in one set of coordinates, this will be true in any coordinates, since a change of coordinates will transform the derivatives by the same linear map. If the partial derivatives of φ and φ' up to k-th order are equal in one set of coordinates, it follows from the chain rule that this remains true in any coordinates (Exercise 3.1). We conclude that having the same partial derivatives at a point m up to a given degree k is an equivalence relation on the space of all local sections on a neighborhood of m. The k-jets are the equivalence classes of this relation.

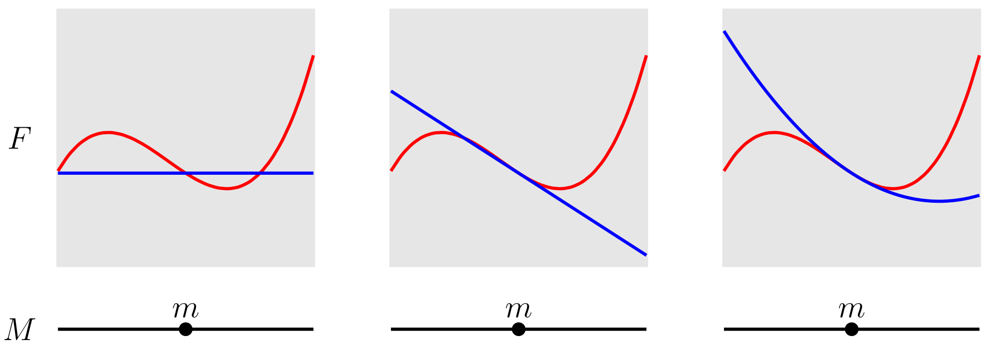


Figure 3.1: Sections of a fiber bundle $F \to M$ that have the same 0-jet, 1-jet, and 2-jet at m.

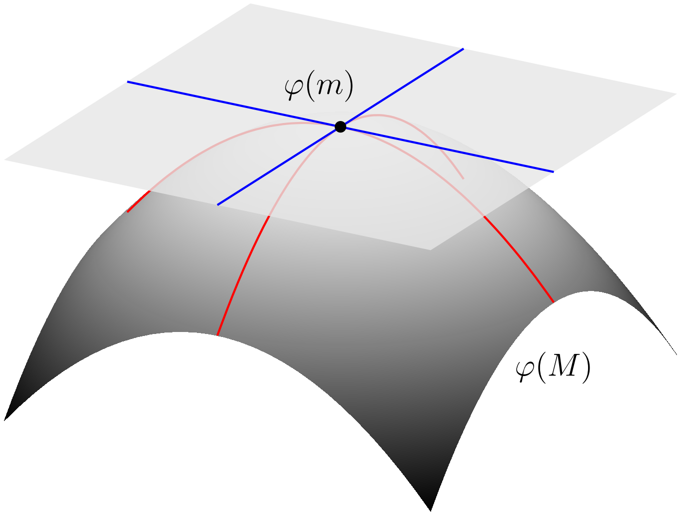


Figure 3.2: The 1-jet at $m \in M$ of a smooth map $\varphi : M \to F$ can be identified with the tangent plane of its image at $\varphi(m)$.

Definition 3.1.2. Two smooth maps $f, g : M \to N$ of manifolds have the same k**-jet** at $m \in M$ if the sections $m \mapsto (m, f(m))$ and $m \mapsto (m, g(m))$ of the trivial bundle $M \times N \to M$ have the same k-jet at m in the sense of Definition 3.1.1.

Remark 3.1.3. Two sections of $F \to M$ have the same k-jet at m in the sense of Definition 3.1.1 if and only if, when viewed as smooth maps $M \to F$, they have the same k-jet at m in the sense of Definition 3.1.2. This shows that the two definitions of jets are equivalent.

Terminology 3.1.4. The natural number k in Definition 3.1.1 and Definition 3.1.2 is called the **order** of the jet.

Example 3.1.5. Two smooth paths $f, g : \mathbb{R} \to M$ have the same 1-jet at $t = 0$ if and only if they represent the same tangent vector.

The last example shows that the concept of jets can be viewed as a generalization of tangent vectors in two ways. First, the domain is generalized from a line $\mathbb{R}$ to a higher dimensional manifold, so that tangent vectors are generalized to tangent planes (Figure 3.2). Second, tangent planes are generalized to surfaces given by higher order polynomials. The geometric meaning of jets is then that two sections have the same jet at m if they have the same value (0-jet), the same tangent plane (1-jet), the same osculating ellipsoid or hyperboloid (2-jet), etc. at m. This is sometimes expressed by saying that, when two sections φ and φ' have the same k-jet at m, they are tangent to k-th order at $\varphi(m)$ (Figure 3.1).

The analogy with tangent vectors can be taken further by generalizing the concept of tangent spaces and tangent bundles. The set of all k-jets at m is denoted by

$$J^k_m F = \{ j^k_m \varphi \mid \text{for all open } U \ni m \text{ and all } \varphi \in \Gamma(U, F) \} \,.$$

The union of all jets at all m will be denoted by

$$J^k F := \bigcup_{m \in M} J^k_m F \,.$$

On the set of k-jets we have the natural projection

$$\mathrm{pr}_{k,-1} : J^k F \longrightarrow M\,, \quad j^k_m\varphi \longmapsto m\,,$$

to the base-point of every jet. The fiber of $\mathrm{pr}_{k,-1}$ over m is $J^k_m F$.

Example 3.1.6. Let $F = \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be the trivial line bundle over $\mathbb{R}$, so that $\mathcal{F} = C^\infty(\mathbb{R})$. The k-jet of a function $\varphi \in C^\infty(\mathbb{R})$ at $m \in \mathbb{R}$ can be identified with the k-th Taylor polynomial of φ at m. This induces an isomorphism

$$J^k_m(\mathbb{R} \times \mathbb{R}) \cong \mathbb{R}[\varepsilon]/(\varepsilon^{k+1})\,.$$

In the language of algebraic geometry, this is the ring of functions on the k-th infinitesimal neighborhood of m.

Example 3.1.7. Let $F = \mathbb{R} \times Q \to \mathbb{R}$ be a trivial bundle over $\mathbb{R}$. A section of F is given by a path $q : \mathbb{R} \to Q$. Its 1-jet at s is given by the tangent vector $v = \frac{d}{dt} q(t)|_s$. This shows that a jet is given by a pair $(s, v) \in \mathbb{R} \times TQ$, so that we have a bijection

$$J^1(\mathbb{R} \times Q) \cong \mathbb{R} \times TQ\,.$$

The bijection of Example 3.1.7 equips $J^1(\mathbb{R} \times Q)$ with the structure of a smooth manifold. Proving that every $J^k F$ is a smooth manifold is analogous to the tangent manifold of a smooth manifold: We choose local bundle coordinates on F and show that these induce local coordinates on $J^k F$.

Let $(x^1, \ldots, x^n, u^1, \ldots, u^r)$ be a system of local bundle coordinates of F, that is, (x^i) are the base coordinates and (u^α) the fiber coordinates of some local trivialization. This induces coordinates $(x^i, u^\alpha, u^\alpha_{i_1}, u^\alpha_{i_1,i_2}, \ldots, u^\alpha_{i_1,\ldots,i_k})$ on $J^k F$ given by

$$\begin{aligned} x^i, u^\alpha_{i_1,i_2,\ldots,i_l} &: J^k F \longrightarrow \mathbb{R}\,, \\ x^i\big(j^k_m\varphi\big) &:= x^i(m)\,, \\ u^\alpha_{i_1,i_2,\ldots,i_l}\big(j^k_m\varphi\big) &:= \frac{\partial^l (u^\alpha \circ \varphi)}{\partial x^{i_1} \partial x^{i_2} \cdots \partial x^{i_l}}\Big|_m\,, \end{aligned} \tag{3.1}$$

for all $l \le k$ and all sequences $i_1, \ldots, i_l$ of indices. In order to handle the indices efficiently we will use multi-index notation.

Notation 3.1.8. Let $(x^1, \ldots, x^n) = (x^i)$ be local coordinates indexed by $1 \le i \le n$. A **multi-index** is an n-tuple $I = (I_1, \ldots, I_n) \in \mathbb{N}_0^n$. It is used for the compact notation

$$x^I := (x^1)^{I_1}(x^2)^{I_2} \cdots (x^n)^{I_n}$$

of monomials in n generators. The number

$$|I| := I_1 + I_2 + \ldots + I_n$$

is called the **length** or **order** of I. Our main use of multi-indices is for higher partial derivatives,

$$\begin{aligned} \frac{\partial^{|I|}}{\partial x^I} &:= \frac{\partial^{|I|}}{(\partial x^1)^{I_1}(\partial x^2)^{I_2} \cdots (\partial x^n)^{I_n}} \\ &= \Big(\frac{\partial}{\partial x^1}\Big)^{I_1}\Big(\frac{\partial}{\partial x^2}\Big)^{I_2} \cdots \Big(\frac{\partial}{\partial x^n}\Big)^{I_n} = \Big(\frac{\partial}{\partial x}\Big)^{I}\,. \end{aligned}$$

This suggests the notation

$$u_I^\alpha(j_m^k\varphi) := \frac{\partial^{|I|}\varphi^\alpha}{\partial x^I}\Big|_m . \tag{3.2}$$

for the jet bundle coordinates.

For every number $1 \leq i \leq n$, we define the **concatenation** of I with i by

$$I, i := (I_1, \ldots, I_{i-1}, I_i + 1, I_{i+1}, \ldots, I_n) .$$

The concatenation of the multi-index $0 = (0, \ldots, 0)$ will be denoted by $0, i = i$. This makes the multi-index notation (3.2) consistent with that of Equation (3.1). That is, if $I = i_1, i_2, \ldots, i_l$ is the concatenated multi-index, then $u_I^\alpha = u_{i_1,\ldots,i_l}^\alpha$. While multi-indices label the coordinates u_I^α uniquely, the concatenation $i_1, \ldots, i_k$ of different sequences can represent the same multi-index. In fact, let I be a multi-index of order k. Then

$$\#\{(i_1, \ldots, i_k) \in \{1, \ldots, n\}^k \mid I = i_1, \ldots, i_k\} = \frac{k!}{I!} ,$$

where the multi-index factorial is defined by

$$I! := I_1! I_2! \cdots I_n! .$$

This combinatorial factor has to be taken into account when changing between the summation over multi-indices I and sequences $i_1, \ldots, i_k$. Let C_I be some finite sequence labelled by the multi-index I, then

$$\sum_I C_I = \sum_k \frac{[i_1, \ldots, i_k]!}{k!} \sum_{1 \leq i_1, \ldots, i_k \leq n} C_{i_1,\ldots,i_k} , \tag{3.3}$$

where $[i_1, \ldots, i_k]!$ denotes the multi-index factorial of the multi-index $I = i_1, \ldots, i_k$. The concatenation of two multi-indices is given by the sum

$$I + J = (I_1 + J_1, \ldots, I_n + J_n) .$$

Splitting the sum over a multi-index into the sum over two concatenated multi-indices we again have to take into account combinatorial factors,

$$\sum_I C_I = \sum_J \sum_K \frac{J!K!}{(J+K)!} C_{J+K} . \tag{3.4}$$

As special case, we have

$$\sum_I C_I = \sum_J \sum_{k=1}^n \frac{1}{(J_k + 1)} C_{J,k} . \tag{3.5}$$

Further usages of multi-indices will be explained as they occur.

Example 3.1.9. The Taylor expansion at the point x_0 of an analytic function $(\varphi^1, \ldots, \varphi^r) : \mathbb{R}^n \to \mathbb{R}^r$ can be written in multi-index notation as

$$\varphi^\alpha(x) = \sum_{|I|=0}^{\infty} \frac{1}{I!} \frac{\partial^{|I|}\varphi^\alpha}{\partial x^I}\Big|_{x_0} (x - x_0)^I \,,$$

which shows that the jet bundle coordinates of $j^k_m\varphi$ can be identified with the k-th Taylor polynomial of φ^α at $x_0 = (x^1(m), \ldots x^n(m))$. In this sense, a k-jet can be viewed as the coordinate independent version of the k-th Taylor polynomial.

It is straight-forward to show that the transition functions from one set of jet bundle coordinates to another are smooth (see Exercise 3.1). The conclusion is the following proposition.

Proposition 3.1.10. *Let $F \to M$ be a smooth fiber bundle. Then J^kF has the natural structure of a smooth manifold and $J^kF \to M$ is a smooth fiber bundle.*

For every $k > l \geq 0$ there is a **forgetful map**

$$\mathrm{pr}_{k,l} : J^kF \longrightarrow J^lF \,, \quad j^k_m\varphi \longmapsto j^l_m\varphi \,,$$

which forgets the partial derivatives of order higher than l. In local jet coordinates it is the projection

$$(x^i, u^\alpha, u^\alpha_{i_1}, \ldots, u^\alpha_{i_1,\ldots,i_k}) \longmapsto (x^i, u^\alpha, u^\alpha_{i_1}, \ldots, u^\alpha_{i_1,\ldots,i_l}) \,, \tag{3.6}$$

which shows that $\mathrm{pr}_{k,l}$ is a surjective submersion and a map of fiber bundles over M.

3.1.2 Jet evaluation and prolongation

Definition 3.1.11. The map

$$\begin{aligned} j^k : \mathcal{F} \times M &\longrightarrow J^kF \\ (\varphi, m) &\longmapsto j^k_m\varphi \end{aligned}$$

is called the k-th **jet evaluation**.

In general, the jet evaluations are not surjective. For example, when $F \to M$ is a non-trivial principal bundle then F has no global sections at all, so the image of j^k is empty. Another important example is the bundle of lorentzian metrics in general relativity, which does not have a global section if the base manifold is closed with non-vanishing Euler characteristic. This is the reason why jets are defined to be represented by local sections. Here is a criterion for the surjectivity of the jet evaluations.

Lemma 3.1.12. *Let $F \to M$ be a smooth fiber bundle. Assume that the evaluation j^0 is surjective, that is, for every point of F there is a global section through that point. Then the jet evaluations j^k are surjective for all $k > 0$.*

Proof. Assume that j^0 is surjective. Then for any k-jet $j^k_m\varphi$ represented by a local section φ, there is a global section $\psi : M \to F$ such that $\psi(m) = \varphi(m)$. We can choose local bundle coordinates (x^i, u^α) on an open neighborhood $U \times V \subset F$ such that φ is defined on U and such that $\varphi(U)$, $\psi(U)$ are both contained in $U \times V$. Furthermore, we can choose the coordinates such that $\psi^\alpha = 0$ on U. Let f be a smooth bump function on U with support contained in U and locally constant value 1 on a small neighborhood of m. Then there is a smooth global section χ defined by $\chi(x) = \psi(x)$ for $x \notin U$ and $\chi^\alpha(x) = f(x)\varphi^\alpha(x)$ for $x \in U$, which satisfies $j^k_m\chi = j^k_m\varphi$. This shows that every k-jet has a preimage under j^k. □

Proposition 3.1.13. *Let $F \to M$ be a smooth fiber bundle with connected fibers. The jet evaluations j^k are surjective for all $k \geq 0$ if and only if $F \to M$ has a global section.*

Proof. Assume that $j^k : \mathcal{F} \times M \to J^kF$ is surjective for all $k \geq 0$. Then the image of j^k is non-empty, so that $\mathcal{F}$ must be non-empty.

Conversely, assume that $\varphi \in \mathcal{F}$. Let $p \in F_m$. Since by assumption F_m is path-connected, there is a smooth path $\gamma : [0,1] \mapsto F_m$ with $\gamma(0) = \varphi(m)$ and $\gamma(1) = p$. Let $U \subset M$ be an open neighborhood of m and $F|_U \cong U \times F_m$ a trivialization in which the section φ is constant, i.e. $\varphi(u) = (u, \varphi(m))$ for all $u \in U$. Let $V \subset U$ be an open ball containing m such that the closure of V is contained in U. Then there is a smooth bump function $f : U \to [0,1]$ such that $f(m) = 1$ and $f(u) = 0$ for all $u \in U \setminus V$. Now we can define a local section $\psi : U \to F$ which is given in the trivialization by $\psi(u) = \big(u, \gamma(f(u))\big)$. By construction, $\psi(m) = p$ and $\psi(u) = \varphi(u)$ for all $u \in U \setminus V$. The section defined by ψ on U and by φ on $M \setminus U$ is a global smooth section of F through p. This shows that j^0 is surjective. It now follows from Lemma 3.1.12 that j^k is surjective for all $k \geq 0$. □

Proposition 3.1.14. *The jet evaluations $\mathcal{F} \times M \to J^kF$ are smooth maps of diffeological spaces.*

Proof. A path $t \mapsto (\varphi_t, m_t) \in \mathcal{F} \times M$ is smooth in the diffeology if $t \mapsto \varphi_t$ is a smooth homotopy of sections given by a smooth map of manifolds $\varphi : \mathbb{R} \times M \to F$ and if $m : \mathbb{R} \to M$ is a smooth map of manifolds.

Let (x^i, u^α) be local bundle coordinates on F. Then $t \mapsto \varphi^\alpha_t = u^\alpha \circ \varphi_t$ and $t \mapsto m^i_t = x^i(m_t)$ are the paths in local coordinates. Let (x^i, u^α_I) be the induced coordinates on J^kF, so that

$$\begin{aligned} x^i\big(j^k(\varphi_t, m_t)\big) &= m^i_t \\ u^\alpha_I\big(j^k(\varphi_t, m_t)\big) &= \frac{\partial^{|I|}\varphi_t}{\partial x^I}(m_t)\,. \end{aligned} \tag{3.7}$$

By assumption m^i_t is a smooth function of t. Since all partial derivatives of the smooth map of manifolds φ are smooth, the maps $t \mapsto u^\alpha_I\big(j^k(\varphi_t, m_t)\big)$ are all smooth. We conclude that $\mathbb{R} \to J^kF$, $t \mapsto j^k(\varphi_t, m_t)$ is a smooth map of manifolds. This argument generalizes from paths to smooth families in $\mathcal{F} \times M$ that are parametrized by open subsets of $\mathbb{R}^n$. □

Proposition 3.1.15. *Let φ be a section of the fiber bundle $F \to M$. The map*

$$j^k\varphi : M \longrightarrow J^kF\,, \quad m \longmapsto j^k_m\varphi\,,$$

*is a section of the k-th jet bundle, called the **k-th jet prolongation** of φ.*

Proof. This is easily checked in local jet coordinates in which $j^k\varphi$ is given by

$$u^\alpha_{i_1,\dots,i_k}(j^k\varphi) = \frac{\partial^k\varphi^\alpha}{\partial x^{i_1}\dots\partial x^{i_k}}\,, \tag{3.8}$$

which is a smooth function of the local base coordinates $(x^1,\dots,x^n)$. □

Notation 3.1.16. In the physics literature, the right side of Equation (3.8) often denotes both, the jet bundle coordinates of the prolongation of a single field φ and the coordinates functions $u^\alpha_{i_1,\dots,i_k}$ themselves. This is analogous to the coordinates $(x^1,\dots,x^n)$ of a manifold, which can denote both, the coordinates of a single point x and the coordinate functions of a chart. For example, consider the action in classical mechanics, $S(q) = \int_{\mathbb{R}} L\big(q^\alpha,\dot q^\alpha\big)dt$. On the one hand, $S(q)$ can be viewed as the action of a single path $q^\alpha \in C^\infty(\mathbb{R},Q)$. In this case, the integrand is a closed 1-form on $\mathbb{R}$, which is always exact. On the other hand, during the derivation of the Euler–Lagrange equation, we discard exact terms under the integral. So for the step "discarding exact terms" to be meaningful, we need to view the arguments of $L(q^\alpha,\dot q^\alpha)$ as jet coordinate functions rather than as the coordinates of the first prolongation of a single path q^α.

Terminology 3.1.17. A section of a jet bundle $J^kF \to M$ that is the prolongation of a section of F is called **holonomic**, and a section that is not a prolongation **non-holonomic**. This language originates historically from the theory of constrained mechanical systems.

Remark 3.1.18. Proposition 3.1.15 allows us to view the k-th jet evaluation equivalently as map

$$j^k : \mathcal{F} \longrightarrow \Gamma(M, J^kF)\,, \quad \varphi \longmapsto j^k\varphi\,,$$

which is a morphism of diffeological spaces.

Proposition 3.1.19. *Let $f : E \to F$ be a morphism of smooth fiber bundles over M. Then*

$$\begin{aligned} J^kf : J^kE &\longrightarrow J^kF \\ j^k_m\varphi &\longmapsto j^k_m(f\circ\varphi)\,, \end{aligned}$$

*is a well-defined morphism of fiber bundles over M called the **k-th jet prolongation** of f.*

Proof. It follows from the chain rule for partial derivatives that $j^k_m(f\circ\varphi)$ depends only on $j^k_m\varphi$, so that j^kf is well-defined. The chain rule also shows that j^kf is smooth. □

Remark 3.1.20. If $E = M$ is the rank 0 fiber bundle over M, a smooth map $E \to F$ covering the identity is a section of F. Its k-th prolongation in the sense of Proposition 3.1.19 is the prolongation in the sense of Proposition 3.1.15.

Let $f : F \to F'$ and $g : F' \to F''$ be morphisms of smooth fiber bundles over M. Let φ be a section of F. Then

$$\begin{aligned}\big(J^k(g \circ f)\big)(j^k_m \varphi) &= j^k_m\big((g \circ f) \circ \varphi\big) = j^k_m\big(g \circ (f \circ \varphi)\big) \\ &= j^k g\big(j^k_m(f \circ \varphi)\big) = j^k g\big((J^k f)(j^k_m \varphi)\big) \\ &= (J^k g \circ J^k f)(j^k_m \varphi)\,,\end{aligned}$$

which shows that the jet prolongation is functorial. This can be stated as follows.

Proposition 3.1.21. *J^k is an endofunctor of the category of smooth fiber bundles over M.*

Example 3.1.22. Let $E = \mathbb{R} \times X$ and $F = \mathbb{R} \times Y$ be trivial bundles over $\mathbb{R}$. A smooth map $f : X \to Y$ of the fibers can be viewed as morphism $\tilde{f} : (t, x) \mapsto (t, f(x))$ of smooth fiber bundles over $\mathbb{R}$. Its first jet prolongation is given by

$$\begin{aligned}J^1\tilde{f} : J^1(\mathbb{R} \times X) \cong \mathbb{R} \times TX &\longrightarrow \mathbb{R} \times TY \cong J^1(\mathbb{R} \times Y) \\ (t, v) &\longmapsto \big(t, Tf(v)\big)\,,\end{aligned}$$

where we have used Example 3.1.7. This shows that the first jet prolongation of f at a fixed time is the tangent map of f.

3.1.3 The affine structure of jet bundles

Two local sections φ and φ' of $\rho : F \to M$ have the same 1-jet at m if they have the same value $\varphi(m) = \varphi'(m)$ and the same derivative $T_m\varphi = T_m\varphi' : T_mM \to T_{\varphi(m)}F$. Since φ is a section of ρ, $T_m\varphi$ is a section of $T_{\varphi(m)}\rho : T_{\varphi(m)}F \to T_mM$. It follows that a 1-jet of F is given by a subspace of a tangent space T_pF that is mapped by $T\rho$ bijectively to $T_{\rho(p)}M$. By definition, an Ehresmann connection is given by the choice of such a subspace of the tangent space, called the horizontal tangent space, at every point of the bundle. We thus arrive at the following observation.

Observation 3.1.23. An Ehresmann connection of $F \to M$ can be identified with a section of the bundle $\mathrm{pr}_{1,0} : J^1F \to F$.

Observation 3.1.23 can be used to express the bundle $J^1F \to F$ in terms of other definitions of connections. An Ehresmann connection can be given by a section of the morphism

$$TF \xrightarrow{(T\rho, \mathrm{pr}_F)} TM \times_M F$$

of fiber bundles over M. Such a section

$$h : TM \times_M F \longrightarrow TF$$

is called a **horizontal lift**. Let h' be another horizontal lift. Then

$$T\rho\big(h'(v_m, f) - h(v_m, f)\big) = 0$$

for all $v_m \in TM$ and $p \in F_m$. It follows that two horizontal lifts differ at each point $p \in F$ by a linear map $T_{\rho(p)}M \to V_pF$, where $VF := \ker T\rho$ is the vertical tangent bundle of F. The vector space of such linear maps can be identified with

$$\mathrm{Hom}(T_{\rho(p)}M, V_pF) \cong T^*_{\rho(p)}M \otimes V_pF .$$

It follows that the difference between two horizontal lifts is given by a section of the vector bundle

$$\rho^*(T^*M) \otimes VF \longrightarrow F ,$$

where $\rho^*(T^*M) := F \times_M T^*M$ denotes the pullback bundle. Returning to observation 3.1.23, we see that the choice of a horizontal lift h, which can be identified with a section of $J^1F \to F$, induces the following isomorphism of bundles over F,

$$\begin{aligned} J^1F &\longrightarrow \rho^*(T^*M) \otimes VF \\ j^1_m\varphi &\longmapsto \big[v_m \mapsto (T_m\varphi)v_m - h\big(v_m, \varphi(m)\big)\big] . \end{aligned}$$

The upshot is summarized in the following proposition.

Proposition 3.1.24. *Let $\rho : F \to M$ be a smooth fiber bundle. The fiber bundle $J^1F \to F$ is an affine bundle modelled on the vector bundle $\rho^*(T^*M) \otimes VF$.*

From Proposition 3.1.24 we recover the well-known fact that the set of connections, which can be identified with the set of sections of $J^1F \to F$, is an affine space (see Proposition 9.3.2 and Proposition 9.3.10). Another consequence is that the sheaf of sections of $J^1F \to F$ is soft, that is, sections on a closed subset can be extended to global sections. Proposition 3.1.24 can be generalized to the following statement.

Proposition 3.1.25. *Let $F \to M$ be a smooth fiber bundle. For every $k > 0$, the forgetful map $\mathrm{pr}_{k,k-1} : J^kF \to J^{k-1}F$ is an affine bundle modelled on the vector bundle $\mathrm{pr}^*_{k-1,-1}(S^kT^*M)) \otimes \mathrm{pr}^*_{k-1,0}(VF)$, where $\mathrm{pr}_{k-1,-1} : J^{k-1}F \to M$ is the bundle map, S^kT^*M the symmetric tensor product of the vector bundle $T^*M \to M$, and $\mathrm{pr}_{k-1,0} : J^{k-1}F \to F$ the forgetful map.*

Proposition 3.1.25 can be proved using jet coordinates, which is somewhat tedious (see e.g. Thm. 5.1.7 and Thm. 6.2.9 in [Sau89]). We will use that $\mathrm{pr}_k : J^kF \to J^{k-1}F$ is naturally embedded as subbundle into the affine bundle $J^1(J^{k-1}F) \to J^{k-1}F$. The embedding is given by the following lemma.

Lemma 3.1.26. *For all $k, l \geq 0$ there is a natural embedding*

$$\iota_{k,l} : J^{k+l}F \longrightarrow J^k(J^lF) , \quad j^{k+l}_m\varphi \longmapsto j^k_m(j^l\varphi) , \tag{3.9}$$

for all local sections φ.

Proof. The k-th order partial derivatives of the l-th prolongation of a local section φ of $F \to M$ are the $(k+l)$-th oder partial derivatives of φ. This implies that the k-jet of $j^l\varphi$ at m depends only on the $(k+l)$-jet of φ at m, which shows that $\iota_{k,l}$ is well-defined. It is easily checked in local jet coordinates that $\iota_{k,l}$ is an embedding. □

It is instructive to spell out the embedding of Lemma 3.1.26 in local coordinates. Let (x^i, u^α) be local fiber bundle coordinates on $F|_U$ for some open $U \subset M$. These induce jet bundle coordinates as in Equation (3.1). A local section $\eta : U \to J^l F$ of the l-th jet bundle is given in local coordinates by

$$\eta = (\eta^\alpha, \eta^\alpha_{i_1}, \ldots, \eta^\alpha_{i_1,\ldots,i_l}) ,$$

where $\eta^\alpha_{i_1,\ldots,i_l} = u^\alpha_{i_1,\ldots,i_l} \circ \eta$. Its k-th jet at m is given in coordinates by

$$j^k_m \eta = \begin{pmatrix} \eta^\alpha, & \eta^\alpha_{i_1}, & \ldots, & \eta^\alpha_{i_1,\ldots,i_l} \\ \frac{\partial \eta^\alpha}{\partial x^{j_1}}, & \frac{\partial \eta^\alpha_{i_1}}{\partial x^{j_1}}, & \ldots, & \frac{\partial \eta^\alpha_{i_1,\ldots,i_l}}{\partial x^{j_1}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^k \eta^\alpha}{\partial x^{j_1} \cdots \partial x^{j_k}}, & \frac{\partial^k \eta^\alpha_{i_1}}{\partial x^{j_1} \cdots \partial x^{j_k}}, & \ldots, & \frac{\partial^k \eta^\alpha_{i_1,\ldots,i_l}}{\partial x^{j_1} \cdots \partial x^{j_k}} \end{pmatrix}_m$$

The embedding $\iota_{k,l}$ maps a $(k+l)$-jet $j^{k+l}_m \varphi$ to

$$\iota_{k,l}(j^{k+l}_m \varphi) = \begin{pmatrix} \varphi^\alpha, & \frac{\partial \varphi^\alpha}{\partial x^{i_1}}, & \ldots, & \frac{\partial^l \varphi^\alpha}{\partial x^{i_1} \cdots \partial x^{i_l}} \\ \frac{\partial \varphi^\alpha}{\partial x^{j_1}}, & \frac{\partial^2 \varphi^\alpha}{\partial x^{i_1} \partial x^{j_1}}, & \ldots, & \frac{\partial^{1+l} \varphi^\alpha}{\partial x^{j_1} \partial x^{i_1} \cdots \partial x^{i_l}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^k \varphi^\alpha}{\partial x^{j_1} \cdots \partial x^{j_k}}, & \frac{\partial^{k+1} \varphi^\alpha}{\partial x^{j_1} \cdots \partial x^{j_k} \partial x^{i_1}}, & \ldots, & \frac{\partial^{k+l} \varphi^\alpha}{\partial x^{j_1} \cdots \partial x^{j_k} \partial x^{i_1} \cdots \partial x^{i_l}} \end{pmatrix}_m$$

The prolongation $J^k \mathrm{pr}_{l,n} : J^k(J^l F) \to J^k(J^n F)$ of the forgetful map $\mathrm{pr}_{l,n} : J^l F \to J^n F$, $n \leq l$, drops the last $l - n$ columns of the coordinate matrix.

Proof of Prop. 3.1.25. The map

$$\begin{tikzcd} J^k F \arrow[r, "\iota_{1,k-1}"] \arrow[dr] & J^1(J^{k-1}F) \arrow[d] \\ & J^{k-1}F \end{tikzcd}$$

embeds the fiber bundle $E := J^k F \to J^{k-1} F$ into the fiber bundle $J^1(J^{k-1}F) \to J^{k-1}F$, which by Proposition 3.1.24 is an affine bundle modelled on the vector bundle $A = \mathrm{pr}^*_{k-1,-1} T^*M \otimes VJ^{k-1}F$. An element $j^1_m \eta \in J^1(J^{k-1}F)$ represented by a local section $\eta : U \to J^{k-1}F$ is in the image of $\iota_{1,k-1}$ if and only if there is a local section $\varphi : U \to F$ such that

$$\begin{aligned} &\begin{pmatrix} \eta^\alpha, & \eta^\alpha_{i_1}, & \ldots, & \eta^\alpha_{i_1,\ldots,i_{k-1}} \\ \frac{\partial \eta^\alpha}{\partial x^{j_1}}, & \frac{\partial \eta^\alpha_{i_1}}{\partial x^{j_1}}, & \ldots, & \frac{\partial \eta^\alpha_{i_1,\ldots,i_{k-1}}}{\partial x^{j_1}} \end{pmatrix}_m \\ = &\begin{pmatrix} \varphi^\alpha, & \frac{\partial \varphi^\alpha}{\partial x^{i_1}}, & \ldots, & \frac{\partial^l \varphi^\alpha}{\partial x^{i_1} \cdots \partial x^{i_{k-1}}} \\ \frac{\partial \varphi^\alpha}{\partial x^{j_1}}, & \frac{\partial^2 \varphi^\alpha}{\partial x^{i_1} \partial x^{j_1}}, & \ldots, & \frac{\partial^k \varphi^\alpha}{\partial x^{j_1} \partial x^{i_1} \cdots \partial x^{i_{k-1}}} \end{pmatrix}_m . \end{aligned} \tag{3.10}$$

We have to show that there is a fiber-wise free and transitive action of the additive group of the vector bundle $B := \mathrm{pr}^*_{k-1,-1}(S^k T^* M)) \otimes \mathrm{pr}^*_{k-1,0}(VF)$ on $\iota_{l,k-1}(J^k F) \subset$

$J^1(J^{k-1}F)$. An element of B is given by a jet $j^{k-1}_m\varphi \in J^{k-1}F$ together with a linear map

$$\theta : S^kTM \longrightarrow V_{\varphi(m)}F\,.$$

Given such a θ, there is a local section $\psi : U \to F$, such that $j^{k-1}_m\psi = j^{k-1}_m\varphi$ and

$$\frac{\partial^k\psi^\alpha}{\partial x^{j_1}\partial x^{i_1}\cdots\partial x^{i_k}}\bigg|_m = \frac{\partial^k\varphi^\alpha}{\partial x^{j_1}\partial x^{i_1}\cdots\partial x^{i_k}}\bigg|_m + \theta^\alpha_{i_1,\dots,i_k}\,.$$

This defines a fiber-wise free and transitive action of $\mathrm{pr}^*_{k-1,-1}(S^kT^*M))\otimes\mathrm{pr}^*_{k-1,0}(VF)$ on J^kF. *** □

3.2 Local maps

3.2.1 Local maps and differential operators

Definition 3.2.1. Let $\mathcal{F} = \Gamma(M, F)$ and $\mathcal{F}' = \Gamma(M, F')$ be the sets of sections of smooth fiber bundles $F \to M$ and $F' \to M$. A map $f : \mathcal{F} \to \mathcal{F}'$ is called **local** of jet order k if there is a smooth map $f_0 : J^kF \to F'$, such that the following diagram commutes:

$$\begin{tikzcd} \mathcal{F}\times M \arrow[r, "f\times\mathrm{id}_M"] \arrow[d, "j^k"'] & \mathcal{F}'\times M \arrow[d, "j^0"] \\ J^kF \arrow[r, "f_0"'] & F' \end{tikzcd} \tag{3.11}$$

Terminology 3.2.2. A local map in the sense of Definition 3.2.1 is also called a **differential operator**, although this terminology is more commonly used when F and F' are trivial vector bundles, so that $\mathcal{F}$ and $\mathcal{F}'$ are function spaces.

Example 3.2.3. The Laplace operator $f = \Delta : C^\infty(\mathbb{R}^3) \to C^\infty(\mathbb{R}^3)$ of Example 1.2.1 descends to the map $f_0 : J^2(\mathbb{R}^3\times\mathbb{R}) \to \mathbb{R}^3\times\mathbb{R}$ given by

$$f_0 = \big((x^1, x^2, x^3), u_{11} + u_{22} + u_{33}\big)$$

in terms of jet bundle coordinates.

Example 3.2.4. Let $F' = TM \to M$, so that $\mathcal{F}' = \mathfrak{X}(M)$ is the space of vector fields. The product of the space of vector fields is the space of sections

$$\mathfrak{X}(M)\times\mathfrak{X}(M) \cong \Gamma(M, TM\times_M TM)\,,$$

of the vector bundle $F := TM\times_M TM$. The Lie bracket of vector fields $\mathfrak{X}(M)\times\mathfrak{X}(M) \to \mathfrak{X}(M)$ is a local map, which descends to J^1F.

Example 3.2.5. A special case for a fiber bundle over M is the trivial bundle $F' = M \xrightarrow{\mathrm{id}} M$, which is the terminal object in fiber bundles over M. The space of fields is given by a point $* = \{\mathrm{id}_M\}$. The terminal map

$$\mathcal{F} \longrightarrow *$$

descends to the bundle map $J^0F = F \to M$, so it is local of jet order 0. Similarly, every point

$$\iota_\varphi : * \hookrightarrow \mathcal{F}$$

mapping $*$ to a field $\varphi \in \mathcal{F}$ descends to the map $\varphi : J^0M = M \to F$, so it is also local of jet order 0.

Example 3.2.6. The map $f : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$ given by

$$f(\varphi) := \sum_{k=0}^{\infty} 2^{-k}\Big(\arctan \circ \frac{\partial^k \varphi}{\partial x^k}\Big)$$

is not local, since the value of $f(\varphi)$ at x depends on derivatives of arbitrarily large order.

Example 3.2.7. A lagrangian $L : \mathcal{F} \to \Omega^n(M)$ is local in the sense of Definition 1.3.5 if it is local in the sense of Definition 3.2.1.

Proposition 3.2.8. *If a map $\mathcal{F} \to \mathcal{F}'$ is local, then it is smooth, that is, a morphism of the diffeological spaces of fields.*

Proof. Let $f_0 : J^kF \to F'$ be the smooth map to which the map $f : \mathcal{F} \to \mathcal{F}'$ descends. Then f is given by

$$f(\varphi) = f_0 \circ j^k\varphi\,, \tag{3.12}$$

for all $\varphi \in \mathcal{F}$. Since j^k is smooth by Proposition 3.1.14, and f_0 is smooth by Definition 3.2.1, so is their composition $f = f_0 \circ j^k$. $\square$

The composition of differential operators on functions on some domain of $\mathbb{R}^n$ is again a differential operator. This suggests that the composition of local maps $f : \mathcal{F} \to \mathcal{F}'$ and $g : \mathcal{F}' \to \mathcal{F}''$ should be local as well. However, the maps $f_0 : J^kF \to F'$ and $g_0 : J^lF' \to F''$, to which f and g descend by Definition 3.2.1, cannot be composed directly, since the target of f_0 and the source of g_0 do not match. Instead we have to use Equation (3.12), which yields

$$\begin{aligned}(g \circ f)(\varphi)\big|_m = g\big(f(\varphi)\big)\big|_m &= g_0\big(j^l_m(f(\varphi)\big) = g_0\big(j^l_m(f_0 \circ j^k\varphi)\big)\\ &= (g_0 \circ j^l f_0)\big(j^l_m(j^k\varphi)\big)\,,\end{aligned}$$

where we have used Proposition 3.1.19. The right side is not yet a function on some jet bundle of F. This issue is resolved by Lemma 3.1.26, which leads to the following proposition.

Proposition 3.2.9. *The composition of two local maps is a local map.*

Proof. Let $f : \mathcal{F} \to \mathcal{F}'$ and $g : \mathcal{F}' \to \mathcal{F}''$ be local maps, which descend to $f_0 : J^kF \to F'$ and $g_0 : J^lF' \to F''$, respectively. Let $\iota_{l,k} : J^{k+l}F \to J^l(J^kF)$ be the injective immersion of Lemma 3.1.26 and $J^l f_0 : J^l(J^kF) \to J^lF'$ the l-th jet prolongation of f_0. Then we have the following commutative diagram,

$$\begin{tikzcd}
\mathcal{F} \times M \arrow[rr, "f\times \mathrm{id}_M"] \arrow[d, "j^{k+l}"'] & & \mathcal{F}' \times M \arrow[r, "g\times \mathrm{id}_M"] \arrow[d, "j^l"] & \mathcal{F}'' \times M \arrow[d, "j^0"] \\
J^{k+l}F \arrow[r, "\iota_{l,k}"] \arrow[d] & J^l(J^kF) \arrow[r, "J^l f_0"] \arrow[dl] & J^lF' \arrow[r, "g_0"'] \arrow[d] & F'' \\
J^kF \arrow[rr, "f_0"'] & & F' &
\end{tikzcd}$$

where $J^{k+l}F \to J^kF$, $J^l(J^kF) \to J^kF$, and $J^lF' \to F'$ are the obvious forgetful maps. If we define $f_l := J^l f_0 \circ \iota_{l,k}$, we see that $(g \circ f) \times \mathrm{id}_M$ descends to $g_0 \circ f_l$. We conclude that $g \circ f$ is local. □

Remark 3.2.10. Proposition 3.2.9 is a generalized version of the fact that the composition of a k-th order differential operator with an l-th order differential operator is a differential operator of order $k + l$.

Corollary 3.2.11. *Spaces of sections of fiber bundles over M and local maps are a subcategory of* $\mathcal{D}$flg.

Let $F \to M$ be a fiber bundle and $F' \to M$ a vector bundle. Let $f : \mathcal{F} \to \mathcal{F}'$ be a local map that descends to $f_0 : J^kF \to F'$. A field $\varphi \in \mathcal{F}$ is a solution of the equation

$$f(\varphi) = 0 \tag{3.13}$$

if and only if

$$M \xrightarrow{j^k\varphi} J^kF \xrightarrow{f_0} F'$$

is the zero map. This shows that Equation (3.13) is a partial differential equation (PDE).

Remark 3.2.12. Finding solutions of a PDE is generally very difficult. It may be easier to first try to find sections $\psi : M \to J^kF$ of the jet bundle such that $f_0 \circ \psi = 0$. Such sections are called **formal solutions** or **non-holonomic solutions** of the PDE. In a second step, we can determine those formal solutions for which $\psi = j^k\varphi$ is the k-th prolongation of a field $\varphi \in \mathcal{F}$, which are sometimes called **holonomic solutions**. The images of the tangent maps of the jet prolongations $Tj^k\varphi : TM \to TJ^kF$ of all fields φ define a distribution on J^kF, called the **Cartan distribution**. If we want to extend a point $x \in f^{-1}(0)$ to a holonomic solution on a neighborhood of m, the tangent space $T_xf^{-1}(0) \subset T_xJ^k_mF$ must be a subspace of the Cartan distribution. Pursuing this approach leads to Cartan-Kähler theory [BCG+91].

Remark 3.2.13. For some PDEs it can be proved that every formal solution is connected by a homotopy to an actual solution. To show that the PDE has a solution it then suffices to solve it formally, which is generally much easier. This approach is called the homotopy principle, or h-principle [EM02].

Proposition 3.2.14. *The tangent map of a local map is local of the same jet order.*

Proof. Let $f : \mathcal{F} \to \mathcal{F}'$ be a morphism of diffeological spaces of fields. Let $t \mapsto \psi_t \in \mathcal{F}$ be a smooth path with $\psi_0 = \varphi$ that represents the tangent vector $\xi_\varphi := \dot{\psi}_0 \in T\mathcal{F} = \Gamma(M, VF)$. Then the smooth path $t \mapsto f(\psi_t)$ represents the tangent vector $(Tf)\xi_\varphi \in T\mathcal{F}' = \Gamma(M, VF')$.

Assume now that f descends to $f_0 : J^kF \to F'$, so that $f(\psi_t) = f_0 \circ j^k\psi_t$. In local coordinates we obtain

$$\begin{aligned}
\big((T_\varphi f)\xi_\varphi\big)^\beta(x) &= \frac{d}{dt}\big(f^\beta(\psi_t)\big)(x)\big|_{t=0} \\
&= \frac{d}{dt} f_0^\beta\big(j_x^k\psi_t\big)\big|_{t=0} \\
&= \sum_{|I|\leq k} \frac{\partial f_0^\beta}{\partial u_I^\alpha}(j_x^k\varphi)\,\frac{d}{dt}u_I^\alpha\big(j_x^k\psi_t\big)\big|_{t=0} \\
&= \sum_{|I|\leq k} \frac{\partial f_0^\beta}{\partial u_I^\alpha}(j_x^k\varphi)\,\frac{\partial^{|I|}\xi_\varphi^\alpha}{\partial x^I}\,.
\end{aligned}$$

The right side depends only on derivatives of φ^α and ξ_φ^α at x up to k-th order, i.e. only on $j_x^k\xi_\varphi$. □

Corollary 3.2.15. *Let $f : \mathcal{F} \to \mathcal{F}'$ be a local map of jet order k. Let $\varphi \in \mathcal{F}$. Then the linear map $T_\varphi f : T_\varphi\mathcal{F} \to T_{f(\varphi)}\mathcal{F}'$ is local of jet order k.*

Terminology 3.2.16. The linear differential operator $T_\varphi f$ is called the **linearization at** φ of the differential operator f.

3.2.2 Local maps of products

Let $E \to M$ and $F \to M$ be smooth fiber bundles. The product of the spaces of fields is itself a space of fields,

$$\mathcal{E} \times \mathcal{F} \cong \Gamma(M, E \times_M F)\,.$$

The k-th jet bundle of $E \times_M F$ is given by

$$J^k(E \times_M F) \cong J^kE \times_M J^kF\,.$$

Lemma 3.2.17. *Let $E \to M$ and $F \to M$ be smooth fiber bundles. Then the projection $\mathcal{E} \times \mathcal{F} \to \mathcal{E}$, the diagonal $\mathcal{E} \to \mathcal{E} \times \mathcal{E}$, and the flip $\mathcal{E} \times \mathcal{F} \to \mathcal{F} \times \mathcal{E}$ descend to smooth maps of the fiber bundles over M, i.e. they are local of jet order* 0.

Proof. The projection is induced by the fiber-wise projection $E \times_M F \to E$, the diagonal by the fiber-wise diagonal $E \to E \times_M E$ and the flip by the fiber-wise flip $E \times_M F \to F \times_M E$. □

Lemma 3.2.18. *Let $E \to M$, $F \to M$, $E' \to M$, and $F' \to M$ be smooth fiber bundles. Let $f : \mathcal{E} \to \mathcal{E}'$ and $g : \mathcal{F} \to \mathcal{F}'$ be a maps of the spaces of fields. If f and g are local, then the product map*

$$f \times g : \mathcal{E} \times \mathcal{F} \longrightarrow \mathcal{E}' \times \mathcal{F}'$$

is local.

Proof. By assumption, f descends to $f_0 : J^kE \to E'$ and g descends to a map $g_0 : J^lF \to F'$. Without loss of generality let $k \geq l$. Then g also descends to the map $g_0' = g_0 \circ \mathrm{pr}_{k,l} : J^kF \to F'$. It follows that $f \times g$ descends to the map $h_0 : J^k(E \times_M F) \to E \times_M F$ defined by

$$h_0\big(j^k_m(\psi, \varphi)\big) = \big(f_0(j^k_m\psi), g_0'(j^k_m\varphi)\big)\,,$$

which shows that $f \times g$ is local. □

Lemma 3.2.19. *Let $E \to M$, $F \to M$, and $F' \to M$ be smooth fiber bundles. Let $f : \mathcal{E} \times \mathcal{F} \to \mathcal{F}'$ be a map of spaces of fields. If f is local then there is a $k < \infty$, such that the maps*

$$f(\_\,, \varphi) : \mathcal{E} \longrightarrow \mathcal{F}'$$
$$f(\psi, \_\,) : \mathcal{F} \longrightarrow \mathcal{F}'$$

are local of jet order k for all $\varphi \in \mathcal{F}$ and $\psi \in \mathcal{E}$.

Proof. The map $f(\_\,, \varphi)$ is given by the composition

$$\mathcal{E} \cong \mathcal{E} \times * \xrightarrow{\mathrm{id}_{\mathcal{E}} \times \iota_\varphi} \mathcal{E} \times \mathcal{F} \xrightarrow{f \times g} \mathcal{F}'\,,$$

where ι_φ is the inclusion of φ of Example 3.2.5. Since $\mathrm{id}_{\mathcal{E}}$ and ι_φ are local, their product is local by Lemma 3.2.18. Since $\mathrm{id}_{\mathcal{E}} \times \iota_\varphi$ and f are local, their composition $f(\_\,, \varphi)$ is local by Proposition 3.2.9. An analogous argument shows that $f(\psi, \_\,)$ is local, too. □

3.2.3 Linear local maps of jet order 0 and 1

Assume that $A \to M$ and $B \to M$ are vector bundles. Let $D : \mathcal{A} \to \mathcal{B}$ be a k-th order local map, so it descends to a map $D_0 : J^kA \to B$ for some $k \geq 0$. D is linear if and only if D_0 is in local jet coordinates of the general form

$$D_0^\beta = \sum_{|I|=0}^{k} D^{\beta I}_\alpha(x)\, u^\alpha_I\,,$$

where (x^i, u^α) are local vector bundle coordinates on $A|_U$ for some $U \subset M$, where (x^i, v^β) are coordinates on $B|_U$, and where the $D^{\beta I}_\alpha$ are smooth functions on U. The linear map D is given in terms of these functions by

$$(Da)^\beta = \sum_{|I|=0}^{k} D^{\beta I}_\alpha \frac{\partial^{|I|} a^\alpha}{\partial x^I}\,. \tag{3.14}$$

Proposition 3.2.20. *A linear map $D : \mathcal{A} \to \mathcal{B}$ of sections of vector bundles is induced by a map $D_0 : A \to B$ of vector bundles if and only if it is $C^\infty(M)$-linear, i.e.*

$$D(fa) = f\,Da$$

for all $a \in \mathcal{A}$ and $f \in C^\infty(M)$.

Proof. The proposition follows from Equation (3.14) for $k = 0$. □

Proposition 3.2.21. *A linear map $D : \mathcal{A} \to \mathcal{B}$ of sections of vector bundles is a first order differential operator if and only if there is a vector bundle map $P : A \to B \otimes TM$, such that*

$$D(fa) = f\,Da + \langle P(a), df \rangle \tag{3.15}$$

for all $a \in \mathcal{A}$ and $f \in C^\infty(M)$.

Proof. Assume that D is a linear first order local map. By Equation (3.14), D is given in local coordinates by

$$(Da)^\beta = D^\beta_\alpha a^\alpha + D^{\beta i}_\alpha \frac{\partial a^\alpha}{\partial x^i}\,. \tag{3.16}$$

It follows that

$$\big(D(fa)\big)^\beta = D^\beta_\alpha f a^\alpha + f\,D^{\beta i}_\alpha \frac{\partial a^\alpha}{\partial x^i} + a^\alpha D^{\beta i}_\alpha \frac{\partial f}{\partial x^i}\,.$$

So if we define P in local coordinates by

$$P(a)^\beta := a^\alpha D^{\beta i}_\alpha \frac{\partial}{\partial x^i}\,, \tag{3.17}$$

then Equation (3.15) follows.

Conversely, assume that Equation (3.15) holds. Let σ_α be the basis of local sections of A such that $u^\alpha(\sigma_{\alpha'}) = \delta^\alpha_{\alpha'}$ and let τ_β be the basis of local sections of B such that $v^\beta(\tau_{\beta'}) = \delta^\beta_{\beta'}$. Let D^α_β be the unique local functions, such that

$$D(\sigma_\alpha) = D^\beta_\alpha \tau_\beta\,.$$

P be given in local coordinates by (3.17) for some local functions $D^{\beta i}_\alpha$. A general local section is of the form $a = a^\alpha \sigma_\alpha$. Using Equation (3.15), we get

$$\begin{aligned} D(a) = D(a^\alpha \sigma_\alpha) &= a^\alpha D(\sigma_\alpha) + \langle P(\sigma_\alpha), a^\alpha \rangle \\ &= a^\alpha D^\beta_\alpha + D^{\beta i}_\alpha \frac{\partial a^\alpha}{\partial x^i}\,, \end{aligned}$$

which has the form of a linear first order local map. □

3.3 The theorems of Peetre and Slovák

3.3.1 Locality in topology

In topology, "local" roughly means "compatible with the restriction to open subsets". In this sense, a map $f : \mathcal{F} \to \mathcal{F}'$ of sections of fiber bundles is considered to be local if the restriction of $f(\varphi)$ to any open subset $U \subset M$ depends only on the restriction of φ to U. Let $\hat{\mathcal{F}}$ denote the sheaf of sections, given by

$$\hat{\mathcal{F}}(U) := \Gamma(U, F|_U)\,,$$

for every open $U \subset M$. The set of global sections is $\mathcal{F} = \hat{\mathcal{F}}(M)$. A morphism of sheaves is given by a map $\hat{f}_U : \hat{\mathcal{F}}(U) \to \hat{\mathcal{F}}'(U)$ for every open subset $U \subset M$

that commutes with the restrictions to every open subset $V \subset U$, i.e. the following diagram commutes.

$$\begin{tikzcd} \hat{\mathcal{F}}(U) \arrow[r, "\hat{f}_U"] \arrow[d, "\mathrm{res}_{U,V}"'] & \hat{\mathcal{F}}'(U) \arrow[d, "\mathrm{res}'_{U,V}"] \\ \hat{\mathcal{F}}(V) \arrow[r, "\hat{f}_V"'] & \hat{\mathcal{F}}'(V) \end{tikzcd}$$

A map $f : \mathcal{F} \to \mathcal{F}'$ is considered to be local in the sense of topology if there is a morphism of sheaves $\hat{f} : \hat{\mathcal{F}} \to \hat{\mathcal{F}}'$ such that $f = \hat{f}_M$.

Proposition 3.3.1. *If $f : \mathcal{F} \to \mathcal{F}'$ is local (in the sense of Definition 3.2.1), then it is induced by a morphisms of sheaves.*

Proof. Let $f_0 : J^k F \to F'$ be the map f descends to. Let

$$\hat{f}_U(\varphi) := f_0 \circ j^k \varphi$$

for all $\varphi \in \Gamma(U, F|_U)$. The restrictions of the jet prolongation $j^k|_U : \Gamma(U, F|_U) \to \Gamma(U, J^k F|_U)$ define a morphism of sheaves; and the morphism of fiber bundles f_0 induces a morphism of the sheaves of sections. Therefore, the composition is a morphism of sheaves. □

Let $f : \mathcal{F} \to \mathcal{F}'$ be induced by a morphism of sheaves. Then for every $m \in M$, the restriction of $f(\varphi)$ to a neighborhood U of m depends only on the restriction of f to U. Since the neighborhood U is arbitrarily small, it follows that the value of $f(\varphi)$ at m depends only on the germ of f at m.

Recall that the germ of a function φ at m is the equivalence class of functions ψ that have a the same restriction $\psi|_U = \varphi|_U$ to some neighborhood U of m. If two functions have the same germ, then they have the same partial derivatives to all orders. The converse is clearly not true. For example, the derivatives of the function $\varphi(x) = \exp(-1/x^2)$ on the real line are all zero at $x = 0$, so it has the same jets as $\psi(x) = 0$, but φ and and ψ do not have the same germ at 0. The germ of a section φ of a fiber bundle at some point m contains more information about the function than the jet $j^k_m \varphi$. Therefore, the condition that $f(\varphi)_m$ depends only on the germ of φ at m is weaker than the condition that it depends on a finite jet, as required by the definition 3.2.1 of locality.

3.3.2 Peetre's theorem

Surprisingly, with rather mild additional assumptions a map $f : \mathcal{F} \to \mathcal{F}'$ that is induced by a morphism of sheaves is local (in the sense of Definition 3.2.1). We first consider the linear case.

Theorem 3.3.2 (Peetre)**.** *Let $A \to M$ and $B \to M$ be vector bundles over a compact base. Let $D : \mathcal{A} \to \mathcal{B}$ be a linear map. If D is induced by a morphism of sheaves of vector spaces, then it is local.*

Lemma 3.1.12 implies that all jet evaluations $j^k : \mathcal{A} \times M \to J^k A$ are surjective. It follows, that if the map $D : \mathcal{A} \to \mathcal{B}$ descends to a map $J^k A \to B$, then this map must be given by

$$\begin{aligned} D_0 : J^k A &\longrightarrow B \\ j^k_m \varphi &\longmapsto (D\varphi)(m) . \end{aligned} \tag{3.18}$$

In the first step, we have to show that the map (3.18) is well defined. For this we will use the following lemma.

Lemma 3.3.3. *Let $D : C^\infty(\mathbb{R}^n, \mathbb{R}^p) \to C^\infty(\mathbb{R}^n, \mathbb{R}^q)$ be a support non-increasing linear map. Then for every point $x \in \mathbb{R}^n$ and every real constant $c > 0$ there is a neighborhood U of x and a natural number $r \geq 0$, such that for all $y \in U \setminus \{x\}$ and $\varphi \in C^\infty(\mathbb{R}^n, \mathbb{R}^p)$ the condition $j^r_y \varphi = 0$ implies $\|(D\varphi)(y)\| \leq c$.*

Proof. Assume that the statement is false. This means that there is a point $x \in \mathbb{R}^n$ and a constant $c > 0$, such that for every neighborhood U of x and every $r \geq 0$ there is a $y \in U$, $y \neq x$ and a $\varphi \in C^\infty(\mathbb{R}^n, \mathbb{R}^p)$, such that $j^k_y \varphi = 0$ and $\|(D\varphi)(x)\| > c$. By choosing a sequence of shrinking neighborhoods $U_0 \supset U_1 \supset \ldots$ with $\bigcap_k U_k = \{x\}$, we can find a sequence $y_k \to x$ and a sequence $\varphi_k \in \mathcal{A}$, such that $j^k_{y_k} \varphi_k = 0$ and $\|(D\varphi_k)(y_k)\| > c$.

By selecting a suitable subsequence, the relations $\|y_k - x\| \leq 4\|y_k - x_j\|$ can be satisfied for all $k > j$. Let us choose smooth maps $\psi_k \in C^\infty(\mathbb{R}^n, \mathbb{R}^p)$ that have the same germ as φ_k at y_k and are zero outside of the ball of radius $\frac{1}{2}$ around y_k. Since the germs are the same, so are the jets $j^k_{y_k} \psi_k = j^k_{y_k} \varphi_k = 0$. Because the jets at y_k are zero, the functions ψ_k can be chosen such that their partial derivatives are bounded in the supremum norm by

$$\left\| \frac{\partial^{|I|} \psi_k}{\partial x^I} \right\|_{\sup} \leq 2^{-k} ,$$

for all multi-indices I of order $|I| \leq k$. Due to this condition, the map defined point-wise by

$$\psi(y) := \sum_{l=0}^{\infty} \psi_{2l}(y)$$

for all $y \in \mathbb{R}^n$ is smooth. By construction, the points y_{2l+1} lie outside of the support of ψ. By assumption, D is support non-increasing so that y_{2l+1} also lies outside of the support of $D\psi$,

$$(D\psi)(y_{2l+1}) = 0 .$$

Since D is support non-increasing, $(D\psi)(y_{2l})$ only depends on the germ of ψ_{2l} at y_{2l} which is equal to the germ of φ_{2l} at y_{2l}, so that

$$(D\psi)(y_{2l}) = (D\varphi)(y_{2l}) .$$

It follows that $y_k \to x$ is a convergent sequence, such that

$$\|(D\psi)(y_{2l})\| > c , \qquad \|(D\psi)(y_{2l+1})\| = 0 ,$$

which shows that $D\psi$ is not continuous at x. This is a contradiction to the assumption that the lemma does not hold. $\square$

In order to show that the D_0 is smooth, we will use Boman's theorem.

Theorem 3.3.4. *Let $f : \mathbb{R}^m \to \mathbb{R}^n$ be a map, such that such that for every smooth path $\gamma : \mathbb{R} \to \mathbb{R}^m$ the path $f \circ \gamma : \mathbb{R} \to \mathbb{R}^n$ is smooth. Then f is smooth.*

Proof. The original proof is in [Bom67]. A more pedagogic proof is found in Thm. 3.4 in [KM97]. □

Proof of Thm. 3.3.2. Choose $c = 1$ and apply Lemma 3.3.3 in a coordinate neighborhood of every point $m \in M$. This yields a cover of neighborhoods U_i with jet orders r_i as in the lemma. Since M is compact, we can choose a finite subcover. Let $r < \infty$ be the maximum of the r_i. Then $j^r_m \varphi = 0$ implies that $\|(Df)(m)\| < 1$ for all $m \in M$.

Let $j^k_m \varphi = 0$ and assume that $\|(D\varphi)(m)\| = \varepsilon > 0$. Then $j^k_m(\frac{\varepsilon}{2}\varphi) = 0$, but $\|(D\frac{2}{\varepsilon}\varphi)(m)\| = 2 > 1$, which is a contradiction, so that $(D\varphi)(m) = 0$. It follows, that (3.18) is a well defined fiber-wise linear map.

It remains to show that D_0 is smooth. As can be easily seen in local coordinates, every smooth path in $J^r A$ can be written as $t \mapsto j^r_{m_t}\varphi_t$, where $t \mapsto \varphi_t$ is a smooth family of sections of A and $t \mapsto m_t$ a smooth path in M. Since D is linear, $D\varphi_t$ is a smooth family of smooth maps. It follows that $t \mapsto (D\varphi_t)(m_t)$ is a smooth path. This shows that every smooth path $j^r_{m_t}\varphi_t$ in $J^r A$ is mapped by D_0 to a smooth path in B. It now follows from Boman's Theorem 3.3.4 that D_0 is smooth. □

3.3.3 The nonlinear case

Theorem 3.3.5 (Slovák)**.** *Let $F \to M$, $F' \to M$ be smooth fiber bundles. Let $f : \mathcal{F} \to \mathcal{F}'$ be induced by a morphism of sheaves of diffeological spaces. Then for every $\varphi \in \mathcal{F}$ and every $m \in M$ there is an open neighborhood $U \ni m$ and an open subbundle $E \subset F|_U$ containing $\varphi(U)$, such that the restricted map $f|_{\mathcal{E}}$ is local (in the sense of Definition 3.2.1).*

The original proof, which is quite involved, can can be found in [Slo88]. A more pedagogic presentation is in [KMS93]. There is a somewhat modernized formulation of the theorem in [NS]. For a recent discussion of the Peetre-Slovák theorem in relation to field theory, we refer the reader to Appendix A in [KM16, Appendix A].

The original statement of Slovák is somewhat more general. It allows for the basis of the target bundle F' to be a different manifold $M' \neq M$ and assumes that there is a map $\eta : M' \to M$ such that $f(\varphi)|_{m'}$ depends only on the germ of φ at $\eta(m')$ for all $m' \in M'$. But this is the same as saying that there is a morphism of sheaves from the pullback sheaf $\eta^*\hat{\mathcal{F}}$ to $\hat{\mathcal{F}}'$. ***

Terminology 3.3.6. The condition that f is a morphism of diffeological spaces is called "regularity" in [Slo88, KMS93].

Corollary 3.3.7. *Let $F \to M$, $F' \to M$ be smooth fiber bundles. Let F be compact. Then a map $f : \mathcal{F} \to \mathcal{F}'$ is local if and only if it is induced by a morphism of sheaves in diffeological spaces.*

A casual way of rephrasing Corollary 3.3.7 is by saying that for sections of compact fiber bundles smooth sheaf-locality is the same as jet-locality. In the non-compact case the jet order may be only locally but not globally finite, so that Definition 3.2.1 is a stronger version of locality. It is debatable, whether global or local finiteness of the jet order is the more appropriate condition in field theory***. Ultimately, this will depend on and be justified by the application.

We will not give a proof of Theorem 3.3.5. But we will state an important technical step, which is interesting in its own right: The Whitney extension theorem gives the exact conditions for a collection of functions on a closed subset of $\mathbb{R}^n$ to be the partial derivatives of a smooth function on $\mathbb{R}^n$.

Theorem 3.3.8. *Let $K \subset \mathbb{R}^n$ be a closed set. Let $\varphi_I : K \to \mathbb{R}$ be continuous functions defined for all multi-indices $I \in \mathbb{N}_0^n$. The following are equivalent:*

(i) For every $r \geq 0$

$$\varphi_I(b) = \sum_{|J| \leq r} \frac{1}{J!} \varphi_{I+J}(a)(b-a)^J + o(|b-a|^r) \tag{3.19}$$

holds uniformly for $|b-a| \to 0$, $a, b \in K$.

(ii) There is a smooth function $\varphi \in C^\infty(\mathbb{R}^n)$ such that

$$\varphi_I = \frac{\partial^{|I|}\varphi}{\partial x^I}\Big|_K .$$

Proof. The original proof where K was assumed to be compact is in [Whi34]. It was first observed in [Bie80] that K being closed is sufficient. For a more pedagogic proof see [Hö3]. □

The condition (3.19) for the functions φ_I imply that $\varphi_I = \frac{\partial^{|I|}\varphi}{\partial x^I}$ in the interior of K. Conversely, if φ is a smooth function and $\varphi_I = \frac{\partial^{|I|}\varphi}{\partial x^I}$, then (3.19) follows from Taylor's theorem. This shows that Equation (3.19) is always satisfied in the interior of K.

When $K = *$ is a point, condition (3.19) is always satisfied, which implies that *any* collection of real numbers c_I for all multi-indices I can be realized as partial derivatives of a smooth function. This is the content of the Borel lemma. In its simplest form it can be stated as follows.

Lemma 3.3.9. *For any infinite sequence of real numbers $c_0, c_1, c_2, \dots$ there is a smooth function $\varphi \in C^\infty(\mathbb{R})$, such that $c_n = \frac{d^n\varphi}{dx^n}\big|_{x=0}$.*

3.4 Infinite jets

A local map of fields descends to a map on the manifold of jets of a finite but arbitrarily large order. When two local maps are composed, their jet orders are added. So even though we can describe a single local map in terms of a map on a finite jet manifolds, we need the jet manifolds of all orders to deal with the category of all local maps. This suggests the following definition.

Definition 3.4.1. Two local sections φ and φ' of a smooth fiber bundle $F \to M$ defined on a neighborhood of m have the same **infinite jet** or **∞-jet at** m, denoted by $j^\infty_m \varphi = j^\infty_m \varphi'$, if they have the same k-jet at m for all $k \geq 0$.

Since having the same k-jet at m is an equivalence relation on the set of local sections, having the same ∞-jet is an equivalence relation as well. An ∞-jet is an equivalence class for this relation. The set of all ∞-jets will be denoted by $J^\infty F$.

Given local bundle coordinates (x^i, u^α), $j^\infty_m \varphi$ is uniquely determined by the coordinates $x^i(m)$ of the base point and the jet coordinates

$$u^\alpha_I(j^\infty_m \varphi) = \frac{\partial^{|I|}\varphi^\alpha}{\partial x^I}\bigg|_m$$

for all α and all multi-indices I. Conversely, the Whitney extension Theorem 3.3.8 tells us that, given numbers c^α_I for all α and I, there is a local section such that $u^\alpha_I(j^\infty_m \varphi) = c^\alpha_I$. In this sense, the infinite collection $\{x^i, u^\alpha, u^\alpha_{i_1}, \ldots\}$ of real valued functions on $J^\infty F$ can be viewed as a set of coordinates.

For every $k \geq 0$, there are natural forgetful maps of sets $\mathrm{pr}_{\infty,k} : J^\infty F \to J^k F$, $j^\infty_m \varphi \mapsto j^k_m \varphi$. The forgetful maps satisfy $\mathrm{pr}_{k,k-1} \circ \mathrm{pr}_{\infty,k} = \mathrm{pr}_{\infty,k-1}$, so they define the commutative diagram

$$\begin{tikzcd}
 & J^\infty F \arrow[dl] \arrow[d] \arrow[dr] \arrow[drr] & & \\
J^0 F & J^1 F \arrow[l] & J^2 F \arrow[l] & \ldots \arrow[l]
\end{tikzcd}$$

As can be easily seen in jet coordinates, any other cone over the diagram $J^0 F \leftarrow J^1 F \leftarrow J^2 F \leftarrow \ldots$ induces a unique map to $J^\infty F$, which shows that $J^\infty F$ is the categorical limit of the sequence of the sets of finite jets.

How do we equip $J^\infty F$ with a differentiable structure? Since the dimension of the jet manifolds $J^k F$ increases with k, the limit of the sequence of the jet manifolds $J^k F$ cannot exist in the category of finite dimensional manifolds. In order to make sense of this limit we, therefore, have to embed $\mathcal{M}\mathrm{fld}$ as subcategory into an ambient category $\mathcal{C}$ in which such limits exist. Let us write down a wish list of some of the properties this category should have.

Wish list 3.4.2. A good category $\mathcal{C}$ for $J^\infty F$ should have the following properties:

(i) There is an injective, full, and faithful functor $I : \mathcal{M}\mathrm{fld} \to \mathcal{C}$.

(ii) For every infinite inverse sequence of manifolds $X_0 \leftarrow X_1 \leftarrow \ldots$ the limit $\check{X} := \lim(I(X_0) \leftarrow I(X_1) \leftarrow \ldots)$ exists in $\mathcal{C}$.

(iii) There is a faithful functor $\check{U} : \mathcal{C} \to \mathcal{S}\mathrm{et}$, such that for every limit $\check{X}$ as in (ii) there is a natural isomorphism $\check{U}(\check{X}) \cong \lim_{i \in \mathcal{I}} \mathcal{M}\mathrm{fld}(*, X_i)$ of sets.

(iv) Given a limit $\check{X}$ as in (ii), every morphism $\check{X} \to I(Y)$ to a manifold Y factors as $\check{X} \to I(X_k) \xrightarrow{I(f)} I(Y)$ through a smooth map $f : X_k \to Y$.

Let us motivate this wish list. Property (i) states that $\mathcal{M}$fld can be embedded as full subcategory into $\mathcal{C}$. Property (ii) ensures that the limit

$$J^\infty F := \lim(I(J^0 F) \leftarrow I(J^1 F) \leftarrow \ldots)$$

exists in $\mathcal{C}$. Property (iii) requires $\mathcal{C}$ to have the structure of a concrete category such that the the underlying set of $J^\infty F$ is the set of infinite jets from Definition 3.4.1. Finally, Property (iv) states that all morphisms out of $J^\infty F$ descend to a finite jet manifold, that is, they are differential operators.

In Chapter 2, we have solved a similar problem with the category diffeological spaces. In fact, $\mathcal{D}$flg satisfies conditions (i), (ii), and (iii) of the wish list 3.4.2. Condition (iv), however, is not satisfied by $\mathcal{D}$flg as the following example shows.

Example 3.4.3. Consider the fiber bundle $F = \mathbb{R} \times \mathbb{R} \to \mathbb{R} = M$ with space of sections $\mathcal{F} = C^\infty(\mathbb{R})$. The map of Example 3.2.6 can be viewed as a map on the infinite jet bundle

$$\begin{aligned} f : J^\infty(\mathbb{R} \times \mathbb{R}) &\longrightarrow \mathbb{R} \\ f(j^\infty_x \varphi) &:= \sum_{k=0}^{\infty} 2^{-k} \arctan\Big(\frac{\partial^k \varphi}{\partial x^k}\Big) , \end{aligned}$$

which does not descend to a map on any finite jet manifold $J^k(\mathbb{R} \times \mathbb{R})$. A map $U \to J^\infty F$, $U \in \mathcal{E}$ucl is a plot of the limit diffeology if and only if the compositions $U \to J^\infty F \to J^l F$ for all $l \geq 0$ are smooth. It follows that all partial sums of $f \circ p : U \to \mathbb{R}$ are smooth. Since the arctangent and all its derivatives are bounded by 1, the convergence of the sum is uniform. It follows that $f \circ p$ is smooth. Since f is a function on $J^\infty F$ that is smooth with respect to the limit diffeology but does not descend to a finite jet manifold, we conclude that $\mathcal{D}$flg does not satisfy condition (iv) of the wish list.

Exercises

Exercise 3.1. Let $f, g : M \to \mathbb{R}$ be functions on a smooth n-dimensional manifold. Let $x = (x^1, \ldots, x^n) : U \to \mathbb{R}^n$ be local coordinates on a neighborhood U of m. Let k be a natural number. Show that if

$$\frac{\partial^l f}{\partial x^{i_1} \cdots \partial x^{i_l}}\Big|_{x(m)} = \frac{\partial^l g}{\partial x^{i_1} \cdots \partial x^{i_l}}\Big|_{x(m)}$$

for all $l \leq k$ and all indices $1 \leq i_1, \ldots, i_l \leq n$, then these equalities hold in any other coordinate system.

Exercise 3.2 (Dimension of jet manifolds)**.** Let $F \to M$ be a smooth fiber bundle with $\dim F = p + q$ and $\dim M = p$. Compute the dimension of $J^k F$.

Exercise 3.3 (Jet bundles of vector bundles)**.** Let $A \to M$ and $B \to M$ be smooth vector bundles. Show the following:

(a) $J^k A \to M$ and $J^k B \to M$ are vector bundles.

(b) $J^k(A \oplus B) \cong J^k A \oplus J^k B$

Exercise 3.4 (Cartan distribution)**.** Let $F \to M$ be a smooth fiber bundle. The **Cartan distribution** $C^k \subset T(J^k F)$ is spanned at every point $j^k_m \varphi \in J^k F$ by the tangent vectors of the form $\xi = T_m(j^k\psi)\, v_m$ for all $v_m \in T_m M$ and all local sections ψ with $j^k_m \psi = j^k_m \varphi$.

(a) Show that C^k is regular.

(b) Compute the rank of C^k.

(c) Show that C^k is not integrable.

Exercise 3.5 (Non-local maps)**.** Show that none of the three maps f, g, and h of Exercise 9 factors through a finite jet manifold of the bundle $M \times \mathbb{R} \to M$.

Exercise 3.6 (Derivations are local)**.** Let $C^\infty(M)$ denote the ring of smooth functions on a manifold M. Show that every derivation $\delta : C^\infty(M) \to C^\infty(M)$ is local.

Exercise 3.7 (Gauge transformations and diffeomorphisms)**.** Let $F = \mathbb{R} \times T^*M \to M$ so that $\mathcal{F} = C^\infty(M) \times \Omega^1(M)$. Let $F' := T^*M \to M$. Show that the map $f : \mathcal{F} \to \mathcal{F}'$ defined by

$$f(\varphi, \omega) = \omega + d\varphi$$

is local. (The map f is called the action of local gauge transformations.) Let $F = TM \times_M \wedge^k T^*M \to M$ and $F' = \wedge^k T^*M$, so that $\mathcal{F} = \mathfrak{X}(M) \times \Omega^k(M)$. Show that the map $f : \mathcal{F} \to \mathcal{F}'$ defined by

$$f(v, \omega) = \mathcal{L}_v \omega \,,$$

where $\mathcal{L}_v$ denotes the Lie derivative with respect to v, is local.

Exercise 3.8 (Jacobi fields)**.** Let $F = \mathbb{R} \times \mathbb{R}^n \to \mathbb{R} = M$ be the trivial bundle. Let $\Gamma^\alpha_{\beta\gamma} : \mathbb{R}^n \to \mathbb{R}$ be a family of smooth functions indexed by $1 \leq \alpha, \beta, \gamma \leq n$ that is symmetric in the lower indices $\Gamma^\alpha_{\beta\gamma} = \Gamma^\alpha_{\gamma\beta}$. Consider the map of fields

$$\begin{aligned} D : C^\infty(\mathbb{R}, \mathbb{R}^n) &\longrightarrow C^\infty(\mathbb{R}, \mathbb{R}^n) \\ q &\longmapsto \ddot{q}^\alpha + \Gamma^\alpha_{\beta\gamma}(q)\dot{q}^\beta \dot{q}^\gamma \,. \end{aligned}$$

Let its zero locus $D^{-1}(0) \subset C^\infty(\mathbb{R}, \mathbb{R}^n)$ be equipped with the subspace diffeology. Show that D is local. Compute the tangent map of D. Show that every tangent vector in $TD^{-1}(0)$ is in the kernel of TD. Is every element of the kernel of $T_q D$ for $q \in D^{-1}(0)$ an element of $TD^{-1}(0)$? (When the $\Gamma^\alpha_{\beta\gamma}$ are the connection coefficients of the Levi-Civita connection of a riemannian metric on $Q = \mathbb{R}^n$, then $D(q) = 0$ is the geodesic equation. The elements in the kernel of TD are called Jacobi fields. They describe the tidal forces of the gravitational field.)

Chapter 4

Pro-manifolds

4.1 Ind-categories and pro-categories

4.1.1 Filtered and cofiltered categories

The guiding example of the infinite jet bundle suggests that we consider limits of diagrams of the form

$$J^0F \longleftarrow J^1F \longleftarrow J^2F \longleftarrow \dots$$

It turns out that it is conceptually easier to first consider the dual situation of colimits of sequential diagrams

$$C_0 \longrightarrow C_1 \longrightarrow C_2 \longrightarrow \dots\,,$$

that is, diagrams $\omega \to \mathcal{C}$ indexed by the smallest transfinite ordinal

$$\omega = (0 \to 1 \to 2 \to \dots)\,.$$

Example 4.1.1. Let $\mathcal{C}$ be the partially ordered set $(\mathbb{R}, \leq)$, viewed as category. A functor $x : \omega \to \mathcal{C}$ is an increasing sequence $x_0 \leq x_1 \leq x_2 \leq \dots$ of real numbers. The functor x has a colimit $y \in \mathbb{R}$ if and only if the sequence of numbers converges to y (Exercise 4.4).

Even if we are primarily interested in diagrams indexed by ω, many categorical constructions involving ω-diagrams will produce diagrams of different shapes. The analogy of Example 4.1.1 also suggests that we may have to consider more general index categories. While every continuous map preserves limits of convergent sequences, the converse is true only if the domain of the map is a first countable topological space. In spaces that are not first countable, we have to consider the convergence of filters instead of sequences. A filter is a family of open subsets of a topological space that is closed under finite intersections. This concept is generalized by filtered categories.

Definition 4.1.2. A category $\mathcal{J}$ is **filtered** if the following three properties are satisfied:

(i) $\mathcal{J}$ is not empty.

(ii) For any two objects $i_1, i_2 \in \mathcal{I}$, there is a diagram,

$$\begin{tikzcd} i_1 \arrow[dr] & \\ & i \\ i_2 \arrow[ur] & \end{tikzcd}$$

(iii) For any two parallel morphisms $f : i_1 \to i_2$ and $g : i_1 \to i_2$, there is a diagram

$$\begin{tikzcd} i_1 \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & i_2 \arrow[r, "h"] & i \end{tikzcd}$$

such that $hf = hg$.

Example 4.1.3. Let $\mathcal{U}$ be a filter of a topological space X, that is, a non-empty collection of open subsets such that for every pair $U, V \in \mathcal{U}$, $U \cap V$ is also contained in $\mathcal{U}$. We can view $\mathcal{U}$ as a full subcategory of $\mathcal{O}\mathrm{pen}(X)^{\mathrm{op}}$. By definition, $\mathcal{U}$ is non-empty, so that (i) is satisfied. Any two elements $U_1, U_2 \in \mathcal{U}$ contain $U_1 \cap U_2$, which is property (ii) of Definition 4.1.2. Since the morphism between any two U_1 and U_2, that is, the inclusion $U_1 \subset U_2$ is unique, two parallel morphisms are always equal, so that we can choose the morphism h of (iii) to be the identity. We conclude that $\mathcal{U}$ is a filtered category.

Proposition 4.1.4. *A category $\mathcal{I}$ is filtered if and only if every finite diagram $D : \mathcal{J} \to \mathcal{I}$ has a cocone.*

Proof. A proof is given in the appendix (Proposition B.0.2). □

Definition 4.1.5. A category $\mathcal{I}$ is **cofiltered** if $\mathcal{I}^{\mathrm{op}}$ is filtered.

Definition 4.1.6. The colimit (limit) of a diagram $D : \mathcal{I} \to \mathcal{C}$ is called **filtered** (**cofiltered**), when $\mathcal{I}$ is.

Example 4.1.7. The sequence

$$\mathbb{R}^0 \longrightarrow \mathbb{R}^1 \longrightarrow \mathbb{R}^2 \longrightarrow \dots$$

of inclusions $\mathbb{R}^n \hookrightarrow \mathbb{R}^n \oplus \mathbb{R} \cong \mathbb{R}^{n+1}$ is a filtered diagram. Its colimit is $\bigoplus_{n=0}^{\infty} \mathbb{R}$, the $\mathbb{R}$-vector space of countably infinite dimension, the elements of which are finite but arbitrarily long sequences of real numbers.

Example 4.1.8. The sequence

$$\mathbb{R}^0 \longleftarrow \mathbb{R}^1 \longleftarrow \mathbb{R}^2 \longleftarrow \dots$$

of the projections $\mathbb{R}^{n+1} \cong \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is a cofiltered diagram. Its limit is $\prod_{n=0}^{\infty} \mathbb{R}$, the countably infinite product of $\mathbb{R}$, the elements of which are infinite sequences of real numbers.

Example 4.1.9. Let $\mathcal{F} : \mathcal{O}\mathrm{pen}(M)^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$ be a presheaf on the topological space M. Let $\mathcal{U}_m \subset \mathcal{O}\mathrm{pen}(M)$ be the subcategory of open sets containing the point $m \in M$. (This is called the neighborhood filter of m.) The colimit of the functor $\mathcal{U}_m^{\mathrm{op}} \hookrightarrow \mathcal{O}\mathrm{pen}(M)^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$,

$$\mathcal{F}_m := \operatorname*{colim}_{U \in \mathcal{U}_m} \mathcal{F}(U)\,,$$

is the **stalk** at m, that is, the set of germs at m. (Recall that two elements $\varphi \in \mathcal{F}(U)$, $\varphi' \in \mathcal{F}(U')$ have the same germ at m if they have the same restriction to some open neighborhood of m.)

Let $\Phi : \mathcal{I} \to \mathcal{J}$ and $X : \mathcal{J} \to \mathcal{C}$ be functors. If the colimit of X exists, the maps to $(X \circ \Phi)_i = X_{\Phi(i)} \to \operatorname{colim} X$ are a cocone of the diagram $X \circ \Phi$. So if the colimit of $X \circ \Phi$ exists as well, the cocone induces, by the universal property of the colimit, a unique morphism

$$\operatorname{colim}(X \circ \Phi) \longrightarrow \operatorname{colim} X\,. \tag{4.1}$$

Definition 4.1.10. A functor $\Phi : \mathcal{I} \to \mathcal{J}$ is **final** if for every functor $X : \mathcal{J} \to \mathcal{C}$ for which $\operatorname{colim}(X \circ \Phi)$ exists, $\operatorname{colim} X$ exists and the morphism (4.1) is an isomorphism.

The following proposition gives a more explicit equivalent characterization of final functors, which is often used as definition. Recall that a category is **connected** if every two objects are connected by a finite zigzag of arrows.

Proposition 4.1.11. *A functor $\Phi : \mathcal{I} \to \mathcal{J}$ is **final** if and only if for every object $j \in \mathcal{J}$ the comma category $j \downarrow \Phi$ is non-empty and connected.*

Proof. See Theorem 1 and Exercise 5 in Section IX.3 of [ML98]. □

Example 4.1.12. Let $\mathcal{I} = \omega = \mathcal{J}$ and $\Phi : \omega \to \omega$ be a functor such that the sequence $(\Phi(0), \Phi(1), \ldots)$ is unbounded. Then for every j in the target, there is some i such that $j \leq \Phi(i)$, which shows that $j \downarrow \Phi$ is non-empty. Moreover, if $j \leq \Phi(i')$ then either $\Phi(i) \leq \Phi(i')$ or $\Phi(i') \leq \Phi(i)$, so that $j \downarrow \Phi$ is connected. We conclude by Proposition 4.1.11 that Φ is final.

Example 4.1.13. Let $\mathcal{I} = \omega$ and $\mathcal{J} = \omega \times \omega$. The diagonal functor $\Phi : \omega \to \omega \times \omega$, $i \to (i, i)$ is final. In order to see this, observe that there is a morphism in ω from (i, j) to (i', j') if and only if $i \leq i'$ and $j \leq j'$. We can then argue as in the last example to show that Φ is final.

Definition 4.1.14. A functor $\Phi : \mathcal{I} \to \mathcal{J}$ is **initial** if for every functor $X : \mathcal{J} \to \mathcal{C}$ for which $\lim(X \circ \Phi)$ exists, $\lim X$ also exists, and the natural morphism

$$\lim X \longrightarrow \lim(X \circ \Phi)$$

is an isomorphism.

Proposition 4.1.15. *A functor $\Phi : \mathcal{I} \to \mathcal{J}$ is initial if and only if for every object $j \in \mathcal{J}$ the comma category $\Phi \downarrow j$ is non-empty and connected.*

Proof. The proposition is dual to Proposition 4.1.11. □

Terminology 4.1.16. Final functors are sometimes called "cofinal" and initial functors are sometimes called "co-cofinal", e.g. in [KS06]. This can be quite confusing, since "cofinal" is sometimes also used as synonym for "initial" in the sense used here. We will generally adhere to the terminology of [ML98]. And besides, in category theory "coco-x" should always mean the same as "x", which is why there is no category theoretical difference between a coconut and a nut.

Let $\mathcal{I}$ and $\mathcal{J}$ be index categories and $X : \mathcal{I} \times \mathcal{J} \to \mathcal{C}$ a functor to a complete and cocomplete category. The morphisms of the limit cone

$$\lim_{j \in \mathcal{J}} X(i,j) \longrightarrow X(i,j)$$

are natural in i, so they induce a morphism of the colimits over i,

$$\operatorname*{colim}_{i \in \mathcal{I}} \lim_{j \in \mathcal{J}} X(i,j) \longrightarrow \operatorname*{colim}_{i \in \mathcal{I}} X(i,j)\,.$$

These morphisms form a cone over the diagram $j \mapsto X(i,j)$, so by the universal property of the limit this induces a unique morphism

$$\operatorname*{colim}_{i \in \mathcal{I}} \lim_{j \in \mathcal{J}} X(i,j) \longrightarrow \lim_{j \in \mathcal{J}} \operatorname*{colim}_{i \in \mathcal{I}} X(i,j)\,. \tag{4.2}$$

Definition 4.1.17. Let $X : \mathcal{I} \times \mathcal{J} \to \mathcal{C}$ be a functor to a complete and cocomplete category. If the morphism (4.2) is an isomorphism then the limit and colimit are said to **commute**.

Proposition 4.1.18. *Let $\mathcal{I}$ be a small category. The following are equivalent:*

(i) $\mathcal{I}$ is filtered.

(ii) For any finite category $\mathcal{J}$ and any functor $X : \mathcal{I} \times \mathcal{J} \to \mathcal{S}\mathrm{et}$ the colimit over $\mathcal{I}$ and the limit over $\mathcal{J}$ commute.

Proof. See Theorem 3.1.6 in [KS06]. Cf. also Theorem 1 in Section IX.2 of [ML98]. □

Proposition 4.1.19. *Let $\mathcal{I}$ be a small category. The following are equivalent:*

(i) $\mathcal{I}$ is cofiltered.

(ii) For any finite category $\mathcal{J}$ and any functor $X : \mathcal{I} \times \mathcal{J} \to \mathcal{S}\mathrm{et}$, the limit over $\mathcal{I}$ and the colimit over $\mathcal{J}$ commute.

Corollary 4.1.20. *Filtered colimits and small limits preserve monomorphisms. Dually, small colimits and cofiltered limits preserve epimorphisms.*

Proof. A morphisms $f : S \to T$ is a monomorphism if and only

$$\begin{tikzcd} S \arrow[r, "\mathrm{id}"] \arrow[d, "\mathrm{id}"'] & S \arrow[d, "f"] \\ S \arrow[r, "f"'] & T \end{tikzcd}$$

is a pullback diagram, which is a finite limit diagram. Since, by Proposition 4.1.19, filtered colimits commute with finite limits, filtered colimits preserve monomorphisms. Since limits commute with limits, limits preserve monomorphisms, as well. □

In short, Proposition 4.1.18 states that filtered colimits commute with finite limits. Dually, Proposition 4.1.19 states that cofiltered limits commute with finite colimits. This is perhaps the most important feature of filtered categories. For more on commuting classes of limits and colimits see [BJLS15].

4.1.2 Definition of ind/pro-categories

Definition 4.1.21. A presheaf is called **ind-representable** if it is isomorphic to a filtered colimit of representable presheaves.

Let us spell out this definition. A presheaf $\hat{X} \in \mathcal{S}\mathrm{et}^{\mathcal{C}^{\mathrm{op}}}$ is ind-representable if $\hat{X} \cong \mathrm{colim}_{i\in\mathcal{J}}\, \mathbb{Y}_{\mathcal{C}}(X_i)$ for some functor $X : \mathcal{J} \to \mathcal{C}$ defined on a small filtered category $\mathcal{J}$.

Definition 4.1.22 (I.8.2 in [Art72])**.** Let $\mathcal{C}$ be a category. The **ind-category** $\mathrm{Ind}(\mathcal{C}) \equiv \mathrm{Ind}\mathcal{C}$ is the full subcategory of $\mathcal{S}\mathrm{et}^{\mathcal{C}^{\mathrm{op}}}$ of ind-representable presheaves.

Let $I : \mathrm{Ind}(\mathcal{C}) \to \mathcal{S}\mathrm{et}^{\mathcal{C}^{\mathrm{op}}}$ denote the inclusion of ind-objects into the category of presheaves. Being ind-representable is a property of a presheaf, so that I is injective. By definition, a morphism of ind-objects is a morphism of presheaves, so that I is full and faithful. Since a representable presheaf is a fortiori ind-representable, the Yoneda embedding $\mathbb{Y}_{\mathcal{C}} : \mathrm{Ind}(\mathcal{C}) \to \mathcal{S}\mathrm{et}^{\mathcal{C}^{\mathrm{op}}}$ takes its values in the image of I, so that we have a commutative diagram

$$\begin{tikzcd} & \mathcal{S}\mathrm{et}^{\mathcal{C}^{\mathrm{op}}} \\ \mathcal{C} \arrow[ur, "\mathbb{Y}_{\mathcal{C}}"] \arrow[r, "y_{\mathcal{C}}"'] & \mathrm{Ind}(\mathcal{C}) \arrow[u, "I"'] \end{tikzcd}$$

where $y_{\mathcal{C}}$ is the Yoneda embedding with restricted codomain. (When the category $\mathcal{C}$ is clear from the context, we will drop the index.) The presheaf $y_{\mathcal{C}}C$ is given by $(y_{\mathcal{C}}C)(A) = (\mathbb{Y}_{\mathcal{C}}C)(A) = \mathcal{C}(A,C)$ for all $A \in \mathcal{C}$. Since $\mathbb{Y}_{\mathcal{C}}$ and I are both full and faithful, so is the functor $y_{\mathcal{C}} : \mathcal{C} \to \mathrm{Ind}(\mathcal{C})$.

The concept dual to ind-categories is that of pro-categories. For the pro-category, we want to enlarge $\mathcal{C}$ by cofiltered limits. Let $X : \mathcal{J} \to \mathcal{C}$ be a cofiltered diagram. Then $X^{\mathrm{op}} : \mathcal{J}^{\mathrm{op}} \to \mathcal{C}^{\mathrm{op}}$ is a filtered diagram. The limit of X is the colimit of X^{op}. So in order to add the limit of X to $\mathcal{C}$ we first embed $\mathcal{C}^{\mathrm{op}}$ in its presheaf category by the Yoneda embedding,

$$\mathbb{Y}_{\mathcal{C}^{\mathrm{op}}} : \mathcal{C}^{\mathrm{op}} \longrightarrow \mathcal{S}\mathrm{et}^{(\mathcal{C}^{\mathrm{op}})^{\mathrm{op}}} \cong \mathcal{S}\mathrm{et}^{\mathcal{C}}\,.$$

An object in $\mathcal{S}\mathrm{et}^{\mathcal{C}}$ is called a **copresheaf** on $\mathcal{C}$. The Yoneda embedding of $C \in \mathcal{C}^{\mathrm{op}}$ is given explicitly by

$$\big(\mathbb{Y}_{\mathcal{C}^{\mathrm{op}}}(C)\big)(A) = \mathcal{C}^{\mathrm{op}}(A,C) = \mathcal{C}(C,A)$$

for all $A \in \mathcal{C}$. The functor $\mathcal{C}(C, \_) : \mathcal{C} \to \mathcal{S}\mathrm{et}$ is called a **representable** copresheaf or the copresheaf **represented by** C. Now we can take the colimit of $\mathbb{Y}_{\mathcal{C}^{\mathrm{op}}} X^{\mathrm{op}} : \mathcal{I}^{\mathrm{op}} \to \mathcal{S}\mathrm{et}^{\mathcal{C}}$ or, equivalently, the limit of $\mathbb{Y}^{\mathrm{op}}_{\mathcal{C}^{\mathrm{op}}} X : \mathcal{I} \to (\mathcal{S}\mathrm{et}^{\mathcal{C}})^{\mathrm{op}}$.

Definition 4.1.23. A copresheaf $\check{X} \in \mathcal{S}\mathrm{et}^{\mathcal{C}}$ is **pro-representable** if there is a cofiltered diagram $X : \mathcal{I} \to \mathcal{C}$ such that $\check{X}$ is isomorphic to the limit of the cofiltered diagram $\mathbb{Y}^{\mathrm{op}}_{\mathcal{C}^{\mathrm{op}}} X : \mathcal{I} \to (\mathcal{S}\mathrm{et}^{\mathcal{C}})^{\mathrm{op}}$.

Definition 4.1.24. Let $\mathcal{C}$ be a category. The **pro-category** $\mathrm{Pro}(\mathcal{C}) \equiv \mathrm{Pro}\mathcal{C}$ is the full subcategory of pro-representable copresheaves in $(\mathcal{S}\mathrm{et}^{\mathcal{C}})^{\mathrm{op}}$.

Proposition 4.1.25. *There is an isomorphism of categories* $\mathrm{Pro}(\mathcal{C}) \cong \big(\mathrm{Ind}(\mathcal{C}^{\mathrm{op}})\big)^{\mathrm{op}}$.

Proof. The isomorphism follows directly from the definition. □

Remark 4.1.26. Proposition 4.1.25 is sometimes taken as definition of pro-categories, e.g. In I.8.10 of [Art72].

Terminology 4.1.27. The prefixes "ind" and "pro" derive from the historic names "inductive limit" for colimit and "projective limit" for limit. By abuse of language, an object $\hat{X} \in \mathrm{Ind}\mathcal{C}$ is called an **ind-object** of $\mathcal{C}$, even though it is not an object of $\mathcal{C}$. Analogously, $\check{X} \in \mathrm{Pro}\mathcal{C}$ is called a **pro-object** of $\mathcal{C}$. When the objects in the category are named, "ind" and "pro" are added as prefixes. For example, a pro-object of the category of finite groups is called a pro-finite group, a pro-object of manifolds a pro-manifold, etc.

Lemma 4.1.28. *Let* $\hat{X} := \mathrm{colim}_{i\in\mathcal{I}} \mathbb{Y}_{\mathcal{C}}(X_i)$ *and* $\hat{Y} := \mathrm{colim}_{j\in\mathcal{J}} \mathbb{Y}_{\mathcal{C}}(Y_j)$ *be presheaves on* $\mathcal{C}$ *represented by the diagrams* $X : \mathcal{I} \to \mathcal{C}$ *and* $Y : \mathcal{J} \to \mathcal{C}$. *Then there is a natural bijection*

$$\mathcal{S}\mathrm{et}^{\mathcal{C}^{\mathrm{op}}}(\hat{X}, \hat{Y}) \cong \lim_{i\in\mathcal{I}} \operatorname*{colim}_{j\in\mathcal{J}} \mathcal{C}(X_i, Y_j)\,.$$

Proof. We have the natural isomorphisms

$$\begin{aligned}
\mathcal{S}\mathrm{et}^{\mathcal{C}^{\mathrm{op}}}(\hat{X}, \hat{Y}) &\cong \mathcal{S}\mathrm{et}^{\mathcal{C}^{\mathrm{op}}}(\operatorname*{colim}_{i\in\mathcal{I}} \mathbb{Y}_{\mathcal{C}}(X_i), \hat{Y}) \\
&\cong \lim_{i\in\mathcal{I}} \mathcal{S}\mathrm{et}^{\mathcal{C}^{\mathrm{op}}}(\mathbb{Y}_{\mathcal{C}}(X_i), \hat{Y}) \\
&\cong \lim_{i\in\mathcal{I}} \hat{Y}(X_i) \\
&= \lim_{i\in\mathcal{I}} \big(\operatorname*{colim}_{j\in\mathcal{J}} \mathbb{Y}_{\mathcal{C}}(Y_j)\big)(X_i) \\
&= \lim_{i\in\mathcal{I}} \operatorname*{colim}_{j\in\mathcal{J}} \big(\mathbb{Y}_{\mathcal{C}}(Y_j)(X_i)\big) \\
&= \lim_{i\in\mathcal{I}} \operatorname*{colim}_{j\in\mathcal{J}} \mathcal{C}(X_i, Y_j)\,.
\end{aligned}$$

In the first step we have used the colimit representation of $\hat{X}$, in the second step the universal property of colimits, in the third step the Yoneda lemma, in the fourth step the colimit representation of $\hat{Y}$, in the fifth step that colimits of presheaves are computed point-wise, and in the last step the definition of the Yoneda embedding. □

Proposition 4.1.29. *Let $\mathcal{C}$ be a category. Let $\hat{X}, \hat{Y} \in \mathrm{Ind}\mathcal{C}$ be represented by the filtered diagrams $X : \mathcal{I} \to \mathcal{C}$ and $Y : \mathcal{J} \to \mathcal{C}$. Then there is a natural isomorphism*

$$\mathrm{Ind}\mathcal{C}(\hat{X}, \hat{Y}) \cong \lim_{i \in \mathcal{I}} \operatorname*{colim}_{j \in \mathcal{J}} \mathcal{C}(X_i, Y_j)\,. \tag{4.3}$$

Proof. $\mathrm{Ind}\mathcal{C}$ is defined to be a *full* subcategory of $\mathcal{S}\mathrm{et}^{\mathcal{C}^{\mathrm{op}}}$, which means that

$$\mathrm{Ind}\mathcal{C}(\hat{X}, \hat{Y}) = \mathcal{S}\mathrm{et}^{\mathcal{C}^{\mathrm{op}}}(\hat{X}, \hat{Y})\,.$$

The proposition now follows from Lemma 4.1.28. □

Corollary 4.1.30. *Let $\mathcal{C}$ be a category. Let $\check{X}, \check{Y} \in \mathrm{Pro}\mathcal{C}$ be represented by the cofiltered diagrams $X : \mathcal{I} \to \mathcal{C}$ and $Y : \mathcal{J} \to \mathcal{C}$. There is a natural isomorphism*

$$\mathrm{Pro}\mathcal{C}(\check{X}, \check{Y}) \cong \lim_{j \in \mathcal{J}} \operatorname*{colim}_{i \in \mathcal{I}} \mathcal{C}(X_i, Y_j)\,. \tag{4.4}$$

Proof. Using Proposition 4.1.25 and Proposition 4.1.29, we can express the hom-set in $\mathrm{Pro}\mathcal{C}$ as

$$\begin{aligned}
\mathrm{Pro}\mathcal{C}(\check{X}, \check{Y}) &\cong \mathrm{Ind}(\mathcal{C}^{\mathrm{op}})^{\mathrm{op}}(\check{X}, \check{Y}) \\
&\cong \mathrm{Ind}(\mathcal{C}^{\mathrm{op}})(\check{Y}, \check{X}) \\
&\cong \lim_{j \in \mathcal{J}} \operatorname*{colim}_{i \in \mathcal{I}} \mathcal{C}^{\mathrm{op}}(Y_j, X_i) \\
&\cong \lim_{j \in \mathcal{J}} \operatorname*{colim}_{i \in \mathcal{I}} \mathcal{C}(X_i, Y_j)\,,
\end{aligned}$$

which proves the corollary. □

4.1.3 Functoriality and naturality of the ind/pro-extension

Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. Since $\mathcal{S}\mathrm{et}^{\mathcal{D}^{\mathrm{op}}}$ is cocomplete, the functor $\mathbb{Y}_{\mathcal{D}}F$ has a left Kan extension along the Yoneda embedding of $\mathcal{C}$, ***

$$\hat{F} := \mathrm{Lan}_{\mathbb{Y}_{\mathcal{C}}}(\mathbb{Y}_{\mathcal{D}}F) : \mathcal{S}\mathrm{et}^{\mathcal{C}^{\mathrm{op}}} \longrightarrow \mathcal{S}\mathrm{et}^{\mathcal{D}^{\mathrm{op}}}\,,$$

which we will call the **Yoneda extension** of F. It is given pointwise on $\hat{X} \in \mathcal{S}\mathrm{et}^{\mathcal{C}^{\mathrm{op}}}$ by the colimit

$$\hat{F}\hat{X} = \operatorname*{colim}_{\mathbb{Y}_{\mathcal{C}}C \to \hat{X}} \mathbb{Y}_{\mathcal{D}}(FC)\,.$$

By the Yoneda lemma, $\mathbb{Y}_{\mathcal{C}}$ is full and faithful. It follows that the diagram

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "F"] \arrow[d, "\mathbb{Y}_{\mathcal{C}}"'] & \mathcal{D} \arrow[d, "\mathbb{Y}_{\mathcal{D}}"] \\
\mathcal{S}\mathrm{et}^{\mathcal{C}^{\mathrm{op}}} \arrow[r, "\hat{F}"'] & \mathcal{S}\mathrm{et}^{\mathcal{D}^{\mathrm{op}}}
\end{tikzcd}$$

commutes. The left Kan extension of any functor along the Yoneda embedding preserves all small colimits (Proposition B.0.1). Let $G : \mathcal{D} \to \mathcal{E}$ be another functor.

It follows from the continuity of $\hat{F}$ that

$$\begin{aligned}\hat{G}\hat{F}(\hat{X}) &= \hat{G}\big(\operatorname*{colim}_{\mathbb{Y}_{\mathcal{C}}C\to\hat{X}} \mathbb{Y}_{\mathcal{D}}(FC)\big)\\ &\cong \operatorname*{colim}_{\mathbb{Y}_{\mathcal{C}}C\to\hat{X}} \hat{G}\mathbb{Y}_{\mathcal{D}}(FC)\\ &\cong \operatorname*{colim}_{\mathbb{Y}_{\mathcal{C}}C\to\hat{X}} \mathbb{Y}_{\mathcal{E}}GF(C)\\ &\cong (\operatorname{Lan}_{\mathbb{Y}_{\mathcal{C}}} \mathbb{Y}_{\mathcal{E}}GF)\hat{X}\\ &\cong \widehat{GF}(\hat{X})\,.\end{aligned}$$

This shows that the Yoneda extension preserves the composition of functors. The left Kan extension $\operatorname{Lan}_{\mathbb{Y}_{\mathcal{C}}} F$ is natural in F. That is, if $\tau : F \to F'$ is a natural transformation of functors $F, F' : \mathcal{C} \to \mathcal{D}$, then there is a natural transformation $\hat{\tau} : \hat{F} \to \hat{F}'$. The upshot is that the maps $\mathcal{C} \mapsto \mathcal{S}\mathrm{et}^{\mathcal{C}^{\mathrm{op}}}$, $F \mapsto \hat{F}$, and $\tau \mapsto \hat{\tau}$ defined an endofunctor of the 2-category of categories, functors, and natural transformations.

Proposition 4.1.31. *The Yoneda extension of a functor $F : \mathcal{C} \to \mathcal{D}$ restricts to a functor of ind-categories*

$$\operatorname{Ind}(F) : \operatorname{Ind}(\mathcal{C}) \longrightarrow \operatorname{Ind}(\mathcal{D})\,.$$

Proof. Let $\hat{X} \in \mathcal{S}\mathrm{et}^{\mathcal{C}^{\mathrm{op}}}$ be an ind-object represented by $X : \mathcal{I} \to \mathcal{C}$. Since $\hat{F}$ preserves colimits, we have

$$\begin{aligned}\hat{F}\hat{X} &= \hat{F}\big(\operatorname*{colim}_{i\in\mathcal{I}} \mathbb{Y}_{\mathcal{C}}(X_i)\big)\\ &\cong \operatorname*{colim}_{i\in\mathcal{I}} \hat{F}\mathbb{Y}_{\mathcal{C}}(X_i)\\ &= \operatorname*{colim}_{i\in\mathcal{I}} \mathbb{Y}_{\mathcal{D}}(FX_i)\,.\end{aligned}$$

This shows that $\hat{F}\hat{X}$ is an ind-object represented by $FX : \mathcal{I} \to \mathcal{D}$. □

Corollary 4.1.32. *The Yoneda extensions of functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{C} \to \mathcal{D}^{\mathrm{op}}$ restrict to functors*

$$\begin{aligned}\operatorname{Pro}(F) &: \operatorname{Pro}(\mathcal{C}) \longrightarrow \operatorname{Pro}(\mathcal{D})\\ \operatorname{Ind}(G) &: \operatorname{Ind}(\mathcal{C}) \longrightarrow \operatorname{Pro}(\mathcal{D})^{\mathrm{op}}\\ \operatorname{Pro}(G) &: \operatorname{Pro}(\mathcal{C}) \longrightarrow \operatorname{Ind}(\mathcal{D})^{\mathrm{op}}\,.\end{aligned}$$

where $\operatorname{Pro}(F) := \operatorname{Ind}(F^{\mathrm{op}})^{\mathrm{op}}$ *and* $\operatorname{Pro}(G) := \operatorname{Ind}(G^{\mathrm{op}})^{\mathrm{op}}$.

Proposition 4.1.33. *Mapping a category to its ind-category extends to an endofunctor* $\operatorname{Ind} : \mathcal{C}\mathrm{at} \to \mathcal{C}\mathrm{at}$ *of the 2-category of categories, functors, and natural transformations. The same is true for pro-categories.*

Proof. We have already explained, that the map from categories to presheaves is a 2-functor $\mathcal{C}\mathrm{at} \to \mathcal{C}\mathrm{at}$. $\operatorname{Ind}(F)$ is the point-wise restriction to ind-objects $\hat{X}$, $\operatorname{Ind}(F)\hat{X} = \hat{F}\hat{X}$. It follows that

$$\begin{aligned}\operatorname{Ind}(GF)\hat{X} &= \widehat{GF}(\hat{X})\\ &\cong \hat{G}\hat{F}(\hat{X})\\ &= \hat{G}(\operatorname{Ind}(F)\hat{X})\\ &= \operatorname{Ind}(G)\operatorname{Ind}(F)\hat{X}\,.\end{aligned}$$

An analogous argument applies to natural transformations. This shows that Ind is a 2-functor. The dual statement for pro-categories follows from Proposition 4.1.25. □

Example 4.1.34. Consider the sequence of vector spaces $\mathbb{R}^0 \to \mathbb{R}^1 \to \dots$ from Example 4.1.7, which represents an ind-object of the category of finite-dimensional vector spaces. The composition with the dual yields the sequence $(\mathbb{R}^0)^* \leftarrow (\mathbb{R}^1)^* \leftarrow \dots$, which represents a pro-object of finite-dimensional vector spaces. Taking the dual again, we get back the ind-object we started with.

The reflexivity of ind/pro-finite dimensional vector spaces is one of the advantages of working in ind- and pro-categories. Taking the algebraic dual of an infinite dimensional vector space always raises the cardinality of the dimension. For example, the dual of the colimit of the sequence $\mathbb{R}^0 \to \mathbb{R}^1 \to \dots$ is $(\coprod_{n=0}^{\infty} \mathbb{R})^* \cong \prod_{n=0}^{\infty} \mathbb{R}^*$, which is the limit of the sequence $(\mathbb{R}^0)^* \leftarrow (\mathbb{R}^1)^* \leftarrow \dots$. But taking the dual again, yields a vector space of the unwieldy dimension $2^{(2^{\aleph_0})}$. Adding a Banach structure and taking bounded duals can make an infinite dimensional vector space reflexive. But when we only have a Fréchet structure, as in the example of smooth sections of a vector bundle, we are out of luck: The dual of a Fréchet space is again a Fréchet space if and only if it was a Banach space to begin with.

Proposition 4.1.35. *For any two categories $\mathcal{C}$ and $\mathcal{D}$, there are natural equivalences*

$$\begin{aligned}\mathrm{Ind}(\mathcal{C} \times \mathcal{D}) &\simeq \mathrm{Ind}(\mathcal{C}) \times \mathrm{Ind}(\mathcal{D})\\ \mathrm{Pro}(\mathcal{C} \times \mathcal{D}) &\simeq \mathrm{Pro}(\mathcal{C}) \times \mathrm{Pro}(\mathcal{D}).\end{aligned}$$

Proof. Let $(\hat{X}, \hat{Y}) \in \mathrm{Ind}(\mathcal{C}) \times \mathrm{Ind}(\mathcal{D})$ be a pair of ind-objects represented by diagrams $X : \mathcal{I} \to \mathcal{C}$ and $Y : \mathcal{J} \to \mathcal{D}$. It is straight-forward to show that the product of two filtered categories is filtered (Proposition 3.2.1 (iii) in [KS06]). Therefore, the product functor $X \times Y : \mathcal{I} \times \mathcal{J} \to \mathcal{C} \times \mathcal{D}$ represents an ind-object of $\mathcal{C} \times \mathcal{D}$. We thus obtain a map

$$\mathrm{Ind}(\mathcal{C}) \times \mathrm{Ind}(\mathcal{D}) \longrightarrow \mathrm{Ind}(\mathcal{C} \times \mathcal{D}). \tag{4.5}$$

Because the product of functors $X \times Y$ is natural in both the domain and the target, the map (4.5) is a functor. And since the Yoneda embedding commutes with products, this functor is full and faithful.

Consider an object $\hat{Z}$ in $\mathrm{Ind}(\mathcal{C} \times \mathcal{D})$ represented by a functor $Z : \mathcal{I} \to \mathcal{C} \times \mathcal{D}$, $i \mapsto X_i \times Y_i$, where $X : \mathcal{I} \to \mathcal{C}$ and $Y : \mathcal{I} \to \mathcal{D}$ are the two components of Z. Since the diagonal functor $\Delta : \mathcal{I} \to \mathcal{I} \times \mathcal{I}$ is final (Exercise 4.2), $X \times Y : \mathcal{I} \times \mathcal{I} \to \mathcal{C} \times \mathcal{D}$ and Z represent isomorphic ind-objects. This shows that the fully faithful functor (4.5) is essentially surjective, so it is an equivalence of categories.

There is an isomorphism $(\mathcal{C} \times \mathcal{D})^{\mathrm{op}} \cong \mathcal{C}^{\mathrm{op}} \times \mathcal{D}^{\mathrm{op}}$ for any pair of categories. We thus obtain

$$\begin{aligned}\mathrm{Pro}(\mathcal{C} \times \mathcal{D}) &\cong \big(\mathrm{Ind}\big((\mathcal{C} \times \mathcal{D})^{\mathrm{op}}\big)\big)^{\mathrm{op}}\\ &\cong \big(\mathrm{Ind}(\mathcal{C}^{\mathrm{op}} \times \mathcal{D}^{\mathrm{op}})\big)^{\mathrm{op}}\\ &\simeq \big(\mathrm{Ind}(\mathcal{C}^{\mathrm{op}}) \times \mathrm{Ind}(\mathcal{D}^{\mathrm{op}})\big)^{\mathrm{op}}\\ &\cong \mathrm{Ind}(\mathcal{C}^{\mathrm{op}})^{\mathrm{op}} \times \mathrm{Ind}(\mathcal{D}^{\mathrm{op}})^{\mathrm{op}}\\ &\cong \mathrm{Pro}(\mathcal{C}) \times \mathrm{Pro}(\mathcal{D}),\end{aligned}$$

which finishes the proof. □

4.1.4 Finite limits and colimits in ind/pro-categories

Even finite limits and colimits in ind/pro-categories can be difficult to compute. Matters become easier if for a diagram $\hat{D} : \mathcal{A} \to \mathrm{Ind}\mathcal{C}$ the objects $\hat{D}(a) \in \mathrm{Ind}\mathcal{C}$ can be represented by diagrams $D(a) : \mathcal{I} \to \mathcal{C}$ indexed by the same filtered category $\mathcal{I}$ and the morphisms of the diagram are all represented by natural transformations $D(a) \to D(b)$. Such a D is called a **level-representation** of $\hat{D}$. If a level-representation of $\hat{D}$ exists, then its limit and colimit can be computed level-wise, which is the statement of the following result, first proved in [AM69].

Proposition 4.1.36. *Let $\mathcal{I}$ be a small filtered category. Then the functor*

$$\begin{aligned} \mathcal{C}^{\mathcal{I}} &\longrightarrow \mathrm{Ind}\mathcal{C} \\ X &\longmapsto \operatorname*{colim}_{i\in\mathcal{I}} \mathbb{Y}_{\mathcal{C}}(X_i) \end{aligned} \tag{4.6}$$

commutes with finite limits and finite colimits.

Proof. Let $D : \mathcal{A} \to \mathcal{C}^{\mathcal{I}}$, $a \mapsto D(a)$ be a diagram indexed by a finite category $\mathcal{A}$. Assume that the colimit of D exists. Since colimits in functor categories are computed point-wise, this means that the colimit of the functor $\mathcal{A} \to \mathcal{C}$, $a \mapsto D(a)_i$ exists for all $i \in \mathcal{I}$.

Let us denote the functor (4.6) by F. The image of this diagram under F is

$$\begin{aligned} FD : \mathcal{A} &\longrightarrow \mathrm{Ind}\mathcal{C} \\ a &\longmapsto \operatorname*{colim}_{i\in\mathcal{I}} \mathbb{Y}_{\mathcal{C}}\big(D(a)_i\big) . \end{aligned}$$

Let $\hat{Y} \in \mathrm{Ind}\mathcal{C}$ be represented by the filtered diagram $Y : \mathcal{J} \to \mathcal{C}$. We have the natural bijections

$$\begin{aligned} \mathrm{Ind}\mathcal{C}(\operatorname*{colim}_{a\in\mathcal{A}} FD(a), \hat{Y}) &\cong \lim_{a\in\mathcal{A}} \mathrm{Ind}\mathcal{C}(FD(a), \hat{Y}) \\ &\cong \lim_{a\in\mathcal{A}} \lim_{i\in\mathcal{I}} \operatorname*{colim}_{j\in\mathcal{J}} \mathcal{C}(D(a)_i, Y_j) \\ &\cong \lim_{i\in\mathcal{I}} \lim_{a\in\mathcal{A}} \operatorname*{colim}_{j\in\mathcal{J}} \mathcal{C}(D(a)_i, Y_j) \\ &\cong \lim_{i\in\mathcal{I}} \operatorname*{colim}_{j\in\mathcal{J}} \lim_{a\in\mathcal{A}} \mathcal{C}(D(a)_i, Y_j) \\ &\cong \lim_{i\in\mathcal{I}} \operatorname*{colim}_{j\in\mathcal{J}} \mathcal{C}(\operatorname*{colim}_{a\in\mathcal{A}} D(a)_i, Y_j) \\ &\cong \mathrm{Ind}\mathcal{C}\big(F(\operatorname*{colim}_{a\in\mathcal{A}} D(a)), \hat{Y}\big) , \end{aligned}$$

where we have used the universal property of the colimit of FD, the commutativity of limits, formula (4.3) for the morphisms in an ind-category, the commutativity of finite limits with filtered colimits stated in Proposition 4.1.18, the universal property of the colimit of the functor $D(\_)_i : \mathcal{A} \to \mathcal{C}$, and formula (4.3) again. Since this bijection holds for all $\hat{Y}$, we conclude that F commutes with the colimits over $\mathcal{A}$.

Assume now that the limit of D exists. Then we have the natural bijections

$$\begin{aligned}
\mathrm{Ind}\mathcal{C}(\hat{Y}, \lim_{a\in\mathcal{A}} FD(a)) &\cong \lim_{a\in\mathcal{A}} \mathrm{Ind}\mathcal{C}(\hat{Y}, FD(a)) \\
&\cong \lim_{a\in\mathcal{A}} \lim_{j\in\mathcal{J}} \operatorname*{colim}_{i\in\mathcal{I}} \mathcal{C}(Y_j, D(a)_i) \\
&\cong \lim_{j\in\mathcal{J}} \lim_{a\in\mathcal{A}} \operatorname*{colim}_{i\in\mathcal{I}} \mathcal{C}(Y_j, D(a)_i) \\
&\cong \lim_{j\in\mathcal{J}} \operatorname*{colim}_{i\in\mathcal{I}} \lim_{a\in\mathcal{A}} \mathcal{C}(Y_j, D(a)_i) \\
&\cong \lim_{j\in\mathcal{J}} \operatorname*{colim}_{i\in\mathcal{I}} \mathcal{C}(Y_j, \lim_{a\in\mathcal{A}} D(a)_i) \\
&\cong \mathrm{Ind}\mathcal{C}\big(\hat{Y}, F(\lim_{a\in\mathcal{A}} D(a))\big)\,,
\end{aligned}$$

which shows that F commutes with the limits over $\mathcal{A}$. □

Example 4.1.37. Let $\hat{X}$, $\hat{Y}$ be ind-objects in a category $\mathcal{C}$ with finite products, that are represented by the filtered diagrams $X, Y : \mathcal{I} \to \mathcal{C}$. Then the product $\hat{X} \times \hat{Y}$ exists and is represented by $\mathcal{I} \to \mathcal{C}$, $i \mapsto X_i \times Y_i$.

Remark 4.1.38. The map $\mathcal{C}^{\mathcal{I}} \to \mathrm{Ind}\mathcal{C}$ does in general not commute with infinite limits or colimits. In fact, it does not even commute with filtered colimits, even though $\mathrm{Ind}\mathcal{C}$ is a cocompletion of $\mathcal{C}$ by filtered colimits (see Example 4.2.5).

A finite diagram $\hat{D}$ can fail to have a level-representation only if $\mathcal{A}$ has "loops", that is, no non-trivial endomorphisms [Isa02]. For example, a level-representation exists for every diagram consisting of a finite number of ind-objects without morphisms between them or for every diagram consisting of a pair of parallel morphisms between a pair of ind-objects [KS06, Cor. 6.3.15]. Since the (co)limits of such diagrams are (co)products and (co)equalizers, Proposition 4.1.36 implies that all finite (co)products and (co)equalizers exist in $\mathrm{Ind}\mathcal{C}$ if they exist in $\mathcal{C}$. Since every finite (co)limit can be obtained by a (co)equalizer of a finite (co)product we arrive at the following proposition.

Proposition 4.1.39. *If $\mathcal{C}$ has all finite coproducts, coequalizers, colimits, products, equalizers, or limits then so does* $\mathrm{Ind}\mathcal{C}$.

This result can be slightly improved. In Proposition 6.1.18 of [KS06] it is shown that having finite coproducts in $\mathcal{C}$ implies that $\mathrm{Ind}\mathcal{C}$ has small coproducts. As a consequence, if $\mathcal{C}$ has finite colimits, then $\mathrm{Ind}\mathcal{C}$ has small colimits. For later reference, we state the dual of Proposition 4.1.36 for pro-categories.

Proposition 4.1.40. *Let $\mathcal{I}$ be a small cofiltered category. Then the functor*

$$\begin{aligned}
\mathcal{C}^{\mathcal{I}} &\longrightarrow \mathrm{Pro}\mathcal{C} \\
X &\longmapsto \lim_{i\in\mathcal{I}} \mathbb{Y}^{\mathrm{op}}_{\mathcal{C}^{\mathrm{op}}}(X_i)
\end{aligned} \tag{4.7}$$

commutes with finite limits and finite colimits.

Proof. This follows from Proposition 4.1.36 by Proposition 4.1.25. The statement was first proved in Proposition 4.1, Appendix A of [AM69]. □

4.1.5 Ind/pro-objects versus colimits/limits

One of the main reasons to introduce the ind-category $\operatorname{Ind}\mathcal{C}$ was to enlarge $\mathcal{C}$ by filtered colimits. What happens if the filtered colimits already exist? Let $\hat{X}$ and $\hat{Y}$ be represented by filtered diagrams $X : \mathcal{I} \to \mathcal{C}$ and $Y : \mathcal{J} \to \mathcal{C}$. Assume that the colimits of X and Y exist in $\mathcal{C}$. From the universal property of the colimit of Y, we obtain a natural map

$$\operatorname*{colim}_{j\in\mathcal{J}} \mathcal{C}(X_i, Y_j) \longrightarrow \mathcal{C}(X_i, \operatorname*{colim}_{j'\in\mathcal{J}} Y_{j'}),$$

for all X_i. Taking the limit over i, we obtain the map

$$\lim_{i\in\mathcal{I}} \operatorname*{colim}_{j\in\mathcal{J}} \mathcal{C}(X_i, Y_j) \longrightarrow \lim_{i\in\mathcal{I}} \mathcal{C}(X_i, \operatorname*{colim}_{j'\in\mathcal{J}} Y_{j'}). \tag{4.8}$$

The domain of this map is $\operatorname{Ind}\mathcal{C}(\hat{X}, \hat{Y})$. The codomain can be written as

$$\lim_{i\in\mathcal{I}} \mathcal{C}(X_i, \operatorname*{colim}_{j\in\mathcal{J}} Y_j) \cong \mathcal{C}(\operatorname*{colim}_{i\in\mathcal{I}} X_i, \operatorname*{colim}_{j\in\mathcal{J}} Y_j).$$

With this, we obtain from (4.8) the natural map

$$\operatorname{Ind}\mathcal{C}(\hat{X}, \hat{Y}) \longrightarrow \mathcal{C}(\operatorname*{colim}_{i\in\mathcal{I}} X_i, \operatorname*{colim}_{j\in\mathcal{J}} Y_j). \tag{4.9}$$

This map is generally neither injective not surjective. It is injective under the following condition on ind-objects, which is satisfied in many applications.

Proposition 4.1.41. *Let $\hat{X}, \hat{Y} \in \operatorname{Ind}\mathcal{C}$ be represented by the diagrams $X : \mathcal{I} \to \mathcal{C}$ and $Y : \mathcal{J} \to \mathcal{C}$ that have colimits in $\mathcal{C}$. Assume that all arrows of the diagrams X and Y are monomorphisms. Then the map* (4.9) *is injective.*

Proof. By Corollary 4.1.20, monomorphisms commute with filtered colimits. Therefore, the morphisms of the colimit cone

$$Y_j \longrightarrow \operatorname*{colim}_{j'\in\mathcal{J}} Y_{j'}$$

are all monomorphisms. It follows that the induced morphisms

$$\mathcal{C}(X_i, Y_j) \longrightarrow \mathcal{C}(X_i, \operatorname*{colim}_{j'\in\mathcal{J}} Y_{j'})$$

are monomorphisms for all $X_i \in \mathcal{C}$. Using again that monomorphisms commute with filtered colimits, we infer that

$$\operatorname*{colim}_{j\in\mathcal{J}} \mathcal{C}(X_i, Y_j) \longrightarrow \mathcal{C}(X_i, \operatorname*{colim}_{j\in\mathcal{J}} Y_j)$$

is a monomorphism. Similarly, monomorphisms commute with limits. Therefore,

$$\lim_{i\in\mathcal{I}} \operatorname*{colim}_{j\in\mathcal{J}} \mathcal{C}(X_i, Y_j) \longrightarrow \lim_{i\in\mathcal{I}} \mathcal{C}(X_i, \operatorname*{colim}_{j\in\mathcal{J}} Y_j) \cong \mathcal{C}(\operatorname*{colim}_{i\in\mathcal{I}} X_i, \operatorname*{colim}_{j\in\mathcal{J}} Y_j)$$

is a monomorphism. By Equation (4.3), we conclude that (4.9) is an injective map. $\square$

Definition 4.1.42. An ind-object (pro-object) in $\mathcal{C}$ is called **strict** if it is represented by a diagram in which every arrow is a monomorphism (epimorphism).

***

Remark 4.1.43. Proposition 4.1.41 states that a morphism $\hat{X} \to \hat{Y}$ of strict ind-objects can be identified with a morphism $\operatorname{colim} X \to \operatorname{colim} Y$ of the colimits (if they exist) of the representing diagrams. However, the map (4.9) is generally not surjective. This means that there may be morphisms of the colimits that do not come from morphisms of the ind-objects. The upshot is that even if all cofiltered limits in $\mathcal{C}$ exist, the objects in Ind$\mathcal{C}$ have a richer structure and, consequently, fewer morphisms than the colimits in $\mathcal{C}$ (see Example 4.1.57 and Example 4.2.5).

Warning 4.1.44. The historic notation in [Art72] for an ind-object $\hat{X}$ represented by the diagram $X : \mathcal{I} \to \mathcal{C}$ is $\varinjlim X$ ($\varinjlim$ is a notation for the colimit). In this notation, the colimit must be taken in the category of presheaves on $\mathcal{C}$, since it is generally different from the colimit in $\mathcal{C}$. To avoid this notational trap, some authors write "$\varinjlim$"X [KS06].

If $\mathcal{C}$ has already all filtered colimits, we can try to define a functor Ind$(\mathcal{C}) \to \mathcal{C}$ that sends an ind-object $\hat{X}$ represented by the functor $X : \mathcal{I} \to \mathcal{C}$ to the colimit

$$\bar{X} := \operatorname*{colim}_{i \in \mathcal{I}} X_i \,.$$

We have to check that this is well defined, that is, up to isomorphism $\bar{X}$ does not depend on the choice of the representing diagram. For every $C \in \mathcal{C}$, we have the isomorphisms

$$\begin{aligned} \mathcal{C}(\operatorname*{colim}_{i \in \mathcal{I}} X_i, C) &\cong \lim_{i \in \mathcal{I}} \mathcal{C}(X_i, C) \\ &\cong \lim_{i \in \mathcal{I}} \mathcal{S}\mathrm{et}^{\mathcal{C}^{\mathrm{op}}}\big(\mathbb{Y}_{\mathcal{C}}(X_i), \mathbb{Y}_{\mathcal{C}}(C)\big) \\ &\cong \mathrm{Ind}\mathcal{C}(\hat{X}, yC) \,. \end{aligned}$$

It follows that if $Y : \mathcal{J} \to \mathcal{C}$ is another diagram representing $\hat{X}$, then

$$\mathcal{C}(\operatorname*{colim}_{j \in \mathcal{J}} Y_j, C) \cong \mathcal{C}(\operatorname*{colim}_{i \in \mathcal{I}} X_i, C)$$

for all $C \in \mathcal{C}$. This implies that $\operatorname{colim}_{j \in \mathcal{J}} Y_j \cong \operatorname{colim}_{i \in \mathcal{I}} X_i$. By choosing the colimits, we obtain a well-defined functor Ind$\mathcal{C} \to \mathcal{C}$, $\hat{X} \to \bar{X}$. It follows from the construction that

$$\overline{yC} \cong C \,.$$

For more details, see [KS06, Prop. 6.3.1].

Definition 4.1.45. An object $C \in \mathcal{C}$ is **compact** or **finitely presented** if for every colimit of a filtered diagram $D : \mathcal{I} \to \mathcal{C}$ the natural morphism

$$\operatorname*{colim}_{i \in \mathcal{I}} \mathcal{C}(C, D_i) \longrightarrow \mathcal{C}(C, \operatorname*{colim}_{i \in \mathcal{I}} D_i)$$

is an isomorphism.

Let $X : \mathcal{I} \to \mathcal{C}$ be a filtered diagram that has a colimit $\bar{X}$. Let $C \in \mathcal{C}$ be compact. Then

$$\begin{aligned}\mathcal{C}(C, \bar{X}) &\cong \mathcal{C}(C, \operatorname*{colim}_{i\in\mathcal{I}} X_i)\\ &\cong \operatorname*{colim}_{i\in\mathcal{I}} \mathcal{C}(C, X_i)\\ &\cong \operatorname{Ind}\mathcal{C}(yC, \hat{X})\,,\end{aligned}$$

where $\hat{X}$ is the ind-object represented by X. We conclude that morphisms from a compact object into a filtered colimit can be identified with morphisms of the ind-objects. If C is not compact, this is generally not true (Example 4.2.5).

4.1.6 Concrete structures

Recall from Terminology 2.1.6 that a faithful functor $U : \mathcal{C} \to \mathcal{S}\mathrm{et}$ is called a concrete structure on $\mathcal{C}$. There may be different concrete structures on the same category (see Remark 4.1.47). In many categories the objects are by definition sets with additional structure, such as groups, rings, algebras, vector spaces, topological spaces, manifolds, etc. In that case, there is the obvious forgetful functor that discards the additional structure.

Proposition 4.1.46. *Let $U : \mathcal{C} \to \mathcal{S}\mathrm{et}$ be a concrete category. Then its left Kan extension to* $\operatorname{Ind}\mathcal{C}$*,*

$$\hat{U} := \operatorname{Lan}_{\mathcal{C}\to\operatorname{Ind}\mathcal{C}} U : \operatorname{Ind}\mathcal{C} \longrightarrow \mathcal{S}\mathrm{et}\,,$$

is a concrete structure.

Proof. Let $\hat{X}, \hat{Y} \in \operatorname{Ind}\mathcal{C}$ be represented by diagrams $X : \mathcal{I} \to \mathcal{C}$ and $Y : \mathcal{J} \to \mathcal{C}$, defined on filtered categories $\mathcal{I}$ and $\mathcal{J}$. First, we observe that the Kan extension of the forgetful functor is given by $\hat{U}\hat{X} = \operatorname{colim}_{i\in I} UX_i$. It follows that

$$\mathcal{S}\mathrm{et}(\hat{U}\hat{X}, \hat{U}\hat{Y}) \cong \lim_{i\in\mathcal{I}} \operatorname*{colim}_{j\in\mathcal{J}} \mathcal{S}\mathrm{et}(UX_i, UY_j)\,. \tag{4.10}$$

Since U is faithful, the forgetful map $\mathcal{C}(X_i, Y_j) \to \mathcal{S}\mathrm{et}(UX_i, UY_j)$ is injective for all $i \in \mathcal{I}$, $j \in \mathcal{J}$. By Corollary 4.1.20 filtered colimits preserve monomorphisms. It follows that the forgetful map

$$\operatorname*{colim}_{j\in\mathcal{J}} \mathcal{C}(X_i, Y_j) \longrightarrow \operatorname*{colim}_{j\in\mathcal{J}} \mathcal{S}\mathrm{et}(UX_i, UY_j) \tag{4.11}$$

is a monomorphism. By Corollary 4.1.20 small limits preserve monomorphisms. It follows that the map

$$\lim_{i\in\mathcal{I}} \operatorname*{colim}_{j\in\mathcal{J}} \mathcal{C}(X_i, Y_j) \longrightarrow \lim_{i\in\mathcal{I}} \operatorname*{colim}_{j\in\mathcal{J}} \mathcal{S}\mathrm{et}(UX_i, UY_j) \tag{4.12}$$

is a monomorphism. Using the isomorphisms (4.3) and (4.10), we conclude that the map

$$\operatorname{Ind}\mathcal{C}(\hat{X}, \hat{Y}) \longrightarrow \mathcal{S}\mathrm{et}(\hat{U}\hat{X}, \hat{U}\hat{Y})$$

is a monomorphism as well. In other words, $\hat{U}$ is faithful. $\square$

Remark 4.1.47. The category of presheaves on any category $\mathcal{C}$ is concrete with the forgetful functor $\hat{X} \mapsto \coprod_{C\in\mathcal{C}} \hat{X}(C)$. But this functor is quite different from the one of Proposition 4.1.46.

Corollary 4.1.48. *Let* $U : \mathcal{C} \to \mathcal{S}\mathrm{et}$ *be a concrete structure. The its right Kan extension to* $\mathrm{Pro}\mathcal{C}$*,*

$$\check{U} := \mathrm{Ran}_{\mathcal{C}\to\mathrm{Pro}\mathcal{C}}\, U : \mathrm{Pro}\mathcal{C} \longrightarrow \mathcal{S}\mathrm{et}\,,$$

is a concrete structure.

Proof. The proof follows from Proposition 4.1.25. □

Corollary 4.1.48 states that if $\mathcal{C}$ is a concrete category then there is a faithful functor $\check{U}$ on $\mathrm{Pro}\mathcal{C}$ such that for every $\check{X} \in \mathrm{Pro}\mathcal{C}$ represented by $X : \mathcal{I} \to \mathcal{C}$ we have

$$\check{U}\check{X} = \lim_{i\in\mathcal{I}} UX_i\,.$$

In many categories the forgetful functor is the functor of morphisms

$$U(C) = \mathcal{C}(S, C)$$

out of a test object S. Such a U is called the **functor of S-points**. The Kan extension of U is now given by

$$\check{U}\check{X} \cong \mathrm{Pro}\mathcal{C}\big(\mathbb{Y}^{\mathrm{op}}_{\mathcal{C}^{\mathrm{op}}} S, \check{X}\big)\,,$$

where we have used formula (4.4) for the hom-sets in $\mathrm{Pro}(\mathcal{C})$. This shows that $\check{U}$ is also the functor of S-points, where we identify S with the presheaf it represents. In the category of vector spaces, the test object is $S = \mathbb{R}$. In geometric categories, such as topological spaces and smooth manifolds, the test object is typically the terminal object $S = *$. Since the Yoneda embedding commutes with limits, $\mathbb{Y}_{\mathcal{C}}(*)$ is the terminal object in $\mathrm{Ind}\mathcal{C}$, which implies that $\mathbb{Y}^{\mathrm{op}}_{\mathcal{C}^{\mathrm{op}}}(*)$ is the terminal object in $\mathrm{Pro}(\mathcal{C})$.

Notation 4.1.49. Having convinced ourselves that a concrete structure on $\mathcal{C}$ extends to concrete structures on both, $\mathrm{Ind}(\mathcal{C})$ and $\mathrm{Pro}(\mathcal{C})$, we will return to the lighter and more intuitive notation $UC \equiv |C|$, $\hat{U}\hat{X} = |\hat{X}|$, and $\check{U}\check{X} \equiv |\check{X}|$.

The upshot is that if the functor of points $C \mapsto |C| := \mathcal{C}(*, C)$ is a concrete structure on $\mathcal{C}$, then so is the functor of points for ind-objects and pro-objects. If $\check{X} \in \mathrm{Pro}\mathcal{C}$ is pro-represented by the diagram $X : \mathcal{I} \longrightarrow \mathcal{C}$ then

$$|\check{X}| := \mathrm{Pro}\mathcal{C}(*, \check{X}) \cong \lim_{i\in\mathcal{I}} |X_i|\,. \tag{4.13}$$

4.1.7 Tensor products, algebras, derivations

The tensor product of vector spaces is an example for a closed symmetric monoidal structure. We recall that a **monoidal structure** on a category $\mathcal{C}$ consists of a functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, called the **tensor product** and an object $1 \in \mathcal{C}$, called the **tensor unit**, that equip $\mathcal{C}$ with a weakly associative and unital multiplication. That means that there are natural isomorphisms $a_{A,B,C} : (A \otimes B) \otimes C \to A \otimes (B \otimes C)$, $l_A : 1 \otimes A \to A$ and $r_A : A \otimes 1 \to A$ satisfying certain coherence axioms. The tensor product is called **symmetric** if there is a natural isomorphism $\tau_{A,B} : A \otimes B \to B \otimes A$ with $\tau_{A,B} \circ \tau_{B,A} = \mathrm{id}_{A\otimes B}$, satisfying additional coherence axioms involving a, l, and r. A monoidal category is called **closed** if for every $B \in \mathcal{C}$ the functor $\_ \otimes B : A \mapsto A \otimes B$ has a right adjoint $C \mapsto \underline{\mathrm{Hom}}(B, C)$, i.e. there is a natural isomorphism

$$\mathcal{C}(A \otimes B, C) \cong \mathcal{C}\big(A, \underline{\mathrm{Hom}}(B, C)\big) .$$

For the full definition of closed symmetric monoidal categories see for example Ch. VII in [ML98] or Section 1 in [Kel05].

Terminology 4.1.50. The object $\underline{\mathrm{Hom}}(A, B)$ is called the **internal** or **inner hom-object**. It is also denoted by $[A, B]$ or A^B.

Example 4.1.51. The category $\mathcal{V} = \mathcal{V}\mathrm{ec}$ with the tensor product $\otimes$ of real vector spaces, the tensor unit $1 = \mathbb{R}$, and the usual vector space of linear maps $\underline{\mathrm{Hom}}(V, W)$ is a closed symmetric monoidal category.

By Proposition 4.1.31 the functor $\otimes$ induces a functor $\mathrm{Ind}(\otimes)$ on $\mathrm{Ind}(\mathcal{C} \times \mathcal{C})$. Composing this functor with the equivalence of Proposition 4.1.35, we obtain a functor

$$\hat{\otimes} : \mathrm{Ind}(\mathcal{C}) \times \mathrm{Ind}(\mathcal{C}) \xrightarrow{\cong} \mathrm{Ind}(\mathcal{C} \times \mathcal{C}) \xrightarrow{\mathrm{Ind}(\otimes)} \mathrm{Ind}(\mathcal{C}) , \tag{4.14}$$

which maps ind-objects $\hat{A}$, $\hat{B}$ represented by diagrams $A : \mathcal{I} \to \mathcal{C}$ and $B : \mathcal{J} \to \mathcal{C}$ to the ind-object represented by $\mathcal{I} \times \mathcal{J} \to \mathcal{C}$, $(i, j) \mapsto A_i \otimes B_j$.

Proposition 4.1.52. *Let* $(\mathcal{C}, \otimes, 1)$ *be a monoidal category. Then the functor* $\hat{\otimes}$ *of Equation* (4.14) *and the object* $\hat{1} := \mathbb{Y}_{\mathcal{C}}(1) \in \mathrm{Ind}\mathcal{C}$ *are a monoidal structure on* $\mathrm{Ind}\mathcal{C}$.

Proof. The associativity of $\hat{\otimes}$ follows from the associativity of $\otimes$ and of the product in categories. Since $\hat{1}$ is represented by the constant diagram $1 : * \to \mathcal{C}$, it follows that $\hat{1} \,\hat{\otimes}\, \hat{X}$ is represented by the diagram $\mathcal{I} \cong * \times \mathcal{I} \to \mathcal{C}$, $i \mapsto 1 \otimes X_i \cong X_i$, which is again X. □

Remark 4.1.53. Equation (4.14) is an example for the **Day convolution product** of functors on a monoidal category [Day70].

A special case for a monoidal structure is the biproduct $\oplus$ of an additive category such as $\mathcal{V}\mathrm{ec}$. In fact, it can be shown that not only the biproduct, but the entire structure of an abelian category extends to the ind-category.

Proposition 4.1.54 (Thm. 8.6.5 in [KS06])**.** *Let* $\mathcal{C}$ *be an abelian category, then* $\mathrm{Ind}\mathcal{C}$ *is an abelian category and the embedding* $\mathcal{C} \to \mathrm{Ind}\mathcal{C}$ *is exact.*

When we have a tensor product on a category, we can define many algebraic structures internal to this category. In fact, any algebraic structure that is given by an operad or a prop can be generalized to any monoidal category. For example a monoid internal to $(\mathcal{C}, \otimes, 1)$ consists of an object $A \in \mathcal{C}$, a multiplication morphism

$\mu : A \otimes A \to A$, and a unit morphism $e : 1 \to A$, such that the following diagrams commute:

$$\begin{tikzcd} A \otimes A \otimes A \arrow[r, "\mathrm{id}\otimes\mu"] \arrow[d, "\mu\otimes\mathrm{id}"'] & A \otimes A \arrow[d, "\mu"] \\ A \otimes A \arrow[r, "\mu"'] & A \end{tikzcd} \qquad \begin{tikzcd} 1 \otimes A \arrow[d, "e\otimes\mathrm{id}"'] & A \arrow[l, "l^{-1}"'] \arrow[r, "r^{-1}"] \arrow[d, "\mathrm{id}"] & A \otimes 1 \arrow[d, "\mathrm{id}\otimes e"] \\ A \otimes A \arrow[r, "\mu"'] & A & A \otimes A \arrow[l, "\mu"] \end{tikzcd}$$

Terminology 4.1.55. A monoid in $(\mathcal{S}\mathrm{et}, \times, *)$ is a monoid in the usual sense, which motivates the terminology. A monoid in $(\mathcal{V}\mathrm{ec}, \otimes, \mathbb{R})$ is an algebra. So when $\mathcal{V}\mathrm{ec}$ or, more generally, the category of modules over a ring is the basic category, a monoid internal to $(\mathcal{C}, \otimes, 1)$ is also called an algebra in $\mathcal{C}$.

Definition 4.1.56. A monoid internal to a monoidal category $(\mathcal{C}, \otimes, 1)$ will be called an **algebra in** $\mathcal{C}$. The category of algebras in $\mathcal{C}$ is denoted by $\mathcal{A}\mathrm{lg}(\mathcal{C})$. When $\mathcal{C} = \mathcal{V}\mathrm{ec}$, we abbreviate $\mathcal{A}\mathrm{lg} \equiv \mathcal{A}\mathrm{lg}(\mathcal{V}\mathrm{ec})$.

Let us spell out the structure of an algebra on an ind-object $\hat{A}$ represented by the diagram $A : \mathcal{I} \to \mathcal{V}\mathrm{ec}$. The tensor square $\hat{A} \,\hat{\otimes}\, \hat{A}$ is represented by the diagram $\mathcal{I} \times \mathcal{I} \to \mathcal{V}\mathrm{ec}$, $(i, j) \mapsto A_i \otimes A_j$. A map $\mu : \hat{A} \,\hat{\otimes}\, \hat{A} \to \hat{A}$ is represented by a family of morphisms

$$\mu_{i,j} : A_i \otimes A_j \longrightarrow A_{k(i,j)} \,. \tag{4.15}$$

This map is an associative multiplication if the families of morphisms

$$\begin{aligned} \mu_{i_1 i_2, i_3} &:= \mu_{k(i_1,i_2),i_3} \circ (\mu_{i_1,i_2} \otimes \mathrm{id}) : A_{i_1} \otimes A_{i_2} \otimes A_{i_3} \longrightarrow A_{k(k(i_1,i_2),i_3)} \\ \mu_{i_1, i_2 i_3} &:= \mu_{i_1,k(i_2,i_3)} \circ (\mathrm{id} \otimes \mu_{i_2,i_3}) : A_{i_1} \otimes A_{i_2} \otimes A_{i_3} \longrightarrow A_{k(i_1,k(i_2,i_3))} \end{aligned}$$

for all $i_1, i_2, i_3 \in \mathcal{I}$ represent the same morphism in $\mathrm{Ind}(\mathcal{V}\mathrm{ec})$. This is the case if there are commutative diagrams

$$\begin{tikzcd} & A_{i_1} \otimes A_{i_2} \otimes A_{i_3} \arrow[dl, "\mu_{i_1 i_2, i_3}"'] \arrow[dr, "\mu_{i_1, i_2 i_3}"] & \\ A_{k(k(i_1,i_2),i_3)} \arrow[dr] & & A_{k(i_1,k(i_2,i_3))} \arrow[dl] \\ & A_l & \end{tikzcd} \tag{4.16}$$

where the unmarked arrows are morphisms of the diagram $A : \mathcal{I} \to \mathcal{V}\mathrm{ec}$. Similarly, the unit of the algebra is given by a map $e : \mathbb{R} \to A_i$, such that there are commutative diagrams

$$\begin{tikzcd} \mathbb{R} \otimes A_j \arrow[d, "e\otimes\mathrm{id}"'] & A_j \arrow[l, "l^{-1}"'] \arrow[r, "r^{-1}"] \arrow[dd] & A_j \otimes \mathbb{R} \arrow[d, "\mathrm{id}\otimes e"] \\ A_i \otimes A_j \arrow[d, "\mu_{i,j}"'] & & A_j \otimes A_i \arrow[d, "\mu_{j,i}"] \\ A_{k(i,j)} \arrow[r] & A_l & A_{k(j,i)} \arrow[l] \end{tikzcd} \tag{4.17}$$

where again the unmarked arrows are some morphisms of the diagram $A : \mathcal{I} \to \mathcal{V}\mathrm{ec}$.

Example 4.1.57. Let $\bar{A}$ be a vector space with a filtration $A_0 \subset A_1 \subset A_2 \subset \ldots \subset \bar{A}$, which can be viewed as a sequence $A : \omega \to \mathcal{V}\mathrm{ec}$ of monomorphisms with colimit $\bar{A}$. An associative multiplication $\bar{\mu} : \bar{A} \otimes \bar{A} \to \bar{A}$ is **filtered** if $\mu(A_i \otimes A_j) \subset A_{i+j}$. Then the restrictions

$$\mu_{i,j} := \bar{\mu}|_{A_i \otimes A_j} : A_i \otimes A_j \longmapsto A_{k(i,j)}$$

for all $i, j \in \omega$ and $k(i,j) = i + j$ represent an associative multiplication on the ind-vector space $\hat{A}$ represented by the diagram A. Moreover, $e \in A_0$ is a unit of the multiplication on $\hat{A}$.

Proposition 4.1.58. *Let $(\mathcal{C}, \otimes, 1)$ be a monoidal category. Let $F_{\mathcal{C}} : \mathcal{A}\mathrm{lg}(\mathcal{C}) \to \mathcal{C}$ denote the natural functor that forgets the structure morphisms of an algebra object. Then there is an injective and faithful functor $I : \mathrm{Ind}\big(\mathcal{A}\mathrm{lg}(\mathcal{C})\big) \to \mathcal{A}\mathrm{lg}\big(\mathrm{Ind}(\mathcal{C})\big)$, such that the diagram*

$$\begin{tikzcd} \mathrm{Ind}\big(\mathcal{A}\mathrm{lg}(\mathcal{C})\big) \arrow[rr, "I"] \arrow[dr, "\mathrm{Ind}(F_{\mathcal{C}})"'] & & \mathcal{A}\mathrm{lg}\big(\mathrm{Ind}(\mathcal{C})\big) \arrow[dl, "F_{\mathrm{Ind}(\mathcal{C})}"] \\ & \mathrm{Ind}(\mathcal{C}) & \end{tikzcd} \tag{4.18}$$

commutes.

Proof. The diagonal functor $\mathcal{I} \to \mathcal{I} \times \mathcal{I}$, $i \to (i,i)$ is final (Exercise 4.2). This implies that the diagram $\mathcal{I} \times \mathcal{I} \to \mathcal{V}\mathrm{ec}$, $(i,j) \mapsto A_i \otimes A_j$ and the diagram $\mathcal{I} \to \mathcal{V}\mathrm{ec}$, $i \mapsto A_i \otimes A_i$ represent the same ind-vector space $\hat{A} \otimes \hat{A}$. More precisely, the family of maps $\mathrm{id} : A_i \otimes A_i \to A_i \otimes A_i$ induces an isomorphism of presheaves

$$\operatorname*{colim}_{i \in \mathcal{I}} y(A_i \otimes A_i) \xrightarrow{\cong} \operatorname*{colim}_{(i,j) \in \mathcal{I} \times \mathcal{I}} y(A_i \otimes A_j)\,. \tag{4.19}$$

For every pair $i, j \in \mathcal{I} \times \mathcal{J}$, let $m(i,j)$ be in $\mathcal{I}$ such that there are maps $i \to m(i,j)$ and $j \to m(i,j)$. Then there are morphisms $A_i \to A_{m(i,j)}$ and $A_j \to A_{m(i,j)}$ in the filtered diagram $A : \mathcal{I} \to \mathcal{V}\mathrm{ec}$. Their tensor product yields a family of morphisms

$$\Delta_{i,j} : A_i \otimes A_j \longrightarrow A_{m(i,j)} \otimes A_{m(i,j)}\,,$$

which represents the inverse of (4.19).

Let $\hat{A}_{\mathrm{alg}} \in \mathrm{Ind}(\mathcal{A}\mathrm{lg}(\mathcal{C}))$ be represented by $\mathcal{I} \to \mathcal{A}\mathrm{lg}(\mathcal{C})$, $i \mapsto (A_i, \mu_i, e_i)$. The family of morphisms $\mu_i : A_i \otimes A_i \to A_i$ defines a morphism $\mu : \hat{A} \otimes \hat{A} \to \hat{A}$ of ind-objects in $\mathcal{C}$. Composing the morphisms with $\Delta_{i,j}$ yields the family of morphisms

$$\mu_{i,j} : A_i \otimes A_j \xrightarrow{\Delta_{i,j}} A_{m(i,j)} \otimes A_{m(i,j)} \xrightarrow{\mu_{m(i,j)}} A_{m(i,j)}\,.$$

which represents μ on the diagram $(i,j) \mapsto A_i \otimes A_j$. From the associativity of μ_i and the fact that all maps in the diagram $A : \mathcal{I} \to \mathcal{C}$ are homomorphisms of algebras, it follows that there is a commutative diagram (4.16) for all $i_1, i_2, i_3 \in \mathcal{I}$. We conclude that μ is an associative multiplication on $\hat{A}$.

Since any arrow $\sigma : A_0 \to A_i$ of the diagram A is a homomorphism of unital algebras, we have $e_i = \sigma(e_0)$. This implies that the morphisms $e : y(1) \to \hat{A}$ of ind-objects in $\mathcal{C}$ that is represented by $e_0 : 1 \to A_0$ makes the diagrams (4.17) commutative, so that e is the unit of μ.

So far, we have shown that the structure morphisms μ_i, e_i of any $\hat{A}_{\mathrm{alg}} \in \mathrm{Ind}(\mathcal{A}\mathrm{lg}(\mathcal{C}))$ represent the morphisms of an algebra structure on the underlying ind-object $\hat{A} \in \mathrm{Ind}(\mathcal{C})$. A morphism $f : \hat{A}_{\mathrm{alg}} \to \hat{B}_{\mathrm{alg}}$ of ind-algebras is represented by a family $f_i : A_i \to B_i$ of morphisms of algebra objects in $\mathcal{C}$. The morphisms f_i induce a morphism $f : \hat{A} \to \hat{B}$ of the underlying ind-objects in $\mathcal{C}$. It is straight-forward to check that f is compatible with the induced algebra structures on $\hat{A}$ and $\hat{B}$, that is, f is a morphisms of algebras in $\mathrm{Ind}(\mathcal{C})$. We conclude that we have a functor $I : \mathrm{Ind}\big(\mathcal{A}\mathrm{lg}(\mathcal{C})\big) \to \mathcal{A}\mathrm{lg}\big(\mathrm{Ind}(\mathcal{C})\big)$.

By definition, $\hat{A}_{\mathrm{alg}}$ and $I(\hat{A}_{\mathrm{alg}})$ have the same underlying $\hat{A} \in \mathrm{Ind}(\mathcal{C})$, which means that the diagram (4.18) commutes. A morphism in $\mathrm{Ind}(\mathcal{A}\mathrm{lg}(\mathcal{C}))$ is given by a morphism in $\mathrm{Ind}(\mathcal{C})$ that satisfies compatibility conditions with the algebra structures. This implies that the forgetful morphism $\mathrm{Ind}(\mathcal{A}\mathrm{lg}(\mathcal{C})) \to \mathrm{Ind}(\mathcal{C})$ is faithful. Since diagram (4.18) commutes, I must be faithful as well. Finally, if the morphisms $\mu_i, \mu_i' : A_i \otimes A_i \to A_i$ and $e_i, e_i' : 1 \to A_i$ represent the same ind-algebra $\hat{A}_{\mathrm{alg}}$, then the induced morphisms $\mu, \mu' : \hat{A} \otimes \hat{A} \to \hat{A}$, $e, e' : \mathbb{Y}_{\mathcal{C}}(1) \to \hat{A}$ of ind-objects in $\mathcal{C}$ are equal. We conclude that I is injective on objects. □

Proposition 4.1.59. *Let $(\mathcal{V}, \otimes, 1)$ be a closed symmetric monoidal category that has all filtered colimits. The functor $\mathrm{Ind}(\mathcal{V}) \to \mathcal{V}$ that maps an ind-object $\hat{A}$ represented by $A : \mathcal{I} \to \mathcal{V}$ to the colimit $\bar{A} = \mathrm{colim}_{i \in \mathcal{I}} A_i$ preserves tensor products.*

Proof. Let $\hat{A}, \hat{B} \in \mathrm{Ind}(\mathcal{V})$ be represented by diagrams $A : \mathcal{I} \to \mathcal{C}$ and $B : \mathcal{J} \to \mathcal{C}$. The tensor product $\hat{A} \hat{\otimes} \hat{B}$ is represented by $\mathcal{I} \times \mathcal{J} \to \mathcal{V}$, $(i,j) \to A_i \otimes B_j$. We have

$$\begin{aligned}
\overline{\hat{A} \otimes \hat{B}} &\cong \operatorname*{colim}_{(i,j) \in \mathcal{I} \times \mathcal{J}} A_i \otimes B_j \\
&\cong \operatorname*{colim}_{i \in \mathcal{I}} \operatorname*{colim}_{j \in \mathcal{J}} (A_i \otimes B_j) \\
&\cong \operatorname*{colim}_{i \in \mathcal{I}} \big(A_i \otimes (\operatorname*{colim}_{j \in \mathcal{J}} B_j)\big) \\
&\cong (\operatorname*{colim}_{i \in \mathcal{I}} A_i) \otimes (\operatorname*{colim}_{j \in \mathcal{J}} B_j) \\
&\cong \bar{A} \otimes \bar{B} \,,
\end{aligned}$$

where we have used that, since $\mathcal{V}$ is closed monoidal so that the tensor product with a fixed object has a right adjoint, the tensor product commutes with colimits. □

Corollary 4.1.60. *The colimit functor $\bar{U} : \mathrm{Ind}(\mathcal{V}) \to \mathcal{V}$ induces a functor*

$$\mathcal{A}\mathrm{lg}(\mathrm{Ind}(\mathcal{V})) \longrightarrow \mathcal{A}\mathrm{lg}(\mathcal{V}) \,.$$

Example 4.1.61. It follows from Corollary 4.1.41 that the colimit functor $\bar{U} : \mathrm{Ind}(\mathcal{V}\mathrm{ec}) \to \mathcal{V}\mathrm{ec}$ is faithful on strict ind-objects. Corollary 4.1.60 then implies that an algebra structure on the strict ind-vector space $\hat{A}$ can be identified with an algebra structure on the colimit vector space $\bar{A}$. Note, however, that the colimit functor $\hat{A} \mapsto \bar{A}$ is neither essentially injective nor full (Remark 4.1.43). This means that non-isomorphic ind-vector spaces $\hat{A} \ncong \hat{B}$ can have isomorphic underlying vector spaces $\bar{A} \cong \bar{B}$, and that there may be algebra structures on $\bar{A}$ that do not arise from an algebra structure on $\hat{A}$.

Example 4.1.62. The category $\mathcal{V} = \mathrm{gr}\mathcal{V}\mathrm{ec}$ of $\mathbb{Z}$-graded vector spaces is closed symmetric monoidal. The tensor product of two graded vector spaces $V_\bullet$ and $W_\bullet$ is given by

$$(V \otimes W)_n = \bigoplus_{p+q=n} V_p \otimes W_q\,.$$

The tensor unit is $\mathbb{R}$ viewed as graded vector space concentrated in degree 0. The symmetric structure is $\tau(v \otimes w) = (-1)^{|v|\,|w|} w \otimes v$. The inner hom-object is the graded vector space

$$\underline{\mathrm{Hom}}_{\mathrm{gr}\mathcal{V}\mathrm{ec}}(V, W)_n = \prod_{p \in \mathbb{Z}} \underline{\mathrm{Hom}}_{\mathcal{V}\mathrm{ec}}(V_p, W_{p+n})\,.$$

By Corollary 4.1.41 the colimit functor $\mathrm{Ind}(\mathrm{gr}\mathcal{V}\mathrm{ec}) \to \mathrm{gr}\mathcal{V}\mathrm{ec}$ that maps the ind-object represented by $A : \mathcal{I} \to \mathrm{gr}\mathcal{V}\mathrm{ec}$ to $\bar{A} = \mathrm{colim}_{i \in \mathcal{I}} A_i$ is faithful on strict ind-objects. Corollary 4.1.60 then shows, that an algebra structure on a strict ind-graded vector space $\hat{A}$ can be identified with an algebra structure on the graded vector space $\bar{A}$.

Definition 4.1.63. Let $(\mathcal{C}, \otimes, 1)$ be an additive monoidal category. Let (A, μ, e) be an algebra object in $\mathcal{C}$. A **derivation** of A is a morphism $\delta : A \to A$ such that the diagram

$$\begin{CD} A \otimes A @>{\mu}>> A \\ @V{\delta\otimes\mathrm{id}+\mathrm{id}\otimes\delta}VV @VV{\delta}V \\ A \otimes A @>>{\mu}> A \end{CD} \tag{4.20}$$

commutes.

Proposition 4.1.64. *Let A be an algebra in an additive monoidal category $\mathcal{C}$. Then* $\mathrm{Der}(A)$ *is closed under the commutator of composition.*

Proof. This is shown by a direct calculation, which is analogous to the case of algebras is $\mathcal{V}\mathrm{ec}$. ☐

4.2 Sequential ind/pro-objects

Definition 4.2.1. An ind-object (pro-object) is called **sequential** if it is represented by a diagram indexed by ω (ω^{op}).

Spelling out this definition, we see that a strict sequential ind-object of $\mathcal{C}$ is represented by a sequence

$$X_0 \xrightarrow{\sigma_0} X_1 \xrightarrow{\sigma_1} X_2 \xrightarrow{\sigma_2} \dots\,,$$

such that every σ_i is a monomorphism. Dually, a strict sequential pro-object of $\mathcal{C}$ is represented by a sequence

$$X_0 \xleftarrow{\sigma_0} X_1 \xleftarrow{\sigma_1} X_2 \xleftarrow{\sigma_2} \dots\,,$$

such that every σ_i is an epimorphism. Many of the ind-objects and pro-objects we are interested in arise from such diagrams, so we will study them in more detail.

4.2.1 Representation of morphisms

There is an explicit description of the set of morphisms between sequential ind-objects.

Proposition 4.2.2. *Let $\hat{X}$ and $\hat{Y}$ be sequential ind-objects in $\mathcal{C}$ represented by the sequences $X_0 \overset{\sigma_0}{\to} X_1 \overset{\sigma_1}{\to} \dots$ and $Y_0 \overset{\tau_0}{\to} Y_1 \overset{\tau_1}{\to} \dots$. A morphism in $\mathrm{Ind}\mathcal{C}(\hat{X}, \hat{Y})$ is represented by a diagram*

$$\begin{tikzcd} X_0 \arrow[r] \arrow[d, "f_0"] & X_1 \arrow[r] \arrow[d, "f_1"] & X_2 \arrow[r] \arrow[d, "f_2"] & \dots \\ Y_{j(0)} \arrow[r] & Y_{j(1)} \arrow[r] & Y_{j(2)} \arrow[r] & \dots \end{tikzcd} \tag{4.21}$$

where $j(i) \leq j(i+1)$ for all $i \in \omega$.

Moreover, if all target indices $j(i)$ are chosen to be minimal in the sense that no f_i factors like

$$\begin{tikzcd} & X_i \arrow[dl, "f_i'"'] \arrow[d, "f_i"] \\ Y_{j(i)-1} \arrow[r, "\tau_{j(i)-1}"'] & Y_{j(i)} \end{tikzcd}$$

and if $\hat{Y}$ is strict, then every f_i is unique.

Proof. In the first step we calculate the inner colimit of Equation (4.3). The set $\operatorname{colim}_j \mathcal{C}(X_i, Y_j)$ is the quotient of the disjoint union of all $\mathcal{C}(X_i, Y_j)$, $j \geq 0$ modulo the equivalence relation that is generated by $f \sim \tau_j \circ f$ for all $f \in \mathcal{C}(X_i, Y_j)$, $j \geq 0$,

$$\operatorname*{colim}_j \mathcal{C}(X_i, Y_j) = \coprod_j \mathcal{C}(X_i, Y_j)/\sim \ . \tag{4.22}$$

Since the index category ω is ordered and bounded from below every element of the quotient has a representative $f_i : X_i \to Y_{j(i)}$ for which $j(i)$ is minimal. From the minimality it follows that $j(i) \leq j(i+1)$.

In the second step we construct the limit of Equation (4.3). The diagram of which we have to compute the limit is

$$C_0 \overset{\sigma_0^*}{\longleftarrow} C_1 \overset{\sigma_1^*}{\longleftarrow} C_2 \overset{\sigma_2^*}{\longleftarrow} \dots ,$$

where $C_i := \operatorname{colim}_j \mathcal{C}(X_i, Y_j)$ and

$$\begin{aligned} \sigma_i^* : \operatorname*{colim}_j \mathcal{C}(X_{i+1}, Y_j) &\longrightarrow \operatorname*{colim}_j \mathcal{C}(X_i, Y_j) \\ [f_{i+1}] &\longmapsto [f_{i+1} \circ \sigma_i] \,. \end{aligned}$$

Every equivalence class in C_i has a representative $f_i : X_i \to Y_{j(i)}$ for which $j(i)$ is minimal. An element in the limit is given by a sequence

$$([f_0], [f_1], [f_2], \dots) \in \prod_i \operatorname*{colim}_j \mathcal{C}(X_i, Y_j)$$

with the property that $\sigma_i^*[f_{i+1}] = [f_i]$ for all i. This means that for every $f_i : X_i \to Y_{j(i)}$ and $f_{i+1} : X_{i+1} \to Y_{j(i+1)}$ we have a commutative square

$$\begin{tikzcd} X_i \arrow[r, "\sigma_i"] \arrow[d, "f_i"'] & X_{i+1} \arrow[d, "f_{i+1}"] \\ Y_{j(i)} \arrow[r, "\tau"'] & Y_{j(i+1)} \end{tikzcd}$$

where

$$\tau : Y_{j(i)} \xrightarrow{\tau_{j(i)}} Y_{j(i)+1} \longrightarrow \dots \xrightarrow{\tau_{j(i+1)-1}} Y_{j(i+1)} .$$

The commutativity of the infinite diagram of the proposition is equivalent to the commutativity of these squares for all i.

We have already seen that the target indices $j(i)$ can be chosen to be minimal. Assume now that $\hat{Y}$ is strict, i.e. all morphisms τ_j are monomorphisms. This implies that if two morphisms $f, f' : X_i \to X_j$ with the same domain and target represent the same equivalence class $[f : X_i \to Y_j] = [f' : X_i \to Y_j]$ in the colimit (4.22), then they are equal $f = f'$. In particular, the morphism $f_i : X_i \to Y_{j(i)}$ that represents $[f_i]$ is unique. □

The composition of an ind-morphism $\hat{X} \to \hat{Y}$ as in Proposition 4.2.2 with another ind-morphism $\hat{g} : \hat{Y} \to \hat{Z}$ of sequential ind-objects represented by a family $g : Y_j \to Y_{k(j)}$ is represented by the family of morphisms obtained by stacking the two diagrams of type (4.21).

$$\begin{tikzcd} X_0 \arrow[r] \arrow[d, "f_0"] & X_1 \arrow[r] \arrow[d, "f_1"] & X_2 \arrow[r] \arrow[d, "f_2"] & \dots \\ Y_{j(0)} \arrow[r] \arrow[d, "g_{j(0)}"] & X_{j(1)} \arrow[r] \arrow[d, "g_{j(1)}"] & X_{j(2)} \arrow[r] \arrow[d, "g_{j(2)}"] & \dots \\ Z_{k(j(0))} \arrow[r] & Z_{k(j(1))} \arrow[r] & Z_{k(j(2))} \arrow[r] & \dots \end{tikzcd} \tag{4.23}$$

Note that, even if $i \mapsto j(i)$ and $j \mapsto k(j)$ are minimal in the sense of Proposition 4.2.2, the numbers $i \mapsto k(j(i))$ may not.

Corollary 4.2.3. *Let $\hat{X}$ be a sequential ind-object of $\mathcal{C}$ represented by the sequence $X_0 \overset{\sigma_0}{\to} X_1 \overset{\sigma_1}{\to} \dots$ and let C be an object in $\mathcal{C}$.*

(i) A morphism in $\operatorname{Ind}\mathcal{C}(\hat{X}, yC)$ is represented by a unique family of morphisms $\{f_i : X_i \to C\}_{i\in\omega}$ satisfying $f_{i+1} \circ \sigma_i = f_i$.

(ii) A morphism in $\operatorname{Ind}\mathcal{C}(yC, \hat{X})$ is represented by a morphism $f : C \to X_i$. Moreover, if i is minimal and $\hat{X}$ is strict, then f is unique.

Warning 4.2.4. The Yoneda embedding commutes with limits but does not commute with colimits, not even with filtered colimits. This means that even if a diagram $X = (X_0 \to X_1 \to \dots)$ does have a colimit $\operatorname{colim}_i X_i$ in $\mathcal{C}$ it is generally not true that $\operatorname{colim}_i X_i$ viewed as constant ind-object is isomorphic to the ind-object represented by X. The next example illustrates this phenomenon.

Example 4.2.5 (Exhaustion of the real line)**.** Let $\mathcal{C} = \mathcal{M}\text{fld}$ be the category of smooth finite-dimensional manifolds. Consider the sequence of embeddings of open intervals,

$$X := \big((-1,1) \hookrightarrow (-2,2) \hookrightarrow (-3,3) \hookrightarrow \ldots\big)\,.$$

On the one hand, a morphism of ind-manifolds from the constant ind-object $\mathbb{R}$ to the ind-manifold $\hat{X}$ represented by this sequence is, according to Proposition 4.2.2, given by a smooth map from $\mathbb{R}$ to one of the intervals $(-n,n)$, in other words, by a bounded function on the real line. On the other hand, the colimit of X is given by the real line $\mathbb{R}$, so that a morphism from $\mathbb{R}$ to the colimit of X is, therefore, a smooth, not necessarily bounded function.

Corollary 4.2.6. *Let $\check{X}$ and $\check{Y}$ be sequential pro-objects in $\mathcal{C}$ represented by the sequences $X_0 \overset{\sigma_0}{\leftarrow} X_1 \overset{\sigma_1}{\leftarrow} \ldots$ and $Y_0 \overset{\tau_0}{\leftarrow} Y_1 \overset{\tau_1}{\leftarrow} \ldots$. A morphism in* $\mathrm{Pro}\mathcal{C}(\check{X},\check{Y})$ *is given by a diagram*

$$\begin{tikzcd}
X_{i(0)} \arrow[d, "f_0"] & X_{i(1)} \arrow[l] \arrow[d, "f_1"] & X_{i(2)} \arrow[l] \arrow[d, "f_2"] & \ldots \arrow[l] \\
Y_0 & Y_1 \arrow[l] & Y_2 \arrow[l] & \ldots \arrow[l]
\end{tikzcd}$$

where all $i(j) \le i(j+1)$ for all $j \in \omega$.

Moreover, if all source indices $i(j)$ are chosen to be minimal and if $\check{X}$ is strict, then every f_i is unique.

Proof. The corollary is obtained from Proposition 4.2.2 by using the isomorphism of Proposition 4.1.25. □

Corollary 4.2.7. *Let $\check{X}$ be as in Corollary 4.2.6 and let C be an object in $\mathcal{C}$.*

(i) A morphism in $\mathrm{Pro}\mathcal{C}(yC,\check{X})$ *is uniquely given by a family of morphisms $\{f_i : C \to X_i\}_{i\in\omega}$ satisfying $\sigma_i \circ f_{i+1} = f_i$.*

(ii) A morphism in $\mathrm{Pro}\mathcal{C}(\check{X},yC)$ *is represented by a morphism $f : X_i \to C$. Moreover, if i is minimal and $\check{X}$ is strict, then f is unique.*

4.2.2 Sections, retracts, isomorphisms, derivations

Choosing the target indices $j(i)$ to be minimal makes the family of morphisms representing an ind-morphism unique, the minimal choice may be difficult or not natural. For example, the identity morphism of a sequential ind-object $\hat{X}$, is naturally represented by the family $\mathrm{id} : X_i \to X_i$, even though $j(i) = i$ is not the minimal choice when $\sigma_{i-1} : X_{i-1} \to X_i$ is an isomorphism. The price we have to pay is that different families of morphisms may represent the same ind-morphism. The next proposition gives a criterion to decide when this is the case.

Proposition 4.2.8. *Let $\hat{X}$ and $\hat{Y}$ be sequential ind-objects as in Proposition 4.2.2. Two families of morphisms $f_i : X_i \to Y_{j(i)}$ and $f'_i : X_i \to Y_{j'(i)}$, with $j(i)$ and $j'(i)$*

not necessarily minimal, represent the same morphism of ind-objects if and only if for every $i \in \omega$ one of the following two diagrams commutes,

$$\begin{tikzcd} X_i \arrow[d, "f_i"'] \arrow[dr, "f'_i"] & \\ Y_{j(i)} \arrow[r] & Y_{j'(i)} \end{tikzcd} \qquad \text{or} \qquad \begin{tikzcd} X_i \arrow[d, "f'_i"'] \arrow[dr, "f_i"] & \\ Y_{j'(i)} \arrow[r] & Y_{j(i)} \end{tikzcd}$$

depending on whether $j(i) \leq j'(i)$ or $j(i) \geq j'(i)$.

Proof. *** □

Corollary 4.2.9. *Let $\hat{X}$ be a strict sequential ind-object of $\mathcal{C}$ represented by the sequence $X_0 \xrightarrow{\sigma_0} X_1 \xrightarrow{\sigma_1} \ldots$. A family of morphisms $f_i : X_i \to X_{j(i)}$ represents the identity morphism of $\hat{X}$ if and only if for every $i \in \omega$ one of the following two conditions is satisfied.*

(i) If $i \leq j(i)$, then f_i is equal to $X_i \xrightarrow{\sigma} X_{j(i)}$.

(ii) If $i > j(i)$, then $X_{j(i)} \xrightarrow{\sigma} X_i$ is an isomorphism and f_i its inverse.

Proof. We apply Proposition 4.2.8 to the case $\hat{Y} = \hat{X}$ and $f'_i := \mathrm{id}_{X_i}$. When $i \leq j(i)$, the second diagram of Proposition 4.2.8 must commute, which is equivalent to condition (i).

When $i > j(i)$, the first diagram of diagram of Proposition 4.2.8 must commute, that is, $\sigma \circ f_i = \mathrm{id}_{X_i}$. Composing on the right with σ yields $\sigma \circ f_i \circ \sigma = \sigma$. By the assumption of strictness of $\hat{X}$, the morphism $\sigma : X_{j(i)} \to X_i$ is a monomorphism, so it follows that $f_i \circ \sigma = \mathrm{id}_{X_{j(i)}}$, i.e. f_i is the left and right inverse of σ, which is condition (ii). □

Corollary 4.2.10. *Let $\hat{X}$ be a strict sequential ind-object of $\mathcal{C}$ represented by the sequence $X_0 \xrightarrow{\sigma_0} X_1 \xrightarrow{\sigma_1} \ldots$ in which none of the arrows is an isomorphism. Then the family of morphisms $\mathrm{id}_{X_i} : X_i \to X_i$ is the unique representative of the identity morphism with minimal target indices.*

With Corollary 4.2.9 and the composition of ind-morphisms in terms of the representing families by diagram (4.23), we can easily determine the conditions for families of morphisms to represent sections, retractions, or isomorphisms in the ind-category. Spelling these conditions out would be highly redundant, though.

Example 4.2.11. Let $\hat{X}$ be the strict sequential ind-object of $\mathcal{C}$ represented by the diagram $X : \omega \to \mathcal{C}$. In Example 4.1.12 we have seen that every unbounded order preserving map $\Phi : \omega \to \omega$ is final, which implies that the ind-object $\hat{X}'$ represented by $X \circ \Phi$ is isomorphic to $\hat{X}$. The isomorphism $f : \hat{X}' \to \hat{X}$ is represented by the family of morphisms $X'_i = X_{\Phi(i)} \xrightarrow{\mathrm{id}} X_{\Phi(i)}$.

As before, we can use the isomorphism of ind- and pro-categories of Proposition 4.1.25 to obtain the dual propositions for pro-objects. We give just one example, because we will need it later for the description of vector fields on pro-manifolds as sections on the pro-tangent bundle.

Proposition 4.2.12. *Let $\check{X}$ and $\check{Y}$ be sequential pro-objects in $\mathcal{C}$ represented by $X_0 \xleftarrow{\sigma_0} X_1 \xleftarrow{\sigma_1} \dots$ and $Y_0 \xleftarrow{\tau_0} X_1 \xleftarrow{\tau_1} \dots$. Let $\check{f} : \check{X} \to \check{Y}$ be a morphism which is represented by the family $(f_i : X_i \to Y_i)_{i\in\omega}$.*

A morphism $\check{g} : \check{Y} \to \check{X}$ represented by a family $(g_i : Y_{j(i)} \to X_i)_{i\in\omega}$ is a section of $\check{f}$ if and only if for every $i \in \omega$ one of the following two conditions is satisfied.

(i) If $i \leq j(i)$, then $f_i \circ g_i$ is equal to $Y_{j(i)} \xrightarrow{\tau} Y_i$.

(ii) If $i > j(i)$, then $Y_i \xrightarrow{\tau} Y_{j(i)}$ is an isomorphism and $f_i \circ g_i$ its inverse.

Remark 4.2.13. When in the sequence $X_0 \xleftarrow{\tau_0} X_1 \xleftarrow{\tau_1} \dots$ a morphism τ_i is an isomorphism, we can skip X_{i+1} and replace τ_i with $\tau_i \circ \tau_{i+1} : X_{i+2} \to X_i$ without changing the pro-object. Unless the sequence is stably constant, i.e. τ_i is an isomorphism for all $i \gg 0$, we obtain by reiterating this procedure a **reduced sequence** for which none of the connecting isomorphisms τ_i is an isomorphisms. If we assume further that the sequence is strict, i.e. all τ_i are epimorphisms, it follows that no composition of connecting morphisms is an isomorphism. In that case, condition (ii) of Proposition 4.2.12 cannot occur.

Example 4.2.14. Let $X_0 \xleftarrow{\sigma_0} X_1 \xleftarrow{\sigma_1} \dots$ be a sequence representing the pro-object $\check{X}$. By condition (i) of Proposition 4.2.12, the morphism $\check{\sigma} : \check{X} \to \check{X}$ represented by the family $\sigma_k : X_{k+1} \to X_k$ is a section of the identity morphism, which is represented by $\mathrm{id}_{X_k} : X_k \to X_k$. We conclude that $\check{\sigma}$ represents the identity morphism of $\check{X}$.

Proposition 4.2.15. *Let $A_0 \xrightarrow{\sigma_0} A_1 \xrightarrow{\sigma_1} \dots$ be a sequence of algebras. Then a derivation of the algebra in ind-vector spaces we obtain from Proposition 4.1.58 is represented by a family of linear maps $\delta_i : A_i \to A_{j(i)}$, $i \in \omega$, such that for all i and all $a, b \in A_i$,*

$$\delta_i(ab) = (\delta_i a)\,\sigma(b) + \sigma(a)\,(\delta_i b)\,,$$

where $\sigma : A_i \to A_{j(i)}$ is the linear map of the diagram A.

Proof. By Proposition 4.2.2 a morphism $\delta : \hat{A} \to \hat{A}$ is represented by a family of morphisms $\delta_i : A_i \mapsto A_{j(i)}$. Let $a, b \in A_i$ and let $\sigma : A_i \to A_{j(i)}$ denote the map of the diagram $A : \omega \to \mathcal{V}\mathrm{ec}$. If the diagram (4.20) commutes, then

$$\begin{aligned} \delta_i(ab) &= (\delta_i \circ \mu_i)(a \otimes b) \\ &= \big(\mu \circ (\delta_i \otimes \mathrm{id} + \mathrm{id} \otimes \delta_i \circ \mu)\big)(a \otimes b) \\ &= (\delta_i a)\,\sigma(b) + \sigma(a)\,(\delta_i b)\,. \end{aligned}$$

Let $a \in A_i$ and $b \in A_j$ be elements that live in different levels of the ind-algebra. The product of a and b in the algebra $\hat{A}$ is given by first mapping them to a higher level A_k, $k \geq i, j$ by the maps $A_i \to A_k$ and $A_j \to A_k$ in the diagram $A : \omega \to \mathcal{V}\mathrm{ec}$ and multiplying them there. □

4.3 Differential geometry on pro-manifolds

A **pro-manifold** is a pro-object of the category $\mathcal{M}\mathrm{fld}$ of smooth finite-dimensional manifolds. In our Wish list 3.4.2, we have given conditions for a category to be a good setting for the differential geometry of infinite jets. Our wishes have been granted.

Proposition 4.3.1. *The category* ProMfld *satisfies the conditions of the Wish list 3.4.2.*

Proof. (i) In Section 4.1.2 we have seen that the embedding $y : \mathcal{M}\mathrm{fld} \to \mathrm{Pro}\mathcal{M}\mathrm{fld}$ is injective, full, and faithful. (ii) An infinite inverse sequence $X_0 \leftarrow X_1 \leftarrow \ldots$ of manifolds is a diagram $X : \omega^{\mathrm{op}} \to \mathcal{M}\mathrm{fld}$ indexed by the cofiltered category ω^{op}. The limit of yX is the copresheaf pro-represented by X. (iii) The functor of points is a concrete structure on $\mathrm{Pro}(\mathcal{M}\mathrm{fld})$ that satisfies Equation (4.13). (iv) was shown in Corollary 4.2.7. $\square$

Proposition 4.3.2. *Let $\check{X}$ be a strict sequential pro-manifold represented by $X_0 \xleftarrow{\sigma_0} X_1 \xleftarrow{\sigma_1} \ldots$. Then every point $x : * \to \check{X}$ is given by a unique sequence $x_0, x_1, x_2, \ldots$ of points $x_i \in X_i$ such that $x_i = \sigma_i(x_{i+1})$ for all $i \geq 0$.*

Proof. The proposition is a special case of Corollary 4.2.6. $\square$

4.3.1 Tangent bundle and vector fields

Proposition 4.1.31 and Corollary 4.1.32 state that covariant and contravariant functors extend to functors between the ind/pro-categories. Therefore, all functorial constructions on smooth manifolds generalize to pro-manifolds in a straight-forward way. The same holds for natural transformations. Since pro-manifolds arise as cofiltered diagrams of manifolds that fail to have a limit in $\mathcal{M}\mathrm{fld}$, we will describe the generalized geometric structures in terms of these diagrams.

First, we consider the tangent functor T, which we view as endofunctor of $\mathcal{M}\mathrm{fld}$. According to Corollary 4.1.32, T induces a functor

$$\mathrm{Pro}(T) : \mathrm{Pro}(\mathcal{M}\mathrm{fld}) \longrightarrow \mathrm{Pro}(\mathcal{M}\mathrm{fld})\,.$$

If $\check{X}$ is a pro-manifold represented by $X : \mathcal{I} \to \mathcal{M}\mathrm{fld}$ then $\mathrm{Pro}(T)\check{X}$ is represented by the diagram $TX : \mathcal{I} \to \mathcal{M}\mathrm{fld}$. The tangent bundle projection of manifolds is a natural transformation $\pi_M : TM \to M$. On the diagram $X : \mathcal{I} \to \mathcal{M}\mathrm{fld}$, the extension $\mathrm{Pro}(\pi)$ is represented by the smooth maps $\pi_{X_i} : TX_i \to X_i$. For example, the tangent bundle projection of a sequential pro-manifold is represented by the diagram

$$\begin{tikzcd}
TX_0 \arrow[d, "\pi_{X_0}"] & TX_1 \arrow[l, "T\sigma_0"'] \arrow[d, "\pi_{X_1}"] & TX_1 \arrow[l, "T\sigma_1"'] \arrow[d, "\pi_{X_2}"] & \ldots \arrow[l] \\
X_0 & X_1 \arrow[l, "\sigma_0"] & X_2 \arrow[l, "\sigma_1"] & \ldots \arrow[l]
\end{tikzcd}$$

The zero section, the addition of tangent vectors, the scalar $\mathbb{R}$-multiplication of tangent vectors, are all represented in an analogous way, by applying the natural transformations of manifolds level-wise to every object of the diagrams.

As we have seen in Proposition 4.1.33, Pro is a functor, that is, it preserves the composition of functors. For the square of the tangent functor we obtain

$$\mathrm{Pro}(T^2) \cong \mathrm{Pro}(T)\ \mathrm{Pro}(T)\,.$$

Moreover, Proposition 4.1.40 implies that the pullback $\mathrm{Pro}(T)\check{X} \times_{\check{X}} \mathrm{Pro}(T)\check{X}$ is represented by the diagram $i \mapsto TX_i \times_{X_i} TX_i$. This justifies the following notation.

Notation 4.3.3. We will use the same notation for the functors and natural transformations on $\mathcal{M}\mathrm{fld}$ as for their extensions to $\mathrm{Pro}\mathcal{M}\mathrm{fld}$, that is, $T \equiv \mathrm{Pro}(T)$, $\pi \equiv \mathrm{Pro}(\pi)$, etc. It will be clear from the context if they are applied to a pro-manifold or a manifold.

The diagram $i \mapsto TX_i$ is equipped with a level-wise vector bundle structure, so that it can be understood as pro-vector bundle. Alternatively, we can view $T\check{X} \to \check{X}$ as bundle of vector spaces in the category $\mathrm{Pro}\mathcal{M}\mathrm{fld}$.

A single **tangent vector** of $\check{X}$ is a point $v : * \to T\check{X}$. Every tangent vector v projects to its base point $\pi_{\check{X}}(v) := \pi_{\check{X}} \circ v : * \to \check{X}$. The **tangent fiber** $T_x\check{X}$ at a point $x : * \to \check{X}$ is defined as the pull-back

$$\begin{tikzcd} T_x\check{X} \arrow[r] \arrow[d] & T\check{X} \arrow[d, "\pi_{\check{X}}"] \\ * \arrow[r, "x"'] & \check{X} \end{tikzcd}$$

For a sequential pro-manifold, a point x is represented by a sequence of points $(x_0, x_1, x_2, \ldots)$ such that $x_i = \sigma_i(x_{i+1})$. A tangent vector at x is represented by a sequence of tangent vectors $(v_0, v_1, v_2, \ldots)$, $v_i \in T_{x_i}X_i$ such that $v_i = T\sigma_i(v_{i+1})$. The tangent fiber $T_x\check{X}$ is the pro-vector space represented by the diagram

$$T_{x_0}X_0 \xleftarrow{T_{x_0}\sigma_0} T_{x_1}X_1 \xleftarrow{T_{x_1}\sigma_1} T_{x_2}X_2 \xleftarrow{T_{x_2}\sigma_2} \ldots$$

Let $\check{Y}$ be a pro-manifold represented by $Y : \mathcal{J} \to \mathcal{M}\mathrm{fld}$ and $f : \check{X} \to \check{Y}$ a morphism of pro-manifolds represented by the family $f_j : X_{k(j)} \to Y_j$. Then the **tangent morphism** $Tf : T\check{X} \to T\check{Y}$ is the morphism of pro-manifolds (or pro-vector bundles) represented by the family $Tf_j : TX_{k(j)} \to TY_j$. It maps a tangent vector $v : * \to T\check{X}$ to the tangent vector $Tf\,v := Tf \circ v : * \to T\check{Y}$.

A **vector field** on $\check{X}$ is a section of $\pi_{\check{X}} : T\check{X} \to \check{X}$. The **value** of a vector field $v : \check{X} \to T\check{X}$ at the point $x : * \to \check{X}$ is the tangent vector $v_x := v \circ x : * \to T\check{X}$. The following proposition describes vector fields on a sequential pro-manifold in terms of the representing sequences.

Proposition 4.3.4. *A vector field v on the sequential pro-manifold represented by $X_0 \overset{\sigma_0}{\leftarrow} X_1 \overset{\sigma_1}{\leftarrow} \ldots$ is represented by a family of smooth maps $(v_i : X_{k(i)} \to TX_i)_{i\in\omega}$ such that the diagram*

$$\begin{tikzcd} TX_0 & TX_1 \arrow[l, "T\sigma_0"'] & TX_2 \arrow[l, "T\sigma_1"'] & \ldots \arrow[l] \\ X_{k(0)} \arrow[u, "v_0"] & X_{k(1)} \arrow[l, "\sigma"] \arrow[u, "v_1"] & X_{k(2)} \arrow[l, "\sigma"] \arrow[u, "v_2"] & \ldots \arrow[l] \end{tikzcd}$$

commutes and for all $i \geq 0$ we have:

(i) If $i \leq k(i)$, then $\pi_{X_i} \circ v_i$ is equal to $X_{k(i)} \overset{\sigma}{\to} X_i$.

(ii) If $i > k(i)$, then $\sigma : X_i \overset{\sigma}{\to} X_{k(i)}$ is an isomorphism and $\pi_{X_i} \circ v_i$ its inverse.

Proof. The proposition follows from Corollary 4.2.6 and Proposition 4.2.12. □

All functors on vector bundles, such as the functors mapping a vector bundle E to the sum $E \oplus E$, the tensor square $E \otimes E$, exterior powers $\wedge^k E$, etc. extend by Corollary 4.1.32 to pro-vector bundles. Composing them with the tangent functor extends these constructions to the tangent bundle of pro-manifolds. For example, $\wedge^k T\check{X}$ is the pro-vector bundle represented by the sequence

$$\wedge^k TX_0 \xleftarrow{\wedge^k T\sigma_0} \wedge^k TX_1 \xleftarrow{\wedge^k T\sigma_1} \wedge^k TX_2 \longleftarrow \ldots$$

A section of $\wedge^k T\check{X}$ is a k**-vector field** on the pro-manifold $\check{X}$.

Remark 4.3.5. Constructions that are not functorial, do generally not extend to pro-vector objects by applying them to every object of a representing diagram. For example, mapping a vector bundle to its dual or to its space of sections is not functorial.

A vector field v on a manifold M can be identified with its action on smooth functions, which is a derivation of the $\mathbb{R}$-algebra of smooth functions $C^\infty(M)$, i.e. a linear map

$$\begin{aligned} C^\infty(M) &\longrightarrow C^\infty(M) \\ f &\longmapsto v \cdot f \,, \end{aligned}$$

that satisfies the Leibniz rule

$$v \cdot (fg) = (v \cdot f)\, g + f\, (v \cdot g)\,.$$

The algebraic description of vector fields is typically the best for working with algebraic structures in differential geometry. For example, it is straight-forward to check that the commutator of two derivations is a derivation, which shows that the space of vector fields is equipped with a Lie bracket. Therefore, we would like to generalize this point of view to the pro-manifold setting.

Mapping a smooth manifold to its algebra of smooth functions is a functor $C^\infty : \mathcal{M}\mathrm{fld} \to \mathcal{A}\mathrm{lg}^{\mathrm{op}}$. By Corollary 4.1.32, we obtain a functor

$$C^\infty \equiv \mathrm{Pro}(C^\infty) : \mathrm{Pro}(\mathcal{M}\mathrm{fld}) \longrightarrow \mathrm{Ind}(\mathcal{A}\mathrm{lg})^{\mathrm{op}}\,,$$

which maps the pro-manifold $\check{X}$ represented by $X : \mathcal{I} \to \mathcal{M}\mathrm{fld}$ to the ind-algebra $C^\infty(\check{X})$ represented by $(C^\infty X)^{\mathrm{op}} : \mathcal{I}^{\mathrm{op}} \to \mathcal{A}\mathrm{lg}$. Since mapping an algebra to its Lie algebra of derivations is *not* functorial, we cannot obtain the derivations of $C^\infty(\check{X})$ in the same way. Instead, we can use Proposition 4.1.58, which shows that an ind-algebra can be viewed as an algebra of ind-objects, that is, a monoid internal to ind-vector spaces. As is the case for any algebra in an additive monoidal category, its derivations (Definition 4.1.63) form a Lie algebra (Proposition 4.1.64).

4.3.2 Vector fields as derivations

Proposition 4.3.6. *Let $\check{X}$ be the pro-manifold represented by the cofiltered diagram $X : \mathcal{I} \to \mathcal{M}\mathrm{fld}$. Then there is a natural bijection between sections of the tangent bundle $T\check{X} \to \check{X}$ in pro-manifolds and the derivations of $C^\infty(\check{X})$ viewed as algebra in the category of ind-vector spaces.*

For ordinary manifolds, the map from vector fields to derivations is obvious, mapping the tangent vector at every point to its directional derivative. The difficult part is to show that this map has an inverse, for which Hadamard's lemma is used. For pro-manifolds the situation is similar. The map from vector fields to derivations is straight-forward, while for the inverse map we need the following lemma.

Lemma 4.3.7. *Let $\tau : Y \to X$ be a smooth map of manifolds. Let $\delta : C^\infty(X) \to C^\infty(Y)$ be a linear map such that $\delta(fg) = (\delta f)\,(\tau^* g) + (\tau^* f)\,(\delta g)$ for all $f, g \in C^\infty(X)$. Then there is a unique map $v : Y \to TX$ making the diagram*

$$\begin{tikzcd} Y \arrow[r, "v"] \arrow[dr, "\tau"'] & TX \arrow[d, "\pi_X"] \\ & X \end{tikzcd}$$

commutative, such that $(\delta f)(y) = v_y \cdot f$ for all $f \in C^\infty(X)$ and $y \in Y$.

Proof. Let $f \in C^\infty(X)$ and $y \in Y$. Let $(x^1, \dots, x^n)$ be local coordinates centered at $\bigl(\tau(y)\bigr)^i = 0$. By Hadamard's lemma $f(x) = f(0) + h_i(x)x^i$, for some functions $h_i \in C^\infty(X)$. At $x = 0$ we have $h_i(0) = \frac{\partial f}{\partial x^i}(0)$. We thus obtain

$$\begin{aligned} (\delta f)(y) &= \bigl\{(\delta h_i)(\tau^* x^i) + (\tau^* h_i)(\delta x^i)\bigr\}_y = (\delta x^i)(y)\,\frac{\partial f}{\partial x^i}(0) \\ &= v_y \cdot f\,, \end{aligned}$$

where $v_y = (\delta x^i)(y)\frac{\partial}{\partial x^i}$. □

Proof of Prop. 4.3.6. We give the proof for a sequential pro-manifold $\check{X}$ represented by the diagram $X_0 \overset{\tau_0}{\leftarrow} X_1 \overset{\tau_1}{\leftarrow} \dots$. Furthermore, we will assume for simplicity that the sequence is strict and reduced, every morphisms τ_i is an epimorphism but not an isomorphism. This is the case we will need later. The proof for a general pro-manifold is analogous.

Let $v : \check{X} \to T\check{X}$ be a vector field on $\check{X}$. By Proposition 4.3.4, v is represented by a family of smooth maps $v_i : X_{k(i)} \to X_i$, $i \in \omega$ such that

$$\begin{tikzcd} X_{k(i)} \arrow[r, "v_i"] \arrow[dr, "\tau_{i\leftarrow k(i)}"'] & TX_i \arrow[d, "\pi_{X_i}"] \\ & X_i \end{tikzcd} \tag{4.24}$$

commutes. This defines a map

$$\begin{aligned} \delta_i : C^\infty(X_i) &\longrightarrow C^\infty(X_{k(i)}) \\ f &\longmapsto (y \mapsto v_y \cdot f)\,. \end{aligned}$$

for every $i \in \omega$, where $\tau_{i\leftarrow k(i)}$ denotes the morphism we get by applying the functor X to $i \leftarrow k(i)$. Since by Proposition 4.3.4 the maps v_i satisfy $T\tau_i \circ v_i = v_{i-1} \circ \tau_{k(i-1)\leftarrow k(i)}$, the maps δ_i satisfy $\delta_i \circ \tau_i^* = \tau^*_{k(i-1)\leftarrow k(i)} \circ \delta_{i-1}$. This shows that the family δ_i represents an endomorphism of the ind-vector space $C^\infty(\check{X})$, which is represented by the diagram

$$C^\infty(X_0) \xrightarrow{\tau_0^*} C^\infty(X_1) \xrightarrow{\tau_1^*} C^\infty(X_2) \xrightarrow{\tau_2^*} \ldots$$

The Leibniz rule for the directional derivative states that

$$v_y \cdot fg = (v_y \cdot f)\, g(\tau(y)) + f(\tau(y))\, (v_y \cdot g)\,,$$

where $\tau = \tau_{i\leftarrow k(i)}$. This shows that $(\delta_i)_{i\in\omega}$ represents a derivation of $C^\infty(\check{X})$.

Conversely, let δ be a derivation of $C^\infty(\check{X})$ represented by maps $\delta_i : C^\infty(X_i) \to C^\infty(X_{k(i)})$. Then lemma 4.3.7 tells us that every δ_i is the directional derivative given by a unique smooth map $v_i : X_{k(i)} \to TX_i$. Since the family δ_i represents a morphism of ind-vector spaces, the family v_i represents a morphism $v : \check{X} \to T\check{X}$ of pro-manifolds. Moreover, since diagram (4.24) commutes, Proposition 4.3.4 implies that v is a section of the bundle projection $T\check{X} \to \check{X}$. □

Corollary 4.3.8. *The set of vector fields on a pro-manifold is a Lie algebra object in* $\mathrm{Ind}(\mathcal{V}\mathrm{ec})$.

Proof. This follows from Proposition 4.3.6 and Proposition 4.1.64. □

To get a better intuition for vector fields on pro-manifolds we will spell out in local coordinates the structures we have on the pro-manifold represented by the diagram

$$\mathbb{R}^0 \longleftarrow \mathbb{R}^1 \longleftarrow \mathbb{R}^2 \longleftarrow \ldots$$

where $\mathbb{R}^{i+1} \to \mathbb{R}^i$ is the projection to the first i-factors (cf. example 4.1.8). Let us denote this pro-manifold by $\check{\mathbb{R}}^\infty$. In local coordinates every submersion is a composition of such projections, so that $\check{\mathbb{R}}^\infty$ is the local model for a large class of pro-manifolds [GP17].

Let $(x^1, \ldots, x^i)$ be the canonical coordinates of $\mathbb{R}^i$. Then a point $p : * \to \check{\mathbb{R}}^\infty$ can be identified with the infinite sequence $(x^1(p), x^2(p), \ldots)$. In fact, by Equation (4.13), the underlying set is

$$|\check{\mathbb{R}}^\infty| = \prod_{i=1}^\infty |\mathbb{R}|\,.$$

A function $f : * \to C^\infty(\check{\mathbb{R}}^\infty)$ is a smooth function $f \in C^\infty(\mathbb{R}^i)$ for some i, that is, a function $f = f(x^1, \ldots, x^i)$ that depends smoothly on a finite number of coordinates. A tangent vector is an element of the set

$$|T\check{\mathbb{R}}^\infty| = \prod_{i=1}^\infty |T\mathbb{R}|\,.$$

Let $(\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^i})$ be the coordinate vector fields on $\mathbb{R}^i$. Then a tangent vector $v_p : * \mapsto T\check{\mathbb{R}}^\infty$ at the point $p = (p^1, p^2, \ldots)$ is given by an infinite sequence

$$\prod_{i=1}^\infty T\mathbb{R} \ni \Big(v_p^1 \frac{\partial}{\partial x^1}\Big|_{p^1}, v_p^2 \frac{\partial}{\partial x^2}\Big|_{p^2}, \ldots \Big) \equiv v_p^1 \frac{\partial}{\partial x^1}\Big|_{p^1} + v_p^2 \frac{\partial}{\partial x^2}\Big|_{p^2} + \ldots$$

for $v_p^i \in \mathbb{R}$, where the infinite sum on the right side is a somewhat abusive but more suggestive notation. A vector field $v \in \mathfrak{X}(\check{\mathbb{R}}^\infty)$ is represented by a family of maps $v_i : \mathbb{R}^{k(i)} \to T\mathbb{R}^i$, where we recall that $k(i) \leq k(i+1)$. In coordinates, it is given by the infinite sum

$$v = v^1 \frac{\partial}{\partial x^1} + v^2 \frac{\partial}{\partial x^2} + \ldots = v^i \frac{\partial}{\partial x^i} ,$$

where $v^i \in C^\infty(\mathbb{R}^{k(i)})$ are the component functions of v. Note that the v^i are very different from the maps v_i, which are given in local coordinates by the partial sums

$$v_i(x^1, \ldots, x^{k(i)}) = v^1(x^1, \ldots, x^{k(1)}) \frac{\partial}{\partial x^1} + \ldots + v^i(x^1, \ldots, x^{k(i)}) \frac{\partial}{\partial x^i} .$$

The action of v on $f \in C^\infty(\mathbb{R}^i)$ is given by

$$v \cdot f = v^1 \frac{\partial f}{\partial x^1} + \ldots + v^i \frac{\partial f}{\partial x^i} ,$$

which is a function in $C^\infty(\mathbb{R}^{k(i)})$. Let w be a vector field represented by the maps $w_i : \mathbb{R}^{l(i)} \to T\mathbb{R}^i$. The Lie bracket of v and w is given by

$$[v, w] = \Big(v^j \frac{\partial w^i}{\partial x^j} - w^j \frac{\partial v^i}{\partial x^j} \Big) \frac{\partial}{\partial x^i} .$$

The difference to the usual formula is that the sum over i is infinite. While the index j runs from 1 to ∞ as well, the condition that all component functions v^i and w^i are smooth functions on a finite-dimensional manifold ensures that the sum over j is finite. ***

4.3.3 Differential forms

Assigning to a manifold the complex of differential forms is a functor $\Omega : \mathcal{M}\mathrm{fld} \to \mathrm{dg}\mathcal{A}\mathrm{lg}^{\mathrm{op}}$ to differential graded algebras. By Corollary 4.1.32 this induces a functor

$$\Omega \equiv \mathrm{Pro}(\Omega) : \mathrm{Pro}(\mathcal{M}\mathrm{fld}) \longrightarrow \mathrm{Ind}(\mathrm{dg}\mathcal{A}\mathrm{lg})^{\mathrm{op}} .$$

When $\check{X} \in \mathrm{Pro}(\mathcal{M}\mathrm{fld})$ is represented by the cofiltered diagram $X : \mathcal{I} \to \mathcal{M}\mathrm{fld}$, then $\Omega(\check{X})$ is represented by the filtered diagram $\mathcal{I}^{\mathrm{op}} \to \mathrm{dg}\mathcal{A}\mathrm{lg}$, $i \mapsto \Omega(X_i)$. By Proposition 4.1.58, we can view the underlying ind-algebra of $\Omega(\check{X})$ as an algebra in the category $\mathrm{Ind}(\mathrm{gr}\mathcal{V}\mathrm{ec})$ of ind-$\mathbb{Z}$ graded vector spaces. The product of $\Omega(\check{X})$ will be denoted as usual by $\wedge$.

A **differential form on** $\check{X}$ is an element of the underlying set of $\Omega(\check{X})$. Every differential form is represented by an element $\alpha \in \Omega^p(X_i)$, where p is the degree of α.

Let $\alpha, \beta \in \Omega(\check{X})$ be represented by $\alpha \in \Omega^p(X_i)$ and $\beta \in \Omega^q(X_j)$. Since the index category $\mathcal{I}$ is cofiltered, there are morphisms $i \leftarrow k \to j$ in $\mathcal{I}$. They are mapped by the functor X to morphisms

$$X_i \xleftarrow{\tau_{i \leftarrow k}} X_k \xrightarrow{\tau_{j \leftarrow k}} X_j .$$

The product $\alpha \wedge \beta$ is then represented by

$$\tau^*_{i\leftarrow k}\alpha \wedge \tau^*_{j\leftarrow k}\beta \in \Omega^{p+q}(X_k)\,. \tag{4.25}$$

This shows that the algebra in $\mathrm{Ind}(\mathrm{gr}\mathcal{V}\mathrm{ec})$ is graded.

Every $\Omega(X_i)$ is equipped with the differential d_i. The differential of the form represented by $\alpha \in \Omega^p(X_i)$ is represented by $d_i\alpha \in \Omega^{p+1}(X_i)$. The family of all de Rham differentials $d_i : \Omega^\bullet(X_i) \to \Omega^{\bullet+1}(X_i)$ represents a degree 1 map d of the graded vector space $\Omega(\check{X})$.

Let $\beta \in \Omega(\check{X})$ be represented by $\beta \in \Omega^q(X_j)$. Their $\wedge$-product is represented by 4.25, so $d(\alpha \wedge \beta)$ is represented by

$$\begin{aligned} d_k(\tau^*_{i\leftarrow k}\alpha \wedge \tau^*_{j\leftarrow k}\beta) &= d_k\tau^*_{i\leftarrow k}\alpha \wedge \tau^*_{j\leftarrow k}\beta + (-1)^p\tau^*_{i\leftarrow k}\alpha \wedge d_k\tau^*_{j\leftarrow k}\beta \\ &= \tau^*_{i\leftarrow k}d_i\alpha \wedge \tau^*_{j\leftarrow k}\beta + (-1)^p\tau^*_{i\leftarrow k}\alpha \wedge \tau^*_{j\leftarrow k}d_j\beta\,, \end{aligned} \tag{4.26}$$

where we have used that the de Rham differentials commute with pullbacks. The right side of Equation (4.26) represents $d\alpha \wedge \beta + (-1)^p\alpha \wedge d\beta$, which shows that d is a derivation.

4.3.4 Inner derivative

For every tangent vector v_m on a manifold M, let $\iota_v : \Omega^1(M) \to \mathbb{R}$, $\iota_{v_m}\alpha = \langle \alpha, v_m\rangle$ denote the evaluation of 1-forms on v_m. Let $f : M \to N$ be a smooth map. Recall that the pullback $f^*\alpha$ of a 1-form $\alpha \in \Omega^1(N)$ is defined by $\iota_{v_m} f^*\alpha = \iota_{Tf\,v_m}\alpha$. This means that for a tangent vector on the pro-manifold $\check{X}$ represented by $v_{x,i} : * \to TX_i$, we have commutative diagrams

$$\begin{tikzcd} \Omega^1(X_i) \arrow[r, "\tau^*"] \arrow[d, "\iota_{v_{x,i}}"'] & \Omega^1(X_j) \arrow[dl, "\iota_{v_{x,j}}"] \\ \mathbb{R} & \end{tikzcd}$$

where $\tau : X_j \to X_i$ is a morphism of the diagram $X : \mathcal{I} \to \mathcal{M}\mathrm{fld}$, so that $v_{x,i} = T\tau\, v_{x,j}$. This shows that the family of maps $\iota_{v,i} : \Omega^1(X_i) \to \mathbb{R}$ represents a morphism of ind-vector spaces

$$\iota_{v_x} : \Omega^1(\check{X}) \longrightarrow \mathbb{R}\,,$$

which is the evaluation of 1-forms on $\check{X}$ on the tangent vector v_x. Let now $v : \check{X} \to T\check{X}$ be a vector field represented by the smooth maps $v_i : X_{k(i)} \to TX_i$. For every $\alpha \in \Omega^1(\check{X})$ we have the family of smooth maps

$$\begin{aligned} (\iota_v\alpha)_i : X_{k(i)} &\longrightarrow \mathbb{R} \\ x &\longmapsto \iota_{v_x}\alpha \end{aligned}$$

which defines a morphism of ind-manifolds $\iota_v\alpha : \check{X} \to \mathbb{R}$. If α is represented by $\alpha \in \Omega^1(X_i)$, then $\iota_v\alpha$ is represented by $(\iota_v\alpha)_i \in C^\infty(X_{k(i)})$, which is given explicitly by

$$(\iota_v\alpha)_i(x) = \langle \alpha_{\tau(x)}, v_{i,x}\rangle\,.$$

where $\tau : X_{k(i)} \to X_i$ is a smooth map of the diagram X. This map depends linearly on α, so we obtain a morphism of ind-vector spaces

$$\iota_v : \Omega^1(\check{X}) \longrightarrow C^\infty(\check{X}) ,$$

which is the pairing of 1-forms with the vector field v in the setting of pro-manifolds.

In order to extend the pairing to the inner derivative on higher degree differential forms we use that $\Omega(\check{X})$ is generated as graded commutative algebra by functions and 1-forms. For every function $f \in C^\infty(\check{X})$ we set

$$\iota_v f := 0 .$$

For $\alpha, \beta \in \Omega^1(\check{X})$ we define

$$\iota_v(\alpha \wedge \beta) := \iota_v\alpha \wedge \beta - \alpha \wedge \iota_v\beta . \tag{4.27}$$

Note that in order to represent the right side by a 1-form on X_l we have to first pull-back all factors along the smooth maps

$$\begin{tikzcd}
 & & X_l \arrow[dll] \arrow[dl] \arrow[dr] \arrow[drr] & & \\
X_{k(i)} & X_j & & X_i & X_{k(j)}
\end{tikzcd}$$

in the diagram X and then multiply and add them in $\Omega(X_l)$. Iterating (4.27), we obtain a derivation of $\Omega(\check{X})$. The result is summarized in the following statment.

Proposition 4.3.9. *Let $v \in \mathfrak{X}(\check{X})$ be a vector field on the pro-manifold $\check{X}$. Then the pairing of vector fields and forms on $\check{X}$ extends to a unique degree -1 derivation of $\Omega(\check{X})$.*

4.3.5 Cartan calculus

Proposition 4.3.10. *In the graded Lie algebra $\underline{\mathrm{Der}}\big(\Omega(\check{X})\big)$ let*

$$\mathcal{L}_v := [\iota_v, d] .$$

denote the ***Lie derivative*** *with respect to the vector field $v \in \mathfrak{X}(\check{X})$. Then*

$$\begin{gathered}
[\mathcal{L}_v, \iota_w] = \iota_{[v,w]} , \quad [\mathcal{L}_v, \mathcal{L}_w] = \mathcal{L}_{[v,w]} , \\
[d, d] = [\iota_v, \iota_w] = [\mathcal{L}_v, d] = 0 ,
\end{gathered}$$

Proof. The proof is completely analogous to the proof for ordinary manifolds. The relations only have to be checked on the generators of the algebra $\Omega(\check{X})$, which are functions f and exact 1-forms $\alpha = df$. Since d is a differential, $[d, d] = 2d^2 = 0$. Since $\iota_v\iota_w f = 0$ and $\iota_v\iota_w\alpha = 0$ for degree reasons, $[\iota_v, \iota_w] = 0$. Using the graded Jacobi identity, we obtain

$$\begin{aligned}
[\mathcal{L}_v, d] &= [[\iota_v, d], d] = [\iota_v, [d, d]] - [[\iota_v, d], d] \\
&= -[\mathcal{L}_v, d] ,
\end{aligned}$$

which implies $[d, \mathcal{L}_v] = 0$. On functions, we have $\mathcal{L}_v f = \iota_v df = v \cdot f$. It follows that

$$\begin{aligned}[\mathcal{L}_v, \iota_w]df &= v \cdot (w \cdot f) - w \cdot (v \cdot f) = [v, w] \cdot f \\ &= \iota_{[v,w]} df\end{aligned}$$

Moreover, for degree reasons we have $[\mathcal{L}_v, \iota_w]f = 0 = \iota_{[v,w]}f$. Together this implies the relation $[\mathcal{L}_v, \iota_w] = \iota_{[v,w]}$. Finally, we compute

$$\begin{aligned}[\mathcal{L}_v, \mathcal{L}_w] &= [\mathcal{L}_v, [\iota_w, d]] = [[\mathcal{L}_v, \iota_w], d] - [\iota_v, [\mathcal{L}_v, d]] = [\iota_{[v,w]}, d] \\ &= \mathcal{L}_{[v,w]} \,,\end{aligned}$$

which finishes the proof. □

Terminology 4.3.11. The graded Lie subalgebra of $\underline{\mathrm{Der}}\big(\Omega(\check{X})\big)$ generated by d, ι_v, $\mathcal{L}_v$ for all $v \in \mathfrak{X}(\check{X})$ is called the **Cartan calculus** on the pro-manifold $\check{X}$.

Let us spell out the Cartan calculus on the pro-manifold represented by $\mathbb{R}^0 \leftarrow \mathbb{R}^1 \leftarrow \ldots$ in terms of local coordinates $(x^1, x^2, \ldots)$ as at the end of 4.3.2. Let dx^i denote the coordinate 1-forms. They are dual to the coordinate vector fields $\iota_{\frac{\partial}{\partial x^i}} dx^j = \delta_i^j$. Every 1-form α is given by a finite sum

$$\alpha = \alpha_1 dx^1 + \ldots + \alpha_n dx^n = \alpha_i dx^i \,,$$

where $\alpha_i \in C^\infty(\mathbb{R}^{k(i)})$. Let l be the maximum of all indices $\{n, k(1), \ldots, k(i)\}$. Then we can view all functions as functions on $C^\infty(\mathbb{R}^l)$ and therefore view α as a 1-form on $\mathbb{R}^l$. Similarly, a general p-form is given by a finite sum

$$\omega = \sum_{0 < i_1 < \ldots < i_p \leq n} \omega_{i_1,\ldots,i_p} dx^{i_1} \wedge \ldots \wedge dx^{i_p} \,,$$

where $\omega_{i_1,\ldots,i_p} \in C^\infty(\mathbb{R}^k)$ for some k. The de Rham differential of a function f on $\mathbb{R}^n$ is given by by the finite sum

$$df = \frac{\partial f}{\partial x^1} dx^1 + \ldots + \frac{\partial f}{\partial x^n} dx^n \,.$$

Since the sums are finite, the inner derivative with respect to a vector field, which is given by an infinite sum $v = v^i \frac{\partial}{\partial x^i}$ is well-defined. For example, the pairing of v with the 1-form α is given by the finite sum

$$\iota_v \alpha = v^1 \alpha_1 + \ldots + v^n \alpha_n \,.$$

The upshot is that in local coordinates the de Rham calculus is given by the usual formulas. The difference is that a vector field is generally given by an infinite sum of partial derivatives. But since functions depend only on a finite number of coordinates and forms are given by finite sums over products of coordinate 1-forms, all operations are well-defined.

Remark 4.3.12. Since the category of dg-algebras has all colimits, it is tempting to consider the dg-algebra

$$\bar{\Omega}(\check{X}) \cong \operatorname*{colim}_{i\in\mathcal{I}} \Omega(X_i)$$

rather than the ind-algebra $\Omega(\check{X})$, which is often the point of view taken in the literature. However, this creates a number of problems. For example, in Proposition 4.3.6, we have seen that pro-vector fields are derivations of the algebra $C^\infty(\check{X})$ in ind-vector spaces. The colimit algebra $\bar{C}^\infty(\check{X})$ in vector spaces has generally more endomorphisms and more derivations (Remark 4.1.43). For such more general derivations the pairing with 1-forms will no longer be defined. Ultimately, it is the ind/pro-categorical framework that guarantees that the Cartan calculus extends nicely from smooth manifolds to pro-manifolds.

Exercises

Exercise 4.1. Let $\mathcal{I}$ be a category with a terminal object.

(a) Show that $\mathcal{I}$ filtered.

(b) Show that the colimit of any diagram $D : \mathcal{I} \to \mathcal{C}$ exists.

Exercise 4.2. Show that for every filtered category the diagonal functor $\mathcal{I} \to \mathcal{I} \times \mathcal{I}$, $i \mapsto (i, i)$ is final.

Exercise 4.3. Let $\Phi : \mathcal{I} \to \mathcal{J}$ be a final functor. Show that if $\mathcal{I}$ is filtered, then $\mathcal{J}$ is filtered.

Exercise 4.4. Let $\mathcal{C}$ be the partially ordered set $(\mathbb{R}, \leq)$, viewed as category. Show that a functor $x : \omega \to \mathcal{C}$, $n \mapsto x_n$ has a colimit $y \in \mathbb{R}$ if and only if the sequence of numbers $x_0, x_1, \ldots$ converges to y. (Recall that ω denotes the category $0 \to 1 \to 2 \to \ldots$.)

Exercise 4.5. Let X be a topological space. For every point $x \in X$, let $\mathcal{U}_x$ denote the set of open neighborhoods of x.

(a) View $\mathcal{U}_x$ as category where there is a unique morphism $U \to V$ if $U \subset V$. Show that $\mathcal{U}_x$ is cofiltered.

(b) Let Y be another topological space. Let $\mathrm{Res}_x : \mathcal{U}_x^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$ be the functor that maps an object U to $\mathcal{T}\mathrm{op}(U, Y)$ and a morphism $U \to V$ to the restriction $\mathcal{T}\mathrm{op}(U, Y) \to \mathcal{T}\mathrm{op}(V, Y)$, $f \mapsto f|_V$. Show that the colimit of Res_x is the set of germs of continuous Y-valued functions at x.

Exercise 4.6. Let $\check{X}$ be a pro-object represented by the diagram $X_0 \xleftarrow{\sigma_0} X_1 \xleftarrow{\sigma_1} X_2 \leftarrow \ldots$. Show that the morphism of pro-objects $\check{X} \to \check{X}$ represented by the family $\{\sigma_i\}_{i\in\omega}$ is the identity.

Exercise 4.7. Let $\hat{X}$ be the strict sequential ind-object of $\mathcal{C}$ represented by the diagram $X : \omega \to \mathcal{C}$. In Example 4.1.12 we have seen that every unbounded order preserving map $\Phi : \omega \to \omega$ is final, which implies that the ind-object $\hat{X}'$ represented by $X\Phi$ is isomorphic to $\hat{X}$. The isomorphism $f : \hat{X}' \to \hat{X}$ is represented by the family of morphisms $X'_i = X_{\Phi(i)} \xrightarrow{\mathrm{id}} X_{\Phi(i)}$. Find a family of morphisms representing the inverse of f.

Exercise 4.8. Let X be a vector space. Let $\mathcal{S}_X$ denote the category that has the finite dimensional subspaces $V \subset X$ as objects and inclusions $V \subset W$ as morphisms. Let $X_{\mathrm{fin}} : \mathcal{S}_X \to \mathcal{V}\mathrm{ec}$, $(V \to X) \mapsto V$ denote the inclusion $\mathcal{S}_X \subset \mathcal{V}\mathrm{ec}$ as subcategory.

(i) Show that X is the colimit of X_{fin}.

(ii) Show that $\mathcal{S}_X$ is filtered.

(iii) Let $\hat{X}_{\mathrm{fin}}$ denote the ind-object in $\mathcal{V}\mathrm{ec}$ represented by the diagram X_{fin}. Let A be a vector space. Show that a morphism $yA \to \hat{X}_{\mathrm{fin}}$ of ind-vector spaces can be identified with a linear map $A \to X$ of finite rank.

(iv) Conclude that yX and $\hat{X}_{\mathrm{fin}}$ are not isomorphic in $\mathrm{Ind}(\mathcal{V}\mathrm{ec})$.

Exercise 4.9. Let $I : \mathcal{M}\mathrm{fld} \to \mathcal{D}\mathrm{flg}$ denote the natural inclusion, which maps a manifold to the smooth diffeology, as in Example 2.1.3 (d). Consider the functor $D : \mathrm{Pro}(\mathcal{M}\mathrm{fld}) \to \mathcal{D}\mathrm{flg}$ that maps a pro-manifold $\check{X}$ represented by $X : \mathcal{I} \to \mathcal{M}\mathrm{fld}$ to its limit in diffeological spaces,

$$D\check{X} := \lim_{i \in \mathcal{I}} IX_i \,.$$

(We call $D\check{X}$ the pro-manifold diffeology.) Show that there is a natural isomorphism

$$\mathrm{Pro}\mathcal{M}\mathrm{fld}(yM, \check{X}) \cong \mathcal{D}\mathrm{flg}(IM, D\check{X}) \,,$$

for all $M \in \mathcal{M}\mathrm{fld}$ and $\check{X} \in \mathrm{Pro}\mathcal{M}\mathrm{fld}$. What happens when we replace yM with a more general pro-manifold?

Exercise 4.10. Let $\tau : S^1 \to S^1$, $e^{2\pi i t} \mapsto e^{4\pi i t}$ denote the double cover of the circle. Consider the sequential pro-manifold $\check{X}$ represented by the diagram $S^1 \xleftarrow{\tau} S^1 \xleftarrow{\tau} S^1 \leftarrow \ldots$. Let $x : * \to \check{X}$ be a point (which is given by a sequence $x_k \in S^1$, $k \geq 0$).

(a) Show that a morphism $\gamma : \mathbb{R} \to \check{X}$ of pro-manifolds with starting point $\gamma(0) = x$ can be identified with a smooth path $\tilde{\gamma} : \mathbb{R} \to S^1$ through some $x_k = \tilde{\gamma}(0)$.

(b) Show that a morphism $\check{X} \to \mathbb{R}$ of pro-manifolds can be identified with a smooth function $f : \mathbb{R} \to \mathbb{R}$ that is 2^{-k}-periodic, $f(t) = f(t + 2^{-k})$, for some natural number $k \geq 0$.

(c) Show that the fiber of $T\check{X}$ at x is isomorphic to the constant pro-vector space $\mathbb{R}$.

(The exercise illustrates that the definition of tangent vectors by paths and by derivations both work in the setting of pro-manifolds.)

Chapter 5

Variational cohomology

5.1 De Rham complex of the pro-manifold of infinite jets

Definition 5.1.1. Let $F \to M$ be a smooth fiber bundle. The pro-manifold represented by the sequence

$$J^0F \xleftarrow{\mathrm{pr}_{1,0}} J^1F \xleftarrow{\mathrm{pr}_{2,1}} J^2F \longleftarrow \dots$$

will be denoted by $J^\infty F$ and called the **pro-manifold of infinite jets**.

The underlying set of $J^\infty F$,

$$|J^\infty F| \cong \lim_{i\in\omega} |J^iF| \,,$$

is the set of infinite jets we have defined in Section 3.4. As we have seen in Chapter 3, the jet manifolds J^iF are equipped with more structure. The projection $\mathrm{pr}_{i,-1}$: $J^iF \to M$ is a smooth fiber bundle and the forgetful maps $\mathrm{pr}_{i+1,i}$ are morphisms of smooth fiber bundles over M, so that we have a commutative diagram

$$\begin{tikzcd}
J^0F \arrow[d, "\mathrm{pr}_{0,-1}"] & J^1F \arrow[l, "\mathrm{pr}_{1,0}"'] \arrow[d, "\mathrm{pr}_{1,-1}"] & J^2F \arrow[l, "\mathrm{pr}_{2,1}"'] \arrow[d, "\mathrm{pr}_{2,-1}"] & \dots \arrow[l] \\
M & M \arrow[l, "\mathrm{id}"] & M \arrow[l, "\mathrm{id}"] & \dots \arrow[l]
\end{tikzcd}$$

This can be viewed as a diagram representing a pro-fiber bundle. Alternatively, the diagram represents a morphism of pro-manifolds $J^\infty F \to M$, where M is identified with the pro-manifold represented by the constant diagram $\omega \to \mathcal{M}\mathrm{fld}$, $i \mapsto M$.

Remark 5.1.2. The forgetful maps $\mathrm{pr}_{k+1,k} : J^{k+1}F \to J^kF$ fit in a commutative diagram

$$\begin{tikzcd}
J^1F \arrow[d, "\mathrm{pr}_{1,0}"] & J^2F \arrow[l] \arrow[d, "\mathrm{pr}_{2,1}"] & J^3F \arrow[l] \arrow[d, "\mathrm{pr}_{3,2}"] & \dots \arrow[l] \\
J^0F & J^1F \arrow[l] & J^2F \arrow[l] & \dots \arrow[l]
\end{tikzcd}$$

which represents an isomorphism pro-manifolds $J^\infty F \to J^\infty F$ (see Example 4.2.11). The diagram also represents a pro-affine bundle. The fiber over a point $j^\infty_m\varphi$ is the pro-affine space represented by the sequence

$$\{j^0_m\varphi\} \times_{J^0F} J^1F \longleftarrow \{j^1_m\varphi\} \times_{J^1F} J^2F \longleftarrow \{j^2_m\varphi\} \times_{J^2F} J^3F \longleftarrow \dots$$

An element of $\{j^k_m \varphi\} \times_{J^k F} J^{k+1}F$ is given by a $(k+1)$-jet $j^{k+1}_m \psi$ such that $j^k_m \psi = j^k_m \varphi$, which shows that the pro-affine space has a single point given by the sequence $j^1_m\varphi, j^2_m\varphi, j^3_m\varphi, \ldots$. This is consistent with Proposition 4.1.40, which implies that the fiber is given by $\{j^\infty_m \varphi\} \times_{J^\infty F} J^\infty F \cong *$.

5.1.1 Vertical and horizontal tangent vectors

We have extended the category of manifolds in two different ways. Diffeological spaces are well suited to describe the differentiable structure of the space of sections of a smooth fiber bundle. Pro-manifolds are useful to describe differential operators as morphisms on the infinite jet manifold. We now combine the two approaches and consider pro-diffeological spaces. The extensions are compatible in the sense that

$$\begin{tikzcd}
\mathcal{M}\mathrm{fld} \arrow[r] \arrow[d] & \mathcal{D}\mathrm{flg} \arrow[d] \\
\mathrm{Pro}(\mathcal{M}\mathrm{fld}) \arrow[r] & \mathrm{Pro}(\mathcal{D}\mathrm{flg})
\end{tikzcd}$$

is a commutative diagram of categories. The diagonal arrow maps a manifold M to the presheaf on $\mathcal{D}\mathrm{flg}$ given by $X \mapsto \mathcal{D}\mathrm{flg}(X, M)$, where M is equipped with the smooth diffeology.

The jet evaluations

$$\begin{tikzcd}
\mathcal{F} \times M \arrow[d, "j^0"'] \arrow[dr, "j^1"] \arrow[drr, "j^2"] & & & \\
J^0F & J^1F \arrow[l] & J^2F \arrow[l] & \ldots \arrow[l]
\end{tikzcd}$$

can, by Proposition 3.1.14, be viewed as smooth maps of diffeological spaces. By Proposition 4.2.7, the collection of all jet evaluations represents a morphism of pro-diffeological spaces.

Definition 5.1.3. The morphism of pro-diffeological spaces

$$j^\infty : \mathcal{F} \times M \longrightarrow J^\infty F$$

represented by the jet evaluations $j^k : \mathcal{F} \times M \to J^k F$ is called the diffeological **infinite jet evaluation**.

The domain of j^∞ is the product of two diffeological spaces. By Proposition 2.2.10, the tangent functor preserves the product,

$$T(\mathcal{F} \times M) \cong T\mathcal{F} \times TM\,. \tag{5.1}$$

It follows that the tangent fiber at $(\varphi, m) \in \mathcal{F} \times M$ is the product of $T_\varphi \mathcal{F} \cong \Gamma(F, \varphi^* VF)$ and $T_m M$, which are both vector spaces. We conclude that the tangent fiber is the product

$$\begin{aligned} T_{(\varphi,m)}(\mathcal{F} \times M) &\cong T_\varphi \mathcal{F} \times T_m M \\ &\cong T_\varphi \mathcal{F} \oplus T_m M\,, \end{aligned} \tag{5.2}$$

which is the same as the direct sum. We will call $T_\varphi \mathcal{F} \cong \Gamma(F, \varphi^* VF)$ the **vertical** tangent space and $T_m M$ the **horizontal** tangent space.

Globally, we have the decomposition

$$\begin{aligned} T\mathcal{F} \times TM &\cong (T\mathcal{F} \times_{\mathcal{F}} \mathcal{F}) \times (M \times_M TM) \\ &\cong (T\mathcal{F} \times M) \times_{\mathcal{F} \times M} (\mathcal{F} \times TM)\,, \end{aligned}$$

where we have used that products commute with pullbacks. The right hand side might be viewed as fiber product of bundles of vector spaces over $\mathcal{F} \times M$ (Terminology 2.2.6), that is, a generalized Whitney sum $(T\mathcal{F} \times M) \oplus (\mathcal{F} \times TM)$. But we do not want to overstretch the analogy, since the $T\mathcal{F} \times M \to \mathcal{F} \times M$ does not have local trivializations.

Since the infinite jet evaluation is a morphism of pro-diffeological spaces, it has a tangent map

$$\begin{tikzcd} T\mathcal{F} \times TM \arrow[r, "Tj^\infty"] \arrow[d] & TJ^\infty F \arrow[d] \\ \mathcal{F} \times M \arrow[r, "j^\infty"'] & J^\infty F \end{tikzcd}$$

which is a morphism of bundles of pro-diffeological vector spaces. Over a point (φ, m), we obtain a morphism of pro-diffeological vector spaces

$$T_{(\varphi,m)} j^\infty : T_\varphi \mathcal{F} \oplus T_m M \longrightarrow T_{j^\infty_m \varphi} J^\infty F\,, \tag{5.3}$$

where the codomain is represented by the diagram

$$T_{j^0_m \varphi} J^0 F \longleftarrow T_{j^1_m \varphi} J^1 F \longleftarrow T_{j^2_m \varphi} J^2 F \longleftarrow \ldots$$

The following theorem states that (5.3) preserves the direct sum and, therefore, induces a splitting of the tangent bundle of $J^\infty F$ into a vertical and a horizontal subbundle.

Theorem 5.1.4. *The tangent map of the infinite jet evaluation* (5.3) *preserves the direct sum of its domain,*

$$Tj^\infty(T_\varphi \mathcal{F} \oplus T_m M) = Tj^\infty(T_\varphi \mathcal{F}) \oplus Tj^\infty(T_m M)\,,$$

for all $(\varphi, m) \in \mathcal{F} \times M$. If $j^0 : \mathcal{F} \times M \to F$ is surjective, then Tj^∞ is surjective and we have the natural isomorphisms

$$\begin{aligned} Tj^\infty(T\mathcal{F} \times M) &\cong J^\infty(VF) \\ Tj^\infty(\mathcal{F} \times TM) &\cong J^\infty F \times_M TM \end{aligned}$$

of bundles over $J^\infty F$. This induces a decomposition

$$TJ^\infty F \cong J^\infty(VF) \times_{J^\infty F} (J^\infty F \times_M TM) \tag{5.4}$$

into a fiber product of bundles of pro-manifolds over $J^\infty F$.

Proof. The proof is constructive and will yield explicit formulas for the decomposition of the tangent spaces of $J^\infty F$. First, we recall from Theorem 2.3.4 that the tangent bundle of $\mathcal{F}$ is given by $T\mathcal{F} \cong \Gamma(M, VF)$, so that a tangent vector in $T_\varphi\mathcal{F}$ consists of a section ξ of $\varphi^* VF$. In local coordinates $\xi(m) = \xi^\alpha(m)\frac{\partial}{\partial u^\alpha}\big|_{\varphi(m)}$, where ξ^α are local functions on M. There are induced jet coordinates $(x^i, u^\alpha_I, \dot{u}^\alpha_I)$ on $J^k VF$, where

$$\dot{u}^\alpha_I(j^k_m \xi) := \frac{\partial^{|I|}\xi^\alpha}{\partial x^I}\Big|_m ,$$

for $|I| \leq k$. The notation is motivated by the jet coordinates of a tangent vector represented by a path $t \mapsto \varphi_t$, which are given by

$$\dot{u}^\alpha_I(j^k_m \dot{\varphi}_0) = \frac{d}{dt}\Big(u^\alpha_I(j^k_m \varphi_t)\Big)_{t=0} .$$

In terms of these jet coordinates we can compute the tangent map of the jet evaluations explicitly.

Every tangent vector on $\mathcal{F} \times M$ is represented by a smooth path $t \mapsto (\varphi_t, m_t)$. As we have seen in Equation (3.7), the coordinates of its k-jet are given by

$$\begin{aligned} x^i\big(j^k(\varphi_t, m_t)\big) &= m^i_t \\ u^\alpha_I\big(j^k(\varphi_t, m_t)\big) &= \frac{\partial^{|I|}\varphi_t}{\partial x^I}(m_t) . \end{aligned}$$

To compute the diffeological tangent map in coordinates, we have to compute the time derivative of the coordinates of these paths. For the base coordinates we get

$$\frac{d}{dt} x^i\big(j^k(\varphi_t, m_t)\big)\Big|_{t=0} = \dot{m}^i_0 . \tag{5.5}$$

For the fiber coordinates of we obtain

$$\begin{aligned} \frac{d}{dt} u^\alpha_I\big(j^k(\varphi_t, m_t)\big)\Big|_{t=0} &= \frac{d}{dt}\Big(\frac{\partial^{|I|}\varphi^\alpha_t}{\partial x^I}(m_t)\Big)_{t=0} \\ &= \Big(\frac{\partial\partial^{|I|}\varphi^\alpha_t}{\partial t \partial x^I}(m_t) + \frac{\partial\partial^{|I|}\varphi^\alpha_t}{\partial x^i \partial x^I}(m_t)\, \dot{m}^i_t\Big)_{t=0} \\ &= \frac{\partial^{|I|}\dot{\varphi}^\alpha_0}{\partial x^I}(m_0) + \frac{\partial^{|I|+1}\varphi^\alpha_0}{\partial x^{I,i}}(m_0)\, \dot{m}^i_0 , \end{aligned}$$

where we have used the chain rule and that partial derivatives commute. (Recall, that we are using the summation convention, so that i in the second term is summed over.) From the last two equations, we read off the tangent map of the k-th jet evaluation

$$\begin{aligned} (T_{(\varphi_0, m_0)} j^k)(\dot{\varphi}_0, \dot{m}_0) &= \dot{m}^i_0 \frac{\partial}{\partial x^i} + \sum_{|I|=0}^{k} \big(\dot{u}^\alpha_I(j^k_{m_0}\dot{\varphi}_0) + \dot{m}^i_0\, u^\alpha_{I,i}(j^{k+1}_{m_0}\varphi_0)\big)\frac{\partial}{\partial u^\alpha_I} \\ &= \dot{u}^\alpha_I(j^k_{m_0}\dot{\varphi}_0)\frac{\partial}{\partial u^\alpha_I} + \dot{m}^i_0\Big(\frac{\partial}{\partial x^i} + \sum_{|I|=0}^{k} u^\alpha_{I,i}(j^{k+1}_{m_0}\varphi_0)\frac{\partial}{\partial u^\alpha_I}\Big) . \end{aligned} \tag{5.6}$$

For the infinite jet evaluation, the sum on the right side becomes infinite. Using the notation $\xi_\varphi := (\varphi_0, \dot\varphi_0)$ for the tangent vector in $T_\varphi \mathcal{F}$ and $v_m := (m_0, \dot m_0)$ for the horizontal tangent vector in $T_m M$, we obtain

$$(Tj^\infty)(\xi_\varphi, v_m) = \sum_{|I|=0}^{\infty} \dot u_I^\alpha(j_m^\infty \xi_\varphi) \frac{\partial}{\partial u_I^\alpha} + v_m^i \Big(\frac{\partial}{\partial x^i} + \sum_{|I|=0}^{\infty} u_{I,i}^\alpha (j_m^\infty \varphi) \frac{\partial}{\partial u_I^\alpha} \Big) . \tag{5.7}$$

The first term is in the image of $T_\varphi \mathcal{F}$, the second term in the image of $T_m M$. Since only the second term contains $\frac{\partial}{\partial x^i}$, the two summands are linearly independent.

Assume now that j^0 is surjective. Every tangent vector $\zeta \in T_{j_m^\infty \varphi} J^\infty F$ is of the form

$$\begin{aligned} \zeta &= v^i \frac{\partial}{\partial x^i} + \sum_{|I|=0}^{\infty} \zeta_I^\alpha \frac{\partial}{\partial u_I^\alpha} \\ &= \sum_{|I|=0}^{\infty} \big(\zeta_I^\alpha - v^i u_{I,i}^\alpha (j_m^\infty \varphi) \big) \frac{\partial}{\partial u_I^\alpha} + v^i \Big(\frac{\partial}{\partial x^i} + \sum_{|I|=0}^{\infty} u_{I,i}^\alpha (j_m^\infty \varphi) \frac{\partial}{\partial u_I^\alpha} \Big) . \end{aligned} \tag{5.8}$$

By Lemma 3.1.12 and the Whitney extension Theorem 3.3.8, j^∞ is surjective. It follows that there is a section $\xi_\varphi \in \Gamma(M, \varphi^* VF)$ such that

$$\dot u_I^\alpha (j_m^\infty \xi_\varphi) = \zeta_I^\alpha - v^i u_{I,i}^\alpha (j_m^\infty \varphi) .$$

Let $v_m = v^i \frac{\partial}{\partial x^i}$. We conclude from (5.7) that $(Tj^\infty)(\xi_\varphi, v_m) = \zeta$. This shows that $T_{(\varphi,m)} j^\infty$ is surjective.

In order to deduce from the fiber-wise splitting a global splitting of the bundle $TJ^\infty F \to J^\infty F$, we consider Equation (5.6). From the first summand on the right side, which is linear in $\dot\varphi_0$, we see that the restriction of Tj^k to $T\mathcal{F} \times M$ factors as

$$\begin{tikzcd} T\mathcal{F} \times M \arrow[d, "j^k_{VF}"'] \arrow[dr, "Tj^k|_{T\mathcal{F}\times M}"] & \\ J^k(VF) \arrow[r, "\tau_k"'] & TJ^k F \end{tikzcd}$$

where j^k_{VF} is the k-th jet evaluation of the bundle $VF \to M$, and where τ_k is the morphism of fiber-wise linear diffeological bundles given by

$$\tau_k \big(j_m^k \dot\varphi_0 \big) = \frac{d}{dt} \big(j_m^k \varphi_t \big)_{t=0} ,$$

for every smooth path $t \mapsto \varphi_t$ of local sections of F. Since the partial derivatives with respect to the coordinates of M commute with the time derivative, τ_k is injective. We have already shown that j^k_{VF} is surjective, which implies that $Tj^k|_{T\mathcal{F}\times M}$ and τ_k have the same image. Since τ_k is injective, we conclude that $Tj^k(T\mathcal{F} \times M)$ is isomorphic to $J^k(VF)$. Since this holds for all k, we obtain an isomorphism of pro-objects $Tj^k(T\mathcal{F} \times M) \cong J^\infty(VF)$.

From the second summand on the right side of Equation (5.6), which is linear in $\dot{m}_0$, we see that the restriction of Tj^k to $\mathcal{F} \times TM$ factors as

$$\begin{tikzcd} \mathcal{F} \times TM \arrow[d, "\beta_{k+1}"'] \arrow[dr, "Tj^k|_{\mathcal{F}\times TM}"] & \\ J^{k+1}F \times_M TM \arrow[r, "\sigma_k"'] & TJ^kF \end{tikzcd}$$

where β_{k+1} sends $(\varphi, v_m) \mapsto (j^{k+1}_m \varphi, v_m)$ and where σ_k is a morphism of fiber-wise linear diffeological bundles given by

$$\sigma_k\big(j^{k+1}_m \varphi, v_m\big) = v^i_m \Big(\frac{\partial}{\partial x^i} + \sum_{|I|=0}^{k} u^\alpha_{I,i}(j^{k+1}_{m_0}\varphi) \frac{\partial}{\partial u^\alpha_I} \Big) .$$

Since j^{k+1} is surjective, β_{k+1} is surjective, so that the commutativity of the diagram implies that σ_k has the same image as $Tj^k|_{\mathcal{F}\times TM}$. However, σ_k is not injective for any k, since the right hand side does not depend on $u^\alpha(j^k_m\varphi) = \varphi^\alpha(m)$. To show that the morphism of pro-objects $\sigma : J^\infty F \times_M TM \to TJ^\infty F$ is a monomorphism, we will construct a left inverse of σ. Let

$$\begin{aligned} \nu_k := (\pi_{J^kF}, T\mathrm{pr}_{k,-1}) : TJ^kF &\longrightarrow J^kF \times_M TM \\ \Big(v^i_m \frac{\partial}{\partial x^i} + \sum_{|I|=0}^{k} \xi^\alpha_I \frac{\partial}{\partial u^\alpha_I} \Big)_{j^k_m\varphi} &\longmapsto \Big(j^k_m\varphi, v^i_m \frac{\partial}{\partial x^i} \Big) , \end{aligned}$$

which defines a morphism of pro-objects $\nu : TJ^\infty F \to J^\infty F \times_M TM$. The composition $\nu \circ \sigma$ is represented by the morphisms

$$\begin{aligned} \nu_k \circ \sigma_k := J^{k+1}F \times_M TM &\longrightarrow J^kF \times_M TM \\ \big(j^{k+1}_m\varphi, v_m\big) &\longmapsto \big(j^k_m\varphi, v_m\big) , \end{aligned}$$

that is, $\nu_k \circ \sigma_k = \mathrm{pr}_{k+1,k} \times \mathrm{id}_{TM}$. It follows from Proposition 4.2.12 that σ is a section of ν. In particular, σ is a monomorphism. We conclude that σ is an isomorphism to its image $j^\infty(\mathcal{F} \times TM)$.

The family of maps $f_k := \tau_k \circ \mathrm{pr}_{k+1,k} + \sigma_k$,

$$f_k : J^{k+1}(VF) \times_{J^{k+1}F} (J^{k+1}F \times_M TM) \longrightarrow TJ^kF ,$$

represent a morphism

$$J^\infty(VF) \times_{J^\infty F} (J^\infty F \times_M TM) \longrightarrow TJ^\infty F .$$

Let $g_k : TJ^{k+1}F \to J^k(VF) \times_M TM$ be defined by

$$g_k\bigg(\Big(\sum_{|I|=0}^{k+1} \xi^\alpha_I \frac{\partial}{\partial u^\alpha_I} + v^i_m \frac{\partial}{\partial x^i} \Big)_{j^{k+1}_m\varphi} \bigg) = \Big(\sum_{|I|=0}^{k} \big(\xi^\alpha_I - v^i_m u^\alpha_{I,i}(j^{k+1}_m\varphi)\big) \frac{\partial}{\partial u^\alpha_I}, v^i_m \frac{\partial}{\partial x^i} \Big)_{j^k_m\varphi} ,$$

where we have used that $J^k(VF) \times_{J^kF} (J^kF \times_M TM) \cong J^k(VF) \times_M TM$. The family g_k represents a morphism of pro-manifolds

$$g : TJ^\infty F \longrightarrow J^\infty(VF) \times_{J^\infty F} (J^\infty F \times_M TM)\,.$$

The composition $g \circ f$ is represented by the family $(g \circ f)_k = g_k \circ f_{k+1}$. In local coordinates this map is given by

$$(g_k \circ f_{k+1})\Big(\Big(\sum_{|I|=0}^{k+2} \xi_I^\alpha \frac{\partial}{\partial u_I^\alpha}, v_m^i \frac{\partial}{\partial x^i}\Big)_{j_m^{k+2}\varphi}\Big) = \Big(\sum_{|I|=0}^{k} \xi_I^\alpha \frac{\partial}{\partial u_I^\alpha}, v_m^i \frac{\partial}{\partial x^i}\Big)_{j_m^k\varphi}\,,$$

that is, $(g \circ f)_k = T\mathrm{pr}_{k+2,k} \times \mathrm{id}_{TM}$. It follows that $g \circ f$ is the identity morphism. In a similar way, we can show that $f_k \circ g_{k+1}$ represents the identity morphism as well. We conclude that f is an isomorphism. □

Warning 5.1.5. The morphisms f_k that represent the splitting f of the pro-vector bundle $TJ^\infty F \to J^\infty F$ are surjective but not injective, so that f_k does not induce a splitting of $TJ^kF \to J^kF$ for any $k < \infty$. This is one of the reasons why we have to work with the infinite jet bundle.

Terminology 5.1.6. $J^\infty(VF) \hookrightarrow TJ^\infty F$ is called the **vertical** tangent bundle and $J^\infty F \times_M TM \hookrightarrow TJ^\infty F$ the **horizontal** tangent bundle of $J^\infty F$. A tangent vector $v : * \to TJ^\infty F$ is called vertical, if it factors as $* \to J^\infty(VF) \to TJ^\infty F$ through the vertical tangent bundle. Analogously, v is called **horizontal** if it factors as $* \to J^\infty F \times_M TM \to TJ^\infty F$ through the horizontal tangent bundle. A vector field is called vertical (horizontal) if all its values are.

Remark 5.1.7. The inclusion of the horizontal subbundle $J^\infty F \times_M TM \to TJ^\infty F$ is a section of the map

$$(\pi_{J^\infty F}, T\mathrm{pr}_{\infty,-1}) : TJ^\infty F \longrightarrow J^\infty F \times_M TM\,,$$

so that it can be interpreted as the horizontal lift of a connection on $TJ^\infty F \to J^\infty F$, which is called the **Cartan connection**.

Remark 5.1.8. A vector $v \in T_{j_m^k\varphi} J^kF$ is in the image $f_k(J^{k+1}F \times_M TM)$ of the $(k+1)$-level of the horizontal tangent bundle if and only if there is a local section ψ such that $v = (T_m j^k\psi)X_m$ for some $X_m \in TM$. (This implies that $j_m^k\psi = j_m^k\varphi$, but v will generally depend on the $(k+1)$-jet of ψ.) In other words, the Cartan distribution is given by the vectors that are tangent to the image of a holonomic section of $J^\infty F \to M$ (Terminology 3.1.17).

As corollary to Theorem 5.1.4 we obtain the following statement.

Corollary 5.1.9. *The vector space of vector fields on $J^\infty F$ decomposes into the direct sum*

$$\mathfrak{X}(J^\infty F) \cong \mathfrak{X}_{\mathrm{vert}}(J^\infty F) \oplus \mathfrak{X}_{\mathrm{hor}}(J^\infty F) \tag{5.9}$$

of the spaces of vertical and horizontal vector fields. Moreover, we have the natural isomorphisms of vector spaces

$$\begin{aligned}\mathfrak{X}_{\mathrm{vert}}(J^\infty F) &\cong \Gamma\big(J^\infty F, J^\infty(VF)\big)\\ \mathfrak{X}_{\mathrm{hor}}(J^\infty F) &\cong \mathrm{Hom}(J^\infty F, TM)\,.\end{aligned}$$

Corollary 5.1.9 means that every vector field $v \in \mathfrak{X}(J^\infty F)$ has a unique decomposition $v = v_{\mathrm{vert}} + v_{\mathrm{hor}}$ into a vertical and a horizontal vector field. In local jet coordinates a vector field $v \in \mathfrak{X}(J^\infty F)$ has the form

$$v = v^i \frac{\partial}{\partial x^i} + \sum_{|I|=0}^{\infty} v_I^\alpha \frac{\partial}{\partial u_I^\alpha} \,, \tag{5.10}$$

where the components v^i and v_I^α are functions on $J^\infty F$, which means that each component is given by a smooth function on a finite jet manifold. The decomposition of v was computed in Equation (5.8). The horizontal component is

$$v_{\mathrm{hor}} = v^i D_i \,,$$

where

$$D_i := \frac{\partial}{\partial x^i} + \sum_{|I|=0}^{\infty} u_{I,i}^\alpha \frac{\partial}{\partial u_I^\alpha} \,. \tag{5.11}$$

The vertical component $v_{\mathrm{vert}} = v - v_{\mathrm{hor}}$ is given by

$$v_{\mathrm{vert}} = \sum_{|I|=0}^{\infty} (v_I^\alpha - v^i u_{I,i}^\alpha) \frac{\partial}{\partial u_I^\alpha} \,.$$

Since v_I^α and v^i are arbitrary, a vertical vector field is of the general form $\sum_{|I|=0}^{\infty} \xi_I^\alpha \frac{\partial}{\partial u_I^\alpha}$ with arbitrary coefficient functions $\xi_I^\alpha \in C^\infty(J^\infty F)$.

Remark 5.1.10. Let $f \in C^\infty(J^k F)$ be a smooth function. Then $D_i f$ is a function defined on a local coordinate neighborhood of $J^{k+1}F$. When we evaluate it at a jet represented by a local section φ, we obtain

$$\begin{aligned} (D_i f)(j_x^{k+1}\varphi) &= \frac{\partial f}{\partial x^i}(j_x^k \varphi) + \sum_{|I|=0}^{k} \left(\frac{\partial}{\partial x^i} \frac{\partial^{|I|}\varphi^\alpha}{\partial x^I} \right) \frac{\partial f}{\partial u_I^\alpha}(j_x^k \varphi) \\ &= \frac{\partial}{\partial x^i} \left(f \circ j^k \varphi \right)\big|_x \,. \end{aligned} \tag{5.12}$$

In other words, D_i acts on holonomic sections of the jet bundle as the partial derivative with respect to x^i.

Remark 5.1.11. The space of vertical vector fields is involutive, i.e. closed under the Lie bracket. A straightforward calculation shows that $[D_i, D_j] = 0$, which implies that the space of horizontal vector fields is involutive, as well. In other words, the Cartan connection is flat.

5.1.2 The variational bicomplex

The splitting of the bundle of pro-manifolds $TJ^\infty F \to J^\infty F$ proved in Theorem 5.1.4 induces a splitting of the ind-vector space of 1-forms as follows.

Let $g_k : TJ^{k+1}F \to J^k(VF) \times_{J^kF} (J^kF \times_M TM)$ be the morphisms of vector bundles defined in the proof of Theorem 5.1.4 that represent the splitting (5.4). The dual of the vector bundle $J^k(VF) \to J^kF$ is given by $J^k(V^*F)$, where $V^*F \to F$ is the dual bundle of $VF \to F$. The dual of the vector bundle $J^kF \times_M TM \to J^kF$ is given by $J^kF \times_M T^*M$. The pullback of sections is

$$g_k^* : \Gamma\big(J^kF, J^k(V^*F)\big) \oplus \Gamma\big(J^kF, J^kF \times_M T^*M\big) \longrightarrow \Omega^1(J^{k+1}F)\,.$$

Since the family of morphisms g_k represents an isomorphism of pro-vector bundles, the pullbacks g_k^* represent an isomorphism of ind-vector spaces,

$$\Omega(J^\infty F) \cong \Gamma\big(J^\infty F, J^\infty(V^*F)\big) \oplus \Gamma\big(J^\infty F, J^\infty F \times_M T^*M\big)\,.$$

The maps g_k are surjective but not injective. Therefore, g_k^* is injective but not surjective, so that g_k^* does not induce a splitting of $\Omega^1(J^{k+1}F)$ for any $k \geq 0$. This is the dual statement to what we have pointed out in Warning 5.1.5 for the tangent bundles. But since g_k^* is injective, we can identify the two summands of the domain of g_k^* with their images under g_k^* in $\Omega^1(J^kF)$.

Definition 5.1.12. The vector spaces

$$\begin{aligned}
\Omega^{1,0}(J^{k+1}F) &:= g_k^*\,\Gamma\big(J^kF, J^k(V^*F)\big)\\
\Omega^{0,1}(J^{k+1}F) &:= g_k^*\,\Gamma(J^kF, J^kF \times_M T^*M)\,.
\end{aligned}$$

for all $k \geq 0$ are the vector spaces of **vertical** and **horizontal** 1-forms.

The subspace of (p,q)-forms is given by

$$\begin{aligned}
\Omega^{p,q}(J^{k+1}F) &= g_k^*\,\Gamma\big(J^kF, \wedge^p J^k(V^*F) \times_{J^kF} (J^kF \times_M \wedge^q T^*M)\big)\\
&= g_k^*\,\Gamma\big(J^kF, \wedge^p J^k(V^*F) \times_M \wedge^q T^*M)\big)\,.
\end{aligned} \tag{5.13}$$

We point out once more that $\Omega^{1,0}(J^kF) \oplus \Omega^{0,1}(J^kF)$ is a proper subspace of $\Omega^1(J^kF)$ for every $k > 0$, so that

$$\bigoplus_{m=p+q} \Omega^{p,q}(J^kF) \subsetneq \Omega^m(J^kF)\,.$$

In other words, there is no natural splitting of the space of 1-forms and no natural bigrading of the space of forms on any of the finite jet manifolds J^kF. For the ind-vector space $\Omega^{p,q}(J^\infty F)$ that is represented by the sequence

$$\Omega^{p,q}(J^1F) \subset \Omega^{p,q}(J^2F) \subset \dots\,,$$

we have the decomposition

$$\Omega^m(J^\infty F) \cong \bigoplus_{m=p+q} \Omega^{p,q}(J^\infty F)\,. \tag{5.14}$$

For calculations we need to determine the local coordinate expression of (p,q)-forms. We begin with the following observation.

Lemma 5.1.13. *A 1-form $\mu \in \Omega^1(J^\infty F)$ is vertical if and only if $\iota_v\mu = 0$ for all $v \in \mathfrak{X}_{\mathrm{hor}}(J^\infty F)$. It is horizontal if and only if $\iota_v\mu = 0$ for all $v \in \mathfrak{X}_{\mathrm{vert}}(J^\infty F)$.*

Proof. This follows from the non-degeneracy of the pairing of vector fields and 1-forms on $J^\infty F$. $\square$

Lemma 5.1.13 can be used to compute the local form of vertical and horizontal 1-forms in jet coordinates. Let $\mathbf{d}$ denote the de Rham differential of $\Omega(J^{k+1}F)$. A 1-form $\mu \in \Omega(J^{k+1}F)$ is given locally by

$$\mu = \mu_i \mathbf{d}x^i + \sum_{|I|=0}^{k+1} \mu_\alpha^I \mathbf{d}u_I^\alpha \,, \tag{5.15}$$

where we have written out the sum to emphasize that it is finite. As $C^\infty(J^\infty F)$-module, $\mathfrak{X}_{\mathrm{hor}}(J^\infty F)$ is locally spanned by the basis of horizontal vector fields $\{D_i\}$ defined in Equation (5.11). The condition for μ to be vertical is therefore

$$0 = \iota_{D_i}\mu = \mu_i + \sum_{|I|=0}^{k+1} u_{I,i}^\alpha \mu_I^\alpha \,.$$

We can write this condition as

$$\mu_i + \sum_{|I|=0}^{k} u_{I,i}^\alpha \mu_I^\alpha = - \sum_{|I|=k+1} u_{I,i}^\alpha \mu_I^\alpha \,.$$

The left side does only depend on jet coordinates up to order $k+1$, whereas the right side also depends linearly on the jet coordinates of order $k+2$. Since the equation must hold for all values of jet coordinates of order $k+2$, it follows that both sides must vanish independently. The right side vanishes if $\mu_I^\alpha = 0$ for $|I| = k+1$. The vanishing of the left side yields an expression for μ_i in terms of μ_I^α. We conclude that μ is vertical if and only if it is of the local form

$$\mu = \sum_{|I|=0}^{k} \mu_I^\alpha (\mathbf{d}u_I^\alpha - u_{I,i}^\alpha \mathbf{d}x^i) = \mu_I^\alpha \theta_\alpha^I \,,$$

where

$$\theta_I^\alpha := \mathbf{d}u_I^\alpha - u_{I,i}^\alpha \mathbf{d}x^i \,.$$

The 1-forms $\theta_\alpha^I \in \Omega^1(J^{|I|+1}F)$ are linearly independent at every point, so that they are a local basis of the $C^\infty(J^{|I|+1}F)$-module $\Omega^{1,0}(J^{|I|+1}F)$.

Terminology 5.1.14. In the language of variational calculus, the 1-forms θ_α^I are called **contact forms**.

As $C^\infty(J^\infty F)$-module, $\mathfrak{X}_{\mathrm{vert}}(J^\infty F)$ is locally spanned by the infinite sums of the vertical coordinate vector fields $\{\frac{\partial}{\partial u_I^\alpha}\}$. This shows that the conditions

$$0 = \iota_{\frac{\partial}{\partial u_I^\alpha}}\mu = \mu_\alpha^I$$

for μ to be horizontal are satisfied if and only if μ is of the form $\mu = \mu_i \mathbf{d}x^i$. We have shown the following.

Lemma 5.1.15. *A local 1-form $\mu \in \Omega^1(J^\infty F)$ given in local coordinates by Equation* (5.15) *decomposes as $\mu = \mu_{\text{vert}} + \mu_{\text{hor}}$ into its vertical and horizontal components*

$$\mu_{\text{vert}} = \mu_\alpha^I \theta_I^\alpha\,, \quad \mu_{\text{hor}} = (\mu_i + \mu_\alpha^I u_{I,i}^\alpha)\mathbf{d}x^i\,. \tag{5.16}$$

A form $\omega \in \Omega^{p,q}(J^\infty F)$ is given in local coordinates by a finite sum

$$\omega = \omega_{\alpha_1,\dots\alpha_p,j_1,\dots,j_q}^{I_1,\dots,I_p} \theta_{I_1}^{\alpha_1} \wedge \dots \wedge \theta_{I_p}^{\alpha_p} \wedge \mathbf{d}x^{j_1} \wedge \dots \wedge \mathbf{d}x^{j_q}\,,$$

where the coefficients $\omega_{\alpha_1,\dots\alpha_p,j_1,\dots,j_q}^{I_1,\dots,I_p}$ are functions in $C^\infty(J^\infty F)$.

Let $\mathrm{pr}_{\Omega^{p,q}} : \Omega(J^\infty F) \to \Omega^{p,q}(J^\infty F)$ denote the projection onto the vector space of (p,q)-forms. The vertical component δ and the horizontal component d of the differential $\mathbf{d}$ are given by the linear maps

$$\begin{aligned} \delta^{p,q} &: \Omega^{p,q}(J^\infty F) \longrightarrow \Omega^{p+1,q}(J^\infty F)\,, \quad \delta^{p,q} := \mathrm{pr}_{\Omega^{p,q+1}} \circ \mathbf{d}|_{\Omega^{p,q}}\,, \\ d^{p,q} &: \Omega^{p,q}(J^\infty F) \longrightarrow \Omega^{p,q+1}(J^\infty F)\,, \quad d^{p,q} := \mathrm{pr}_{\Omega^{p,q+1}} \circ \mathbf{d}|_{\Omega^{p,q}}\,. \end{aligned}$$

Proposition 5.1.16. *The bigraded vector space with the vertical differential δ and the horizontal differential d is a differential bicomplex.*

Proof. This is a standard argument. We must show that $\mathbf{d} = \delta + d$ which implies that $\delta^2 = 0$, $d^2 = 0$, and $\delta d = -d\delta$. For $\mathbf{d}$ acting on functions this is clear by definition. For $\mathbf{d}|_{\Omega^{0,1}}$ we have

$$\begin{aligned} \mathbf{d}|_{\Omega^{0,1}} &= (\mathrm{pr}_{\Omega^{2,0}} + \mathrm{pr}_{\Omega^{1,1}} + \mathrm{pr}_{\Omega^{0,2}}) \circ \mathbf{d}|_{\Omega^{0,1}} \\ &= \mathrm{pr}_{\Omega^{2,0}} \circ \mathbf{d}|_{\Omega^{0,1}} + \delta + d\,, \end{aligned}$$

so we have to show that $\mathrm{pr}_{\Omega^{2,0}} \circ \mathbf{d}|_{\Omega^{0,1}} = 0$. Let $\mu \in \Omega^{0,1}(J^\infty F)$. Evaluated on two vertical vector fields $v, w \in \mathfrak{X}(J^\infty F)_{\text{vert}}$ the differential can be written as

$$\begin{aligned} (\mathbf{d}\mu)(v,w) &= v \cdot \mu(w) - w \cdot \mu(v) - \mu([v,w]) \\ &= -\mu([v,w])\,, \end{aligned}$$

where we have used that $\mu(v) = 0 = \mu(w)$ because μ is horizontal and v, w vertical. We see that $\mathrm{pr}_{\Omega^{2,0}} \circ \mathbf{d}|_{\Omega^{0,1}} = 0$ if and only if $\mathfrak{X}(J^\infty F)_{\text{vert}}$ is involutive. Analogously, $\mathrm{pr}_{\Omega^{0,2}} \circ \mathbf{d}|_{\Omega^{0,1}} = 0$ if and only if $\mathfrak{X}(J^\infty F)_{\text{hor}}$ is involutive. The spaces of vertical and horizontal vector fields are both involutive (Remark 5.1.11), so that $\mathbf{d}\omega = \delta\omega + d\omega$ for an arbitrary 1-form ω. Since functions and 1-forms generate the graded algebra $\Omega(J^\infty F)$, it follows that $\mathbf{d} = \delta + d$. $\square$

We can depict the variational bicomplex by the diagram

$$\begin{tikzcd} \vdots & \vdots & & \vdots \\ \Omega^{1,0}(J^\infty F) \arrow[u, "\delta"'] \arrow[r, "d"] & \Omega^{1,1}(J^\infty F) \arrow[u, "\delta"'] \arrow[r, "d"] & \cdots \arrow[r, "d"] & \Omega^{1,n}(J^\infty F) \arrow[u, "\delta"'] \\ \Omega^{0,0}(J^\infty F) \arrow[u, "\delta"'] \arrow[r, "d"] & \Omega^{0,1}(J^\infty F) \arrow[u, "\delta"'] \arrow[r, "d"] & \cdots \arrow[r, "d"] & \Omega^{0,n}(J^\infty F) \arrow[u, "\delta"'] \end{tikzcd} \tag{5.17}$$

where $n = \dim M$.

Terminology 5.1.17. The vertical differential δ is also called the **variation**. The horizontal differential d is also called the **spacetime differential**.

Let us compute the differentials in local coordinates. From Equation (5.16) we get

$$\begin{aligned}
\delta x^i &= (\mathbf{d}x^i)_{\text{vert}} = 0 \\
dx^i &= (\mathbf{d}x^i)_{\text{hor}} = \mathbf{d}x^i \\
\delta u^\alpha_I &= (\mathbf{d}u^\alpha_I)_{\text{vert}} = \theta^\alpha_I \\
du^\alpha_I &= (\mathbf{d}u^\alpha_I)_{\text{hor}} = u^\alpha_{I,i} dx^i \,.
\end{aligned}$$

For a function $f \in \Omega^{0,0}(J^\infty F)$ we thus obtain

$$\delta f = \Big(\frac{\partial f}{\partial x^i}\mathbf{d}x^i + \frac{\partial f}{\partial u^\alpha_I}\mathbf{d}u^\alpha_I\Big)_{\text{vert}} = \frac{\partial f}{\partial u^\alpha_I}\delta u^\alpha_I \,, \tag{5.18a}$$

$$df = \Big(\frac{\partial f}{\partial x^i}\mathbf{d}x^i + \frac{\partial f}{\partial u^\alpha_I}\mathbf{d}u^\alpha_I\Big)_{\text{hor}} = \frac{\partial f}{\partial x^i}dx^i + u^\alpha_{I,i}\frac{\partial f}{\partial u^\alpha_I}dx^i = (D_i f)\, dx^i \,. \tag{5.18b}$$

Using the relations $\delta^2 = 0$, $d^2 = 0$, and $\delta d = -d\delta$, we can easily compute the differentials of the coordinate 1-forms,

$$\begin{aligned}
\delta(dx^i) &= -d\delta x^i = 0 \\
d(dx^i) &= 0 \\
\delta(\delta u^\alpha_I) &= 0 \\
d(\delta u^\alpha_I) &= -\delta(du^\alpha_I) = -\delta(u^\alpha_{I,i}dx^i) = -\delta u^\alpha_{I,i} \wedge dx^i \,.
\end{aligned}$$

Using the formulas for the differentials of functions and coordinate 1-forms, as well as the fact that δ and d are derivations, we can compute the differentials of an arbitrary form $\omega \in \Omega^{p,q}(J^\infty F)$, which can be expressed in local coordinates as

$$\omega = \omega^{I_1\ldots I_p}_{\alpha_1\ldots\alpha_p i_1\ldots i_q} \delta u^{\alpha_1}_{I_1} \wedge \ldots \wedge \delta u^{\alpha_p}_{I_p} \wedge dx^{i_1} \wedge \ldots \wedge dx^{i_q} \,. \tag{5.19}$$

Here the coefficients $\omega^{I_1\ldots I_p}_{\alpha_1\ldots\alpha_p i_1\ldots i_q}$ are functions on $J^\infty F$. Note that the sum is finite, that is, there is a k such that the terms vanish for $|I| > k$.

The inner derivatives of the differentials with respect to the coordinate vector fields are

$$\begin{aligned}
\iota_{\frac{\partial}{\partial x^j}} dx^i &= \delta^i_j \\
\iota_{\frac{\partial}{\partial u^\beta_J}} dx^i &= 0 \\
\iota_{\frac{\partial}{\partial x^j}} \delta u^\alpha_I &= -u^\alpha_{I,j} \\
\iota_{\frac{\partial}{\partial u^\beta_J}} \delta u^\alpha_I &= \delta^\alpha_\beta \delta^J_I \,.
\end{aligned}$$

5.1.3 Strictly vertical and horizontal vector fields

We have seen in Section 5.1.2 that the product structure of $\mathcal{F} \times M$ induces a splitting of the tangent bundle of $J^\infty F$ into a horizontal and vertical subspace. The product structure $\mathcal{F} \times M$ enables us also to lift vector fields on $\mathcal{F}$ and vector fields on M to vector fields on $\mathcal{F} \times M$ by using the trivial connection of the bundles $\mathcal{F} \times M \to \mathcal{F}$ and $\mathcal{F} \times M \to M$, respectively. On finite-dimensional manifolds such lifts can be characterized infinitesimally as follows.

Proposition 5.1.18. *Let $X \times Y$ be a product of manifolds. Let d_X and d_Y be the differentials of the bicomplex $\Omega(X \times Y)$. A vector field $v \in \mathfrak{X}(X \times Y)$ is the lift of a vector field on X if and only if $[\iota_v, d_Y] = 0$.*

Proof. In local coordinates $(x^1, \ldots, x^p, y^1, \ldots, y^q)$ a vector field v is of the form

$$v = a^i(x,y)\frac{\partial}{\partial x^i} + b^i(x,y)\frac{\partial}{\partial y^i}\,,$$

which is the lift of a vector field on X if and only if the functions $\frac{\partial a^i}{\partial y^k} = 0$ and $b^i = 0$. For any function $f \in C^\infty(X \times Y)$ we have

$$[\iota_v, d_Y]f = \iota_v d_Y f = b^i \frac{\partial f}{\partial y^i}\,.$$

This shows that $[\iota_v, d_Y]f = 0$ for all functions f if and only if $b^i = 0$. For a 1-form $\mu = \alpha_i(x,y)dx^i + \beta_i(x,y)dy^i$ we have

$$\begin{aligned}
[\iota_v, d_Y]\mu &= (\iota_v d_Y + d_Y \iota_v)\mu \\
&= \iota_v\Big(\frac{\partial \alpha_i}{\partial y^j}dy^j \wedge dx^i + \frac{\partial \beta_i}{\partial y^j}dy^j \wedge dy^i\Big) + d_Y(a^i\alpha_i + b^i\beta_i) \\
&= \Big(\frac{\partial \alpha_i}{\partial y^j}(b^j dx^i - a^i dy^j) + \frac{\partial \beta_i}{\partial y^j}(b^j dy^i - b^i dy^j)\Big) \\
&\quad + \Big(\frac{\partial a^i}{\partial y^j}\alpha_i + a^i\frac{\partial \alpha_i}{\partial y^j} + \frac{\partial b^i}{\partial y^j}\beta_i + b^i\frac{\partial \beta_i}{\partial y^j}\Big)dy^j \\
&= \frac{\partial a^i}{\partial y^j}\alpha_i dy^j + \Big(\frac{\partial \alpha_i}{\partial y^j}b^j dx^i - \frac{\partial \beta_j}{\partial y^i}b^i dy^j + \frac{\partial b^i}{\partial y^j}\beta_i dy^j\Big)\,.
\end{aligned}$$

The first term vanishes for all 1-forms μ if and only if a^i does not depend on the y^i. The second term vanishes if and only if $b^i = 0$.

We conclude that v is a lift of a vector field on X if and only if $[i_v, d_Y]$ annihilates all functions and 1-forms. Since functions and 1-forms generate $\Omega(X \times Y)$ as $\mathbb{R}$-algebra and since $[\iota_v, d_Y]$ is a derivation, this is the case if and only if $[\iota_v, d_Y] = 0$. □

Definition 5.1.19. A vector field $v \in \mathfrak{X}(J^\infty F)$ will be called **strictly vertical** if $[\iota_v, d] = 0$ and **strictly horizontal** if $[\iota_v, \delta] = 0$.

Remark 5.1.20. A strictly vertical vector field v satisfies $\iota_v dx^\alpha = [\iota_v, d]x^\alpha = 0$, which shows that it is vertical. Analogously, a strictly horizontal vector field v satisfies $\iota_v \delta u_I^\alpha = [\iota_v, \delta]u_I^\alpha = 0$, which shows that it is horizontal.

Proposition 5.1.21. *For all strictly vertical vector fields ξ and ξ', we have the following graded Lie brackets:*

$$[\iota_\xi, \delta] = \mathcal{L}_\xi\,, \quad [\mathcal{L}_\xi, \iota_{\xi'}] = \iota_{[\xi,\xi']}\,, \quad [\mathcal{L}_\xi, \mathcal{L}_{\xi'}] = \mathcal{L}_{[\xi,\xi']}\,,$$
$$[\delta, \delta] = [\iota_\xi, \iota_{\xi'}] = [\mathcal{L}_\xi, \delta] = 0\,,$$

For all strictly horizontal vector fields X, X', we have

$$[\iota_X, d] = \mathcal{L}_X\,, \quad [\mathcal{L}_X, \iota_{X'}] = \iota_{[X,X']}\,, \quad [\mathcal{L}_X, \mathcal{L}_{X'}] = \mathcal{L}_{[X,X']}\,,$$
$$[d, d] = [\iota_X, \iota_{X'}] = [\mathcal{L}_X, d] = 0\,.$$

Moreover, we have the relations

$$\begin{aligned}
[\delta, d] = [\delta, \iota_X] = [\delta, \mathcal{L}_X] = 0\\
[\iota_\xi, d] = [\iota_\xi, \iota_X] = [\iota_\xi, \mathcal{L}_X] = 0\\
[\mathcal{L}_\xi, d] = [\mathcal{L}_\xi, \iota_X] = [\mathcal{L}_\xi, \mathcal{L}_X] = 0\,.
\end{aligned}$$

In other words, we have two commuting Cartan calculi, the vertical and the horizontal Cartan calculus on $\Omega(J^\infty F)$, each satisfying the relations of Proposition 4.3.10.

Proof. The relations follow directly from the relations of Proposition 4.3.10, from the fact that we have a bicomplex (Proposition 5.1.16), and from the Definition 5.1.19 of strictly vertical and horizontal vector fields. □

Lemma 5.1.22. *A vector field $v \in \mathfrak{X}(J^\infty F)$ is strictly horizontal if and only if it is of the local form*

$$v = v^i(x) D_i\,,$$

for smooth functions $v^i \in C^\infty(M)$.

Proof. Since $[\iota_v, \delta]$ is a derivation, it is zero if it vanishes on functions f and the coordinate 1-forms dx^i and δu_i^α, which generate the algebra $\Omega(J^\infty F)$ locally. In local coordinates, v is given by Equation (5.10), so we obtain

$$\begin{aligned}
[\iota_v, \delta] f &= \iota_v \frac{\partial f}{\partial u_I^\alpha} \delta u_I^\alpha\\
&= \frac{\partial f}{\partial u_I^\alpha}(v_I^\alpha - u_{I,i}^\alpha v^i)\,,
\end{aligned}$$

where we have used that $\delta u_I^\alpha = \theta_I^\alpha = \mathbf{d}u_I^\alpha - u_{I,i}^\alpha \mathbf{d}x^i$. This vanishes for all functions if and only if $v_I^\alpha = u_{I,i}^\alpha v^i$, i.e. if and only if v is of the form

$$v = v^i \frac{\partial}{\partial x^i} + u_{I,i}^\alpha v^i \frac{\partial}{\partial u_I^\alpha} = v^i D_i\,,$$

which means that v is horizontal. Next, we obtain

$$\begin{aligned}
[\iota_v, \delta] dx^i &= \iota_v \delta dx^i + \delta \iota_v dx^i\\
&= \frac{\partial v^i}{\partial u_I^\alpha} \delta u_I^\alpha\,,
\end{aligned}$$

which vanishes if and only if v^i does not depend on the fiber coordinates u_I^α. Finally, we get

$$[\iota_v, \delta]\delta u_I^\alpha = \delta \iota_v u_I^\alpha + \delta(\iota_v \delta u_I^\alpha)\,,$$

which vanishes when v is horizontal such that the expression in parentheses vanishes. This shows that the last equation does not yield an additional condition. We conclude that v is strictly horizontal if it is horizontal with the coefficient functions v^i depending only on the base coordinates x^i. □

Conceptually, strictly horizontal vector fields in $\mathfrak{X}(J^\infty F)$ play the role of the lifts of vector fields on M to vector fields on $\mathcal{F}\times M$. In fact, the strictly horizontal vector fields are the lifts

$$\begin{aligned}\mathfrak{X}(M) &\longrightarrow \mathfrak{X}(J^\infty F)\\ v^i(x)\frac{\partial}{\partial x^i} &\longmapsto v^i(x)D_i\,.\end{aligned}$$

of vector fields on M by the Cartan connection. An analogous interpretation of strictly vertical vector fields is not possible, since $J^\infty F$ is not a bundle over $\mathcal{F}$.

5.1.4 Equivalence of strictly vertical and local vector fields

In Theorem 2.3.4, we have shown that $T\mathcal{F} \cong \Gamma(M, VF)$. A vector field on $\mathcal{F}$ is given by a map

$$\xi : \Gamma(M,F) \longrightarrow \Gamma(M,VF)\,,$$

such that $(\pi_F)_*\xi = \mathrm{id}_{\mathcal{F}}$, where $\pi_F : VF \to F$ is the bundle projection. Since ξ is a map of fields, it makes sense to talk about local vector fields in the sense of Definition 3.2.1, a local vector field $\xi : \mathcal{F} \to T\mathcal{F}$ descends to a smooth map $v_0 : J^kF \to VF$ covering the identity on M, such that the diagram

$$\begin{array}{ccc}\mathcal{F}\times M & \xrightarrow{\xi\times\mathrm{id}_M} & T\mathcal{F}\times M\\ {\scriptstyle j^k_F}\big\downarrow & & \big\downarrow{\scriptstyle j^0_{VF}}\\ J^kF & \xrightarrow[v_0]{} & VF\end{array}$$

commutes. Since $(\pi_F)_*\xi = \mathrm{id}_{\mathcal{F}}$, the map v_0 covers the identity on F.

Terminology 5.1.23. *** A smooth map $v_0 : J^kF \to VF$, for some k, covering the identity of F is called an **evolutionary "vector field"**. Since there is no coordinate independent lift $VF \to TJ^kF$, it cannot be viewed as vector field on J^kF, which is why we put quotes around "vector field".

Remark 5.1.24. Every evolutionary "vector field" $v_0 : J^kF \to VF$ induces a local vector field ξ on $\mathcal{F}$ given by $\xi_\varphi := v_0 \circ j^k\varphi$ for all $\varphi \in \mathcal{F}$.

In order to view a local vector field on $\mathcal{F}$ as a vector field on $J^\infty F$, we have to prolong the corresponding evolutionary "vector field" $v_0 : J^kF \to VF$ to the map

$$v_l : J^{k+l}F \xrightarrow{\iota_{l,k}} J^l(J^kF) \xrightarrow{J^lv_0} J^lVF \xrightarrow{\tau_l} TJ^lF\,, \tag{5.20}$$

where $\iota_{l,k}$ is the embedding (3.9) of Lemma 3.1.26, where $J^lv_0 : J^l(J^k) \to J^lVF$ is the l-th prolongation of v_0 defined in Proposition 3.1.19, and where τ_l is the map defined in the proof of Theorem 5.1.4.

Proposition 5.1.25. *Let $v_0 : J^kF \to VF$ be a smooth map covering the identity of F, that is, an evolutionary "vector field". Then the smooth maps $v_l : J^{k+l}F \to TJ^lF$ of* (5.20) *represent a vector field $v : J^\infty F \to TJ^\infty F$, which is called the* ***infinite prolongation*** *of v_0.*

Proof. We have the following row of commutative squares

$$\begin{tikzcd}
J^{k+l+1}F \arrow[r, "\iota_{l+1,k}"] \arrow[d] & J^{l+1}(J^kF) \arrow[r, "j^{l+1}v_0"] \arrow[d] & J^{l+1}VF \arrow[r, "\tau_{l+1}"] \arrow[d] & TJ^{l+1}F \arrow[d, "T\mathrm{pr}_{l+1,l}"] \\
J^{k+l}F \arrow[r, "\iota_{l,k}"'] & J^l(J^kF) \arrow[r, "j^lv_0"'] & J^lVF \arrow[r, "\tau_l"'] & TJ^lF
\end{tikzcd}$$

where the unmarked vertical arrows are the obvious forgetful maps. The commutativity of the outer rectangle shows that the prolongations v_l represent a morphism $v : J^\infty F \to TJ^\infty F$ of pro-manifolds.

In order to show that v is a section of $TJ^\infty F \to J^\infty F$ we consider the following diagram:

$$\begin{tikzcd}
J^{k+l}F \arrow[r, "\iota_{l,k}"] \arrow[d, "\mathrm{pr}_{k+l,l}"] & J^l(J^kF) \arrow[r, "J^lv_0"] \arrow[d, "J^l\mathrm{pr}_{k,0}"] & J^lVF \arrow[r, "\tau_l"] \arrow[d, "J^l\pi_F"] & TJ^lF \arrow[d, "\pi_{J^lF}"] \\
J^lF \arrow[r, "\mathrm{id}"'] & J^lF \arrow[r, "\mathrm{id}"'] & J^lF \arrow[r, "\mathrm{id}"'] & J^lF
\end{tikzcd}$$

It follows from the definition of $\iota_{l,k}$ in Lemma 3.1.26 that the first square commutes. By assumption, v_0 covers the identity, $\pi_F \circ v_0 = \mathrm{pr}_{k,0}$. By applying the l-th prolongation functor we obtain $J^l\pi_F \circ J^lv_0 = J^l\mathrm{pr}_{k,0}$, which is the commutativity of the second square. The commutativity of the third square follows from the definition of τ_l. We conclude that the outer rectangle commutes, that is,

$$\pi_{J^lF} \circ v_l = \mathrm{pr}_{k+l,l}\,.$$

It follows from Proposition 4.3.4 that the maps v_l represent a section of $TJ^\infty F \to J^\infty F$. □

Theorem 5.1.26. *Let $F \to M$ be a smooth fiber bundle. Let $v : J^\infty F \to TJ^\infty F$ be a vector field on the pro-manifold $J^\infty F$. The following are equivalent:*

(i) v is strictly vertical.

(ii) v is the infinite prolongation of an evolutionary "vector field".

(iii) There is a unique local vector field on $\mathfrak{F}$ that projects to v.

The situation of Theorem 5.1.26 can be summarized in the following diagram of pro-diffeological spaces:

$$\begin{tikzcd}
\mathfrak{F} \times M \arrow[r, "\xi \times \mathrm{id}_M"] \arrow[d, "j^\infty_F"'] & T\mathfrak{F} \times M \arrow[d, "j^\infty_{VF}"] \\
J^\infty F \arrow[r, "v"] \arrow[d] & J^\infty(VF) \arrow[d] \\
J^kF \arrow[r, "v_0"'] & VF
\end{tikzcd}$$

Here, we have used that a vertical vector field $v : J^\infty F \to TJ^\infty F$ takes its values in the vertical tangent space $J^\infty(VF) \hookrightarrow TJ^\infty F$ as defined in Theorem 5.1.4. Theorem 5.1.26 states that given a strictly vertical vector field v, there is a unique ξ that makes this diagram commutative. The map v_0 is not determined uniquely by v. It is unique only if we require the jet order k to be minimal. In general, a local vector field ξ does not determine v or v_0 uniquely. In fact, if $\mathcal{F} = \emptyset$, then *any* v_0 and its prolongation v will make the diagram commutative. If we assume the jet evaluations to be surjective (see Lemma 3.1.12), then v is uniquely determined by ξ if we require k to be minimal. The proof of Theorem 5.1.26 relies on the following technical lemmas.

Notation 5.1.27. For every multi-index $I = (I_1, \dots, I_n)$ and $n = \dim M$, we denote

$$D_I := D_1^{I_1} D_2^{I_2} \cdots D_n^{I_n} \,.$$

In particular, $D_{i_1,\dots,i_k} = D_{i_1} \cdots D_{i_k}$.

Lemma 5.1.28. *A vector field $v \in \mathfrak{X}(J^\infty F)$ is strictly vertical if and only if it is of the form*

$$v = \sum_{|I|=0}^{\infty} (D_I v^\alpha) \frac{\partial}{\partial u_I^\alpha} \,,$$

for some functions $v^\alpha \in C^\infty(J^\infty F)$.

Proof. Let $v = \sum_{|I|=0}^{\infty} v_I^\alpha \frac{\partial}{\partial u_I^\alpha}$ be an arbitrary vector field on $J^\infty F$. Locally, the variational bicomplex is generated by the coordinate functions x^i, u_I^α and the coordinate 1-forms dx^i, δu_I^α. The operator $[\iota_v, d]$ is a derivation, so that it suffices to check the relation $[\iota_v, d] = 0$ on the generators. On x^i we obtain the condition

$$[\iota_v, d] x^i = \iota_v dx^i = v^i = 0 \,,$$

so that v must be vertical, as already noted. On u_I^α we obtain $[\iota_v, d]u_I^\alpha = \iota_v du_I^\alpha = \iota_v u_{I,i}^\alpha dx^i = u_{Ii}^\alpha v^i = 0$, which follows from the first condition. On the horizontal coordinate one forms we have $[\iota_v, d]dx^i = d\iota_v dx^i = dv^i = 0$ which also follows from the first equation. On the vertical coordinate 1-forms we get

$$\begin{aligned} [\iota_v, d]\delta u_I^\alpha &= \iota_v d\delta u_I^\alpha + d(\iota_v \delta u_I^\alpha) \\ &= \iota_v(-\delta u_{I,i}^\alpha \wedge dx^i) + dv_I^\alpha \\ &= -v_{I,i}^\alpha \, dx^i + v^i \delta u_{I,i}^\alpha + (D_i v_I^\alpha) \, dx^i \,. \end{aligned}$$

Assuming that $v^i = 0$ we obtain the condition

$$v_{I,i}^\alpha = D_i v_I^\alpha \,.$$

By induction, this implies that $v_{i_1,\dots,i_n}^\alpha = D_{i_1} \cdots D_{i_n} v^\alpha = D_{i_1,\dots i_k} v^\alpha$. This proves the lemma. □

Lemma 5.1.29. *Let $f : F \to \tilde{F}$ be a map of smooth fiber bundles over M covering the identity of M. Let x^i be local coordinates on a neighborhood U of $m \in M$, u^α fiber coordinates of F, and $\tilde{u}^\beta$ fiber coordinates of $\tilde{F}$, both over U. Then the k-th prolongation $J^k f : J^k F \to J^k \tilde{F}$ is given in the induced jet bundle coordinates by*

$$f_I^\beta = D_I f^\beta \,,$$

for all multi-indices I with $|I| \leq k$, where $f_I^\beta = \tilde{u}_I^\beta \circ J^k f$.

Proof. In Proposition 3.1.19 the k-th prolongation $J^k f$ was defined as the map that sends $j_m^k \varphi$ to $j_m^k(f \circ \varphi)$. In local coordinates we have

$$\begin{aligned}
(\tilde{u}^\beta_{i_1,\dots,i_l} \circ J^k f)(j_x^k \varphi) &= \tilde{u}^\beta_{i_1,\dots,i_l}\big((J^k f)(j_x^k \varphi)\big) \\
&= \tilde{u}^\beta_{i_1,\dots,i_l}\big(j_x^k (f \circ \varphi)\big) \\
&= \frac{\partial^l (f^\beta \circ \varphi)}{\partial x^{i_1} \cdots \partial x^{i_l}} \\
&= \frac{\partial^{l-1}}{\partial x^{i_1} \cdots \partial x^{i_{l-1}}} \frac{\partial (f^\beta \circ \varphi)}{\partial x^{i_l}} \\
&= \frac{\partial^{l-1}}{\partial x^{i_1} \cdots \partial x^{i_{l-1}}} \big[(D_{i_l} f^\beta) \circ j^1 \varphi\big] \\
&= \frac{\partial^{l-2}}{\partial x^{i_1} \cdots \partial x^{i_{l-2}}} \big[(D_{i_{l-1}} D_{i_l} f^\beta) \circ j^2 \varphi\big] \\
&= (D_{i_1} \cdots D_{i_l} f^\beta)(j_x^l \varphi) \,,
\end{aligned}$$

where in the last step we have repeatedly applied Equation (5.12). Note, that while the right side depends only on the l-jet of φ, it can be viewed as function on the k-jet. □

Lemma 5.1.30. *Let $\xi : \mathcal{F} \to T\mathcal{F}$ be a local vector field that descends to a smooth map $v_0 : J^k F \to VF$. Then ξ projects to the infinite prolongation $v : J^\infty F \to TJ^\infty F$ of v_0.*

Proof. Since v_0 is an evolutionary "vector field"(Terminology 5.1.23), it covers the identity of F. Moreover, as we have noted in Remark **??**, ξ is given in terms of v_0 by the relation

$$\xi_\varphi(m) = v_0(j_m^k \varphi) \,, \tag{5.21}$$

for all $(\varphi, m) \in \mathcal{F} \times M$. Let $\xi_\varphi \in T_\varphi \mathcal{F}$ be represented by the path $t \mapsto \varphi_t$ in $\mathcal{F}$, i.e. $\xi_\varphi = \dot{\varphi}_0$. Then the tangent map of $j^l : \Gamma(M, VF) \to VF$ is given by

$$\begin{aligned}
(Tj^l)(\xi_\varphi, m) = (Tj^l)(\dot{\varphi}_0, m) &= \frac{d}{dt}(j_m^l \varphi_t)\Big|_{t=0} \\
&= \tau_l(j_m^l \dot{\varphi}_0) = \tau_l(j_m^l \xi_\varphi) \\
&= \tau_l\big(j_m^l (v_0 \circ j^k \varphi)\big) \\
&= \big(\tau_l \circ J^l v_0 \circ j^l (j^k \varphi)\big)(m) \\
&= \big(\tau_l \circ J^l v_0 \circ \iota_{l,k} \circ j^{k+l}\big)(\varphi, m) \\
&= v_l(j_m^{k+l} \varphi) \,,
\end{aligned}$$

where we have used the definition of τ_l from the proof of Theorem 5.1.4 and the definition of $\iota_{l,k}$ from Lemma 3.1.26. This shows that the diagram

$$\begin{array}{ccc} \mathcal{F} \times M & \xrightarrow{\xi \times \mathrm{id}_M} & T\mathcal{F} \times M \\ {\scriptstyle j^{k+l}}\downarrow & & \downarrow{\scriptstyle Tj^l} \\ J^{k+l}F & \xrightarrow[v_l]{} & TJ^lF \end{array}$$

commutes for all $l \geq 0$. We conclude that ξ descends to the vector field on $J^\infty F$ that is represented by the prolongations v_l. $\square$

Proof of Theorem 5.1.26. Let $v_0 : J^kF \to VF$ be an evolutionary "vector field" given in local bundle coordinates by $v_0 = v_0^\alpha \frac{\partial}{\partial y^\alpha}$. It follows from Lemma 5.1.29 that the infinite prolongation $v = \sum_{|I|=0}^\infty v_I^\alpha \frac{\partial}{\partial u_I^\alpha}$ of v_0 is given by $v_I^\alpha = D_I v_0^\alpha$. Lemma 5.1.28 now implies that (i) and (ii) are equivalent.

Let $\xi : \mathcal{F} \to T\mathcal{F}$, $\xi \mapsto \xi_\varphi$ be a local vector field that descends to the smooth map $v_0 : J^kF \to VF$, that is, to an evolutionary "vector field" (Terminology 5.1.23). Conversely, we have noted in Remark 5.1.24 that for every evolutionary "vector field" v_0, there is a unique vector field ξ on $\mathcal{F}$ that descends to v_0. Moreover, we have shown in Lemma 5.1.30 that ξ projects to the infinite prolongation of v_0. We conclude that (ii) and (iii) are equivalent. $\square$

5.1.5 Basic forms

Definition 5.1.31. A differential form $\omega \in \Omega(J^\infty F)$ is called **horizontal** if $\iota_\xi \omega = 0$ for all vertical vector fields $\xi \in \mathfrak{X}(J^\infty F)$. It is called **vertically invariant** if $\mathcal{L}_\xi \omega = 0$ for all vertical vector fields ξ. A form that is horizontal and vertically invariant is called **basic**.

Proposition 5.1.32. *A differential form $\omega \in \Omega(J^\infty F)$ is basic if and only if it is the pullback of a form on the base manifold M by the projection $J^\infty F \to M$.*

Proof. Let $\omega \in \Omega^{0,q}(J^\infty F)$ be a horizontal form. In local coordinates we have $\omega = \omega_{i_1,\ldots,i_q}\, dx^{i_1} \wedge \ldots \wedge dx^{i_q}$, where the $\omega_{i_1,\ldots,i_q}$ are functions on $J^\infty F$. For the action of the Lie derivative with respect to a vertical coordinate vector field we get

$$\begin{aligned} \mathcal{L}_{\frac{\partial}{\partial u_I^\alpha}} \omega &= \frac{\partial}{\partial u_I^\alpha} \lrcorner (d+\delta)\omega \\ &= \frac{\partial}{\partial u_I^\alpha} \lrcorner \Big((D_j \omega_{i_1,\ldots,i_q}) dx^j \wedge dx^{i_1} \wedge \ldots dx^{i_q} \\ &\qquad + \sum_{|J|=0}^\infty \frac{\partial^{|J|} \omega_{i_1,\ldots,i_q}}{\partial u_J^\beta}\, \delta u_J^\beta \wedge dx^{i_1} \wedge \ldots dx^{i_q} \Big) \\ &= \frac{\partial^{|I|} \omega_{i_1,\ldots,i_q}}{\partial u_I^\alpha}\, dx^{i_1} \wedge \ldots \wedge dx^{i_q} . \end{aligned}$$

We conclude that, in local coordinates, $\omega = \omega_{i_1,\ldots,i_q}(x)\, dx^{i_1} \wedge \ldots \wedge dx^{i_q}$, that is, ω is the pullback of a form on M. For a general vertical vector field $\xi = \sum_{|I|=0}^\infty \xi_I^\alpha \frac{\partial}{\partial u_I^\alpha}$, we have $\mathcal{L}_\xi \omega = \iota_\xi \mathbf{d}\omega = \sum_{|I|=0}^{k+1} \xi_I^\alpha (\frac{\partial}{\partial u_I^\alpha} \lrcorner \mathbf{d}\omega) = 0$, where k is the maximal jet order of the coefficient functions. □

Remark 5.1.33. We can define a form $\omega \in \Omega(J^\infty F)$ to be horizontally basic if $\iota_v \omega = 0$ and $\mathcal{L}_v \omega = 0$ for all horizontal vector fields $v \in \mathfrak{X}(J^\infty F)$. However, it turns out that this condition is only satisfied by locally constant functions, so that it is not a useful concept.

5.2 Cohomology of the variational bicomplex

In our setup, the variational bicomplex consists of a bigraded commutative ind-algebra $\Omega(J^\infty F)$ with the vertical and horizontal derivations δ, which are elements of the graded Lie algebra of internal derivations $\underline{\mathrm{Der}}(\Omega(J^\infty F))$. In cohomology, it is more common to view the ind-bigraded algebra, which is represented by the sequence $\Omega(J^0 F) \to \Omega(J^1 F) \to \Omega(J^2 F) \to \ldots$, as filtration

$$\Omega(J^0 F) \subset \Omega(J^1 F) \subset \Omega(J^2 F) \subset \ldots \subset \bar{\Omega}(J^\infty F)\,,$$

of bigraded algebras, where

$$\bar{\Omega}(J^\infty F) := \operatorname*{colim}_{k \in \omega} \Omega(J^k F)$$

is the colimit in bigraded algebras. The multiplication of the algebra satisfies

$$\Omega(J^k F)\,\Omega(J^l F) \subset \Omega(J^{\max(j,l)} F)\,,$$

and the differentials satisfy

$$\delta \Omega^{p,q}(J^k F) \subset \Omega^{p+1,q}(J^k F)\,, \quad d\Omega^{p,q}(J^k F) \subset \Omega^{p,q+1}(J^{k+1} F)\,,$$

as can be deduced from the local coordinate expressions for δ and d. Viewing the variational ind-bicomplex as filtered bicomplex allows us to apply the method of spectral sequences without modification, although we will need only a very simple version of it.

5.2.1 Cohomological partial integration

Let $\alpha, \beta \in \Omega(M)$ be compactly supported differential forms, such that $d\alpha \wedge \beta \in \Omega^n(M)$ is a form of degree $n = \dim M$ that can be integrated over M. Then $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^{|\alpha|} \alpha \wedge d\beta$, so that by Stokes' theorem

$$\int_M d\alpha \wedge \beta = -\int_M (-1)^{|\alpha|} \alpha \wedge d\beta + \int_{\partial M} \alpha \wedge \beta\,.$$

If $\partial M = 0$, then the second term on the right side vanishes, so that we obtain the coordinate free version of partial integration. The procedure does not depend on taking the integrals and can be stated in terms of the integrands as

$$[d\alpha \wedge \beta] = -[(-1)^{|\alpha|} \alpha \wedge d\beta]\,,$$

where the brackets denote the cohomology classes. This formula, which holds for forms with arbitrary support and in all degrees, can be viewed as cohomological version of partial integration. It generalizes to the d-cohomology classes of the variational bicomplex and is an important step in the computation of its horizontal cohomology classes.

Using the local coordinate formulas for d, we get

$$\begin{aligned}\mathcal{L}_{D_i}\delta u^\alpha_I &= (\iota_{D_i}d + d\iota_{D_i})\delta u^\alpha_I = \iota_{D_i}(-\delta u^\alpha_{I,j}\wedge dx^j)\\ &= \delta u^\alpha_{I,i}\,.\end{aligned}\tag{5.22}$$

From Equation (5.22) we deduce the formula

$$\delta u^\alpha_I = \mathcal{L}_{D_I}\delta u^\alpha\,.$$

A form $\omega\in\Omega^{p,n}(J^\infty F)$ for $p>0$ can be written locally as

$$\omega = \delta u^\alpha_I\wedge\tau^I_\alpha\,,$$

where the $(p-1,n)$-forms τ^I_α are given by

$$\tau^I_\alpha = \frac{1}{p}\Big(\frac{\partial}{\partial u^\alpha_I}\lrcorner\,\omega\Big)\,,\tag{5.23}$$

Using the derivation property of the Lie derivative we get

$$\begin{aligned}\delta u^\alpha_{i_1,\dots,i_k}\wedge\tau^{i_1,\dots,i_k}_\alpha &= (\mathcal{L}_{D_{i_k}}\delta u^\alpha_{i_1,\dots,i_{k-1}})\wedge\tau^{i_1,\dots,i_k}_\alpha\\ &= -\delta u^\alpha_{i_1,\dots,i_{k-1}}\wedge\mathcal{L}_{D_{i_k}}\tau^{i_1,\dots,i_k}_\alpha + \mathcal{L}_{D_{i_k}}(\delta u^\alpha_{i_1,\dots,i_{k-1}}\wedge\tau^{i_1,\dots,i_k}_\alpha)\,,\end{aligned}\tag{5.24}$$

where there is no summation over repeated indices. Since τ^I_α is of top horizontal degree, the second term on the right side is exact, so that Equation (5.24) can be viewed as a cohomological version of partial integration. Applying Equation (5.24) recursively to the first term on the right side, we obtain

$$\begin{aligned}\delta u^\alpha_{i_1,\dots,i_k}\wedge\tau^{i_1,\dots,i_k}_\alpha &= \delta u^\alpha\wedge(-1)^k(\mathcal{L}_{D_{i_1}}\cdots\mathcal{L}_{D_{i_k}}\tau^{i_1,\dots,i_k}_\alpha)\\ &\quad+\sum_{l=1}^k(-1)^{k-l}\mathcal{L}_{D_{i_l}}\big(\delta u^\alpha_{i_1,\dots,i_{l-1}}\wedge(\mathcal{L}_{D_{i_{l+1}}}\cdots\mathcal{L}_{D_{i_k}}\tau^{i_1,\dots,i_k}_\alpha)\big)\,.\end{aligned}\tag{5.25}$$

We will now rewrite this equation in multi-index notation. Using Equation (3.3), we get

$$\sum_k\sum_{i_1,\dots,i_k}\frac{[i_1,\dots,i_k]!}{k!}\,\delta u^\alpha_{i_1,\dots,i_k}\wedge\tau^{i_1,\dots,i_k}_\alpha = \omega\,.$$

The sum of the first term on the right side of Equation (5.25) is given by

$$\begin{aligned}P\omega :&= \sum_k\sum_{i_1,\dots,i_k}\frac{[i_1,\dots,i_k]!}{k!}\,(-1)^k\delta u^\alpha\wedge(\mathcal{L}_{D_{i_1}}\cdots\mathcal{L}_{D_{i_k}}\tau^{i_1,\dots,i_k}_\alpha)\\ &= \delta u^\alpha\wedge\sum_I(-1)^{|I|}\mathcal{L}_{D_I}\tau^I_\alpha\,.\end{aligned}$$

Using Equation (5.23), we can write this as

$$P\omega := \delta u^\alpha \wedge \frac{1}{p}\sum_I (-1)^{|I|}\mathcal{L}_{D_I}\Big(\frac{\partial}{\partial u^\alpha_I} \lrcorner\, \omega\Big)\,. \tag{5.26}$$

Since the second term of the right side of Equation (5.24) is exact, the sum is also exact. We conclude that in local coordinates every form $\omega \in \Omega^{p,n}(J^\infty F)$, $p > 0$, can be written as

$$\omega = P\omega + d\eta\,,$$

for some $\eta \in \Omega^{p,n-1}(J^\infty F)$.

Theorem 5.2.1 (Thm. 2.12 in [And89]). *Let $F \to M$ be a smooth fiber bundle over an n-dimensional manifold. There is a unique family of linear operators $P : \Omega^{p,n}(J^\infty F) \to \Omega^{p,n}(J^\infty F)$, $p > 0$, which is defined in local coordinates by Equation* (5.26). *It has the following properties:*

(i) *$\omega - P\omega$ is locally d-exact for all $\omega \in \Omega^{p,n}(J^\infty F)$, $p > 0$.*

(ii) *P is a projection, $P^2 = P$.*

(iii) $Pd = 0$.

(iv) $(P\delta)^2 = 0$.

Definition 5.2.2. The operator $\Omega^{p,n}(J^\infty F) \to \Omega^{p+1,n}(J^\infty F)$, $\omega \mapsto P\delta\omega$ is called the **Euler operator** and denoted by $E := P\delta$.

Property (iv) states that E is a differential operator. Forms in $P\Omega^{1,n}(J^\infty E)$ are called **source forms**. More generally, forms in the image of P are sometimes called **functional forms** [And89]. Properties (i)-(iv) are local.

5.2.2 The acyclicity theorem

Theorem 5.2.3 (Thm. 5.1 in [And89]). *For $p > 0$, the augmented horizontal complex*

$$0 \to \Omega^{p,0}(J^\infty F) \xrightarrow{d} \Omega^{p,1}(J^\infty F) \xrightarrow{d} \dots \xrightarrow{d} \Omega^{p,n}(J^\infty F) \xrightarrow{P} \Omega^{p,n}_{\text{fun}}(J^\infty F) \to 0$$

is exact.

Corollary 5.2.4. *Let P be the partial integration operator of Theorem 5.2.1; let $\omega \in \Omega^{p,n}(J^\infty F)$ for $p > 0$. Then $\omega - P\omega$ is d-exact.*

The rest of this section is devoted to the proof of this theorem. We first prove local exactness by the construction of explicit homotopy operators. In a second step we use a partition of unity and the generalized Mayer-Vietoris sequence to deduce global exactness.

Proposition 5.2.5. *Let $F = \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n = M$ a trivial vector bundle. Then the complex of Theorem 5.2.3 is exact.*

5.2.3 The cohomology of the Euler–Lagrange complex

Theorem 5.2.6. *The cohomology of the Euler–Lagrange complex*

$$\begin{gathered}0 \longrightarrow \Omega^{0,0}(J^\infty F) \xrightarrow{d} \Omega^{0,1}(J^\infty F) \xrightarrow{d} \dots \\ \dots \xrightarrow{d} \Omega^{0,n-1}(J^\infty F) \xrightarrow{d} \Omega^{0,n}(J^\infty F) \xrightarrow{P\delta} \Omega^{1,n}_{\mathrm{fun}}(J^\infty F) \xrightarrow{P\delta} \Omega^{2,n}_{\mathrm{fun}}(J^\infty F) \longrightarrow \dots\end{gathered}$$

where $n = \dim M$*, is isomorphic to the de Rham cohomology of the manifold* F*, that is,*

$$H^q\big(\Omega^{0,\bullet}(J^\infty F), d\big) \cong H^q(F)\,, \qquad 0 \le q \le n-1 \tag{5.27a}$$

$$\frac{\ker\big(P\delta : \Omega^{0,n}(J^\infty F) \to \Omega^{1,n}_{\mathrm{fun}}(J^\infty F)\big)}{d\big(\Omega^{0,n-1}(J^\infty F)\big)} \cong H^n(F) \tag{5.27b}$$

$$H^p\big(\Omega^{\bullet,n}_{\mathrm{fun}}(J^\infty F), P\delta\big) \cong H^{n+p}(F)\,, \qquad p \ge 1\,. \tag{5.27c}$$

Warning 5.2.7. In Equation (5.26a) of [And89, Thm. 5.9], it is erroneously claimed that (5.27a) holds for $q = n$. (This would imply that the horizontal cohomology of closed forms in $\Omega^{0,n}(J^\infty F)$ for a vector bundle F over a non-compact manifold M vanishes.) The correct statement is Equation (5.27b).

Exercises

In Exercises 5.1 through 5.4 we consider the following situation: Let V and H be smooth manifolds. Recall, that every vector field $X \in \mathfrak{X}(V \times H)$ splits as $X = X_V + X_H$ into a vector field X_V in the direction of V and a vector field X_H in the direction of H. The de Rham complex $\Omega(V \times H)$ is a bicomplex, that is, the ring has a bigrading and the de Rham differential splits as $d = d_V + d_H$ into a differential d_V of bidegree $(1,0)$ and a differential d_H of bidegree $(0,1)$, which graded commute $d_V d_H = -d_H d_V$. We will call V the vertical and H the horizontal manifold, X_V a vertical and X_H a horizontal vector field, d_V the vertical and d_H the horizontal differential, etc.

Exercise 5.1. Let $(x^1, \dots, x^m)$ be local coordinates on V and $(y^1, \dots, y^n)$ local coordinates on H.

(i) Express a vector field $X \in \mathfrak{X}(V \times H)$, its vertical component X_V, and its horizontal component X_H in local coordinates.

(ii) Let α be a (p,q)-form in $\Omega(V \times H)$. Express α, $d\alpha$, $d_V\alpha$, and $d_H\alpha$ in local coordinates.

Exercise 5.2. A form $\alpha \in \Omega(V \times H)$ is called **horizontal** if $\iota_X\alpha = 0$ for all vertical vector fields X. It is called **vertically invariant** if $\mathcal{L}_X\alpha = 0$ for all vertical vector fields X. It is called **horizontally basic** if it is both, horizontal and vertically invariant.

Show that α is horizontally basic if and only if it is the pullback of a form on H by the projection $\mathrm{pr}_H : V \times H \to H$.

Exercise 5.3. The trivial fiber bundle $\mathrm{pr}_H : V \times H \to H$ is equipped with the trivial connection, so that every vector field on H has a lift to a vector field on $V \times H$. A vector field X on $V \times H$ will be called **strictly horizontal** if $[\iota_X, d_V] = 0$. (Here ι_X is the inner derivative and the bracket denotes the graded commutator.)

Show that X is strictly horizontal if and only if it is the lift of a vector field on H.

Exercise 5.4. Recall that the Cartan calculus on $\Omega(V \times H)$ consists of the graded derivations d, ι_X, and $\mathcal{L}_X$ for all vector fields X on $V \times H$, satisfying the usual commutation relations.

(i) Show that the graded derivations d_H, ι_X, and $\mathcal{L}_X$ for strictly horizontal vector fields X satisfy the commutation relations of a Cartan calculus. (We call this the horizontal Cartan calculus. There is an analogous vertical Cartan calculus.)

(ii) Show that the graded commutator of any derivation of the horizontal Cartan calculus with any derivation of the vertical Cartan calculus vanishes.

Exercise 5.5. Let $\omega \in \Omega(J^\infty F)$ be a vertical form such that $\mathcal{L}_v \omega = 0$ for all horizontal vector fields $v \in \mathfrak{X}(J^\infty F)$. Show that ω is a locally constant function.

Exercise 5.6. Let $C^k \subset TJ^kF$ be the Cartan distribution of Exercise 14. Let C be the pro-manifold represented by $C^0 \leftarrow C^1 \leftarrow \ldots$ where the arrows are the tangent maps of the forgetful maps.

(a) Show that the inclusions $C^k \to TJ^kF$ represent a morphism of bundles of pro-manifolds over $J^\infty F$.

(b) Show that C is a vector subbundle, that is, at every point $x : * \to J^\infty F$, the fiber C_x is a vector subspace of $T_x J^\infty F$.

(c) Compute the rank of C, that is, the dimension of the fibers C_x.

(d) Show that a vector field $v : J^\infty F \to TJ^\infty F$ is horizontal if and only if it factors through C.

(e) Show that C is integrable, that is, an involutive subbundle of $TJ^\infty F$.

Exercise 5.7. Show that every vector field $v \in \mathfrak{X}(J^\infty F)$ that leaves the Cartan distribution invariant is of the form $v = \xi + X$ where ξ is strictly vertical (Definition 5.1.19) and X is horizontal.

Chapter 6

The cohomological action principle

Recall from Definition 1.3.2 that a lagrangian is a smooth map $L : \mathcal{F} \to \Omega^{\text{top}}(M)$. When M is closed we can define the action integral by

$$S(\varphi) := \int_M L(\varphi)\,, \tag{6.1}$$

The action principle states that the critical points of S are the solutions of the equations of motion. If L is a local map, then the critical points of the action are the solutions of a PDE, the Euler–Lagrange equation. We will give a proof of this statement in Theorem 6.2.8, using the diffeological framework for the differential geometry of the space $\mathcal{F}$.

When M is not compact, the action integral will generally not be defined for all fields. We might hope that we can circumvent this problem by restricting the action to the subspace of fields for which it is defined. However, this restriction will generally not be a smooth map (Exercise 2.7). Moreover, the condition that the action integral is defined may exclude almost all solutions of the field equations, as is the case in classical mechanics.

For a better approach, we observe that for the derivation of the Euler–Lagrange equation we only need to be able to discard d-exact terms under the integral. This suggests that the action principle may be reformulated as a cohomological statement about the integrand. In a first attempt at such a cohomological formuation, we could look at the map

$$\begin{aligned} \mathcal{F} &\longrightarrow H^n(M) \\ \varphi &\longmapsto [L(\varphi)]\,, \end{aligned} \tag{6.2}$$

where n is the dimension of M and where the bracket denotes the de Rham cohomology class in $H^n(M)$. When M is a closed, connected, and orientable manifold, then $H^n(M) \cong \mathbb{R}$. Once we have chosen a volume form as generator of $H^n(M)$, the map (6.2) is the action divided by the total volume of M. When M is non-compact, however, $H^n(M) = 0$ so that (6.2) is the zero map. We conclude that we cannot simply replace the integral of the action by the cohomology class in $\Omega(M)$.

In order to obtain a mathematically rigorous and general action principle that holds for M non-compact, we have to reformulate the notions of lagrangian, action, critical point, symmetry, etc. within the cohomology of $\mathcal{F} \times M$ and, locally, the cohomology of $J^\infty F$. It is straightforward to interpret the lagrangian as a $(0,n)$-form on $\mathcal{F} \times M$. The integration over M should then be replaced by the cohomology in the direction of M. This suggests the following dictionary:

	Analysis	Cohomology
Lagrangian	$L : \mathcal{F} \to \Omega^n(M)$	$L \in \Omega^{0,n}(\mathcal{F} \times M)$
Action	$S = \int_M L : \mathcal{F} \to \mathbb{R}$	$[L]_d \in H_d^{0,\mathrm{n}}(\mathcal{F} \times M)$
Symmetry $\Phi \in \mathrm{Diff}(\mathcal{F})$	$\Phi^* S = S$	$\Phi^* L = L + d\alpha$
Inf. symmetry $\xi \in \mathfrak{X}(\mathcal{F})$	$\mathcal{L}_\xi S = 0$	$\mathcal{L}_\xi L = d\alpha$
critical point $\varphi \in \mathcal{F}$	$\delta_\varphi S = 0$	?

Here d is the horizontal differential, $\mathcal{L}_\xi S$ the Lie derivative, and $\delta S : T\mathcal{F} \to \mathbb{R}$ the differential, which can all be understood rigorously in terms of the diffeological Cartan calculus. What is still missing is the cohomological version of the notion of critical point of the action.

6.1 Local diffeological forms

6.1.1 Differential forms on elastic diffeological spaces

A differential k-form on an elastic diffeological space X can be viewed as multilinear and antisymmetric morphism

$$\alpha : \underbrace{TX \times_X \ldots \times_X TX}_{=:T_k X} \longrightarrow \mathbb{R}\,.$$

It is straightforward to define the **inner derivative** $\iota_v \alpha$ with respect to a vector field $v : X \to TX$ by precomposing with $v \times \mathrm{id} : T_{k-1}X \to T_k X$. The evaluation of the resulting $(k-1)$-form at the tangent vectors $w_x^1, \ldots, w_x^{k-1} \in T_x X$ is given by

$$(\iota_v \alpha)(w_x^1, \ldots, w_x^{k-1}) = \alpha\big(v(x), w_x^1, \ldots, w_x^{k-1}\big)\,.$$

Similarly, the **evaluation of α at** $x \in X$ is given by the restriction

$$\alpha_x : (T_x X)^k \longrightarrow \mathbb{R}\,,$$

to the fiber $\{x\} \times_X T_k X \cong (T_x X)^k$.

The differential of a 0-form, that is, a function $f : X \to \mathbb{R}$ is given by

$$df : TX \xrightarrow{Tf} T\mathbb{R} \cong \mathbb{R} \times \mathbb{R} \xrightarrow{\mathrm{pr}_2} \mathbb{R}\,,$$

where pr_2 is the projection to the tangent fiber of $T\mathbb{R}$. The differential of a higher form $\alpha : T_k X \to \mathbb{R}$ is more difficult to describe ***. It is easier to use the equivalent description of the form as a family of differential forms $\{\alpha_p \in \Omega(U)\}$ on all plots $p : U \to X$ that is compatible with the pullbacks along morphisms of plots, $f^*\alpha_p = \alpha_{f^*p}$ where $f : V \to U$ is a smooth map. The differential of α is now given by the family of differentials $\{d\alpha_p\}$.

The de Rham complex of a product $X \times Y$ of elastic diffeological spaces is a bicomplex. A (p,q)-form is given by a morphism

$$\alpha : T_p X \times T_q Y \longrightarrow \mathbb{R}$$

that is multilinear and antisymmetric with respect to the action of the product $S_p \times S_q$ of the symmetric groups. Using that $\mathcal{D}\mathrm{flg}$ has all exponential objects, we can view α as a morphism $T_p X \to \underline{\mathcal{D}\mathrm{flg}}(T_q Y, \mathbb{R})$. The adjunction between the product and the exponential space preserves multilinearity and antisymmetry, so that a (p, q)-form can be equivalently viewed as a multilinear and antisymmetric morphism

$$\alpha : T_p X \longrightarrow \Omega^q(Y)\,, \tag{6.3}$$

where $\Omega^q(Y)$ is equipped with the functional diffeology. In other words, a (p, q)-form on $X \times Y$ can be viewed as a p-form on X with values in $\Omega^q(Y)$,

$$\Omega^{p,q}(X \times Y) \cong \Omega^p\big(X, \Omega^q(Y)\big)\,.$$

The evaluation of the $\Omega^q(Y)$-valued p-form (6.3) at $x \in X$ will be called the **evaluation of α at $x \in X$** and denoted by α_x. If $\alpha_x : (T_x X)^p \to \Omega^q(Y)$ is the zero map, α will be said to **vanish at** x.

Remark 6.1.1. The evaluation of $\alpha \in \Omega^{0,q}(X \times Y)$ at $x \in X$ is the pullback of α to $\{x\} \times Y \hookrightarrow X \times Y$.

The Y-differential of a (p, q)-form is given in terms of the morphism (6.3) by

$$d_Y \alpha : T_p X \xrightarrow{\alpha} \Omega^q(Y) \xrightarrow{(-1)^p d_Y} \Omega^{q+1}(Y)\,.$$

The X-differential of a $(0, q)$-form is given by

$$d_X \alpha : TX \xrightarrow{T\alpha} T\Omega^q(Y) \cong \Omega^q(Y) \times \Omega^q(Y) \xrightarrow{\mathrm{pr}_2} \Omega^q(Y)\,,$$

where pr_2 is the projection to the fiber of $T\Omega^q(Y) \to \Omega^q(Y)$.

Definition 6.1.2. A form $\alpha \in \Omega^{p,q}(X \times Y)$ will be called **d_Y-closed at** $x \in X$ if $d_Y \alpha$ vanishes at x. It will be called **d_Y-exact at** x if there is a $(p, q - 1)$-form β such that $\alpha - d_Y \beta$ vanishes at x.

6.1.2 Local forms on $\mathcal{F} \times M$

A (p, q)-form $\alpha \in \mathcal{F} \times M$ can be viewed equivalently as a morphism (6.3) of diffeological spaces,

$$\alpha : T_p \mathcal{F} \longrightarrow \Omega^q(M)\,. \tag{6.4}$$

Proposition 6.1.3. *The multilinear and antisymmetric map* (6.4) *is local (Definition 3.2.1) if and only if, viewed as (p, q)-form on $\mathcal{F} \times M$, it is the pullback along j^∞ of a (p, q)-form on $J^\infty F$.*

Proof. By definition, the map (6.4) is local if and only if it descends to a map

$$\alpha_0 : J^k(V_p F) \longrightarrow \wedge^q T^* M\,,$$

where $V_p F = VF \times_M \ldots \times_M VF$ is the p-fold fiber product. Identifying T^*M with a fiber-wise linear map $TM \to \mathbb{R}$ and then using the adjunction between $\mathcal{M}\mathrm{fld}(TM, \_)$ and $\_ \times TM$, we can identify α_0 with a map

$$\tilde{\alpha} : J^k(V_p F) \times_M T_q M \longrightarrow \mathbb{R}$$

that is fiber-wise linear and antisymmetric in the components of V_pF and T_qM. The domain of $\tilde{\alpha}$ can be written as

$$J^k(V_pF) \times_M T_qM \cong J^k(V_pF) \times_{J^kF} (J^kF \times_M T_qM)\,.$$

From (5.13) we see that $\tilde{\alpha}$ can be viewed as a (p,q)-form on $J^{k+1}F$. By Theorem 5.1.4, we obtain a commutative diagram

$$\begin{tikzcd}
(T_p\mathcal{F} \times M) \times_{\mathcal{F}\times M} (\mathcal{F} \times T_qM) \arrow[r, "\alpha"] \arrow[d, "T_pj^\infty \times T_qj^\infty"'] & \mathbb{R} \\
J^\infty(V_pF) \times_{J^\infty F} (J^\infty F \times_M T_qM) \arrow[ur, "\tilde{\alpha}"'] &
\end{tikzcd}$$

This shows that α is the pullback of $\tilde{\alpha}$ by j^∞. □

We will denote the space of local (p,q)-forms on $\mathcal{F} \times M$ by

$$\Omega^{p,q}_{\mathrm{loc}}(\mathcal{F} \times M) := (j^\infty)^*\Omega^{p,q}(J^\infty F)\,.$$

As is the case for any pullback of differential forms, $(j^\infty)^*$ commutes with the differentials. Moreover, since by Theorem 5.1.4 $(j^\infty)^*$ preserves the bigrading, it commutes with the vertical and horizontal differential separately,

$$(j^\infty)^*\delta\alpha = \delta(j^\infty)^*\alpha\,, \qquad (j^\infty)^*d\alpha = d(j^\infty)^*\alpha\,,$$

for all $\alpha \in \Omega(J^\infty F)$. This can be stated as follows.

Proposition 6.1.4. *The pullback $(j^\infty)^* : \Omega(J^\infty F) \to \Omega(\mathcal{F} \times M)$ is a morphism of bicomplexes.*

Remark 6.1.5. If the evaluation j^0 is surjective, then it follows from Theorem 5.1.4 that $(j^\infty)^*$ is injective so that we can identify $\Omega_{\mathrm{loc}}(\mathcal{F} \times M)$ with the variational bicomplex $\Omega(J^\infty F)$. In general, however, the bicomplex of local forms is a quotient of the variational bicomplex.

The evaluation of a (p,q)-form at $\varphi \in \mathcal{F}$ is given by the restriction of the map $\alpha : T_p\mathcal{F} \to \Omega^q(M)$ to the fiber $(T_\varphi\mathcal{F})^p \cong \{\varphi\} \times_{\mathcal{F}} T_p\mathcal{F}$. If α is local, so that it descends to $\alpha_0 : J^k(VF) \to \wedge^qT^*M$, we have the commutative diagram

$$\begin{tikzcd}
(T_\varphi\mathcal{F})^p \times M \arrow[r] \arrow[d, "j^k"] & T_p\mathcal{F} \times M \arrow[r, "\alpha\times\mathrm{id}_M"] \arrow[d, "j^k"] & \Omega^q(M) \times M \arrow[d, "j^0"] \\
M \times_{J^kF} J^k(V_pF) \arrow[r] & J^k(V_pF) \arrow[r, "\alpha_0"'] & \wedge^qT^*M
\end{tikzcd}$$

where the first vertical arrow is the jet evaluation of the bundle $\varphi^*V_pF \to M$, the second vertical arrow the jet evaluation of $V_pF \to M$, and the third vertical arrow the evaluation of $\wedge^qT^*M \to M$. This suggests the following notion.

Definition 6.1.6. The **evaluation at** $\varphi \in \mathcal{F}$ of a (p,q)-form on $J^\infty F$ given by a multilinear and antisymmetric map $\omega : J^\infty(V_p F) \to \wedge^q T^*M$ is the restriction

$$\omega_\varphi : M \times_{J^\infty F} J^\infty(V_p F) \longrightarrow \wedge^q T^*M\,.$$

We say that ω is **zero at** φ or **vanishes at** φ if ω_φ is the zero map. The condition $\omega_\varphi = 0$ is the **PDE of the form** ω.

In local coordinates, ω is given by (5.19) so that the PDE of ω is the system of equations

$$\omega^{I_1 \dots I_p}_{\alpha_1 \dots \alpha_p i_1 \dots i_q}\Big(\varphi^\alpha, \frac{\partial \varphi^\alpha}{\partial x^{i_1}}, \dots, \frac{\partial^k \varphi^\alpha}{\partial x^{i_1} \cdots \partial x^{i_k}}\Big) = 0\,.$$

Definition 6.1.7. A form $\omega \in \Omega^{p,q}(J^\infty F)$ will be called d**-closed at** $\varphi \in \mathcal{F}$ if $d\omega$ vanishes at φ. It will be called d**-exact at** φ if there is a $(p,q-1)$-form β such that $\omega - d\beta$ vanishes at φ.

Proposition 6.1.8. *Let $\omega \in \Omega^{p,q}(J^\infty F)$ and $\varphi \in \mathcal{F}$. Then:*

(i) ω vanishes at φ if and only if $(j^\infty)^*\omega \in \Omega^{p,q}(\mathcal{F} \times M)$ vanishes at φ.

(ii) ω is d-closed at φ if and only if $(j^\infty)^*\omega$ is d-closed at φ.

(iii) If ω is d-exact at φ, then $(j^\infty)^*\omega$ is d-exact at φ.

Proof. Let $\tilde{v} \in T_{j^k_m\varphi} J^k F$. By working in a tubular neighborhood of $\varphi(M) \subset F$ we can find a path $t \mapsto (\psi_t, m_t) \in \mathcal{F} \times M$ such that $\psi_0 = \varphi$ and $\frac{d}{dt} j^k_{m(t)} \psi_t = \tilde{v}$. This shows that $v := (\dot{\psi}_0, \dot{m}_0) \in T_\varphi \mathcal{F} \times M$ is mapped by Tj^k to $\tilde{v}$. We conclude that $Tj^\infty : T_\varphi\mathcal{F} \times T_m M \to T_{j^\infty_m\varphi} J^\infty F$ is surjective. By definition of the pullback,

$$\big((j^\infty)^*\omega\big)_\varphi(v^1, \dots, v^{p+q}) = \omega_\varphi(Tj^\infty v^1, \dots, Tj^\infty v^{p+2})$$

for all $v^1, \dots, v^{p+q} \in T_\varphi\mathcal{F} \times T_m M$. Since Tj^∞ is surjective at (φ, m) for all $m \in M$, the left side vanishes for all $v^1, \dots, v^{p+q}$ if and only if the right side does. This proves (i).

Since $(j^\infty)^*$ is a morphism of bicomplexes,

$$d(j^\infty)^*\omega = (j^\infty)^* d\omega\,.$$

By definition, $(j^\infty)^*\omega$ is d-closed at φ if and only if $d(j^\infty)^*\omega$ vanishes at φ. By (i), $(j^\infty)^* d\omega$ vanishes at φ if and only if $d\omega$ vanishes at φ. By definition, this is the case if ω is d-closed, which proves (ii).

Assume that there is a form $\alpha \in \Omega^{p,q-1}(J^\infty F)$, such that $\omega - d\alpha$ vanishes at φ. Since

$$(j^\infty)^*\omega - d(j^\infty)^*\alpha = (j^\infty)^*(\omega - d\alpha)\,,$$

it follows from (i) that the left side vanishes at φ, so that $(j^\infty)^*\omega$ is d-exact at φ. This proves (iii). □

Proposition 6.1.9. *Let $\omega \in \Omega^{p,q}(J^\infty F)$ for $p > 0$ and $q < n = \dim M$. If ω is d-closed at $\varphi \in \mathcal{F}$, then ω is d-exact at φ.*

Proof. *** □

6.2 The action principle

6.2.1 Euler–Lagrange form

It follows from Proposition 6.1.3 that a lagrangian $\tilde{L} : \mathcal{F} \to \Omega^n(M)$ is local if and only if it is the pullback of a form $L \in \Omega^{0,n}(J^\infty F)$. It is convenient to formulate the notion of local lagrangian field theory in terms of L.

Definition 6.2.1. A **local lagrangian field theory** is given by a manifold M, a fiber bundle $F \to M$, and a form $L \in \Omega^{0,n}(J^\infty F)$, where $n = \dim M$, called the **lagrangian form**.

A lagrangian form is given in local coordinates by

$$L = L(x^i, u^\alpha, \dots, u^\alpha_{i_1,\dots,i_k})\, dx^1 \wedge \dots \wedge dx^n\,,$$

where k is the jet order of L. When we evaluate the lagrangian $\tilde{L} := (j^\infty)L$ at $\varphi \in \mathcal{F}$, we obtain

$$\tilde{L}(\varphi) = L\Big(x^i, \varphi^\alpha, \frac{\partial \varphi^\alpha}{\partial x^{i_1}}, \dots, \frac{\partial^k \varphi^\alpha}{\partial x^{i_1} \cdots \partial x^{i_k}}\Big)\, dx^1 \wedge \dots \wedge dx^n\,,$$

which is the usual expression for the integrand of the action integral found in physics textbooks.

Definition 6.2.2. Let L be a lagrangian form. The form

$$EL \in \Omega^{1,n}(J^\infty F)\,,$$

where $E = P\delta$ is the Euler operator (Definition 5.2.2), is called the **Euler–Lagrange form** of L. The PDE

$$EL_\varphi = 0 \tag{6.5}$$

is called the **Euler–Lagrange equation**.

In local coordinates, the Euler–Lagrange form is given by

$$EL = E_\alpha \delta u^\alpha \wedge dx^1 \wedge \dots \wedge dx^n\,, \tag{6.6}$$

where $E_\alpha = E_\alpha(x^i, u^\beta, u^\beta_{i_1}, \dots, u^\beta_{i_1,\dots,i_k})$ are functions on some finite jet manifold $J^k F$. The Euler–Lagrange equation is the k-th order PDE given in local coordinates by

$$E_\alpha\Big(x^i, \varphi^\beta, \frac{\partial \varphi^\beta}{\partial x^{i_1}}, \dots, \frac{\partial^k \varphi^\beta}{\partial x^{i_1} \cdots \partial x^{i_k}}\Big) = 0\,.$$

Using the local coordinate formula (5.18a) for the vertical differential δ and the formula (5.26) for the interior Euler operator P, we see that E_α is given in terms of L by

$$E_\alpha = \sum_{|I| \le k} (-1)^{|I|} D_I\Big(\frac{\partial L}{\partial u^\alpha_I}\Big)\,.$$

The Euler–Lagrange equation then takes the local coordinate form

$$\sum_{|I| \le k} (-1)^{|I|} \frac{\partial^{|I|}}{\partial x^I}\Big(\frac{\partial L}{\partial u^\alpha_I} \circ j^k \varphi\Big) = 0\,.$$

Notation 6.2.3. In the physics literature it is customary to use the same notation $u_I^\alpha \equiv \frac{\partial^{|I|}\varphi^\alpha}{\partial x^I}$ for the jet coordinate functions u_I^α and their evaluation at a field. With this notation, the Euler–Lagrange equation is written as

$$\sum_{|I|\leq k}(-1)^{|I|}\frac{\partial^{|I|}}{\partial x^I}\left(\frac{\partial L}{\partial(\frac{\partial^{|I|}\varphi^\alpha}{\partial x^I})}\right) = 0\,.$$

Definition 6.2.4. Let (M, F, L) be a local LFT. The diffeological subspace

$$\mathcal{F}_{\text{shell}} = \{\varphi \in \mathcal{F} \mid EL_\varphi = 0\} \subset \mathcal{F}$$

will be called the **diffeological space of solutions** of the Euler–Lagrange equation.

Terminology 6.2.5. Let (M, F, L) be a local LFT. The horizontal cohomology class $[L]_d \in H_d^{0,\text{top}}(J^\infty F)$ will be called the **action cohomology class** or, short, the **action class**.

Proposition 6.2.6. *If two lagrangian forms $L, L' \in \Omega^{0,n}(J^\infty F)$ represent the same action class $[L]_d = [L']_d$, then they have the same Euler–Lagrange form $EL = EL'$.*

Proof. By definition, two lagrangian forms L and L' represent the same action class if and only if they differ by a d-exact form, $L - L' = d\alpha$ for $\alpha \in \Omega^{0,n}(J^\infty F)$. It follows that

$$EL - EL' = Ed\alpha = P\delta d\alpha = -Pd(\delta\alpha) = 0\,,$$

where in the last step we have used Theorem 5.2.1 (iii). □

Remark 6.2.7. The converse of Proposition 6.2.1 is not true in general. By Theorem 5.2.6, the obstruction lies in $H^n(F)$. That is, the converse holds if and only if this cohomology vanishes, for example when $F \to M$ is a vector bundle and M is non-compact.

6.2.2 The Euler–Lagrange theorem

When M is closed and oriented, the action is defined by

$$S(\varphi) := \int_M L_\varphi\,,$$

which is a smooth map of diffeological spaces $S : \mathcal{F} \to \mathbb{R}$.

Theorem 6.2.8 (Euler–Lagrange)**.** *Let (M, F, L) be a local lagrangian field theory over a closed manifold M. Then $\varphi \in \mathcal{F}$ is a diffeological critical point of the action if and only if φ is a solution of the Euler–Lagrange equation.*

Proof. Let $t \mapsto \varphi_t \in \mathcal{F}$ be a smooth path, which represents the tangent vector $\dot\varphi_0 \in T_{\varphi_0}\mathcal{F}$. We get

$$\begin{aligned}
\iota_{\dot\varphi_0}\delta S &= \frac{d}{dt}S(\varphi_t)\Big|_{t=0} = \int_M \frac{\partial}{\partial t}L_{\varphi_t}\Big|_{t=0} = \int_M \frac{\partial}{\partial t}\big((j^\infty)^*L\big)_{\varphi_t}\Big|_{t=0} \\
&= \int_M \iota_{\dot\varphi_0}\delta(j^\infty)^*L = \int_M \iota_{\dot\varphi_0}(j^\infty)^*\delta L = \int_M \iota_{\dot\varphi_0}(j^\infty)^*(P\delta L - d\alpha) \\
&= \int_M \iota_{\dot\varphi_0}(j^\infty)^*EL - \int_M \iota_{\dot\varphi_0}d(j^\infty)^*L \\
&= \int_M \iota_{\dot\varphi_0}(j^\infty)^*EL - \int_M d\iota_{\dot\varphi_0}(j^\infty)^*L \\
&= \int_M \iota_{Tj^\infty\dot\varphi_0}EL\,,
\end{aligned}$$

where we have used the definition of the diffeological derivative, that for a smooth integrand we can commute differentiation and integration, the definition of the evaluation of a form at φ_t, that the vertical differential δ and $(j^\infty)^*$ commute by Proposition 6.1.4, that $\omega - P\omega$ is d-exact by the acyclicity Theorem 5.2.3, the definition of the Euler–Lagrange form $EL = P\delta L$, that the horizontal differential d an $(j^\infty)^*$ commute by Proposition 6.1.4, and $\iota_{\dot\varphi_0}$ and d commute since $\dot\varphi_0$ is vertical, and that the integral over a d-exact integrand vanishes.

The integrand on the right hand side is of the form

$$\iota_{Tj^\infty\dot\varphi_0}EL = (\dot\varphi_0^\alpha)(E_\alpha \circ j^k\varphi_0) \wedge dx^1 \wedge \ldots \wedge dx^n\,,$$

where the E_α are smooth functions on some jet manifold J^kF. The integral of the right side vanishes for all functions $\dot\varphi_0^\alpha$ if and only $E_\alpha \circ j^k\varphi_0 = 0$, that is, if and only if φ_0 satisfies the Euler–Lagrange equation. □

6.2.3 The cohomological Euler–Lagrange theorem

Theorem 6.2.9. *Let (M,F,L) be a local LFT. Then δL is exact at $\varphi \in \mathcal{F}$ if and only if φ is a solution of the Euler–Lagrange equation.*

Theorem 6.2.9 will follow immediately from the following, more general result, which we will also need for the theory of generalized Jacobi fields.

Proposition 6.2.10. *Let $\omega \in \Omega^{p,\mathrm{top}}(J^\infty F)$ where $p > 0$, let $\varphi \in \mathcal{F}$, and let P be the interior Euler operator. The following are equivalent:*

(i) ω is d-exact at φ

(ii) $P\omega$ vanishes at φ.

For the proof of Proposition 6.2.10, we need the following two technical lemmas.

Lemma 6.2.11. *Let $\omega \in \Omega(J^\infty F)$ and let $v \in \mathfrak{X}(J^\infty F)$ be a horizontal vector field. If ω vanishes at $\varphi \in \mathcal{F}$, then $\mathcal{L}_v\omega$ vanishes at φ.*

Proof. The condition $(\mathcal{L}_v\omega)_\varphi = 0$ is local, so it can be checked in local coordinates in which the vector field is of the form $v = v^i D_i$ for some functions $v^i \in C^\infty(J^\infty F)$. First, consider the case of a function $f \in \Omega^0(J^\infty F)$. Then

$$(\mathcal{L}_v f)_\varphi = \big(v^i(D_i f)\big)_\varphi = (v^i \circ j^\infty\varphi)\frac{\partial}{\partial x^i}(f \circ j^\infty\varphi)\,,$$

where we have used Remark 5.1.10. If $f_\varphi = f \circ j^\infty\varphi \in C^\infty(M)$ is zero, then the right side is zero, which proves the statement for 0-forms. Let now $\omega \in \Omega^{p,q}(J^\infty F)$. In local coordinates

$$\begin{aligned}\omega &= \omega^{I_1\dots I_p}_{\alpha_1\dots\alpha_p i_1\dots i_q}\delta u^{\alpha_1}_{I_1} \wedge \dots \wedge \delta u^{\alpha_p}_{I_p} \wedge dx^{i_1} \wedge \dots \wedge dx^{i_q}\\ &= \omega^{I_1\dots I_p}_{\alpha_1\dots\alpha_p i_1\dots i_q}\tau^{\alpha_1\dots\alpha_p i_1\dots i_q}_{I_1\dots I_p}\,,\end{aligned}$$

where

$$\tau^{\alpha_1\dots\alpha_p i_1\dots i_q}_{I_1\dots I_p} := \delta u^{\alpha_1}_{I_1} \wedge \dots \wedge \delta u^{\alpha_p}_{I_p} \wedge dx^{i_1} \wedge \dots \wedge dx^{i_q}\,.$$

The form ω vanishes at φ if and only if the functions $\omega^{I_1\dots I_p}_{\alpha_1\dots\alpha_p i_1\dots i_q}$ vanish at φ. For the Lie derivative with respect to v we obtain

$$\mathcal{L}_v\omega = (\mathcal{L}_v\omega^{I_1\dots I_p}_{\alpha_1\dots\alpha_p i_1\dots i_q})\tau^{\alpha_1\dots\alpha_p i_1\dots i_q}_{I_1\dots I_p} + \omega^{I_1\dots I_p}_{\alpha_1\dots\alpha_p i_1\dots i_q}(\mathcal{L}_v\tau^{\alpha_1\dots\alpha_p i_1\dots i_q}_{I_1\dots I_p})\,.$$

Assume that the functions $\omega^{I_1\dots I_p}_{\alpha_1\dots\alpha_p i_1\dots i_q}$ vanish at φ. We have already shown that their Lie derivatives with respect to v vanish at φ, so that both terms on the right hand side vanish at φ. □

Lemma 6.2.12. *Let $\omega \in \Omega^{p,n}(J^\infty F)$ where $p > 0$, let $\varphi \in \mathcal{F}$, and let P be the interior Euler operator. If ω vanishes at φ, then $P\omega$ vanishes at φ.*

Proof. The condition $(P\omega)_\varphi = 0$ is local, so it can be checked in local coordinates, in which $P\omega$ is given by Equation (5.26), that is,

$$P\omega = \delta u^\alpha \wedge \frac{1}{p}\sum_I (-1)^{|I|}\mathcal{L}_{D_I}\Big(\frac{\partial}{\partial u^\alpha_I} \lrcorner\, \omega\Big)\,. \tag{6.7}$$

Assume that ω vanishes at φ. Then $\frac{\partial}{\partial u^\alpha_I} \lrcorner\, \omega$ vanishes at φ. It follows from Lemma 6.2.11 that

$$\mathcal{L}_{D_I}\Big(\frac{\partial}{\partial u^\alpha_I} \lrcorner\, \omega\Big) = (\mathcal{L}_{D_1})^{I_1}\cdots(\mathcal{L}_{D_n})^{I_n}\Big(\frac{\partial}{\partial u^\alpha_I} \lrcorner\, \omega\Big)$$

vanishes at φ. Since each summand on the right hand side of Equation (6.7) vanishes at φ, so does the sum $P\omega$. □

Proof of Proposition 6.2.10. Assume (i). Then there is a form $\alpha \in \Omega^{p,q-1}(J^\infty F)$, so that $\omega - d\alpha$ vanishes at φ. By Lemmma 6.2.12, it follows that $P(\omega - d\alpha) = P\omega$ vanishes at φ. □

Proof of Theorem 6.2.9. By Proposition 6.2.10, δL is exact at φ if and only if $P\delta L = EL$ vanishes at φ, that is, if and only if $EL_\varphi = 0$. □

The proof of Theorem 6.2.9 sidesteps integration altogether. It only uses, via Proposition 6.2.10, basic properties of the interior Euler operator P, which is the cohomological replacement for partial integration.

6.2.4 The inverse problem of Lagrangian Field Theory

Given a PDE, how can we decide whether it is the Euler–Lagrange equation of an LFT? This is the inverse problem of Lagrangian Field Theory. In our setup, a k-th order PDE is to be given by local functions $\omega_\alpha : J^kF\big|_U \to \mathbb{R}$, $1 \le \alpha \le \dim F_m$ on an open cover $\{U \to M\}$ that define by the local expression (6.6) a $(1, n)$-form ω of source type. The inverse problem now consists of finding a lagrangian L such that

$$\omega = P\delta L\,.$$

A necessary condition is that ω is closed in the Euler–Lagrange complex (Theorem 5.2.6),

$$P\delta\omega = 0\,,$$

called the **Helmholtz condition**. It can be checked in local coordinates, using the formulas for δ and P. *** If it is satisfied, then the obstruction to the existence of a lagrangian lies in the cohomology

$$H^1\big(\Omega^{\bullet,n}_{\text{fun}}(J^\infty F), P\delta\big) \cong H^{n+1}(F)\,.$$

For example, if $F \to M$ is a vector bundle, then $H^{n+1}(F) = 0$, so that the obstruction vanishes. In this case, every form that satisfies the Helmholtz condition is the Euler–Lagrange equation of some lagrangian.

Chapter 7

Symmetries

7.1 Noether's theorems

Noether's first theorem relates symmetries of the action and conserved currents. Before we state the theorem we will define these concepts.

7.1.1 Symmetries of the action class

Assume that M is closed, so that the action function $S : \mathcal{F} \to \mathbb{R}$ is defined as in (6.1). An automorphism $\Phi \in \mathrm{Aut}(\mathcal{F})$ is a global symmetry of the action if $\Phi^* S = S$. A vector field ξ on $\mathcal{F}$ is an infinitesimal symmetry of the action if $\iota_\xi \delta S = 0$.

If M is not closed, we replace the integration over M by the cohomology in the direction of M. A **symmetry** of an LFT with lagrangian $\tilde{L} : \mathcal{F} \to \Omega^n(M)$ is an automorphism $\Phi \in \mathrm{Aut}(\mathcal{F})$ such that

$$\Phi^* \tilde{L} = \tilde{L} + d\tilde{\alpha}$$

for some $\tilde{\alpha} \in \Omega^{0,n-1}(\mathcal{F} \times M)$. An **infinitesimal symmetry** is a vector field $\tilde{\xi} \in \mathfrak{X}(\mathcal{X})$ such that

$$\mathcal{L}_{\tilde{\xi}} \tilde{L} = d\tilde{\alpha} \,. \tag{7.1}$$

If the lagrangian is local, $\tilde{L} = (j^\infty)^* L$, the symmetry is called **local** if both $\tilde{\xi}$ and $\tilde{\alpha}$ are local. That is, by Theorem 5.1.26, $\tilde{\xi}$ descends to a strictly vertical vector field $\xi \in \mathfrak{X}(J^\infty F)$ and $\tilde{\alpha} = (j^\infty)^* \alpha$ for some $\alpha \in \Omega^{0,n-1}(J^\infty F)$. The condition (7.1) then becomes

$$(j^\infty)^* \mathcal{L}_\xi L = (j^\infty)^* d\alpha \,.$$

This condition is satisfied if $\mathcal{L}_\xi L = d\alpha$, which suggests the following definition.

Definition 7.1.1. Let (M, F, L) be a local LFT. A strictly vertical vector field $\xi \in \mathfrak{X}(J^\infty F)$ such that $\mathcal{L}_\xi L = d\alpha$ for some $\alpha \in \Omega^{0,n-1}(J^\infty F)$ will be called a **Noether symmetry**.

Remark 7.1.2. If we assume that the evaluation $j^0 : \mathcal{F} \times M \to F$ is surjective, so that $(j^\infty)^*$ is injective, a strictly vertical vector field $\xi \in \mathfrak{X}(J^\infty F)$ is a Noether symmetry if and only if the corresponding local vector field $\tilde{\xi} \in \mathfrak{X}(\mathcal{F})$ is a local symmetry.

Remark 7.1.3. In Proposition 5.1.21 we have shown that the vertical and the horizontal Cartan calculi on the infinite jet bundle commute. As a consequence, the action of the vertical differential, the inner, and the Lie derivatives with respect to a strictly vertical vector field $\xi \in \mathfrak{X}(J^\infty F)$ on a form $\omega \in \Omega(J^\infty F)$ descend to actions on its d-cohomology class by

$$\delta[\omega]_d := [\delta\omega]_d\,, \quad \iota_\xi[\omega]_d := [\iota_\xi\omega]_d\,, \quad \mathcal{L}_\xi[\omega]_d := [\mathcal{L}_\xi\omega]_d\,.$$

In this notation, ξ is a Noether symmetry of if and only if $\mathcal{L}_\xi[L]_d = 0$, that is, if the action class is invariant.

Terminology 7.1.4. Definition 7.1.1 is the notion of symmetry used in Emmy Noether's original paper [Noe18]. (For an account of the history, see [KS11].) A Noether symmetry for $d\alpha \neq 0$ is often called a generalized symmetry of the lagrangian. A non-generalized symmetry may be defined by $\mathcal{L}_\xi L = 0$ or as infinite prolongation of a vertical vector field $\eta = \xi^\alpha(x^i, u^\alpha)\frac{\partial}{\partial u^\alpha}$ on F [Olv93].

Remark 7.1.5. The Lie derivative with respect to a strictly horizontal vector field X on a lagrangian form L is given by

$$\mathcal{L}_X L = [\iota_X, d+\delta]L = [\iota_X, d]L = d(\iota_X L)\,,$$

so that $\mathcal{L}_X[L]_d = 0$. In this sense every strictly horizontal vector field X can be viewed as a symmetry.

7.1.2 Currents and charges

Definition 7.1.6. A differential form $j \in \Omega^{0,n-1}(\mathcal{F}\times M)$ is called a **current**.

Integrating a current j over a closed oriented and cooriented codimension 1 submanifold $S \subset M$ yields a smooth map

$$q_S : \mathcal{F} \longrightarrow \mathbb{R}\,, \quad q_S(\varphi) := \int_S j_\varphi\,, \tag{7.2}$$

is called the corresponding **charge** on S. Assume that the spacetime manifold M is locally split into time and space, that is, there is an embedding

$$\sigma : \mathbb{R}\times\Sigma \hookrightarrow M\,,$$

where Σ is a closed oriented manifold. The integration of the current over the time slices,

$$q_t(\varphi) := q_{\sigma_t(\Sigma)}(\varphi)\,, \tag{7.3}$$

can be viewed as the total charge on Σ as a function of time. Let $t, x^1, \dots, x^{n-1}$ be local coordinates of $\mathbb{R}\times\Sigma$. Then a local current has the local coordinate form

$$j_\varphi = \rho_\varphi(t,x)\,\mathrm{vol}_\Sigma + j^k_\varphi(t,x)\,dt\wedge\Big(\frac{\partial}{\partial x^k} \lrcorner \mathrm{vol}_\Sigma\Big)\,,$$

where $\mathrm{vol}_\Sigma = dx^1\wedge\ldots\wedge dx^{n-1}$ is the volume form on Σ, and where ρ_φ and j^k_φ are smooth local functions on M.

Terminology 7.1.7. The smooth function $\rho_\varphi \in C^\infty(M)$ is called the **charge density** and the vector field $j^k_\varphi \frac{\partial}{\partial x^k} \in \mathfrak{X}(M)$ the **current density**, e.g. the electric charge density and the electric current density in Maxwell theory, or the mass density and the material flow density in fluid dynamics.

Local currents can be viewed as representing in a coordinate independent way local observables, that is, locally defined physical quantities like charge densities and their flows. If two currents differ by an exact current, $j - j' = d\beta$, the corresponding charges are the same, $q_S = q'_S$ (assuming that S is closed). In this case j and j' represent the same physical quantity. As before, we also define the notion of current in terms of the infinite jet bundle.

Definition 7.1.8. A form $j \in \Omega^{0,n-1}(J^\infty F)$ is called a **current**.

The evaluation j_φ of a current $j \in \Omega^{0,n-1}(J^\infty F)$ at $\varphi \in \mathcal{F}$ is an $(n-1)$-form. The integral of j_φ over a closed oriented codimension 1 submanifold $S \subset M$ is the charge on S.

Definition 7.1.9. Let (M, F, L) be a local LFT. A form in $\Omega(\mathcal{F} \times M)$ or in $\Omega(J^\infty F)$ is called **conserved** if it is d-closed at all solutions $\varphi \in \mathcal{F}_{\text{shell}}$ of the Euler-Lagrange equation.

When a current $j \in \Omega^{0,n-1}(\mathcal{F} \times M)$ is conserved, then the corresponding charge $q_S(\varphi)$ for $\varphi \in \mathcal{F}_{\text{shell}}$ depends only on the homology class of S. In particular, $t \mapsto q_t(\varphi)$ as defined in Equation (7.3) is constant. More generally, let M be a cobordism and $f : M \to [0,1]$ a Morse function such that $f^{-1}(0) = (\partial M)_{\text{in}}$ and $f^{-1}(1) = (\partial M)_{\text{out}}$. This can be viewed as time parametrization of M where $S_t := f^{-1}(t)$ is the time t slice. As before, $q_t(\varphi) := q_{S_t}(\varphi)$ is constant for $\varphi \in \mathcal{F}_{\text{shell}}$.

Proposition 7.1.10. *Let j in $\Omega^{0,n-1}(J^\infty F)$ be a current and $q_S : \mathcal{F} \to \mathbb{R}$ the corresponding charge on the closed codimension 1 submanifold $S \subset M$. If $dj = 0$, then q_S is constant along any smooth path in $\mathcal{F}$.*

Proof. From $dj = 0$ it follows that $d\delta j = -\delta dj = 0$. It follows from the acyclicity Theorem 5.2.3 that $\delta j \in \Omega^{1,n-1}(J^\infty F)$ is exact, $\delta j = d\beta$. Let $\varphi : \mathbb{R} \to \mathcal{F}$, $t \mapsto \varphi_t$ be a smooth path. Then

$$\begin{aligned}
\frac{d}{dt} q_S(\varphi_t) &= \frac{d}{dt} \int_S j_{\varphi_t} = \int_S \frac{\partial}{\partial t} \big((j^\infty)^* j\big)_{\varphi_t} = \int_S \iota_{\dot\varphi_t} \delta (j^\infty)^* j \\
&= \int_S \iota_{\dot\varphi_t} (j^\infty)^* \delta j = \int_S \iota_{\dot\varphi_t} (j^\infty)^* d\beta = \int_S d\big(\iota_{\dot\varphi_t} (j^\infty)^* \beta\big) \\
&= 0\,,
\end{aligned}$$

where we have used the same rules as in the proof of Theorem 6.2.8. It follows that the function $\mathbb{R} \to \mathbb{R}$, $t \mapsto q_S(\varphi_t)$ is constant. □

As a corollary to Proposition 7.1.10 we conclude that if $\mathcal{F}$ is connected by piecewise smooth paths, then the charge of a d-closed current is a constant function on $\mathcal{F}$. From the viewpoint of physics, this tells us that d-closed currents do not represent particularly interesting observables, which is why conserved currents are required to be d-closed on $\mathcal{F}_{\text{shell}}$ only.

7.1.3 Noether's first theorem

Proposition 7.1.11. *Let (M, F, L) be a local LFT. Then there is a $\gamma \in \Omega^{1,\mathrm{top}-1}(J^\infty F)$ such that*

$$\delta L = EL - d\gamma \,. \tag{7.4}$$

Proof. This is Corollary 5.2.4 for $\omega = L$. □

We will call γ a **boundary form** of the LFT. It follows from the acyclicity theorem 5.2.3 that the boundary form is determined by the LFT up to an exact form.

Theorem 7.1.12 (Noether's first theorem)**.** *Let (M, F, L) be a local LFT and γ a boundary form. Let $\xi \in \mathfrak{X}(J^\infty F)$ be a Noether symmetry, such that $\mathcal{L}_\xi L = d\alpha$. Then the current*

$$j := \alpha - \iota_\xi \gamma$$

is conserved.

Proof. Since ξ is strictly vertical, we have $\iota_\xi L = 0$ and $d\iota_\xi\gamma = -\iota_\xi d\gamma$. We obtain

$$dj = d(\alpha - \iota_\xi\gamma) = \mathcal{L}_\xi L + \iota_\xi d\gamma = \iota_\xi(\delta L + d\gamma) = \iota_\xi EL \,,$$

which vanishes on shell. □

Terminology 7.1.13. The current j of Thm. 7.1.12 is called a **Noether current** and the corresponding charge a **Noether charge** of the symmetry ξ.

Remark 7.1.14. The Noether current of Thmeorem 7.1.12 depends on the choice of both α and γ. Two Noether currents j and j' for the same symmetry ξ differ by a closed current, $d(j_\xi - j'_\xi) = 0$. It then follows from Proposition 7.1.10 that the Noether charge is unique up to a charge that is constant on the path-connected components of $\mathcal{F}$.

Definition 7.1.15. Let L be a lagrangian form, $j \in \Omega^{0,\mathrm{top}-1}(J^\infty F)$ a current, and $\xi \in \mathfrak{X}(J^\infty F)$ a strictly vertical vector field. If

$$dj = \iota_\xi EL \,,$$

then (j, ξ) is called a **Noether pair**.

Let (j, ξ) be a Noether pair. Then

$$\begin{aligned} \mathcal{L}_\xi L &= \iota_\xi \delta L = \iota_\xi(EL - d\gamma) = dj + d\iota_\xi\gamma \\ &= d\alpha \,, \end{aligned}$$

where $\alpha = j + \iota_\xi\gamma$. This shows that ξ is a Noether symmetry and j its Noether current.

While the proof of Theorem 7.1.12 makes Noether's first theorem look deceptively simple, it is the mathematical implementation of one of the most important principles in physics, the relation between Lie group symmetries and the fundamental physical quantities like momentum, energy, and charge. Here is a table:

external (spacetime) symmetry	conserved quantity
space translations	linear momentum
space rotations	angular momentum
time translation	energy
velocity transf. (Galilei group) $x \mapsto x - vt$	center of mass
Boosts (Lorentz group) $x \mapsto \frac{x-vt}{\sqrt{1-(v/c)^2}}$	center of mass

internal (gauge) symmetry	conserved quantity
U(1) gauge symmetry	electric charge
SU(2) gauge symmetry	hypercharge
SU(3) gauge symmetry	color charge

7.1.4 Noether's second theorem

Definition 7.1.16. Let $\tilde{L} \in \Omega^{0,n}(\mathcal{F} \times M)$ be a lagrangian form and $\mathcal{E}$ a set. A **family of symmetries** is given by maps

$$\begin{aligned}\tilde{\xi} &: \mathcal{E} \longrightarrow \mathfrak{X}(\mathcal{F})\\ \tilde{\alpha} &: \mathcal{E} \longrightarrow \Omega^{0,n-1}(\mathcal{F} \times M)\end{aligned}$$

satisfying $\iota_{\tilde{\xi}_\varepsilon} \delta \tilde{L} = d\tilde{\alpha}_\varepsilon$ for all $e \in \mathcal{E}$. The family is **local** if $\mathcal{E} = \Gamma(M, E)$ is the space of sections of a fiber bundle $E \to M$ and the maps

$$\begin{aligned}\mathcal{E} \times \mathcal{F} &\longrightarrow T\mathcal{F}\\ (\varepsilon, \varphi) &\longmapsto (\tilde{\xi}_\varepsilon)_\varphi\end{aligned}$$

and

$$\begin{aligned}\mathcal{E} \times \mathcal{F} &\longrightarrow \Omega^{n-1}(M)\\ (\varepsilon, \varphi) &\longmapsto (\tilde{\alpha}_\varepsilon)_\varphi\end{aligned}$$

are local in the sense of Definition 3.2.1. A local family is **linear** if $E \to M$ is a vector bundle and both $\tilde{\xi}$ and $\tilde{\alpha}$ are local maps.

Remark 7.1.17. Since neither $\mathfrak{X}(\mathcal{F})$ nor $\Omega^{0,n-1}(\mathcal{F} \times M)$ are spaces of sections of a smooth fiber bundle, the locality of a family of symmetries has to be defined in term of the maps on $\mathcal{E} \times \mathcal{F}$. This uses the exponential property of mapping spaces in $\mathcal{D}$flg and is analogous to defining smooth families $U \to \mathcal{F}$ as differentiable maps $U \times M \to F$.

If we unpack the condition of locality, we obtain maps

$$\begin{aligned}\xi_0 &: J^k E \times_M J^k F \longrightarrow VF\\ \alpha &: J^k E \times_M J^k F \longrightarrow \wedge^{n-1} T^* M\,.\end{aligned} \tag{7.5}$$

Precomposing ξ_0 with the infinite jet prolongation of a section $e \in \mathcal{E}$, we obtain the evolutionary "vector field"

$$(\xi_0)_\varepsilon := \xi_0(j^\infty \varepsilon, \_) : J^k F \longrightarrow VF\,.$$

Its infinite prolongation

$$\xi_\varepsilon : J^\infty F \longrightarrow J^\infty(VF) \tag{7.6}$$

is the strictly vertical vector field to which $\tilde{\xi}_\varepsilon \in \mathfrak{X}(\mathcal{F})$ projects by Theorem 5.1.26. Precomposing α with $j^\infty\varepsilon$, we obtain a map

$$\alpha_\varepsilon := \alpha(j^\infty\varepsilon, \_) : J^\infty F \longrightarrow \wedge^{n-1}T^*M\,, \tag{7.7}$$

which we can view as a $(0, n-1)$-form in $\Omega^{0,n-1}(J^kF) \subset \Omega^{0,n-1}(J^\infty F)$. The form $\tilde{\alpha}_\varepsilon \in \Omega^{0,n-1}(J^\infty F)$ is the pullback of $\tilde{\alpha}_\varepsilon = (j^\infty)^*\alpha_\varepsilon$.

Assume now that the lagrangian $\tilde{L}$ is local, that is, a pullback of a lagrangian form $L \in \Omega^{0,n}(J^\infty F)$. Then the condition $\iota_{\tilde{\xi}_\varepsilon}\delta\tilde{L} = d\tilde{\alpha}_\varepsilon$ can be written as

$$(j^\infty)^*(\iota_{\xi_\varepsilon}\delta L - d\alpha_\varepsilon) = 0$$

If $E \to M$ is a vector bundle, then j^0 is surjective. It follows by Lemma 3.1.12 that j^∞ is surjective, so that $(j^\infty)^*$ is injective. In this case, it follows that

$$\mathcal{L}_{\xi_\varepsilon}L = d\alpha_\varepsilon$$

for all $e \in \mathcal{E}$. The upshot is the following result.

Proposition 7.1.18. *Let (M, F, L) be a local LFT. Let $\tilde{\xi} : \mathcal{E} \to \mathfrak{X}(\mathcal{F})$, $\tilde{\alpha} : \mathcal{E} \to \Omega^{n-1}(\mathcal{F} \times M)$ be a local linear family of symmetries of $\tilde{L} = (j^\infty)^*L$. Then there are unique maps (7.5) such that $\tilde{\xi}_\varepsilon$ projects to the strictly vertical vector field ξ_ε defined in (7.6), $\tilde{\alpha}$ is the pullback of $\alpha_\varepsilon \in \Omega^{0,n-1}(J^\infty F)$ defined in (7.7), and $\mathcal{L}_{\xi_\varepsilon}L = d\alpha_\varepsilon$ for all $\varepsilon \in \mathcal{E}$, for all $\varepsilon \in \mathcal{E}$.*

Theorem 7.1.19. *Let (M, F, L) be a local LFT with boundary form γ; let $E \to M$ be a smooth vector bundle of nonzero rank; let $\xi_0 : J^kE \times_M J^kF \to VF$ and $\alpha : J^kE \times_M J^kF \to \wedge^{n-1}T^*M$ be a linear local family of symmetries. Then for every $\varepsilon \in \mathcal{E}$ the Noether current $j_\varepsilon = \alpha_\varepsilon - \iota_{\xi_\varepsilon}\gamma$ is d-exact at every $\varphi \in \mathcal{F}_{\mathrm{shell}}$.*

Proof. Let $\varphi \in \mathcal{F}_{\mathrm{shell}}$. Consider the map

$$\begin{aligned} \nu : \mathcal{E} &\longrightarrow \Omega^{n-1}(M) \\ \varepsilon &\longmapsto (j_\varepsilon)_\varphi\,. \end{aligned}$$

Since α_ε and ξ_ε are local in ε, and γ is a local form, ν is a local map. Since both ξ_ε and α_ε are linear in ε, ν is a linear map in $\mathcal{E}$.

Since $E \to M$ is a vector bundle, the vertical bundle is given by $VE = E \times_M E$, so that $T\mathcal{E} \cong \mathcal{E} \times \mathcal{E}$. Precomposing ν with the projection $\mathrm{pr}_2 : T\mathcal{E} \to \mathcal{E}$ to the fiber, we obtain a map

$$\tilde{\mu} := \nu \circ \mathrm{pr}_2 : T\mathcal{E} \longrightarrow \Omega^{n-1}(M)\,.$$

Since the projection $\mathrm{pr}_2 : \mathcal{E} \times \mathcal{E} \to \mathcal{E}$ descends to the projection $E \times_M E \to E$, it is local of degree 0. Since ν is local and the composition of local maps is local, it follows that $\tilde{\mu}$ is local. Since ν is linear, $\tilde{\mu}$ is fiber-wise linear, so that it can be viewed as a local $(1, n-1)$-form. That is, there is a $\mu \in \Omega^{1,n-1}(J^\infty E)$ such that $\tilde{\mu} = (j^\infty)^*\mu$.

By Noether's first Theorem 7.1.12 j_ε is d-closed at all $\varphi \in \mathcal{F}_{\text{shell}}$, so that $d \circ \nu = 0$. This implies that $d \circ \tilde{\mu} = d \circ \nu \circ \mathrm{pr}_2 = 0$. If we view $\tilde{\mu}$ as local $(1, n-1)$-form, it follows that $d\tilde{\mu} = d(j^\infty)^*\mu = (j^\infty)^* d\mu = 0$. Since $E \to M$ is a vector bundle, $(j^\infty)^*$ is injective, so that it follows $d\mu = 0$. We can now apply the acyclicity Theorem 5.2.3, which shows that $\mu = d\sigma$ for some $\sigma \in \Omega^{1,n-2}(J^\infty E)$.

The local form $\tilde{\sigma} = (j^\infty)^*\sigma$ on $\mathcal{E} \times M$ can be viewed as a map

$$\tilde{\sigma} : T\mathcal{E} \longrightarrow \Omega^{1,n-2}(\mathcal{E} \times M)\,,$$

which satisfies $d \circ \tilde{\sigma} = \tilde{\mu} = \nu \circ \mathrm{pr}_2$. Let $\varepsilon_0 \in \mathcal{E}$. The embedding of the fiber of $T\mathcal{E} \to \mathcal{E}$ over $\varepsilon_0 \in \mathcal{E}$,

$$\begin{aligned} i : \mathcal{E} &\longrightarrow \mathcal{E} \times \mathcal{E} \cong T\mathcal{E} \\ \varepsilon &\longmapsto (\varepsilon_0, \varepsilon)\,, \end{aligned}$$

is a linear map and a section of the projection, $\mathrm{pr}_2 \circ i = \mathrm{id}_{\mathcal{E}}$. It follows that

$$d \circ (\tilde{\sigma} \circ i) = \nu \circ \mathrm{pr}_2 \circ i = \nu\,.$$

We conclude that

$$(j_\varepsilon)_\varphi = \nu(\varepsilon) = d\big(\tilde{\sigma}\big(i(\varepsilon)\big)\big)\,,$$

which finishes the proof. □

Remark 7.1.20. Theorem 7.1.19 was stated without proof in Theorem 15 b) of [Zuc87], where it was attributed to E. Noether's original article [Noe18] of 1918. However, while the general idea may be extrapolated from §6 of [Noe18], the proof of Theorem 7.1.19 for general background manifolds M and general fiber bundles F relies on concepts and technical results that were not available at the time. The proof uses diffeology and variational cohomology, crucially the acyclicity Theorem 5.2.3. The proof given here can be found in [Ber19].

Remark 7.1.21. The form $\beta = \tilde{\sigma} \circ i$ from the proof of Theorem 7.1.19 that satisfies $(j_\varepsilon - d\beta_\varepsilon)_\varphi = 0$ generally depends on $\varphi \in \mathcal{F}_{\text{shell}}$. Therefore, β_ε cannot be viewed as $(0, n-2)$-form on $\mathcal{F}_{\text{shell}} \times M$. In other words, Theorem 7.1.19 does *not* state that the pullback of j_ε to $\mathcal{F}_{\text{shell}} \times M$ is d-exact.

Remark 7.1.22. By adding to j_ε of Theorem 7.1.19 a current η that is d-closed but not d-exact at φ, we obtain a Noether pair $(\xi_\varepsilon, j_\varepsilon + \eta)$. This shows that there may be other Noether currents for ξ_ε that are not d-exact at all $\varphi \in \mathcal{F}_{\text{shell}}$.

Corollary 7.1.23. *Assume the situation of Theorem 7.1.19. Let $S \subset M$ be a closed oriented codimension 1 submanifold. Then the charge $q_\varepsilon(\varphi) := \int_S j_\varepsilon(\varphi)$ vanishes for all $\varphi \in \mathcal{F}_{\text{shell}}$ and all $\varepsilon \in \mathcal{E}$.*

7.2 Jacobi fields

7.2.1 Linearization of the Euler-Lagrange equation

A from $\omega \in \Omega^{p,q}(J^kF)$ can be viewed as a section $\omega : J^kF \to \wedge^{p+q}T^*J^kF$. We have the following commutative diagram,

$$\begin{tikzcd}
M \arrow[drr, bend left, "\omega\circ j^k\varphi"] \arrow[dr, dashed] \arrow[ddr, bend right, "\mathrm{id}"'] & & \\
& M \times_{J^kF} (\wedge^{p+q}T^*J^kF) \arrow[r] \arrow[d] & \wedge^{p+q}T^*J^kF \arrow[d, shift left, "\pi_{J^kF}"] \\
& M \arrow[r, "j^k\varphi"] \arrow[dr, "\mathrm{id}"'] & J^kF \arrow[u, shift left, "\omega"] \arrow[d, "\mathrm{pr}_M"] \\
& & M
\end{tikzcd}$$

which is analogous to (2.40). This shows that $\omega \circ j^k\varphi$ is a section of the bundle $\wedge^n T^*J^kF \to M$ and that $\pi_{J^kF} \circ \omega_\varphi = j^k\varphi$. This shows that ω gives rise to the local map

$$\begin{aligned} D_\omega : \mathcal{F} &\longrightarrow \Gamma(M, \wedge^{p+q}T^*J^kF) \\ \varphi &\longmapsto \omega \circ j^k\varphi\,. \end{aligned} \tag{7.8}$$

The equation $D_\omega(\varphi) = 0$ is equivalent to the PDE $\omega_\varphi = 0$.

Warning 7.2.1. If $F \to M$ is a vector bundle, the bundle $\wedge^{p+q}T^*J^kF \to M$ is a vector bundle, so the target $\Gamma(M, \wedge^{p+q}T^*J^kF)$ of the differential operator $\varphi \mapsto \omega_\varphi$ is a vector space. However, the 0 on the right side of the PDE $D_\omega(\varphi) = 0$ must not be viewed as the zero in this vector space but in $\Gamma(M, \varphi^* \wedge^{p+q} T^*J^kF)$.

We recall from Terminology 3.2.16 that the tangent map

$$T_\varphi D_\omega : T_\varphi \mathcal{F} \to T_{D_\omega(\varphi)}\Gamma(M, \wedge^{p+q}T^*J^kF)$$

is called the linearization of D_ω at φ. The PDE

$$(T_\varphi D_\omega)\xi_\varphi = 0$$

for $\xi_\varphi \in \Gamma(M, \varphi^*VF)$ is the **linearization at** φ of the PDE $D_\omega\varphi = 0$ or, equivalently, of the PDE $\omega_\varphi = 0$.

Definition 7.2.2. A solution of the linearization of the Euler-Lagrange equation at some $\varphi \in \mathcal{F}_{\mathrm{shell}}$ is called a **Jacobi field**.

We can describe Jacobi fields in terms of the Cartan calculus on $J^\infty F$. For this we need the following straight-forward concept.

Definition 7.2.3. Let $v \in \mathfrak{X}(J^\infty F)$ and $\varphi \in \mathcal{F}$. The map

$$v_\varphi := v \circ j^\infty\varphi : M \longrightarrow TJ^\infty F$$

will be called the **evaluation of** v **at** φ.

The evaluation vector fields $v \in \mathfrak{X}(J^\infty F)$ at φ is compatible with the evaluation of a forms $\omega \in \Omega(J^\infty F)$ at φ, that is,

$$\iota_{v_\varphi}\omega = (\iota_v \omega)_\varphi\,.$$

Assume now that v is a strictly vertical vector field on $J^\infty F$ and ξ the corresponding local vector field on $\mathcal{F}$ given by Theorem 5.1.26. Then the evaluation of ξ at $\varphi : * \to \mathcal{F}$ descends to the evaluation of v at φ. That is, the diagram

$$\begin{tikzcd} * \times M \arrow[r, "\varphi\times \mathrm{id}_M"] \arrow[dr, "j^\infty\varphi"'] & \mathcal{F}\times M \arrow[r, "\xi\times\mathrm{id}_M"] \arrow[d, "j^\infty_F"] & T\mathcal{F}\times M \arrow[d, "j^\infty_{VF}"] \\ & J^\infty F \arrow[r, "v"'] & J^\infty(VF) \end{tikzcd}$$

is commutative, where we have used that a vertical vector field takes its values in the vertical tangent space $J^\infty(VF) \hookrightarrow TJ^\infty F$ defined in Theorem 5.1.4. The tangent vector $\xi_\varphi \in T_\varphi\mathcal{F}$ is given by a map $\xi_\varphi : M \to VF$. It follows from the commutative diagram that $v_\varphi : M \to J^\infty(VF)$ is the infinite prolongation

$$v_\varphi = j^\infty \xi_\varphi\,. \tag{7.9}$$

Lemma 7.2.4. *Let $v \in \mathfrak{X}(J^\infty F)$ be a vector field, $\omega \in \Omega^{p,q}(J^\infty F)$ a form, and $\varphi \in \mathcal{F}$ a field, such that $\omega_\varphi = 0$. Then the evaluation of the Lie derivative $\mathcal{L}_v\omega$ at φ depends only on v_φ.*

Proof. The form ω on $J^\infty F$ is represented by a form on a finite jet bundle J^kF, $k < \infty$, which we also denote by ω. The sheaf of differential (p,q)-forms on J^kF is locally free, which means that locally $\omega = \omega_a \tau^a$, where $\{\tau_a\}$ is a local frame of (p,q)-forms and $\omega_a \in C^\infty(J^kF)$ the coefficient functions. The explicit form of $\{\tau_l\}$ can be deduced from the local coordinate form (5.19), but does not matter here. The assumption $\omega_\varphi = 0$ is equivalent to $\omega_a \circ j^k\varphi = 0$ for all a.

Since $\mathcal{L}_v$ is a derivation, the evaluation of $\mathcal{L}_v\omega$ at φ is given locally by

$$\begin{aligned} \big(\mathcal{L}_v(\omega_a\tau^a)\big)_\varphi &= \big((\mathcal{L}_v\omega_a)\tau^a + \omega_a(\mathcal{L}_v\tau^a)\big)_\varphi \\ &= (\mathcal{L}_v\omega_a)_\varphi\,\tau^a_\varphi + (\omega_a)_\varphi\,(\mathcal{L}_v\tau^a)_\varphi \\ &= \big(\iota_v(\delta+d)\omega_a)_\varphi\,\tau^a_\varphi \\ &= \big(\iota_{v_\varphi}(\delta+d)\omega_a\big)\tau^a_\varphi\,. \end{aligned} \tag{7.10}$$

This shows that the right side only depends on v_φ. □

Notation 7.2.5. We will denote the expression defined on the right side of Equation (7.10) by

$$\mathcal{L}_{v_\varphi}\omega = (\mathcal{L}_v\omega)_\varphi\,.$$

Lemma 7.2.6. *Let $\omega \in \Omega^{p,q}(J^\infty F)$ and $D_\omega : \mathcal{F} \to \Gamma(M, \wedge^{p+q}T^*J^\infty F)$ the associated differential operator* (7.8)*. Let $\varphi \in \mathcal{F}$ be such that $D_\omega(\varphi) = 0$. Then*

$$(T_\varphi D_\omega)\xi_\varphi = \mathcal{L}_{j^\infty\xi_\varphi}\omega\,. \tag{7.11}$$

for all $\xi_\varphi \in T_\varphi\mathcal{F}$.

Proof. In local coordinates we have $\omega = \omega_l \tau^l$, where $\{\tau_l\}$ is a local frame of (p,q)-forms as in the proof of Lemma 7.2.4. By assumption, $\varphi \in \mathcal{F}$ satisfies $D_\omega(\varphi) = 0$, which is the case if and only if $\omega_l \circ j^\infty\varphi = 0$. Let $t \mapsto \psi_t \in \mathcal{F}$ be a smooth path, such that $\psi_0 = \varphi$. The image of $\xi_\varphi := \dot\psi_0 \in T_\varphi\mathcal{F}$ under the diffeological tangent map D_ω is given by

$$\begin{aligned}
(T_\varphi D_\omega)\xi_\varphi &= \frac{d}{dt} D_\omega \psi_t\big|_{t=0} \\
&= \frac{d}{dt}\omega_{\psi_t}\big|_{t=0} \\
&= \frac{d}{dt}\Big((\omega_l \circ j^\infty\psi_t)\,\tau^l_{\psi_t}\Big)_{t=0} \\
&= \Big(\frac{d}{dt}(\omega_l \circ j^\infty\psi_t)\Big)_{t=0}\tau^l_\varphi + (\omega_l \circ j^\infty\varphi)\Big(\frac{d}{dt}\tau^l_{\psi_t}\Big)_{t=0} \\
&= \frac{d}{dt}(\omega_l \circ j^\infty\psi_t)\big|_{t=0}\,\tau^l_\varphi\,,
\end{aligned}$$

where we have used that $\omega \circ j^\infty\varphi = 0$. In local coordinates, the first factor of the right side can be written as

$$\begin{aligned}
\frac{d}{dt}(\omega_l \circ j^\infty\psi_t)\big|_{t=0} &= \frac{d}{dt}(u_I^\alpha \circ \psi_t)\big|_{t=0}\Big(\frac{\partial\omega_l}{\partial u_I^\alpha} \circ j^\infty\psi_0\Big) \\
&= \frac{d}{dt}\frac{\partial^I\psi_t^\alpha}{\partial u_I^\alpha}\big|_{t=0}\Big(\frac{\partial\omega_l}{\partial u_I^\alpha} \circ j^\infty\psi_0\Big) \\
&= \frac{\partial^{|I|}\xi_\varphi^\alpha}{\partial x^I}\Big(\frac{\partial\omega_l}{\partial u_I^\alpha} \circ j^\infty\varphi\Big) \\
&= \iota_{j^\infty\xi_\varphi}\delta\omega_l\,.
\end{aligned}$$

We conclude that, locally, we have

$$(T_\varphi D_\omega)\xi_\varphi = (\iota_{v_\varphi}\delta\omega_l)\,\tau^l\,. \tag{7.12}$$

The right sides of Equation (7.12) and Equation (7.10) for $v_\varphi = j^\infty\xi_\varphi$ are equal, which implies Equation (7.11). □

Proposition 7.2.7. *Let (M,F,L) be a local LFT and $\varphi \in \mathcal{F}_{\text{shell}}$. Then $\xi_\varphi \in T_\varphi\mathcal{F}$ is a Jacobi field if and only if*

$$\mathcal{L}_{j^\infty\xi_\varphi}EL = 0\,.$$

Proof. Let $\varphi \in \mathcal{F}_{\text{shell}}$. By definition, $\xi_\varphi \in T_\varphi\mathcal{F}$ is a Jacobi field if it lies in the kernel of $T_\varphi D_{EL}$. Lemma 7.2.6 for $\omega = EL$ states that $(T_\varphi D_{EL})\xi_\varphi = \mathcal{L}_{j^\infty\xi_\varphi}EL$. □

In local coordinates $EL = E_\alpha\delta u^\alpha \wedge \mathrm{vol}$, where $\mathrm{vol} = dx^1 \wedge \ldots \wedge dx^n$, $n = \dim M$. Let k be the jet order of EL. The Lie derivative with respect to ξ_φ, where $\varphi \in \mathcal{F}_{\text{shell}}$, is given by

$$\begin{aligned}
\mathcal{L}_{j^\infty\xi_\varphi}EL &= (\iota_{j^\infty\xi_\varphi}\delta E_\alpha)\delta u^\alpha \wedge \mathrm{vol} \\
&= \sum_{|J|=0}^{k}\frac{\partial E_\alpha}{\partial u_J^\beta}(j^k\varphi)\,\frac{\partial^{|J|}\xi_\varphi^\beta}{\partial x^J}\,\delta u^\alpha \wedge \mathrm{vol}\,,
\end{aligned}$$

which vanishes if and only if the **Jacobi equation**

$$\sum_{|J|=0}^{k} \frac{\partial E_\alpha}{\partial u_J^\beta}(j^k\varphi)\,\frac{\partial^{|J|}\xi_\varphi^\beta}{\partial x^J} = 0 \tag{7.13}$$

is satisfied.

7.2.2 Tangent vectors on shell

Proposition 7.2.8. *Let (M, F, L) be a local LFT. If $\xi_\varphi \in T\mathcal{F}$ is tangent to the diffeological space $\mathcal{F}_{\text{shell}}$ of solutions of the Euler-Lagrange equation, then it is a Jacobi field.*

Proof. By definition of the subspace diffeology of $\mathcal{F}_{\text{shell}} \subset \mathcal{F}$, every tangent vector in $\xi_\varphi \in T_\varphi\mathcal{F}_{\text{shell}}$ is represented by a smooth path $t \mapsto \psi_t \in \mathcal{F}_{\text{shell}}$. This means that $EL_{\psi_t} = 0$ for all t. It follows that

$$(T_\varphi D_{EL})\xi_\varphi = (T_\varphi D_{EL})\dot{\psi}_0 = \frac{d}{dt} EL_{\psi_t}\big|_{t=0} = 0\,.$$

It follows from Proposition 7.2.7 that ξ_φ is a Jacobi field. □

The converse of Proposition 7.2.8 is not true in general, since not every solution of the Jacobi equation is represented by a path in $\mathcal{F}_{\text{shell}}$. The first obstruction to extending Jacobi fields to such paths arises when the Euler-Lagrange equation viewed as function on J^kF is degenerate as in the following example.

Example 7.2.9. Let $M = \mathbb{R}$ and $F = \mathbb{R} \times Q \to \mathbb{R}$ with $Q = \mathbb{R}$, so that the space of fields is $\mathcal{F} \cong C^\infty(\mathbb{R})$. Consider the lagrangian $L = \frac{1}{3}\dot{q}^3 dt$. The Euler-Lagrange form is $EL = -2\dot{q}\ddot{q}\delta q \wedge dt$, so the Euler-Lagrange equation is

$$2\dot{q}\ddot{q} = 0\,,$$

that is $\frac{d}{dt}\dot{q}^2 = 0$, which is equivalent to $\dot{q}^2 = v^2$ for some constant velocity $v \in \mathbb{R}$. The solutions are the constant velocity paths,

$$\mathcal{F}_{\text{shell}} = \{q \in C^\infty(\mathbb{R}) \mid q(t) = x + vt\,,\ x, v \in \mathbb{R}\}\,.$$

The tangent space of $\mathcal{F}$ at the constant path $q(t) = 0$ is given by $T_0\mathcal{F} = C^\infty(\mathbb{R})$. The subspace of vectors tangent to $\mathcal{F}_{\text{shell}}$ is given by

$$T_0\mathcal{F}_{\text{shell}} = \{\xi \in C^\infty(\mathbb{R}) \mid \xi(t) = \alpha + \beta t\,,\ \alpha, \beta \in \mathbb{R}\}\,.$$

The Jacobi equation (7.13) for $\xi \in T_{x+vt}\mathcal{F} \cong C^\infty(\mathbb{R})$ is

$$v\ddot{\xi} = 0\,.$$

When $v = 0$, the equation is trivially satisfied for every $\xi \in C^\infty(\mathbb{R})$. This shows that *every* tangent vector at a constant path $q(t) = x$ is a Jacobi field. We conclude that there are Jacobi fields that are not represented by a path in $\mathcal{F}_{\text{shell}}$.

The essential property of Example 7.2.9 is that the component functions $E_\alpha : J^kF \to \mathbb{R}$ of the Euler-Lagrange are degenerate in the sense that there are tangent vectors to J^kF that annihilate all E_α but are not tangent to the zero locus of the E_α. As a consequence, the obstruction to extending Jacobi fields to paths in $\mathcal{F}_{\text{shell}}$ is local. More precisely, there are Jacobi fields ξ and points $m \in M$ where the restriction $\xi|_U$ to any neighborhood U of m cannot be represented by a path of local solutions of the Euler-Lagrange equation. There are also global obstructions as the next example shows.

Example 7.2.10. Let $M = \mathbb{R}$ and $F = \mathbb{R} \times Q \to \mathbb{R}$ with $Q = (-1, 1)$, so that the space of fields is $\mathcal{F} \cong C^\infty\big(\mathbb{R}, (-1, 1)\big)$, the space of smooth paths in the open interval $(-1, 1)$. Let $L = \frac{1}{2}\dot{q}^2 dt$ be the lagrangian of the free particle in Q. The Euler-Lagrange form is $EL = -\ddot{q}\delta q \wedge dt$. Since any path of non-zero constant velocity would have to leave $(-1, 1)$ eventually, the space of solutions is the space of constant paths,

$$\mathcal{F}_{\text{shell}} = \{q \in \mathcal{F} \mid q(t) = x\,,\ x \in (-1, 1)\}\,.$$

The tangent space at the constant path $q = 0$ is given by

$$T_0\mathcal{F} = \Gamma\big(\mathbb{R}, \mathbb{R} \times T_0(-1, 1)\big) \cong C^\infty(\mathbb{R})\,.$$

The subspace of vectors tangent to $\mathcal{F}_{\text{shell}}$ is given by constant functions,

$$T_0\mathcal{F}_{\text{shell}} = \{\xi \in C^\infty(\mathbb{R}) \mid \xi(t) = \alpha\,,\ \alpha \in \mathbb{R}\}\,.$$

The Jacobi equation (7.13) for $\xi \in C^\infty(\mathbb{R})$ is $\ddot{\xi} = 0$, the solutions of which are of the form $\xi = \alpha + \beta t$ for $\alpha, \beta \in \mathbb{R}$. We conclude that there are Jacobi fields with $\beta \neq 0$ that are not tangent vectors to $\mathcal{F}_{\text{shell}}$.

Remark 7.2.11. We could *define* the tangent spaces of the variety $\mathcal{F}_{\text{shell}}$ to be given by all Jacobi fields, as it is done in [Zuc87, Def. 7]. In other words, we could use the Zariski tangent space of algebraic geometry for the variety $\mathcal{F}_{\text{shell}}$ rather than the diffeological tangent space. However, this would be inconsistent with the diffeological description of the spaces of fields and obscure the interesting geometric phenomena exhibited by examples 7.2.9 and 7.2.10.

Definition 7.2.12. Let (M, F, L) be a local LFT. A solution $\varphi \in \mathcal{F}_{\text{shell}}$ of the Euler-Lagrange equation for which $T_\varphi\mathcal{F}_{\text{shell}}$ is equal to the space of Jacobi fields will be called **non-degenerate**.

7.2.3 Jacobi fields in riemannian geometry

7.2.4 Symmetries and Jacobi fields

Proposition 7.2.13. *Let (M, F, L) be a local LFT. Let $\xi \in \mathfrak{X}(J^\infty F)$ be a strictly vertical vector field and $\tilde{\xi} \in \mathfrak{X}(\mathcal{F})$ the corresponding local vector field given by Theorem 5.1.26. If ξ is a Noether symmetry, then $\tilde{\xi}_\varphi$ is a Jacobi field for all $\varphi \in \mathcal{F}_{\text{shell}}$.*

Lemma 7.2.14. *Let $\omega \in \Omega^{1,n}(J^\infty F)$ be a source form and $\xi \in \mathfrak{X}(J^\infty F)$ a strictly vertical vector field. If $\varphi \in \mathcal{F}$ satisfies $\omega_\varphi = 0$, then*

$$(\mathcal{L}_\xi \omega)_\varphi = \big(P(\mathcal{L}_\xi \omega)\big)_\varphi \,. \tag{7.14}$$

Proof. Equation (7.14) is local, so it can be checked in local coordinates. By assumption, ω is a source form, so we have in local coordinates

$$\omega = \delta u^\alpha \wedge \omega_\alpha \tau \,,$$

where $\tau = dx^1 \wedge \ldots \wedge dx^n$ for the volume form of the local coordinates. For the Lie derivative we get

$$\mathcal{L}_\xi \omega = \delta u^\alpha \wedge (\mathcal{L}_\xi \omega_\alpha)\tau + \delta \xi^\alpha \wedge \omega_\alpha \tau \,,$$

where we have used that ξ is strictly vertical. Using formula (5.26) for P we see that the first term on the right side satisfies

$$P(\delta u^\alpha \wedge (\mathcal{L}_\xi \omega_\alpha)\tau) = \delta u^\alpha \wedge (\mathcal{L}_\xi \omega_\alpha)\tau \,.$$

For the second term we obtain

$$P(\delta \xi^\alpha \wedge \omega_\alpha \tau) = \delta u^\beta \wedge \tau \, \frac{1}{p} \sum_I (-1)^{|I|} D_I \Big(\frac{\partial \xi^\alpha}{\partial u^\beta_I} \, \omega_\alpha \Big) \,.$$

For any $\varphi \in \mathcal{F}$ we have

$$\begin{aligned} \Big(D_I \Big(\frac{\partial \xi^\alpha}{\partial u^\beta_I} \, \omega_\alpha \Big) \Big)_\varphi &= \frac{\partial^{|I|}}{\partial x^I} \Big(\Big(\frac{\partial \xi^\alpha}{\partial u^\beta_I} \, \omega_\alpha \Big) \circ j^\infty \varphi \Big) \\ &= \frac{\partial^{|I|}}{\partial x^I} \Big(\Big(\frac{\partial \xi^\alpha}{\partial u^\beta_I} \circ j^\infty \varphi \Big) \, (\omega_\alpha \circ j^\infty \varphi) \Big) \,, \end{aligned} \tag{7.15}$$

where we have used Remark 5.1.10 Assume now that $\omega_\varphi = 0$, which is the case if and only if $\omega_\alpha \circ j^\infty \varphi = 0$, so that the right side of (7.15) vanishes. This implies that

$$\big(P(\delta \xi^\alpha \wedge \omega_\alpha \tau)\big)_\varphi = 0 \,.$$

Putting things together, we obtain

$$\begin{aligned} \big(P(\mathcal{L}_\xi \omega)\big)_\varphi &= \big(P(\delta u^\alpha \wedge (\mathcal{L}_\xi \omega_\alpha)\tau)\big)_\varphi + \big(P(\delta \xi^\alpha \wedge \omega_\alpha \tau)\big)_\varphi \\ &= \big(\delta u^\alpha \wedge (\mathcal{L}_\xi \omega_\alpha)\tau\big)_\varphi \\ &= \big(\delta u^\alpha \wedge (\mathcal{L}_\xi \omega_\alpha)\tau\big)_{j^\infty \varphi} + (\delta \xi^\alpha \wedge \omega_\alpha \tau)_\varphi \\ &= (\mathcal{L}_\xi \omega)_\varphi \,, \end{aligned}$$

where we have used that $(\delta \xi^\alpha \wedge \omega_\alpha \tau)_{j^\infty \varphi} = 0$ since $\omega_\alpha \circ j^\infty \varphi = 0$. □

Proof of Proposition 7.2.13. Assume that ξ is a Noether symmetry, so that $\mathcal{L}_\xi L = d\alpha$. Then

$$\begin{aligned} \mathcal{L}_\xi EL &= \mathcal{L}_\xi(\delta L + d\gamma) \\ &= \delta \mathcal{L}_\xi L + d\mathcal{L}_\xi \gamma \\ &= \delta d\alpha + d\mathcal{L}_\xi \gamma \\ &= d(-\delta\alpha + \delta \iota_\xi \gamma + \iota_\xi \delta\gamma) \\ &= d(-\delta j + \iota_\xi \delta\gamma) \,, \end{aligned}$$

where $j = \alpha - \iota_\xi \gamma$ is the Noether current. It then follows from Theorem 5.2.1 that $P(\mathcal{L}_\xi EL) = 0$. Assume that φ is a solution of the Euler-Lagrange equation, $EL_\varphi = 0$. We now apply Lemma 7.2.14 to $\omega = EL$, which shows that

$$\mathcal{L}_{\tilde{\xi}_\varphi} EL = (\mathcal{L}_\xi EL)_\varphi = \big(P(\mathcal{L}_\xi EL)\big)_\varphi = 0\,.$$

From Proposition 7.2.7 we conclude that $\tilde{\xi}_\varphi$ is a Jacobi field. □

Remark 7.2.15. Proposition 7.2.13 was stated in Proposition 13 a) of [Zuc87] which refers to a forthcoming paper for the proof. To my best knowledge this announced paper has never appeared. Nonetheless, the statement of Proposition 13 a) has been used subsequently in the literature. For example, it is used as the first step in the proof of Proposition 2.76 of [DF99].

7.2.5 Presymplectic structures

The boundary form γ is determined by the lagrangian only up to a closed form.

Proposition 7.2.16. *Let (M, F, L) be a local LFT. The form $\delta\gamma$, where γ is a boundary form, is unique up to a d-exact form.*

Proof. The boundary form γ is unique up to a closed form. So if γ' is another boundary form $\tau := \gamma' - \gamma$ is d-closed. For $\delta\tau = \delta\gamma' - \delta\gamma$ we obtain $d\delta\tau = -\delta d\tau = 0$. It now follows from the acyclicity Theorem 5.2.3 that $\delta\tau$ is exact. □

Terminology 7.2.17. In [Zuc87] $\delta\gamma$ is called the **universal current**.

Lemma 7.2.18. *Let $\varphi \in \mathcal{F}_{\mathrm{shell}}$. Let $\xi_\varphi, \chi_\varphi \in T_\varphi\mathcal{F}$ be Jacobi fields. Then*

$$\iota_{j^\infty\xi_\varphi}\iota_{j^\infty\chi_\varphi}\delta EL = 0\,.$$

Proof. Let $\hat{\xi}$, $\hat{\chi}$ be strictly vertical vector fields extending $j^\infty\xi_\varphi$ and $j^\infty\chi_\varphi$. We have

$$\begin{aligned}\iota_{\hat\xi}\iota_{\hat\chi}\delta EL &= (\iota_{\hat\xi}\mathcal{L}_{\hat\chi} - \iota_{\hat\xi}\delta\iota_{\hat\chi})EL \\ &= (\iota_{\hat\xi}\mathcal{L}_{\hat\chi} - \mathcal{L}_{\hat\xi}\iota_{\hat\chi})EL \\ &= (\iota_{\hat\xi}\mathcal{L}_{\hat\chi} - \iota_{\hat\chi}\mathcal{L}_{\hat\xi} - \iota_{[\hat\xi,\hat\chi]})EL\,.\end{aligned}$$

When we evaluate the right side at φ, the first two terms vanish because $\hat{\xi}$ and $\hat{\chi}$ are infinite prolongations of Jacobi fields. The last term vanishes because φ is a solution of the Euler-Lagrange equation. □

Proposition 7.2.19. *Let $\varphi \in \mathcal{F}$, let $\xi_\varphi, \chi_\varphi \in T_\varphi\mathcal{F}$ be Jacobi fields. Then*

$$d(\iota_{j^\infty\xi_\varphi}\iota_{j^\infty\chi_\varphi}\delta\gamma) = 0\,.$$

Proof. It follows from the proof of Theorem 5.1.26, that

$$d(\iota_{j^\infty\xi_\varphi}\iota_{j^\infty\chi_\varphi}\delta\gamma) = \iota_{j^\infty\xi_\varphi}\iota_{j^\infty\chi_\varphi}d\delta\gamma\,.$$

Moreover, we have

$$\begin{aligned}d(\delta\gamma) &= -\delta d\gamma = \delta(EL - \delta L) \\ &= \delta EL\,.\end{aligned}$$

The proposition now follows from Lemma 7.2.18. □

Let $S \subset M$ be a closed oriented codimension 1 submanifold. Integrating $\delta\gamma$ over S yields a 2-form ω_S on $\mathcal{F}$, which is defined by

$$\omega_S(\xi_\varphi, \chi_\varphi) := \int_S \iota_{\chi_\varphi}\iota_{\xi_\varphi}(j^\infty)^*\delta\gamma = \int_S \iota_{j^\infty\chi_\varphi}\iota_{j^\infty\xi_\varphi}\delta\gamma$$

for all $\xi_\varphi, \chi_\varphi \in T\mathcal{F}$. It follows from Proposition 7.2.16 that ω_S is independent of the choice of the boundary form γ. While ω_S does depend on S, it follows from Proposition 7.2.19 that for two Jacobi fields ξ_φ and χ_φ, $\omega_S(\xi_\varphi, \chi_\varphi)$ depends only on the homology class of S.

Proposition 7.2.20. *Let (M, F, L) be a local LFT. Let $(j, \hat{\xi})$ be a Noether pair. Let $\tilde{\xi}$ be the unique local vector field on $\mathcal{F}$ that projects to ξ from Theorem 5.1.26. Let $S \subset M$ a closed oriented codimension 1 submanifold and q_S the charge of j on S. Then*

$$(\iota_{\tilde{\xi}}\omega_S - \delta q_S)_\varphi = 0$$

for all $\varphi \in \mathcal{F}_{\text{shell}}$.

Proof. As shown in the proof of Proposition 7.2.13, we have

$$d(\iota_\xi\delta\gamma - \delta j) = \mathcal{L}_\xi EL\,.$$

Proposition 7.2.13 states that the right side vanishes at $\varphi \in \mathcal{F}_{\text{shell}}$. In other words, $\iota_\xi\delta\gamma - \delta j$ is d-closed at φ. Proposition 6.1.9 then implies that $\iota_\xi\delta\gamma - \delta j$ is d-exact at φ. We conclude that $(\iota_\xi\omega_S - \delta q_S)_\varphi = \int_S(\iota_{\tilde{\xi}}\delta\gamma - \delta j)_\varphi = 0$. □

Proposition 7.2.21. *Let (M, F, L) be a local LFT with boundary form γ. Let $E \to M$ be a smooth vector bundle of non-zero rank. Let $\xi : \mathcal{E} \to \mathfrak{X}(\mathcal{F})$ and $\alpha : \mathcal{E} \to \Omega^{0,n-1}(\mathcal{F} \times M)$ be a local linear family of symmetries. Let $S \subset M$ be a closed codimension 1 submanifold. Then*

$$\omega_S(\xi_{\varepsilon,\varphi}, \chi_\varphi) = 0$$

for all $\varepsilon \in \mathcal{E}$, all $\varphi \in \mathcal{F}_{\text{shell}}$, and all Jacobi fields $\chi_\varphi \in T_\varphi\mathcal{F}$.

Proof. The map

$$\begin{aligned} \nu : \mathcal{E} &\longrightarrow \Omega^{1,n-1}(M) \\ \varepsilon &\longmapsto \iota_{j^\infty\chi_\varphi}\iota_{j^\infty\xi_{\varepsilon,\varphi}}\delta\gamma \end{aligned}$$

is local and linear. By Proposition (7.2.13), $\xi_{\varepsilon,\varphi}$ is a Jacobi field. Proposition 7.2.19 implies that

$$\begin{aligned} (d \circ \nu)(\varepsilon) = d(\iota_{j^\infty\chi_\varphi}\iota_{j^\infty\xi_{\varepsilon,\varphi}}\delta\gamma) &= \iota_{j^\infty\chi_\varphi}\iota_{\hat{\xi}_{\varepsilon,\varphi}}d(\delta\gamma) \\ &= 0\,. \end{aligned}$$

It follows from the same arguments as in the proof of Noether's second Theorem 7.1.19, that $\nu = d \circ \mu$ for some local linear map $\mu : \mathcal{E} \to \Omega^{1,n-1}(M)$. We conclude that

$$\omega_S(\xi_{\varepsilon,\varphi}, \chi_\varphi) = \int_S d\big(\mu(\varepsilon)\big) = 0\,,$$

which finishes the proof. □

Remark 7.2.22. Proposition 7.2.21 is stated as Thm. 13 b) in [Zuc87]. The sketch of a proof given there is not correct, however, as it requires the assumption that χ_φ be a diffeological tangent vector and not only a Jacobi field. With this stronger assumption, we can give the following short proof as proposed in [Zuc87]:

By Theorem 7.1.19, the Noether charge $q_{\varepsilon,S}$ of ξ_ε vanishes on $\mathfrak{F}_{\text{shell}}$. It follows, that $\iota_{\chi_\varphi}\delta q_{\varepsilon,S} = 0$ for any tangent vector $\chi_\varphi \in T\mathfrak{F}_{\text{shell}}$. From Proposition 7.2.20 we deduce that $\omega_S(\xi_{\varepsilon,\varphi}, \chi_\varphi) = \iota_{\chi_\varphi}\iota_{\xi_{\varepsilon,\varphi}}\omega_S = \iota_{\chi_\varphi}\delta q_{\varepsilon,S} = 0$.

Chapter 8

Multisymplectic structure

8.1 Operads and homotopy algebraic structures

8.1.1 Linear operads

The construction and structure of homotopy algebraic structures is best understood in terms of operads. An operad describes an algebraic structure given by operations that take several inputs and produce one output. An operation with k inputs is called a k-ary operation, generalizing the terms "unary" ($k = 1$), "binary" ($k = 2$), "ternary" ($k = 3$), etc. An operation with three inputs can be depicted by a rooted tree with three branches. The output of operations can be inserted into the inputs of other operations. In the representation by trees, a tree can be grafted onto every branch of another tree (Figure 8.1).

When we iterate the grafting of trees the result does not depend on whether we first graft on the top or on the bottom. In this sense, the grafting operation is associative. If we graft single branches to the branch of a tree, the tree does not change. In this sense, the grafting operation is unital.

Since we generally have more than one operation with k inputs, the vertices of the trees are labelled by a set of k-ary operations. If this set is a vector space and the insertion of operations linear, we call the operad linear. This is the case we will consider, so let us formalize it. For every $k \geq 0$, let $\mathcal{P}(k)$ denote the vector space of k-ary operations. The insertion of an i_l-ary operation into the l-th input of an k-ary operation for all $1 \leq l \leq k$ gives rise to the linear map of composition

$$\gamma_{i_1,\ldots,i_k} : \mathcal{P}(k) \otimes \mathcal{P}(i_1) \otimes \ldots \otimes \mathcal{P}(i_k) \longrightarrow \mathcal{P}(i_1 + \ldots + i_k)\,,$$

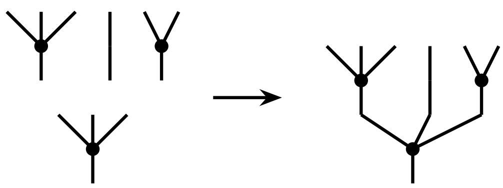


Figure 8.1: Grafting trees with 3, 1, and 2 branches onto a tree with 3 branches. In the operad, this represents the composition of operations with 3, 1, and 3 inputs with an operation with 3 inputs, which yields an operation with 6 inputs.

for all $k \geq 0$, $i_1, \dots i_k \geq k$. Moreover, there is the identity unary operation, given by a linear map $\mathrm{id} \in \mathcal{P}(1)$ that acts as identity,

$$\gamma_{1,\dots,1}(\alpha \otimes \mathrm{id}^{\otimes k}) = \alpha$$

for all $\alpha \in \mathcal{P}(k)$. The family of all operations

$$\mathcal{P} = \big(\mathcal{P}(0), \mathcal{P}(1), \dots\big)$$

can be viewed as an $\mathbb{N}$-graded vector space. The category of $\mathbb{N}$-graded vector spaces will be denoted by $\mathcal{V}\mathrm{ec}^{\mathbb{N}}$.

Terminology 8.1.1. The degree k of the vector space $\mathcal{P}(k)$ is called the **arity**.

The **composite** of the $\mathbb{N}$-graded vector space $\mathcal{P}$ and another $\mathbb{N}$-graded vector space $\mathcal{Q} = \big(\mathcal{Q}(0), \mathcal{Q}(1), \dots\big)$ is the $\mathbb{N}$-graded vector space defined by

$$\begin{aligned}(\mathcal{P} \circ \mathcal{Q})(l) &:= \coprod_{k \geq 0} \mathcal{P}(k) \otimes (\mathcal{Q}^{\otimes k})(l) \\ &= \coprod_{\substack{k \geq 0 \\ i_1 + \dots + i_k = l}} \mathcal{P}(k) \otimes \big(\mathcal{Q}(i_1) \otimes \dots \otimes \mathcal{Q}(i_k)\big),\end{aligned}$$

where $\mathcal{Q}^{\otimes k}$ denotes the k-fold tensor power of the graded vector space $\mathcal{Q}$ (Appendix A.1).

Notation 8.1.2. We do not use the notation $\bigoplus$ and $\oplus$ for direct sums in this section. Firstly, we reserve $\bigoplus$ for the biproduct, which does not exists for an infinite number of vector spaces. Secondly, by using the coproduct, the statements in this section remain true in many other categories.

Remark 8.1.3. The composite is not symmetric, $\mathcal{P} \circ \mathcal{Q} \not\cong \mathcal{Q} \circ \mathcal{P}$.

Let

$$I_\circ := (0, \mathbb{R}, 0, \dots)$$

denote the one dimensional vector space concentrated in arity 1.

Proposition 8.1.4. *The category $\mathcal{V}\mathrm{ec}^{\mathbb{N}}$ is equipped by the composite $\circ$ and unit $I_\circ$ with a right closed monoidal structure.*

Proof. Let $\mathcal{P}$, $\mathcal{Q}$, and $\mathcal{R}$ be $\mathbb{N}$-graded vector spaces. Using the symmetric structure of the tensor product, we obtain the following natural isomorphism:

$$\begin{aligned}(\mathcal{P} \circ \mathcal{Q}) \circ \mathcal{R} &= \coprod_{l \geq 0} \Big(\coprod_{k \geq 0} \mathcal{P}(k) \otimes (\mathcal{Q}^{\otimes k})(l) \Big) \otimes \mathcal{R}^{\otimes l} \\ &= \coprod_{l \geq 0} \Big(\coprod_{\substack{k \geq 0 \\ i_1 + \dots + i_k = l}} \mathcal{P}(k) \otimes \big(\mathcal{Q}(i_1) \otimes \dots \otimes \mathcal{Q}(i_k)\big) \Big) \otimes \mathcal{R}^{\otimes (i_1 + \dots + i_k)} \\ &\cong \coprod_{k \geq 0} \mathcal{P}(k) \otimes \Big(\coprod_{i_1 \geq 0} \mathcal{Q}(i_1) \otimes \mathcal{R}^{\otimes i_1} \Big) \otimes \dots \otimes \Big(\coprod_{i_k \geq 0} \mathcal{Q}(i_k) \otimes \mathcal{R}^{\otimes i_k} \Big) \\ &= \coprod_{k \geq 0} \mathcal{P}(k) \otimes (\mathcal{Q} \circ \mathcal{R})^{\otimes k} \\ &= \mathcal{P} \circ (\mathcal{Q} \circ \mathcal{R}).\end{aligned}$$

In the third step, we have used the symmetric structure to change the order of the factors in the tensor product and that the summation over $l \geq 0$ and $i_1 + \ldots + i_k = l$ is the same as the summation over all $i_1 \geq 0, \ldots, i_k \geq 0$. This shows that the composite is an associative product of graded vector spaces.

The product with I is given by

$$\begin{aligned}(\mathcal{P} \circ I_\circ)(l) &= \coprod_{\substack{k\geq 0\\ i_1+\ldots+i_k=l}} \mathcal{P}(k) \otimes \big(I_\circ(i_1) \otimes \ldots \otimes I_\circ(i_k)\big) = \mathcal{P}(l) \otimes \mathbb{R}^{\otimes l} \\ &\cong \mathcal{P}(l)\,,\end{aligned}$$

which shows that we have a natural isomorphism $\mathcal{P} \circ I_\circ \cong \mathcal{P}$. It is straightforward to show that $I_\circ \circ \mathcal{P} \cong \mathcal{P}$, as well.

The pentagon relation of the associator follows from the pentagon relation of the direct sum and the tensor product. The compatibility of the unit with the associator follows from an analogous argument.

We still have to show that $\circ$ is a functor. Given morphisms $f : \mathcal{P} \to \mathcal{P}'$ and $g : \mathcal{Q} \to \mathcal{Q}'$, we define their composite $f \circ g : \mathcal{P} \circ \mathcal{Q} \to \mathcal{P}' \circ \mathcal{Q}'$ by

$$(f \circ g)_l := \coprod_{k\geq 0} f_k \otimes (g^{\otimes k})_l$$

for all $l \geq 0$. Since the direct sum and the tensor product are functorial, it follows that $\circ$ is a functor. That is, for composable pairs of morphisms (f, f') and (g, g'), we have

$$ff' \circ gg' = (f \circ g)(f' \circ g')\,,$$

where we denote the composition of morphisms by juxtaposition to distinguish it from the composite. This shows that $(\mathcal{V}\mathrm{ec}^{\mathbb{N}}, \circ, I_\circ)$ is a monoidal category.

The category of vector spaces is closed monoidal. This means that for all vector spaces V and W there is an inner hom $\underline{\mathcal{V}\mathrm{ec}}(V, W) \in \mathcal{V}\mathrm{ec}$, such that $\mathcal{V}\mathrm{ec}(U \otimes V, W) \cong \mathcal{V}\mathrm{ec}(U, \underline{\mathcal{V}\mathrm{ec}}(V, W))$. We obtain

$$\begin{aligned}\mathcal{V}\mathrm{ec}^{\mathbb{N}}(\mathcal{P} \circ \mathcal{Q}, \mathcal{R}) &= \prod_{l\geq 0} \mathcal{V}\mathrm{ec}\big((\mathcal{P} \circ \mathcal{Q})(l), \mathcal{R}(l)\big) \\ &\cong \prod_{l\geq 0} \mathcal{V}\mathrm{ec}\Big(\coprod_{\substack{k\geq 0\\ i_1+\ldots+i_k=l}} \mathcal{P}(k) \otimes \big(\mathcal{Q}(i_1) \otimes \ldots \otimes \mathcal{Q}(i_k)\big), \mathcal{R}(l)\Big) \\ &\cong \prod_{l\geq 0} \prod_{\substack{k\geq 0\\ i_1+\ldots+i_k=l}} \mathcal{V}\mathrm{ec}\big(\mathcal{P}(k) \otimes \big(\mathcal{Q}(i_1) \otimes \ldots \otimes \mathcal{Q}(i_k)\big), \mathcal{R}(l)\big) \\ &\cong \prod_{l\geq 0} \prod_{\substack{k\geq 0\\ i_1+\ldots+i_k=l}} \mathcal{V}\mathrm{ec}\big(\mathcal{P}(k), \underline{\mathcal{V}\mathrm{ec}}\big(\mathcal{Q}(i_1) \otimes \ldots \otimes \mathcal{Q}(i_k), \mathcal{R}(l)\big)\big) \\ &\cong \prod_{k\geq 0} \mathcal{V}\mathrm{ec}\Big(\mathcal{P}(k), \prod_{l\geq 0} \underline{\mathcal{V}\mathrm{ec}}\Big(\coprod_{i_1+\ldots+i_k=l} \mathcal{Q}(i_1) \otimes \ldots \otimes \mathcal{Q}(i_k), \mathcal{R}(l)\Big)\Big) \\ &\cong \mathcal{V}\mathrm{ec}^{\mathbb{N}}\Big(\mathcal{P}, \prod_{l\geq 0} \underline{\mathcal{V}\mathrm{ec}}\big((\mathcal{Q}^{\otimes k})(l), \mathcal{R}(l)\big)\Big)\,.\end{aligned}$$

We conclude that the right composite product $\_\circ\mathcal{Q}$ has a right adjoint $\mathrm{Map}(Q, \_)$ given by the $\mathbb{N}$-graded vector space with arity k component

$$\mathrm{Map}(\mathcal{Q},\mathcal{R})(k) = \prod_{l\geq 0} \underline{\mathcal{V}\mathrm{ec}}\big((\mathcal{Q}^{\otimes k})(l), \mathcal{R}(l)\big)\,,$$

which shows that the monoidal category is right closed. □

Definition 8.1.5. A **linear operad** is a monoid internal to the monoidal category $(\mathcal{V}\mathrm{ec}^{\mathbb{N}}, \circ, I_\circ)$.

Explicitly, a linear operad is given by a graded vector space $\mathcal{P} = \big(\mathcal{P}(0), \mathcal{P}(1), \ldots\big)$ together with a morphisms

$$\begin{aligned} \gamma : \mathcal{P}\circ\mathcal{P} &\longrightarrow \mathcal{P} \\ \eta : I_\circ &\longrightarrow \mathcal{P}\,, \end{aligned}$$

called the composition and the unit, that satisfy the usual conditions of associativity and unitality. A **morphism** of linear operads is a morphism of monoids.

The arity l component of $\mathcal{P}\circ\mathcal{P}$ is the coproduct of the domains of all operadic compositions that map to $\mathcal{P}(l)$. This means that the composition γ can be decomposed into components

$$\gamma_{i_1,\ldots,i_k} : \mathcal{P}(k)\otimes\mathcal{P}(i_1)\otimes\ldots\otimes\mathcal{P}(i_k) \longrightarrow \mathcal{P}(i_1+\ldots+i_k)\,.$$

Spelling out the associativity of the composition in terms of all these components $\gamma_{i_1,\ldots,i_k}$, we obtain the following commutative diagram:

$$\begin{tikzcd}
 & \mathcal{P}_k\otimes(\mathcal{P}_{i_1}\otimes\mathcal{P}_{j_{1,1}}\otimes\ldots\otimes\mathcal{P}_{j_{1,i_1}})\otimes\ldots\otimes(\mathcal{P}_{i_k}\otimes\mathcal{P}_{j_{k,1}}\otimes\ldots\otimes\mathcal{P}_{j_{k,i_k}}) \arrow[ddl, "\mathrm{id}_{\mathcal{P}_k}\otimes\gamma_{i_1;j_{1,1},\ldots,j_{1,i_1}}\otimes\ldots\otimes\gamma_{i_k;j_{k,1},\ldots,j_{k,i_k}}"'] \arrow[dr, "\cong"] & \\
 & & (\mathcal{P}_k\otimes\mathcal{P}_{i_1}\otimes\ldots\otimes\mathcal{P}_{i_k})\otimes\mathcal{P}_{j_{1,1}}\otimes\ldots\otimes\mathcal{P}_{j_{k,i_k}} \arrow[dd, "\gamma_{k;i_1,\ldots,i_k}\otimes\mathrm{id}_{\mathcal{P}_{j_{1,1}}}\otimes\ldots\otimes\mathrm{id}_{\mathcal{P}_{j_{k,i_k}}}"] \\
\mathcal{P}_k\otimes\mathcal{P}_{j_{1,1}+\ldots+j_{1,i_1}}\otimes\ldots\otimes\mathcal{P}_{j_{k,1}+\ldots+j_{k,i_k}} \arrow[ddr, "\gamma_{i_1;j_{1,1}+\ldots+j_{1,i_1},\ldots j_{k,1}+\ldots+j_{k,i_k}},"'] & & \\
 & & \mathcal{P}_{i_1+\ldots+i_k}\otimes\mathcal{P}_{j_{1,1}}\otimes\ldots\otimes\mathcal{P}_{j_{k,i_k}} \arrow[dl, "\gamma_{i_1+\ldots+i_k;j_{1,1},\ldots,j_{k,i_k}}"] \\
 & \mathcal{P}_{j_{1,1}+\ldots+j_{k,i_k}} &
\end{tikzcd}$$

Here we have used subscripts for the arity to conserve space. The left vertical arrows corresponds to grafting the trees first on the top and then on the bottom, the right vertical arrows to grafting first at the bottom and then at the top. The unnamed isomorphism on the top right rearranges the order of the tensor factors using the symmetric structure. We see that, while the associativity is very natural in terms of the graphic representation by trees and very concise in terms of the monoidal structure of Definition 8.1.5, it is rather complicated when expressed in terms of the composition maps in each arity.

Notation 8.1.6. A more intuitive notation for the operadic composition is

$$\gamma_{i_1,\ldots,i_k}(\alpha \otimes \beta_1 \otimes \ldots \otimes \beta_k) =: \alpha \circ (\beta_1 \otimes \ldots \otimes \beta_k)\,,$$

which expresses the idea that we compose the k-ary operation α with the operations $\beta_1, \ldots, \beta_k$. The composition in the j-th argument is defined by

$$\alpha \circ_j \beta := \alpha \circ (\mathrm{id}^{\otimes(j-1)} \otimes \beta \otimes \mathrm{id}^{\otimes(k-j)})\,.$$

Example 8.1.7. The prime example of an operad is that of associative algebras. Here we have the operation of multiplication $m \in \mathcal{P}(2)$. By definition of an operad, we also have the identity $\mathrm{id} \in \mathcal{P}(1)$ and finite compositions of these operations. Since m is associative the multiplication satisfies $m \circ (m \otimes \mathrm{id}) = m \circ (\mathrm{id} \otimes m) =: m_3$, the multiplication of three arguments. In the same vein, we obtain by composing the multiplication repeatedly a single k-ary operation m_k which corresponds to the multiplication of k arguments. The upshot is that the linear operad that describes the operations of an associative algebra is given by

$$\mathcal{P}(0) = 0\,, \quad \mathcal{P}(1) = \mathbb{R}\,\mathrm{id}\,, \quad \mathcal{P}(2) = \mathbb{R}\,m\,, \quad \mathcal{P}(3) = \mathbb{R}\,m_3\,, \quad \ldots$$

If we leave off the basis elements of each vector space, we simply have $\mathcal{P}(0) = 0$ and $\mathcal{P}(k) = \mathbb{R}$ for all $k \geq 1$. This operad is called the **associative operad** and will be denoted by Ass.

Example 8.1.8. In order to encode unital algebras in terms of an operad we have to modify the associative operad by setting

$$\mathcal{P}(0) := \mathbb{R}\,e\,,$$

where e is the unit. In arity $k \geq 1$ we keep the spaces $\mathcal{P}(k) = \mathbb{R}$ of the associative operad. The unitality condition states that $m \circ (e \otimes \mathrm{id}) = \mathrm{id} = m \circ (\mathrm{id} \otimes e)$. Similarly, $m_3 \circ (e \otimes \mathrm{id} \otimes \mathrm{id}) = m$, etc. This operad is called the **unital associative operad**.

Example 8.1.9. By Proposition 8.1.4, $(\mathcal{V}\mathrm{ec}^{\mathbb{N}}, \circ, I_\circ)$ is right closed monoidal. The right exponential objects $\mathrm{Map}(\mathcal{V}, \mathcal{W})$ equip $\mathcal{V}\mathrm{ec}^{\mathbb{N}}$ with an enrichment over itself. The enriched composition $\mathrm{Map}(\mathcal{V}, \mathcal{W}) \circ \mathrm{Map}(\mathcal{U}, \mathcal{V}) \to \mathrm{Map}(\mathcal{U}, \mathcal{W})$ equips the endomorphism object $\mathrm{Map}(\mathcal{V}, \mathcal{V})$ with a monoidal structure. In other words,

$$\mathrm{End}_{\mathcal{V}} := \mathrm{Map}(\mathcal{V}, \mathcal{V})$$

is an operad, called the **endomorphism operad** of $\mathcal{V}$. Its arity k component is given by

$$(\mathrm{End}_{\mathcal{V}})(k) = \mathrm{Map}(\mathcal{V}^{\otimes k}, \mathcal{V})\,.$$

Since an operad is a monoid internal to a monoidal category, we have a natural notion of **left module** of an operad. Explicitly, a left module over an operad $\mathcal{P}$ is given by an $\mathbb{N}$-graded vector space $\mathcal{V} = \big(\mathcal{V}(0), \mathcal{V}(1), \ldots\big)$ together with a structure morphism, the action

$$\mathcal{P} \circ \mathcal{V} \longrightarrow \mathcal{V}\,,$$

that satisfies the conditions of associativity and unitality. As is the case for any right closed monoidal category, we can use the natural isomorphism

$$\mathcal{V}\mathrm{ec}^{\mathbb{N}}(\mathcal{P} \circ \mathcal{V}, \mathcal{V}) \cong \mathcal{V}\mathrm{ec}^{\mathbb{N}}\big(\mathcal{P}, \mathrm{Map}(\mathcal{V}, \mathcal{V})\big)$$

to identify a left module with a homomorphism of monoids, that is, a morphism of operads

$$\rho : \mathcal{P} \longrightarrow \mathrm{End}_{\mathcal{V}} \, .$$

Definition 8.1.10. A module of the linear operad $\mathcal{P}$ on the graded vector space $\mathcal{V} = (V, 0, \ldots)$ is called an **algebra** of $\mathcal{P}$ on the vector space V.

The endomorphism operad of a vector space V viewed as $\mathbb{N}$-graded vector space concentrated in degree 0 is given by

$$\mathrm{End}_V(k) = \underline{\mathcal{V}\mathrm{ec}}(V^{\otimes k}, V) \, .$$

A morphism of operads $\rho : \mathcal{P} \to \mathrm{End}_V$ is given by a family of linear maps

$$\rho_k : \mathcal{P}(k) \longrightarrow \underline{\mathcal{V}\mathrm{ec}}(V^{\otimes k}, V)$$

for $k \geq 0$. In terms of these maps, the associativity of the module structure is given by the equation

$$\rho_{i_1 + \ldots + i_k}\big(\alpha \circ (\beta_1 \otimes \ldots \otimes \beta_k)\big) = \rho_k(\alpha) \circ \big(\rho_{i_1}(\beta_1) \otimes \ldots \otimes \rho_{i_k}(\beta_k)\big)$$

for all $\alpha \in \mathcal{P}(k)$, $\beta_l \in \mathcal{P}(i_l)$. In other words, the morphism of graded vector spaces $\rho = (\rho_0, \rho_1, \ldots)$ maps the operadic composition to the composition of linear maps. Similarly, the condition of unitality is equivalent to

$$\rho_1(\mathrm{id}) = \mathrm{id}_V \, .$$

In other words, ρ maps the operadic unit to the identity morphism of V.

8.1.2 Symmetric linear operads

To define many algebraic structures we have to be able to permute the inputs of its operations. For example, the antisymmetry of a Lie bracket $l : V \otimes V \to V$, $a \otimes b \mapsto [a, b]$ is expressed by $l \circ \tau_{12} = -l$, where $\tau_{12} : V \otimes V \to V \otimes V$ is the symmetric structure of the tensor product. More generally, the symmetric structure of any monoidal structure $\otimes$ on some category defines an action of the symmetric group S_k on $V^{\otimes k}$. For example, an associative algebra is commutative if the multiplication $m_k : V^{\otimes k} \to V$, $a_1 \otimes \ldots \otimes a_k \mapsto a_1 \cdots a_k$ of k elements is invariant under precomposition with the action of the symmetric group, that is, $m_k \circ \sigma = m_k$ for all $\sigma \in S_k$. To implement these kind of properties on the level of an operad, we have to equip the space $\mathcal{P}(k)$ of k-ary operations with a linear action of the symmetric group S_k.

Definition 8.1.11. A **collection** is a family $\mathcal{P} = \big(\mathcal{P}(0), \mathcal{P}(1), \ldots\big)$, where $\mathcal{P}(k)$ is an $\mathbb{R}$-vector space with a right action of the symmetric group S_k. A morphism $f : \mathcal{P} \to \mathcal{Q}$ of collections is a family $f_k : \mathcal{P}(k) \to \mathcal{Q}(k)$, $k \geq 0$ of S_k-equivariant linear maps. The category of collections will be denoted by $\mathcal{C}\mathrm{oll}$.

Terminology 8.1.12. Loday and Vallette use the term $\mathbb{S}$-module for a collection [LV12, Section 5.1.1].

We would like to define an operad in which we can permute the inputs of the operations by replacing $\mathbb{N}$-graded vector spaces by collections. For this we have to define a monoidal structure on collections. However, the tensor product of vector spaces $\mathcal{P}(p) \otimes_{\mathbb{R}} \mathcal{P}(q)$ carries only an $S_p \times S_q$-action, but not an S_{p+q}-action as needed. To fix this we observe that there is a natural inclusion of groups,

$$i_{p,q} : S_p \times S_q \hookrightarrow S_{p+q} \,. \tag{8.1}$$

where S_p is included as permutation group of the first p elements and S_q as the permutation group of the last q elements of $\{1, \ldots, p+q\}$. This can be used to induce an S_{p+q} action from the $S_p \times S_q$ action as follows.

For any group G, let $\mathbb{R}G$ denote its **group $\mathbb{R}$-algebra**. The elements of $\mathbb{R}G$ consist of finite linear combinations $c_1 g_1 + \ldots c_k g_k$, for $k \geq 0$, $c_1, \ldots, c_k \in \mathbb{R}$, and $g_1, \ldots, g_k \in G$. The multiplication is the linear extension of the group multiplication. The group algebra is the free $\mathbb{R}$-algebra generated by G. More precisely, it is the left adjoint of the functor that sends a unital $\mathbb{R}$-algebra to its group of units. From the universal property of the left adjoint it follows that a right representation of the group G can can be naturally identified with a right representation of the group algebra $\mathbb{R}G$.

The inclusion (8.1) induces a left action of $S_p \times S_q$ on S_q. This left action commutes with the right action of S_{p+q} on itself by right multiplication. It follows that $\mathbb{R}S_{p+q}$ is an $\mathbb{R}(S_p \times S_q)$-$\mathbb{R}S_{p+q}$ bimodule. Let $\mathcal{P}$ and $\mathcal{Q}$ be collections. By tensoring the right $\mathbb{R}(S_p \times S_q)$-module $\mathcal{P}(p) \otimes_{\mathbb{R}} \mathcal{Q}(q)$ with this bimodule we obtain an $\mathbb{R}S_{p+q}$-module. This procedure is called induction of representations or modules. It enables us to define the tensor product of collections by

$$(\mathcal{P} \otimes \mathcal{Q})(k) := \bigoplus_{p+q=k} \big(\mathcal{P}(p) \otimes_{\mathbb{R}} \mathcal{Q}(q)\big) \otimes_{S_p \times S_q} \mathbb{R}S_k \,, \tag{8.2}$$

where $\otimes_{S_p \times S_q}$ is shorthand for the tensor product over the group algebra $\mathbb{R}(S_p \times S_q)$.

Proposition 8.1.13. *The tensor product* (8.2) *is associative and unital.*

Proof. The inclusion of three factors $S_p \times S_q \times S_r \hookrightarrow S_{p+q+r}$ is associative in the sense that the the square

$$\begin{tikzcd} S_p \times S_q \times S_r \arrow[r, "\mathrm{id} \times i_{q,r}"] \arrow[d, "i_{p,q} \times id"'] & S_p \times S_{q+r} \arrow[d, "i_{p,q+r}"] \\ S_{p+q} \times S_r \arrow[r, "i_{p+q,r}"'] & S_{p+q+r} \end{tikzcd}$$

commutes. Using that $\mathbb{R}(S_p \times S_q) \cong \mathbb{R}S_p \otimes_{\mathbb{R}} \mathbb{R}S_q$, we obtain an analogous statement for the group algebras, which implies that the tensor product (8.2) is associative. The tensor unit is given by $(\mathbb{R}, 0, \ldots)$. $\square$

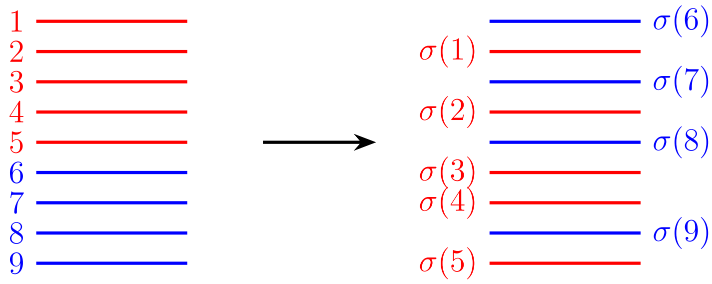


Figure 8.2: The $(5,4)$-shuffle $\sigma = \left(\begin{smallmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2 & 4 & 6 & 7 & 9 & 1 & 3 & 5 & 8 \end{smallmatrix}\right)$ visualized as a red and a blue stack of cards pushed into each other.

In order to describe the right side of Equation (8.2) more explicitly, we observe that since the left regular action of the subgroup $S_q \times S_p$ on S_{p+q} is free, so is the left action of $\mathbb{R}(S_p \times S_q)$ on $\mathbb{R}S_{p+q}$. It follows that

$$\begin{aligned} M \otimes_{S_p \times S_q} \mathbb{R}S_{p+q} &\cong M \otimes_{\mathbb{R}} \big(\mathbb{R}(S_p \times S_q) \backslash \mathbb{R}S_{p+q}\big) \\ &\cong M \otimes_{\mathbb{R}} \mathbb{R}(S_p \times S_q \backslash S_{p+q}) \end{aligned}$$

for every right $\mathbb{R}(S_p \times S_q)$-module M, where $\mathbb{R}(S_p \times S_q \backslash S_{p+q})$ denotes the free $\mathbb{R}$-vector space generated by the left residual classes $S_p \times S_q \backslash S_{p+q}$. A useful explicit set of representatives of the residual classes is given by shuffles.

Definition 8.1.14. Let p and q be natural numbers. A (p,q)**-shuffle** is a permutation $\sigma \in S_{p+q}$, such that $\sigma(1) < \sigma(2) < \ldots < \sigma(p)$ and $\sigma(p+1) < \sigma(p+2) < \ldots < \sigma(p+q)$. The inverse of a shuffle is called an **unshuffle**. The set of (p,q)-shuffles will be denoted by $\mathrm{Sh}(p,q)$, the set of unshuffles by $\mathrm{Sh}(p,q)^{-1}$.

The terminology is suggested by the shuffling of two stacks of cards into one stack (Figure 8.2). The sign of a shuffle is the sign of the permutation.

Proposition 8.1.15. *The set of (p,q)-shuffles is a set of representatives of the right residual classes $S_{p+q}/S_p \times S_q$.*

Proof. For every $\sigma \in S_{p+q}$, let $I_p(\sigma) := \{\sigma(1), \ldots, \sigma(p)\}$ denote the image of $\{1, \ldots, p\}$ and $I_q(\sigma) := \{\sigma(p+1), \ldots, \sigma(p+q)\}$ the image of $\{p+1, \ldots, p+q\}$.

Every pair (J_p, J_q) of sets of order $\#J_p = p$ and $\#J_q = q$ such that $J_p \cup J_q = \{1, \ldots, p+q\}$ determines a unique shuffle σ such that $I_p(\sigma) = J_p$ and $I_q(\sigma) = J_q$. Moreover, for an arbitrary permutation σ, the pair $\big(I_p(\sigma), I_q(\sigma)\big)$ is invariant under precomposition with $S_p \times S_q$. It follows that we have a map

$$\begin{aligned} S_{p+q}/S_p \times S_q &\longrightarrow \mathrm{Sh}(p,q) \\ [\sigma] &\longmapsto \big(I_p(\sigma), I_q(\sigma)\big)\,. \end{aligned}$$

By construction, this map is a section of the quotient map $S_{p+q} \to S_{p+q}/S_p \times S_q$. $\square$

Remark 8.1.16. Proposition 8.1.15 shows that the number of (p,q)-shuffles is $\frac{(p+q)!}{p!q!}$. The left action of $\tau \in S_{p+q}$ on $[\sigma] \in S_{p+q}/S_p \times S_q$ is given in terms of the ordered partitions by $\tau \cdot [\sigma] = \big(\tau(I_p(\sigma)), \tau(I_q(\sigma))\big)$.

The group inverse maps right residual classes bijectively to left residual classes,

$$S_{p+q}/S_p \times S_q \xrightarrow{\cong} S_p \times S_q \backslash S_{p+q}\,,$$

so that we obtain an bijection

$$S_p \times S_q \backslash S_{p+q} \cong \mathrm{Sh}(p,q)^{-1}\,.$$

The right action of $\sigma \in S_{p+q}$ on $S_p \times S_q \backslash S_{p+q}$ is given by the left action of the inverse σ^{-1} on $\mathrm{Sh}(p,q)$. We arrive at the following conclusion.

Proposition 8.1.17. *The induction of a right $\mathbb{R}(S_q \times S_p)$-module to an S_{p+q}-module is isomorphic to*

$$M \otimes_{S_p \times S_q} \mathbb{R}S_{p+q} \cong M \otimes_{\mathbb{R}} \mathbb{R}\,\mathrm{Sh}(p,q)^{-1}\,,$$

where the right action of $\tau \in S_{p+q}$ on $\sigma \in \mathrm{Sh}(p,q)^{-1} \subset S_{p+q}$ is given by $\sigma \cdot \tau = \tau^{-1}\sigma$.

By applying the isomorphism of Proposition 8.1.17 to the tensor product of collections (8.2), we obtain the isomorphism of right S_k-modules

$$(\mathcal{P} \otimes \mathcal{Q})(k) \cong \bigoplus_{p+q=k} \big(\mathcal{P}(p) \otimes_{\mathbb{R}} \mathcal{Q}(q)\big) \otimes_{\mathbb{R}} \mathbb{R}\,\mathrm{Sh}(p,q)^{-1}\,, \tag{8.3}$$

which is the k-th component of an isomorphism of collections.

Proposition 8.1.18. *The tensor product of collections* (8.2) *is symmetric.*

Proof. The subgroups $S_p \times S_q$ and $S_q \times S_p$ in S_{p+q} are conjugate to each other. It follows that their residual classes are the same. We thus obtain an isomorphism $\varphi_{p,q} : \mathrm{Sh}(p,q)^{-1} \xrightarrow{\cong} \mathrm{Sh}(q,p)$ of S_{p+q}-modules. Using that the tensor product of vector spaces is symmetric, we obtain the natural transformation

$$\big(\mathcal{P}(p) \otimes_{\mathbb{R}} \mathcal{Q}(q)\big) \otimes_{\mathbb{R}} \mathbb{R}\,\mathrm{Sh}(p,q)^{-1} \xrightarrow{\tau_{\mathcal{P}(p),\mathcal{Q}(q)} \otimes \varphi_{p,q}} \big(\mathcal{Q}(q) \otimes_{\mathbb{R}} \mathcal{P}(q)\big) \otimes_{\mathbb{R}} \mathbb{R}\,\mathrm{Sh}(q,p)^{-1}\,,$$

where τ is the symmetric structure of the tensor product of vector spaces. Since the natural transformations τ and φ both square to zero, so does their tensor product. The proof now follows from the isomorphism (8.3). $\square$

Since the tensor product of collections is symmetric by Proposition 8.1.18, the k-fold tensor power $\mathcal{Q}^{\otimes k}$ of an collection $\mathcal{Q}$ carries a natural representation of S_k. The composite of collections is now defined by

$$\mathcal{P} \circ \mathcal{Q} := \coprod_{k \geq 0} \mathcal{P}(k) \otimes_{S_k} \mathcal{Q}^{\otimes k}\,. \tag{8.4}$$

Explicitly, the l-th component of this collection is given by

$$(\mathcal{P} \circ \mathcal{Q})(l) \cong \coprod_{k \geq 0} \mathcal{P}(k) \otimes_{S_k} \Big(\coprod_{i_1+\ldots+i_k=l} \big(\mathcal{Q}(i_1) \otimes_{\mathbb{R}} \ldots \otimes_{\mathbb{R}} \mathcal{Q}(i_k)\big) \otimes_{S_{i_1} \times \ldots \times S_{i_k}} \mathbb{R}S_l\Big)\,.$$

Remark 8.1.19. There is no natural S_k-action on $\mathcal{Q}(i_1) \otimes_{\mathbb{R}} \ldots \otimes_{\mathbb{R}} \mathcal{Q}(i_k)$. But the symmetric structure of vector space defines an S_k-action on the direct sum over all $i_1, \ldots, i_k$, since it contains the tensor product in every order.

As before, let $I_\circ = (0, \mathbb{R}, 0, \ldots)$, where now $\mathbb{R}$ is the trivial S_1-module.

Proposition 8.1.20. *The category* $\mathcal{C}\mathrm{oll}$ *is equipped by the composite* $\circ$ *and* $I_\circ$ *with a right closed monoidal structure.*

Definition 8.1.21. A **symmetric linear operad** is a monoid internal to the monoidal category $(\mathcal{C}\mathrm{oll}, \circ, I_\circ)$.

Remark 8.1.22. The definition of the tensor product (8.2) and composite (8.4) of collections works for any category with a symmetric monoidal structure $\otimes$ that distributes over the coproduct, $A \otimes (B \sqcup C) \cong (A \otimes B) \sqcup (A \otimes B)$. The Definition 8.1.21 of operads then carries over verbatim.

Just as for non-symmetric linear operads, a module of a symmetric operad $\mathcal{P}$ is defined to be a module $\mathcal{M}$ over the monoid $\mathcal{P}$ internal to the monoidal category $(\mathcal{C}\mathrm{oll}, \circ, I_\circ)$. An algebra of $\mathcal{P}$ is a $\mathcal{P}$-module $\mathcal{M} = (V, 0, \ldots)$ concentrated in arity 0.

Example 8.1.23. Let V be a vector space. The symmetric structure defines a left S_k-action on $V^{\otimes k}$. This induces a right S_k-action on the vector space $\underline{\mathrm{Hom}}(V^{\otimes k}, V)$. This shows that the endomorphism operad is naturally an symmetric linear operad.

Proposition 8.1.24. *An algebra of a linear symmetric operad* $\mathcal{P}$ *on the vector space* V *can be naturally identified with a morphism of symmetric linear operads* $\mathcal{P} \to \mathrm{End}_V$*.*

Example 8.1.25. The symmetric linear operad given by $\mathcal{P}(k) = \mathbb{R}$ with the trivial S_k-action is called the **commutative operad** denoted by Com. The algebras of Comm are the commutative algebras. Commutative algebras cannot be described by a non-symmetric operad.

For every non-symmetric linear operad $\mathcal{P}$ we get a free symmetric linear operad $\mathcal{P}'$, by replacing the vector space $\mathcal{P}(k)$ by the free right $\mathbb{R}S_k$-module generated by $\mathcal{P}(k)$,

$$\mathcal{P}'(k) := \mathcal{P}(k) \otimes_{\mathbb{R}} \mathbb{R}S_k\,.$$

The algebras of $\mathcal{P}$ can be identified with the algebras of $\mathcal{P}'$.

Example 8.1.26. The symmetric operad associated to the associative operad of Example 8.1.7 is given by $\mathrm{Ass}'(0) = 0$ and $\mathrm{Ass}'(k) := \mathbb{R}S_k$ for $k > 0$. Its algebras are the associative algebras.

8.1.3 Homotopy algebras

The definition of algebras of a symmetric linear operad makes sense in any symmetric monoidal category that is enriched over vector spaces. In real homological algebra, we consider the category $\mathcal{C}\mathrm{h}$ of chain complexes of $\mathbb{R}$-modules, that is, $\mathbb{Z}$-graded vector spaces with a differential of degree -1. A summary of this category and its symmetric monoidal structure is given in Appendix A.1.

For two chain complexes W and V, the set $\mathcal{C}\mathrm{h}(W, V)$ of chain maps has the natural structure of a vector space. Then

$$(\mathrm{End}_V)(k) := \mathcal{C}\mathrm{h}(V^{\otimes k}, V)$$

is a vector space for all $k \geq 0$. The symmetric structure of graded vector spaces defines a left S_k-action on $V^{\otimes l}$, which yields a right S_k-action on $(\mathrm{End}_V)(k)$. The composition of chain maps defines the composition of a symmetric linear operad End_V.

Definition 8.1.27. Let $\mathcal{P}$ be a symmetric linear operad and V a chain complex. A morphism of operads $\mathcal{P} \to \mathrm{End}_V$ is called a **differential graded $\mathcal{P}$-algebra** or **dg$\mathcal{P}$-algebra** in short.

Example 8.1.28. A differential graded algebra of the associative operad is a graded algebra A together with a degree -1 differential d satisfying,

$$d(ab) = (da)b + (-1)^{|a|}a(db)\,,$$

for all $a, b \in A$.

Example 8.1.29. Let M be an n-dimensional manifold. The graded vector space $V_k := \Omega^{k-n}(M)$ with the wedge product of differential forms and the de Rham differential is a differential graded commutative algebra, that is, a differential graded algebra of the commutative operad Com.

Recall that a morphism $\varphi : V \to W$ of chain complexes is a quasi-isomorphism if it induces an isomorphism $H_k(V) \cong H_k(W)$ on all homology groups. Since we are working in vector spaces, we can always find a homotopy inverse $\bar{\varphi} : V \to W$, that is, a morphism such that $\bar{\varphi}\varphi$ is homotopy equivalent to id_V and $\varphi\bar{\varphi}$ homotopy equivalent to id_W. We can define linear maps

$$\begin{aligned} \tau_k : \mathcal{C}\mathrm{h}(V^{\otimes k}, V) &\longrightarrow \mathcal{C}\mathrm{h}(W^{\otimes k}, W) \\ \alpha &\longmapsto \varphi\alpha\bar{\varphi}^{\otimes k}\,, \end{aligned}$$

for all $k \geq 0$. However, this is not a morphism of operads unless $\bar{\varphi}$ is the actual inverse of φ. Therefore, the composition with the structure morphism $\rho : \mathcal{P} \to \mathrm{End}_V$ of a dg$\mathcal{P}$-algebra yields linear maps

$$\mathcal{P}(k) \xrightarrow{\rho_k} \mathcal{C}\mathrm{h}(V^{\otimes k}, V) \xrightarrow{\tau_k} \mathcal{C}\mathrm{h}(W^{\otimes k}, W) \tag{8.5}$$

for all $k \geq 0$, but not a morphism of operads. In this sense, dg$\mathcal{P}$-algebras are generally not homotopy invariant.

Questions 8.1.30. Is there a way to make algebras of operads compatible with homotopies? More precisely:

(Q1) Are there operads $\mathcal{Q}$ with algebras that are stable under homotopy, in the sense that (8.5) is a $\mathcal{Q}$-algebra for all quasi-isomorphisms $V \xrightarrow{\sim} W$?

(Q2) Can we resolve every operad $\mathcal{P}$ by an operad $\mathcal{Q} \to \mathcal{P}$ with homotopy stable algebras as in Question (Q1)?

(Q3) Can we choose the $\mathcal{Q}$ of Question (Q2) minimally, in the sense that every $\mathcal{Q}$-algebra arises by (8.5) from a $\mathcal{P}$-algebra?

The answer to all the Questions (8.1.30) is yes. The problem has a long and rich history starting in the 1960's, which led to the development of homotopy algebraic structures. At the beginning of the 2000's, a concise, conceptual, and general answer was given in terms of model categories by Berger and Moerdijk [BM03,BM06,BM07] (see also [Spi03]). For this, we have to consider the homotopy theory of the operads themselves, internalizing them to the category of chain complexes. The reader who has not been exposed to model categories may skip this part and continue with Section 8.1.5.

A collection in chain complexes is a family $\mathcal{P} = \big(\mathcal{P}(0), \mathcal{P}(1), \ldots\big)$, where $\mathcal{P}(k)$ is a chain complex with a right action of the symmetric group S_k. A morphism $f : \mathcal{P} \to \mathcal{Q}$ of collections is a family $f_k : \mathcal{P}(k) \to \mathcal{Q}(k)$, $k \geq 0$ of S_k-equivariant chain maps. The category of collections in chain complexes will be denoted by $\mathcal{C}\mathrm{oll}(\mathcal{C}\mathrm{h})$. Formula (8.2) defines a symmetric monoidal product on $\mathcal{C}\mathrm{oll}(\mathcal{C}\mathrm{h})$. Formula (8.4) defines another monoidal structure $\circ$ on $\mathcal{C}\mathrm{oll}(\mathcal{C}\mathrm{h})$. The unit of $\circ$ is the differential graded collection $I_\circ = (0, \mathbb{R}, 0, \ldots)$, where $\mathbb{R} = (\mathbb{R}, 0, \ldots)$ denotes the chain complex concentrated in degree 0 with trivial S_k-action.

Definition 8.1.31. A **differential graded operad** is a monoid internal to the monoidal category $(\mathcal{C}\mathrm{oll}(\mathcal{C}\mathrm{h}), \circ, I_\circ)$.

Just as for symmetric linear operads, a module of a differential graded operad $\mathcal{P}$ is a module $\mathcal{M}$ of the monoid of Definition 8.1.31. If the module is concentrated in arity 0, $\mathcal{M} = (V, 0, \ldots)$, it is called an algebra of $\mathcal{P}$. An algebra can be identified with a morphism of operads $\mathcal{P} \to \mathrm{End}_V$, where V is a chain complex and End_V its endomorphism operad, given by

$$(\mathrm{End}_V)(k) = \underline{\mathcal{C}\mathrm{h}}(V^{\otimes k}, V)\,.$$

Here $\underline{\mathcal{C}\mathrm{h}}$ is the inner hom of chain complexes (Appendix A.1).

We have an adjunction

$$\mathrm{dg} : \mathcal{V}\mathrm{ec} \rightleftarrows \mathcal{C}\mathrm{h} : Z_0 \quad ,$$

where dg maps a vector space to a chain complex concentrated in degree 0, and Z_0 maps a chain complex to the space of 0-cycles. Since dg is monoidal, applying it to an operad $\mathcal{P}$ yields a differential graded operad $\mathrm{dg}\mathcal{P}$. By the adjunction, we have

$$\begin{aligned}\mathcal{C}\mathrm{h}\big(\mathrm{dg}\mathcal{P}(k), \underline{\mathcal{C}\mathrm{h}}(V^{\otimes k}, V)\big) &\cong \mathcal{V}\mathrm{ec}\big(\mathcal{P}(k), Z_0\underline{\mathcal{C}\mathrm{h}}(V^{\otimes k}, V)\big)\\ &\cong \mathcal{V}\mathrm{ec}\big(\mathcal{P}(k), \mathcal{C}\mathrm{h}(V^{\otimes k}, V)\big)\,,\end{aligned}$$

where $\mathcal{C}\mathrm{h}$ is viewed as enriched over $\mathcal{V}\mathrm{ec}$. This leads to the following observation.

Remark 8.1.32. A differential graded algebra of the (symmetric) linear operad $\mathcal{P}$, which is a $\mathcal{P}$-algebra in $\mathcal{C}\mathrm{h}$, can be identified with an algebra of the (symmetric) differential graded algebra dg$\mathcal{P}$. This is consistent with the terminology of Definition 8.1.27.

Warning 8.1.33. There are two gradings in play now. The *arity* that indexes the collection $\mathcal{P} = \big(\mathcal{P}(0), \mathcal{P}(1), \ldots\big)$ and the *homological degree* of the chain complex

$$\mathcal{P}(k) = (\ldots \xrightarrow{d_{i+1}} \mathcal{P}(k)_i \xrightarrow{d_i} \mathcal{P}(k)_{i-1} \xrightarrow{d_{i-1}} \ldots)\,.$$

The two degrees play a very different role and have to be distinguished carefully.

We recall that $\mathcal{C}\mathrm{h}$ is equipped with the structure of a model category as follows. Weak equivalences are quasi-isomorphisms. Fibrations are chain maps that are surjective in every degree. This implies that all objects are fibrant and that, trivially, the fibrant replacement functor is symmetric monoidal. Cofibrations are chain maps that are in every degree split injective with projective cokernel. Since we are working over a field, all injective maps are split with projective cokernel, so that this condition is implied. It follows that every object is cofibrant. The model structure is cofibrantly generated. For a model structure with these properties, Berger and Moerdijk have proved the following results.

Theorem 8.1.34. *There is a model structure on the category of differential graded operads such that a morphism $\mathcal{Q} \to \mathcal{P}$ is a weak equivalence (fibration) if $\mathcal{Q}(k) \to \mathcal{P}(k)$ is a weak equivalence (fibration) for all $k \geq 0$.*

Proof. This is Theorem 3.2 in [BM03] for the category of chain complexes of $\mathbb{R}$-vector spaces. □

Theorem 8.1.35 (Thm. 3.5 (c) in [BM03])**.** *Let $\mathcal{Q}$ be a cofibrant differential graded operad. Let V be a Q-algebra in* $\mathcal{C}\mathrm{h}$*. Then for every quasi-isomorphism $\varphi : V \to W$, Equation* (8.5) *defines a Q-algebra on W.*

Proof. This follows from Theorem 3.5 (c) in [BM03] for the category of chain complexes of $\mathbb{R}$-vector spaces. □

For the next statement the algebras of an operad are equipped with the structure of a model category using the free-forgetful adjunction between $\mathcal{C}\mathrm{h}$ and the category of $\mathcal{P}$-algebras [BM03, Section 2.5]. A morphism of $\mathcal{P}$-algebras is a weak equivalence (fibration) if the underlying morphism of chain complexes is.

Theorem 8.1.36. *Let $\mathcal{P}_\infty \xrightarrow{\sim} \mathcal{P}$ be a cofibrant replacement of the differential graded operad $\mathcal{P}$. Then the category of $\mathcal{P}_\infty$-algebras and the category of $\mathcal{P}$-algebras are Quillen equivalent.*

Proof. This is Corollary 4.5 in [BM03] for the category of chain complexes of $\mathbb{R}$-vector spaces. □

Answers 8.1.37. We can now give satisfactory answers to Questions 8.1.30.

(A1) Theorem 8.1.35 states that the algebras of a cofibrant operad are transferred by quasi-isomorphisms. This is called the **transfer theorem**. It shows that the category of algebras of a cofibrant operad is stable under homotopies.

(A2) As is the case in any model category, every differential graded operad $\mathcal{P}$ has a cofibrant replacement $\mathcal{P}_\infty \xrightarrow{\sim} \mathcal{P}$. $\mathcal{P}_\infty$-algebras are also called **homotopy $\mathcal{P}$-algebras**.

(A3) Theorem 8.1.36 implies that every $\mathcal{P}_\infty$-algebra is weakly equivalent to a $\mathcal{P}$-algebra, that is, it arises via (8.5) from a $\mathcal{P}$-algebra. This $\mathcal{P}$-algebra is called a **rectification** of the $\mathcal{P}_\infty$-algebra.

The upshot is that in order to construct the homotopical generalization of an algebraic structure given by an operad we have to construct its cofibrant replacement.

8.1.4 Morphisms of homotopy algebras

Let $\mathcal{P}$ a differential graded symmetric operad. By definition, an algebra of $\mathcal{P}$ is a module in $\mathcal{C}\mathrm{oll}(\mathcal{C}\mathrm{h})$ on a collection $V = (V, 0, \ldots)$ concentrated in arity zero, so a morphism of $\mathcal{P}$-algebras is a morphism of modules given by a commutative square

$$\begin{tikzcd} \mathcal{P} \circ V_1 \arrow[r, "\mathrm{id} \circ f"] \arrow[d, "\rho_1"'] & \mathcal{P} \circ V_2 \arrow[d, "\rho_2"] \\ V \arrow[r, "f"'] & V_2 \end{tikzcd}$$

where $f : V_1 \to V_2$ is a morphism of chain complexes and where the vertical arrows are the module actions.

The commutativity of the square is equivalent to the equality of the diagonal morphisms $f\rho_1 = \rho_2(\mathrm{id} \circ f) : \mathcal{P} \circ V_1 \to V_2$, where we use juxtaposition for the composition of morphisms to avoid confusion with the composite $\circ$. In the homotopical setting, we should not require strict commutativity but only up to homotopy,

$$f\rho_1 - \rho_2(\mathrm{id} \circ f) = d_1 h - h d_2 \,,$$

for some degree 1 map $h : \mathcal{P} \circ V_1 \to V_2$ of graded vector spaces. However, similar to the algebras of the operad $\mathcal{P}$, this notion of weak homomorphism is not fully compatible with homotopies. Moreover, the issues are not fixed by replacing $\mathcal{P}$ with a cofibrant resolution $\mathcal{P}_\infty$.

In order arrive at a fully homotopy invariant notion of morphisms of algebras, we need to generalize the notion of operads further. The homotopy h can be viewed as a morphism of collections $h : \mathcal{P} \to \mathrm{Map}(V_1, V_2)$. The target does not form an operad, but can be composed with endomorphisms of V_1 from the right and endomorphisms of V_2 from the right. That is we have operations that take inputs of type V_i and produce an output of type V_j for $j, j \in \{1, 2\}$. These operations can only be composed if the type of the outputs and inputs match. In the language of operads the types of inputs are called **colors**.

For the definition of an operad colored by the set (or class) of colors C we replace the collection $(\mathcal{P}(1), \mathcal{P}(2), \ldots)$ with a family of objects (vector spaces, chain complexes) $\mathcal{P}(c; c_1, \ldots, c_k)$ indexed by $k \geq 0$ and colors $c, c_1, \ldots, c_k \in C$. The composite with another such family $\mathcal{Q}$ is defined by

$$(\mathcal{P} \circ \mathcal{Q})(c; c_1, \ldots, c_l) := \coprod_{\substack{k \geq 0 \\ d_1, \ldots, d_k \in C}} \mathcal{P}(d_1, \ldots, d_k; c) \otimes \coprod_{i_1 + \ldots + i_k = l} \big(\mathcal{Q}(c_1, \ldots, c_{i_1}; d_1) \otimes \ldots \\ \ldots \otimes \mathcal{Q}(c_{i_1 + \ldots + i_{k-1}+1}, \ldots, c_l; d_k)\big)$$

The identity is replaced by the family given by $I_\circ(c; c) = \mathbb{R}$ for all $c \in C$ and 0 otherwise. In other words, there is an identity operation for every color. This defines a monoidal structure on colored families [BM07]. A **colored operad** is a monoid in this category. Colored operads are also called multicategories [Lei04].

For symmetric operads we have to replace the action of the symmetric group by an isomorphism

$$\sigma(c_1, \ldots, c_k; c) : \mathcal{P}(c_1, \ldots, c_k; c) \xrightarrow{\cong} \mathcal{P}(c_{\sigma(1)}, \ldots, c_{\sigma(k)}; c)$$

for all arities $k \geq 0$, all colors $c, c_1, \ldots, c_k \in C$ and all permutations $\sigma \in S_k$, that induces an action of the symmetric group on $\coprod_{c_1, \ldots, c_k \in C} \mathcal{P}(c; c_{\sigma(1)}, \ldots, c_{\sigma(k)})$.

Example 8.1.38. Let $V_1, \ldots, V_n$ be a family of vector spaces. Then we can define the **colored endomorphism operad** by the family

$$\mathrm{End}_{V_1, \ldots, V_n}(c_1, \ldots, c_k; c) = \mathcal{V}\mathrm{ec}(V_{c_1} \otimes \ldots \otimes V_{c_1}, V_c)$$

for all $c_1, \ldots, c_k, c \in C = \{1, \ldots, n\}$ with the tensor product and composition of linear maps. As an extremal case, we can take as colors all vector spaces $C = \mathrm{Obj}(\mathcal{V}\mathrm{ec})$. We will denote its colored endomorphism operad by $\mathrm{End}_{\mathcal{V}\mathrm{ec}}$.

Every operad $\mathcal{P}$ can be viewed as a colored operad with a single color. A $\mathcal{P}$-algebra is then a morphism of colored operads

$$\mathcal{P} \longrightarrow \mathrm{End}_{\mathcal{V}\mathrm{ec}} \, .$$

In this sense, the use of colored operads relieves us of choosing the vector space V before defining a $\mathcal{P}$-algebra on V. Now, we can describe strict morphisms of algebras of an operad $\mathcal{P}$ by a morphism of operads.

***

8.1.5 L_∞-algebras

The forgetful functor that maps a linear operad $\mathcal{P}$ to the underlying collection $U(\mathcal{P}) \in \mathcal{C}\mathrm{oll}(\mathcal{V}\mathrm{ec})$ has a left adjoint, the free linear operad $\mathcal{F}(E)$ generated by a collection E. Explicitly, $\mathcal{F}(E)_k$ is the vector space spanned by all rooted trees with k leaves and each $(l+1)$-vertex labelled by an element of E_l. The operadic composition is given by the grafting of trees. For the free symmetric linear operad the leaves of trees are labelled by natural numbers. In other words, we replace $\mathcal{F}(E)_k$ with the free S_k-module $\mathcal{F}(E)_k \otimes \mathbb{R}\, S_k$.

Many operads are given by a presentation, that is, by a free operad modulo relations. A good example is the Lie operad Lie which encodes the operations of a

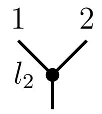
Lie algebra. It is the free operad generated by a single binary operation l_2 modulo the relations

$$0 = l_2 + l_2 \cdot \left(\begin{smallmatrix} 1 & 2 \\ 2 & 1 \end{smallmatrix}\right) \tag{8.6a}$$

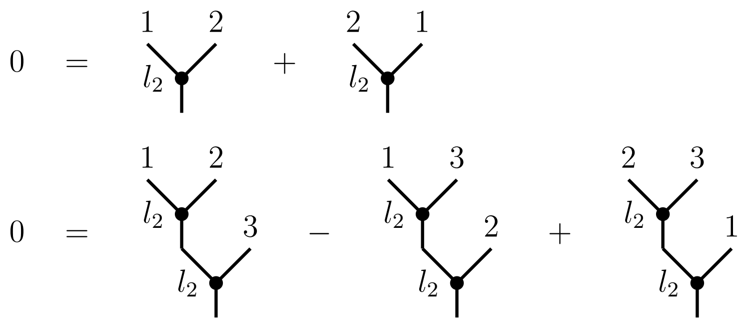
$$\begin{aligned} 0 &= l_2 \circ (l_2 \otimes \mathrm{id}) - \big(l_2 \circ (l_2 \otimes \mathrm{id})\big) \cdot \left(\begin{smallmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{smallmatrix}\right) + \big(l_2 \circ (l_2 \otimes \mathrm{id})\big) \cdot \left(\begin{smallmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{smallmatrix}\right) \\ &=: \tilde{l}_3 \,. \end{aligned} \tag{8.6b}$$

The right side of (8.6b) is not the usual Jacobiator, but will be seen to be a special case of the shuffle sum on the right side of (8.7). It is rather cumbersome to give an explicit basis of the k-ary spaces $\mathrm{Lie}(k)$.

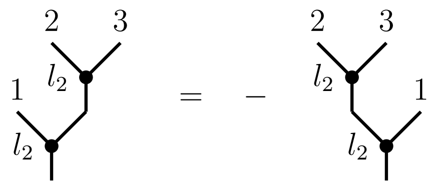
Represented by trees, the generator is



and the relations are



The relations can be applied at all subtrees. For example, we obtain the relation



by applying the antisymmetry relation to the lower vertex. The Lie operad is the quotient of the free symmetric linear operad generated by l_2 modulo relations (8.6).

To obtain the operad describing homotopy Lie algebras, we have to view the Lie operad as differential graded operad concentrated in degree 0 and then construct a cofibrant replacement $L_\infty \xrightarrow{\sim} \mathrm{Lie}$. (We now identify $\mathrm{Lie} \equiv \mathrm{dgLie}$ as is customary.) There is a number of methods to construct cofibrant resolutions of operads, such as the cobar-bar construction [GJ] and Koszul duality [GK94]. Here, we will use a version of the Boardman-Vogt resolution [BV73, BM06].

A differential graded symmetric operad is cofibrant when the underlying graded symmetric operad is free [BM06, Lemma 3.1]. (When an operad $\mathcal{Q}$ is free, then there is no obstruction to lift a morphism $\mathcal{Q} \to \mathcal{P}$ along an acyclic fibration $\mathcal{R} \xrightarrow{\sim} \mathcal{P}$.) So in order to construct a cofibrant replacement of dgLie, we have to construct an operad L_∞ that is free as graded symmetric operad together with a surjective quasi-isomorphism of chain complexes $L_\infty \to \mathrm{Lie}$. This can be done step by step.

We start with the quotient map $\varepsilon : \mathrm{Lie}^{(0)} \to \mathrm{dgLie}$, where

$$\mathrm{Lie}^{(0)} := \mathcal{F}(0, 0, \mathbb{R}l_2 \otimes \mathbb{R}S_2, 0, \dots)\,,$$

with l_2 in degree 0. The kernel of ε is the operadic ideal generated by the Relations (8.6), which is not free as graded symmetric operad.

In the next step, we add to $\mathrm{Lie}^{(0)}$ a basis of the kernel of ε in degree 1. The two Relations (8.6) are quite different in nature. Relation (8.6a) does not involve the operadic composition but only the action of the symmetric group. We can implement the antisymmetry of l_2 by replacing the free S_2-module $\mathbb{R}l_2 \otimes \mathbb{R}S_2$ by the vector space $\mathbb{R}l_2$ with the antisymmetric S_2-action $l_2 \cdot \sigma = (-1)^{|\sigma|} l_2$. The kernel of the morphism

$$\mathcal{F}(0, \mathbb{R}\,\mathrm{id}, \mathbb{R}l_2, 0, \dots) \longrightarrow \mathrm{dgLie}$$

is the operadic ideal generated by the right hand side of Relation (8.6b). So we add a generator l_3 in arity 3 and degree 1 to define the free graded symmetric operad

$$\mathrm{Lie}^{(1)} := \mathcal{F}(0, \mathbb{R}\,\mathrm{id}, \mathbb{R}l_2, \mathbb{R}l_3 \otimes \mathbb{R}S_3, 0, \dots)\,,$$

together with a differential defined by

$$dl_3 = -\tilde{l}_3\,,$$

where $\tilde{l}_3$ was defined in (8.6b). The minus sign is introduced for convenience. Note that for degree reasons $dl_2 = 0$.

The kernel of d is generated by $l_3 + (-1)^{|\sigma|} l_3 \cdot \sigma$ for all $\sigma \in S_3$ and

$$\tilde{l}_4 := \sum_{\sigma \in \mathrm{Sh}(2,2)} (-1)^{|\sigma|} \big(l_3 \circ (l_2 \otimes \mathrm{id}^{\otimes 2})\big) \cdot \sigma - \sum_{\sigma \in \mathrm{Sh}(3,1)} (-1)^{|\sigma|} \big(l_2 \circ (l_3 \otimes \mathrm{id})\big) \cdot \sigma\,.$$

The first set of generators of the kernel can be taken care of by replacing the free S_3-module $\mathbb{R}l_3 \otimes \mathbb{R}S_3$ by the vector space $\mathbb{R}l_3$ with the antisymmetric S_3-action $l_3 \cdot \sigma = (-1)^{|\sigma|} l_3$. We add to the generators an element l_4 of degree 2 and get the free operad

$$\mathrm{Lie}^{(2)} := \mathcal{F}(0, \mathbb{R}\,\mathrm{id}, \mathbb{R}l_2, \mathbb{R}l_3, \mathbb{R}l_4 \otimes \mathbb{R}S_4, 0, \dots)\,,$$

together with the differential extended to the new generator by

$$dl_4 := -\tilde{l}_4\,.$$

Repeating this procedure, we obtain the following result.

Proposition 8.1.39. *The free graded symmetric operad generated by antisymmetric generators l_k of arity k and degree $k-2$ for all $k \geq 2$, together with the differential*

$$dl_k = -\sum_{\substack{i+j-1=k \\ \sigma \in \mathrm{Sh}(i,j-1)}} (-1)^{i(j-1)} (-1)^{|\sigma|} \big(l_j \circ (l_i \otimes \mathrm{id}^{\otimes(j-1)})\big) \cdot \sigma \tag{8.7}$$

is a cofibrant replacement of the Lie operad, denoted by L_∞.

An L_∞-algebra is given by a chain complex L and antisymmetric morphisms $l_k \in \underline{\mathcal{C}\mathrm{h}}(V^{\otimes k}, V)$ of degree $k-2$ with differentials (8.7). Let us spell this out in terms of elements. First, we recall from Equation (A.3) that the left action of the symmetric group S_k is given by

$$\sigma \cdot (a_1 \otimes \ldots \otimes a_k) = \varepsilon(\sigma; a_1, \ldots, a_k)(a_{\sigma^{-1}(1)} \otimes \ldots \otimes a_{\sigma^{-1}(k)}),$$

where $\varepsilon(\sigma; a_1, \ldots, a_k) \equiv \varepsilon(\sigma)$ is the Koszul sign. For the right action of S_k we obtain

$$(a_1 \otimes \ldots \otimes a_k) \cdot \sigma = \varepsilon(\sigma)(a_{\sigma(1)} \otimes \ldots \otimes a_{\sigma(k)}).$$

We arrive at the following description of an L_∞-algebra.

Proposition 8.1.40. *An L_∞-algebra is a chain complex L together with graded antisymmetric linear maps*

$$l_k : L^{\otimes k} \longrightarrow L$$

of degree $|l_k| = 2-k$ for all $k \geq 2$, such

$$\begin{aligned}(dl_k)&(a_1, \ldots, a_k) \\ &= -\sum_{\substack{i+j-1=k \\ \sigma\in \mathrm{Sh}(i,j-1)}} (-1)^{i(j-1)}(-1)^{|\sigma|}\varepsilon(\sigma) l_j\big(l_i(a_{\sigma(1)}, \ldots, a_{\sigma(i)}), a_{\sigma(i+1)}, \ldots, a_{\sigma(k)}\big)\end{aligned} \tag{8.8}$$

for all $k \geq 2$ and all $a_1, \ldots, a_k \in L$.

The operation l_k of an L_∞-algebra is called the **k-bracket**, also denoted by

$$\begin{aligned}\{a_1, \ldots, a_k\} &\equiv l_k(a_1, \ldots, a_k) \\ &\equiv l_k(a_1 \otimes \ldots \otimes a_k).\end{aligned}$$

The differential of $l_k \in \underline{\mathcal{C}\mathrm{h}}(V^{\otimes k}, V)$ is given by Equations (A.8) and (A.9),

$$\begin{aligned}(dl_k)(a_1 \otimes \ldots \otimes a_k) &= \big(d \circ l_k - (-1)^{|l_k|} l_k \circ d\big)(a_1 \otimes \ldots \otimes a_k) \\ &= d\big(l_k(a_1 \otimes \ldots \otimes a_k)\big) \\ &\quad - (-1)^k l_k\big(da_1 \otimes \ldots \otimes a_k + (-1)^{|a_1|} a_1 \otimes da_2 \otimes \ldots \otimes a_k \\ &\quad + \ldots + (-1)^{|a_1|+\ldots+|a_{k-1}|} a_1 \otimes \ldots \otimes da_k\big).\end{aligned}$$

With this notation, Equation (8.8) for $k=2$ can be written as

$$d\{a_1, a_2\} = \{da_1, a_2\} + (-1)^{|a_1|}\{a_1, da_2\},$$

which is the condition that d is a graded derivation of the 2-bracket. For $k=3$ we obtain the condition

$$\begin{aligned}(dl_3)(a_1, a_2, a_3) = &-\{\{a_1, a_2\}, a_3\} - (-1)^{|a_1|(|a_2|+|a_3|)}\{\{a_2, a_3\}, a_1\} \\ &- (-1)^{|a_3|(|a_1|+|a_2|)}\{\{a_3, a_1\}, a_2\}.\end{aligned}$$

The right side of this equation is minus the graded Jacobiator of the 2-bracket. This shows that the failure of the 2-bracket to satisfy the Jacobi relation is given by the differential of the 3-bracket. For $k > 3$ we obtain similar relations, which can be interpreted as higher generalizations of the Jacobi relations.

Example 8.1.41. A Lie algebra $\mathfrak{g}$, viewed as graded vector space concentrated in degree 0, with 2-bracket $l_2 = [\_\,,\_]$ and all other brackets vanishing, $l_k = 0$, $k \neq 2$, is an L_∞-algebra.

Example 8.1.42. A differential graded Lie algebra is a differential complex (L, d) with a graded Lie bracket such that d is a graded derivation, $d[a, b] = [da, b] + (-1)^{|a|}[a, db]$. It is an L_∞-algebra with $l_1 = d$, $l_2 = [\_\,,\_]$, and $l_k = 0$ for $k > 2$.

It is convenient to denote the differential of an L_∞-algebra by

$$l_1 \equiv d$$

and call it the 1-bracket. Using the antisymmetry of l_k, we can write the differential of l_k as

$$\begin{aligned}(dl_k)(a_1, \dots, a_k) =& l_1\big(l_k(a_1, \dots, a_k)\big) \\ &+ (-1)^{k-1} \sum_{\sigma \in \mathrm{Sh}(1,k-1)} (-1)^{|\sigma|} \varepsilon(\sigma) l_k\big(l_1(a_{\sigma(1)}), a_{\sigma(2)}, \dots, a_{\sigma(k)}\big)\,.\end{aligned} \tag{8.9}$$

The right side of this equation has, up to the sign, the same form as the right side of (8.8). In fact, the sign was introduced such that Equations (8.8) can be written in terms of the operations l_k as

$$0 = \sum_{\substack{i+j-1=k \\ \sigma \in \mathrm{Sh}(i,j-1)}} (-1)^{i(j-1)} (-1)^{|\sigma|} \varepsilon(\sigma) l_j\big(l_i(a_{\sigma(1)}, \dots, a_{\sigma(i)}), a_{\sigma(i+1)}, \dots, a_{\sigma(k)}\big) \tag{8.10}$$

for all $k \geq 1$, where now the indices of the summation start at $i, j = 1$. Moreover, in some situations it can be more natural to consider cochain complexes so that the differential is of degree 1. We thus obtain the following equivalent form of L_∞-algebras.

Proposition 8.1.43. *An L_∞-algebra is a $\mathbb{Z}$-graded vector space L together with graded linear maps*

$$l_k : \wedge^k L \longrightarrow L$$

of degree $|l_k| = 2 - k$ for all $k \geq 1$, satisfying Equations (8.10).

An ∞-morphism $f : L \to L'$ of L_∞-algebras is given by a family of linear maps $f_k : \wedge^k L \to L$, $k \geq 1$ of degree $1 - k$, subject to relations that are best expressed either in terms of the L_∞-operad or in the language of formal pointed manifolds. If the domain $L = \mathfrak{g}$ is a Lie algebra, as is the case for a homotopy momentum map, the conditions for f to be a morphism simplify greatly. They can be expressed succinctly in terms of the boundary operator $\delta_{\mathrm{CE}} : \wedge^\bullet \mathfrak{g} \to \wedge^{\bullet-1}\mathfrak{g}$ of the Chevalley-Eilenberg complex for Lie homology, defined by

$$\delta_{\mathrm{CE}}(a_1 \wedge \ldots \wedge a_k) = \sum_{1 \leq i < j \leq k} (-1)^{i+j} [a_i, a_j] \wedge a_1 \wedge \ldots \hat{a}_i \ldots \hat{a}_j \wedge \ldots \wedge a_k$$

for all $a_1, \dots, a_k \in \mathfrak{g}$.

Proposition 8.1.44 (Proposition 3.8 in [CFRZ16]). *Let $\mathfrak{g}$ be a Lie algebra viewed as L_∞-algebra; let L be an L_∞-algebra such that*

$$l_k(a_1, \ldots, a_k) = 0$$

for $k \geq 2$ and $|a_1| + \ldots |a_k| < 0$. Then a morphism $f : \mathfrak{g} \to L$ of L_∞-algebras is given by a family of maps

$$f_k : \wedge^k \mathfrak{g} \longrightarrow L\,,$$

for $k \geq 1$ of degree $|f_k| = 1 - k$, such that

$$l_1 f_k + f_{k-1}\,\delta_{\mathrm{CE}} = -l_k\,.$$

8.2 The L_∞-algebra of conserved currents

8.2.1 The Poisson algebra of a presymplectic manifold

In the context of symplectic geometry, a closed 2-form ω on a manifold X is called a **presymplectic** form. It is called **symplectic** if it is non-degenerate, that is, if the map $TX \to T^*X$, $v \mapsto \iota_v \omega$ is invertible.

Terminology 8.2.1. Some authors call a closed 2-form presymplectic only if it has constant rank. We will not make this assumption.

A vector field $v \in \mathfrak{X}(X)$ is called presymplectic if it leaves ω invariant, $\mathcal{L}_v \omega = 0$. Since ω is closed, v is presymplectic if and only if $\iota_v \omega$ is a closed 1-form. A stronger condition is that $\iota_v \omega$ is exact. This leads to the following concept.

Definition 8.2.2. Let (X, ω) be a presymplectic manifold. A pair (v, f) consisting of a vector field $v \in \mathfrak{X}(X)$ and a function $f \in C^\infty(X)$ is called **hamiltonian** if

$$\iota_v \omega = -df\,.$$

A vector field or a form is called hamiltonian if it is part of a hamiltonian pair.

Remark 8.2.3. ***

We denote the space of hamiltonian vector fields by $\mathfrak{X}_{\mathrm{ham}}(X)$ and the space of hamiltonian functions by $C^\infty_{\mathrm{ham}}(X)$. If (f, v) and (f, v') are both hamiltonian pairs, then $v' - v$ lies in the kernel of ω, $\iota_{v'-v}\omega = 0$. Similarly, if (f, v) and (f', v) are hamiltonian pairs, then $f' - f$ is a closed, that is, locally constant. Every hamiltonian vector field is presymplectic.

Proposition 8.2.4. *$C^\infty_{\mathrm{ham}}(X)$ is an $\mathbb{R}$-subalgebra of $C^\infty(X)$.*

Proof. Let (v, f) and (w, g) be hamiltonian pairs and c a real number. Since the inner derivative and the de Rham differential are both $\mathbb{R}$-linear, $(v + w, f + g)$ and (cv, cf) are hamiltonian pairs. It follows that cf and $f + g$ are hamiltonian. From

$$\begin{aligned} \iota_{gv+fw}\omega &= g\iota_v\omega + f\iota_w\omega = -gdf - fdg \\ &= -d(fg)\,. \end{aligned}$$

it follows that $(gv + fw, fg)$ is a hamiltonian pair, so that fg is hamiltonian. $\square$

For hamiltonian pairs (v, f), (w, g) we define the bracket

$$\{f, g\} := \iota_w \iota_v \omega \,. \tag{8.11}$$

If (f, v') is another hamiltonian pair, then $\{f, g\} = \iota_w \iota_{v'} \omega = \iota_w \iota_v \omega$, since $v' - v$ lies in the kernel of ω. This shows that the bracket (8.11) depends only on the hamiltonian function f and not on the choice of its hamiltonian vector field. The same argument applies to g. In other words, the bracket is well-defined on hamiltonian functions.

Using the Cartan calculus, we can derive the relation

$$\begin{aligned} d\iota_w \iota_v &= \mathcal{L}_w \iota_v - \iota_w d\iota_v \\ &= \iota_{[w,v]} + \iota_v \mathcal{L}_w - \iota_w d\iota_v \\ &= -\iota_{[v,w]} + \iota_v \iota_w d + \iota_v d\iota_w - \iota_w d\iota_v \,. \end{aligned} \tag{8.12}$$

By applying this relation to ω and using that ω, $\iota_w\omega$, and $\iota_v\omega$ are closed, we obtain

$$d\{f, g\} = d\iota_w \iota_v \omega = -\iota_{[v,w]} \omega \,.$$

This shows that $([v, w], \{f, g\})$ is a hamiltonian pair, which implies that the bracket of hamiltonian functions is hamiltonian. We conclude that (8.11) defines a map

$$\{\_\,,\_\} : C^\infty_{\mathrm{ham}}(X) \times C^\infty_{\mathrm{ham}}(X) \longrightarrow C^\infty_{\mathrm{ham}}(X) \,.$$

Proposition 8.2.5. *The bracket* (8.11) *of hamiltonian functions is a Lie bracket and a derivation in each argument.*

Proof. By construction, the bracket is bilinear and antisymmetric. Let (u, f), (v, g), and (w, h) be hamiltonian pairs. Then

$$\begin{aligned} \{f, gh\} &= \iota_{hv+gw} \iota_u \omega = h\iota_u \iota_v \omega + g\iota_u \iota_w \omega \\ &= \{f, g\}h + g\{f, h\} \,, \end{aligned}$$

which shows that the bracket is a derivation in the second argument. It follows from the antisymmetry of the bracket that it is a derivation in the first argument, too.

The iterated bracket can be written as

$$\begin{aligned} \{f, \{g, h\}\} &= \iota_u d\{g, h\} = \iota_u d\iota_w \iota_v \omega \\ &= \mathcal{L}_u \iota_w \iota_v \omega - d\iota_u \iota_w \iota_v \omega \,. \end{aligned} \tag{8.13}$$

Using the relations of the Cartan calculus, we can express the first term as

$$\begin{aligned} \mathcal{L}_u \iota_w \iota_v \omega &= \iota_{[u,w]} \iota_v \omega + \iota_w \mathcal{L}_u \iota_v \omega \\ &= \iota_{[u,w]} \iota_v \omega + \iota_w \iota_{[u,v]} \omega + \iota_w \iota_v \mathcal{L}_u \omega \\ &= \{g, \{f, h\}\} + \{\{f, g\}, h\} \,, \end{aligned} \tag{8.14}$$

where in the last step we have used that the hamiltonian vector field u is symplectic, $\mathcal{L}_u \omega = 0$. Inserting Equation (8.14) into (8.13) and moving all terms with Poisson brackets to the left side, we obtain

$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = -d\iota_w \iota_v \iota_u \omega \,. \tag{8.15}$$

The left hand side is the Jacobiator. The right hand side vanishes because ω is a 2-form. We conclude that the bracket satisfies the Jacobi identity. □

8.2.2 The L_∞-algebra of a premultisymplectic form

Definition 8.2.6. Let M be a manifold with a closed $(n+1)$-form ω. A pair (X, α) consisting of a vector field $X \in \mathfrak{X}(M)$ and a form $\alpha \in \Omega^{n-1}(M)$ is called **hamiltonian** if

$$\iota_X \omega = -d\alpha\,.$$

A vector field or a form is called hamiltonian if it is part of a hamiltonian pair.

We denote the space of hamiltonian vector fields by $\mathfrak{X}_{\mathrm{ham}}(M)$ and the space of hamiltonian forms by $\Omega^{n-1}_{\mathrm{ham}}(M)$. Given the pair (M, ω), we can construct an L_∞-algebra $L_\infty(M, \omega)$ defined as follows.

Theorem 8.2.7 (Theorem 5.2 in [Rog12])**.** *The $\mathbb{Z}$-graded vector space*

$$L_\infty(M,\omega)_i = \begin{cases} \Omega^{n-1}_{\mathrm{ham}}(M) & ; i = 0 \\ \Omega^{n-1+i}(M) & ; 1-n \leq i < 0 \\ 0 & ; \textit{otherwise}\,, \end{cases}$$

with the linear maps $l_k : \wedge^k L_\infty(M,\omega) \to L_\infty(M,\omega)$ defined by

$$l_1(\alpha_1) = d\alpha_1$$

for $|\alpha_1| < 0$,

$$l_k(\alpha_1 \wedge \ldots \wedge \alpha_k) = -(-1)^k \iota_{X_1} \iota_{X_2} \cdots \iota_{X_k} \omega\,,$$

for $|\alpha_1| = \ldots = |\alpha_k| = 0$ where (X_i, α_i) are hamiltonian pairs, and zero in all other cases is an L_∞-algebra.

8.2.3 Premultisymplectic structure of a Lagrangian Field Theory

Let γ be a boundary form. The form

$$\lambda := L + \gamma \tag{8.16}$$

of total degree n will be called the **Lepage form**. Let the total differential of $J^\infty F$ be denoted by $\mathbf{d} = \delta + d$. The total differential of the Lepage form is

$$\begin{aligned} \omega &:= \mathbf{d}\lambda \\ &= EL + \delta\gamma\,, \end{aligned} \tag{8.17}$$

which is the premultisymplectic structure we are interested in.

Terminology 8.2.8. The term "Lepage form" is used for example in [Kru83] or [And89, p. 199]. Deligne and Freed call λ the "total Lagrangian" [DF99, p. 161].

On $J^\infty F$ we have the splitting of vector fields into a vertical and horizontal component which leads to the bigrading on the de Rham complex. Moreover, we have the acyclicity theorem of the variational bicomplex. This leads to the following description of hamiltonian vector fields.

Proposition 8.2.9. *Let X be a vector field on $J^\infty F$ with vertical component $X^\perp$. Then X is hamiltonian with respect to the premultisymplectic form $\omega = EL + \delta\gamma$ if and only if*

(i) $\mathcal{L}_X \omega = 0$, *and*

(ii) $\iota_{X^\perp} EL = dj$ *for some* $j \in \Omega^{0,n-1}(J^\infty F)$.

In the proof, we will use the following lemma [Del18, Thm. 11.1.6].

Lemma 8.2.10. *Let $\beta \in \Omega^n(J^\infty F)$ be a* $\mathbf{d}$*-closed form and*

$$\beta = \beta_0 + \ldots + \beta_n$$

its decomposition into summands of bidegree $\deg \beta_k = (k, n-k)$*. Then β is* $\mathbf{d}$*-exact if and only if β_0 is d-exact.*

Proof. A form $\alpha \in \Omega^{n-1}(J^\infty F)$ can be decomposed as

$$\alpha = \alpha_0 + \ldots + \alpha_{n-1}\,,$$

into components of bidegree $\deg \alpha_k = (k, n-1-k)$. The total differential decomposes as

$$\mathbf{d}\alpha = d\alpha_0 + (\delta\alpha_0 + d\alpha_1) + \ldots + (\delta\alpha_{n-2} + d\alpha_{n-1}) + \delta\alpha_{n-1}\,,$$

into summands of homogeneous bidegree, where the first summand has bidegree $(0, n)$ and the last $(n, 0)$. Assume that $\beta = \mathbf{d}\alpha$. This condition must hold in each bidegree individually. In particular we have $\beta_0 = d\alpha_0$.

Conversely, assume that $\beta_0 = d\alpha_0$ for some $\alpha_0 \in \Omega^{0,n-1}(J^\infty F)$. The total differential of β decomposes as

$$\mathbf{d}\beta = (\delta\beta_0 + d\beta_1) + \ldots + (\delta\beta_{n-1} + d\beta_n) + \delta\beta_n$$

into summands of homogeneous bidegree, where the first summand has bidegree $(1, n)$ and the last $(n+1, 0)$. By assumption $\mathbf{d}\beta = 0$, which has to hold in each bidegree separately,

$$\begin{aligned}
0 &= \delta\beta_0 + d\beta_1 \\
0 &= \delta\beta_1 + d\beta_2 \\
&\;\;\vdots \\
0 &= \delta\beta_{n-1} + d\beta_n \\
0 &= \delta\beta_n\,.
\end{aligned}$$

From the first equation we get

$$\begin{aligned}
0 = \delta\beta_0 + d\beta_1 &= \delta d\alpha_0 + d\beta_1 \\
&= d(-\delta\alpha_0 + \beta_1)\,.
\end{aligned}$$

It follows from the acyclicity theorem for the variational bicomplex [Tak79, Thm. 4.6] that $-\delta\alpha_0+\beta_1 = d\alpha_1$ for some $\alpha_1 \in \Omega^{1,n-2}(J^\infty F)$. The bidegree $(2, n-1)$ component of $\mathbf{d}\beta = 0$ can now be written as

$$\begin{aligned} 0 &= \delta\beta_1 + d\beta_2 = \delta(\delta\alpha_0 + d\alpha_1) + d\beta_2 \\ &= d(-\delta\alpha_1 + \beta_2)\,. \end{aligned}$$

As before, it follows from the acyclicity theorem that $\beta_2 = \delta\alpha_1 + d\alpha_2$ for some $\alpha_2 \in \Omega^{2,n-3}(J^\infty F)$. By induction, we obtain forms $\alpha_0, \ldots, \alpha_{n-1}$ such that $\mathbf{d}\alpha = \beta$ for $\alpha = \alpha_0 + \ldots + \alpha_{n-1}$. □

Proof of Proposition 8.2.9. Let $X^\perp$ be the vertical and $X^\parallel$ the horizontal component of X. Assume that $\iota_X\omega = -\mathbf{d}\alpha$. The left hand side decomposes as

$$\begin{aligned} \iota_X\omega &= (\iota_{X^\perp} + \iota_{X^\parallel})(EL + \delta\gamma) \\ &= \iota_{X^\perp}EL + \big(\iota_{X^\parallel}EL + \iota_{X^\perp}\delta\gamma\big) + \iota_{X^\parallel}\delta\gamma\,, \end{aligned}$$

into summands of bidegree $(0, n)$, $(1, n-1)$, and $(2, n-2)$. We conclude that the bidegree $(0, n)$ component of the hamiltonian condition is $\iota_{X^\perp}EL = -d\alpha_0$, which is condition (ii) for $\alpha_0 = -j$. Since ω is closed, we have $\mathcal{L}_X\omega = \mathbf{d}\iota_X\omega = 0$ which is condition (i).

Conversely, assume that (i) and (ii) hold. This means that $-\beta = \iota_X\omega$ is $\mathbf{d}$-closed and that $\beta_0 = d\alpha_0$, where $\alpha_0 = -j$. It now follows from Lemma 8.2.10 that $\iota_X\omega = -\beta = -\mathbf{d}\alpha$. □

A form $j \in \Omega^{0,n-1}(J^\infty F)$ is also called a **current**. A current is called **conserved** if it is d-closed on shell, that is, if dj vanishes at every solution of the Euler–Lagrange equation. Proposition 8.2.9 shows that the degree $(0, n-1)$ component of a hamiltonian form is a conserved current. In this sense, $L_\infty(J^\infty F, \omega)$ can be viewed as higher current algebra.

The k-bracket of hamiltonian forms $\alpha_1, \ldots, \alpha_k$ is given by

$$\begin{aligned} \{\alpha_1, \ldots, \alpha_k\} &= -(-1)^k(\iota_{X_1^\perp} + \iota_{X_1^\parallel}) \cdots (\iota_{X_k^\perp} + \iota_{X_k^\parallel})(EL + \delta\gamma) \\ &= \sum_{1\le i<j\le k} (-1)^{k-i-j}(\iota_{X_1^\parallel} \cdots \widehat{\iota_{X_i^\parallel}} \cdots \widehat{\iota_{X_j^\parallel}} \cdots \iota_{X_k^\parallel})(\iota_{X_i^\perp}\iota_{X_j^\perp}\delta\gamma) \\ &\quad + \sum_{1\le i\le k} (-1)^{i-1}(\iota_{X_1^\parallel} \cdots \widehat{\iota_{X_i^\parallel}} \cdots \iota_{X_k^\parallel})(\iota_{X_i^\perp}EL) \\ &\quad + \sum_{1\le i\le k} (-1)^{i-1}(\iota_{X_1^\parallel} \cdots \widehat{\iota_{X_i^\parallel}} \cdots \iota_{X_k^\parallel})(\iota_{X_i^\perp}\delta\gamma) - (-1)^k(\iota_{X_1^\parallel} \cdots \iota_{X_k^\parallel})EL \\ &\quad - (-1)^k(\iota_{X_1^\parallel} \cdots \iota_{X_k^\parallel})\delta\gamma\,, \end{aligned}$$

where $X_1, \ldots, X_k$ are the hamiltonian vector fields. The 2-bracket is given by

$$\begin{aligned} \{\alpha, \beta\} &= (\iota_{Y^\perp} + \iota_{Y^\parallel})(\iota_{X^\perp} + \iota_{X^\parallel})(EL + \delta\gamma) \\ &= \iota_{Y^\perp}\iota_{X^\perp}\delta\gamma + (\iota_{Y^\parallel}\iota_{X^\perp} - \iota_{X^\parallel}\iota_{Y^\perp})EL \\ &\quad + (\iota_{Y^\parallel}\iota_{X^\perp} - \iota_{X^\parallel}\iota_{Y^\perp})\delta\gamma + \iota_{X^\parallel}\iota_{X^\parallel}EL \\ &\quad + \iota_{Y^\parallel}\iota_{X^\parallel}\delta\gamma\,, \end{aligned} \tag{8.18}$$

where (X, α) and (Y, β) are hamiltonian pairs.

8.3 Homotopy momentum maps

8.3.1 Hamiltonian actions on presymplectic manifolds

Let $\mathfrak{g}$ be a Lie algebra. An action of $\mathfrak{g}$ on a manifold X is given by a homomorphism of Lie algebras $\rho : \mathfrak{g} \to \mathfrak{X}(X)$. There are several conditions of compatibility of the action with the presymplectic form. The first is that the vector fields $\rho(a)$ are symmetries of the presymplectic form,

$$\begin{aligned} 0 &= \mathcal{L}_{\rho(a)}\omega = (d\iota_{\rho(a)} + \iota_{\rho(a)}d)\omega \\ &= d(\iota_{\rho(a)}\omega)\,, \end{aligned}$$

for all $a \in \mathfrak{g}$. If this condition is satisfied, the action is called **presymplectic**.

A stronger condition is that $\iota_{\rho(a)}\omega$ is exact,

$$\iota_{\rho(a)}\omega = -d\mu_a$$

for some smooth function $\mu_a \in C^\infty(M)$. In other words, the vector field $\rho(a)$ is hamiltonian.

So far, the compatibility conditions have not involved the Lie bracket on $\mathfrak{g}$. The natural condition to require is that the map $\mathfrak{g} \to C^\infty_{\mathrm{ham}}(X)$, $a \to \mu_a$ is a homomorphism of Lie algebras. This leads to the following notion.

Definition 8.3.1. Let $\rho : \mathfrak{g} \to C^\infty(X)$ an action of the Lie algebra $\mathfrak{g}$ on the manifold X that is equipped with a presymplectic form ω. A linear map

$$\mu : \mathfrak{g} \longrightarrow C^\infty(X)\,,$$

such that $\iota_{\rho(a)}\omega = -d\mu(a)$ is called a **momentum map**. An action with an momentum map is called **weakly hamiltonian**. It is called **hamiltonian** if the momentum map is a homomorphism of Lie algebras.

Terminology 8.3.2. Instead of "momentum map", many authors use the term "moment map", which derives from a mistranslation of the French term "moment" as in "moment cinétique" (angular momentum) or "application moment" [Sou70]. In mathematical physics, this creates a confusing clash of terminology, since "moment" is used to denote the integral of a physical quantity times (powers of) the distance from the origin or from an axis, such as the moment of inertia, the moment of force (torque), the electric dipole moment, etc.

Using the universal property of exponential objects in sets, we have the natural bijections

$$\mathcal{S}\mathrm{et}\big(\mathfrak{g}, \mathcal{S}\mathrm{et}(X,\mathbb{R})\big) \cong \mathcal{S}\mathrm{et}(\mathfrak{g} \times X, \mathbb{R}) \cong \mathcal{S}\mathrm{et}\big(X, \mathcal{S}\mathrm{et}(\mathfrak{g},\mathbb{R})\big)\,.$$

Adding the condition that the momentum map is linear in $\mathfrak{g}$ with values in smooth maps $C^\infty(X) \subset \mathcal{S}\mathrm{et}(X,\mathbb{R})$, we obtain the isomorphisms

$$\mathrm{Hom}_{\mathbb{R}}\big(\mathfrak{g}, C^\infty(X)\big) \cong \mathrm{Hom}_{\mathbb{R}}(\mathfrak{g} \times X, \mathbb{R}) \cong \mathcal{M}\mathrm{fld}\big(X, \mathfrak{g}^*\big)\,, \tag{8.19}$$

where $\mathrm{Hom}_{\mathbb{R}}(\mathfrak{g} \times X, \mathbb{R})$ denotes smooth maps that are linear in $\mathfrak{g}$. These isomorphisms allow us to express the momentum map in three equivalent ways.

On the right side of Equation (8.19), we have the map

$$\begin{aligned} \tilde{\mu} : X &\longrightarrow \mathfrak{g}^* \\ p &\mapsto \big(a \mapsto \mu(a)(m)\big) \,. \end{aligned}$$

This map has the technical advantage that it does not involve the infinite dimensional space $C^\infty(X)$. When $\mathfrak{g}$ is finite-dimensional, then μ is a smooth map of finite dimensional manifolds that can be studied with the usual tools of differential geometry. The target $\mathfrak{g}^*$ carries the coadjoint action of $\mathfrak{g}$, defined by $(a \cdot \varphi)(b) = -\varphi([a,b])$ for all $a, b \in \mathfrak{g}$, $\varphi \in \mathfrak{g}^*$. It is straightforward to show that μ is a homomorphism of Lie algebras if and only if $\tilde{\mu}$ is $\mathfrak{g}$-equivariant.

Terminology 8.3.3. ***

In the center of Relations (8.19), the momentum map is given by a map

$$\begin{aligned} \bar{\mu} : \mathfrak{g} \times X &\longrightarrow \mathbb{R} \\ (a,p) &\longmapsto \mu(a)(p) \,, \end{aligned}$$

that is linear in $\mathfrak{g}$ and smooth in X. This point of view is particularly useful for our purposes.

8.3.2 Obstruction cohomology

We can view the momentum map as an element in the bigraded vector space

$$\Omega^{p,q}(\mathfrak{g}, X) := \mathrm{Hom}_{\mathbb{R}}\big(\wedge^p \mathfrak{g}, \Omega^q(X)\big) \,,$$

where $\mathrm{Hom}_{\mathbb{R}}$ denotes linear maps. This can be equipped with the structure of a differential bicomplex by the differentials of Lie algebra cohomology and de Rham cohomology as follows.

Let $\mathcal{U}\mathfrak{g}$ denote the universal enveloping algebra of $\mathfrak{g}$. Its **counit** is a homomorphism of $\mathbb{R}$-algebras $\varepsilon : \mathcal{U}\mathfrak{g} \to \mathbb{R}$ given by $\varepsilon(1) = 1$ and $\varepsilon(a) = 0$ for all $a \in \mathfrak{g}$. With the counit, every vector space V can be equipped with the **trivial action** $u \cdot v = \varepsilon(u)v$ for all $u \in \mathcal{U}\mathfrak{g}$ and $v \in V$. The Lie algebra cohomology of $\mathfrak{g}$ is the cohomology of the trivial $\mathcal{U}\mathfrak{g}$-module $\mathbb{R}$. The standard resolution of this module is given by the Chevalley-Eilenberg complex

$$\mathbb{R} \xleftarrow{\varepsilon} \mathcal{U}\mathfrak{g} \otimes \mathfrak{g} \xleftarrow{\delta_{\mathrm{CE}}} \mathcal{U}\mathfrak{g} \otimes \wedge^2 \mathfrak{g} \xleftarrow{\delta_{\mathrm{CE}}} \ldots \,,$$

with the differential δ_{CE} defined by

$$\begin{aligned} \delta_{\mathrm{CE}}(u \otimes a_0 \wedge \ldots \wedge a_p) &= \sum_{i=0}^{p} (-1)^i u a_i \otimes a_0 \wedge \ldots \wedge \hat{a}_i \wedge \ldots \wedge a_p \\ &\quad + \sum_{1 \le i < j \le p} (-1)^{i+j} u \otimes [a_i, a_j] \wedge \ldots \wedge \hat{a}_i \wedge \ldots \wedge \hat{a}_j \wedge \ldots \wedge a_p \,. \end{aligned}$$

The space of q-forms $\Omega^q(X)$ is equipped with the left action by the Lie algebra $\mathfrak{g}$ given by $a \cdot \varphi = \mathcal{L}_{\rho(a)}\varphi$ for all $a \in \mathfrak{g}$. Equivalently, this equips $\Omega^q(X)$ with a left action by the algebra $\mathcal{U}\mathfrak{g}$. The de Rham differential commutes with Lie derivatives, so $\Omega(X)$ is a complex of $\mathcal{U}\mathfrak{g}$-modules.

Since the Chevalley-Eilenberg complex consists of free $\mathcal{U}\mathfrak{g}$-modules, we have

$$\Omega^{p,q}(\mathfrak{g}, X) = \mathrm{Hom}_{\mathcal{U}\mathfrak{g}}\big(\mathcal{U}\mathfrak{g} \otimes \wedge^p \mathfrak{g}, \Omega^q(X)\big)\,,$$

which is equipped with the differentials

$$\begin{aligned} d_{\mathfrak{g}}\varphi &:= \varphi \circ \delta_{\mathrm{CE}} \\ d_X \varphi &:= (-1)^p d \circ \varphi\,, \end{aligned}$$

for $\varphi \in \Omega^{p,q}(\mathfrak{g}, X)$.

Proposition 8.3.4. *If $\mathfrak{g}$ is finite-dimensional, then*

$$H^k\big(\mathrm{Tot}(\Omega^{\bullet,\bullet}(\mathfrak{g}, X))\big) = \bigoplus_{p=0}^{k} H^p(\mathfrak{g}) \otimes H^{k-p}(X)\,.$$

where $H(\mathfrak{g}) = H(\mathfrak{g}, \mathbb{R})$ denotes Lie algebra cohomology with trivial coefficients.

Proof. Since $\mathfrak{g}$ is finite-dimensional, we have the isomorphism

$$\Omega^{p,q}(\mathfrak{g}, X) \cong \big(\wedge^p \mathfrak{g}^* \otimes \mathcal{U}\mathfrak{g}\big) \otimes_{\mathcal{U}\mathfrak{g}} \Omega^q(X)\,,$$

which shows that $\Omega^{p,q}(\mathfrak{g}, X)$ is the tensor product of complexes of $\mathcal{U}\mathfrak{g}$-modules, so that we can apply the Künneth formula. Since one of the factors consists of free modules, there is no torsion term. *** □

We will denote the evaluation of $\varphi : \wedge^p \mathfrak{g} \to \Omega^q(X)$ also by

$$\begin{aligned} \varphi(a_1, \ldots, a_p, v_0, \ldots, v_q) &\equiv \varphi(a_1 \wedge \ldots \wedge a_p)(v_1, \ldots, v_q) \\ &\equiv \iota_{v_q} \cdots \iota_{v_1} \varphi(a_1 \wedge \ldots \wedge a_p)\,. \end{aligned}$$

With this notation, we view φ as a bilinear map

$$\varphi : \wedge^p \mathfrak{g} \times \wedge^q TX \longrightarrow \mathbb{R}\,.$$

The Lie algebroid differential of φ is given by

$$\begin{aligned} (d_{\mathfrak{g}}\varphi)(a_0, \ldots, a_p) = &\sum_{0 \le i \le p} (-1)^i \mathcal{L}_{\rho(a_i)} \varphi(a_0, \ldots, \hat{a}_i, \ldots, a_p) \\ &+ \sum_{0 \le i < j \le p} (-1)^{i+j} \varphi([a_i, a_j], a_0, \ldots, \hat{a}_i, \ldots, \hat{a}_j, \ldots, a_p)\,, \end{aligned}$$

where the hat on $\hat{a}_i$ denotes the omission of the argument a_i. When we evaluate this formula at vector fields $v_1, \ldots, v_q$, we have to move the inner derivatives past the

Lie derivative, using the relation $\iota_v \mathcal{L}_{\rho(a)} = \mathcal{L}_{\rho(a)} \iota_v - \iota_{[\rho(a),v]}$. By iteration, we obtain the formula

$$\begin{aligned}
\iota_{v_q} \cdots \iota_{v_1} \mathcal{L}_{\rho(a_i)} &= \iota_{v_q} \cdots \iota_{v_2} \mathcal{L}_{\rho(a_i)} \iota_{v_1} - \iota_{v_q} \cdots \iota_{v_2} \iota_{[\rho(a_i),v_1]} \\
&= \iota_{v_q} \cdots \iota_{v_3} \mathcal{L}_{\rho(a_i)} \iota_{v_2} \iota_{v_1} - \iota_{v_q} \cdots \iota_{[\rho(a_i),v_2]} \iota_{v_1} - \iota_{v_q} \cdots \iota_{v_2} \iota_{[\rho(a_i),v_1]} \\
&\vdots \\
&= \mathcal{L}_{\rho(a_i)} \iota_{v_p} \cdots \iota_{v_1} - \sum_{1 \le k \le q} \iota_{v_q} \cdots \iota_{v_{k+1}} \iota_{[\rho(a_i),v_k]} \iota_{v_{k-1}} \cdots \iota_{v_1} \,.
\end{aligned}$$

With this relation, we obtain

$$\begin{aligned}
&(d_{\mathfrak{g}}\varphi)(a_0, \ldots, a_p, v_1, \ldots, v_q) \\
&\quad = \sum_{0 \le i \le q} (-1)^i \rho(a_i) \cdot \varphi(a_0, \ldots, \hat{a}_i, \ldots, a_p, v_1, \ldots, v_q) \\
&\qquad - \sum_{\substack{0 \le i \le p \\ 0 \le k \le q}} (-1)^i \bar{\varphi}(a_0, \ldots, \hat{a}_i, \ldots, a_p, v_1, \ldots, v_{k-1}, [\rho(a_i), v_k], v_{k+1}, \ldots v_q) \\
&\qquad + \sum_{0 \le i < j \le p} (-1)^{i+j} \varphi([a_i, a_j], a_0, \ldots, \hat{a}_i, \ldots, \hat{a}_j, \ldots, a_p, v_1, \ldots, v_q)
\end{aligned}$$

Similarly, the de Rham differential is given by

$$\begin{aligned}
&(d_X\varphi)(a_1, \ldots, a_p, v_0, \ldots v_q) \\
&\quad = (-1)^p \sum_{0 \le i \le q} (-1)^i v_i \cdot \varphi(a_1, \ldots, a_p, v_1, \ldots, \hat{v}_i, \ldots, v_p) \\
&\qquad + (-1)^p \sum_{0 \le i < j \le q} (-1)^{i+j} \varphi(a_1, \ldots, a_p, [v_i, v_j], v_0, \ldots, \hat{v}_i, \ldots, \hat{v}_j, \ldots, v_q) \,.
\end{aligned}$$

We will now express the structure of a hamiltonian action in terms of the bicomplex $\Omega^{p,q}(\mathfrak{g}, X)$. For this, we will use the forms $\bar{\omega}_1 \in \Omega^{1,1}(\mathfrak{g}, X)$ and $\bar{\omega}_2 \in \Omega^{2,0}(\mathfrak{g}, X)$ that are defined by

$$\begin{aligned}
\bar{\omega}_1(a, v) &:= \omega\big(\rho(a), v\big) \\
\bar{\omega}_2(a, b) &:= \omega\big(\rho(a), \rho(b)\big) \,,
\end{aligned}$$

for all $a, b \in \mathfrak{g}$ and $v \in TX$. Let

$$\bar{\omega} := \bar{\omega}_1 + \bar{\omega}_2 \in \Omega^2(\mathfrak{g}, X) \,.$$

Theorem 8.3.5. *Let $\rho : \mathfrak{g} \to C^\infty(X)$ be an action of the Lie algebra $\mathfrak{g}$ on the presymplectic manifold (X, ω). Then:*

(i) The action is presymplectic if and only if $\bar{\omega}$ is closed under the total differential $d_{\mathrm{tot}} = d_{\mathfrak{g}} + d_X$.

(ii) $\mu : \mathfrak{g} \to C^\infty(M)$ is a momentum map if and only if

$$\bar{\omega}_1 = d_X \mu \,.$$

(iii) A momentum map μ is a homomorphism of Lie algebras if and only if

$$\bar{\omega}_2 = d_{\mathfrak{g}}\mu \, .$$

Proof. The de Rham differential of $\bar{\omega}_1$ is given by

$$\begin{aligned}(d_X\bar{\omega}_1)(a,v,w) &= -\iota_w\iota_v d\iota_{\rho(a)}\omega \\ &= -\iota_w\iota_v\mathcal{L}_{\rho(a)}\omega + \iota_w\iota_v\iota_{\rho(a)}d\omega \\ &= -\iota_w\iota_v\mathcal{L}_{\rho(a)}\omega \, ,\end{aligned} \tag{8.20}$$

where we have used Cartan's magic formula and that ω is d_X-closed. The Lie algebra differential of $\bar{\omega}_1$ is given by

$$\begin{aligned}(d_{\mathfrak{g}}\bar{\omega}_1)(a,b,v) &= \rho(a)\cdot\bar{\omega}_1(b,v) - \bar{\omega}_1\big(b,[\rho(a),v]\big) \\ &\quad - \rho(b)\cdot\bar{\omega}_1(b,v) + \bar{\omega}_1\big(a,[\rho(b),v]\big) - \bar{\omega}_1([a,b],v) \\ &= -\rho(a)\cdot\omega\big(\rho(b),v\big) + \omega\big(a,[\rho(b),v]\big) \\ &\quad - \rho(b)\cdot\omega\big(\rho(a),v\big) + \omega\big(a,[\rho(b),v]\big) - \omega\big(\rho([a,b]),v\big) \\ &= (d\omega)\big(\rho(a),\rho(b),v\big) - v\cdot\omega\big(\rho(a),\rho(b)\big) \\ &= -v\cdot\omega\big(\rho(a),\rho(b)\big) \, .\end{aligned}$$

The de Rham differential of $\bar{\omega}_2$ is given by

$$\begin{aligned}(d_X\bar{\omega}_2)(a,b,v) &= v\cdot\bar{\omega}_2(a,b) \\ &= v\cdot\omega\big(\rho(a),\rho(b)\big) \, .\end{aligned}$$

From the last two equations we deduce that

$$d_{\mathfrak{g}}\bar{\omega}_1 + d_X\bar{\omega}_2 = 0 \, .$$

The Lie algebra differential of $\bar{\omega}_1$ is given by

$$\begin{aligned}(d_{\mathfrak{g}}\bar{\omega}_2)(a,b,c) &= \rho(a)\cdot\bar{\omega}_2(b,c) - \rho(b)\cdot\bar{\omega}_2(a,c) + \rho(c)\cdot\bar{\omega}_2(a,b) \\ &\quad - \bar{\omega}_2\big([a,b],c\big) + \bar{\omega}_2\big([a,c],b\big) - \bar{\omega}_2\big([b,c],a\big) \\ &= \rho(a)\cdot\omega\big(\rho(b),\rho(c)\big) - \rho(b)\cdot\omega\big(\rho(a),\rho(c)\big) + \rho(c)\cdot\omega\big(\rho(a),\rho(b)\big) \\ &\quad - \omega\big(\rho([a,b]),c\big) + \omega\big(\rho([a,c]),b\big) - \omega\big(\rho([b,c]),a\big) \\ &= (d\omega)\big(\rho(a),\rho(b),\rho(c)\big) \\ &= 0 \, .\end{aligned}$$

Putting everything together, we obtain

$$\begin{aligned}d_{\mathrm{tot}}\bar{\omega} &= (d_{\mathfrak{g}} + d_X)(\bar{\omega}_1 + \bar{\omega}_2) \\ &= d_X\bar{\omega}_1 + (d_{\mathfrak{g}}\bar{\omega}_1 + d_X\bar{\omega}_2) + d_{\mathfrak{g}}\bar{\omega}_2 \\ &= d_X\bar{\omega}_1 \, .\end{aligned}$$

From Equation (8.20) we see that the right side vanishes if and only if $\mathcal{L}_{\rho(a)}\omega = 0$ for all $a \in \mathfrak{g}$. We conclude that $\bar{\omega}$ is d_{tot}-closed if and only if the action is presymplectic. This proves (i).

Let $\mu : \mathfrak{g} \to C^\infty(M)$ be a linear map, which we view as an element $\bar{\mu} \in \Omega^{1,0}(\mathfrak{g}, X)$. Its de Rham differential is given by

$$(d_X\mu)(a, v) = -v \cdot \mu(a) = -\iota_v d\big(\mu(a)\big)\,.$$

We conclude that $d_X\bar{\mu} = \bar{\omega}_1$ if and only if μ is a momentum map. This proves (ii).

The Lie algebra differential of $\bar{\mu}$ is given by

$$(d_{\mathfrak{g}}\mu)(a, b) = \rho(a) \cdot \mu(b) - \rho(b) \cdot \mu(a) - \mu([a, b])\,.$$

Assuming that μ is a momentum map, we obtain

$$\begin{aligned}(d_{\mathfrak{g}}\mu)(a, b) &= \iota_{\rho(a)} d\mu(b) - \iota_{\rho(b)} d\mu(a) - \mu([a, b]) \\ &= -\iota_{\rho(a)}\iota_{\rho(b)}\omega + \iota_{\rho(b)}\iota_{\rho(a)}\omega - \mu([a, b]) \\ &= 2\omega\big(\rho(a), \rho(b)\big) - \mu([a, b])\,.\end{aligned}$$

We conclude that $d_{\mathfrak{g}}\mu = \bar{\omega}_2$ if and only if

$$\mu([a, b]) = \omega\big(\rho(a), \rho(b)\big) = \big\{\mu(a), \mu(b)\big\}\,,$$

that is, if the momentum map is a homomorphism of Lie algebras. This proves (iii). □

Proposition 8.3.6. *An action $\rho : \mathfrak{g} \to C^\infty(X)$ of the Lie algebra $\mathfrak{g}$ on the presymplectic manifold (X, ω) is hamiltonian if and only if $\bar{\omega}$ is exact in the total complex of the bicomplex $\Omega^{\bullet,\bullet}(\mathfrak{g}, X)$.*

Proof. Let $\theta \in \Omega^{0,1}(\mathfrak{g} \times X)$ and $\mu \in \Omega^{1,0}(\mathfrak{g} \times X)$. The condition $\bar{\omega} = d_{\mathrm{tot}}(\theta + \mu)$ is equivalent to the three equations

$$\begin{aligned}0 &= d_X\theta \\ \bar{\omega}_1 &= d_X\mu + d_{\mathfrak{g}}\theta \\ \bar{\omega}_2 &= d_{\mathfrak{g}}\mu\,,\end{aligned}$$

in bidegrees $(0, 2)$, $(1, 1)$, and $(2, 0)$. The differentials of θ are given by

$$d_X\theta = d\theta$$

and

$$(d_{\mathfrak{g}}\theta)(a, v) = \iota_v \mathcal{L}_{\rho(a)}\theta\,.$$

The other differentials were computed in the proof of Theorem 8.3.5. This yields the three conditions

$$\begin{aligned}0 &= d\theta \\ \omega\big(\rho(a), v\big) &= -\iota_v d\mu(a) + \iota_v d\iota_{\rho(a)}\theta + \iota_v\iota_{\rho(a)} d\theta \\ \omega\big(\rho(a), \rho(b)\big) &= 2\omega\big(\rho(a), \rho(b)\big) - \mu([a, b])\,,\end{aligned}$$

for all $v \in \mathfrak{X}(X)$, $a, b \in \mathfrak{g}$. Inserting the first equation into the second, and rearranging the terms, we obtain the equivalent conditions

$$\begin{aligned} d\theta &= 0 && (8.21a) \\ \iota_{\rho(a)}\omega &= -d\big(\mu(a) - \iota_{\rho(a)}\theta\big) && (8.21b) \\ \mu([a,b]) &= \omega\big(\rho(a), \rho(b)\big) \,. && (8.21c) \end{aligned}$$

Assume that Equations (8.21) are satisfied. Equation (8.21a) states that θ is a closed 1-form. Let $\bar\theta \in \Omega^{1,0}(\mathfrak{g} \times X)$ be defined by $\bar\theta(a) = \iota_{\rho(a)}\theta$. Its Lie algebra differential is given by

$$\begin{aligned} (d_{\mathfrak{g}}\bar\theta)(a,b) &= \rho(a) \cdot \theta\big(\rho(b)\big) - \rho(b) \cdot \theta\big(\rho(a)\big) - \rho(a) \cdot \theta\big(\rho([a,b])\big) \\ &= \rho(a) \cdot \theta\big(\rho(b)\big) - \rho(b) \cdot \theta\big(\rho(a)\big) - \rho(a) \cdot \theta\big(\rho([a,b])\big) \\ &= (d\theta)\big(\rho(a), \rho(b)\big) \\ &= 0 \,. \end{aligned}$$

Let $\mu' := \mu(a) - \bar\theta(\mu)$. Equation (8.21a) states that μ' is a momentum map. Since $d_{\mathfrak{g}}\mu = d_{\mathfrak{g}}(\mu' + \bar\theta) = d\mu'$, Equation (8.21c) states that μ' is a homomorphism of Lie algebras.

Conversely, assume that μ is the momentum map of a hamiltonian action. Then we can choose $\theta = 0$ to satisfy Equations (8.21). □

In other words, Proposition 8.3.6 tells us that the cohomology class

$$[\bar\omega] \in H^2\big(\mathrm{Tot}(\Omega^{\bullet,\bullet}(\mathfrak{g}, X))\big)$$

is the obstruction to a presymplectic action being hamiltonian. More precisely, with the Künneth formula of Proposition 8.3.4 we conclude that

$$[\bar\omega_1] \in H^1(\mathfrak{g}) \otimes H^1(X)$$

is the obstruction to the existence of a momentum map and

$$[\bar\omega_2] \in H^2(\mathfrak{g}) \otimes H^0(X)$$

the obstruction for the momentum map to be a homomorphism of Lie algebras.

Corollary 8.3.7. *Let $\mathfrak{g}$ be a semisimple Lie algebra. Then every presymplectic $\mathfrak{g}$-action is hamiltonian.*

Proof. The first Whitehead lemma states that $H^1(\mathfrak{g}) = 0$, the second Whitehead lemma that $H^2(\mathfrak{g}) = 0$. Therefore, both obstructions to the action being hamiltonian vanish. □

8.3.3 Homotopy momentum maps

A collection of linear maps $\mu_k : \wedge^k\mathfrak{g} \to L_\infty(M, \omega)$ is a homotopy momentum map if and only if [CFRZ16, Prop. 3.8]

$$d\mu_k(a_1 \wedge \ldots \wedge a_k) + \mu_{k-1}\, \delta_{\mathrm{CE}}(a_1 \wedge \ldots \wedge a_k) = (-1)^k \iota_{\rho(a_1)} \cdots \iota_{\rho(a_k)}\omega \,,$$

for all $1 \leq k \leq n+1$, where we set $\mu_0 := 0$ and $\mu_{n+1} := 0$. This relation can be interpreted homotopically as follows. Shifting the degree of $\mathfrak{g}$ by 1 and shifting the degree of the de Rham complex by $n+1$, the right hand side can be expressed in terms of the degree 0 map

$$\begin{aligned} \nu : S(\mathfrak{g}[1]) &\longrightarrow \Omega(M)[n+1] \\ \nu(a_1 \wedge \cdots \wedge a_k) &:= (-1)^k \iota_{\rho(a_1)} \cdots \iota_{\rho(a_k)} \omega \,, \end{aligned}$$

where we have used that $S(\mathfrak{g}[1])_{-k} \cong \wedge^k \mathfrak{g}$. The maps μ_k have degree -1. The condition for μ to be a morphism of L_∞-algebras can be written succinctly as [CFRZ16, Sec. 6.2]

$$d\mu_k + \mu_{k-1}\, \delta_{\mathrm{CE}} = \nu_k \,, \tag{8.22}$$

that is, a homotopy momentum map μ is a null-homotopy of the map of cochain complexes ν.

Definition 8.3.8 (Definition/Proposition 5.1 in [CFRZ16])**.** Let M be a manifold with a closed $(n+1)$-form ω. Let $\rho : \mathfrak{g} \to \mathfrak{X}(M)$ be a homomorphism of Lie algebras. A **homotopy momentum map** of the action ρ is a homomorphism of L_∞-algebras

$$\mu : \mathfrak{g} \longrightarrow L_\infty(M, \omega) \,,$$

such that

$$\iota_{\rho(a)}\omega = -d\, \mu_1(a)$$

for all $a \in \mathfrak{g}$.

In degree $k = 1$ the condition (8.22) reads $d\mu_1(a_1) = -\iota_{\rho(a_1)}\omega$, that is $\big(\rho(a_1), \mu_1(a_1)\big)$ is a hamiltonian pair. With this relation, ν can be expressed in terms of the L_∞-brackets as

$$\nu(a_1 \wedge \ldots \wedge a_k) = -l_k\big(\mu_1(a_1) \wedge \ldots \wedge \mu_1(a_k)\big) \,.$$

For $k = 2$, Equation (8.22) is spelled out as

$$l_2\big(\mu_1(a_1) \wedge \mu_1(a_2)\big) = \mu_1\big([a_1, a_2]\big) - d\mu_2(a_1 \wedge a_2) \,, \tag{8.23}$$

which shows that the failure of μ_1 to be a homomorphism of Lie algebras is a d-exact term. For $k = 3$ we obtain

$$\begin{aligned} l_3\big(\mu_1(a_1) \wedge \mu_1(a_2) \wedge \mu_1(a_3)\big) = {} & \mu_2([a_1, a_2] \wedge a_3 + [a_2, a_3] \wedge a_1 + [a_3, a_1] \wedge a_2) \\ & - d\mu_3(a_1 \wedge a_2 \wedge a_3) \,. \end{aligned}$$

Proposition 8.3.9 (Sec. 8.1 in [CFRZ16])**.** *Let $\omega = d\lambda$ for some $\lambda \in \Omega^n(M)$. If λ is invariant under the action $\rho : \mathfrak{g} \to \mathfrak{X}(M)$, i.e.*

$$\mathcal{L}_{\rho(a)}\lambda = 0$$

for all $a \in \mathfrak{g}$, then it has a homotopy momentum map $\mu : \mathfrak{g} \to L_\infty(M, \omega)$ given by

$$\mu_k(a_1 \wedge \ldots \wedge a_k) = \iota_{\rho(a_1)} \cdots \iota_{\rho(a_k)}\lambda \,.$$

Notation 8.3.10. For shorter notation we will write the k-bracket also as

$$\begin{aligned} l_k(\alpha_1 \wedge \ldots \wedge \alpha_k) &\equiv l_k(\alpha_1, \ldots, \alpha_k) \\ &\equiv \{\alpha_1, \ldots, \alpha_k\} \end{aligned}$$

Analogously, we will write the momentum map as

$$\mu_k(a_1 \wedge \ldots \wedge a_k) \equiv \mu_k(a_1, \ldots, a_k)\,.$$

In this notation, Equation (8.23) is written as

$$\{\mu_1(a_1), \mu_1(a_2)\} = \mu_1\big([a_1, a_2]\big) - d\mu_2(a_1, a_2)\,. \tag{8.24}$$

8.3.4 Manifest diffeomorphism symmetries

In [DF99, p. 169], a **manifest symmetry** was defined to be a vector field $X \in \mathfrak{X}(J^\infty F)$ such that:

(i) $X = \xi + \hat{v}$ is the sum of a strictly vertical vector field ξ and a strictly horizontal vector field $\hat{v}$.

(ii) $\mathcal{L}_{\xi+\hat{v}}(L + \gamma) = 0$.

This suggests the following terminology:

Definition 8.3.11. Let (M, F, L) be a LFT with boundary form γ. An action

$$\begin{aligned} \rho : \mathfrak{X}(M) &\longrightarrow \mathfrak{X}(J^\infty F) \\ v &\longmapsto \rho(v) := \xi_v + \hat{v}\,. \end{aligned}$$

by manifest symmetries will be called a **manifest diffeomorphism symmetry**.

Remark 8.3.12. The Cartan lift $v \mapsto \hat{v}$ of vector fields on M is a homomorphism of Lie algebras. Since strictly vertical and strictly horizontal vector fields commute, it follows that the map $v \mapsto \xi_v$ is a homomorphism of Lie algebras, too.

Remark 8.3.13. If $F \to M$ is a natural bundle, that is diffeomorphisms between open subsets of M lift functorially to diffeomorphisms between local sections, then it follows from [ET79] that we have an action of vector fields on $J^\infty F$. The diffeomorphism symmetries of LFTs often arise in this way.

Proposition 8.3.14. *Let (M, F, L) be an LFT with boundary form γ. Then every manifest diffeomorphism symmetry $\rho : \mathfrak{X}(M) \to \mathfrak{X}(J^\infty F)$ has a homotopy momentum map*

$$\mu : \mathfrak{X}(M) \longrightarrow L_\infty(J^\infty F, EL + \delta\gamma)\,.$$

given by

$$\mu_k(v_1, \ldots, v_k) := \iota_{\rho(v_1)} \cdots \iota_{\rho(v_k)}(L + \gamma)\,.$$

Proof. This is a special case of Proposition 8.3.9. ∎

The homotopy momentum map of a single vector field is split into a bidegree $(0, n-1)$ and a bidegree $(1, n-2)$ summand as

$$\begin{aligned}\mu_1(v) &= (\iota_{\xi_v} + \iota_{\hat{v}})(L+\gamma) = (\iota_{\hat{v}} L + \iota_{\xi_v}\gamma) + \iota_{\hat{v}}\gamma \\ &= -j_v + \iota_{\hat{v}}\gamma\,,\end{aligned} \tag{8.25}$$

where

$$j_v = -\iota_{\hat{v}} L - \iota_{\xi_v}\gamma \tag{8.26}$$

is the Noether current of ξ_v. In general, the map μ_k splits into a $(0, n-k)$ and a $(1, n-k-1)$ component given by the two lines of the right hand side of the equation

$$\begin{aligned}\mu_k(v_1, \ldots, v_k) = &-\sum_{i=1}^{k}(-1)^{k-i}(\iota_{\hat{v}_1}\cdots\widehat{\iota_{\hat{v}_i}}\cdots\iota_{\hat{v}_k})j_{v_i} + (1-k)(\iota_{\hat{v}_1}\cdots\iota_{\hat{v}_k})L \\ &+ (\iota_{\hat{v}_1}\cdots\iota_{\hat{v}_k})\gamma\,.\end{aligned}$$

For example, we have

$$\mu_2(v,w) = (\iota_{\hat{v}} j_w - \iota_{\hat{w}} j_v + \iota_{\hat{v}}\iota_{\hat{w}} L) + \iota_{\hat{v}}\iota_{\hat{w}}\gamma$$

Using Equation (8.18), we can write the l_2-bracket of the momenta as

$$\begin{aligned}\{\mu_1(v), \mu_1(w)\} = &\ \iota_{\xi_w}\iota_{\xi_v}\delta\gamma + (\iota_{\hat{w}}\iota_{\xi_v} - \iota_{\hat{v}}\iota_{\xi_w})EL \\ &+ (\iota_{\hat{w}}\iota_{\xi_v} - \iota_{\hat{v}}\iota_{\xi_w})\delta\gamma + \iota_{\hat{w}}\iota_{\hat{v}} EL \\ &+ \iota_{\hat{w}}\iota_{\hat{v}}\delta\gamma\,,\end{aligned} \tag{8.27}$$

where the three lines of the right hand side are of bidegrees $(0, n-1)$, $(1, n-2)$, and $(2, n-3)$. The right hand side of Equation (8.24) is expressed in terms of the Noether current as

$$\begin{aligned}\mu_1([v,w]) - \mathbf{d}\mu_2(v,w) = &-j_{[v,w]} + d(\iota_{\hat{v}} j_w - \iota_{\hat{w}} j_v + \iota_{\hat{v}}\iota_{\hat{w}} L) \\ &+ \iota_{\widehat{[v,w]}}\gamma - \delta(\iota_{\hat{v}} j_w - \iota_{\hat{w}} j_v + \iota_{\hat{v}}\iota_{\hat{w}} L) - d\iota_{\hat{v}}\iota_{\hat{w}}\gamma \\ &+ \iota_{\hat{w}}\iota_{\hat{v}}\delta\gamma\,.\end{aligned}$$

Remark 8.3.15. If we integrate $\mu_1(a)$ over a closed codimension 1 submanifold $\Sigma \subset M$, we see from Equation (8.25) that we obtain, up to a sign, the usual Noether charge $\int_\Sigma \mu_1(v) = -\int_\Sigma j_v$. This is no longer true for the brackets. The integral $\int_\Sigma \iota_{\xi_w}\iota_{\xi_v}\delta\gamma$ of the first summand on the right hand side of Equation (8.27) is the usual bracket of charges. The integral of the second summand, however, is an additional contribution, which is not present in the multimomentum map of [BHL10, Sec. 4.1]. The integrals of all other terms on the right hand side of Equation (8.27) vanish for degree reasons.

Example 8.3.16 (Classical mechanics)**.** In classical mechanics spacetime is time $M = \mathbb{R}$ and the configuration bundle is trivial, $F = \mathbb{R} \times Q \to \mathbb{R}$, so that $\mathcal{F} = C^\infty(\mathbb{R}, Q)$ is the space of smooth paths in Q. Let us consider the lagrangian of a particle of mass 1 in a potential V,

$$L = \big(\tfrac{1}{2}\dot{q}^i\dot{q}^i - V(q)\big)dt\,.$$

Here $t, q^i, \dot{q}^i, \ddot{q}^i, \ldots$ are coordinates on the infinite jet bundle, given by

$$\dot{q}^i(j_0^\infty x) = \frac{dx^i}{dt}\Big|_{t=0}$$

for a path $x : \mathbb{R} \to Q$. Using the relations $d\delta q^i = -\delta\dot{q}^i \wedge dt$, $dq^i = \dot{q}^i dt$, and $d\dot{q}^i = \ddot{q}^i dt$, we find that $\delta L = EL - d\gamma$ with

$$\begin{aligned} EL &= -\Big(\ddot{q}^i + \frac{\partial V}{\partial q^i}\Big)\delta q^i \wedge dt \\ \gamma &= \dot{q}^i \delta q^i \,. \end{aligned}$$

For the presymplectic form ω we obtain

$$\omega = -\Big(\ddot{q}^i + \frac{\partial V}{\partial q^i}\Big)\delta q^i \wedge dt + \delta\dot{q}^i \wedge \delta q^i \,,$$

which is a form on $J^2(\mathbb{R} \times Q)$. The Cartan lift of the infinitesimal generator of time translation, i.e of the coordinate vector field $\partial_t \equiv \frac{\partial}{\partial t} \in \mathfrak{X}(\mathbb{R})$ is

$$\hat{\partial}_t = \frac{\partial}{\partial t} + \dot{q}^i \frac{\partial}{\partial q^i} + \ddot{q}^i \frac{\partial}{\partial \dot{q}^i} + \ldots$$

The time translation $x(\tau) \mapsto x(\tau - t)$ of paths descends to the strictly vertical vector field

$$\xi_{\partial_t} = -\dot{q}^i \frac{\partial}{\partial q^i} - \ddot{q}^i \frac{\partial}{\partial \dot{q}^i} - \ldots$$

The fundamental vector field of the diagonal action of time translation on $J^\infty F$ is therefore given by

$$\rho(\partial_t) = \xi_{\partial_t} + \hat{\partial}_t = \frac{\partial}{\partial t} \,. \tag{8.28}$$

This equation looks like a tautology, but the vector field $\frac{\partial}{\partial t}$ on the right hand side is not horizontal and must not be identified with the vector field in the time direction. Moreover, ρ is not $C^\infty(M)$-linear.

Equation (8.28) implies that $\mathcal{L}_{\rho(\partial_t)}(L + \gamma) = 0$, so that time translation is a manifest symmetry. The corresponding momentum map is given by

$$\mu_1(\partial_t) = -j_{\partial_t} \,,$$

since for degree reasons the term $\iota_{\hat{\partial}_t}\gamma$ vanishes. The Noether momentum

$$j_{\partial_t} = -\iota_{\hat{\partial}_t} L - \iota_{\xi_{\partial_t}} \gamma = \tfrac{1}{2}\dot{q}^i \dot{q}^i + V(q)$$

is the energy.

Chapter 9

Examples

9.1 Particle in a potential

Let us consider classical mechanics, where we had

$$i_\varepsilon(\delta L - EL) = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}^i}\varepsilon^i\right) dt = d\left(i_\varepsilon \frac{\partial \mathcal{L}}{\partial \dot{q}^i}\delta q^i\right) = -i_\varepsilon d\gamma\,,$$

where

$$\gamma = \frac{\partial L}{\partial \dot{q}^i}\delta q^i\,.$$

Let us consider the lagrangian $L = \left(\frac{1}{2}m\dot{q}^i\dot{q}^i - V(q)\right)dt = (T - V)dt$ of a particle of mass m moving in a potential V, which is defined on the first jet bundle $J^1(Q \times \mathbb{R}) \cong TQ \times \mathbb{R}$.

Consider time translation, i.e. the diffeomorphism $\Phi : \mathbb{R} \to \mathbb{R}$, $t \mapsto t + \varepsilon$, which acts on a path by push-forward. Since

$$\big(\Phi_*(q^i)\big)(t) = q^i\big(\Phi^{-1}(t)\big) = q^i(t - \varepsilon) = q^i(t) - \varepsilon\dot{q}^i(t) + \mathcal{O}(\varepsilon^2)\,,$$

The corresponding vector field on $\mathcal{F} = C^\infty(\mathbb{R}, Q)$ is given in jet coordinates by

$$\xi = -\dot{q}^i\frac{\partial}{\partial q^i} - \ddot{q}^i\frac{\partial}{\partial \dot{q}^i} - \dddot{q}^i\frac{\partial}{\partial \ddot{q}^i} - \dots$$

We can check that this is a symmetry of the homological action,

$$\mathcal{L}_\xi L = -\left(m\dot{q}^i\ddot{q}^i - \frac{\partial V}{\partial q^i}\dot{q}^i\right) dt = -dL = d\alpha_\xi\,.$$

From Noether's theorem we now obtain

$$j_\xi = \alpha_\xi - i_\xi\gamma = -T + V + m\dot{q}^i\dot{q}^i = T + V\,,$$

which is the total energy of the mechanical system.

Example 9.1.1. Consider the lagrangian given by the length of a path in a riemannian manifold Q. Every vector field $f(t)\frac{\partial}{\partial t}$ of $\mathbb{R}$ induces a symmetry on paths by reparametrization. The conserved current is given by $j_f = 0$. [Exercise]

Example 9.1.2. To the 1-form $A = A_i dx^i$ can add an exact 1-form df without changing the lagrangian of Maxwell theory. In coordinates the vector field is

$$\xi_f = \frac{\partial f}{\partial x^i}\frac{\partial}{\partial A_i} + \frac{\partial^2 f}{\partial x^i x^j}\frac{\partial}{\partial(\partial_i A_j)}\,.$$

The conserved current is given by $j = *(\partial^i F_{ij} dx^j)$. [Exercise]

It can be shown that an infinitesimal symmetry of the homological action is tangent to the variety of solutions $\mathcal{F}_{\text{shell}} = EL^{-1}(0)$. This means that on shell

$$0 = \int_S (-\delta j_\xi + i_\xi \delta\gamma) = -\delta q_\xi + i_\xi \delta\omega_S\,.$$

In other words, on shell the infinitesimal symmetry ξ is a hamiltonian vector field generated by the Noether charge q_ξ. If the Noether charge vanishes on-shell then ξ lies in the kernel of ω_S. We conclude that if there is a family of symmetries $\{\xi_f\}_{f\in C^\infty(M)}$ to which Noether's second theorem applies then every ξ_f lies in the kernel of ω_S.

9.2 Free particle on a curved background

9.3 Gauge theory

9.3.1 Review of connections on principal bundles

In Yang-Mills gauge theory the fields are connections on a principal bundle. We will first review this concept.

Definition of connections on principal bundles Let G be a Lie group and $\pi : P \to M$ a right principal G-bundle. We denote the free and proper right G-action by $P \times G \to P$, $(p, g) \mapsto p \cdot g = R_g p$, where $R : G \to \mathrm{Diff}(P)$ denotes the structure homomorphism of the action. A connection on the fibre bundle $P \to M$ is given by a **horizontal lift** $h : TM \times_M P \to TP$, i.e. a right splitting of the short exact sequence of vector bundles over P,

$$\begin{tikzcd} 0 \arrow[r] & VP \arrow[r] & TP \arrow[r, "{(T\pi,\mathrm{pr}_P)}"] & TM \times_M P \arrow[l, bend left, "h"] \arrow[r] & 0 \end{tikzcd}\ . \tag{9.1}$$

The group G acts on $\xi_p \in TP$ by

$$\xi_p \cdot g := TR_g \xi_p\,.$$

Since the bundle projection $\pi : P \to M$ is G-invariant, $\pi(R_g p) = \pi(p)$, its tangent map $T\pi : TP \to TM$ is invariant as well, $T\pi(TR_g\xi_p) = T\pi\,\xi_p$. Since TR_g is a map of vector bundles covering R_g, $\mathrm{pr}_P(TR_g\xi_p) = R_g(\mathrm{pr}_P\xi_p)$, the tangent projection $\mathrm{pr}_P : TP \to P$ is G-equivariant. It follows that, when we equip $TM \times_M P$ with the right G-action defined by

$$(v, p) \cdot g := (v, p \cdot g)\,,$$

then the map $(T\pi, \mathrm{pr}_P)$ is G-equivariant. From the G-invariance of $T\pi$ it follows that $VP = \ker T\pi$ is also G-invariant, so that the inclusion $VP \subset TP$ is G-equivariant. The upshot is that the short exact sequence (9.1) is a sequence of G-equivariant maps of vector bundles over P. Therefore, we should require the splitting h of a connection to be G-equivariant as well.

Definition 9.3.1. A **connection on a principal bundle** P or a **principal connection on** P is an equivariant splitting h of the short exact sequence (9.1) of vector bundles over P.

The affine space of connections The set of connections on P is a subset of the vector space of all maps of vector bundles $TM \times_M P \to TP$. However, since the zero map is not a section of $(T\pi, \mathrm{pr}_P)$, connections are not a vector subspace. The difference of two connections h and h' satisfies

$$T\pi\big(h'(v,p) - h(v,p)\big) = 0\,,$$

that is, $\mu := h' - h$ takes its values in $\ker T\pi = VP$. Conversely, let $\mu : TM \times_M P \to VP$ be a G-equivariant map of vector bundles over P. Then $h + \mu$ satisfies $T\pi\big(h(v,p) + \mu(v,p)\big) = v$, so that h is a G-equivariant splitting of (9.1). The conclusion is that the set of connections on the principal bundle P is an affine space modelled on the vector space of G-equivariant maps $TM \times_M P \to VP$ of vector bundles.

Since the G-actions are free and proper, such a G-equivariant map can be identified with a map on the G-quotients, $(TM \times_M P)/G \to VP/G$. Since G acts trivially on TM, the quotient of the domain is

$$(TM \times_M P)/G \cong TM \times_M (P/G) \cong TM\,.$$

The quotient VP/G of the target has a nice description, too. Every vertical tangent vector of P can be represented by a smooth path $t \mapsto p_t \in P$ with constant base point $\pi(p_t) = \pi(p_0)$. Since the fibre over $\pi(p_0)$ is isomorphic to G, there is a unique smooth path $t \mapsto g_t \in G$ with $g_0 = e$, such that $p_t = p_0 \cdot g_t$. It follows that we can identify the tangent space at every point with $\mathfrak{g}$, which means that we have an isomorphism of vector bundles

$$VP \cong P \times \mathfrak{g}\,.$$

The action of $h \in G$ on the vertical path p_t is given by

$$p_t \cdot h = (p_0 \cdot g_t) \cdot h = (p_0 \cdot h) \cdot h^{-1} g_t h\,.$$

Differentiating with respect to t, we see that the action of G on VP is given on the isomorphic vector bundle $P \times \mathfrak{g}$ by

$$(p, X) \cdot g = (p \cdot g, \mathrm{Ad}_{g^{-1}} X)\,.$$

It follows that the quotient

$$VP/G \cong (P \times \mathfrak{g})/G = P \times_{\mathrm{Ad}} \mathfrak{g}$$

is the vector bundle associated to the principal bundle by the adjoint representation $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$, which is called the **adjoint bundle**. We summarize our findings in the following proposition.

Proposition 9.3.2. *Let P be a principal G-bundle and $P \times_{\mathrm{Ad}} \mathfrak{g}$ the associated adjoint bundle. The set of principal connections on P is an affine space modelled on the vector space*

$$\Gamma^\infty\big(M, T^*M \otimes (P \times_{\mathrm{Ad}} \mathfrak{g})\big) \cong \Omega^1(M) \otimes_{C^\infty(M)} \Gamma^\infty(M, P \times_{\mathrm{Ad}} \mathfrak{g}) .$$

Corollary 9.3.3. *When the adjoint bundle of P is trivial, $P \times_{\mathrm{Ad}} \mathfrak{g} \cong M \times \mathfrak{g}$, then the affine space of connections is modelled on the vector space of $\mathfrak{g}$-valued 1-forms $\Omega^1(M) \otimes \mathfrak{g}$.*

There are two basic cases, in which the adjoint bundle is trivial, so that Cor. 9.3.3 applies. In the first case P is a trivial bundle. An important example for this is when $M = \mathbb{R}^4$ is Minkowski space. Another example is, when we restrict M to a coordinate ball $U \subset M$. This implies that locally, connections are modelled on the space of $\mathfrak{g}$-valued 1-forms. These forms are called **local connection 1-forms**.

The second case is that G is abelian, so that the adjoint representation is trivial. For example when $G = \mathrm{U}(1)$, so that $\mathfrak{g} = \mathrm{u}(1) = \mathbb{R}$, principal connections are modelled on the vector space of 1-forms on M. This is the case we have in Maxwell theory.

Curvature Taking the quotient by G of the sequence (9.1), we obtain a short sequence of vector bundles over M,

$$0 \longrightarrow P \times_{\mathrm{Ad}} \mathfrak{g} \longrightarrow TP/G \xrightarrow{T\pi} TM \longrightarrow 0 \ , \tag{9.2}$$

which is called the **Atiyah sequence** of the principal bundle P. This sequence of vector bundles induces a sequence of the vector spaces of sections,

$$0 \longrightarrow \Gamma^\infty(M, P \times_{\mathrm{Ad}} \mathfrak{g}) \longrightarrow \mathfrak{X}(P)^G \xrightarrow{\pi_*} \mathfrak{X}(M) \longrightarrow 0 \ , \tag{9.3}$$

where $\mathfrak{X}(P)^G$ denotes the space of G-invariant vector fields on P.

Remark 9.3.4. The right G-action on P induces a left G-action on vector fields by pullback, $g \cdot \xi = R_g^* \xi$. A vector field ξ is G-invariant if it is a fixed point under this action. Observe, that the map $\xi : P \to TP$ of a G-invariant vector field is G-equivariant.

A splitting of (9.2) induces a splitting $h : \mathfrak{X}(M) \to \mathfrak{X}(P)^G$ of (9.3). The **curvature** of the connection is given by

$$F(v, w) := [h(v), h(w)] - h\big([v, w]\big) , \tag{9.4}$$

for all $v, w \in \mathfrak{X}(M)$. The curvature vanishes if and only if h is a homomorphism of Lie algebras. If this is the case, the connection is called **flat**.

Remark 9.3.5. Sequence (9.3) can be viewed as an extension of Lie algebras. Then F is the 2-cocycle in the Lie algebroid cohomology that classifies extensions up to isomorphism.

The horizontal lift of every $v \in \mathfrak{X}(M)$ projects to v, $\pi_* h(v) = v$. In other words, $h(v)$ is **π-related** to v. Since the Lie brackets of π-related vector fields are π-related (see e.g. Prop. 8.30 in [Lee13]), the curvature satisfies

$$\begin{aligned} \pi_* F(v,w) &= \pi_*[h(v), h(w)] - \pi_* h\big([v,w]\big) \\ &= [\pi_* h(v), \pi_* h(w)] - [v,w] \\ &= 0\,. \end{aligned}$$

Moreover, using that $h(fv) = (\pi^* f) h(v)$ and $h(v) \cdot \pi^* f = \pi^*(v \cdot f)$ for every $f \in C^\infty(M)$, a similar calculation shows that F is $C^\infty(M)$-linear in both arguments, which implies that F is a bundle map on $\wedge^2 T^* M$. We conclude that the curvature is a 2-form with values in the adjoint bundle,

$$\begin{aligned} F &\in \Gamma^\infty\big(M, \wedge^2 T^* M \otimes (P \times_{\mathrm{Ad}} \mathfrak{g})\big) \\ &\cong \Omega^2(M) \otimes_{C^\infty(M)} \Gamma^\infty(M, P \times_{\mathrm{Ad}} \mathfrak{g})\,. \end{aligned}$$

Remark 9.3.6. According to Prop. 9.3.2, the set of connections is an *affine space* modelled on the vector space of 1-forms on M with values in the adjoint bundle. The curvature, however, takes values in the *vector space* of 2-forms on M with values in the adjoint bundle. The reason is that the curvature is defined as difference of two terms in an affine space.

Connection and curvature as invariant forms As it is the case for short exact sequences in any abelian category, a right splitting of the sequence (9.1) induces a left splitting and vice versa. In fact, given a horizontal lift $h : TM \times_M P \to TP$, we obtain a map

$$\begin{aligned} \theta : TP &\longrightarrow VP \\ \xi_p &\longmapsto \theta(\xi_p) := \xi_p - h\big(T\pi\, \xi_p, p\big)\,, \end{aligned}$$

which maps $\xi_p \in VP$ to ξ_p, so it is a retract of the inclusion $VP \to TP$. Moreover, if h is G-equivariant, then so is θ. Using the natural trivialization $VP \cong P \times \mathfrak{g}$, this retract can be viewed as a linear map $\theta : TP \to \mathfrak{g}$, which is equivariant with respect to the action $\xi_p \cdot g = TR_g \xi_p$ on TP and $X \cdot g = \mathrm{Ad}_{g^{-1}} X$ on $\mathfrak{g}$. These observations are summarized in the following proposition.

Proposition 9.3.7. *Let $\Omega^\bullet(P) \otimes \mathfrak{g}$ be equipped with the left G-action defined by*

$$g \cdot (\alpha \otimes X) := R_g^* \alpha \otimes \mathrm{Ad}_g X\,,$$

for all $\alpha \otimes X \in \Omega^\bullet(P) \otimes \mathfrak{g}$. Then a principal connection on P is given by a unique G-invariant form $\theta \in \Omega^1(P) \otimes \mathfrak{g}$ that acts as the identity $\theta(\xi_p) = \xi_p$ on all vertical vectors $\xi_p \in V_p P \cong \mathfrak{g}$, $p \in P$.

Terminology 9.3.8. In view of Prop. 9.3.7, an invariant $\mathfrak{g}$-valued 1-form on P that restricts to the identity on vertical vectors is called a **connection 1-form**.

Given a connection 1-form θ or, equivalently, a horizontal lift h, the **horizontal tangent space** at $p \in P$ is defined as

$$H_p := \ker \theta_p = h(TM \times_M \{p\}) \subset T_pP \,.$$

The **horizontal distribution** $H = \ker \theta \subset TP$ is the Ehresmann connection given by θ. Since θ is G-invariant, so is H. In fact, a connection on a principal bundle can be identified with a G-invariant Ehresmann connection.

Terminology 9.3.9. A form in $\Omega^\bullet(P) \otimes \mathfrak{g}$ is called **horizontal** if it annihilates the vertical tangent bundle VP. A form that is horizontal and G-invariant is called **basic**.

The vector space of all G-invariant forms will be denoted by $(\Omega^\bullet(P) \otimes \mathfrak{g})^G$ and the space of horizontal forms by $\Omega^\bullet(P)_{\mathrm{hor}}$. So the space of basic forms is denoted by $(\Omega^\bullet(P)_{\mathrm{hor}} \otimes \mathfrak{g})^G$.

Proposition 9.3.10. *The set of connection 1-forms is an affine space modelled on the vector space of basic 1-forms.*

Proof. Let θ be a connection 1-form. If θ' another connection 1-form, then $\mu := \theta' - \theta$ is a G-invariant 1-form, such that for all $\xi_p \in VP$ we have $\mu(\xi_p) = \theta'(\xi_p) - \theta(\xi_p) = \xi_p - \xi_p = 0$, so that μ is horizontal. Conversely, if μ is a basic 1-form on P, then $\theta' := \theta + \mu$ is a G-invariant 1-form on P, such that for all $\xi_p \in VP$ we have $\theta'(\xi_p) = \theta(\xi_p) + \mu(\xi_p) = \xi_p + 0 = \xi_p$, so that θ' is a connection 1-form. □

Both Prop. 9.3.10 and Prop. 9.3.2 establish that the set of connections has the natural structure of an affine space, which implies that the affine spaces of the two propositions must be isomorphic. The following lemma makes this explicit.

Lemma 9.3.11. *A connection on the principal bundle P induces an isomorphism of $C^\infty(M)$-modules*

$$\Gamma^\infty\big(M, \wedge^\bullet T^*M \otimes (P \times_{\mathrm{Ad}} \mathfrak{g})\big) \cong (\Omega^\bullet(P)_{\mathrm{hor}} \otimes \mathfrak{g})^G \,. \tag{9.5}$$

Proof. A section σ of $\wedge^k T^*M \otimes (P \times_{\mathrm{Ad}} \mathfrak{g}) \to M$ can be identified with a map

$$\wedge^k TM \longrightarrow P \times_{\mathrm{Ad}} \mathfrak{g} \,,$$

of vector bundles over M, which in turn can be identified with a G-equivariant map

$$\wedge^k TM \times_M P \longrightarrow P \times \mathfrak{g}$$

of vector bundles over P. The horizontal lift induces a G-equivariant isomorphism

$$h : TM \times_M P \xrightarrow{\cong} H$$

of vector bundles over P. This shows that σ can be identified with a G-equivariant linear map

$$\wedge^k H \longrightarrow \mathfrak{g} \,,$$

which can be identified with a G-invariant section of $\wedge^k H^* \otimes \mathfrak{g} \to P$, which in turn can be identified with a basic form

$$\mu_\sigma \in (\Omega^k(P)_{\mathrm{hor}} \otimes \mathfrak{g})^G \cong \Gamma^\infty(M, \wedge^k H \otimes \mathfrak{g})^G .$$

From μ_σ we retrieve σ by

$$\sigma(v_1 \wedge \ldots \wedge v_k, p) = \big[p, \mu\big(h(v_1, p) \wedge \ldots \wedge h(v_k, p)\big)\big] ,$$

for all $v_1, \ldots, v_k \in T_m M$, all $m \in M$, and all p in the fibre over m, where $[p, X]$ for $p \in P$, $X \in \mathfrak{g}$ denotes an equivalence class in $P \times_{\mathrm{Ad}} \mathfrak{g} = (P \times \mathfrak{g})/G$. □

Remark 9.3.12. A local trivialization $P|_U \cong U \times G$ induces an isomorphism of each side of Eq. (9.5) with $\Omega^\bullet(U) \otimes \mathfrak{g}$.

A G-invariant vector field ξ is vertical if and only if it projects to the zero vector field, $\pi_*\xi = 0$. If ξ is vertical and χ an arbitrary G-invariant vector field, then

$$\begin{aligned} \pi_*[\xi, \chi] &= [\pi_*\xi, \pi_*\chi] = [0, \pi_*\chi] \\ &= 0 , \end{aligned}$$

that is, the Lie bracket of a vertical G-invariant vector field with any other G-invariant vector is again vertical.

A connection is flat if the horizontal distribution H is integrable, which by the Frobenius theorem is the case if and only if the space of horizontal vector fields is closed under the Lie bracket. Every vector field $\xi \in \mathfrak{X}(P)^G$ can be decomposed as $\xi = \xi_V + \xi_H$ into its vertical and horizontal parts,

$$\xi_V = \theta(\xi) , \quad \xi_H = \xi - \theta(\xi) .$$

Since a vector field is horizontal if and only if it is annihilated by θ, the distribution H is involutive if and only if

$$\tilde{F}(\xi, \chi) := \theta\big([\xi_H, \chi_H]\big) \tag{9.6}$$

vanishes for all $\xi, \chi \in \mathfrak{X}(P)^G$. It is straightforward to check that $\tilde{F}(\xi, \chi)$ is $C^\infty(P)$-linear in both arguments, so it is a two form on P. Moreover, $\tilde{F}$ is vertical and annihilates horizontal vector fields, so it can be viewed as a basic 2-form,

$$\tilde{F} \in (\Omega^2(P)_{\mathrm{hor}} \otimes \mathfrak{g})^G .$$

Proposition 9.3.13. *The basic 2-form $\tilde{F}$ is identified by the isomorphism of Lem. 9.3.11 with the curvature form F defined in Eq.* (9.4).

Proof. For every $v \in \mathfrak{X}(M)$, the horizontal lift $h(v) \in \mathfrak{X}(P)^G$ is the unique horizontal G-invariant vector field that projects to $\pi_*h(v) = v$. When we evaluate $\tilde{F}$ on the horizontal lifts of two vector fields $v, w \in \mathfrak{X}(M)$, we obtain

$$\begin{aligned} \tilde{F}\big(h(v), h(w)\big) &= \theta\big([h(v), h(w)]\big) \\ &= \theta\big([h(v), h(w)] - h([v, w])\big) \\ &= [h(v), h(w)] - h([v, w]) \\ &= F(v, w) , \end{aligned}$$

which proves the proposition. □

Notation 9.3.14. In view of Prop. 9.3.13, we will from now on denote the 2-form $\tilde{F}$ defined in Eq. (9.6) also by $F \equiv \tilde{F}$.

The DGLA of invariant forms The de Rham differential on $\Omega^\bullet(P)$ and the Lie bracket on $\mathfrak{g}$ can be extended to the graded vector space $\Omega^\bullet(P) \otimes \mathfrak{g}$, by

$$\begin{aligned} d(\alpha \otimes X) &:= d\alpha \otimes X \\ [\alpha \otimes X, \beta \otimes Y] &:= (\alpha \wedge \beta) \otimes [X, Y] \,, \end{aligned} \tag{9.7}$$

for all $\alpha \otimes X, \beta \otimes Y \in \Omega^\bullet(P) \otimes \mathfrak{g}$. The following proposition is straightforward to prove.

Proposition 9.3.15. *The differential and bracket* (9.7) *equip the graded vector space* $\Omega^\bullet(P) \otimes \mathfrak{g}$ *with the structure of a differential graded Lie algebra (DGLA).*

Proposition 9.3.16. *The graded subspace* $(\Omega^\bullet(P) \otimes \mathfrak{g})^G \subset \Omega^\bullet(P) \otimes \mathfrak{g}$ *of G-invariant forms is a sub-DGLA, i.e. it is closed under the differential and the Lie bracket.*

Proof. Every pullback commutes with the differential, $R_g^* d\alpha = d(R_g^*\alpha)$, and with the product, $R_g^*(\alpha \wedge \beta) = R_g^*\alpha \wedge R_g^*\beta$, for all $\alpha, \beta \in \Omega^\bullet(P)$. The adjoint action commutes with the Lie bracket $\mathrm{Ad}_g[X, Y] = [\mathrm{Ad}_g X, \mathrm{Ad}_g Y]$. With these relations it is easy to show that the bracket of invariant forms $\alpha \otimes X, \beta \otimes Y \in (\Omega^\bullet(P) \otimes \mathfrak{g})^G$ satisfies

$$\begin{aligned} g \cdot [\alpha \otimes X, \beta \otimes Y] &= g \cdot \big((\alpha \wedge \beta) \otimes [X, Y]\big) \\ &= R_g^*(\alpha \wedge \beta) \otimes \mathrm{Ad}_g[X, Y] \\ &= (R_g^*\alpha \wedge R_g^*\beta) \otimes [\mathrm{Ad}_g X, \mathrm{Ad}_g Y] \\ &= [R_g^*\alpha \otimes \mathrm{Ad}_g X, R_g^*\beta \otimes \mathrm{Ad}_g Y] \\ &= [g \cdot (\alpha \otimes X), g \cdot (\beta \otimes Y)] \\ &= [\alpha \otimes X, \beta \otimes Y] \,, \end{aligned}$$

so it is G-invariant. Similarly, we obtain for the differential of a G-invariant form

$$\begin{aligned} g \cdot d(\alpha \otimes X) &= g \cdot (d\alpha \otimes X) \\ &= R_g^* d\alpha \otimes \mathrm{Ad}_g X \\ &= d(R_g^*\alpha) \otimes \mathrm{Ad}_g X \\ &= d(R_g^*\alpha \otimes \mathrm{Ad}_g X) \\ &= d\big(g \cdot (\alpha \otimes X)\big) \\ &= d(\alpha \otimes X) \,, \end{aligned}$$

so it is G-invariant, as well. □

Proposition 9.3.17. *The curvature of a connection 1-form* $\theta \in (\Omega^1(P) \otimes \mathfrak{g})^G$ *is given by*

$$F = -d\theta + \frac{1}{2}[\theta, \theta] \,.$$

Proof. The curvature can be written as

$$\begin{aligned} F(\xi, \chi) &= \theta\big([\xi - \theta(\xi), \chi - \theta(\chi)]\big) \\ &= \theta\big([\xi, \chi] - [\xi, \theta(\chi)] - [\theta(\xi), \chi] + [\theta(\xi), \theta(\chi)]\big) \\ &= \theta([\xi, \chi]) - [\xi, \theta(\chi)] + [\chi, \theta(\xi)] + [\theta(\xi), \theta(\chi)] \end{aligned} \tag{9.8}$$

for all G-invariant vector fields ξ, χ.

By the identification $VP = P \times \mathfrak{g}$, the elements of $\mathfrak{g}$ are the fundamental vector fields of the G-action on P. So if a vector field $\xi \in \mathfrak{X}(P)$ is G-invariant, $R_g^* \xi = \xi$, then the Lie derivative of ξ with respect to all $X \in \mathfrak{g}$ must vanish,

$$[\xi, X] = 0\,. \tag{9.9}$$

Let $\{X_\alpha\} \subset \mathfrak{g}$ be a basis. Then the connection 1-form can be written as $\theta = \theta^\alpha \otimes X_\alpha$. It follows from (9.9), that for G-invariant vector fields $\xi, \chi \in \mathfrak{X}(P)^G$ we have

$$[\xi, \theta(\chi)] = [\xi, \theta^\alpha(\chi) X_\alpha] = \big(\xi \cdot \theta^\alpha(\chi)\big) X_\alpha\,.$$

With this relation we obtain

$$\begin{aligned}(d\theta)(\xi, \chi) &= (d\theta^\alpha)(\xi, \chi)\, X_\alpha \\ &= \big(\xi \cdot \theta^\alpha(\chi) - \chi \cdot \theta^\alpha(\xi) - \theta^\alpha([\xi, \chi])\big)\, X_\alpha \\ &= [\xi, \theta(\chi)] - [\chi, \theta(\xi)] - \theta([\xi, \chi])\,,\end{aligned}$$

which is minus the first three terms of the right hand side of Eq. (9.8). For the last term, we have

$$\begin{aligned}[\theta, \theta](\xi, \chi) &= \iota_\chi \iota_\xi [\theta, \theta] \\ &= \iota_\chi \iota_\xi [\theta^\alpha \otimes X_\alpha, \theta^\beta \otimes X_\beta] \\ &= \iota_\chi \iota_\xi (\theta^\alpha \wedge \theta^\beta) \otimes [X_\alpha, X_\beta] \\ &= \iota_\chi \big(\theta^\alpha(\xi)\, \theta^\beta - \theta^\alpha\, \theta^\beta(\xi)\big) \otimes [X_\alpha, X_\alpha] \\ &= \big(\theta^\alpha(\xi)\, \theta^\beta(\chi) - \theta^a \alpha(\chi)\, \theta^\beta(\xi)\big) \otimes [X_\alpha, X_\beta] \\ &= 2[\theta^\alpha(\xi) X_\alpha, \theta^\beta(\chi) X_\beta] \\ &= 2[\theta(\xi), \theta(\chi)]\,,\end{aligned}$$

from which is follows that $\frac{1}{2}[\theta, \theta](\xi, \chi) = [\theta(\xi), \theta(\chi)]$. We conclude that the sum of $(-d\theta)(\xi, \chi)$ and $\frac{1}{2}[\theta, \theta](\xi, \chi)$ is the right hand side of (9.8). □

Terminology 9.3.18. An element A of a DGLA is called **Maurer-Cartan element** if $dA + \frac{1}{2}[A, A] = 0$. In this terminology, a connection 1-form defines a flat connection if $A = -\theta$ is a Maurer-Cartan element of the DGLA $(\Omega^\bullet(P) \otimes \mathfrak{g})^G$.

Given a connection 1-form θ, we define a linear map by

$$\begin{aligned} d_\theta : \Omega^\bullet(P) \otimes \mathfrak{g} &\longrightarrow \Omega^{\bullet+1}(P) \otimes \mathfrak{g} \\ d_\theta \eta &:= d\eta - [\theta, \eta] \end{aligned} \tag{9.10}$$

for all $\eta \in \Omega^\bullet(P) \otimes \mathfrak{g}$. The map d_θ is a derivation, i.e. it satisfies

$$d_\theta[\eta, \zeta] = [d_\theta \eta, \zeta] + (-1)^{|\eta|}[\eta, d_\theta \zeta]$$

for all $\eta, \zeta \in \Omega^\bullet(P) \otimes \mathfrak{g}$, which can be checked by a straightforward calculation. Since θ is G-invariant, d_θ maps G-invariant forms to G-invariant forms, so it induces a degree 1 derivation on G-invariant forms. Moreover, d_θ is a differential, $d_\theta^2 = 0$, if and only if θ defines a flat connection.

Proposition 9.3.19. *Let θ be a connection 1-form and F its curvature 2-form. Then $d_\theta F = 0$.*

Proof. We have

$$\begin{aligned} d_\theta F &= d\big(-d\theta + \tfrac{1}{2}[\theta,\theta]\big) - \big[\theta, -d\theta + \tfrac{1}{2}[\theta,\theta]\big] \\ &= -d^2\theta + \tfrac{1}{2}\big([d\theta,\theta] - [\theta,d\theta]\big) + [\theta,d\theta] - \tfrac{1}{2}\big[\theta,[\theta,\theta]\big] \\ &= -\tfrac{1}{2}\big[\theta,[\theta,\theta]\big]\,, \end{aligned}$$

where we have used $d^2 = 0$ and $[d\theta,\theta] = -[\theta,d\theta]$. For the remaining term we get from the graded Jacobi identity

$$\begin{aligned} [\theta,[\theta,\theta]] &= [[\theta,\theta],\theta]] - [\theta,[\theta,\theta]] \\ &= -2[\theta,[\theta,\theta]]\,, \end{aligned}$$

which implies that $[\theta,[\theta,\theta]] = 0$. □

9.3.2 Yang-Mills gauge theory

In Yang-Mills gauge theory, the fields are the connections on a principal G-bundle P over a lorentzian 4-manifold M. As we have seen in Prop. 9.3.2, connections are sections of an affine bundle, so they are really fields in the sense of Def. 1.1.1. We define the **gauge field**

$$A := -\theta$$

to be the negative of the connection 1-form. The curvature is given in terms of A by

$$F(A) = dA + \tfrac{1}{2}[A,A]\,. \tag{9.11}$$

The product of fields So far, we have equipped $(\Omega^\bullet(P)\otimes\mathfrak{g})^G$ with the structure of a DGLA. In order to make sense of terms like $F(A)\wedge\star F(A)$, which appear in the lagrangian (1.5) of Maxwell theory, we have to be able to multiply elements of $(\Omega^\bullet(P)\otimes\mathfrak{g})^G$. This is achieved by embedding $\mathfrak{g}$ into its universal enveloping algebra $\mathfrak{U}\mathfrak{g}$, which is the free associative algebra generated by $\mathfrak{g}$ modulo the relations $XY - YX = [X,Y]$ for all $X,Y\in\mathfrak{g}$. The adjoint action of G on $\mathfrak{g}$ extends uniquely to the adjoint action on $\mathfrak{U}\mathfrak{g}$, so we obtain a map

$$(\Omega^\bullet(P)\otimes\mathfrak{g})^G \longrightarrow \big(\Omega^\bullet(P)\otimes\mathfrak{U}\mathfrak{g}\big)^G\,.$$

The right hand side is a differential graded algebra (DGA). The associative product of $\alpha\otimes a$ and $\beta\otimes b$ in $\Omega^\bullet(P)\otimes\mathfrak{U}\mathfrak{g}$ is denoted by

$$(\alpha\otimes a)\wedge(\beta\otimes b) := (\alpha\wedge\beta)\otimes ab\,.$$

Warning 9.3.20. The product in $\Omega^\bullet(P)\otimes\mathfrak{U}\mathfrak{g}$ is denoted by $\wedge$, even though it is not graded anti-commutative.

The trace Let $\Phi : G \to \mathrm{GL}(k, \mathbb{R})$ be a finite-dimensional representation of G and $\rho : \mathfrak{U}\mathfrak{g} \to \mathrm{Mat}(k, \mathbb{R})$ the corresponding representation of the universal enveloping algebra. Let $\mathrm{Tr} : \mathrm{Mat}(k, \mathbb{R}) \to \mathbb{R}$ denote the trace. We define

$$\mathrm{Tr}_\rho : \mathfrak{U}\mathfrak{g} \xrightarrow{\rho} \mathrm{Mat}(k, \mathbb{R}) \xrightarrow{\mathrm{Tr}} \mathbb{R}\,.$$

Note that Tr_ρ inherits the trace property $\mathrm{Tr}_\rho(XY) = \mathrm{Tr}_\rho(YX)$ from the trace of matrices. The action of G induces an action of G on $\mathrm{Mat}(k, \mathbb{R})$ given by $g \cdot B := \Phi(g)B\Phi(g)^{-1}$. The map ρ is G-equivariant with respect to this action and the adjoint action on G. The matrix trace is invariant with respect to the adjoint action, so that Tr_ρ is G-invariant. It follows that the map

$$\Omega^\bullet(P) \otimes \mathfrak{U}\mathfrak{g} \xrightarrow{\mathrm{id}\otimes\mathrm{Tr}_\rho} \Omega^\bullet(P) \otimes \mathbb{R}$$

is G-invariant, so that it descends to a map on equivariant forms,

$$\begin{aligned} \mathrm{Tr}_\rho : \big(\Omega^\bullet(P)_{\mathrm{hor}} \otimes \mathfrak{U}\mathfrak{g}\big)^G &\longrightarrow \Omega^\bullet(M) \\ \eta \otimes a &\longmapsto \mathrm{Tr}_\rho(a)\,\eta\,, \end{aligned} \tag{9.12}$$

where we have used the isomorphism

$$(\Omega(P)_{\mathrm{hor}})^G \cong \Omega(M)\,.$$

From (9.12) we can deduce that the trace is graded cyclic,

$$\mathrm{Tr}_\rho(\eta_1 \wedge \ldots \wedge \eta_k) = (-1)^{|\eta_1|(|\eta_2|+\ldots+|\eta_k|)}\mathrm{Tr}_\rho(\eta_2 \wedge \ldots \wedge \eta_k \wedge \eta_1)\,. \tag{9.13}$$

Remark 9.3.21. For every Lie algebra there is the adjoint representation on the vector space $\mathfrak{g}$, given by $\mathrm{ad}(X)\,Y = [X, Y]$. The bilinear form $\mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$, $(X, Y) \mapsto \mathrm{Tr}_{\mathrm{ad}}(X, Y)$ is called the **Killing form**. A real Lie algebra is semi-simple if the Killing form is non-degenerate, and it is the Lie algebra of a compact Lie group if the Killing form is negative definite. So when G is semi-simple compact, like SU(2) and SU(3), the trace is taken with respect to the adjoint action. For $G = \mathrm{U}(1)$, however, the adjoint action is trivial, so that the trace has to be taken with respect to a non-zero character of u(1).

Lagrangian and field equations We now have all the technical ingredients to write down the vacuum Yang-Mills lagrangian, which is given by

$$L(A) = \mathrm{Tr}_\rho\big(\tfrac{1}{2}\,F(A) \wedge \star F(A)\big)\,, \tag{9.14}$$

where ρ is the adjoint representation for G semisimple and a non-zero element of $\mathfrak{g}^*$ for $G = \mathrm{U}(1)$. The Euler-Lagrange equation is

$$d_A \star F(A) = 0\,, \tag{9.15}$$

where $d_A = d_\theta$ is the gauge equivariant extension of d, which was introduced in Eq. (9.10).

If we view Eq. (9.15) as equation for the field F, we have to add the equation

$$d_A F = 0 \tag{9.16}$$

to the field equations. In Prop. 9.3.19 we have seen that this equation is satisfied for the field $F = F(A)$ that arises as curvature of A. Eqs. (9.15) and (9.16) together are called the the **Yang-Mills equations**.

Example 9.3.22. Let $G = \mathrm{U}(1)$, so that $\mathfrak{g} = \mathbb{R}$. Since $\mathrm{U}(1)$ is abelian, the adjoint action is trivial, which implies the isomorphism

$$\big(\Omega^\bullet(P)_{\mathrm{hor}} \otimes \mathbb{R}\big)^{\mathrm{U}(1)} \cong \Omega^\bullet(M) \otimes \mathfrak{g}\,.$$

It follows that if we choose some connection as the origin in the affine space of connections, then connections can be identified with 1-forms on M. The trace can be taken with respect to the representation $\rho = \mathrm{id}_\mathbb{R}$, so that $\mathrm{Tr}_\rho = \mathrm{id}$. We thus retrieve the Maxwell lagrangian (1.5) with $j = 0$ from the Yang-Mills lagrangian (9.14).

Remark 9.3.23. At first sight, the Yang-Mills equations look no more complicated than the Maxwell equations. Note, however, that the expression (9.11) for the curvature $F(A)$ contains a quadratic term, so that the field equations contain cubic terms in A. This makes solving the Yang-Mills equation a very difficult non-linear problem. In fact, one of the Millennium prize problems in mathematics is about the solutions of the Yang-Mills equations.

9.4 Chern-Simons theory

Chern-Simons form There are other interesting lagrangians on the space of connections on a principal G-bundle $P \to M$.

Definition 9.4.1. Let A be a gauge field, i.e. $A = -\theta$ for a connection 1-form θ. The 3-form on M given by

$$\omega_{\mathrm{CS}}(A) := \mathrm{Tr}_{\mathrm{ad}}\big(F(A) \wedge A - \tfrac{1}{3} A \wedge A \wedge A\big)$$

is called the **Chern-Simons 3-form** for A.

Proposition 9.4.2. *The Chern-Simons 3-form satisfies*

$$d\omega_{\mathrm{CS}}(A) = \mathrm{Tr}_{\mathrm{ad}}\big(F(A) \wedge F(A)\big)\,.$$

Proof. Let $\{X_\alpha\} \subset \mathfrak{g}$ be a basis. The square of $A = A^\alpha \otimes X_\alpha$ is given by

$$\begin{aligned}
A \wedge A &= (A^\alpha \otimes X_\alpha) \wedge (A^\beta \otimes X_\beta)\\
&= (A^\alpha \wedge A^\beta) \otimes X_\alpha X_\beta\\
&= \tfrac{1}{2}(A^\alpha \wedge A^\beta - A^\beta \wedge A^\alpha) \otimes X_\alpha X_\beta\\
&= \tfrac{1}{2} A^\alpha \wedge A^\beta \otimes (X_\alpha X_\beta - X_\beta X_\alpha)\\
&= \tfrac{1}{2} A^\alpha \wedge A^\beta \otimes [X_\alpha, X_\beta]\\
&= \tfrac{1}{2}[A, A]\,.
\end{aligned}$$

It follows that the curvature form of A can be written as

$$F(A) = dA + A \wedge A\,. \tag{9.17}$$

Inserting this expression for $F(A)$ into the definition of the Chern-Simons 3-form, we obtain

$$\omega_{\mathrm{CS}}(A) = \mathrm{Tr}_{\mathrm{ad}}\big(dA \wedge A + \tfrac{2}{3} A \wedge A \wedge A\big)\,.$$

We have to compute the differential of $\omega_{\mathrm{CS}}(A)$. First, we observe that since the trace satisfies Eq. (9.13), we have the relations

$$\mathrm{Tr}(dA \wedge A \wedge A) = \mathrm{Tr}(A \wedge A \wedge dA) = -\mathrm{Tr}(A \wedge dA \wedge A)\,,$$

where from now on we write Tr for the trace. In fact, the computations do not depend on the representation with respect to which we take the trace. Eq. (9.13) also yields

$$\mathrm{Tr}(A \wedge A \wedge A \wedge A) = -\mathrm{Tr}(A \wedge A \wedge A \wedge A)\,,$$

which implies that

$$\mathrm{Tr}(A \wedge A \wedge A \wedge A) = 0\,.$$

With these relations we obtain

$$\begin{aligned} d\omega_{\mathrm{CS}}(A) &= \mathrm{Tr}\big\{d(dA \wedge A) + \tfrac{2}{3} d(A \wedge A \wedge A)\big\} \\ &= \mathrm{Tr}\big\{dA \wedge dA + \tfrac{2}{3}(dA \wedge A \wedge A - A \wedge dA \wedge A + A \wedge A \wedge dA)\big\} \\ &= \mathrm{Tr}\big\{dA \wedge dA + 2\, dA \wedge A \wedge A + A \wedge A \wedge A \wedge A\big\} \\ &= \mathrm{Tr}\big\{F(A) \wedge F(A)\big\}\,, \end{aligned}$$

where we have used that Tr and d commute, and that d is a derivation. □

Lagrangian and field equation Let M be 3-manifold. The Chern-Simons lagrangian is given by the Chern-Simons 3-form,

$$L(A) := \omega_{\mathrm{CS}}(A)\,.$$

The Euler-Lagrange equation is

$$F(A) = 0\,.$$

In other words, Chern-Simons theory is the theory of flat connections on principal fibre bundles. In fact, it is closely related to secondary characteristic classes in Chern-Weil theory. ***

9.5 General relativity

We turn to general relativity. Here, the fields are lorentzian metrics on the spacetime manifold M. Vector fields on M act on metrics by the Lie derivative. This action is local, so that it descends to the infinite jet bundle, inducing an action on the variational bicomplex. In order to study this action, we introduce in Def. 9.5.3 the concept of covariant and contravariant families of forms in the variational bicomplex, which generalizes the concept of tensor fields. In Sec. 9.5.5 we generalize the notion of covariant derivative to such families of forms. In Sec. 9.5.6 we derive divergence formulas that express the horizontal differential of a form in terms of the covariant derivative and the metric volume form. While the computations are similar to those with tensor fields, there are also differences. For example, the metric volume form is invariant (Lem. 9.5.11), rather than transforming as a density.

9.5.1 The action of spacetime vector fields

Assume that M is a manifold of finite dimension n. The configuration bundle of general relativity is the bundle of fibre-wise lorentzian metrics on the tangent spaces of the spacetime manifold M, which we denote by $\mathrm{Lor} \to M$. We use the "east coast" sign convention in which the signature of the metric is $(-1, 1, \ldots, 1)$. The diffeological space of lorentzian metrics on M will be denoted by $\mathcal{L}\mathrm{or}$.

Remark 9.5.1. In many papers on LFTs and the variational bicomplex one of the the following simplifying assumptions about the configuration bundle $F \to M$ is made: F is a vector bundle; the fibres of F are connected; the space of sections $\mathcal{F} = \Gamma(M, F)$ is non-empty; the jet evaluations $j^k : \mathcal{F} \times M \to J^k F$ are surjective. All these assumptions generally fail for the bundle of lorentzian metrics.

The configuration bundle is natural, which means that local diffeomorphisms on M lift functorially to the sheaf of sections. In particular, we have a left action of the diffeomorphism group $\mathrm{Diff}(M)$ on the space of fields $\mathcal{L}\mathrm{or}$ by pushforward. Infinitesimally, we have a left action of the Lie algebra of vector fields,

$$\begin{aligned} \Xi : \mathfrak{X}(M) &\longrightarrow \mathfrak{X}(\mathcal{L}\mathrm{or}) \\ v &\longmapsto (\Xi_v : \eta \mapsto -\mathcal{L}_v \eta)\,, \end{aligned}$$

where the symmetric 2-form $-\mathcal{L}_v\eta$ represents a tangent vector in $T_\eta\mathcal{L}\mathrm{or}$. This action is local, so that it descends to an action of $\mathfrak{X}(M)$ on $J^\infty\mathrm{Lor}$ by strictly vertical vector fields,

$$\begin{aligned} \xi : \mathfrak{X}(M) &\longrightarrow \mathfrak{X}(J^\infty\mathrm{Lor}) \\ v &\longmapsto \xi_v\,. \end{aligned}$$

Together with the Cartan lift of the vector field in $\mathfrak{X}(M)$, we obtain a homomorphism of Lie algebras

$$\begin{aligned} \rho : \mathfrak{X}(M) &\longrightarrow \mathfrak{X}(J^\infty\mathrm{Lor}) \\ v &\longmapsto \rho(v) := \xi_v + \hat{v}\,. \end{aligned} \tag{9.18}$$

Our ultimate goal is to show that ρ is a manifest symmetry of general relativity for a natural choice of boundary form. In this section we will gather the necessary tools.

9.5.2 Jet coordinates

Let $(x^1, \ldots, x^n)$ be a system of local spacetime coordinates on an open subset $U \subset M$. The coordinate vector fields will be denoted by $\partial_a = \frac{\partial}{\partial x^a}$, the coordinate 1-forms by dx^a. A lorentzian metric $\eta \in \mathcal{L}\mathrm{or}$ is written in local coordinates as $\eta = \frac{1}{2}\eta_{ab} dx^a \wedge dx^b$, where $\eta_{ab} = \iota_{\partial_b}\iota_{\partial_a}\eta \in C^\infty(M)$ are the matrix components of the metric. (Recall that we use the Einstein summation convention throughout the paper.)

The local coordinates on M induce local jet coordinates given by

$$\begin{aligned} g_{ab,c_1\cdots c_k} : J^\infty\mathrm{Lor} &\longrightarrow \mathbb{R} \\ j^\infty_x \eta &\longmapsto \frac{\partial^k \eta_{ab}}{\partial x^{c_1} \cdots \partial x^{c_k}}\Big|_x . \end{aligned}$$

Since the partial derivatives commute, $g_{ab,c_1\cdots c_k}$ is invariant under permutations of the indices $c_1, \ldots, c_k$. To avoid overcounting in summation formulas it is convenient to use the multi-index notation of multi-variable analysis: A multi-index is a tuple $C = (C_1, \ldots, C_n)$ of natural numbers $C_k \geq 0$. The number $|C| = C_1 + \ldots + C_n$ is called the length of the index. The concatenation of a multi-index with a single index is given by

$$Cd = (C_1, \ldots, C_d + 1, \ldots, C_n) .$$

The jet coordinate function labeled by a multi-index is given by

$$g_{ab,C}(j^\infty_x \eta) = \frac{\partial^{|C|} \eta_{ab}}{(\partial x^1)^{C_1} \cdots (\partial x^n)^{C_n}}\Big|_x .$$

The collection of functions $\{x^a, g_{ab,C}\}$ for $1 \leq a \leq b \leq n$ and $C \in \mathbb{N}_0^n$ is a system of local coordinates on $J^\infty\mathrm{Lor}$.

Remark 9.5.2. In the physics literature, the same notation is usually used for both the jet coordinates and their evaluation on a field, which can be confusing. For example, if M is non-compact, every n-form is exact, in particular the integrand $L(\eta)$ of the action. So for the step "discarding exact terms" during the derivation of the Euler-Lagrange equation to be meaningful, we must view the integrand as an element $L \in \Omega^{0,n}(J^\infty\mathrm{Lor})$, i.e. as an expression of the jet coordinates like $g_{ab,c}$ and not of the derivatives $\frac{\partial \eta_{ab}}{\partial x^c}$ of a particular metric η.

The variational bicomplex is generated as bigraded algebra by the coordinate functions, the vertical coordinate 1-forms $\delta g_{ab,C}$ in degree $(1,0)$, and the horizontal coordinate 1-forms dx^a in degree $(0,1)$. A (p,q)-form is given in local coordinates by

$$\omega = \omega^{a_1,b_1,\ldots,a_p,b_p,C_1,\ldots,C_p}_{e_1,\ldots,e_q} \delta g_{a_1 b_1, C_1} \wedge \ldots \wedge \delta g_{a_p b_p, C_p} \wedge dx^{e_1} \wedge \ldots \wedge dx^{e_q} ,$$

where the coefficients are functions on $J^\infty\mathrm{Lor}$. The other differentials of the jet coordinates are given by [And89, p. 18]

$$\begin{aligned} \delta x^a &= 0 \\ d g_{ab,C} &= g_{ab,Ce}\, dx^e . \end{aligned}$$

It follows that the differentials of the coordinate 1-forms are given by $ddx^a = 0$, $\delta\delta g_{ab,C} = 0$, $\delta dx^a = 0$, and

$$d\delta g_{ab,C} = -\delta g_{ab,Ce} \wedge dx^e .$$

Dually, the $C^\infty(J^\infty\mathrm{Lor})$-module of vertical vector fields is spanned by the coordinate vector fields $\frac{\partial}{\partial g_{ab,C}}$, which satisfy

$$\iota_{\frac{\partial}{\partial g_{ab,C}}}\delta g_{a'b',C'} = \delta^a_{a'}\delta^b_{b'}\delta^C_{C'}$$
$$\iota_{\frac{\partial}{\partial g_{ab,C}}}dx^e = 0\,.$$

The module of horizontal vector fields, called the Cartan distribution, is spanned by the vector fields

$$\hat\partial_a = \frac{\partial}{\partial x^a} + \sum_{|D|=0}^{\infty} g_{bc,Da}\frac{\partial}{\partial g_{bc,D}}\,,$$

which satisfy

$$\iota_{\hat\partial_a}\delta g_{bc,D} = 0$$
$$\iota_{\hat\partial_a}dx^{a'} = \delta^{a'}_a\,.$$

The Cartan distribution can be viewed as an Ehresmann connection on the bundle $J^\infty\mathrm{Lor} \to M$. The horizontal lift of a vector field $v = v^a(x)\frac{\partial}{\partial x^a}$ on M to $J^\infty\mathrm{Lor}$ is given by

$$\hat v = v^a(x)\hat\partial_a\,.$$

The vertical and horizontal differentials of a function $f \in C^\infty(J^\infty\mathrm{Lor})$ are given by [And89, pp. 18-19]

$$\delta f = \sum_{|C|=0}^{\infty}\frac{\partial f}{\partial g_{ab,C}}\delta g_{ab,C}$$
$$df = (\hat\partial_a f)dx^a\,.$$

The horizontal differential of a form $\omega \in \Omega^{p,q}(J^\infty\mathrm{Lor})$ is given in local coordinates by

$$d\omega = (-1)^{p+q}(\mathcal{L}_{\hat\partial_a}\omega)\wedge dx^a\,. \tag{9.19}$$

A vector field is strictly horizontal if and only if it is the horizontal lift $\hat v$ of a vector field v on M by the Cartan connection. A vector field ξ is strictly vertical if and only if it is the infinite prolongation of an evolutionary "vector field", i.e. of a map $\xi_0 : J^\infty\mathrm{Lor} \to V\mathrm{Lor}$ of bundles over Lor, where $V\mathrm{Lor} \subset T\mathrm{Lor}$ is the vertical tangent bundle. In local coordinates it is of the form

$$\xi = \sum_{|C|=0}^{\infty}(\hat\partial_C\xi_{ab})\frac{\partial}{\partial g_{ab,C}}\,, \tag{9.20}$$

where the ξ_{ab} are functions on $J^\infty\mathrm{Lor}$ and where $\hat\partial_C = (\hat\partial_1)^{C_1}\cdots(\hat\partial_n)^{C_n}$ is the multi-index notation for the iterated application of the horizontal lifts of the coordinate vector fields.

9.5.3 Action of spacetime vector fields on infinite jets

The action of a vector field $v \in \mathfrak{X}(M)$ on a lorentzian metric $\eta \in \mathcal{L}\mathrm{or}$ by the negative Lie derivative, $\eta \mapsto -\mathcal{L}_v\eta$, is given in local coordinates by

$$\eta_{ab}dx^a dx^b \longmapsto -\Big(v^c\frac{\partial\eta_{ab}}{\partial x^c} + \frac{\partial v^{a'}}{\partial x^a}\eta_{a'b} + \frac{\partial v^{b'}}{\partial x^b}\eta_{ab'}\Big)dx^a dx^b\,.$$

We can view this as transformation of the coordinate functions

$$g_{ab} \longmapsto -\Big(v^c g_{ab,c} + \frac{\partial v^{a'}}{\partial x^a} g_{a'b} + \frac{\partial v^{b'}}{\partial x^b} g_{ab'}\Big) =: \xi_{ab}\,, \tag{9.21}$$

which are the components of the evolutionary "vector field" $\xi_{ab}\frac{\partial}{\partial g_{ab}}$. Its infinite prolongation is the strictly vertical vector field

$$\xi_v = \sum_{|C|=0}^{\infty} (\hat{\partial}_C \xi_{ab}) \frac{\partial}{\partial g_{ab,C}}\,,$$

which defines the action (9.18) of vector fields on the infinite jet bundle.

9.5.4 Covariant and contravariant families of forms

The Lie derivative of a coordinate function with respect to a strictly horizontal vector field is given by

$$\mathcal{L}_{\hat{v}} g_{ab,C} = \iota_{v^e \hat{\partial}_e} d g_{ab,C} = v^e g_{ab,Ce}\,.$$

In particular, we have

$$\mathcal{L}_{\hat{\partial}_e} g_{ab} = g_{ab,e}\,.$$

Note that this is the Lie derivative of a single function $g_{ab} \in C^\infty(J^\infty \mathrm{Lor})$ and must not be confused with the Lie derivative of a metric 2-form on M. The formula (9.21) for the 0-jet component ξ_{ab} of ξ_v can now be written as

$$\mathcal{L}_{\xi_v} g_{ab} = -\mathcal{L}_{\hat{v}} g_{ab} - \frac{\partial v^{a'}}{\partial x^a} g_{a'b} - \frac{\partial v^{b'}}{\partial x^b} g_{ab'}\,.$$

This can be expressed in terms of the diagonal action ρ as

$$\mathcal{L}_{\rho(v)} g_{ab} = -\frac{\partial v^{a'}}{\partial x^a} g_{a'b} - \frac{\partial v^{b'}}{\partial x^b} g_{ab'}\,. \tag{9.22}$$

Since δ commutes with both $\mathcal{L}_{\xi_v}$ and $\mathcal{L}_{\hat{v}}$, it commutes with $\mathcal{L}_{\rho(v)}$. This implies that

$$\mathcal{L}_{\rho(v)} \delta g_{ab} = -\frac{\partial v^{a'}}{\partial x^a} \delta g_{a'b} - \frac{\partial v^{b'}}{\partial x^b} \delta g_{ab'}\,. \tag{9.23}$$

Using $g^{ab} g_{bc} = \delta^a_c$, we get

$$\mathcal{L}_{\rho(v)} g^{ab} = \frac{\partial v^a}{\partial x^{a'}} g^{a'b} + \frac{\partial v^b}{\partial x^{b'}} g^{ab'}\,. \tag{9.24}$$

These calculations suggest the following definition.

Definition 9.5.3. A family of forms $\chi^{b_1\cdots b_q}_{a_1\cdots a_p} \in \Omega(J^\infty \mathrm{Lor})$, $1 \leq a_1, \ldots, b_q \leq n$ is called **covariant** in $a_1, \ldots, a_p$ and **contravariant** in $b_1, \ldots, b_q$ if

$$\mathcal{L}_{\rho(v)} \chi^{b_1\cdots b_q}_{a_1\cdots a_p} = -\sum_{i=1}^{p} \frac{\partial v^{a'_i}}{\partial x^{a_i}} \chi^{b_1\cdots b_q}_{a_1\cdots a'_i \cdots a_p} + \sum_{i=1}^{q} \frac{\partial v^{b_i}}{\partial x^{b'_i}} \chi^{b_1\cdots b'_i\cdots b_q}_{a_1\cdots a_p}\,.$$

A form $\chi \in \Omega(J^\infty \mathrm{Lor})$ is called **invariant** if $\mathcal{L}_{\rho(v)} \chi = 0$.

Def. 9.5.3 generalizes the notion of covariant and contravariant tensors to families of forms in $\Omega(J^\infty \mathrm{Lor})$. In this terminology Eqs. (9.22), (9.23), and (9.24) show that the indices of g_{ab} and δg_{ab} are covariant, while those of g^{ab} and δg^{ab} are contravariant. Covariant and contravariant families behave in many ways as tensors.

Lemma 9.5.4. *Let χ_a be a covariant and ψ^b a contravariant family of forms. Then the family $\chi_a \wedge \psi^b$ is covariant in a and contravariant in b.*

Proof. This follows immediately from the fact that $\mathcal{L}_{\rho(v)}$ is a degree 0 derivation of the algebra $\Omega(J^\infty \mathrm{Lor})$. □

Lemma 9.5.5. *Let χ_a^b be a family of forms that is covariant in a and contravariant in b, then the contracted form χ_a^a (summation over a) is invariant.*

The last two lemmas generalize in an obvious way to families with several indices. An immediate consequence of Lem. 9.5.4 and Lem. 9.5.5 is that we can raise and lower indices with the metric coordinate functions in the usual way: If χ_a is covariant, then $\chi^a = g^{aa'}\chi_{a'}$ is contravariant. If χ^a is contravariant, then $\chi_a = g_{aa'}\chi^{a'}$ is covariant.

Lemma 9.5.6. *If the family $\chi_b \in \Omega(J^\infty \mathrm{Lor})$ is covariant, then the family $\iota_{\hat{\partial}_a}\chi_b$ is covariant in a and b.*

Proof. Let $\psi_{ab} = \iota_{\hat{\partial}_a}\chi_b$ We have

$$\begin{aligned}
\mathcal{L}_{\rho(v)}\psi_{ab} &= \mathcal{L}_{\xi_v+\hat{v}}(\iota_{\hat{\partial}_a}\chi_b) \\
&= \big(\iota_{\hat{\partial}_a}\mathcal{L}_{\xi_v} + \iota_{\hat{\partial}_a}\mathcal{L}_{\hat{v}} + \iota_{[\hat{v},\hat{\partial}_a]}\big)\chi_b \\
&= \iota_{\hat{\partial}_a}\mathcal{L}_{\rho(v)}\chi_b - \frac{\partial v^{a'}}{\partial x^a}(\iota_{\hat{\partial}_{a'}}\chi_b) \\
&= -\frac{\partial v^{b'}}{\partial x^b}\psi_{ab'} - \frac{\partial v^{a'}}{\partial x^a}\psi_{a'b}\,,
\end{aligned}$$

which shows that ψ_{ab} is covariant in a and b. □

Lemma 9.5.7. *If the family $\chi_a \in \Omega(J^\infty \mathrm{Lor})$ is covariant, then the family $\delta\chi_b$ is covariant.*

Proof. We have

$$\begin{aligned}
\mathcal{L}_{\rho(v)}\delta\chi_a &= \delta\mathcal{L}_{\rho(v)}\chi_a \\
&= \delta\Big(-\frac{\partial v^{a'}}{\partial x^a}\chi_{a'}\Big) \\
&= -\frac{\partial v^{a'}}{\partial x^a}\delta\chi_{a'}\,,
\end{aligned}$$

which shows that χ_a is covariant. □

The last lemma generalizes in an obvious way to families of forms with covariant and contravariant indices. The analogous statement for the horizontal differential works only for invariant forms:

Lemma 9.5.8. *If $\chi \in \Omega(J^\infty \mathrm{Lor})$ is invariant, then $d\chi$ is invariant.*

Proof. The differential d commutes with $\mathcal{L}_{\rho(v)}$, so that $\mathcal{L}_{\rho(v)} d\chi = d\mathcal{L}_{\rho(v)}\chi = 0$. □

Lemma 9.5.9. *If the form $\chi \in \Omega(J^\infty \mathrm{Lor})$ is invariant, then the family $\mathcal{L}_{\hat{\partial}_a}\chi$ is covariant.*

Proof. We have

$$\begin{aligned}
\mathcal{L}_{\rho(v)}(\mathcal{L}_{\hat{\partial}_a}\chi) &= \mathcal{L}_{\xi_v+\hat{v}}(\mathcal{L}_{\hat{\partial}_a}\chi) \\
&= \big(\mathcal{L}_{\hat{\partial}_a}\mathcal{L}_{\xi_v} + \mathcal{L}_{\hat{\partial}_a}\mathcal{L}_{\hat{v}} + \mathcal{L}_{[\hat{v},\hat{\partial}_a]}\big)\chi \\
&= \mathcal{L}_{\hat{\partial}_a}\mathcal{L}_{\xi_v+\hat{v}}\chi - \frac{\partial v^{a'}}{\partial x^a}(\mathcal{L}_{\hat{\partial}_{a'}}\chi) \\
&= -\frac{\partial v^{a'}}{\partial x^a}(\mathcal{L}_{\hat{\partial}_{a'}}\chi)\,,
\end{aligned}$$

which shows that $\mathcal{L}_{\hat{\partial}_a}\chi$ is a covariant family. □

Lemma 9.5.9 holds only for an invariant form χ. If χ_b is a covariant family, then $\mathcal{L}_{\hat{\partial}_a}\chi_b$ is *not* covariant. In order to obtain a covariant family by differentiation we have to generalize the concept of covariant derivative to families of forms in the variational bicomplex.

9.5.5 Covariant derivative of families of forms

In the cohomological approach to general relativity, we have to interpret the connection coefficients, the covariant derivative, the curvature, the volume form, etc. as expressions in the variational bicomplex. The connection coefficients of the Levi-Civita connection have to be viewed as functions on $J^\infty\mathrm{Lor}$ that are given in local coordinates by the expression

$$\Gamma^a_{bc} = \tfrac{1}{2} g^{ad}(g_{db,c} + g_{dc,b} - g_{bc,d})\,. \tag{9.25}$$

The covariant derivative has to be defined in the variational bicomplex as follows. For a family of forms $\chi^{b_1\cdots b_q}_{a_1\cdots a_p}$ that is covariant in the lower indices and contravariant in the upper indices we define

$$\nabla_c \chi^{b_1\cdots b_q}_{a_1\cdots a_p} = \mathcal{L}_{\hat{\partial}_c}\chi^{b_1\cdots b_q}_{a_1\cdots a_p} - \sum_{i=1}^{p} \Gamma^{a'_i}_{ca_i}\,\chi^{b_1\cdots b_q}_{a_1\cdots a'_i\cdots a_p} + \sum_{i=1}^{q} \Gamma^{b_i}_{cb'_i}\,\chi^{b_1\cdots b'_i\cdots b_q}_{a_1\cdots a_p}\,.$$

Using this definition, we can check by the usual calculation that the connection coefficients (9.25) of the Levi-Civita connection is the unique family of functions symmetric in b and c, such that $\nabla_c g_{ab} = 0$. The Riemann curvature tensor is given by $\mathrm{Riem}_{abc}{}^{d}\,\chi_d = (\nabla_a\nabla_b - \nabla_b\nabla_a)\chi_c$, which has now to be viewed as a family of functions on $J^\infty\mathrm{Lor}$.

Lemma 9.5.10. *Let χ_b be a covariant family of vertical forms. Then the family $\nabla_a\chi_b$ is covariant in a and b.*

Proof. We have to compute the Lie derivative of $\nabla_a \chi_b = \mathcal{L}_{\hat\partial_a}\chi_b - \Gamma^c_{ab}\chi_c$ with respect to $\rho(v) = \xi_v + \hat v$. For the first summand we get

$$\begin{aligned}
\mathcal{L}_{\rho(v)}(\mathcal{L}_{\hat\partial_a}\chi_b) &= \mathcal{L}_{\hat\partial_a}(\mathcal{L}_{\xi_v+\hat v}\chi_b) + \mathcal{L}_{[\hat v,\hat\partial_a]}\chi_b \\
&= \mathcal{L}_{\hat\partial_a}\Big(-\frac{\partial v^{b'}}{\partial x^b}\chi_{b'}\Big) + \iota_{[\hat v,\hat\partial_a]} d\chi_b \\
&= -\frac{\partial^2 v^{b'}}{\partial x^a \partial x^b}\chi_{b'} - \frac{\partial v^{b'}}{\partial x^b}(\mathcal{L}_{\hat\partial_a}\chi_{b'}) - \frac{\partial v^{a'}}{\partial x^a}\iota_{\hat\partial_{a'}} d\chi_b \\
&= -\frac{\partial v^{b'}}{\partial x^b}(\mathcal{L}_{\hat\partial_a}\chi_{b'}) - \frac{\partial v^{a'}}{\partial x^a}(\mathcal{L}_{\hat\partial_{a'}}\chi_b) - \frac{\partial^2 v^c}{\partial x^a \partial x^b}\chi_c\,.
\end{aligned}$$

For the second summand we must compute the Lie derivative of the connection coefficients. For this we need the following formula.

$$\begin{aligned}
\mathcal{L}_{\xi_v} g_{ab,c} &= \mathcal{L}_{\xi_v}\mathcal{L}_{\hat\partial_c} g_{ab} \\
&= \mathcal{L}_{\hat\partial_c}\mathcal{L}_{\xi_v} g_{ab} \\
&= -\mathcal{L}_{\hat\partial_c}\Big(\mathcal{L}_{\hat v} g_{ab} + \frac{\partial v^{a'}}{\partial x^a} g_{a'b} + \frac{\partial v^{b'}}{\partial x^b} g_{ab'}\Big) \\
&= -(\mathcal{L}_{[\hat\partial_c,\hat v]} + \mathcal{L}_{\hat v}\mathcal{L}_{\hat\partial_c}) g_{ab} \\
&\quad - \frac{\partial^2 v^{a'}}{\partial x^c \partial x^a} g_{a'b} - \frac{\partial v^{a'}}{\partial x^a}\delta g_{a'b,c} - \frac{\partial^2 v^{b'}}{\partial x^c \partial x^b} g_{ab'} - \frac{\partial v^{b'}}{\partial x^b} g_{ab',c} \\
&= -\mathcal{L}_{\hat v} g_{ab,c} - \frac{\partial v^{a'}}{\partial x^a} g_{a'b,c} - \frac{\partial v^{b'}}{\partial x^b} g_{ab',c} - \frac{\partial v^{c'}}{\partial x^c} g_{ab,c'} \\
&\quad - \frac{\partial^2 v^{a'}}{\partial x^c \partial x^a} g_{a'b} - \frac{\partial^2 v^{b'}}{\partial x^c \partial x^b} g_{ab'}
\end{aligned}$$

With this relation, we can compute the action of vector fields on the connection coefficients, which yields

$$\mathcal{L}_{\rho(v)}\Gamma^c_{ab} = \frac{\partial v^c}{\partial x^{c'}}\Gamma^{c'}_{ab} - \frac{\partial v^{a'}}{\partial x^a}\Gamma^c_{a'b} - \frac{\partial v^{b'}}{\partial x^b}\Gamma^c_{ab'} - \frac{\partial^2 v^c}{\partial x^a \partial x^b}\,.$$

Putting everything together, we obtain

$$\begin{aligned}
\mathcal{L}_{\rho(v)}(\nabla_a\chi_b) &= \mathcal{L}_{\rho(v)}\mathcal{L}_{\hat\partial_a}\chi_b - (\mathcal{L}_{\rho(v)}\Gamma^c_{ab})\chi_c - \Gamma^c_{ab}(\mathcal{L}_{\rho(v)}\chi_c) \\
&= \frac{\partial v^{a'}}{\partial x^a}(\nabla_{a'}\chi_b) + \frac{\partial v^{b'}}{\partial x^a}(\nabla_a\chi_{b'})\,,
\end{aligned}$$

where the terms containing the second order derivatives of v^a cancel. This finishes the proof. □

9.5.6 Divergence formulas

In the variational bicomplex, the metric volume form is the $(0,n)$-form on $J^\infty\mathrm{Lor}$ defined as

$$\mathrm{vol}_g = \sqrt{-\det g}\; dx^1\wedge\ldots\wedge dx^n\,. \tag{9.26}$$

We recall that we have adopted the "east coast" sign convention for Lorentz metrics with 1 negative and $n-1$ positive signs, so that $\det g$ is negative. The partial derivative of the square root of the determinant with respect to the 0-jet coordinates is given by

$$\frac{\partial}{\partial g_{ab}}\sqrt{-\det g} = \tfrac{1}{2} g^{ab}\sqrt{-\det g}\,.$$

The partial derivatives with respect to x^a and all higher jet coordinates $g_{ab,C}$ vanish. It follows that the vertical and the horizontal differentials are given by

$$\begin{aligned}\delta\sqrt{-\det g} &= \tfrac{1}{2} g^{ab}\delta g_{ab}\sqrt{-\det g}\\ d\sqrt{-\det g} &= \tfrac{1}{2} g^{ab} g_{ab,c}\sqrt{-\det g}\; dx^c\,.\end{aligned}$$

For the vertical differential of the volume form we obtain

$$\delta \mathrm{vol}_g = \tfrac{1}{2} g^{ab}\delta g_{ab}\mathrm{vol}_g\,. \tag{9.27}$$

Although vol_g is not a volume form on $J^\infty\mathrm{Lor}$, every $(0,n)$-form τ can be written as

$$\tau = f dx^1\wedge\ldots\wedge dx^n = \frac{f}{\sqrt{-\det g}}\mathrm{vol}_g\,,$$

for a unique function $f \in C^\infty(J^\infty\mathrm{Lor})$. Therefore, we can define the divergence of a vector field $X \in \mathfrak{X}(J^\infty\mathrm{Lor})$ by the relation

$$\mathcal{L}_X \mathrm{vol}_g = (\operatorname{div} X)\mathrm{vol}_g\,.$$

For a strictly horizontal vector field $\hat{v}$ we have

$$\begin{aligned}\mathcal{L}_{\hat v}\mathrm{vol}_g &= (\mathcal{L}_{\hat v}\sqrt{-\det g})\; dx^1\wedge\ldots\wedge dx^n + \sqrt{-\det g}\;\mathcal{L}_{\hat v}(dx^1\wedge\ldots\wedge dx^n)\\ &= \Big(\tfrac{1}{2} g^{ab} g_{ab,c} v^c + \frac{\partial v^c}{\partial x^c}\Big)\mathrm{vol}_g = \Big(\Gamma^a_{ac} v^c + \frac{\partial v^c}{\partial x^c}\Big)\mathrm{vol}_g\\ &= (\nabla_a v^a)\mathrm{vol}_g\,.\end{aligned} \tag{9.28}$$

We conclude that

$$\operatorname{div}\hat v = \nabla_a v^a\,.$$

While this looks like the usual expression, we point out that the divergence $\operatorname{div}\hat v$ is now a function on $J^1\mathrm{Lor}$.

Lemma 9.5.11. *The metric volume form is invariant.*

Proof. For the Lie derivative of the volume form with respect to the vertical vector field we obtain

$$\begin{aligned}\mathcal{L}_{\xi_v}\mathrm{vol}_g &= \iota_{\xi_v}\delta\mathrm{vol}_g\\ &= -\tfrac{1}{2} g^{ab}\Big(\mathcal{L}_{\hat v} g_{ab} + \frac{\partial v^{a'}}{\partial x^a} g_{a'b} + \frac{\partial v^{b'}}{\partial x^b} g_{ab'}\Big)\mathrm{vol}_g\\ &= -\Big(v^c \tfrac{1}{2} g^{ab} g_{ab,c} + \frac{\partial v^c}{\partial x^c}\Big)\mathrm{vol}_g\\ &= -\mathcal{L}_{\hat v}\mathrm{vol}_g\,,\end{aligned}$$

where in the last step we have used Eq. (9.28). We conclude that $\mathcal{L}_{\rho(v)}\mathrm{vol}_g = 0$ for all $v \in \mathfrak{X}(M)$. □

Remark 9.5.12. Lem. 9.5.11 can be stated by saying that $\xi_v + \hat{v}$ is divergence free.

From the formula for the divergence of a vector field we deduce

$$(\nabla_a v^a)\mathrm{vol}_g = d\big(v^a \iota_{\hat{\partial}_a} \mathrm{vol}_g\big)\,.$$

This formula generalizes to higher vertical forms, as we will show next.

Proposition 9.5.13. *Let χ^a be a family of $(p,0)$-forms on $J^\infty\mathrm{Lor}$. Then*

$$\nabla_a \chi^a \wedge \mathrm{vol}_g = (-1)^p d(\chi^a \wedge \iota_{\hat{\partial}_a} \mathrm{vol}_g)\,. \tag{9.29}$$

Proof. Consider the $(p, n-1)$-form

$$\chi = (-1)^p \chi^a \wedge \iota_{\hat{\partial}_a} \mathrm{vol}_g\,,$$

where the χ^a are $(p,0)$-forms. The horizontal differential of χ is given by

$$\begin{aligned}
d\chi &= (-1)^{p+n-1}(\mathcal{L}_{\hat{\partial}_c}\chi) \wedge dx^c \\
&= (-1)^{n-1}\mathcal{L}_{\hat{\partial}_c}(\chi^a \wedge \iota_{\hat{\partial}_a}\mathrm{vol}_g) \wedge dx^c \\
&= (\mathcal{L}_{\hat{\partial}_c}\chi^a) \wedge (-1)^{n-1}(\iota_{\hat{\partial}_a}\mathrm{vol}_g) \wedge dx^c + \chi^a \wedge (-1)^{n-1}(\mathcal{L}_{\hat{\partial}_c}\iota_{\hat{\partial}_a}\mathrm{vol}_g) \wedge dx^c \\
&= (\mathcal{L}_{\hat{\partial}_a}\chi^a) \wedge \mathrm{vol}_g + \chi^a \wedge (-1)^{n-1}(\iota_{\hat{\partial}_a}\mathcal{L}_{\hat{\partial}_c}\mathrm{vol}_g) \wedge dx^c \\
&= (\mathcal{L}_{\hat{\partial}_a}\chi^a) \wedge \mathrm{vol}_g + \chi^a \wedge (-1)^{n-1}\iota_{\hat{\partial}_a}(\Gamma^b_{bc}\mathrm{vol}_g) \wedge dx^c \\
&= (\mathcal{L}_{\hat{\partial}_a}\chi^a + \Gamma^b_{ba}\chi^a) \wedge \mathrm{vol}_g \\
&= (\nabla_a \chi^a) \wedge \mathrm{vol}_g\,,
\end{aligned}$$

where we have used Eq. (9.19), the Leibniz rule, the relation

$$(\iota_{\hat{\partial}_a}\mathrm{vol}_g) \wedge dx^c = (-1)^{n-1}\delta^c_a \mathrm{vol}_g\,,$$

and Eq. (9.28). □

For later use, we generalize the formula (9.29) further to families of $(p,1)$-forms.

Proposition 9.5.14. *Let χ^{ab} be a family of $(p,1)$-forms on $J^\infty\mathrm{Lor}$ such that $\chi^{ab} = -\chi^{ba}$. Then*

$$\nabla_a \chi^{ab} \wedge \iota_{\hat{\partial}_b}\mathrm{vol}_g = (-1)^p d(\tfrac{1}{2}\chi^{ab} \wedge \iota_{\hat{\partial}_a}\iota_{\hat{\partial}_b}\mathrm{vol}_g)\,. \tag{9.30}$$

Proof. Consider the $(p, n-2)$-form

$$\chi = \tfrac{1}{2}(-1)^p \chi^{ab} \wedge \iota_{\hat{\partial}_a}\iota_{\hat{\partial}_b}\mathrm{vol}_g\,.$$

We have the relation

$$\begin{aligned}
(\iota_{\hat{\partial}_a}\iota_{\hat{\partial}_b}\mathrm{vol}_g) \wedge dx^c &= \iota_{\hat{\partial}_a}[(\iota_{\hat{\partial}_b}\mathrm{vol}_g) \wedge dx^c] - (-1)^{n-1}(\iota_{\hat{\partial}_b}\mathrm{vol}_g) \wedge (\iota_{\hat{\partial}_a} dx^c) \\
&= [\iota_{\hat{\partial}_a}(-1)^{n-1}\delta^c_b \mathrm{vol}_g] - (-1)^{n-1}(\iota_{\hat{\partial}_b}\mathrm{vol}_g)\delta^c_a \\
&= (-1)^{n-1}(\delta^c_b \iota_{\hat{\partial}_a} - \delta^c_a \iota_{\hat{\partial}_b})\,\mathrm{vol}_g\,.
\end{aligned}$$

Moreover, since $\chi^{ab} = -\chi^{ba}$, we have

$$\begin{aligned}\nabla_a \chi^{ab} &= \mathcal{L}_{\hat{\partial}_a} \chi^{ab} + \Gamma^a_{ad}\chi^{db} + \Gamma^b_{ad}\chi^{ad} \\ &= \mathcal{L}_{\hat{\partial}_a} \chi^{ab} + \Gamma^a_{ad}\chi^{db} \,.\end{aligned}$$

Using these relations, we can compute the horizontal differential of χ as

$$\begin{aligned} d\chi &= \tfrac{1}{2}(-1)^{p+n-2}(\mathcal{L}_{\hat{\partial}_c}\chi) \wedge dx^c \\ &= \tfrac{1}{2}(\mathcal{L}_{\hat{\partial}_c}\chi^{ab}) \wedge (-1)^{n-2}(\iota_{\hat{\partial}_a}\iota_{\hat{\partial}_b}\mathrm{vol}_g) \wedge dx^c \\ &\quad + \tfrac{1}{2}\chi^{ab} \wedge (-1)^{n-2}\mathcal{L}_{\hat{\partial}_c}(\iota_{\hat{\partial}_a}\iota_{\hat{\partial}_b}\mathrm{vol}_g) \wedge dx^c \\ &= \tfrac{1}{2}(\mathcal{L}_{\hat{\partial}_c}\chi^{ab} + \chi^{ab}\Gamma^d_{dc}) \wedge (-1)^{n-2}(\iota_{\hat{\partial}_a}\iota_{\hat{\partial}_b}\mathrm{vol}_g) \wedge dx^c \\ &= \tfrac{1}{2}(\mathcal{L}_{\hat{\partial}_c}\chi^{ab} + \chi^{ab}\Gamma^d_{dc}) \wedge (-1)(\delta^c_b\iota_{\hat{\partial}_a} - \delta^c_a\iota_{\hat{\partial}_b})\mathrm{vol}_g \\ &= (\mathcal{L}_{\hat{\partial}_a}\chi^{ab} + \chi^{ab}\Gamma^d_{da}) \wedge \iota_{\hat{\partial}_b}\mathrm{vol}_g \\ &= (\nabla_a\chi^{ab}) \wedge \iota_{\hat{\partial}_b}\mathrm{vol}_g \,, \end{aligned}$$

which finishes the proof. □

9.5.7 Euler-Lagrange and boundary form

The lagrangian form of the Hilbert–Einstein action is

$$L = R\,\mathrm{vol}_g \,, \tag{9.31}$$

where R is the scalar curvature, which has to be interpreted within the variational bicomplex as a function on $J^\infty\mathrm{Lor}$ as follows: The Riemann curvature tensor is given in local coordinates in terms of the connection coefficients (9.25) by

$$\mathrm{Riem}_{abc}{}^{d} = \hat{\partial}_b\Gamma^d_{ac} - \hat{\partial}_a\Gamma^d_{bc} + \Gamma^e_{ac}\Gamma^d_{eb} - \Gamma^e_{bc}\Gamma^d_{ea} \,.$$

This is the usual formula [Wal84, Eq. (3.4.4)] with the partial coordinate derivatives replaced by the Cartan lifts $\hat{\partial}_a$ and $\hat{\partial}_b$. The Ricci curvature is given by the contraction $\mathrm{Ric}_{ab} := \mathrm{Riem}_{aeb}{}^{e}$ and the scalar curvature by the trace of the Ricci curvature $R = g^{ab}\,\mathrm{Ric}_{ab}$.

The vertical differential of the scalar curvature $R = g^{ab}\,\mathrm{Ric}_{ab}$ is given by

$$\delta(g^{ab}\,\mathrm{Ric}_{ab}) = \delta g^{ab}\,\mathrm{Ric}_{ab} + g^{ab}\delta\,\mathrm{Ric}_{ab} \,.$$

The first term can be written as

$$\delta g^{ab}\,\mathrm{Ric}_{ab} = -\,\mathrm{Ric}^{ab}\,\delta g_{ab}$$

The second term is given by [Wal84, Eq. (E.1.15)]

$$g^{ab}\delta\,\mathrm{Ric}_{ab} = \nabla^a\gamma_a$$

where

$$\gamma_a = g^{bc}(\nabla_c\delta g_{ab} - \nabla_a\delta g_{bc}) \,,$$

and where the covariant derivative is to be understood as

$$\nabla^a \gamma_a = g^{ab}(\mathcal{L}_{\hat{\partial}_a}\gamma_b - \Gamma^c_{ab}\gamma_c)\,,$$

as explained in Sec. 9.5.5. The vertical differential of the volume form was computed in Eq. (9.27). Putting everything together, we get

$$\delta L = -\big(\mathrm{Ric}^{ab} - \tfrac{1}{2}Rg^{ab}\big)\delta g_{ab} \wedge \mathrm{vol}_g + \nabla^a\gamma_a \wedge \mathrm{vol}_g\,.$$

The first term is the Euler-Lagrange form

$$EL = -G^{ab}\delta g_{ab} \wedge \mathrm{vol}_g\,,$$

where

$$G^{ab} = \mathrm{Ric}^{ab} - \tfrac{1}{2}Rg^{ab}$$

is the Einstein tensor. The Einstein tensor is divergence-free, i.e.

$$\begin{aligned}\nabla_a G^{ab} &= \mathcal{L}_{\hat{\partial}_a} G^{ab} + \Gamma^a_{ac}G^{cb} + \Gamma^b_{ac}G^{ac} \\ &= 0\,.\end{aligned}$$

Using Eq. (9.29), the second term can be written as a d-exact term

$$(\nabla^a\gamma_a) \wedge \mathrm{vol}_g = -d\gamma\,,$$

where

$$\begin{aligned}\gamma &= \gamma^a \wedge \iota_{\hat{\partial}_a}\mathrm{vol}_g \\ &= g^{ad}g^{bc}(\nabla_c\delta g_{ab} - \nabla_a\delta g_{bc}) \wedge \iota_{\hat{\partial}_d}\mathrm{vol}_g\end{aligned} \tag{9.32}$$

is the boundary form.

9.5.8 Invariance of the Lepage form

Theorem 9.5.15. *The Lepage form $L+\gamma$ given by the sum of the Hilbert–Einstein lagrangian* (9.31) *and the boundary form* (9.32) *is invariant under the action* (9.18) *of spacetime vector fields. In other words, the action is a manifest diffeomorphism symmetry in the sense of Def. 8.3.11.*

Proof. The invariance must hold independently in every bidegree, so that we need to prove the two equations

$$\mathcal{L}_{\xi_v+\hat{v}}L = 0\,, \qquad \mathcal{L}_{\xi_v+\hat{v}}\gamma = 0\,.$$

We start by proving the invariance of L. We have

$$\begin{aligned}\mathcal{L}_{\xi_v}L = \iota_{\xi_v}\delta L &= \iota_{\xi_v}(EL - d\gamma) \\ &= \iota_{\xi_v}EL + d\iota_{\xi_v}\gamma\,.\end{aligned} \tag{9.33}$$

We will compute both summands separately. First we use (9.21) to compute

$$\begin{aligned}
\iota_{\xi_v}\delta g_{ab} &= -\Big(v^c g_{ab,c} + \frac{\partial v^{a'}}{\partial x^a} g_{a'b} + \frac{\partial v^{b'}}{\partial x^b} g_{ab'}\Big) \\
&= -\big(v^c g_{ab,c} + \partial_{\hat{\partial}_a}(v^c g_{cb}) - v^c g_{cb,a} + \partial_{\hat{\partial}_b}(v^c g_{ac}) - v^c g_{ac,b}\big) \\
&= -\big(\partial_{\hat{\partial}_a} v_b + \partial_{\hat{\partial}_b} v_a - v_c g^{ce}(g_{ca,b} + g_{cb,a} - g_{ab,c})\big) \\
&= -\big(\partial_{\hat{\partial}_a} v_b + \partial_{\hat{\partial}_b} v_a - v_c 2\Gamma^c_{ab}\big) \\
&= -(\nabla_a v_b + \nabla_b v_a)\,,
\end{aligned}$$

where we have used (9.25). With this formula we obtain

$$\begin{aligned}
\iota_{\xi_v} EL &= G^{ab}(\nabla_a v_b + \nabla_b v_a)\mathrm{vol}_g \\
&= 2\big(\nabla_a(G^{ab} v_b)\big)\,\mathrm{vol}_g \\
&= d\big(2G^{ab} v_b \iota_{\hat{\partial}_a}\mathrm{vol}_g\big)\,,
\end{aligned} \tag{9.34}$$

where in the last step we have used the divergence formula (9.29). For the second term we compute

$$\begin{aligned}
\iota_{\xi_v}\gamma &= [\iota_{\xi_v} g^{ad} g^{bc}(\nabla_c \delta g_{ab} - \nabla_a \delta g_{bc})] \wedge \iota_{\hat{\partial}_d}\mathrm{vol}_g \\
&= g^{ad} g^{bc}[-\nabla_c(\nabla_a v_b + \nabla_b v_a) + \nabla_a(\nabla_b v_c + \nabla_c v_b)]\,\iota_{\hat{\partial}_d}\mathrm{vol}_g \\
&= g^{ad} g^{bc}[\nabla_c\nabla_a v_b - \nabla_c\nabla_b v_a - 2\nabla_c\nabla_a v_b + \nabla_a(\nabla_b v_c + \nabla_c v_b)]\,\iota_{\hat{\partial}_d}\mathrm{vol}_g \\
&= [\nabla_c(\nabla^d v^c - \nabla^c v^d) - 2g^{ad} g^{bc}(\nabla_c\nabla_a - \nabla_a\nabla_c)v_b]\,\iota_{\hat{\partial}_d}\mathrm{vol}_g \\
&= [\nabla_c(\nabla^d v^c - \nabla^c v^d) - 2\,\mathrm{Ric}^{bd}\, v_b]\,\iota_{\hat{\partial}_d}\mathrm{vol}_g \\
&= -2\,\mathrm{Ric}^{ab}\, v_a\,\iota_{\hat{\partial}_b}\mathrm{vol}_g + [\nabla_a(\nabla^b v^a - \nabla^a v^b)]\,\iota_{\hat{\partial}_b}\mathrm{vol}_g \\
&= -2\,\mathrm{Ric}^{ab}\, v_a\,\iota_{\hat{\partial}_b}\mathrm{vol}_g - d\big(\tfrac{1}{2}(\nabla^a v^b - \nabla^b v^a)\,\iota_{\hat{\partial}_a}\iota_{\hat{\partial}_b}\mathrm{vol}_g\big)\,,
\end{aligned} \tag{9.35}$$

where in the last step we have used the divergence formula (9.30). Inserting (9.34) and (9.35) into the right hand side of (9.33), we obtain

$$\begin{aligned}
\mathcal{L}_{\xi_v} L &= 2d\big(G^{ab} v_b \iota_{\hat{\partial}_a}\mathrm{vol}_g - \mathrm{Ric}^{ab}\, v_a\,\iota_{\hat{\partial}_b}\mathrm{vol}_g\big) \\
&= -d\big(R v^a \iota_{\hat{\partial}_a}\mathrm{vol}_g\big) \\
&= -\mathcal{L}_{\hat{v}} L\,,
\end{aligned}$$

which finishes the proof of the invariance of L.

It remains to prove the invariance of γ. The strategy of the proof is to show that all indices appearing in

$$\gamma = g^{ad} g^{bc}(\nabla_c \delta g_{ab} - \nabla_a \delta g_{bc}) \wedge \iota_{\hat{\partial}_d}\mathrm{vol}_g$$

are covariant or contravariant in the sense of Def. 9.5.3, so that their contraction is invariant by Lem. 9.5.5.

We have shown in Lem. 9.5.11 that the volume form is invariant. It follows from Lem. 9.5.6 that the index d of $\iota_{\hat{\partial}_d}\mathrm{vol}_g$ is covariant. We have shown in Eq. (9.24) that the indices of g^{ad} and g^{bc} are contravariant. In Eq. (9.23) we have seen that the indices of δg_{bc} are covariant. It follows from Lem. 9.5.10 that the indices of the covariant derivatives ∇_c and ∇_a are covariant. Lem. 9.5.4 shows that the wedge product is contravariant in all upper and covariant in all lower indices. With Lem. 9.5.5 we conclude that γ is invariant. □

Theorem 9.5.16. *The action of spacetime vector fields on the infinite jet bundle of Lorentz metrics defined in* (9.18) *has a homotopy momentum map*

$$\mu : \mathfrak{X}(M) \longrightarrow L_\infty(J^\infty \mathrm{Lor}, EL + \delta\gamma)\,,$$

given by

$$\begin{aligned} \mu_k : \wedge^k \mathfrak{X}(M) &\longrightarrow L_\infty(J^\infty \mathrm{Lor}, EL + \delta\gamma) \\ \mu_k(v_1, \dots, v_k) &:= \iota_{\rho(v_1)} \cdots \iota_{\rho(v_k)}(L + \gamma)\,. \end{aligned}$$

Proof. The proof follows from Thm. 9.5.15 and Prop. 8.3.9. □

The Noether current, which was given in (8.26) by the general formula $j_v = -\iota_{\hat{v}} L - \iota_{\xi_v}\gamma$, can be computed with (9.35) to

$$j_v = 2G^{ab} v_a \wedge \iota_{\hat{\partial}_b} \mathrm{vol}_g + d\big(\tfrac{1}{2}(\nabla^a v^b - \nabla^b v^a)\, \iota_{\hat{\partial}_a} \iota_{\hat{\partial}_b} \mathrm{vol}_g\big)\,. \tag{9.36}$$

The $k = 1$ component of the homotopy momentum map, which was given in (8.25) by the general formula $\mu_1(v) = -j_v + \iota_{\hat{v}}\gamma$, is

$$\begin{aligned} \mu_1(v) = &-2G^{ab} v_a \wedge \iota_{\hat{\partial}_b} \mathrm{vol}_g - d\big(\tfrac{1}{2}(\nabla^a v^b - \nabla^b v^a)\, \iota_{\hat{\partial}_a} \iota_{\hat{\partial}_b} \mathrm{vol}_g\big) \\ &+ g^{ad} g^{bc} (\nabla_c \delta g_{ab} - \nabla_a \delta g_{bc}) v^e \wedge \iota_{\hat{\partial}_d} \iota_{\hat{\partial}_e} \mathrm{vol}_g\,. \end{aligned}$$

Remark 9.5.17. The Noether current of a symmetry is determined only up to a d-closed form. Usually, the second summand of (9.36) is dropped, so that the Noether current is $C^\infty(M)$-linear in v and can be interpreted as the energy-momentum tensor G^{ab}. Here, we must take (9.36) as Noether current so that μ is a homomorphism of L_∞-algebras.

9.6 Poisson sigma models

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Appendix A

Graded algebra

A.1 Graded vector spaces

A.1.1 Tensor product

A **$\mathbb{Z}$-graded vector space** V is a family of vector spaces $\{V_i \mid i \in \mathbb{Z}\}$ indexed by $\mathbb{Z}$. An **element of** V is an element $v \in V_i$ of one of the vector spaces in the family. Its degree is $|v| = i$. A morphism $f : V \to W$ of graded vector spaces is given by a family of linear maps $\{f_i : V_i \to W_i \mid i \in \mathbb{Z}\}$. That is,

$$\mathcal{V}\mathrm{ec}^{\mathbb{Z}}(V, W) = \prod_{i \in \mathbb{Z}} \mathcal{V}\mathrm{ec}(V_i, W_i)\,. \tag{A.1}$$

The composition is given by the composition of all components $(f \circ g)_i = f_i \circ g_i$. The category of $\mathbb{Z}$-graded vector spaces will be denoted by $\mathcal{V}\mathrm{ec}^{\mathbb{Z}}$.

Remark A.1.1. If we view $\mathbb{Z}$ as a discrete category (with only identity morphisms), then $\mathcal{V}\mathrm{ec}^{\mathbb{Z}}$ is the category of functors $\mathbb{Z} \to \mathcal{V}\mathrm{ec}$.

The **tensor product** of two graded vector spaces V and W is given by the family

$$(V \otimes W)_i = \coprod_{p+q=i} V_p \otimes W_q\,, \tag{A.2}$$

where on the right side $\otimes$ denotes the ordinary tensor product of vector spaces.

Notation A.1.2. The sum on the right side of Equation (A.2) is generally infinite. This is why we are careful to use the notation $\coprod$ for the coproduct in (A.2), reserving $\bigoplus$ for the *biproduct*. Only if the sum is finite, the coproduct and the biproduct are the same.

The **graded symmetric structure** of this tensor product is defined by

$$\begin{aligned} \tau_{V,W} : V \otimes W &\longrightarrow W \otimes V \\ v \otimes w &\longmapsto (-1)^{|v|\,|w|} w \otimes v\,, \end{aligned}$$

for all elements v and w of V. The symmetric structure defines an action of the permutation group S_k on $V^{\otimes k}$. The action of $\sigma \in S_k$ on $v_1 \otimes \ldots \otimes v_k$ will pick up the **Koszul sign** ε defined by

$$\sigma \cdot (v_1 \otimes \cdots \otimes v_k) = \varepsilon(\sigma; v_1, \ldots, v_k)(v_{\sigma^{-1}(1)} \otimes \ldots \otimes v_{\sigma^{-1}(k)})\,. \tag{A.3}$$

Remark A.1.3. The inverse of the permutation appearing in Equation (A.3) is a common source of confusion. The reader is invited to check this in specific cases. For example $\tau_{12} \cdot (a \otimes b \otimes c) = (b \otimes a \otimes c)$ and $\tau_{23} \cdot (a \otimes b \otimes c) = (b \otimes a \otimes c)$. For $\sigma = \tau_{23}\tau_{12} = \left(\begin{smallmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{smallmatrix}\right)$ we obtain

$$\begin{aligned}\sigma \cdot (v_1 \otimes v_2 \otimes v_3) &= \tau_{23}\tau 12 \cdot (v_1 \otimes v_2 \otimes v_3) \\ &= \tau_{23} \cdot (v_2 \otimes v_1 \otimes v_3) \\ &= v_2 \otimes v_3 \otimes v_1 \\ &= v_{\sigma^{-1}(1)} \otimes v_{\sigma^{-1}(2)} \otimes v_{\sigma^{-1}(3)} .\end{aligned}$$

Conceptually, the family (v_1, v_2, v_3) can be viewed as a map $\{1, 2, 3\} \to V$. We have to act with the inverse of $\sigma \in S_3$ on the domain of this map to obtain a left action.

The **graded antisymmetric structure** is given by $-\tau$. It defines an action of the permutation group S_k on $V^{\otimes k}$ by

$$(v_1 \otimes \cdots \otimes v_k) \longmapsto (-1)^{|\sigma|}\sigma \cdot (v_1 \otimes \cdots \otimes v_k)$$

for all $\sigma \in S_k$, where $|\sigma|$ is the order and $(-1)^{|\sigma|}$ the sign of the permutation.

The quotient of the vector space $V^{\otimes k}$ by the graded symmetric S_k-action is a graded vector space called the **graded symmetric tensor product** and denoted by S^kV. For example,

$$S^2V = (V \otimes V)/\langle v \otimes w - (-1)^{|v|\,|w|} w \otimes v \mid v, w \in V\rangle .$$

The quotient of the vector space $V^{\otimes k}$ by the graded antisymmetric S_k-action is a graded vector space called the **graded antisymmetric tensor product** or the **graded exterior product** and denoted by $\wedge^k V$. For example,

$$\wedge^2 V = (V \otimes V)/\langle v \otimes w + (-1)^{|v|\,|w|} w \otimes v \mid v, w \in V\rangle .$$

A linear map $f : \wedge^k V \to W$ can be identified with a multilinear map $f : V \times \ldots \times V \to W$ that is graded antisymmetric,

$$f(v_1, \ldots, v_k) = (-1)^{|\sigma|}\varepsilon(\sigma; v_1, \ldots, v_k)(v_{\sigma(1)}, \ldots, v_{\sigma(k)}) .$$

A.1.2 Décalage isomorphism

The **degree shift** of a graded vector space V by $k \in \mathbb{Z}$ is given by the family $V[k]_i := V_{k+i}$. Note that this means that an element $v \in V$ of degree $|v| = i$ has degree $i - k$ in $V[k]$. In particular, $\mathbb{R}[k]$ is concentrated in degree $-k$. We have

$$\begin{aligned}\mathcal{V}\mathrm{ec}^{\mathbb{Z}}(V, W[k]) &= \prod_{i\in\mathbb{Z}} \mathcal{V}\mathrm{ec}(V_i, W[k]_i) \\ &= \prod_{i\in\mathbb{Z}} \mathcal{V}\mathrm{ec}(V_i, W_{k+i}) \\ &= \prod_{j\in\mathbb{Z}} \mathcal{V}\mathrm{ec}(V_{-k+j}, W_j) \\ &= \mathcal{V}\mathrm{ec}^{\mathbb{Z}}(V[-k], W) .\end{aligned}$$

In particular, we have

$$\mathcal{V}\mathrm{ec}^{\mathbb{Z}}(\mathbb{R}[-k], V) = \mathcal{V}\mathrm{ec}(\mathbb{R}, W_k) = W_k\,.$$

The tensor product satisfies $V \otimes \mathbb{R}[k] \cong V[k]$. In particular, $\mathbb{R}[1]^{\otimes k} \cong \mathbb{R}[k]$. Using the symmetric structure, we obtain a natural isomorphism from

$$V[1]^{\otimes k} \cong V \otimes \mathbb{R}[1] \otimes V \otimes \mathbb{R}[1] \otimes \ldots \otimes V \otimes \mathbb{R}[1]$$

to

$$V[k] \cong V \otimes \ldots \otimes V \otimes \mathbb{R}[1] \otimes \ldots \mathbb{R}[1]\,,$$

which is given by

$$\begin{aligned} \mathrm{dec} : V[1]^{\otimes k} &\longrightarrow V[k] \\ v_1[1] \otimes \ldots \otimes v_k[1] &\longmapsto (-1)^{\sum_{l=1}^{k}(l-1)|v_l|}(v_1 \otimes \ldots \otimes v_k)[k]\,, \end{aligned}$$

and called the **décalage** isomorphism. By the naturality of the décalage isomorphism, we have the following commutative diagram:

$$\begin{tikzcd} V[1]^{\otimes k} \arrow[r, "\mathrm{dec}_k"] \arrow[d, "\sigma"'] & V^{\otimes k}[k] \arrow[d, "(-1)^{|\sigma|}\sigma"] \\ V[1]^{\otimes k} \arrow[r, "\mathrm{dec}_k"'] & V^{\otimes k}[k] \end{tikzcd}$$

In other words, the décalage isomorphism intertwines the symmetric and the anti-symmetric action of the permutation group. It follows that it induces an isomorphism

$$S^k(V[1]) \xrightarrow{\cong} (\wedge^k V)[k]\,. \tag{A.4}$$

Let $TV = \mathbb{R} \oplus V \oplus V^{\otimes 2} \oplus \ldots$ be the tensor algebra generated by V, which is the free graded algebra generated by V. By taking the quotient of every summand $V^{\otimes k}$ with respect to the symmetric S_k action, we obtain the graded symmetric algebra $SV = \bigoplus_{k=0}^{\infty} S^k V$, which is the free graded commutative algebra generated by V. By taking the quotient with respect to the the antisymmetric S_k-action, we obtain the graded antisymmetric algebra $\wedge S = \bigoplus_{k=0}^{\infty} \wedge^k S$, which is the free graded anticommutative algebra generated by V.

Warning A.1.4. The décalage isomorphisms (A.4) do *not* define an isomorphism of graded algebras.

A.1.3 Graded mapping space

The category of graded vector spaces has all exponential objects, denoted by $\underline{\mathcal{V}\mathrm{ec}}^{\mathbb{Z}}(V, W)$ and usually called the **graded mapping space** or **inner hom**. They are determined by the universal property,

$$\begin{aligned} \underline{\mathcal{V}\mathrm{ec}}^{\mathbb{Z}}(V, W)_k &= \mathcal{V}\mathrm{ec}^{\mathbb{Z}}\big(\mathbb{R}[-k], \underline{\mathcal{V}\mathrm{ec}}^{\mathbb{Z}}(V, W)\big) \\ &\cong \mathcal{V}\mathrm{ec}^{\mathbb{Z}}\big(\mathbb{R}[-k] \otimes V, W)\big) \\ &\cong \mathcal{V}\mathrm{ec}^{\mathbb{Z}}\big(V[-k], \underline{\mathcal{V}\mathrm{ec}}^{\mathbb{Z}}(V, W)\big) \\ &= \prod_i \underline{\mathcal{V}\mathrm{ec}}(V_i, W_{i+k})\,, \end{aligned} \tag{A.5}$$

where $\underline{\mathcal{V}\mathrm{ec}}$ is the inner hom of vector spaces. An element in $\underline{\mathcal{V}\mathrm{ec}}^{\mathbb{Z}}(V,W)_k$ is called a linear map of degree k, which can be equivalently viewed as a map of graded vector spaces

$$V \longrightarrow W[k]\,.$$

The **dual** of V is the graded vector space

$$V^* := \underline{\mathcal{V}\mathrm{ec}}^{\mathbb{Z}}(V,\mathbb{R})\,,$$

which is given explicitly by the family

$$(V^*)_i = (V_{-i})^*\,,$$

where V^* is the dual vector space of V.

A.1.4 Total vector space of a graded vector space

By definition, a $\mathbb{Z}$-graded vector space is a family $\{V_i\}$ of vector spaces indexed by $\mathbb{Z}$. It is often convenient to consider the vector space

$$V^{\mathrm{tot}} := \coprod_{i\in\mathbb{Z}} V_i\,,$$

which is called the **total vector space** of the graded vector spaces. In V^{tot} we can add elements of different degrees. Note however, that now that degree is not defined for arbitrary elements. For example, the zero element in V^{tot} does not have a degree. The elements that lie in $V_i \subset V^{\mathrm{tot}}$ are called **homogeneous** of degree i.

The tensor product of vector spaces has a right adjoint, the mapping space, so that it preserves colimits. It follows that the tensor product of the total vector spaces of two graded vector spaces is given by

$$\begin{aligned} V^{\mathrm{tot}} \otimes W^{\mathrm{tot}} = \Big(\coprod_{i\in\mathbb{Z}} V_i\Big) \otimes \Big(\coprod_{j\in\mathbb{Z}} W_j\Big) &\cong \coprod_{i\in\mathbb{Z}}\coprod_{j\in\mathbb{Z}} (V_i \otimes W_j) \cong \coprod_{k\in\mathbb{Z}}\coprod_{p\in\mathbb{Z}} (V_p \otimes W_{k-p}) \\ &\cong (V\otimes W)^{\mathrm{tot}}\,. \end{aligned}$$

We conclude that the tensor product of the total vector spaces of two graded vector spaces is the total vector space of their tensor.

The mapping space between the total vector spaces of two graded vector spaces is given by

$$\underline{\mathcal{V}\mathrm{ec}}(V^{\mathrm{tot}}, W^{\mathrm{tot}}) = \underline{\mathcal{V}\mathrm{ec}}\Big(\coprod_{i\in\mathbb{Z}} V_i, \coprod_{j\in\mathbb{Z}} W_j\Big) \cong \prod_{i\in\mathbb{Z}} \underline{\mathcal{V}\mathrm{ec}}\Big(V_i, \coprod_{j\in\mathbb{Z}} W_j\Big)\,.$$

A map $V_i \to \amalg_{j\in\mathbb{Z}} W_j$ has **degree** k if it factors through $W_{i+k} \hookrightarrow \amalg_{j\in\mathbb{Z}} W_j$. It follows that the degree k maps are given by the vector space

$$\begin{aligned} \underline{\mathcal{V}\mathrm{ec}}(V^{\mathrm{tot}}, W^{\mathrm{tot}})_k &\cong \prod_{i\in\mathbb{Z}} \underline{\mathcal{V}\mathrm{ec}}(V_i, W_{i+l}) \\ &\cong \underline{\mathcal{V}\mathrm{ec}}^{\mathbb{Z}}(V,W)\,, \end{aligned}$$

where we have used formula (A.5) for the graded mapping space. The upshot is that we can view a $\mathbb{Z}$-graded vector space equivalently as a vector space given by a coproduct of vector spaces indexed by $\mathbb{Z}$.

A.1.5 Bigraded mapping space

We would like to equip the graded mapping space with a bigrading given by both indices of a linear map $V_i \to W_{i+k}$. For this, we have to replace the product on the right side of (A.5) by the coproduct, which yields the vector sapce

$$\underline{\mathcal{V}\mathrm{ec}}^{\mathbb{Z}}_{\mathrm{fin}}(V,W)_k = \coprod_{i\in\mathbb{Z}} \underline{\mathcal{V}\mathrm{ec}}(V_i, W_{i+k})\,. \tag{A.6}$$

Let $\underline{\mathcal{V}\mathrm{ec}}^{\mathbb{Z}}_{\mathrm{fin}}(V,W)_k \subset \underline{\mathcal{V}\mathrm{ec}}^{\mathbb{Z}}(V,W)_k$ is the subspace of morphisms $\mu : V \to W[k]$ that only have a finite number of components $\mu_i : V_i \to W_{i+k}$ that are non-zero.

The right side of (A.6) can be viewed as the total space of a graded vector space. This suggest to define a bigraded vector space by

$$\underline{\mathcal{V}\mathrm{ec}}^{\mathbb{Z}}_{\mathrm{fin}}(V,W)_{p,q} := \underline{\mathcal{V}\mathrm{ec}}(V_{-p}, W_q)\,, \tag{A.7}$$

so that

$$\underline{\mathcal{V}\mathrm{ec}}^{\mathbb{Z}}_{\mathrm{fin}}(V,W)_k = \coprod_{k=p+q} \underline{\mathcal{V}\mathrm{ec}}^{\mathbb{Z}}_{\mathrm{fin}}(V,W)_{p,q}$$

is the coproduct of the components of total degree k.

Warning A.1.5. If we attempt to put a bigrading on all of $\underline{\mathcal{V}\mathrm{ec}}^{\mathbb{Z}}(V,W)$, the infinite product on the right side of (A.5) will cause problems. For example, it does not commute with the tensor product.

A.2 Graded algebras

Terminology A.2.1. The following terminology is analogous to the Terminology 2.2.6. Let Wibble be an algebraic theory that is commonly represented in vector spaces, such as algebras, commutative algebras, modules, derivations, or Lie algebras. A Wibble object in the monoidal category of graded vector spaces is called a **graded Wibble**.

A.2.1 Graded associative algebras

A **$\mathbb{Z}$-graded algebra** is a monoid internal to the monoidal category $\mathcal{V}\mathrm{ec}^{\mathbb{Z}}$. Explicitly, an algebra is given by an graded vector space $A \in \mathcal{V}\mathrm{ec}^{\mathbb{Z}}$ with a morphisms of multiplication $\mu : A \otimes A \to A$ and a morphism of the unit $\eta : \mathbb{R} \to A$, the unit, satisfying the usual relations of associativity and unitality. Since μ preserves the degrees, the degree of a product is given by $|ab| = |\mu(a\otimes b)| = |a \otimes b| = |a| + |b|$.

Example A.2.2. The graded endomorphism space $A = \underline{\mathcal{V}\mathrm{ec}}^{\mathbb{Z}}(V,V)$ of any graded vector space V is a graded algebra with the composition of endomorphisms as product and the identity as unit.

A graded algebra is **commutative** if $\mu \circ \tau = \tau$. This means explicitly, that

$$ab = (-1)^{|a|\,|b|} ba\,,$$

for all $a, b \in A$. Similarly, a graded algebra is **anticommutative** if $\mu \circ \tau = -\mu$. Explicitly, this means that

$$ab = -(-1)^{|a|\,|b|} ba\,,$$

for all $a, b \in A$.

Warning A.2.3. An anticommutative graded algebra on A is *not* the same as a commutative algebra on $A[1]$. This is only the case if A is concentrated in degree 0.

A map $\varphi \in \underline{\mathcal{V}\mathrm{ec}}^{\mathbb{Z}}(A, A)$ is a **graded derivation** of the graded algebra A, if it is a derivation internal to $\mathcal{V}\mathrm{ec}^{\mathbb{Z}}$. Explicitly, this means that

$$\varphi(ab) = \varphi(a)\, b + (-1)^{|a|\,|\varphi|}$$

for all $a, b \in A$.

A.2.2 Graded Lie algebras

A **graded Lie algebra** is a Lie algebra object in $\mathcal{V}\mathrm{ec}^{\mathbb{Z}}$. Explicitly, a Lie algebra is given by a graded vector space $L \in \mathcal{V}\mathrm{ec}^{\mathbb{Z}}$, with a morphism of the Lie bracket $l_2 \equiv [\_\,,\_\,] : L \otimes L \to L$ that is graded anticommutative, $l_2 \circ \tau = -l_2$, and satisfies the Jacobi identity,

$$\sum_{\sigma \in S_3} l_2 \circ (\mathrm{id} \otimes l_2) \circ (-1)^{|\sigma|} \sigma = 0\,,$$

where S_3 is the symmetric group. Explicitly, this means that

$$[a, b] = -(-1)^{|a|\,|b|}[b, a]$$

and

$$[a, [b, c]] + (-1)^{|a|\,(|b|+|c|)}[b, [c, a]] + (-1)^{|b|\,(|c|+|a|)}[c, [a, b]] = 0$$

for all $a, b, c \in L$.

Example A.2.4. Let A be a graded algebra. Then the graded commutator,

$$[a, b] := ab - (-1)^{|a|\,|b|} ba$$

equips A with the structure of a graded Lie algebra.

Example A.2.5. Let A be a graded algebra. The graded vector space of derivations of A is a graded Lie subalgebra of the commutator Lie algebra of the algebra of graded endomorphisms $\underline{\mathcal{V}\mathrm{ec}}^{\mathbb{Z}}(A, A)$.

A.3 Differential graded vector spaces

A.3.1 Differentials

Let $V \in \mathcal{V}\mathrm{ec}^{\mathbb{Z}}$ be a graded vector space. Let $k \in \mathbb{Z}$ be odd. A degree k map

$$d : V \longrightarrow V[k]$$

that squares to zero $d^2 = 0$ is called a **differential** of degree k. A graded vector space together with a differential is called **differential graded vector space**. If $k = -1$ we obtain chain complexes and cochain complexes if $k = 1$. These are the two cases usually considered.

Let V be a graded vector space with a differential d_V of odd degree k and W a graded vector spaces with a differential d_W also of degree k. A morphism $(V, d_V) \to (W, d_W)$ of differential graded vector spaces is a morphism $\varphi : V \to W$ of graded vector spaces that intertwines the differentials, $\varphi \circ d_W = d_V \circ \varphi$.

The morphism $d : V \otimes W \to (V \otimes W)[k]$ of odd degree k defined by

$$d(v \otimes w) := (d_V v) \otimes w + (-1)^{|v|} v \otimes (d_W w) \tag{A.8}$$

is a differential on the tensor product. It is straightforward to check that the symmetric structure commutes with this differential.

The morphism $d : \underline{\mathcal{V}\mathrm{ec}}^{\mathbb{Z}}(V, W) \to \underline{\mathcal{V}\mathrm{ec}}^{\mathbb{Z}}(V, W)[k]$ of odd degree k defined by

$$d\varphi = d_W \circ \varphi - (-1)^{|\varphi|} \varphi \circ d_V \tag{A.9}$$

is a differential on the graded mapping space.

Notation A.3.1. Since it is clear from the context, which of the differentials d_V and d_W is meant in Equations (A.8) and (A.9), the subscripts are usually omitted from the notation.

A morphism $\varphi : V \to W$ of graded vector spaces is a morphism of differential graded vector spaces if and only if $\varphi - \psi \in \underline{\mathcal{V}\mathrm{ec}}^{\mathbb{Z}}(V, W)$ is closed. Two closed morphisms $\varphi, \psi : V \to W$ of graded vector spaces are **homotopy equivalent** if their difference $\varphi - \psi \in \underline{\mathcal{V}\mathrm{ec}}^{\mathbb{Z}}(V, W)$ is exact. That is, there is some $h \in \underline{\mathcal{V}\mathrm{ec}}^{\mathbb{Z}}(V, W)_{-k}$ such that

$$\begin{aligned} \varphi - \psi &= dh \\ &= d \circ h - (-1)^{k(-k)} h \circ d \\ &= d \circ h + h \circ d\,. \end{aligned}$$

A morphism $\tilde{\varphi} : W \to V$ is called a **homotopy inverse** of φ if $\tilde{\varphi} \circ \varphi$ is homotopy equivalent to id_V and $\varphi \circ \tilde{\varphi}$ is homotopy equivalent to id_W. If this is the case, φ is called a **weak equivalence** of differential graded vector spaces.

Proposition A.3.2. *The category of $\mathbb{Z}$-graded vector spaces with a differential of odd degree k is symmetric closed monoidal, with the graded tensor product with differential* (A.8)*, the graded mapping space with differential* (A.9)*, and the symmetric structure of graded vector spaces.*

Remark A.3.3. The graded space $\underline{\mathcal{V}\mathrm{ec}}^{\mathbb{Z}}_{\mathrm{fin}}(V, W)$ defined in (A.6) has a bigrading defined in (A.7). The map $\varphi \mapsto \varphi \circ d_V$ of bidegree $(k, 0)$ and the map $\varphi \mapsto d_W \circ \varphi$ of bidegree $(0, k)$ equip $\underline{\mathcal{V}\mathrm{ec}}^{\mathbb{Z}}_{\mathrm{fin}}(V, W)$ with the structure of a differential bicomplex.

A.3.2 Chain complexes

When the differentials have degree $k = -1$, we obtain the category $\mathcal{C}\mathrm{h}$ of chain complexes of the field $\mathbb{R}$. The inner hom or mapping complex will be denoted by $\underline{\mathcal{C}\mathrm{h}}$. The set of chain maps is retrieved from the mapping complex by

$$\begin{aligned} \mathcal{C}\mathrm{h}(V,W) &\cong \mathcal{C}\mathrm{h}(\mathbb{R}\otimes V, W) \\ &\cong \mathcal{C}\mathrm{h}\big(\mathbb{R}, \underline{\mathcal{C}\mathrm{h}}(V,W)\big) \\ &\cong \{\varphi \in \underline{\mathcal{C}\mathrm{h}}(V,W)_0 \mid d\varphi = 0\} \\ &= Z_0\big(\underline{\mathcal{C}\mathrm{h}}(V,W)\big)\,, \end{aligned}$$

where Z_0 denotes the 0-cycles of a chain complex.

Every vector space can be viewed as a chain complex concentrated in degree 0. This defines a full, faithful, and monoidal functor

$$\begin{aligned} \mathrm{dg} : \mathcal{V}\mathrm{ec} &\longrightarrow \mathcal{C}\mathrm{h} \\ V &\longmapsto (\ldots \to 0 \to V \to 0 \to \ldots)\,. \end{aligned}$$

It is straightforward to check that a chain map $\mathrm{dg}(V) \to W$ is the same as a linear map $V \to Z_0(W)$. This shows that the functor of 0-cycles

$$Z_0 : \mathcal{C}\mathrm{h} \longrightarrow \mathcal{V}\mathrm{ec}$$

is the right adjoint of dg.

Remark A.3.4. If we restrict $\mathcal{C}\mathrm{h}$ to the full subcategory $\mathcal{C}\mathrm{h}_{\geq 0}$ of non-negatively graded chain complexes, the space of 0-cycles is the same as the zeroth homology $Z_0 = H_0$. We thus obtain an adjunction

$$\mathrm{dg} : \mathcal{V}\mathrm{ec} \rightleftarrows \mathcal{C}\mathrm{h}_{\geq 0} : H_0 \quad .$$

Appendix B

Useful facts

Given a functor $\Phi : \mathcal{I} \to \mathcal{J}$ and an object $j \in \mathcal{J}$, the **comma category** $j \downarrow \Phi$ has as objects pairs $(i, j \to \Phi(i))$ and as morphisms commutative triangles $j \to \Phi(i) \to \Phi(i')$.

Proposition B.0.1. *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor and assume that $\mathcal{D}$ has all colimits. Then the left Kan extension of F along the Yoneda embedding $Y : \mathcal{C} \to \mathcal{S}\mathrm{et}^{\mathcal{C}^{\mathrm{op}}}$ preserves all colimits.*

Proof. The left Kan extension to a complete category is pointwise, so that it can be expressed as coend

$$(\mathrm{Lan}_Y F)(S) \cong \int^C \mathcal{S}\mathrm{et}^{\mathcal{C}^{\mathrm{op}}}(YC, X) \otimes FC\,,$$

for all $X \in \mathcal{S}\mathrm{et}^{\mathcal{C}^{\mathrm{op}}}$. The copower functor $\otimes : \mathcal{S}\mathrm{et} \times \mathcal{D} \to \mathcal{D}$ is defined by the natural isomorphism

$$\mathcal{D}\big(S \otimes D, D'\big) \cong \mathcal{S}\mathrm{et}\big(S, \mathcal{D}(D, D')\big)\,,$$

which implies that $\_ \otimes D : \mathcal{S}\mathrm{et} \to \mathcal{D}$ is left adjoint to $\mathcal{D}(D, \_)$. The left Kan extension is the composition of the functor

$$\mathcal{S}\mathrm{et}^{\mathcal{C}^{\mathrm{op}}}(YC, \_) : \mathcal{S}\mathrm{et}^{\mathcal{C}^{\mathrm{op}}} \longrightarrow \mathcal{S}\mathrm{et}$$

with the functor

$$\_ \otimes FC : \mathcal{S}\mathrm{et} \longrightarrow \mathcal{D}\,,$$

followed by the coend. By the Yoneda lemma, the first functor is

$$\mathcal{S}\mathrm{et}^{\mathcal{C}^{\mathrm{op}}}(YC, X) \cong X(C)\,.$$

Since colimits in functor categories are computed pointwise, this functor commutes with colimits. The second functor preserves colimits because it is a left adjoint. Finally, the coend is itself given by a colimit, so that it, too, preserves colimits. We conclude that the left Kan extension preserves colimits. $\square$

Proposition B.0.2. *A category $\mathcal{I}$ is filtered if and only if every finite diagram $D : \mathcal{J} \to \mathcal{I}$ has a cocone.*

Proof. Recall that a cocone over a diagram D is an object $i \in \mathcal{I}$ and a natural transformation $\tau : D \to \Delta^i$, where $\Delta^i : \mathcal{J} \to \mathcal{I}$, $j \mapsto i$ denotes the constant functor with value i. This means that for every $j \in \mathcal{J}$ there is a morphism $\tau_j : D_j \to i$ such that for every $f : j \to j'$ in $\mathcal{J}$ we have $\tau_{j'} \circ Df = \tau_j$. There are three basic examples for cocones:

When $\mathcal{J} = \emptyset$, then a cocone is an object i in $\mathcal{I}$, so that $\mathcal{I}$ is non-empty. When $\mathcal{J}$ has two objects with no arrows between them, then a $\mathcal{J}$-diagram consists of a diagram of type (ii) in Def. 4.1.2. When $\mathcal{J}$ consists of two parallel morphisms from j_1 to j_2, then a cocone is a diagram of type (iii) in Def. 4.1.2. We conclude that if $\mathcal{I}$ has cocones on all finite diagrams, then $\mathcal{I}$ is filtered.

Conversely, assume that $\mathcal{I}$ is filtered and let $D : \mathcal{J} \to \mathcal{I}$ be a finite diagram. If $\mathcal{J} = \emptyset$, then D has a cocone since $\mathcal{I}$ is not empty by property (i) in Def. 4.1.2. Now, assume that $\mathcal{J}$ is not empty and let $\{j_1, \ldots, j_n\}$ be its set of objects. Then, for every j_k, j_l in $\mathcal{J}$, there is a diagram

$$\begin{tikzcd} D(j_k) \arrow[dr] & & D(j_l) \arrow[dl] \\ & i_{kl} & \end{tikzcd}$$

in $\mathcal{I}$ by property (ii) in Def. 4.1.2. Furthermore, for every $r \leftarrow i \to s$ in $\mathcal{I}$, there exists an element $t \in \mathcal{I}$ and morphisms $r \to t$ and $s \to t$ such that the diagram

$$\begin{tikzcd} & i \arrow[dl] \arrow[dr] & \\ r \arrow[dr] & & s \arrow[dl] \\ & t & \end{tikzcd}$$

commutes by properties (ii) and (iii) of Def. 4.1.2. All in all, we get the following commutative diagram

$$\begin{tikzcd}
D(j_1) \arrow[dr] & & D(j_2) \arrow[dl] \arrow[dr] & & \cdots \arrow[dl] & & \cdots \arrow[dr] & & D(j_n) \arrow[dl] \\
& i_{12} \arrow[dr] & & i_{23} \arrow[dl] \arrow[dr] & & \cdots & & i_{(n-1)n} \arrow[dl] & \\
& & i_{13} \arrow[dr] & & \cdots \arrow[dl] \arrow[dr] & & \cdots \arrow[dl] & & \\
& & & i_{14} \arrow[dr] & & \cdots \arrow[dl] & & & \\
& & & & i & & & &
\end{tikzcd}$$

Lastly, for all $f : j_k \to j_l$ in $\mathcal{J}$, one can choose the element i_{kl} such that the diagram

$$\begin{tikzcd} D(j_k) \arrow[rr, "D(f)"] \arrow[dr] & & D(j_l) \arrow[dl] \\ & i_{kl} & \end{tikzcd}$$

commutes again by the properties of a filtered category. It follows that $i \in \mathfrak{I}$ is a cocone for the finite diagram D.

*** $\square$

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