LECTURES ON DIFFEOLOGICAL GROUPOIDS

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Abstract. Diffeological groupoids appear in many areas of mathematics, such as infinite-dimensional Lie theory, classical field theory, deformation theory, and moduli spaces. The category of diffeological spaces, however, is too general and does not have a good differential calculus, which would be needed for a Lie theory of diffeological groupoids. I will introduce the notion of elastic diffeological spaces and show that these form a subcategory with an abstract tangent structure in the sense of Rosicky. The tangent structure yields a Cartan calculus consisting of vector fields, differential forms, the de Rham differential, inner derivatives, and Lie derivatives, satisfying the usual relations. Surprisingly, all diffeological groups are elastic. I then introduce the notion of diffeological Lie algebroids, which is the structure of the space of invariant vector fields of a strongly elastic diffeological groupoid form a diffeological Lie algebroid. As application, I will revisit a diffeological groupoid that arises in lorentzian geometry whose diffeological Lie algebroid encodes the Poisson brackets of the Gauss-Codazzi constraint functions. These lectures were given as a minicourse at the "finite and infinite-dimensional meeting on Lie groupoids, Poisson geometry and integrability" from August 16-20, 2021 in Vienna.

Contents

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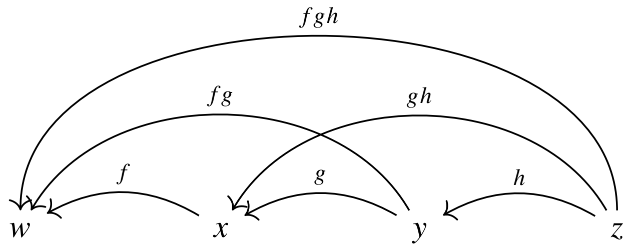
1. Beyond Lie groupoids

1.1. **Groupoids.** A groupoid is a category whose morphisms or arrows are all invertible. Representing the morphisms by arrows, we can depict the basic structure in the following way:



We will denote the maps from an arrow to its left and right endpoints by $G_0 \xleftarrow{l} G_1 \xrightarrow{r} G_0$. Some people draw the arrows to the left, some to the right, but everybody agrees what is left and right. (This is notation is due to A. Weinstein and would save the community a lot of headaches about whether source and target are on the left or the right, e.g. in the fibre product $G_1 \times_{G_0} G_1$.) Groupoids encompass and, therefore, generalize a number of familiar concepts.

Example 1.1. The most important guiding examples are the following:

- An equivalence relation $G_1 \subset G_0 \times G_0$ on the set G_0 is a groupoid with endpoint maps $l = \mathrm{pr}_1$, $r = \mathrm{pr}_2$, composition $(x, y)(y, z) = (x, z)$, identity $1_x = (x, x)$, and inverse $(x, y)^{-1} = (y, x)$.
- A group is a groupoid with one object.
- A left group action $\alpha : H \times X \to X$ is the left endpoint map $l = \alpha$ of a groupoid $G_1 := H \times X \rightrightarrows X =: G_0$, with right endpoint map $r = \mathrm{pr}_2$, composition $(g, h \cdot x)(h, x) = (gh, x)$, and inverse $(g, x)^{-1} = (g^{-1}, g \cdot x)$.

Depending on which of these three classes, relations, groups, or group actions, is considered to be the guiding example for groupoids, we are often led to different concepts and terminology for groupoids.

1.2. **Lie groupoids.** Just as for groups, many interesting groupoids are equipped with a geometric structure. For example, we could equip a groupoid with a topology, such that all structure maps are continuous. Such a groupoid is a groupoid internal to the category of topological spaces. It is tempting do define a Lie groupoid in an analogous way as groupoid internal to smooth (finite-dimensional) manifolds, but this is not quite right. The first issue is that the composition is defined on the pullback $G_2 = G_1 \times^{r,l}_{G_0} G_1$ which does not generally exist in smooth manifolds. The second issue concerns the construction of the Lie algebroid. We need an isomorphism

$$T\big(G_1 \times_{G_0} G_1\big) \cong TG_1 \times_{TG_0} TG_1 \tag{1}$$

in order to have a smooth right G_1-action on the r-vertical tangent bundle which we use to smoothly identify right invariant vector fields with sections of the Lie algebroid. Both issues are solved if we require the endpoint maps to be submersions. We thus arrive at the following definition.

Definition 1.2. A Lie groupoid G is a groupoid such that G_1, and G_0 are smooth manifolds, $l, r : G_1 \to G_0$ are submersions, and all structure maps (multiplication, identity, multiplication, and inverse) are smooth.

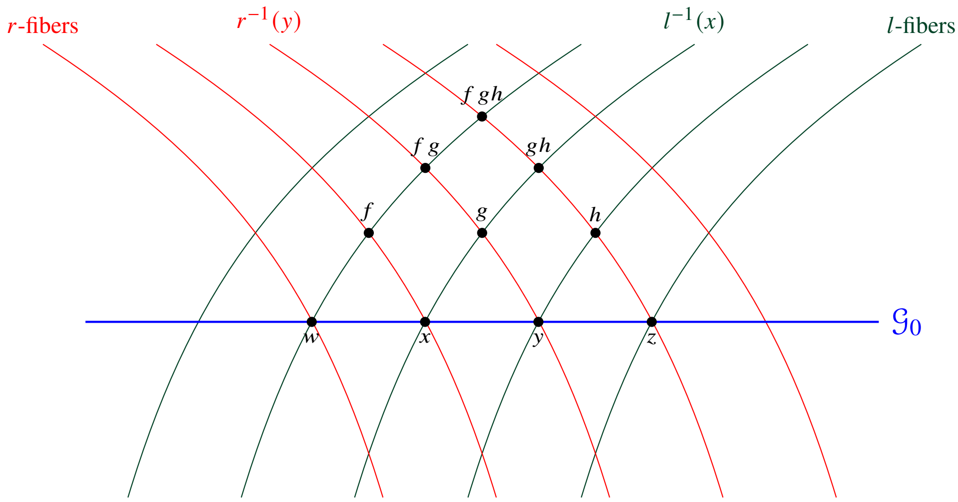


FIGURE 1. A Lie groupoid $G_1 \rightrightarrows G_0$

Example 1.3. Here are some examples of Lie groupoids:

- A **Lie group** is a Lie groupoid over a point.
- The smooth action of a Lie group on a smooth manifold gives rise to an **action Lie groupoid**.
- An **orbifold** is a Lie groupoid such that the characteristic map $(l, r) : G_1 \to G_0 \times G_0$ is proper and $r : G_1 \to G_0$ is a local diffeomorphism. Such Lie groupoids are called proper étale.
- Every **differentiable stack** is presented by a Lie groupoid. For example, the action Lie groupoid presents the quotient stack of the action.
- Let $H \times M \to M$ be the smooth action of a Lie group H on the manifold M and $K \subset H$ a closed subgroup such that its induced action on M is free and proper. Then $K \backslash H \times_K M \rightrightarrows M/K$ is a Lie groupoid, called the **K-reduced Lie groupoid** [BW].
- By applying the tangent functor $T : \mathcal{M}\text{fld} \to \mathcal{M}\text{fld}$ to a Lie groupoid, we obtain the **tangent groupoid** $TG_1 \rightrightarrows TG_0$. In a similar way we can define **jet groupoids**.

To a Lie groupoid we can associate an infinitesimal object as follows. First, we generalize the notion of right invariant vector fields from groups to groupoids. Using Eq. (1), we can restrict the tangent map of the groupoid multiplication as

$$\begin{array}{ccc} TG_1 \times_{TG_0} TG_1 & \longrightarrow & TG_1 \\ {\scriptstyle (\mathrm{id},0)}\uparrow & & \uparrow{\scriptstyle \mathrm{id}\circ \mathrm{pr}_1} \\ TG_1 \times^{Tr,Tl\circ 0}_{TG_0} G_1 & \longrightarrow & TG_1 \times^{Tr,0}_{TG_0} G_0 \end{array}$$

Using

$$TG_1 \times_{TG_0} G_0 \cong \ker Tr$$
$$TG_1 \times_{TG_0} G_1 \cong \ker Tr \times_{G_0} G_1 \,,$$

we obtain a smooth right G_1-action

$$\ker Tr \times_{G_0} G_1 \longrightarrow \ker Tr \,,$$

which is the right groupoid translation on the r-vertical tangent bundle. The smooth sections of $\ker Tr \to G_1$ that are invariant under the right translation are called right invariant vector fields on G_1. The following two observations are in analogy to the case of Lie groups:

First, the right invariant vector fields can be identified with the sections of the r-vertical bundle at the identity bisection,

$$A := \ker Tr \times_{G_1} G_0 \,,$$

which is a vector bundle over G_0. Second, the right invariant vector fields are invariant under the bracket of vector fields. In this way we obtain a bracket on the sections of A. The new phenomenon is that the bracket satisfies the Leibniz rule

$$[a, fb] = (\rho(a) \cdot f)b + f[a, b] \,,$$

for $a, b \in \Gamma(G_0, A)$, $f \in C^\infty(G_0)$, where $\rho := Tl|_{G_0} : A \to TG_0$ is called the anchor.

1.3. **Groupoids modeled on function spaces.** There are many interesting and relevant groups and groupoids that have some kind of differentiable structure but are not modeled on finite-dimensional smooth manifolds. The first example that comes to mind is the diffeomorphism group $\mathrm{Diff}(M)$ of a smooth manifold M. Another example is the group of automorphisms of a H-principal bundle over M, which is the set $C^\infty(M, H)$ with pointwise group structure. Both are examples of bisection groups of groupoids.

Definition 1.4. Let $G_1 \rightrightarrows G_0$ be a Lie groupoid. A smooth section $\sigma : G_0 \to G_1$ of $r : G_1 \to G_0$ is called a **bisection** if $\Phi_\sigma := l \circ \sigma : G_0 \to G_0$ is a diffeomorphism.

The bisections of a groupoid form a group with multiplication, identity, and inverse defined by

$$\begin{aligned}(\sigma\tau)(x) &= \sigma\big(\Phi_\tau(x)\big)\tau(x) \\ e_x &:= 1_x \\ \sigma^{-1}(x) &= \sigma\big(\Phi_\sigma^{-1}(x)\big)^{-1}.\end{aligned} \tag{2}$$

We will denote the bisection group of a groupoid G by $\mathcal{G}$.

Example 1.5. Let $G_1 := M \times M \rightrightarrows M =: G_0$ be the pair groupoid of a smooth manifold. A section of $r = \mathrm{pr}_2$ is of the form $\sigma(m) = \big(\Phi(m), m\big)$, where $\Phi : M \to M$ is a smooth map, which is a bisection if and only if Φ is a diffeomorphism. This shows that the bisection group is isomorphic to $\mathrm{Diff}(M)$.

Definition 1.6. Let $G_1 \rightrightarrows G_0$ be a Lie groupoid. Let $U \subset G_0$ be an open subset. A local section $\sigma : U \to G_1, r\circ\sigma = \mathrm{id}_U$ is called a **local bisection** if $\Phi_\sigma := l\circ\sigma|_U : U \to l\big(\sigma(U)\big)$ is a diffeomorphism.

The set of all local bisections of a groupoid is itself a groupoid over the set $\mathcal{O}\mathrm{pen}(M)$ of open subsets of M. The structure maps are given by the formulas (2).

Example 1.7. The groupoid of local bisections of the pair groupoid $M \times M$ is the groupoid of local diffeomorphisms of M. A pseudogroup is a subgroupoid with sheaf-like properties.

We have already seen that bisection groups of groupoids include diffeomorphism groups and the groups of local gauge transformations. In fact, most local symmetries in classical field theory are given by the action of a bisection group $\mathcal{G}$ of a groupoid $G \rightrightarrows M$ on the "space" of fields $\mathcal{F} = \Gamma(M, F)$, where $F \to M$ is the configuration bundle of the field theory and M the spacetime manifold. The action groupoid $\mathcal{G} \times \mathcal{F} \rightrightarrows \mathcal{F}$ can be viewed as the basic symmetry structure of the field theory. Since fields are considered to be physically equivalent if they are related by a symmetry, this groupoid can also be viewed as the differentiable stack of the physical (but off-shell) degrees of freedom.

Example 1.8. Let $\mathcal{L}\mathrm{or}(M)$ be the "space" of Lorentz metrics on M. The groupoid of the action of $\mathrm{Diff}(M)$ on $\mathcal{L}\mathrm{or}(M)$ by pushforward is the basic symmetry structure of general relativity on M.

The groups and groupoids in this section have as objects sets of smooth functions between smooth manifolds, such as a subset of $C^\infty(G_0, G_1)$ for the bisection group. These sets come equipped with a natural notion of smooth families given by smooth homotopies. For example, a smooth family in $\mathrm{Diff}(M)$ parametrized by an open subset $U \subset \mathbb{R}^n$ is a smooth map

$$\Phi : U \times M \longrightarrow M$$

such that $\Phi_u : M \to M, m \mapsto \Phi(u,m)$ is a diffeomorphism. A smooth path $\Phi : \mathbb{R} \times M \to M$ is a flow generated by a vector field v given by

$$v : m \longmapsto \frac{d}{dt}\Phi(t,m)\big|_{t=0} \in TM\,,$$

where the right hand side is to be understood by the kinematic notion of tangent vectors on M as equivalence classes of smooth paths. We can also derive the Lie algebra of $\mathrm{Diff}(M)$ by considering the right action on $C^\infty(M)$ by pullback. By differentiating the action of the flow Φ^v generated by v, we obtain the Lie derivative

$$\frac{d}{dt}\Phi_t^* f\,\big|_{t=0} = \mathcal{L}_v f\,. \tag{3}$$

By taking the second derivative of the group commutator we obtain

$$\frac{d}{ds}\frac{d}{dt}\big(\Phi_s^v \Phi_t^w (\Phi_s^v)^{-1} (\Phi_t^w)^{-1}\big)^* f\,\big|_{t,s=0} = -[\mathcal{L}_v, \mathcal{L}_w] f\,. \tag{4}$$

(The minus sign on the right hand side comes from the fact that the pullback is a right action. It is the origin of a number of annoying sign issues in differential geometry.) The upshot is the following proposition:

Proposition 1.9. *The Lie algebra of* $\mathrm{Diff}(M)$ *is* $\mathfrak{X}(M)^{\mathrm{op}}$.

Proof. The Lie derivative $\mathcal{L} : \mathfrak{X}(M) \to \mathrm{End}\big(C^\infty(M)\big), v \mapsto \mathcal{L}_v$ is a faithful representation of the Lie algebra of vector fields $\mathfrak{X}$. From Eq. (3) we conclude that the tangent space of $\mathrm{Diff}(M)$ at id_M can is given by $\mathfrak{X}(M)$. Eq. (4) together with $[\mathcal{L}_v, \mathcal{L}_w] = \mathcal{L}_{[v,w]}$ shows that $-[v,w]$ is the Lie bracket of $\mathrm{Diff}(M)$. □

This can be generalized to the following statement, which shows that the relation between Lie groupoids and Lie algebroids, can be viewed as relation between infinite dimensional Lie groups and Lie algebras.

Proposition 1.10 ([CdSW99, SW15]). *The Lie algebra of the bisection group of a Lie groupoid is the Lie algebra of sections of its Lie algebroid.*

At the first glance it seems that the derivation of the Lie algebra of $\mathrm{Diff}(M)$ relies only on the notion of smooth families, which are given in functions spaces by smooth homotopies. At second thought, however, we get the uneasy feeling that all this may have worked so well due to the fact, that we have some underlying structure on $\mathrm{Diff}(M)$ that we ignored but profited from unwittingly:

- The smooth families of $\mathrm{Diff}(M)$ seem to have additional sheaf-like properties. For example the restriction of smooth family of diffeomorphisms is a again a smooth family. Does our computation rely on this?
- $\mathrm{Diff}(M)$ has the structure of a Fréchet manifold modeled on $\mathfrak{X}(M)$. Is it necessary to ensure that the tangent space at the identity is a vector space?

- We have used that $\mathrm{Diff}(M)$ has an action on $C^\infty(M)$ and that the induced infinitesimal representation of $\mathfrak{X}(M)$ is faithful. In the case of bisection groups we have a faithful representation on $C^\infty(G_1)$. Do we need that such a representation exists?
- How does this construction work for groupoids?

One goal of the following lectures is to show that these questions have satisfactory answers in the framework of diffeological spaces.

2. Differential calculus on elastic diffeological spaces

2.1. **Diffeological spaces.** The notion of diffeological spaces formalizes the sheaf properties of smooth families parametrized by real parameters, such as smooth homotopies of function spaces.

Definition 2.1 (e.g. Def. 1.5 in [IZ13])**.** Let X be a set. A **diffeology** on X is a map D that assigns to every open subset $U \subset \mathbb{R}^n$ for all $n \geq 0$ a set $D(U) \subset \mathrm{Hom}_{\mathcal{S}\mathrm{et}}(U, X)$ of maps called **plots**, such that the following conditions are satisfied:

(D1) Every constant map $p : U \to * \to X$ is a plot.
(D2) A map $p : U \to X$ is a plot if all restrictions $p|_{U_i} : U_i \to X$ to an open cover $\{U_i \hookrightarrow U\}_{i \in I}$ are plots.
(D3) The composition $p \circ f$ of a plot $p : U \to X$ with a smooth map $f : V \to U$ is a plot.

A set with a diffeology is called a **diffeological space**. A map of sets $f : X \to Y$ is **smooth** if for every plot $p : U \to X$ the map $f \circ p : U \to Y$ is a plot. The category of diffeological spaces and smooth maps will be denoted by $\mathcal{D}\mathrm{flg}$.

Example 2.2. The following are natural and useful examples of diffeological spaces derived from differentiable structures:

(a) Every smooth finite-dimensional manifold M is equipped with the **manifold diffeology** given by $D(U) = C^\infty(U, M)$, i.e. the plots are the infinitely often differentiable maps.
(b) Let $\iota : S \hookrightarrow M$ a subset of a smooth manifold. A plot of the **subspace diffeology** on S is a map $p : U \to S$ such that $\iota \circ p : U \to S \hookrightarrow M$ is smooth. The subspace diffeology is the largest diffeology such that ι is smooth.
(c) Let $\pi : M \to S$ a surjective map from a manifold to a set. A plot of the **quotient diffeology** on S is a map $p : U \to S$ such that every $u \in U$ has a neighborhood $V \ni u$, such that $p|_V$ has a smooth lift to M, i.e. $p|_V = \pi \circ q$ for as smooth map $q : V \to M$. The quotient diffeology is the largest diffeology such that π is smooth.
(d) Let M and N be smooth manifolds. The **functional diffeology** on $C^\infty(M, N)$ is defined by
$$D(U) = C^\infty(U \times M, N) \subset \mathcal{S}\mathrm{et}\big(U, C^\infty(M, N)\big)\,,$$
i.e. the plots are smooth homotopies of maps.

Example 2.3. The cases that first piqued my interest in diffeological structures are the following:

(e) The bisection group of a Lie groupoid $G_1 \rightrightarrows G_0$ equipped with the subspace diffeology of the functional diffeology $C^\infty(G_0, G_1)$ is a diffeological group, i.e. a group internal to the category of diffeological spaces.

(f) The action groupoid of a local action of the bisection group of $G_1 \rightrightarrows G_0$ on a space of sections of a smooth bundle $F \to G_0$ is a diffeological groupoid, i.e. a groupoid internal to the category of diffeological spaces.

Example 2.4. The following examples show that the diffeological notion of smoothness is very general:

(g) The **discrete diffeology** or **fine diffeology** on a set S is the diffeology for which the plots are the locally constant maps.

(h) The **trivial diffeology** or **coarse diffeology** on a set S is given by $D(U) = \mathcal{S}\mathrm{et}\big(U, S\big)$, i.e. all maps are plots.

(i) Every topological space X is equipped with the **continuous diffeology** given by $D(U) = \mathcal{T}\mathrm{op}(U, X)$, i.e. the plots are the continuous maps.

(j) The constructions of subspace diffeologies, quotient diffeologies, and functional diffeologies still works if we replace the smooth manifolds M and N with arbitrary diffeological spaces.

The last set of examples shows that the category of diffeological spaces is too general as to allow for strong geometric results. From my experience diffeology is the most useful in applications to geometry if we start from a genuinely differential geometric situation and allow from incremental generalizations. For example the diffeology on the quotient of the torus by a free $\mathbb{R}$-action, i.e. the leaf space of the Kronecker foliation, contains as much information as the C^*-algebras of the corresponding noncommutative torus [DI85].

If we want to develop a general theory on diffeological spaces, such as a Lie theory of diffeological groupoids, the task is to identify additional properties that enable us to define the structures and prove the propositions that we need. The difficult part is to find the weakest assumptions possible, so that we do not simply come back to standard differential geometry. For our purposes, we want to figure out on what kind of spaces we have a Cartan calculus, i.e. the tangent functor, vector fields, differential forms, the de Rham differential, the inner derivative, and the Lie derivative, all satisfying the usual relations.

2.2. **The natural tangent functor.** The first structure we need for a differential calculus on diffeological spaces is a tangent functor. Unfortunately, a whole zoo of inequivalent definitions can be found in the literature. (For a an overview see [CW16]). Why?

The first reason for this variety is that equivalent definitions for manifolds become inequivalent for diffeological spaces. On manifolds, there are three common definitions of tangent vectors: a) In terms of local coordinates. On $U \subset \mathbb{R}^n$ a tangent vector is an element of $TU = U \times \mathbb{R}^n$, transforming appropriately under a coordinate change. b) In terms of equivalence classes of paths. This is usually called the kinematic definition. c) In terms of derivations of the ring of smooth functions. This is often called the algebraic definition. Moving between these definitions requires bona fide analytic tools like Hadamard's lemma and the mean value theorem, which we no longer have for general diffeological spaces.

The second reason is that some authors force the fibres of the tangent bundle to be vector spaces. Since a diffeological space is not locally modeled on a vector space, such constructions usually consist of the construction of a free diffeological vector space generated by the real cone of tangent vectors. Only the generating cone reflects the infinitesimal structure of the diffeological space. Moreover, different cones can generate the same vector space, so that we even lose geometric information by this construction.

The third reason for the variety of tangent functors is that some authors start by defining tangent fibres at points and then equip the union of all tangent spaces with a diffeology in rather ad hoc ways. This can lead to inequivalent definitions and even to outright mistakes.

In order to find a definition of the tangent functor that serves our purpose, the construction of a differential calculus, it is helpful to move from the toad perspective, defining a diffeology by maps between sets with properties, to the eagle perspective of the following categorical definition:

Proposition 2.5 ([BH11]). *A diffeological space is a concrete sheaf on the site* $\mathcal{E}\mathrm{ucl}$ *of open subsets of all* $\mathbb{R}^n$*,* $n \geq 0$ *and open covers.*

The only term which needs some explanation here is "concrete". A concrete structure on a category $\mathcal{C}$ is a faithful forgetful functor $\mathcal{C} \to \mathcal{S}\mathrm{et}$, $C \to |C|$ by which every object in $\mathcal{C}$ can be viewed as a set with structure and every morphism as a map of sets with properties. For many categories with terminal object $*$, the concrete structure is the functor of points, $|C| = \mathcal{C}(*, C)$. In geometry and topology it is common to not spell out the forgetful functor. For example, when in the definition of diffeologies we say that $p : U \to X$ is a map of sets, we really mean $p : |U| \to X$.

A site is concrete if the functor of points is faithful, hence a concrete structure, and if it maps covers to surjective maps. $\mathcal{E}\mathrm{ucl}$ is concrete. A sheaf $D : \mathcal{E}\mathrm{ucl}^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$ is concrete if $D(U)$ is a subset of the set of all maps $\{|U| \to D(*)\}$. From this, it is straightforward to recover the definition 2.1 in terms of plots.

Once we have established that $\mathcal{D}\mathrm{flg}$ is a category of concrete sheaves, we can turn to a handbook of category theory [Joh02] and find that it has a number of nice properties, which would be really hard to show explicitly with plots:

Proposition 2.6. *The category of sets is a quasitopos with a strict initial object i.e. every morphism* $X \to \emptyset$ *is an isomorphism. In particular, it has the following properties:*

- $\mathcal{D}\mathrm{flg}$ *has all small colimits.*
- $\mathcal{D}\mathrm{flg}$ *has all limits.*
- $\mathcal{D}\mathrm{flg}$ *has all exponential objects, i.e. internal homs or mapping spaces.*

- $\mathcal{D}\mathrm{flg}$ *is locally cartesian closed, i.e. for every object* X *in* $\mathcal{D}\mathrm{flg}$ *the overcategory* $\mathcal{D}\mathrm{flg}\downarrow X$ *is cartesian closed.*
- *Strong monomorphisms (inductions) and strong epimorphisms (subductions) are effective.*
- *Strong monomorphisms and strong epimorphisms are stable under pullback.*
- $\mathcal{D}\mathrm{flg}$ *is quasiadhesive, i.e. the pushout of a strong monomorphism is a strong monomorphism and the pushout square is a pullback square.*
- *Coproducts are disjoint, i.e.* $X \to X \sqcup Y \leftarrow Y$ *are monomorphisms and* $X \times_{X\sqcup Y} Y \cong \emptyset$.
- *The faithful functor of points* $\mathcal{D}\mathrm{flg} \to \mathcal{S}\mathrm{et}$, $X \to \mathcal{D}\mathrm{flg}(*, X)$ *has a left and a right adjoint.*

Moreover, we have the Yoneda embedding

$$y : \mathcal{E}\mathrm{ucl} \longrightarrow \mathcal{D}\mathrm{flg}$$

which is faithful and dense, i.e. every diffeological space X is the colimit

$$X \cong \operatorname*{colim}_{yU\to X} yU$$

of its plots. This is a generalization of the statement that every smooth manifold is the colimit of the charts of an atlas, the colimit being the categorical formulation of the gluing of the charts.

Let $F : \mathcal{E}\mathrm{ucl} \to \mathcal{E}\mathrm{ucl}$ be an endofunctor. The (pointwise) left Kan extension of yF along the Yoneda embedding,

$$(\mathrm{Lan}_y\, yF)X = \operatorname*{colim}_{yU\to X} yFU\,. \tag{5}$$

is the unique extension of F that preserves these colimits. Because y is faithful, the diagram

$$\begin{array}{ccc} \mathcal{E}\mathrm{ucl} & \xrightarrow{\;F\;} & \mathcal{E}\mathrm{ucl} \\ {\scriptstyle y}\downarrow & & \downarrow{\scriptstyle y} \\ \mathcal{D}\mathrm{flg} & \xrightarrow{\mathrm{Lan}_y\, yF} & \mathcal{D}\mathrm{flg} \end{array}$$

commutes. This means that if we identify the elements of $\mathcal{E}\mathrm{ucl}$ with their images under the Yoneda embedding, then the Kan extension of F restricts to F on $\mathcal{E}\mathrm{ucl}$. Moreover, the left Kan extension maps natural transformations to natural transformations and is monoidal with respect to the composition of endofunctors. The nice properties of the Kan extension can be summarized as follows:

Proposition 2.7. *The functor of categories of endofunctors,*

$$\mathrm{Lan}_y\, y(\_\_) : \mathrm{End}(\mathcal{E}\mathrm{ucl}) \longrightarrow \mathrm{End}(\mathcal{D}\mathrm{flg})\,, \tag{6}$$

is fully faithful and monoidal.

Let us give an interpretation of this proposition in more explicit terms. A construction on manifolds is local if it can be defined a functor $F : \mathcal{E}\mathrm{ucl} \to \mathcal{E}\mathrm{ucl}$ on the charts. A good example is the tangent functor $TU = U \times \mathbb{R}^n$. Its left Kan extension along $\mathcal{E}\mathrm{ucl} \hookrightarrow \mathcal{M}\mathrm{fld}$ is the tangent functor of manifolds. A map like the tangent projection $\pi_U : TU \to U$ must be compatible with coordinate changes in order to be well defined on manifolds. In categorical language it must be natural in U, i.e.

$$\begin{array}{ccc} TU & \xrightarrow{Tf} & TV \\ {\scriptstyle \pi_U}\downarrow & & \downarrow{\scriptstyle \pi_V} \\ U & \xrightarrow{f} & V \end{array}$$

must commute for all smooth maps $f : U \to V$. The functor (6) generalizes this local to global principle from charts of a manifold to the plots of a diffeological space.

Definition 2.8. The left Kan extension of the tangent functor $\hat{T} : \mathcal{E}\mathrm{ucl} \to \mathcal{E}\mathrm{ucl}$ of euclidean spaces will be denoted by

$$T := \mathrm{Lan}_y\, y\hat{T} : \mathcal{D}\mathrm{flg} \to \mathcal{D}\mathrm{flg}$$

and called the **natural tangent functor** of diffeological spaces.

How do we describe and compute TX of a diffeological space explicitly? We have to spell out the colimit formula (5) for the Kan extension. The upshot is the following: The set TX is the set of equivalence classes

$$TX = \coprod_{p:U\to X} (TU)_p \;/\sim\,, \tag{7}$$

where the index p distinguishes the different copies of TU for plots with the same domain. The equivalence relation is given as follows. We say that a vector $\zeta_u \in (TU)_p$ is **T-related** to a vector $\eta_v \in (TV)_q$ if there is a smooth map $f : U \to V$ such that $q \circ f = p$ and $Tf\,\zeta_u = \eta_v$. Two vectors are related by $\sim$ iff they are connected by a finite zigzag of T-relations. The diffeology of TX is the quotient diffeology of (7)

Moreover, since every tangent vector $\eta_u \in TU$ can be represented by a smooth path, every tangent vector in TX is also represented by a smooth path, i.e. by a plot $\mathbb{R} \to X$. The conclusion is that elements of TX are kinematic tangent vectors. However, the fibres T_xX do not generally have the structure of a vector space (figure 2.2).

Proposition 2.9. *Let M and N be smooth manifolds. Then*

$$TC^\infty(M,N) \cong C^\infty(M,TN)\,,$$

where the right hand side is equipped with the functional diffeology.

Every reasonable differential geometer would expect this result, so that this proposition seems to confirm what we already knew. However, the proof of the proposition 2.9 is surprisingly hard. In fact, in the literature it had only been proved under the assumption that M is compact [CW16].

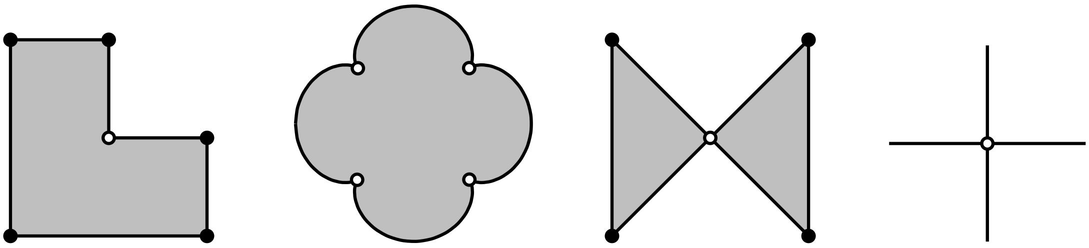


FIGURE 2. Diffeological subspaces of $\mathbb{R}^2$ with points marked in white where the tangent fibre is not a vector space.

2.3. **Abstract tangent functors.** Let us take stock of the natural structure of the tangent functor of euclidean spaces. We will denote the k-fold fibre product of the tangent bundle by

$$\hat{T}_k U := \underbrace{\hat{T}U \times_U \hat{T}U \times_U \ldots \times_U \hat{T}U}_{k \text{ factors}}$$

for every open subset $U \subset \mathbb{R}^n$. The tangent functor, its powers, and fibre products is given explicitly by

$$\begin{aligned}
\hat{T}U &= U \times \mathbb{R}^n \\
\hat{T}^2 U &= U \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \\
\hat{T}^k U &= U \times (\mathbb{R}^n)^{2^k - 1} \\
\hat{T}_2 U &= U \times \mathbb{R}^n \times \mathbb{R}^n \\
\hat{T}_k U &= U \times (\mathbb{R}^n)^k \,.
\end{aligned}$$

On a smooth map $f : U \to V \subset \mathbb{R}^m$ the functors are given by

$$\hat{T}f : (u, u_1^i) \longmapsto \Big(f(u), \frac{\partial f^a}{\partial x^i} u_1^i \Big) \tag{8}$$

$$\hat{T}^2 f : (u, u_1^i, u_2^i, u_{12}^i) \longmapsto \Big(f(u), \frac{\partial f^a}{\partial x^i} u_1^i, \frac{\partial f^a}{\partial x^i} u_2^i, \frac{\partial f^a}{\partial x^i} u_{12}^i + \frac{\partial^2 f^a}{\partial x^i \partial x^j} u_1^i u_2^j \Big) \tag{9}$$

$$\hat{T}_2 f : (u, u_1^i, u_2^i) \longmapsto \Big(f(u), \frac{\partial f^a}{\partial x^i} u_1^i, \frac{\partial f^a}{\partial x^i} u_2^i \Big) \,.$$

The formulas for $\hat{T}^k$ and $\hat{T}_k$ are analogous. We have the following natural transformations between these functors:

bundle projection:

$$\begin{aligned}
\hat{\pi}_U : \hat{T}U &\longrightarrow U \\
(u, u_1) &\longmapsto u
\end{aligned}$$

zero section:

$$\begin{aligned}
\hat{0}_U : U &\longrightarrow \hat{T}U \\
u &\longmapsto (u, 0)
\end{aligned}$$

addition:

$$\begin{aligned} \hat{+}_U : \hat{T}_2 U &\longrightarrow \hat{T} U \\ (u, u_1, v_1) &\longmapsto (u, u_1 + v_1) \end{aligned}$$

inverse:

$$\begin{aligned} \hat{-}_U : \hat{T} U &\longrightarrow \hat{T} U \\ (u, u_1) &\longmapsto (u, -u_1) \end{aligned}$$

vertical lift:

$$\begin{aligned} \hat{\lambda}_U : \hat{T} U &\longrightarrow \hat{T}^2 U \\ (u, u_1) &\longmapsto (u, 0, 0, u_1) \end{aligned}$$

flip of differentiation

$$\begin{aligned} \hat{\tau}_U : \hat{T}^2 U &\longrightarrow \hat{T}^2 U \\ (u, u_1, u_2, u_{12}) &\longmapsto (u, u_2, u_1, u_{12}) . \end{aligned}$$

Out of these natural transformations we can construct new ones. For example, T can be equipped with the structure of a monad whose algebras are an interesting class of singular foliations [Jub12]. The natural transformations satisfy a number of compatibility relations, which are encoded in the definition of an abstract tangent functor:

Definition 2.10 ([Ros84, CC14]). A **tangent structure** (with inverses) consists of a functor $T : \mathcal{C} \to \mathcal{C}$ together with natural transformations $\pi_X : TX \to X$, $0_X : X \to TX$, $+_X : T_2 X \to TX$, $\lambda_X : TX \to TX^2$, and $\tau_X : T^2 X \to T^2 X$, such that the following axioms holds:

- **Fibre products:** The fibre products $T_k X$ exist for all $k \geq 0$ and are preserved by T, i.e. $TT_k X \cong T_k TX$.
- **Group structure:** $\pi_X : TX \to X$ with neutral element 0_X and addition $+_X$ is a natural group bundle, i.e. an abelian group object in $\mathcal{C} \downarrow X$.
- **Symmetric structure:** $\tau : T^2 \to T^2$ is a symmetric structure on T, such that

$$\begin{array}{ccc} T^2 X & \xrightarrow{\tau_X} & T^2 X \\ {\scriptstyle \pi_{TX}} \downarrow & & \downarrow {\scriptstyle T\pi_X} \\ TX & \xrightarrow[\text{id}]{} & TX \end{array} \tag{10}$$

is a morphism of natural group bundles, i.e. $(T+) \circ (\tau \times \tau) = \tau \circ (+T)$.

- **Vertical lift:** The diagrams

(11)
$$\begin{CD} TX @>{\lambda_X}>> T^2X \\ @V{\pi_X}VV @VV{T\pi_X}V \\ X @>>{0}> TX \end{CD} \qquad \begin{CD} TX @>{\lambda_X}>> T^2X \\ @V{\lambda_X}VV @VV{\lambda_{TX}}V \\ T^2X @>>{T\lambda_X}> T^3X \end{CD}$$

  commute. Moreover, the first diagram is a morphism of natural group bundles, i.e. $(T+) \circ (\lambda \times \lambda) = \lambda \circ +$.
- **Vertical lift is a kernel:** The diagram

(12)
$$\begin{CD} TX @>{\lambda_X}>> T^2X \\ @V{\pi_X}VV @VV{(\pi_{TX}, T\pi_X)}V \\ X @>>{(0_X, 0_X)}> T_2X \end{CD}$$

  is a pullback.
- **Compatibility of vertical lift and symmetric structure:** The diagrams

(13)
$$\begin{array}{ccc} & TX & \\ {}^{\lambda_X}\swarrow & & \searrow^{\lambda_X} \\ T^2X & \xrightarrow[\tau_X]{} & T^2X \end{array} \qquad \begin{CD} T^2X @>{T\lambda_X}>> T^3X @>{\tau_{TX}}>> T^3X \\ @V{\tau_X}VV @. @VV{T\tau_X}V \\ T^2X @>>{\lambda_{TX}}> @. T^3X \end{CD}$$

  commute.

A category together with a tangent functor is called a **tangent category** (with inverses).

It is easy to check that the tangent functor of $\mathcal{E}$ucl satisfies the conditions of the definition 2.10. In addition we have the fibrewise scalar multiplication $\mathbb{R} \times TX \to TX$, which is was not made part of the definition since most categories do not have $\mathbb{R}$ as object.

2.4. **Elastic diffeological spaces.** We will now Kan extend the tangent structure of $\mathcal{E}$ucl to $\mathcal{D}$flg. Due to the proposition 2.7, taking powers of $\hat{T}$ is compatible with the left Kan extension, so that we have the natural isomorphism

$$T^k X \cong (\mathrm{Lan}_y\, y\hat{T}^k)X$$

By applying the functor (6), we obtain the natural transformations

$$\begin{aligned} \pi_X &:= (\mathrm{Lan}_y\, y\hat{\pi})_X : TX \longrightarrow X \\ 0_X &:= (\mathrm{Lan}_y\, y\hat{0})_X : X \longrightarrow TX \\ -_X &:= (\mathrm{Lan}_y\, y\hat{-})_X : TX \longrightarrow TX \\ \lambda_X &:= (\mathrm{Lan}_y\, y\hat{\lambda})_X : TX \longrightarrow T^2X \\ \tau_X &:= (\mathrm{Lan}_y\, y\hat{\tau})_X : T^2X \longrightarrow T^2X\,. \end{aligned}$$

Due to the functoriality of the Kan extension, these satisfy all the commutative diagrams as before: the diagrams of the monoidal structure, diagram (10), (11), and (13). The Kan extension of the fibre product $\hat{T}_k$, $k \geq 0$ will be denoted by

$$\Theta_k := \mathrm{Lan}_y\, y\hat{T}_k : \mathcal{D}\mathrm{flg} \longrightarrow \mathcal{D}\mathrm{flg}\,.$$

The left Kan extension does not commute with pullbacks, so that $\Theta_k X$ is not naturally isomorphic to $T_k X$. More precisely, the Kan extension of the projections $\mathrm{pr}_i : \hat{T}_k X \to TX$ are morphisms $\Theta_k X \to TX$, which induce a morphism

$$(\theta_k)_X : \Theta_k X \longrightarrow T_k X\,,$$

which is generally not an isomorphism.

The natural transformation of the tangent structure of $\mathcal{E}$ucl involving $\hat{T}_k U$ are, therefore, mapped by the Kan extension to diagrams involving $\Theta_k X$: The fibrewise addition $\hat{+}_U : \hat{T}_2 U \to \hat{T} U$ is mapped to a natural transformation

$$(\mathrm{Lan}_y\, y\hat{+})_X : \Theta_2 X \longrightarrow TX\,.$$

For this reason, the natural tangent bundle TX of a diffeological space does generally not have the structure of an abelian monoid or group. The natural transformation $(\hat{\pi}_{\hat{T}U}, \hat{T}\hat{\pi}_U) : \hat{T}^2 U \to \hat{T}_2 U$ that appears in the diagram (12) is mapped to a natural transformation

$$T^2 X \longrightarrow \Theta_2 X\,.$$

The map $(\pi_{TX}, T\pi_X) : T^2 X \to T_2 X$ still exists due to the universal property of the pullback $T_2 X = TX \times_X TX$, but the diagram (12) is no longer commutative in $\mathcal{D}$flg. The search for the conditions that ensure that the natural tangent of diffeological spaces satisfies the properties of an abstract tangent structure in the sense of Def. 2.10 leads to the following concept.

Definition 2.11. A diffeological space X is called **elastic** if

$$(\theta_k)_{T^l X} : \Theta_k T^l X \longrightarrow T_k T^l X$$

is an isomorphism for all $k, l \geq 0$. The full subcategory of elastic diffeological spaces will be denoted by $\mathcal{E}$lst.

The geometric intuition of elastic spaces and the reason for the terminology is the following. Every tangent vector $v_x \in T_x X$ is represented by a path. One can picture this by stretching out x in the direction of v_x to a smooth path $\gamma : (-\varepsilon, \varepsilon) \to X$ of short but non-zero length through $\gamma(0) = x$, such that the coordinate tangent vector $\frac{\partial}{\partial t}$ at the origin of the interval is mapped by $T_0\gamma$ to v_x. In this sense, every point of X is elastic in every infinitesimal direction.

However, we generally cannot stretch out x in the direction of several tangent vectors $v_x^1, \ldots, v_x^k \in TX_x$. That is, we cannot always find a plot $p : U \to X$ with $p(0) = x$ such that $(T_0 p)\frac{\partial}{\partial t^i} = v_x^i$, where $(t^1, \ldots, t^k)$ are the canonical coordinates of $U \subset \mathbb{R}^k$. And even if we can find such a plot, it may happen that the tangent map Tp is not injective at 0, such that we cannot identify the tangent vectors on X with the coordinate vectors on U. This identification is possible if and only if $(\Theta_k X)_x \to (TX_x)^k \cong (T_k X)_x$ is a bijection. If this is the case for all $k \geq 0$, we call the point x elastic.

If all points of a diffeological space are elastic, then $(\theta_k)_X$ is a bimorphism. Since the category of diffeological spaces is not balanced, however, this does not imply that $(\theta_k)_X$ is an isomorphism. For a good definition of elastic diffeological spaces, it must be required by definition that $(theta_k)_X$ has a smooth inverse. Moreover, we have to require that the $(\theta_k)_{T^l X}$ has the same property, so that elastic spaces are closed under the tangent functor.

Theorem 2.12. *Elastic diffeological spaces are closed under the following operations:*

(i) Coproducts of elastic spaces are elastic.
(ii) Finite products of elastic spaces are elastic.
(iii) Retracts of elastic spaces are elastic.

Theorem 2.13. *The category of elastic diffeological spaces is a tangent category (with inverses).*

2.5. **Examples of elastic diffeological spaces.**

2.5.1. *Manifolds.* For finite dimensional vector spaces we have

$$\Theta_k y\mathbb{R}^n \cong y\hat{T}_k\mathbb{R}^n \cong (y\hat{T})_k\mathbb{R}^n \cong T_k y\mathbb{R}^n \,,$$

where we have used that the Yoneda embedding commutes with limits. Since being elastic is a property that is local with respect to the diffeological topology on a manifold, it follows that every space locally modeled on $\mathbb{R}^n$ is elastic.

2.5.2. *Manifolds with corners.* Consider the set $[0,\mathbb{R}) \subset \infty$ equipped with the subspace diffeology. Every smooth map $p \in U \to \mathbb{R}$ with image $p(U) \subset [0,\infty)$ has vanishing derivatives to all orders at every point u_0 with $p(u_0) = 0$. It follows that

$$T_0[0,\infty) = 0\,.$$

The tangent space at an interior point $x \in (0,\infty)$ is given by $T_x[0,\infty) = \mathbb{R}$.

We want to show that $[0,\infty)$ is elastic. For this, we consider the maps

$$\begin{aligned} \pi : \mathbb{R} &\longrightarrow [0,\infty) \\ x &\longmapsto x^2 \end{aligned}$$

and

$$\begin{aligned} \sigma : [0,\infty) &\longrightarrow \mathbb{R} \\ x &\longmapsto \sqrt{x}\,, \end{aligned}$$

which satisfy $\pi \circ \sigma = \mathrm{id}_{[0,\infty)}$ as maps of sets. We have to show that σ is smooth. Let $p : U \mapsto [0,\infty)$ be a plot. Since σ is smooth on the interior $(0,\infty)$, $\sigma \circ p$ is smooth at a all points in U that are mapped to the interior $(0,\infty)$. Assume that $p(u_0) = 0$. Since all

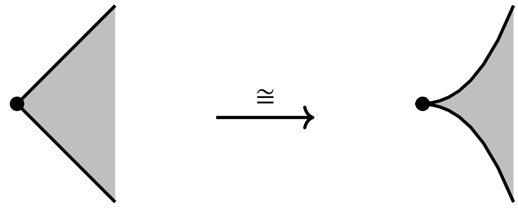


FIGURE 3. Squeezing a corner by multiplying the y-coordinate with the function $f(x) = x^{\frac{3}{2}}$. The boundary of the resulting diffeological subspace of $\mathbb{R}^2$ is the curve $x^2 = y^5$ with a cusp at $(0,0)$.

derivatives of a plot p vanish at u_0, p vanishes to all orders at u_0, i.e.

$$\frac{p(u)}{\|u - u_0\|^k} \xrightarrow{u \to u_0} 0\,,$$

for all $k \geq 0$. This implies that $\sqrt{p(u)} = (\sigma \circ p)(u)$ vanishes to all orders at u_0 as well, so that $\sigma \circ p$ is differentiable at u_0. We conclude that σ is smooth, so that $[0,\infty)$ is a smooth retract of $\mathbb{R}$.

By Thm. 2.12, we conclude that the retract $[0,\infty)$ is elastic and that any finite product $\mathbb{R}^n_k := [0,\infty)^k \times \mathbb{R}^{n-k}$ is elastic. Since the diffeological tangent functor commutes with products, the tangent spaces are given by

$$T_{(x_1,\dots,x_n)}\mathbb{R}^n_k = T_{x_1}[0,\infty) \times \dots T_{x_k}[0,\infty) \times \mathbb{R}^{n-k}\,. \tag{14}$$

Finally, we conclude that every diffeological space modeled locally on $\mathbb{R}^n_k$ is elastic. In other words, manifolds with corners are elastic.

2.5.3. *Manifolds with cusps.* Consider the following subset of $\mathbb{R}^2$,

$$X := \big\{(x,y) \mid x \geq 0 \ \wedge \ |y| \leq x\big\}\,,$$

with the subspace diffeology. We can squeeze or stretch the corner at $(0,0)$ by multiplying the y coordinate by a smooth function $f \in C^\infty\big([0,\infty)\big)$ (see Fig. 3),

$$\begin{aligned} \varphi : X &\longrightarrow X_f \\ (x,y) &\longmapsto \big(x, f(x)\,y\big)\,, \end{aligned}$$

where

$$X_f := \big\{(x,y) \mid x \geq 0 \ \wedge \ |y| \leq f(x)\big\}\,.$$

Assume that $f(x) > 0$ for $x > 0$. Then φ has an inverse map given by $\varphi^{-1}(0,0) = (0,0)$ and

$$\varphi^{-1}(x,y) = \Big(x, \frac{y}{f(x)}\Big)$$

otherwise. Let us equip X_f with the pullback diffeology of φ^{-1}. Since φ^{-1} is surjective, φ is an isomorphism of diffeological spaces. X is $[0,\infty) \times [0,\infty)$ rotated by minus 45 degrees, so it is elastic. Since φ is an isomorphism X_f is elastic, too.

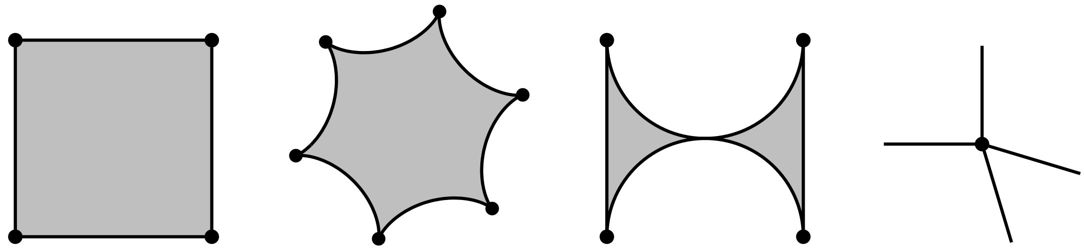


FIGURE 4. Elastic diffeological subspaces of $\mathbb{R}^2$. The tangent spaces are 0 at the marked points, $\mathbb{R}$ at points on the black lines, and $\mathbb{R}^2$ at gray points in the interior.

2.5.4. *Function spaces.*

Theorem 2.14. *Let $F \to M$ be a smooth fibre bundle. The set of sections $\mathcal{F} = \Gamma(M, F)$ equipped with the subspace diffeology of the functional diffeology on $C^\infty(M, F)$ is elastic.*

Corollary 2.15. *Let M and N be smooth manifolds. Then $C^\infty(M, F)$ with the functional diffeology is elastic.*

2.5.5. *Pro-finite dimensional manifolds.* Pro-objects in $\mathcal{M}\mathrm{fld}$ are elastic, e.g. the infinite jet bundle $J^\infty F$.

2.6. **The Cartan calculus on elastic spaces.** A vector field on a diffeological space is a section $v : X \to TX$ of the bundle projection $\pi_X : TX \to X$. For differential forms, there are two natural definitions. The first is as the left Kan extension of the de Rham functor,

$$\Omega(X) := \operatorname*{colim}_{yU \to X} \Omega(U) .$$

A form $\omega \in \Omega(X)$ is a family of forms

$$\omega_p \in \Omega(U)$$

for every plot $p : U \to X$, such that $\omega_p = f^* \omega_q$ for every plot $q : V \to X$ and smooth map $f : U \to V$ satisfying $p = q \circ f$. $\Omega(X)$ is equipped with a differential, which is given by applying the de Rham differential to every element of the family, $(d\omega)_p = d\omega_p$.

The second definition of a differential form is as antisymmetric, fibrewise multilinear map

$$\omega : T_k X \longrightarrow \mathbb{R} .$$

For these kind of forms we have the inner derivative with respect to a vector field which is given by precomposition

$$\iota_v \omega : T_{k-1} X \xrightarrow{(v, \mathrm{id}_{T_{k-1}X})} T_k X \xrightarrow{\omega} \mathbb{R} .$$

Proposition 2.16. *The two definitions of differential forms are naturally isomorphic if and only if $\theta_k : \Theta_k X \to T_k X$ is an isomorphism for all $k \geq 0$.*

If X is elastic, so that T satisfies the properties of a tangent structure, then we can define the Lie bracket of two vector fields $v, w : X \to TX$ as follows. First we construct the morphism

$$\begin{array}{c} X \\ \downarrow{\scriptstyle \Delta} \\ X \times X \\ \downarrow{\scriptstyle v\times w} \\ TX \times_X TX \\ \downarrow{\scriptstyle Tw\times Tv} \\ T^2X \times_{TX} T^2X \\ \downarrow{\scriptstyle \mathrm{id}\times T-} \\ T^2X \times_{TX} T^2X \\ \downarrow{\scriptstyle T+} \\ T^2X \end{array}$$

which we denote by $\beta(v,w) : X \to T^2X$. Then we check that this map takes values in the kernel of $(\pi_{TX}, T\pi_X) : T^2X \to T_2X$. It follows from the universal property of the pullback diagram (12) of an abstract tangent structure, that there is a unique vector field $[v,w] : X \to TX$ such that $\beta(v,w) = \lambda \circ [v,w]$. It is quite involved to show that $[v,w]$ is is indeed a Lie bracket satisfying the Jacobi identity [CC15].

Theorem 2.17. *Let X be an elastic diffeological space. Then the differential calculus consisting of differential forms, the de Rham differential, vector fields, the inner derivative, the Lie bracket of vector fields, and the Lie derivative defined by Cartan's magic formula satisfies the relations of a Cartan calculus.*

3. Towards a Lie theory for diffeological groupoids

Definition 3.1. A **diffeological groupoid** is a groupoid internal to the category of diffeological spaces.

To my best knowledge, there is no Lie theory for general diffeological groupoids, nor can there be. The category of diffeological groupoids is too general as to allow for a good infinitesimal object, its yet to be defined diffeological Lie algebroid, that would retain useful structural information. This could be fixed if we require our diffeological groupoids to be diffeological manifolds, i.e. to be locally modeled on diffeological vector spaces, but this would exclude many interesting examples and give away too much of the convenience of the category of diffeological spaces. (An example is the action groupoid of the action of a symmetry bisection group on the space of solutions of Euler-Lagrange equation of a field theory, which usually has singularities.) It seems to me that what is really needed is a good differential calculus, as provided by the tangent structure of elastic diffeological spaces.

3.1. **The infinitesimal object of an elastic diffeological groupoid.**

Definition 3.2. A diffeological groupoid is called **elastic** if

$$G_k \cong \underbrace{G_1 \times^{r,l}_{G_0} G_1 \times^{r,l}_{G_0} \ldots \times^{r,l}_{G_0} G_1}_{k \text{ factors}}$$

is elastic for all $k \geq 0$.

How strong is this condition? Let us first consider the group case:

Theorem 3.3. *Every diffeological group is elastic.*

I find this result quite surprising. It explains in hindsight why we never ran into technical problems computing the Lie algebras of bisection groups such as $\mathrm{Diff}(M)$ using only the diffeological structure. As a corollary we obtain the following proposition:

Proposition 3.4. *The action groupoid of the smooth action of a diffeological group on an elastic diffeological space is elastic.*

Action groupoids over elastic spaces are the starting point for the construction of many groupoids that are relevant in classical field theory. The tangent structure of elastic diffeological spaces is sufficient to construct a well-behaved infinitesimal object of any higher elastic diffeological groupoid given by a Kan simplicial elastic diffeological space $G : \mathbf{\Delta}^{\mathrm{op}} \to \mathcal{D}\mathrm{flg}$.

If we want to generalize the usual construction of Lie algebroids in terms of invariant vector fields, we have to strengthen the conditions on our groupoids somewhat. The following definition is tentative:

Definition 3.5. An diffeological groupoid is called **strongly elastic** if it is elastic and if

$$TG_k \cong TG_1 \times_{TG_0} TG_1 \times_{TG_0} \ldots \times_{TG_0} TG_1$$

for all $k \geq 0$.

Definition 3.6. A diffeological $\mathbb{R}$**-module bundle** over M is an $\mathbb{R}$-module object $A \to M$ in $\mathcal{D}\mathrm{flg} \downarrow M$.

An $\mathbb{R}$-module bundle consists of morphisms $\pi : A \to M, 0 : M \to A, + : A \times_M A \to A$, $- : A \times_M A \to A$, and $\cdot : \mathbb{R} \times A \to A$, all covering id_M, that satisfy the usual relations expressed by commutative diagrams. The diffeological space of sections $\mathcal{A} := \Gamma(M, A)$ is an abelian group with pointwise group structure and a $\mathcal{D}\mathrm{flg}(M, \mathbb{R})$-module given by $(fa)(m) = f(m) \cdot a(m)$.

Definition 3.7. A diffeological Lie algebroid consists of an $\mathbb{R}$-module bundle $A \to M$ such that A is elastic and a smooth bilinear Lie bracket on the space of sections $\mathcal{A} = \Gamma(M, A)$, and a morphism $\rho : A \to TM$ of $\mathbb{R}$-module bundles such that

$$[a, fb] = (\iota_{\rho(a)} df) + f[a, b] \,.$$

The definition looks almost identical to that of a Lie algebroid, so let us point out the differences. The bundle $A \to M$ has no local trivializations and the $\mathcal{D}\mathrm{flg}(M, \mathbb{R})$-module $\mathcal{A}$ is generally not locally free. Therefore, it seems to me that it cannot be deduced from the Leibniz rule that ρ is a homomorphism of Lie algebras. I am not sure at this point whether this property should be added to the definition, as in the old definitions of a Lie algebroid.

Proposition 3.8. *The right invariant vector fields of a strongly elastic diffeological groupoid form a diffeological Lie algebroid in the sense of Def. 3.7.*

3.2. **Application: Diffeological groupoids in general relativity.** The following application of diffeological groupoids arose in the study of the initial value problem of general relativity [BFW13]. The basic symmetry of (vacuum) general relativity, i.e. the theory of Ricci flat lorentzian metrics on a given background manifold M is $\mathrm{Diff}(M)$. This is a symmetry in the sense of lagrangian field theory, i.e. $\mathrm{Diff}(M)$ leaves the Hilbert-Einstein action invariant and, therefore, maps solutions of the field equation to solutions. As is the case for any classical field theory, the initial value formulation of general relativity requires the choice of an initial time slice $\Sigma \hookrightarrow M$, an embedded codimension 1 submanifold. The issue is, that such a submanifold is not invariant under $\mathrm{Diff}(M)$. In physics terminology, it breaks the symmetry.

This problem is the source of fundamental problems of the theory and leads to a number of undesirable phenomena. For example, the constraint functions of the initial value problem, which arise from Noether's theorem and can be viewed as functions on $T^*\mathrm{Riem}(M)$ have Poisson brackets that do not close. This shows that the constraints cannot be interpreted as momenta or charges of a hamiltonian action of $\mathrm{Diff}(M)$ or any other group. The whole problem "smells" like groupoids and Lie algebroids, at least for someone used to working with them.

There are several groupoids that we can associate to the geometric situation at hand. The first groupoid has as objects all lorentzian manifolds that admit an embedding of Σ as spacelike codimension 1 submanifold. (Let us call these Σ-adapted.) The arrows are the isometries between Σ-adapted lorentzian manifolds. Note that this groupoid is not small. This can be interpreted as the groupoid presenting the moduli stack of Σ-adapted spacetimes. We will denote it by $G_{\mathrm{stack}}(\Sigma)$.

The second groupoid encodes the choice of initial time slice and is constructed as follows. Let $(M, g, i : \Sigma \hookrightarrow M)$ be a lorentzian manifold with a (spacelike cooriented) embedding of Σ. Two such triples (M_1, g_1, i_1) and (M_2, g_2, i_2) are isomorphic if there is a diffeomorphism $\varphi : M_1 \to M_2$ such that $\varphi_* g_1 = g_2$ and $\varphi \circ i_1 = i_2$. An object of the groupoid is an isomorphism class of such triples. The arrows consist of isomorphism classes of (M, g, i, j), where (i, j) is a pair of embeddings. The groupoid structure is induced by the pair groupoid structure of the pairs of embeddings. We have called this the groupoid of evolutions and denoted by $G_{\mathrm{evol}}(\Sigma)$.

The third groupoid arises in the following way. Let $\mathrm{Diff}(M) \ltimes \mathcal{L}\mathrm{or}(M)$ be the action groupoid of the action of diffeomorphisms on lorentzian manifolds by pushforward. Let $\Sigma \subset M$ be a submanifold. Let us denote by $\mathrm{Diff}(M)^\Sigma$ the subgroup of diffeomorphisms that fix all points of Σ. Note that the subgroup is not normal. The third groupoid is the reduction

$$G_{\mathrm{red}}(\Sigma, M) = \mathrm{Diff}(M)^\Sigma \backslash \mathrm{Diff}(M) \times_{\mathrm{Diff}(M)^\Sigma} \mathcal{L}\mathrm{or}(M)/\mathrm{Diff}(M)^\Sigma$$

of the action groupoid by $\mathrm{Diff}(M)^\Sigma$ [BW].

All three groupoids are equipped with a diffeological structure that comes from the functional diffeology on the space of metrics and the spaces of Σ-embeddings.

Proposition 3.9. *Let $G_{\mathrm{stack}}(\Sigma)$, $G_{\mathrm{evol}}(\Sigma)$, and $G_{\mathrm{red}}(\Sigma, M)$ be the diffeological groupoids we have just defined.*

- *$G_{\mathrm{stack}}(\Sigma)$ and $G_{\mathrm{evol}}(\Sigma)$ are Morita equivalent.*
- *$G_{\mathrm{red}}(\Sigma, M)$ is a diffeologically connected component and subgroupoid of $G_{\mathrm{evol}}(\Sigma)$.*
- *The objects of $G_{\mathrm{red}}(\Sigma, M)$ and $G_{\mathrm{evol}}(\Sigma)$ are represented by unique gaussian metrics on a neighborhood of $\Sigma \times \{0\}$ in $\Sigma \times \mathbb{R}$.*
- *The Lie algebroid of $G_{\mathrm{evol}}(\Sigma)$ is a trivial vector bundle with fibre $\mathfrak{X}(\Sigma) \times C^\infty(\Sigma)$. The bracket of constant sections is the Poisson bracket of the constraint functions.*

  $$[(X, \varphi), (Y, \psi)] = \big([X, Y], X \cdot \psi - Y \cdot \varphi + \varphi \,\mathrm{grad}_h \psi - \psi \,\mathrm{grad}_h \varphi\big),$$

  where $g = -\frac{1}{2}dt^2 + h_t$ for $h_t \in \mathcal{R}\mathrm{iem}(\Sigma)$ is a gaussian metric.

Remark 3.10. The construction of G_{red} generalizes to the action of any bisection groupoid on a space of sections $\mathcal{F} = \Gamma(M, F)$.

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