Modular embeddings of Teichmüller curves

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Abstract

Fuchsian groups with a modular embedding have the richest arithmetic properties among non-arithmetic Fuchsian groups. But they are very rare, all known examples being related either to triangle groups or to Teichmüller curves. In Part I of this paper we study the arithmetic properties of the modular embedding and develop from scratch a theory of twisted modular forms for Fuchsian groups with a modular embedding, proving dimension formulas, coefficient growth estimates and differential equations. In Part II we provide a modular proof for an Apéry-like integrality statement for solutions of Picard–Fuchs equations. We illustrate the theory on a worked example, giving explicit Fourier expansions of twisted modular forms and the equation of a Teichmüller curve in a Hilbert modular surface. In Part III we show that genus two Teichmüller curves are cut out in Hilbert modular surfaces by a product of theta derivatives. We rederive most of the known properties of those Teichmüller curves from this viewpoint, without using the theory of flat surfaces. As a consequence we give the modular embeddings for all genus two Teichmüller curves and prove that the Fourier developments of their twisted modular forms are algebraic up to one transcendental scaling constant. Moreover, we prove that Bainbridge’s compactification of Hilbert modular surfaces is toroidal. The strategy to compactify can be expressed using continued fractions and resembles Hirzebruch’s in form, but every detail is different.

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Introduction

Modular forms are certainly best understood for the full modular group $SL(2,\mathbb{Z})$, closely followed by those for its congruence subgroups and other arithmetic groups. Among the non-arithmetic Fuchsian groups, the groups having the best arithmetic properties are those admitting a modular
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embedding. Here, modular embedding refers to the existence of a map \( \varphi : \mathbb{H} \rightarrow \mathbb{H} \) intertwining the action of a Fuchsian group \( \Gamma \) and its Galois conjugate. The notion of modular embedding (in this sense) appears for the first time in the work of Cohen and Wolfart [CW90]. They show that triangle groups admit modular embeddings, and for more than a decade these remained the only examples. An infinite collection of new examples were found with the discovery of new Teichmüller curves by Calta [Cal04] and McMullen [McM03]. To find the modular embeddings for them is one of the motivations for this paper.

For a reader whose main focus is modular forms this paper wants to advertise an interesting new class of modular forms. For example, we explain an integrality phenomenon for the coefficients of a solution of a Picard–Fuchs differential equation, like Beukers’s modular proof of the corresponding phenomenon for Apéry’s famous differential equations, except that this time the explanation requires using a pair of ‘\( q \)-coordinates’. For a reader with a main focus on Teichmüller curves, we show how to rediscover many of their properties without referring to the theory of flat surfaces.

The paper has three parts, linked by the aim to describe modular embeddings. In Part I we set up a general theory of modular forms for Fuchsian groups admitting a modular embedding. We call these twisted modular forms and we prove the twisted analogues of the properties that appear in most textbooks about modular forms in the untwisted case. In Part II we start from an example of a Fuchsian group with modular embedding where the Picard–Fuchs differential equations can be explicitly computed. We invite the reader to discover the properties of Fourier coefficients of the modular embeddings and of twisted modular forms via this worked example. In Part III we give explicitly the Fourier expansions of the modular embedding for the genus two Teichmüller curves found by Calta [Cal04] and McMullen [McM03]. In the rest of the introduction we describe the results in more detail, highlighting the main theorems (not necessarily in the same order) with bullet points.

**Modular embeddings and twisted modular forms.** Suppose that the Fuchsian group \( \Gamma \subset \text{SL}(2, \mathbb{R}) \) has coefficients in a quadratic number field \( K \) with Galois group generated by \( \sigma \). The essential ingredient of a modular embedding for \( \Gamma \) is a holomorphic function \( \varphi : \mathbb{H} \rightarrow \mathbb{H} \) with the \( \Gamma \)-equivariance property

\[
\varphi \left( \frac{az+b}{cz+d} \right) = a^\sigma \varphi(z) + b^\sigma c^\sigma \varphi(z) + d^\sigma.
\]

We show that such a modular embedding always has a ‘Fourier expansion’, and an old theorem of Carathéodory gives us a good estimate for its Fourier coefficients. Analyzing \( \varphi \) leads us to the definition of a new kind of modular form that we call a \( \varphi \)-twisted modular form of bi-weight \((k, \ell)\). This is a holomorphic function \( f : \mathbb{H} \rightarrow \mathbb{C} \) with the transformation property

\[
f \left( \frac{az+b}{cz+d} \right) = (cz+d)^k (c^\sigma \varphi(z) + d^\sigma)^\ell f(z).
\]

For example, direct calculation shows that \( \varphi'(z) \) is a twisted modular form of bi-weight \((2, -2)\).

We develop a theory of twisted modular forms from scratch, analyzing to which extent classical topics of modular forms generalize to this new notion. Our first topic is the coefficient growth.

- For \( \ell > 0 \) and \( k + \ell > 2 \) the Fourier coefficients of a twisted modular form \( f(z) = \sum_{n \geq 0} a_n q^n \) of \((k, \ell)\) satisfy the estimate \( a_n = O(n^{k+\ell-1}) \).

Similar estimates are given for other bi-weights as well; see Theorem 2.1 for the complete statement. The proofs combine the well-known Hecke argument in the untwisted case and the mechanism underlying the equidistribution of long horocycles.
The next classical topic is the dimension of the space of modular forms. A modular embedding comes with one basic invariant \( \lambda_2 \), that one can view is several ways: as an integral over a conformal density \((10)\), as a ratio of the degrees of the natural line bundles whose sections are twisted modular forms, or as a Lyapunov exponent for the Teichmüller geodesic flow in the case of Teichmüller curves.

- For \( k + \ell \) even and \( k + \lambda_2 \ell > 2 \) the dimension of the space of twisted modular forms of bi-weight \((k, \ell)\) is the sum of \((k + \lambda_2 \ell - 1)(g - 1)\) and contributions from the cusps and elliptic fixed points. Here \( g \) denotes the genus of \( \mathbb{H}/\Gamma \).

For a torsion-free Fuchsian group this is of course a classical Riemann–Roch calculation. Hence the main point is to determine the elliptic fixed point contributions in the twisted case. See \((17)\) for the definition of the characteristic quantities of elliptic fixed points and Theorem 3.2 for the complete statement.

Finally, the statement that modular forms expressed in terms of a modular function satisfy differential equations also carries over to the twisted case.

- If \( f(z) \) is a twisted modular form of bi-weight \((k, \ell) \geq (0, 0)\) and \( t \) a modular function, then the function \( y(t) \) defined locally by \( y(t(z)) = f(z) \) satisfies a linear differential equation of order \((k + 1)(\ell + 1)\) with algebraic coefficients (Theorem 4.1).

**Modular embeddings via differential equations.** The starting point of the whole project was a worked example, the Teichmüller curves for \( D = 17 \), that we present in §§6–8. (The definition of Teichmüller curves along with a summary of the classification results for Teichmüller curves in genus two is given in §§5.3 and 5.4.) Starting from the flat geometry definition we briefly explain the derivation of the equation of the Teichmüller curve as family of hyperelliptic curves and computation of the Picard–Fuchs differential equations, following [BM10a].

In this way, we present in §7.1 the Fourier expansion of twisted modular forms explicitly. The corresponding group \( \Gamma \) is in this case neither arithmetic nor commensurable to a triangle group, and the differential equations expressing the modular forms in terms of a hauptmodul for \( \Gamma \) (i.e. a suitably scaled modular function \( t : \mathbb{H}/\Gamma \to \mathbb{P}^1 \) of degree one) is not hypergeometric.

The twisted modular forms have two curious properties.

- The twisted modular forms do not have a power series expansion in \( K[[q]] \) for the standard modular parameter \( q = e^{2\pi iz}/\alpha \), where \( \alpha \) is the width of the cusp, but lie instead in \( K[[Aq]] \), where \( A \) is a transcendental number of Gelfond–Schneider type (i.e. a number of the form \( \beta_1^\beta_2 \) with all \( \beta_i \) algebraic).

- If \( f \) is a twisted modular form and \( t \) a suitably scaled modular function, then the function \( y(t) \) locally defined by \( y(t(z)) = f(z) \), with \( t(z) \) as above, has \( O_K \)-integral Taylor coefficients.

The second of these observations was already proved in [BM10a] using \( p \)-adic differential equations. In §8 we will give a ‘modular’ proof of both statements. The surprising feature here is that, while the classical proof by Beukers of the integrality of the Apéry coefficients using modularity relies on the integrality of the Fourier coefficients of the \( q \)-expansions of modular forms on arithmetic groups, here the expansions of both \( f(z) \) and \( t(z) \) with respect to \( Aq \) have coefficients with infinitely many prime factors in their denominators, and yet the integrality of \( f \) with respect to \( t \) still holds. To give a modular argument for this integrality, we have to use the relationship between twisted modular forms on the Teichmüller curve and Hilbert modular forms on the ambient surface.

For \( D = 17 \) the full ring of symmetric Hilbert modular forms has been determined by Hermann [Her81]. We recall and use his construction to write down explicitly the equation...
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of the Teichmüller curves for $D = 17$ on the (rational) symmetric Hilbert modular surface in Theorem 8.3.

- There exist coordinates $U, V$ on the Hilbert modular surface $X_{17}$, explicitly given in terms of theta functions, such that the two Teichmüller curves on $X_{17}$ are cut out by the quadratic equations (70) and (71).

Modular embeddings via derivatives of theta functions. The concrete example $D = 17$ led us to the discovery of a general construction of the modular form cutting out Teichmüller curves.

- The vanishing locus of the Hilbert modular form $D \theta$ of weight $(3,9)$, given as a product of derivatives of odd theta functions, is precisely the union of one or two Teichmüller curves on the Hilbert modular surface $X_D$ (Theorem 9.1).

Starting from the theta function viewpoint we prove the characterizing properties of genus two Teichmüller curves without relying either on the geodesic definition or on any flat surface properties. Maybe these ideas can be used to construct new Teichmüller curves. Given the length of the paper, we simplify our task and prove the following results only for fundamental discriminants $D$. With appropriate care, the proofs can certainly be adapted to the general case.

- The vanishing locus of $D \theta$ is transversal to one of the two natural foliations of the Hilbert modular surface $X_D$ (Theorem 12.1).

- The vanishing locus of $D \theta$ is disjoint from the reducible locus (Theorem 12.2).

On the compactified Hilbert modular surface, the reducible locus and the vanishing locus of a Hilbert modular form always intersect and the number of intersection points is proportional to the volume of the Hilbert modular surface. So the claim is that all of these intersection points lie on the boundary of the Hilbert modular surface, hence at cusps of the vanishing locus of $D \theta$. While for the second statement we also have a proof using theta functions, we give proofs of both these statements relying on the following description of the cusps.

- The cusps of the vanishing locus of $D \theta$ are in bijection with pairs consisting of a standard quadratic form $ax^2 + bxy + cy^2$ of discriminant $D$ and a class $r \in \mathbb{Z}/\gcd(a,c)\mathbb{Z}$ (Theorem 10.1).

Here an indefinite quadratic form $ax^2 + bxy + cy^2$ is called standard if $a > 0 > c$ and $a + b + c < 0$. As a statement about cusps of Teichmüller curves, this result already appears in [McM06a] and [Bai07]. Our proof, however, is completely different. We now explain the main idea.

Suppose that a power series $f = \sum c_\nu q_1^\nu q_2^{2\nu}$ has to vanish on a branch of a curve parametrized by $q_1 = q^{a_1} e^{s_1(q)}$ and $q_2 = q^{a_2} e^{s_2(q)}$. After these parametrizations are plugged into $f$, the lowest order exponent (in $q$) has to appear twice, since otherwise the terms cannot cancel. In the concrete case of $f = D \theta$ we are led to the following notion. Given a fractional $O_D$-ideal $a$, we say that a primitive element $\alpha \in (a^2)^\vee$ is a multiminimizer for $a$ if the quadratic form $x \mapsto \text{tr}(\alpha x^2)$ on $a$ takes its minimum on one of the three non-trivial cosets $\frac{1}{2}a/a$ at least (and then, as we show, exactly) twice (with $x$ and $-x$ not distinguished). We show on the one hand that multiminimizers for $a$ are in bijection with standard quadratic forms in the wide ideal class. (See e.g. [Zag81] for the correspondence between ideal classes and quadratic forms.) On the other hand, on any branch of the vanishing locus the local parameter can be chosen such that $\alpha = \alpha_1 = \sigma(\alpha_2)$ is a multiminimizer and that the multiminimizers (up to multiplication by the square of a unit) determine the branch uniquely up to an element of $\mathbb{Z}/\gcd(a,c)\mathbb{Z}$.

We have given the definition of multiminimizers and the description of branches of the vanishing locus in detail since this notion and construction reappears twice in the rest of the paper. First, multiminimizers appear prominently in the discussion of Bainbridge’s compactification below and, second, this description of the branches immediately gives the Fourier expansion of the modular embedding of the uniformizing group of the genus two...
Teichmüller curves for any $D$ (see Theorem 13.1). Moreover, both ‘curious properties’ mentioned in the case $D = 17$ hold in general. In particular, we have the following result.

- For any $D$, any cusp of the vanishing locus of $D\theta$ with corresponding Fuchsian group $\Gamma$ and modular embedding $\varphi$, the $\varphi$-twisted modular forms of bi-weight $(k, \ell)$ have a basis with Fourier expansions of the form $\sum_{n \geq 0} a_n (Aq)^n$ with $a_n$ algebraic and $A$ transcendental of Gelfond–Schneider type (Theorem 13.2).

As another application of the description of Teichmüller curves via theta derivatives, we give in Theorem 13.3 a description of the quadratic differentials on the leaves of the natural foliation of a Hilbert modular surface whose integral measures the flat distance between the two zeros of the eigenform for real multiplication. These quadratic differentials can be packaged together to a meromorphic modular form of weight $(-2, 4)$ that we give as the quotient of theta series and their derivatives. Our result has been used by McMullen [McM12] to describe the beautiful and complicated flat structure on the leaves visually (‘snow falling on cedars’).

**Hirzebruch’s compactification and Bainbridge’s compactification.** Hirzebruch constructed a minimal smooth compactification of Hilbert modular surfaces. His compactification is remarkable in many ways. First, it is the prototype of what is nowadays a called a toroidal compactification, i.e. it is given by a fan of decreasing slopes, periodic under the action of the squares of units. The fan is given for each cusp $\alpha$ of the Hilbert modular surface by the corners of the lower convex hull of $(\alpha^2)^\vee$ in $\mathbb{R}_+^2$.

Bainbridge [Bai07] observed that the closure in the Deligne–Mumford compactification of the Torelli preimage of Hilbert modular surfaces provides another compactification. This compactification is in general neither smooth nor minimal, but it was useful in his calculation of Euler characteristics of Teichmüller curves. It is amusing to compare the two types of compactifications and to discover that they are parallel in spirit, but different in every concrete detail.

- Bainbridge’s compactification is a toroidal compactification, given for each cusp $\alpha$ by the fan of multiminimizers (lying in $(\alpha^2)^\vee$) for $\alpha$ (Theorem 11.5).

The second remarkable property of Hirzebruch’s compactification is that it can easily be computed using a continued fraction algorithm.

- Hirzebruch’s compactification is driven by the ‘fast minus’ continued fraction algorithm, while Bainbridge’s compactification is driven by a ‘slow plus’ continued fraction algorithm.

The reader will find the precise description of the algorithms in §11.2 and §10.4, respectively. The bijection between standard and reduced indefinite quadratic forms induces a subtle relationship between the number of boundary components of Hirzebruch’s and Bainbridge’s compactification. In particular, the number of curves in the Bainbridge compactification of any cusp is always the same as the number for the Hirzebruch compactification of some cusp, but not necessarily the same one. The definitions and details, and several examples, are given in §11.

**Part I. Modular embeddings and twisted modular forms**

The notion of modular embedding in the sense used here appears for the first time in a paper by Cohen and Wolfart [CW90]. They study holomorphic maps $\mathbb{H} \to \mathbb{H}$ equivariant with respect to a Fuchsian group and its Galois conjugate.

In particular, Cohen and Wolfart show that all triangle groups admit modular embeddings. Subsequent work of Schmutz-Schaller and Wolfart [SW00] gave some necessary conditions for a group to admit a modular embedding. Some Fuchsian quadrangle groups were shown in [Ric02]...
not to admit modular embeddings, but it took more than a decade until new examples of modular embeddings were discovered.

The first new examples arose from the Teichmüller curves discovered by Calta and McMullen (see [Cal04] and [McM03], and [Mö10a] for the modular viewpoint). All Teichmüller curves give rise to modular embeddings. We summarize the known results of Teichmüller curves (and thus the known groups admitting a modular embedding) briefly at the end of §1.

Here, in Part I, we think of the group Fuchsian group \( \Gamma \) as given (e.g. in terms of a presentation) and study the properties of the modular embeddings as holomorphic maps. We define an extension of the notion of modular forms to this context that we call twisted modular forms. The aim of the first part is to study this new object and to derive the analogues of the standard results on modular forms (Fourier coefficients, dimension, differential equations) from scratch in the context of twisted modular forms.

1. Hilbert modular embeddings

The term **modular embedding** is used in the literature both for equivariant maps from \( H \rightarrow H^g \) (starting with [CW90]) and from \( H^g \rightarrow H \) (already in [Ham66]). To distinguish between them, we call them ‘Hilbert modular embeddings’ and ‘Siegel modular embeddings’, respectively, according to the range of the corresponding map. We will be interested mostly in the quadratic case \( g = 2 \) and refer to [SW00] for basic notions in the general case.

Throughout this paper we denote by \( K \) a real quadratic field, with a fixed embedding \( K \subseteq \mathbb{R} \).

We use the letter \( \sigma \) to denote the Galois conjugation of \( K \) or the second embedding of \( K \) into \( \mathbb{R} \), writing \( \sigma(x) \) or \( x^\sigma \) interchangeably for \( x \in K \).

By a **Hilbert modular group** for \( K \) we will mean any subgroup \( \Gamma_K \) of \( \text{SL}(2, K) \) commensurable with \( \text{SL}(2, O) \) for some order \( O \subseteq K \). (Later we will make specific choices.) Such a group acts discretely and cofinitely on \( H^2 \) by \((z_1, z_2) \mapsto ((az_1 + b)/(cz_1 + d), (a^\sigma z_2 + b^\sigma)/(c^\sigma z_2 + d^\sigma))\). Here \( H \) denotes the upper half-plane.

We will be interested only in **Hilbert modular embeddings** for which the first coordinate in \( H^2 \) is a local coordinate everywhere. A modular embedding of this type is described by the data \((\Gamma, \varphi)\), where:

- \( \Gamma \) is a subgroup of some Hilbert modular group \( \Gamma_K \subseteq \text{SL}(2, K) \) which, viewed as a subgroup of \( \text{SL}(2, \mathbb{R}) \), is Fuchsian (i.e. discrete and cofinite);
- \( \varphi : H \rightarrow H \) is a holomorphic map satisfying \( \varphi \circ \gamma = \gamma^\sigma \circ \varphi \) for all \( \gamma \in \Gamma \).

For such a pair \((\Gamma, \varphi)\), the map \( z \mapsto (z, \varphi(z)) \) defines a map from the curve \( H/\Gamma \) to the Hilbert modular surface \( H^2/\Gamma_K \).

Written out explicitly, the condition on \( \varphi \) means that we have

\[
\varphi \left( \begin{array}{c} az + b \\ cz + d \end{array} \right) = \frac{a^\sigma \varphi(z) + b^\sigma}{c^\sigma \varphi(z) + d^\sigma}
\]

for all \((a, b, c, d) \in \Gamma \) and \( z \in H \). We will use this transformation property in §2 to define a 1-cocycle on \( \Gamma \) and hence a new type of modular form (‘twisted’ form) on \( \Gamma \). Just as for usual modular forms, these must satisfy suitable growth conditions at the cusps of \( \Gamma \), and to formulate these we need to know how \( \varphi \) behaves near the cusps. Assume first that one of these cusps is at \( \infty \), with the stabilizer of \( \infty \) in the image \( \bar{\Gamma} \) of \( \Gamma \) in \( \text{PSL}(2, \mathbb{R}) \) being generated by \( \pm \left( \begin{array}{cc} 1 & \alpha \\ 0 & 1 \end{array} \right) \) with \( \alpha \in K \cap \mathbb{R}_+ \).

Then we have the following proposition.
Proposition 1.1. Suppose that the stabilizer of $\infty$ in $\Gamma$ is generated by $z \mapsto z + \alpha$ with $\alpha \in K$, $\alpha > 0$. Then $\alpha^\sigma$ is also positive and $\varphi(z)$ has an expansion of the form

$$\varphi(z) = \frac{\alpha^\sigma}{\alpha} z + \sum_{n=0}^{\infty} B_n e\left(\frac{n z}{\alpha}\right) \quad (\forall z \in \mathbb{H}),$$

where $e(x) := e^{2\pi i x}$ and where the coefficients $B_n$ satisfy the inequalities

$$|B_n| \leq 2 \Im(B_0) \quad \text{for all } n \geq 1. \quad (3)$$

Proof. From (1) we have $\varphi(z + \alpha) = \varphi(z) + \alpha^\sigma$, so the function $\varphi(z) - \alpha^\sigma z/\alpha$ is invariant under $z \mapsto z + \alpha$ and hence equals $f(e(z/\alpha))$ for some holomorphic function $f(q)$ in the punctured disk $\mathbb{D}^* = \{q : 0 < |q| < 1\}$. Define a second holomorphic function $F$ in $\mathbb{D}^*$ by $F(q) = e(f(q)/|\alpha^\sigma|)$. From $\varphi(\mathbb{H}) \subseteq \mathbb{H}$ we deduce that $|q^{\pm 1}F(q)| < 1$ in $\mathbb{D}^*$, where the sign is chosen so that $\pm \alpha^\sigma > 0$. It follows that $F(q)$ extends to a meromorphic function in $\mathbb{D}$ with at most a simple pole at $q = 0$. But then the fact that $F$ has a single-valued logarithm in $\mathbb{D}^*$ implies that its order of vanishing at 0 must be zero, so $\alpha^\sigma$ must be positive and $f$ extends holomorphically to the full disk and hence has a convergent Taylor expansion $\sum_{n=0}^{\infty} B_n q^n$, proving the first claim. For the second, we note first that the estimate $|F(q)| \leq |1/q|$ and the holomorphy of $F$ at 0 imply by the maximum principle that $|F(q)| \leq 1$ in the disk $\mathbb{D}$ (this is just the Schwarz lemma, applied to the function $qF(q)$), so $B_0$ has positive imaginary part and the function $f(q)/B_0$ takes on values in the right half-plane. An elementary argument then gives the estimate $|B_n| \leq 2n\Im(B_0)$. (Write $f(q) = (B_0 - B_0 q/\lambda(q))/(1 - \lambda(q))$ where $\lambda$ sends $\mathbb{D}$ to $\mathbb{D}$ and 0 to 0; then $B_n = 2n\Im(B_0) \sum_{m=1}^{\infty} |\lambda^m|_n$, where $|\lambda^m|_n$ denotes the coefficient of $q^n$ in $\lambda(q)^m$, which is bounded in absolute value by 1 because $\lambda$ is.) The stronger estimate $|B_n| \leq 2\Im(B_0)$ follows from a theorem of Carathéodory [Car07], which says precisely that a holomorphic function mapping $\mathbb{D}$ to the right half-plane and sending 0 to 1 has all its Taylor coefficients at 0 bounded by 2 in absolute value. $\square$

Corollary 1.2. The imaginary part of $\varphi(z)$ satisfies the inequalities

$$\frac{\alpha^\sigma}{\alpha} y \leq \Im(\varphi(z)) \leq \frac{\alpha^\sigma}{\alpha} y + C \left(1 + \frac{1}{y}\right)$$

for all $z = x + iy \in \mathbb{H}$, where $C$ is a constant independent of $z$.

Proof. The first statement is just the inequality $|F(q)| \leq 1$ established in the course of the above proof, and because $\sum_{n=0}^{\infty} |q|^n \ll 1 + 1/y$, the second statement follows from (3). $\square$

Exactly similar statements hold for all of the other cusps of $\Gamma$. Recall that, by definition, such a cusp is an element $\nu \in \mathbb{P}^1(K)$ whose stabilizer $\Gamma_\nu$ in $\Gamma$ is infinite cyclic, say $\Gamma_\nu = \langle \pm \gamma_\nu \rangle$. Choose $g \in \text{SL}(2, K)$ with $g(\infty) = \nu$. Then $g^{-1}\gamma_\nu g = \pm \left(\begin{smallmatrix} \frac{1}{\alpha} & 0 \\ 0 & \alpha \end{smallmatrix}\right)$ for some positive element $\alpha$ of $K$. Equation (1) implies that the function $\varphi_g = g^{\sigma - 1} \circ \varphi \circ g$ satisfies $\varphi_g(z + \alpha) = \varphi_g(z) + \alpha^\sigma$, because $\varphi_g \circ \left(\begin{smallmatrix} 1 \\ \alpha \end{smallmatrix}\right) = \varphi_g g^{-1}\gamma_\nu g = g^{\sigma - 1}\varphi_g \gamma_\nu g = g^{\sigma - 1}\gamma_\nu^\sigma \varphi_g = g^{\sigma - 1}\gamma_\nu^\sigma \varphi g = (g^{-1}\gamma_\nu g)^\sigma \varphi_g = \left(\begin{smallmatrix} 1 \\ \alpha^\sigma \end{smallmatrix}\right)^\sigma \varphi_g$. Then the same proof as for the case $g = \text{Id}$ shows that $\alpha^\sigma$ is positive and that

$$\varphi_g(z) = \frac{\alpha^\sigma}{\alpha} z + \sum_{n=0}^{\infty} B_n e\left(\frac{n z}{\alpha}\right)$$

for some constants $B_n \in \mathbb{C}$, satisfying the same estimate (3) as before, and for all $z \in \mathbb{H}$. Of course $\alpha$ and $B_n$ depend on $\nu$, and also on the choice of $g$, but the expansions for $\Gamma$-equivalent

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cusps are the same up to trivial rescalings, because \( \varphi_{\gamma g} = \varphi_g \) for \( \gamma \in \Gamma \), so that there are only finitely many essentially distinct expansions.

Another basic property of \( \varphi \), obtained by applying the Schwarz lemma to the map \( I_\varphi(z) \circ \varphi \circ I_{\varphi}^{-1} : \mathbb{D} \to \mathbb{D} \), where \( I_a \) for \( a \in \mathbb{H} \) denotes the standard isomorphism \( (\mathbb{H}, a) \to (\mathbb{D}, 0) \) given by \( z \mapsto (z - a)/(1 - \bar{a}z) \), is that one has the inequalities

\[
\left| \frac{\varphi(z) - \varphi(z')}{|\varphi(z) - \varphi(z')} \right| \leq \left| \frac{z - z'}{|z - z'|} \right|, \quad \left| \frac{\varphi(z) - \varphi(z')}{3(\varphi(z))} \right| \leq \left| \frac{z - z'}{3(z)} \right|
\]

(6)
or equivalently \( d(\varphi(z), \varphi(z')) \leq d(z, z') \) for any \( z, z' \in \mathbb{H} \), where \( d : \mathbb{H} \times \mathbb{H} \to \mathbb{R}_{\geq 0} \) denotes the Poincaré metric. (This is of course a standard property of any holomorphic map from the upper half-plane to itself.) Fixing \( z \) and letting \( \Im(z') \) tend to infinity, or fixing \( z' \) and letting \( \Im(z) \) tend to zero, we obtain second proofs of the two inequalities in (4), while letting \( z' \) tend to \( z \) we obtain the estimate

\[
|\varphi'(z)| \leq \frac{\Im(\varphi(z))}{\Im(z)} \quad (z \in \mathbb{H})
\]

(7)
or equivalently \( |\kappa(z)| \leq 1 \), where \( \kappa : \mathbb{H} \to \mathbb{C} \) is the map defined by

\[
\kappa(z) = \frac{\Im(z)}{\Im(\varphi(z))} \varphi'(z).
\]

(8)

From the equivariance property (1) we obtain the formulas

\[
\Im(\varphi(\gamma z)) = \frac{\Im(\varphi(z))}{|cz + d|^2}, \quad \varphi'(az + b) = \frac{(cz + d)^2}{(cz + d)^2 + |c|^2} \varphi'(z),
\]

(9)

and these together with the standard formula \( \Im(\gamma z) = \Im(z)/|cz + d|^2 \) imply that the function \( \kappa \) is \( \Gamma \)-invariant. We can therefore introduce a basic invariant \( \lambda_2 = \lambda_2(\Gamma, \varphi) \) of the pair \((\Gamma, \varphi)\) by

\[
\lambda_2 = \frac{1}{\text{vol}(\Gamma \backslash \mathbb{H})} \int_{\Gamma \backslash \mathbb{H}} |\varphi'(z)|^2 d\mu = \frac{\int_{\mathbb{H}} |\kappa|^2 d\mu}{\int_{\mathbb{H}} d\mu} \in (0, 1],
\]

(10)

where \( d\mu = y^{-2} dx dy \) (with \( z = x + iy \) as usual) is the standard \( \text{SL}(2, \mathbb{R}) \)-invariant measure on \( \mathbb{H} \), and where the integral can be taken over any fundamental domain for the action of \( \Gamma \) on \( \mathbb{H} \) and is absolutely convergent because \( |\kappa| \leq 1 \). The invariant \( \lambda_2 \), whose values are always rational numbers, can be interpreted either as a ratio of the intersection numbers on the Hilbert modular surface with the line bundles \( \mathcal{L} \) and \( \tilde{\mathcal{L}} \) defined in §3, or as the second Lyapunov exponent [Möll14], which explains the notation. It is a commensurability invariant of the Fuchsian group \( \Gamma \).

We close this section by describing briefly the four known types of Hilbert modular embeddings. The first two will play a role in this paper and will be discussed in more detail in §5. The other two are mentioned only for the sake of completeness.

**Type 1. Modular curves.** Here \( \Gamma = \Gamma_A = \{ \gamma \in \Gamma_K \mid A\gamma = \gamma A \} \) where \( A \) is a suitable element of \( \text{GL}^+(2, K) \), and the map \( \varphi \) is defined by \( \varphi(z) = Az \). The corresponding curves \( (1, \varphi)/(\mathbb{H}/\Gamma) \subset \mathbb{H}^2/\Gamma_K \) in this case are the irreducible components of the curves \( T_N \) studied in [HZ76] and [HZ77] and reviewed in §5.2. In particular, there are infinitely many curves of this type on each Hilbert modular surface, and conversely each of these curves lies on infinitely many Hilbert modular surfaces in \( \mathbb{H}^2/\text{Sp}(4, \mathbb{Z}) \).

**Type 2. Teichmüller curves.** These are defined abstractly as the algebraic curves in the moduli space \( \mathcal{M}_g \) of curves of genus \( g \) that are totally geodesic submanifolds for the Teichmüller metric.
In genus two, they always lie on Hilbert modular surfaces and (apart from one exception for the field \( \mathbb{Q}(\sqrt{5}) \)) have a modular interpretation as the components of the moduli space of genus two curves whose Jacobian has real multiplication by an order in a real quadratic field and such that the unique (up to a scalar) holomorphic form on the curve that is equivariant with respect to this action\(^1\) has a double zero. There are at most two curves of this type on each Hilbert modular surface \( X_K \), and conversely each Teichmüller curve lies on exactly one \( X_K \). The proof that these curves have a modular embedding comes from [Möl06a]. (See also the proof of Proposition 5.6.)

There exists a variant of these curves, not used in this paper but studied in detail by Weiß in [Wei14], called ‘twisted Teichmüller curves’, obtained as the images of Teichmüller curves under the action of elements of \( \text{GL}(2, K)^+ \). They are still geodesic for the Kobayashi metric in \( X_K \), but no longer for the Kobayashi (= Teichmüller) metric in \( M_2 \). There are in general infinitely many of these curves on any Hilbert modular surface.

**Types 3 and 4. Curves related to Prym varieties.** Recall that a Prym variety is the kernel of the map \( \text{Jac}(C) \to \text{Jac}(C_0) \) induced by a double cover \( C \to C_0 \) of curves. By the Riemann–Hurwitz formula, it is two-dimensional if and only if the genus \( g \) of \( C \) lies between two and five. For the cases \( g = 3 \) or \( 4 \), there is a construction of Teichmüller curves in the moduli space \( M_g \) corresponding to certain Prym varieties having real multiplication by an order in a real quadratic field [McM06b]. Our cases 3 and 4 are these two cases, in the order \( g = 4, g = 3 \).

The four types 1–4 are distinguished by the invariant \( \lambda_2 \), which takes on the values 1 for Type 1, \( \frac{1}{3} \) for Type 2, \( \frac{1}{2} \) for Type 3, and \( \frac{1}{4} \) for Type 4 [Bai07, Theorem 15.1] and [Möl14, Proposition 5.1]. For each of these values, there is an infinite number of commensurability classes of Fuchsian groups with this invariant.

The exceptional Teichmüller curve over \( \mathbb{Q}(\sqrt{5}) \) corresponds to \( \lambda_2 = 1/2 \), and at the time of writing this is the only known commensurability class of a Fuchsian group with this value of the invariant \( \lambda_2 \).

### 2. Twisted modular forms

For any function \( \varphi : \mathbb{H} \to \mathbb{H} \) satisfying (1) we can define two factors \( J(g, z) \) and \( \tilde{J}(g, z) \) for \( g \in \text{SL}(2, K) \) and \( z \in \mathbb{H} \) by

\[
J(g, z) = cz + d, \quad \tilde{J}(g, z) = c^\sigma \varphi(z) + d^\sigma \quad \text{if } g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.
\]

The calculation

\[
\tilde{J}(\gamma_1 \gamma_2, z) = (c_1 a_2 + d_1 c_2)^\sigma \varphi(z) + (c_1 b_2 + d_1 d_2)^\sigma
\]

\[
= \left( c_1^\sigma \frac{a_2^\sigma \varphi(z) + b_2^\sigma}{c_2^\sigma \varphi(z) + d_2^\sigma} + d_1^\sigma \right) \left( c_2^\sigma \varphi(z) + d_2^\sigma \right)
\]

\[
= \left( c_1^\sigma \varphi(z) \frac{a_2 + b_2}{c_2 z + d_2} + d_1^\sigma \right) \left( c_2^\sigma \varphi(z) + d_2^\sigma \right) \quad \text{(by equation (1))}
\]

\[
= \tilde{J}(\gamma_1, \gamma_2 z) \tilde{J}(\gamma_2, z) \quad \text{for } \gamma_1, \gamma_2 \in \Gamma
\]

shows that \( \tilde{J} \) is a cocycle for \( \Gamma \). (The corresponding statement for \( J \), which follows from the same calculation with \( \varphi \equiv \text{Id}, \) is, of course, standard.) It follows that the map \( f \mapsto f|_{(k, \ell) g} \) of the space of holomorphic functions in \( \mathbb{H} \) to itself defined for \( k, \ell \in \mathbb{Z} \) and \( g \in \text{GL}^+(2, K) \) by

\[
(f|_{(k, \ell) g})(z) = J(g, z)^{-k} \tilde{J}(g, z)^{-\ell} f(g z)
\]

is a group action when restricted to \( \Gamma \).

\(^1\)That is, \((\mu \varphi)^* = \mu \varphi\) for all \( \mu \) in the order; cf. § 5.1 (‘first eigendifferential’) for details.
Theorem derivatives

We now define a \( \varphi \)-twisted modular form of bi-weight \((k, \ell)\) on \( \Gamma \) to be a holomorphic function \( f : \mathbb{H} \to \mathbb{C} \) satisfying \( f|_{(k,\ell)} \gamma = f \) for all \( \gamma \in \Gamma \) together with the growth requirement that the function \( f_g \) is bounded as \( \Im(z) \to \infty \) for every \( g \in \SL(2, K) \), where \( f_g(z) = (cz + d)^{-k}(e^r \varphi(z) + d^r)^{-\ell} \). The function \( f_g \) depends only on the coset \( \Gamma g \).

To describe its Fourier expansion, we need to distinguish cases. If \(-1 \in \Gamma\), then the modular transformation property implies that \( k + \ell \) is even if \( M_{k,\ell}(\Gamma) \) is to be non-zero. If \(-1 \notin \Gamma\), then there may exist non-zero twisted modular forms for both parities of \( k + \ell \). Let \( \alpha \) as before be the totally positive element of \( K \) with \((g^{-1}\Gamma g)_\infty = \langle \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \rangle \). Recall that the cusp \( g\infty \) is called irregular if \( \Gamma \) does not contain \(-1\) and if \((g^{-1}\Gamma g)_\infty = \langle -\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \rangle \). Now there are two cases. If the cusp is regular or if \( k + \ell \) is even, then \( f_g(z + \alpha) = f_g(z) \) and the Fourier expansion of \( f_g \) has the form

\[
f_g(z) = \sum_{n \geq 0} a_n e(nz/\alpha) \quad \text{as } \Im(z) \to \infty,
\]

where \( n \) ranges over non-negative integers. Only if the cusp is irregular and \( k + \ell \) is odd, then \( f_g(z + \alpha) = -f_g(z) \) and the Fourier expansion is as in (13), but now with \( n \) ranging over \( \mathbb{Z}_{\geq 0} + \frac{1}{2} \) instead of \( \mathbb{Z}_{\geq 0} \).

If the Fourier coefficient \( a_0 \) is zero for all cusps (a condition that is automatically satisfied at regular cusps if \( k + \ell \) is odd), we call \( f \) a \textit{cusp form}. The spaces of \( \varphi \)-twisted modular forms and cusp forms of bi-weight \((k, \ell)\) will be denoted by \( M_{k,\ell}(\Gamma, \varphi) \) and \( S_{k,\ell}(\Gamma, \varphi) \), respectively. We will often omit the \( \varphi \) when no confusion can arise.

Obviously, ordinary modular forms of weight \( k \) on \( \Gamma \) are \( \varphi \)-twisted modular forms of bi-weight \((k, 0)\) for any \( \varphi \), and in fact \( M_{k,0}(\Gamma, \varphi) = M_k(\Gamma) \). We give three examples of twisted modular forms with \( \ell \neq 0 \).

(i) We always have \( \varphi' \in M_{2,-2}(\Gamma, \varphi) \), by virtue of the second equation in (9) and the expansions of \( \varphi \) at the cusps given in \$1\). This example shows that the weights \( k \) and \( \ell \) of a holomorphic twisted modular form do not both have to be positive. A similar example, which follows from the calculations given at the end of \$4, is that \( 2\varphi'\varphi'' - 3\varphi'^2 \) belongs to \( M_{8,-4}(\Gamma, \varphi) \). Note the quotient of this by \( \varphi'^2 \) is the Schwarzian derivative of \( \varphi \).

(ii) In the case of modular curves, when the map \( \varphi : \mathbb{H} \to \mathbb{H} \) is given by a fractional linear transformation \( A \in \SL^+(2, K) \), the calculation

\[
\tilde{J}(\gamma, z) = e^r Az + d^r = J(\gamma^\sigma, A \varphi(z)) = \frac{J(A^\gamma, z)}{J(A, z)} = \frac{J(A, \gamma z)}{J(A, z)} = J(\gamma, z)
\]

for \( \gamma \in \Gamma \) shows that if \( f \) belongs to \( M_{k,\ell}(\Gamma, \varphi) \), then the function \( f_{A\gamma}(z) = J(A, z)^{-\ell} f(z) \) belongs to \( M_{k+\ell}(\Gamma) \) in the usual sense, so that here we do not get a new kind of modular forms. For Teichmüller curves, on the other hand, the automorphy factor \( \tilde{J} \) cannot be reduced to an automorphy factor of the classical sort, and the twisted forms are a genuinely new type of modular form.

(iii) If \( (\Gamma, K) \) is a Hilbert modular group containing \( \Gamma \) and \( F \) is a Hilbert modular form of weight \((k, \ell)\) on \( (\Gamma, K) \) (i.e. \( F : \mathbb{H}^2 \to \mathbb{C} \) is a holomorphic map satisfying \( F(\gamma z_1, \gamma^\sigma z_2) = (cz_1 + d)^{-k}(e^r z_2 + d^r)^{-\ell} F(z_1, z_2) \) for all \( \gamma \in (\Gamma, K) \) and all \( z_1, z_2 \in \mathbb{H} \)), then the restriction of \( F \) to \( \mathbb{H} \) under the embedding \((1, \varphi) : \mathbb{H}^2 \to \mathbb{H}^2 \) is an element of \( M_{k,\ell}(\Gamma, \varphi) \).

The last example provides many twisted modular forms for any \((\Gamma, \varphi)\). But not all twisted modular forms arise in this way, and it makes sense to study the twisted forms independently of the two-variable theory. In particular, one can ask for the dimensions of the spaces \( M_{k,\ell}(\Gamma, \varphi) \) and \( S_{k,\ell}(\Gamma, \varphi) \) and for the structure of the bigraded ring \( M_{k,\ell}(\Gamma, \varphi) = \bigoplus_{k,\ell} M_{k,\ell}(\Gamma, \varphi) \), just as is

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usually done for classical modular forms when $\ell = 0$, and we can also study the classical topic of growth of Fourier coefficients. We shall give a general formula for the dimensions in the next section and a description of the ring of twisted forms in a special example in § 7, while the rest of this section is devoted to the study of the coefficient growth.

**Theorem 2.1.** Let $f(z) = \sum a_n q^n$ be a twisted modular form of bi-weight $(k, \ell)$, and set $K = k + |\ell|$. Then the Fourier coefficients of $f$ satisfy the estimates

$$a_n = \begin{cases} O(n^{K/2}) & \text{if } k + \ell < 2 \text{ or } f \text{ is a cusp form,} \\ O(n^{K/2} \log n) & \text{if } k + \ell = 2, \\ O(n^{k-1+\max(0,\ell)}) & \text{if } k + \ell > 2. \end{cases}$$

**Proof.** Suppose first that $f$ is cuspidal. Here we use a modification of the well-known argument given by Hecke in the untwisted case. We construct the real-valued continuous function

$$F(z) = |f(z)| y^{k/2} \tilde{y}^{\ell/2},$$

where $y = y(z) = \Im(z)$ and $\tilde{y} = \tilde{y}(z) = \Im(\varphi(z))$. This function is $\Gamma$-invariant by the definition of a twisted modular form. Since $f$ is a cusp form, $F$ decays rapidly at cusps and hence is bounded on the whole upper half-plane, so $f(z) = O(y^{-k/2} \tilde{y}^{\ell/2})$. On the other hand, $a_n = (1/\alpha) \int_{0}^{\alpha} f(x + iy) e(-n(x + iy)/\alpha) \, dx$ for any $y$. Specializing to $y = 1/n$ and using the first or the second inequality in (4) depending on the sign of $\ell$, we obtain the estimate stated.

In the remaining cases, still

$$|a_n| \leq \frac{1}{\alpha} \int_{0}^{\alpha} F(x + i/n) n^{k/2} \Im(\varphi(x + 1/n))^{-\ell/2} \, dx \ll n^{K/2} \int_{0}^{\alpha} F(x + i/n) \, dx,$$

by (4) (where the constant implied by $\ll$ depends only on $\Gamma$), but now $F(z) = F(\gamma(z)) = O(H(z)^{(k+\ell)/2})$ instead of $O(1)$, where $H(z)$ is defined in the lemma below. Since the exponent $k - 1 + \max(0, \ell)$ is equal to $K/2 + (k + \ell)/2 - 1$, the remaining statement is precisely the content of the following lemma. \hfill $\square$

**Lemma 2.2.** Let $\Gamma$ be a non-cocompact Fuchsian group, with the width of the cusp at $\infty$ equal to 1, and define the height function $H(z) = \max_{\gamma \in \Gamma} \Im(\gamma z)$. Then for $n > 1$ and $\lambda > 0$ one has the estimates

$$\int_{0}^{1} H\left(x + \frac{i}{n}\right) \lambda \, dx = \begin{cases} O(1) & \text{if } 0 < \lambda < 1, \\ O(\log n) & \text{if } \lambda = 1, \\ O(n^{\lambda-1}) & \text{if } \lambda > 1, \end{cases}$$

where the implied constant does not depend on $n$.

**Proof.** The case $\lambda = \frac{1}{2}$ is [Stö04, Proposition 2.2]. Essentially the same method can be used to give all cases. We provide the details only for $\lambda > \frac{1}{2}$. We let $T = \left( \begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix} \right)$ and choose for each of the $h$ cusps $\eta_j$ of $\Gamma$ a matrix $N_j \in \SL(2, \mathbb{R})$ such that $N_j \eta_j = \infty$, and such that the stabilizer of $\infty$ in $\Gamma_j = N_j \Gamma N_j^{-1}$ is always $\langle T \rangle$. Let $\mathcal{F}$ be a closed fundamental domain for $\Gamma$, which we may choose so that the cusp neighborhoods have the shape

$$N_j(\mathcal{F}) \cap \{ \zeta \in \mathbb{H} : \Im(\zeta) > B \} = [0, 1] \times [B, \infty) \quad (j = 1, \ldots, h)$$

for some $B > 1$, and are disjoint. We define the truncation function $[x]_B$ to be $x$ if $x > B$ and 0 otherwise. Since the complement in the fundamental domain of the cusp neighborhoods
is compact, it suffices to prove the statement of the lemma with $|H|_B$ in the place of $H$. Note that
\[
\int_0^1 \left| H\left(x + \frac{i}{n}\right)\right|^\lambda_B \, dx = \sum_{j=1}^h \sum_{\gamma \in (T)\setminus \Gamma_j} \int_0^1 \left| \Im\left(\gamma\left(x + \frac{i}{n}\right)\right)\right|^\lambda_B \, dx.
\] (14)

Suppose that $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ gives a non-zero contribution to the right-hand side. Recall that Shimizu’s lemma states that in a Fuchsian group, normalized so that the cusp $\infty$ has width 1, either $c = 0$ or $1 \leq |c|$. Here the truncation implies that $c \neq 0$ and that
\[
1 \leq \frac{1/n}{|c(x + i/n) + d|^2},
\]
from which
\[
1 \leq |c| \leq \sqrt{n} \quad \text{and} \quad -\frac{d}{c} \in [-1, 2].
\]

On the other hand, with the substitution $x = -d/c + t/n$ we get
\[
\int_0^1 \left| \Im\left(\gamma\left(x + \frac{i}{n}\right)\right)\right|^\lambda_B \, dx \leq \int_{-\infty}^\infty \left(\frac{1/n}{(cx + d)^2 + c^2/n^2}\right)^\lambda \, dx
\]
\[
\leq n^{\lambda - 1} / |c^{2\lambda} \int_{-\infty}^\infty (t^2 + 1)^\lambda = O\left(\frac{n^\lambda - 1}{|c|^{2\lambda}}\right).
\]

(It is this estimate which has to be changed, taking into account $|\cdot|_B$, when $\lambda \leq \frac{1}{2}$.) We define
\[
C_{\mu,\nu}^j(x, X) = \left\{ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in (T)\setminus \Gamma_j : x < |c| \leq X, -\frac{d}{c} \in [\mu, \nu] \right\}
\]
and $C_{\mu,\nu}^j(X) = C_{\mu,\nu}^j(0, X)$. The crucial observation now is that the cardinality of this set is bounded for $X > 1$, any $j$ and $\mu < \nu$
\[
\#C_{\mu,\nu}^j(X) \leq (\nu - \mu)X^2 + 1.
\]

This again follows from Shimizu’s Lemma, which implies that for two matrices $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\gamma' = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$ in $C_{\mu,\nu}^j(X)$ we have $|d'/c' - d/c| \geq |cc'|^{-1} \geq X^{-2}$ (see [Iwa95, Proposition 2.8], for details).

If $\lambda = 1$, then the contribution of the $j$th cusp to the right-hand side of (14) is bounded above by a constant times
\[
\sum_{\gamma \in C_{\mu,\nu}^j(\sqrt{n})} \frac{1}{|c|^2} \leq \sum_{k=0}^{(1/2) \log_2 n} \sum_{\gamma \in C_{\mu,\nu}^j(2^{k-1}, 2^k)} \frac{1}{|c|^2}
\]
\[
\leq \sum_{k=0}^{(1/2) \log_2 n} 3 \cdot 2^k = O(\log n).
\]

The other cases with $\lambda > 1/2$ are calculated in the same way, the estimate for the left-hand side of (14) now being $O(n^{\lambda - 1} \sum_{k=0}^{(1/2) \log_2 n} 2^k(1-\lambda))$, which is $O(n^{\lambda - 1})$ for $\lambda > 1$ and $O(1)$ for $\lambda < 1$. \qed
The bounds in the preceding theorem are not sharp for the classical case $\ell = 0$, since the methods of Rankin and Selberg give an improvement for any Fuchsian group. Note, however, that the deeper results of Deligne cannot be applied here, since on Teichmüller curves one does not dispose of Hecke operators.

In the more interesting case of strictly twisted modular forms (i.e. $\ell \neq 0$) we know in the example of $\phi'$ that the coefficients grow like $O(n)$ by Caratheodory’s estimate (3), while the preceding theorem with $k = 2, \ell = -2$ gives only the bound $O(n^2)$. However, we do not know if the behavior of the coefficients of $\phi'$ is typical for a twisted modular form of bi-weight $(2, -2)$ or if there exist other elements whose coefficient growth is closer to $O(n^2)$.

### 3. Dimensions and degrees

Twisted modular forms of bi-weight $(k, \ell)$ can be thought of as sections of the bundle $L^k \otimes \tilde{L}^\ell$ with appropriate growth conditions at the cusps, where $L$ and $\tilde{L}$ are the line bundles over $\mathbb{H}/\Gamma$ defined as the quotients of $\mathbb{H} \times \mathbb{C}$ by the equivalence relations $(z, u) \sim (\gamma z, J(\gamma, z)u)$ and $(z, u) \sim (\gamma z, \tilde{J}(\gamma, z)u)$ for $\gamma \in \Gamma$. The dimension of the space of such forms for a given group can therefore be computed by the Riemann–Roch theorem for curves, just as in the case of classical modular forms, if we know the degrees of the two bundles $L$ and $\tilde{L}$ and the numbers of cusps and elliptic fixed points of various orders of the group. In particular, if $\Gamma$ had no cusps and no fixed points (a situation of which, so far as we know, there is no example), then the Riemann–Roch theorem would give

\[ \dim M_{k, \ell}(\Gamma) = (k \deg(L) + \ell \deg(\tilde{L}) - 1)(g - 1), \]

where $g$ is the genus of $\mathbb{H}/\Gamma$.

The presence of cusps (including possibly irregular ones) and elliptic fixed points will make the actual formula more complicated.

Let $\Pi$ be a torsion-free subgroup of finite index in $\Gamma$. Such a group always exists, since the level-three subgroup of a Hilbert modular group is torsion-free and $\Gamma$ is a subgroup of the Hilbert modular group. We define $\Pi_0$ to be a subgroup of finite index such that the eigenvalues of all parabolic elements are one (i.e. all cusps are regular). Such a subgroup exists, since $\Pi$ is free if it is has a cusp. By passing to a smaller subgroup if necessary, we may suppose $\Pi_0 \subset \Gamma$ to be normal. We let $L_0$ and $\tilde{L}_0$ be the line bundles over $\mathbb{H}/\Pi_0$ defined by the automorphy factors $J$ and $\tilde{J}$, respectively.

The basic invariant we attach to a Hilbert modular embedding is the ratio

\[ \lambda_2 = \frac{\deg(\tilde{L}_0)}{\deg(L_0)}. \]  

As a consequence of the proof of Theorem 3.2 below, we see that this number does not depend on the choice of $\Pi_0$ among torsion-free subgroups of $\Gamma$ with regular cusps. The value of the invariant $\lambda_2$ is given by $1, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}$ on the four classes of Hilbert modular embeddings described in §1.

The definition of $\lambda_2$ and the classical result that ordinary modular forms of weight two are differential forms on $\mathbb{H}/\Pi_0$ imply the following proposition.

**Proposition 3.1.** Let $g_0$ and $s_0$ denote the genus and the number of cusps of $\mathbb{H}/\Pi_0$, respectively. Then

\[ \deg(L_0) = g_0 - 1 + s_0/2 \quad \text{and} \quad \deg(\tilde{L}_0) = \lambda_2 \left( g_0 - 1 + s_0/2 \right). \]

Now we have to pass from $\Pi_0$ to $\Gamma$, which may have both elliptic fixed points and irregular cusps. If $\Gamma$ contains $-I$, then we will assume that $k + \ell$ is even, since otherwise the equation $f|_{(k, \ell)}(-I) = -f$ implies that the space of twisted modular forms of bi-weight $(k, \ell)$ is zero.
We define characteristic numbers at elliptic fixed points and cusps for the bundle of twisted modular forms of bi-weight \((k, \ell)\) in the following way. Suppose that \(x\) is an elliptic fixed point and that the isotropy group \(\Gamma_x\) is of order \(n_x\) in \(\text{SL}(2, \mathbb{R})\). We take a generator \(\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_x\) that acts on the tangent space at \(x\) by a rotation by \(2\pi/n_x\) in the positive direction, i.e. such that \(\arccos(\text{tr}(\gamma)/2) = 2\pi/n_x\) and \(\text{c sin}(2\pi/n_x) \leq 0\). We let \(r_x = 1/n_x\). Since \(\gamma\) is of finite order, so is \(\gamma^\sigma\) and we can define \(r_x^\sigma \in 1/n_x\mathbb{Z}\) by

\[
\cos(2\pi r_x^\sigma) = \text{tr}(\gamma^\sigma)/2 \quad \text{and} \quad c^\sigma \sin(2\pi r_x^\sigma) \leq 0.
\]

Then the characteristic number at \(x\) is defined as

\[
b_x(k, \ell) = \{-kr_x - \ell r_x^\sigma\},
\]

where the curly braces denote the fractional part (in \([0, 1)\)) of the rational number. If \(x\) is a cusp, we define the characteristic number \(b_x(k, \ell)\) to be \(1/2\) if the cusp is irregular, \(-1/2\) if \(x\) is odd, and we let \(b_x(k, \ell) = 0\) in all other cases.

We remark that characteristic numbers are a finer information than the usual type of the elliptic fixed point [vdG87, §I.5 and V.7], since there are two possibilities even for fixed points of order two (in \(\text{PSL}(2, \mathbb{R})\)). We let \(\Delta = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)\) and \(S = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)\). Then the contribution of \(\Delta S \Delta^{-1} \in \text{SL}(O_D^\times)\) is \(b_x(k, \ell) = \{-kr_x - \ell r_x^\sigma\}\), whereas the contribution of \(S\) is \(b_x(k, \ell) = \{-kr_x - \ell r_x^\sigma\}\). Note that in all these calculations we consider modular embeddings to \(\mathbb{H}^2\). If we consider a modular embedding to \(\mathbb{H} \times \mathbb{H}^\gamma\), the fixed point of \(S\) is \(-i\) in the second factor, so its contribution is \(b_x(k, \ell) = \{-kr_x - \ell r_x^\sigma\}\).

We can now give the dimension of the space of modular forms in terms of the topology of \(\mathbb{H}/\Gamma\) and those characteristic numbers. We let \(n_x\) be the order of the isotropy group \(\Gamma_x\) in \(\text{PSL}(2, \mathbb{R})\).

**Theorem 3.2.** Let \(k\) and \(\ell\) be integers. If \(k+\ell\) is odd and \(\Gamma\) contains \(-1\), then \(\dim M_{k,\ell}(\Gamma, \varphi) = 0\). If \(-1 \not\in \Gamma\) or \(k + \ell\) is even, then twisted modular forms of bi-weight \((k, \ell)\) are precisely the global sections of a line bundle \(L_{k,\ell}\) of degree

\[
\deg(L_{k,\ell}) = (k + \lambda_2\ell) \left( g - 1 + s + \frac{1}{2} \sum_{x \in \mathbb{H}/\Gamma} \left( 1 - \frac{1}{n_x} \right) \right) - \sum_{x \in \mathbb{H}/\Gamma} b_x(k, \ell),
\]

where \(g\) and \(s\) denote the genus and the number of cusps of \(\Gamma\). If also \(k + \lambda_2\ell \geq 2\), then

\[
\dim M_{k,\ell}(\Gamma, \varphi) = \deg(L_{k,\ell}) + (1 - g).
\]

**Proof.** We mimic the standard argument for ordinary modular forms and describe \(L_{k,\ell}\) as a subsheaf of \(L^{\otimes k} \otimes \mathcal{L}^{\otimes \ell}\). If \(t\) is a local parameter at \(x\), for \(x\) both in \(\mathbb{H}/\Gamma\) or being a cusp, the stalk \((L_{k,\ell})_x\) at consists of all germs of holomorphic functions \(f\) with \(f(\gamma t) = J(\gamma, t)^k \tilde{J}(\gamma, \varphi(t))\ell f(t)\) for all \(\gamma \in \Gamma_x\) in the stabilizer of \(x\). With this definition, twisted modular forms of bi-weight \((k, \ell)\) for the modular embedding \(\varphi\) are obviously the global sections of \(L_{k,\ell}\).

In order to compute the degree of \(L_{k,\ell}\) we use the map \(\pi : \mathbb{H}/\Pi_0 \to \mathbb{H}/\Gamma\). This induces an inclusion \(\pi^*(L_{k,\ell}) \to L^{\otimes k}_0 \otimes \mathcal{L}^{\otimes \ell}_0\). Since we know the degree of the image in terms of \(g, s\) and \(\lambda_2\), it suffices to compute the degree of its cokernel \(K\). This cokernel is supported at the elliptic fixed points and at the cusps.

Suppose first that \(z_\gamma \in \mathbb{H}\) is the fixed point of \(\gamma \in \Gamma\). The point \(\varphi(z_\gamma)\) is fixed by \(\gamma^\sigma\), so that by the cocycle condition both \(J(\gamma, z_\gamma)\) and \(\tilde{J}(\gamma, \varphi(z_\gamma))\) are roots of unity of some order that divides the order of the isotropy group \(n_x\).
Now let $y$ be one of the preimages of $x$ and let $u$ be a local parameter at $y$, so that $t = u^{\pi x}$. Then

$$(L_0^k \tilde{\mathcal{L}}_0^\ell)^y \cong \mathbb{C}[u] \quad \text{and} \quad (\pi^*(L_{k,\ell}))^y = u^{\pi x} B_x(k,\ell) \mathbb{C}[u],$$

where $B_x(k,\ell) \in [0,1)$ is the unique rational number such that

$$J(\gamma, z_x)^k \tilde{J}(\gamma, \varphi(z_x))^\ell = \mathbf{e}(B_x(k,\ell)).$$

Consequently, $\dim \mathcal{K}_y = \pi x B_x(k,\ell)$ for each of the $\deg(\pi)/\pi x$ points $y$ above $x$.

Next we want to show that $B_x(k,\ell) = b_x(k,\ell)$. If $x = i$, then the generator of $\Gamma_x$ specified above is $\gamma = \left(\begin{smallmatrix} \cos(2\pi/n_x) & \sin(2\pi/n_x) \\ -\sin(2\pi/n_x) & \cos(2\pi/n_x) \end{smallmatrix}\right)$ and hence $J(\gamma_1, i)^k = \mathbf{e}(-kr_x)$ and $\tilde{J}(\gamma_1, i)^\ell = \mathbf{e}(r_x^\sigma)$ by the definition of $r_x$ and $r_x^\sigma$. This proves the claim in the special case $x = i$. For the general case note first that for any $\alpha \in \text{SL}(2, \mathbb{R})$ the cocycle property implies $J(\gamma, z_x) = J(\alpha \gamma \alpha^{-1}, z_x)$, where $z_x \alpha \gamma \alpha^{-1} = \alpha z_x$ is the fixed point of $\alpha \gamma \alpha^{-1}$. The equivariance property (1) implies that $\varphi(z_x) = \gamma \varphi(z_x)$ and hence $\tilde{J}(\gamma, z_x) = J(\gamma, \varphi(z_x))$. If $\alpha$ takes the fixed point of $\gamma$ to $i$, then $\alpha \gamma \alpha^{-1}$ takes the fixed point of $\gamma^\sigma$ to $i$ and so

$$J(\gamma, z_x)^k \tilde{J}(\gamma, \varphi(z_x))^\ell = J(\alpha \gamma \alpha, i)^k J((\alpha \gamma \alpha)^\sigma, i)^\ell$$

reduces to the case already considered.

Now suppose that $x$ is a cusp and let $y$ be one of the cusps of $\Pi_0$ above $x$. If $\pi x$ denotes the degree of the covering $\pi$ at $y$, then there are $\deg(\pi)/\pi x$ cusps above $x$ since $\pi$ is Galois.

We start with the case $-I \notin \Gamma$. Then the stabilizer $\Gamma_x$ is infinite cyclic. Let $\gamma$ be a generator. The same argument as for (18) allows us to assume that the fixed point $z_x = \infty$. Note that $\gamma^\sigma$ also fixes $\infty$, so that $J(\gamma, \infty) = \tilde{J}(\gamma, \varphi(\infty)) = \gamma_2,2$, the lower right entry of $\gamma$. Since the cusp is irregular if and only if the generator $\gamma$ has $\gamma_{2,2} = -1$, we deduce

$$J(\gamma, \infty)^k \tilde{J}(\gamma, \varphi(\infty))^\ell = \mathbf{e}(b_x(k,\ell))$$

for $b_x(k,\ell)$ defined above. On the other hand, let $a$ be the width of the cusp $\infty$ of $\Gamma_0$, so that $q = \mathbf{e}(z/a)$ is a local parameter at $y$. Then

$$(L_0^k \tilde{\mathcal{L}}_0^\ell)^y \cong \mathbb{C}[q] \quad \text{and} \quad (\pi^*(L_{k,\ell}))^y = q^{\pi x} b_x(k,\ell) \mathbb{C}[q],$$

so that in total $\dim \mathcal{K}_y = \pi x b_x(k,\ell)$.

With the same local calculation one checks that if $-I \in \Gamma$ then we always have $\pi^*(L_{k,\ell})^y = (L_0^k \tilde{\mathcal{L}}_0^\ell)^y$. Hence, in this case, too, $\dim \mathcal{K}_y = \pi x b_x(k,\ell) = 0$ holds by definition.

Altogether, this implies

$$\deg(L_{k,\ell}) = \frac{1}{\deg(\pi)} ((k + \lambda_2 \ell) \deg(L_0)) - \sum_{x \in \mathbb{H}/\Gamma} b_x(k,\ell).$$

The number of cusps of $\Gamma$ is $s_0 = \deg(\pi) \sum_{x \in \partial(\mathbb{H}/\Gamma)} 1/\pi x$. Together with (16) and the Riemann–Hurwitz formula

$$g(\mathbb{H}/\Pi_0) - 1 = g(\mathbb{H}/\Gamma) - 1 + \frac{1}{2} \sum_{x \in \mathbb{H}/\Gamma} \left(1 - \frac{1}{\pi x}\right)$$

this implies the degree claim. The dimension statement then follows from Riemann–Roch since the $H^1$-term vanishes for $\deg(L_{k,\ell}) > 2g - 2$, which is guaranteed by the hypotheses on $k + \lambda_2 \ell$ and the fact that $b_x(k,\ell) \leq 1 - 1/\pi x$ for all cusps and elliptic fixed points.
4. Differential equations coming from twisted modular forms

A basic fact about modular forms, whose proof will be recalled below, is that for any Fuchsian group \( \Gamma \subset \text{SL}(2, \mathbb{R}) \), any modular function \( t \) on \( \Gamma \) and any modular form \( f \) of integral weight \( k \geq 1 \) on \( \Gamma \), the function \( y(t) \) defined locally by \( y(z) = y(t(z)) \) satisfies a linear differential equation of order \( k + 1 \) with algebraic coefficients (and even with polynomial coefficients if \( \mathbb{H}/\Gamma \) has genus zero and \( t \) is a hauptmodule\(^2\)). In this subsection we prove the corresponding statement for twisted modular forms. This statement will give one of the two approaches used in this paper to describe Teichmüller curves explicitly on Hilbert modular surfaces, by comparing the differential equations coming from their geometric definition (Picard–Fuchs differential equations) with the differential equations satisfied by suitable twisted modular forms on them.

**Theorem 4.1.** Let \( f(z) \) be a twisted modular form on \( (\Gamma, \varphi) \) of bi-weight \( (k, \ell) \), with \( k, \ell \geq 0 \), and \( t(z) \) a modular function with respect to the same group \( \Gamma \). Then the function \( y(t) \) defined locally by \( y(z) = y(t(z)) \) satisfies a linear differential equation of order \((k+1)(\ell+1)\) with algebraic coefficients.

**Proof.** It suffices to prove this for the two cases \((k, \ell) = (1, 0)\) and \((0, 1)\), since the general case follows from these. (The number \((k+1)(\ell+1)\) arises as the dimension of \( \text{Sym}^k(V_1) \otimes \text{Sym}^\ell(V_2) \) where \( \dim V_1 = \dim V_2 = 2 \).) The first case is the classical theorem mentioned above, of which several proofs are known (see e.g. [BGZ09, \S 5.3]). We reproduce one of them here since it generalizes directly to the more complicated case of bi-weight \((0, 1)\).

Let, then, \( f(z) \) be an ordinary modular form of weight one and \( t(z) \) a modular function on \( \Gamma \). By definition we have the two transformation equations

\[
\begin{align*}
t \left( \frac{az + b}{cz + d} \right) &= t(z), \\
f \left( \frac{az + b}{cz + d} \right) &= (cz + d) f(z)
\end{align*}
\]

for all matrices \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \). Differentiating these equations gives the further transformation equations

\[
\begin{align*}
\frac{d t}{d z} \left( \frac{az + b}{cz + d} \right) &= (cz + d)^2 \frac{d t}{d z}(z), \\
\frac{d f}{dz} \left( \frac{az + b}{cz + d} \right) &= (cz + d)^3 \frac{d f}{d z}(z) + c(cz + d)^2 f(z), \\
\frac{d^2 f}{d z^2} \left( \frac{az + b}{cz + d} \right) &= (cz + d)^5 \frac{d^2 f}{d z^2}(z) + 4c(cz + d)^4 \frac{d f}{d z}(z) + 2c^2(cz + d)^3 f(z).
\end{align*}
\]

The first of these equations says that \( t' \) is a (meromorphic) modular form of weight two, and by combining the others we find that the expression \( 2t'^2 - f f'' \) is a modular form of weight six. It follows that

\[
\frac{t'(z)}{f(z)^2} = A(t(z)), \quad \frac{2t'(z)^2 - f(z)f''(z)}{t'(z)f(z)^4} = B(t(z))
\]

for some rational (or, if \( t \) is not a hauptmodule, algebraic) functions \( A(t) \) and \( B(t) \). A direct calculation shows that

\[
\frac{1}{t^4} \left( \frac{t'}{t^2} \frac{1}{t'} \frac{f''}{f'} \right)' + \frac{2t'^2 - f f''}{t' f^4} = 0,
\]

\(^2\)Recall that a ‘hauptmodule’ (or ‘Hauptmodul’ if one retains the German spelling) is a modular function \( t \) giving an isomorphism between \( \mathbb{H}/\Gamma \) and \( \mathbb{P}^1(\mathbb{C}) \) if the former has genus zero.
It follows that the function $y(t)$ defined parametrically by the equation $y(t(z)) = f(z)$ (which of course can only hold locally, since $t(z)$ is $\Gamma$-invariant and $f(z)$ is not) satisfies the second-order linear differential equations

$$\left(A(t) y'(t)\right)' + B(t) y(t) = 0,$$  \hfill (20)

or $Ay'' + A'y' + By = 0$. This proves the theorem in the case $(k, \ell) = (1, 0)$.

Now suppose that $f$ is a twisted modular form of bi-weight $(0,1)$, i.e. $f$ satisfies the transformation equation $f((az+b)/(cz+d)) = (c^\sigma \varphi(z) + d^\sigma)f(z)$ for $(a, b, c, d) \in \Gamma$. From this equation, and from (1) and its derivative (the second equation in (9)), we find by further differentiating the transformation equations

$$\varphi'' \left( \frac{az+b}{cz+d} \right) = \frac{(cz+d)^4}{(c^\sigma \varphi(z) + d^\sigma)^2} \varphi''(z) + \frac{2c(cz+d)^3}{(c^\sigma \varphi(z) + d^\sigma)^2} \varphi'(z)$$

$$- \frac{2c^\sigma(cz+d)^4}{(c^\sigma \varphi(z) + d^\sigma)^3} \varphi'(z)^2,$$

$$f' \left( \frac{az+b}{cz+d} \right) = (cz+d)^2(c^\sigma \varphi(z) + d^\sigma)f'(z) + c^\sigma(cz+d)^2 \varphi'(z)f(z),$$

$$f'' \left( \frac{az+b}{cz+d} \right) = (cz+d)^4(c^\sigma \varphi(z) + d^\sigma)f''(z)$$

$$+ [2c(cz+d)^3(c^\sigma \varphi(z) + d^\sigma) + 2c^\sigma(cz+d)^4 \varphi'(z)]f'(z)$$

$$+ [c^\sigma(cz+d)^4 \varphi''(z) + 2cc^\sigma(cz+d)^3 \varphi'(z)]f(z).$$

From these equations it follows that the combination $(2f'^2 - ff'')\varphi' + ff''\varphi''$ is a modular form of weight six. But we have already seen that $t'$ and $\varphi'$ are twisted modular of bi-weights $(2,0)$ and $(2,-2)$, respectively. It follows that

$$\frac{t'(z)}{\varphi'(z)f(z)^2} = A(t(z)),$$

$$\frac{(2f'(z)^2 - f(z)f''(z))\varphi'(z) + f(z)f'(z)\varphi''(z)}{t'(z)\varphi'(z)^2f(z)^4} = B(t(z))$$  \hfill (21)

for some algebraic (respectively rational if $t$ is a hauptmodule) functions $A(t)$ and $B(t)$, and since by direct calculation we have

$$\frac{1}{t'} \left( \frac{t'}{\varphi''f^2} \frac{1}{t'} \frac{f'}{f} \right)' + \frac{(2f'^2 - ff'')\varphi' - ff''\varphi''}{t'\varphi''f^4} f = 0$$

in this case, we deduce that $f$ satisfies a second-order linear differential equation of the same form (20) as before.

\[ \square \]

**Remark.** The two weight six modular forms $2f'^2 - ff''$ (for $f \in M_{1,0}(\Gamma)$) and $(2f'^2 - ff'')\varphi' + ff''\varphi''$ (for $f \in M_{0,1}(\Gamma)$) used above, which are easily checked to be holomorphic at the cusps, are special cases of the classical Rankin–Cohen bracket and of twisted versions of it, respectively. Without going into details, we mention that the twisted Rankin–Cohen brackets of two twisted modular forms $f_i \in M_{k_i,\ell_i}(\Gamma, \varphi)$ ($i = 1, 2$) can be defined as the product of the usual Rankin–Cohen brackets of $\varphi^{\ell_i/2}f_1$ and $\varphi^{\ell_i/2}f_2$ (which by example (i) of §2 are ordinary modular forms of weight $k_1 + \ell_1$ and $k_2 + \ell_2$ on $\Gamma$) with a suitable power of $\varphi'$. 

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In §4 we have seen abstractly how classical or twisted modular forms give rise to differential equations. In this Part II we show conversely, in a specific example, how to obtain from these differential equations the Hilbert modular embedding $\varphi$. The example that we will consider in detail is $D = 17$, for which the differential equations needed were computed in [BM10a]. In §6 we will sketch how these were obtained, referring to that paper for the full details. In §7 we discuss the arithmetical properties of the solutions of these differential equations and compute the Fourier expansions of the corresponding modular forms at all cusps. We turn Theorem 3.2 into a concrete description of the ring of modular forms (Theorem 5.5 for the general result and Propositions 7.1 and 7.2 for the special case $D = 17$), since the corresponding local invariants can be computed for Teichmüller curves. In §8 we will show how to use these solutions to obtain an explicit embedding of the Teichmüller curve in the Hilbert modular surface. The introductory §5 provides the necessary background on Hilbert modular surfaces and Teichmüller curves in genus two.

5. Curves on Hilbert modular surfaces

As we have already said, there are two basic examples of the situation described in §1: modular curves and Teichmüller curves. In this section we describe both of these, the first relatively briefly since it is well known and the second in more detail. We begin with a preliminary subsection specifying more precisely the Hilbert modular surfaces that will be used in this paper. The main new result in this section is the dimension formula Theorem 5.5.

5.1 Hilbert modular groups and Hilbert modular surfaces

As before, we denote by $K$ be a real quadratic field, together with a fixed embedding $K \subset \mathbb{R}$, and denote by $\sigma$ both the Galois conjugation and the second embedding of $K$ into $\mathbb{R}$. In §1 we briefly defined Hilbert modular groups and Hilbert modular surfaces, denoting them generically by $\Gamma_K$ and $H_2/\Gamma_K$. Now we want to be more specific. Our general reference are Hirzebruch’s seminal paper [Hir73] and the book [vdG87] by van der Geer.

Usually when one speaks of ‘the’ Hilbert modular group for $K$ one means the group $\text{SL}(2, \mathcal{O}_D)$, where $D$ is the discriminant of an order $\mathcal{O} = \mathcal{O}_D \subset K$. However, since we want principally polarized abelian surfaces, we need to work instead with the modified Hilbert modular group

$$\text{SL}(\mathcal{O}^\vee \oplus \mathcal{O}) = \begin{pmatrix} \mathcal{O} & \mathcal{O}^\vee \mathcal{O}^{-1} \\ (\mathcal{O}^\vee)^{-1} & \mathcal{O} \end{pmatrix} \cap \text{SL}(2, K),$$

where $\mathcal{O}^\vee$ denotes the set of $x \in K$ for which $xy$ has integral trace for all $y \in \mathcal{O}$. One has $\mathcal{O}^\vee = \mathcal{d}^{-1}$, where in the case of quadratic fields the ideal $\mathcal{d}$, called the different of $K$, is simply the principal ideal $(\sqrt{D}) = \sqrt{D} \mathcal{O}$. Note that the two groups $\text{SL}(2, \mathcal{O})$ and $\text{SL}(\mathcal{O}^\vee \oplus \mathcal{O})$ are conjugate in $\text{GL}(2, K)$ by the action of the diagonal matrix $\Delta = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{D} \end{pmatrix}$, and in particular are isomorphic as abstract groups. But the action of $\Delta$ exchanges the upper and lower half-planes in the second factor (since the Hilbert modular group acts on the second factor via its Galois conjugates and the Galois conjugate of the determinant of $\Delta$ is negative), so the quotient is the Hilbert modular surface

$$X_D = H^2/\text{SL}(\mathcal{O}^\vee \oplus \mathcal{O}),$$

which is isomorphic to $X_D^- = H \times H^-/\text{SL}(2, \mathcal{O})$ and not in general isomorphic to the ‘standard’ Hilbert modular surface $X\mathcal{O} = H^2/\text{SL}(2, \mathcal{O})$. (They do not even necessarily have the same Euler characteristic.) If $\mathcal{O}$ contains a unit $\varepsilon$ of negative norm, which happens, for instance, when $D$ is...
prime, then $\mathfrak{o}$ is principal in the narrow sense and the varieties $X_{\mathfrak{o}}$ and $X_{\mathfrak{o}^{-}}$ are isomorphic via $(z_1, z_2) \mapsto (\varepsilon z_1, \varepsilon^{-2} z_2)$. To a point $z = (z_1, z_2) \in \mathbb{H}^2$ we associate the polarized abelian surface $A_z = \mathbb{C}^2 / \mathcal{L}_z$, where $\mathcal{L}_z \subset \mathbb{C}^2$ is the lattice

$$\mathcal{L}_z = \{(a z_1 + b z_2 + b^*), a \in \mathfrak{o}^\vee, b \in \mathfrak{o}\}, \quad (22)$$

with the action of $\mathfrak{o}$ on $A_z$ induced from the action $\lambda(v_1, v_2) = (\lambda v_1, \lambda^* v_2)$ of $\mathfrak{o}$ on $\mathbb{C}^2$ and with the polarization induced from the antisymmetric pairing

$$((a, b), (a', b')) = \text{tr}_{K/Q}(ab' - a'b) \quad (a, a' \in (\mathfrak{o}^\vee)^{-1}, \ b, b' \in \mathfrak{o}). \quad (23)$$

This pairing is unimodular and the polarization is principal, which is why that case is of special interest.

We observe that the action of $\mathfrak{o}$ on $A_z$ gives a canonical splitting of the two-dimensional space of holomorphic one-forms on $A$ into two one-dimensional eigenspaces, generated by the differential forms $\omega = dv_1$ and $\tilde{\omega} = dv_2$, which we will call the first and second eigendifferential, respectively. If $A_x$ is the Jacobian of a curve $C$ of genus two, then by the canonical identification of the spaces of holomorphic one-forms on $C$ and on $A_x$ we obtain corresponding eigendifferentials on $C$. These will be used in the definition of Teichmüller curves in § 5.3.

Since the isomorphism class of $A_x$ depends only on the image of $z$ in $X_D$, and since polarized abelian surfaces are parametrized by points in the quotient of the Siegel upper half-space $\mathbb{H}_2$ by $\text{Sp}(4, \mathbb{Z})$ we get an embedding of the Hilbert modular surface $X_D$ into the quotient $\mathbb{H}_2 / \text{Sp}(4, \mathbb{Z})$, a Siegel modular embedding. Explicitly, let $\psi$ be the map from $\mathbb{H}_2^2$ to $\mathbb{H}_2$ given by

$$\psi : z = (z_1, z_2) \mapsto B \begin{pmatrix} z_1 & 0 \\ 0 & z_2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (24)$$

where, for some $\mathbb{Z}$-basis $(\omega_1, \omega_2)$ of $\mathfrak{o}$ we let

$$B = \begin{pmatrix} \omega_1 & \omega_2^\phi \\ \omega_2 & \omega_2^\phi \end{pmatrix} \quad \text{and} \quad A = B^{-1}. \quad (25)$$

We define a homomorphism $\Psi : \text{SL}(\mathfrak{o}^\vee \oplus \mathfrak{o}) \to \text{Sp}(4, \mathbb{Z})$ by

$$\Psi : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} B^T & 0 \\ 0 & A \end{pmatrix} \begin{pmatrix} \delta & \delta \phi \\ \delta^{-1} & \delta^{-1} \phi \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad (26)$$

where $\delta$ for $a \in K$ denotes the diagonal matrix $\text{diag}(a, a^\phi)$. Then the map $\psi$ is equivariant with respect to the actions of $\text{SL}(\mathfrak{o}^\vee \oplus \mathfrak{o})$ on $\mathbb{H}_2^2$ and of $\Psi(\text{SL}(\mathfrak{o}^\vee \oplus \mathfrak{o}))$ on $\mathbb{H}_2$, so it induces a map, also denoted by $\psi$, on the level of quotient spaces.

Remark on notation. We will use the letters $\psi$ and $\Psi$ for Siegel modular embeddings and $\varphi$ and $\Phi$ for Hilbert modular embeddings. The capital letter will denote the map on the level of modular groups and the small letter the map on the level of symmetric spaces or quotient spaces.

More generally, for any invertible $\mathfrak{o}$-ideal $a$ the trace pairing (23) on the lattice $\mathfrak{o}^\vee \oplus a$ is unimodular and, consequently, the abelian surface $\mathcal{L}_{z,a}$, defined as in (22) with $\mathfrak{o}$ replaced by $a$, is principally polarized. This implies that the Hilbert modular surfaces

$$X_{D,a} = \mathbb{H} / \text{SL}(\mathfrak{o}^\vee \oplus a), \quad (27)$$

where

$$\text{SL}(\mathfrak{o}^\vee \oplus a) = \left(\mathfrak{o}(a^\vee)^{-1} \ a^\vee(a)^{-1} \right) \cap \text{SL}(2, K),$$

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also parametrize principally polarized abelian varieties with real multiplication by $\mathcal{O}$. The only difference is that now the cusp at $\infty$ of $X_D, a$ is in general a different one than the cusp at $\infty$ for $X_D$. We will use these variants $X_{D, a}$ when we discuss cusps of Hilbert modular surface in §10. If we construct $B$ using some $\mathcal{O}$-basis $(\omega_1, \omega_2)$ of $a$, then (24) defines a map $\psi$ that is equivariant with respect to a homomorphism

$$\Psi : \text{SL}(a^\vee \oplus a) \to \text{Sp}(4, \mathbb{Z})$$

(28)

given by the same definition (25). Hence, the pair $(\psi, \Psi)$ defines a Siegel modular embedding of $X_{D, a}$.

### 5.2 Modular curves

We already defined the modular curves in §1 as the quotients of $\mathbb{H}$ by subgroups of $\text{SL}(2, \mathbb{R})$ of the form $\Gamma_A = \{ \gamma \in \Gamma_K \mid A\gamma = \gamma^a A \}$ where $A$ is a ‘suitable element’ of $\text{GL}(2, K)$, embedded into appropriate Hilbert modular surfaces via $z \mapsto (z, A z)$. Here ‘suitable’ means that the adjoint of $A$ equals its Galois conjugate, so that $A = \left( \begin{array}{cc} \lambda & -b \sqrt{D} \\ a \sqrt{D} & \lambda \end{array} \right)$ for some $(a, b, \lambda) \in \mathbb{Q} \times \mathbb{Q} \times K$, which after multiplying $A$ by a suitable scalar in $\mathbb{Q}^\times$ (which does not change the definition of $\Gamma_A$) we can assume belongs to $\mathbb{Z} \times \mathbb{Z} \times \mathcal{O}_K$. The corresponding embedded curve in $\mathbb{H}^2$ is defined as $a z_1 z_2 + \nu^a z_1 + \nu z_2 + b = 0$, where $\nu = \lambda / \sqrt{D} \in \mathcal{O}^\vee = \mathcal{O}^{-1}$, and the union of these curves (or, rather, of their images in the Hilbert modular surface) when $A$ ranges over all matrices as above with given determinant $N \in \mathbb{N}$ is denoted by $T_N$. These curves $T_N$ were studied in detail (for the Hilbert modular surface $X_0$) in [HZ76] and [HZ77]. We recall a few results that we will use. The curve $T_N$ is non-empty if and only if $N$ is congruent modulo $D$ to the norm of an element of $\mathcal{O}$, and is non-compact (i.e. passes through the cusps) if and only if $N$ is the norm of an integral ideal $a$ of $K$, in which case each of its components is non-compact. It is not in general irreducible, for three reasons. First, we have $T_N = \bigcup_{d \mid N} F_N/d^2$, where $F_N$ is defined like $T_N$ but with the additional requirement that $(a, b, \nu)$ is primitive in the lattice $\mathbb{Z} \times \mathbb{Z} \times \mathcal{O}^{-1}$. Second, the $F_N$ are in general not irreducible either, but decompose as $\bigcup_{\alpha} F_N(\alpha)$, where $\alpha$ ranges over the elements of $\mathcal{O}^{-1}/\mathcal{O}$ with $N(\alpha) \equiv N/D \pmod{1}$ and $F_N(\alpha)$ is defined by requiring $\nu \equiv \alpha (\text{mod } \mathcal{O})$ (see [Zag75, p. 4, Remark 1]). Finally, even the $F_N(\alpha)$ need not be irreducible. (For instance, if $D = p$ is prime and $p^2 \mid N$, then the two Legendre symbols $(a/p)$ and $(b/p)$ cannot both vanish or have opposite values, so $F_N = F_N(0)$ has two components distinguished by the invariant $\varepsilon \in \{ \pm 1 \}$ defined by $(a/p) = \varepsilon$ or $(b/p) = \varepsilon$; cf. [Fra77].) However, if $N$ is ‘admissible’ in the sense of the proposition on [HZ77, p. 57] (i.e. $N$ is the norm of a primitive ideal in the principal genus), then that proposition says that each $F_N(\alpha)$ is irreducible, as one shows by counting the number of cusps of each component separately and of the whole curve $F_N$.

The same results apply to the curves in $X_D^-$ defined (and denoted) in the same way but with the determinant of $A$ now being $-N$ rather than $N$. Using the identification $(z_1, z_2) \mapsto (\sqrt{D} z_1, -\sqrt{D} z_2)$ of $X_D^-$ with $X_D$, we can consider these as curves in $X_D$, the defining equation now being

$$AD z_1 z_2 + \lambda^a z_1 + \lambda z_2 + B = 0 \quad (A, B \in \mathbb{Z}, \lambda \in \mathcal{O}, ABD - \lambda \lambda^a = N).$$

(29)

A special union of these curves will be play a role a in characterization of Teichmüller curves below. In the moduli space of principally polarized abelian surfaces $A_2$ we denote by $P$ the product locus (also called reducible locus), i.e. the locus of abelian varieties that split, as polarized abelian varieties, into a product of elliptic curves. The Torelli map gives an isomorphism

$$t : M_2 \to A_2 \setminus P.$$
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The intersection of \( P \) with the Hilbert modular surface \( X_D \) will be denoted by \( P_D \). It is a union of modular curves, as described in the following proposition.

**Proposition 5.1.** The decomposition into irreducible components of \( P_D \) is given by

\[
P_D = \bigcup_{\nu \in \mathbb{D}^{-1}, \nu \not\equiv 0, \text{Tr}(\nu) = 1} P_{D,\nu} = \bigcup_{r \equiv D \pmod{2}, |r| < 2\sqrt{D}} P_{D, (r+\sqrt{D})/2\sqrt{D}}
\]

where \( P_{D,\nu} \) is the image in \( X_D \) of the curve \((\nu, \nu')\) \( \mathbb{H} \subseteq \mathbb{H}^2 \).

**Proof.** This is essentially [McM07, Corollary 3.5], which states that

\[
T_N \left( \frac{r + \sqrt{D}}{2} \right).
\]

Since each \( N \) occurring is admissible (it is the norm of the primitive principal ideal generated by \((r+\sqrt{D})/2\)), we have that \( T_N(\nu) = F_N(\nu) \) is irreducible for each \( \nu = (r+\sqrt{D})/2\sqrt{D} \) and hence coincides with its subcurve \( P_{D,\nu} \). \( \square \)

We would like to say a few words to explain where the equation \( D = r^2 + 4N \) in (30) comes from. A point of \( P_D \) corresponds to a product \( E \times E' \) of elliptic curves having real multiplication by \( \mathcal{O} \), i.e., for which there is an endomorphism \( \Phi = \left( \alpha \begin{smallmatrix} \lambda \\ \lambda' \end{smallmatrix} \right) \) of \( E \times E' \) satisfying a quadratic equation of discriminant \( D \) over \( \mathbb{Z} \). Since for generic points the elliptic curves \( E \) and \( E' \) do not have complex multiplication, we have \( \alpha \in \text{End}(E) = \mathbb{Z} \), \( b \in \text{End}(E') = \mathbb{Z} \), and \( \lambda' = \lambda' \lambda = \lambda = N = \deg \lambda \in \mathbb{N} \) and hence \( \Phi^2 - (a+b)\Phi + (ab-N) = 0 \), \( D = (a+b)^2 - 4(ab-N) = (a-b)^2 + 4N \).

We should also mention that the statement \( P_D \subseteq \bigcup_{r+4N=D} T_N \) is just the special case \( D' = 1 \) of the general statement that the intersection of two Humbert surfaces \( \mathcal{H}_D \) and \( \mathcal{H}_{D'} \) in the moduli space \( \mathcal{A}_2 = \mathbb{H}_2/\text{Sp}(4, \mathbb{Z}) \) is contained in the union of \( T_N \) with \( DD' = r^2 + 4N \) for some \( r \in \mathbb{Z}, \ N \in \mathbb{N} \). This statement is well known, and is given implicitly in [vdG87, p. 215, proof of Proposition XI.2.8], but since we could not find a convenient reference and since the proof is easy, we give it here. We recall (cf. [vdG87, ch. IX]) that the Humbert surface \( \mathcal{H}_D \) is defined as the image of \( \mathcal{A}_2 \) of the union of the curves

\[
\left\{ \begin{array}{c}
\tau_1 \\
\tau_2 \\
\tau_3
\end{array} \right\} \in \mathbb{H}_2 \bigg| a\tau_1 + b\tau_2 + c\tau_3 + d(\tau_2^2 - \tau_1\tau_3) + e = 0
\]

with \((a, b, c, d, e) \in \mathbb{Z}^5, b^2 - 4ac - 4de = D\). If \( D \) is a fundamental discriminant, then \( \mathcal{H}_D \) is irreducible and hence can be given by any one of the equations in (31). The locus of products of elliptic curves in \( \mathcal{A}_2 \) is \( \mathcal{H}_1 \), because the standard embedding \((\mathbb{H}/\text{SL}_2(\mathbb{Z}))^2 \to \mathbb{H}_2/\text{Sp}_4(\mathbb{Z}) \) is given by the equation \( \tau_2 = 0 \) in \( \mathbb{H}_2 \), which has the form (31) with \((a, b, c, d, e) = (0, 1, 0, 0, 0)\), \( b^2 - 4ac - 4de = 1 \). The Hilbert modular surface \( X_D \) can be identified with \( \mathcal{H}_D \), because if we write \( N(x\omega_1 + y\omega_2) = Ax^2 + Bxy + Cy^2 \) \((A, B, C \in \mathbb{Z}, B^2 - 4AC = D)\), then the map (24) is given by

\[
\left( \begin{array}{c}
\tau_1 \\
\tau_2 \\
\tau_3
\end{array} \right) = \left( \begin{array}{ccc}
\omega_1^2 z_1 + \omega_1^2 z_2 \\
\omega_1^2 z_1 + \omega_1^2 z_2 \\
\omega_2^2 z_1 + \omega_2^2 z_2
\end{array} \right),
\]

which satisfies an equation of the form (31) with \((a, b, c, d, e) = (C, -B, A, 0, 0)\), \( b^2 - 4ac - 4de = D \).

In general, to compute the intersection \( \mathcal{H}_D \cap \mathcal{H}_{D'} \) we substitute the expression in (32) into the an equation of the form (31) with \( b^2 - 4ac - 4de = D' \). This gives the equation

\[-dDz_1z_2 + (aw_1^2 + bw_1\omega_2 + cw_2^2)z_1 + (aw_1^2 + bw_1\omega_2 + cw_2^2)z_2 + e = 0,
\]
Theta derivatives

which has the form (29) with

\[ N = -deD - N(a\omega_1^2 + b\omega_1\omega_2 + c\omega_2^2) = \frac{DD' - (2Aa + Bb + 2Cc)^2}{4} \]

as asserted. In the special case \( D' = 1 \), we recover the equation \( D = r^2 + 4N \) and also see that we are on the component \( F_N((r + \sqrt{D})/2\sqrt{D}) \) of \( F_N \), as claimed in (30), since it is easily seen that \( a\omega_1^2 + b\omega_1\omega_2 + c\omega_2^2 \equiv (r + \sqrt{D})/2 \pmod{d} \).

5.3 Teichmüller curves and Veech groups

A Teichmüller curve is an irreducible algebraic curve \( W \) in the moduli space \( \mathcal{M}_g \) of curves of genus \( g \) which is a totally geodesic submanifold for the Teichmüller metric. Teichmüller curves are generated by a pair consisting of a curve \( C \) and a non-zero holomorphic one-form \( \omega \in H^1(C, \Omega^1_C) \). Such pairs are called flat surfaces. An introductory text to flat surfaces is the survey [Zor06], for example. On the set of flat surfaces there is an action of \( \text{GL}(2, \mathbb{R}) \) and Teichmüller curves are the projection to \( \mathcal{M}_g \) of the orbit \( \text{GL}(2, \mathbb{R}) \cdot (C, \omega) \). The uniformizing group \( \Gamma \) such that \( W = \mathbb{H}/\Gamma \), called a Veech group, can be read off from the flat geometry of the pair \((C, \omega)\). Let \( K_\Gamma \) be the trace field of \( \Gamma \) and \( r = [K_\Gamma : \mathbb{Q}] \). Teichmüller curves with \( r = g \) are called algebraically primitive.

Under the Torelli map, algebraically primitive Teichmüller curves map to the locus of abelian varieties with real multiplication by \( K \) (see [Möl06a, Theorem 2.6]). In particular for \( g = 2 \) the universal covering of an algebraically primitive Teichmüller curve defines a map

\[ (\varphi_0, \varphi) : \mathbb{H} \to \mathbb{H}^2 \]

equivariant with respect to the action of the Veech group (acting on the left in the obvious way and on the right via its embedding into \( \text{SL}(2, \mathbb{R}) \to \text{SL}(2, \mathbb{R})^2 \)). The geodesic definition of Teichmüller curves implies that \( \varphi_0 \) is a Möbius transformation. Moreover we may suppose \( \varphi_0 = \text{id} \) using appropriate choices in the universal covering map. Consequently, Teichmüller curves define Hilbert modular embeddings in the above sense [Möl06a, §3].

The space of flat surfaces \((C, \omega)\) is naturally stratified by the number and multiplicities of the zeros of \( \omega \). In particular, for \( g = 2 \) we have two strata \( \Omega \mathcal{M}_2(1, 1) \) and \( \Omega \mathcal{M}_2(2) \), corresponding to \( \omega \) having two distinct zeros or one double zero, respectively. For \( g = 2 \) we have the following classification for algebraically primitive Teichmüller curves [McM03, McM05, Möl06b, McM06a].

**Theorem 5.2.** There is only one Teichmüller curve in the stratum \( \Omega \mathcal{M}_2(1, 1) \), called the decagon curve. It lies in the Hilbert modular surface \( X_5 \).

The stratum \( \Omega \mathcal{M}_2(2) \) contains infinitely many algebraically primitive Teichmüller curves \( W_D \), each lying in a unique Hilbert modular surface. For each non-square discriminant \( D \geq 5 \) the Hilbert modular surface \( X_D \) contains exactly one Teichmüller curve if \( D \neq 1 \pmod{8} \) and exactly two if \( D \equiv 1 \pmod{8} \).

The union \( W_D \) of the Teichmüller curves in \( X_D \) other than the decagon curve is the locus in \( \mathcal{M}_2 \) of curves whose Jacobians have real multiplication by \( \mathcal{O}_D \) and such that the eigendifferential on which \( \mathcal{O}_D \) acts via the embedding \( K \subset \mathbb{R} \) has a double zero.

The two components in the case \( D \equiv 1 \pmod{8} \) are distinguished by a spin invariant \( \delta \in \{0, 1\} \) and will be denoted by \( W_D^\delta = \mathbb{H}/\Gamma_D^\delta \), so that \( W_D = W_D^0 \cup W_D^1 \) in this case. The definition of the spin invariant is given in [McM05] and will not be repeated here, but in §9.2 we will be able to give a new and equivalent definition in terms of our description of Teichmüller curves via theta functions.
The Teichmüller curves in $\Omega M_2(2)$ admit the following characterization, which is an adaptation of the criterion in [Möl06a, Theorem 5.3]. Let $\mathcal{F}_i$ ($i = 1$ or $2$) be the two natural foliations of a Hilbert modular surface $X_D$ for which the $i$th coordinate is locally constant in the uniformization.

**Theorem 5.3.** An algebraic curve $W \subset X_D$ is a union of Teichmüller curves if and only if:

(i) $W$ is disjoint from the reducible locus; and

(ii) $W$ is everywhere transversal to $\mathcal{F}_1$.

**Sketch of proof.** If $W$ is a Teichmüller curve, then conditions (i) and (ii) hold by definition and by the fact that we can use the first coordinate as a parameter, respectively.

For the converse recall that over a Hilbert modular surface the relative first cohomology with coefficients in $K$ splits into two eigenspaces, two local systems over $K$ that we denote by $L$ and $\tilde{L}$ and that are interchanged by the Galois group of $K$. Consequently, over any curve in a Hilbert modular surface the cohomology splits in the same way.

Condition (i) is equivalent to $W$ being in the image of the locus of Jacobians with real multiplication under the Torelli map. To apply the criterion of [Möl06a, Theorem 5.3], we need to show that the Kodaira–Spencer map for $L$ or $\tilde{L}$ vanishes nowhere on $W$. Condition (ii) implies the non-vanishing of the Kodaira–Spencer map for the corresponding $L$ in the interior of $X_D$, while at the cusps non-vanishing is automatic, by a local calculation as in [BM10b, Proposition 2.2]. □

One can generalize this setup using algebraic curves in $A_9$ that are totally geodesic for the Kobayashi metric. See [MV10] for a characterization of these Kobayashi geodesics.

### 5.4 Twisted modular forms for $W_D$

The topology of $W_D$ and the ratio $\lambda_2$ are completely determined, combining the work of several authors. We summarize the results and combine them with Theorem 3.2 to determine the dimension of the space of twisted modular forms.

**Theorem 5.4.** For any non-square discriminant $D$, the fundamental invariants of the curves $W_D$ are as follows.

(i) The orbifold Euler characteristic of $W_D$ equals

$$\chi(W_D) = -\frac{9}{2} \chi(X_D),$$

where $X_D$ is the Hilbert modular surface $\mathbb{H}^2/\text{SL}(O_D^+ \oplus O_D)$.

(ii) The cusps of $W_D$ are in bijection with standard quadratic forms of discriminant $D$ (see §10).

(iii) For $D = 5$, the curve $W_D$ has two elliptic fixed points, one of order two and one of order five. For $D \neq 5$, there are $e_2(D)$ elliptic fixed points of order two on $W_D$ and no other fixed points, where $e_2(D)$ is a sum of class numbers of imaginary quadratic orders [Muk14, Table 1]. In particular, for $D \equiv 1 \mod 8$, there are $e_2(D) = \frac{1}{2} h(-4D)$ elliptic fixed points of order two.

(iv) The curves $W^0_D$ and $W^1_D$ are defined over $\mathbb{Q}(\sqrt{D})$ and are Galois conjugate.

(v) The curves $W^0_D$ and $W^1_D$ are homeomorphic.

(vi) For a torsion-free subgroup of the Veech group of any component $W^i_D$ of $W_D$ the ratio $\lambda_2$ of the degrees of the line bundles $L$ and $\tilde{L}$ equals $\frac{1}{3}$. 

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Proof. Statement (i) is the main result of [Bai07, Theorem 1.1]. Statement (ii) is implicit in [McM05] and explicit in [Bai07, Theorem 8.7(5)]. Statement (iii) is the main result of [Muk14]. Statement (iv) is [BM10a, Theorem 3.3(b)] and statement (v) follows directly. Statement (vi) was shown in [Bai07, Corollary 12.4], and with a different proof in [BM10a, Corollary 2.4]. □

We recall that the value of $\chi(X_D)$ is known, and is given for a fundamental discriminant $D$ by

$$\chi(X_D) = 2 \zeta_K(-1) = \frac{1}{30} \sum_{D=b^2+4ac} a$$

(34)

(see [Hir73]), and in general by a similar explicit formula.

Every curve of genus two is hyperelliptic and consequently, $-I$ is in the Veech group for every Teichmüller curve in genus two. The dimension of the space of twisted modular forms can now be deduced from Theorem 3.2.

**Theorem 5.5.** For $D > 5$ the space of twisted modular forms $M_{k,\ell}$ on $W_D$ is zero for $k + \ell$ odd. For $k + \ell$ even and $D \not\equiv 1 \mod 8$

$$\dim M_{k,\ell}(\Gamma) = -\frac{1}{2} \left( k + \ell \right) \chi(W_D) - \left\{ \frac{-k + \ell}{4} \right\} e_2(D),$$

where $\{x\}$ is the fractional part of $x$, and for each of the two components for $D \equiv 1 \mod 8$

$$\dim M_{k,\ell}(\Gamma) = -\frac{1}{4} \left( k + \ell \right) \chi(W_D) - \frac{1}{4} \left\{ \frac{-k + \ell}{4} \right\} e_2(D),$$

where $\chi(W_D)$ and $e_2(D)$ is given in Theorem 5.4.

Proof. The first statement holds because the Veech group contains $-I$. Given the general dimension calculation in Theorem 3.2 and the Euler characteristic in (33) it remains to show that for all the fixed points $x$ of order two the local contribution $b_x(k,\ell)$ is $\{(-k + \ell)/4\}$, not $\{(-k - \ell)/4\}$ for some of them.

Suppose that $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(\mathbb{Z})$ is of order 4 and stabilizes $z = (z_1, z_2) \in \mathbb{H}^2$. Then multiplication by the diagonal matrix with diagonal entries $((cz_1 + d)^{-1}, (c^2z_2 + d^2)^{-1})$ defines a linear map $J$ of $\mathbb{C}^2$ that stabilizes the lattice $\mathbb{Z}_z$ from (22) (i.e. the corresponding abelian surface has complex multiplication by the ring generated by $\mathcal{O}_D$ and $J$). To show that $b_x(k,\ell) = \{(-k + \ell)/4\}$ is hence equivalent to showing that $J^{-1}$ (or $J$) has two eigenspaces of dimension one, rather than a two-dimensional eigenspace.

Mukamel [Muk14] studies, along with his classification of fixed points of Teichmüller curves, the locus $\mathcal{M}_2(D_8)$ of genus two surfaces with automorphism group containing the dihedral group of order eight. He shows that all of the fixed points of order two on the Teichmüller curves $W_D$ lie on the intersection (in $A_2$) of the Hilbert modular surface $X_D$ with $\mathcal{M}_2(D_8)$. The family of curves over $\mathcal{M}_2(D_8)$ is given by the hyperelliptic equation

$$Y^2 = (X^2 - 1)(X^2 + aX + 1), \quad a \in \mathbb{C}\setminus\{\pm 2\}.$$

The automorphism of order four is $J(X,Y) = \left(1/X, iY/X^3\right)$ and the eigendifferentials are $dX/Y + XdX/Y$ and $dX/Y - XdX/Y$, which lie in the eigenspace for $+1$ and for $-1$, respectively. This proves the claim on the $J$-eigenspaces. □
5.5 Gauss–Manin connection and Picard–Fuchs equation
Here we explain why Teichmüller curves give rise to twisted modular forms and how to obtain
the differential equations we attached to them in §4 geometrically. For the moment, let \(W\) be
any curve in \(M_2\) such that the corresponding family of Jacobians has real multiplication by an
order in \(K\). Then the vector bundle with fiber \(H^1(C, \mathbb{C})\) over the point \([C] \in M_2\) splits (over \(\mathbb{R},\)
and in fact over \(K\)), as in the proof of Theorem 5.3, into rank-two subbundles \(L\) and \(\bar{L}\). This
vector bundle also comes with a flat (Gauss–Manin) connection \(\nabla\). The bundles \(L\) and \(\bar{L}\) come
with holomorphic subbundles \(L\) and \(\bar{L}\), respectively, whose fibers over \(X\) are the holomorphic
one-forms on \(C\) that are eigenforms for the real multiplication. The bundles \(L\) and \(\bar{L}\) naturally
extend over the cusps \(\mathbb{W}\backslash\mathbb{W}\), where the fibers are stable forms. We denote them by the same
letters. (We recall that a form is called stable if in the limit as \(t \to t_0\), where the genus two curve parametrized by \(t \neq t_0\) degenerates to a curve of genus zero with double points, the corresponding
differential on the normalization of this curve has simple poles with opposite residues at the points
that get identified.)

Suppose for simplicity that \(W\) is a rational curve with parameter \(t\). If we choose sections
\(\omega(t)\) of \(\mathcal{L}\) and \(\tilde{\omega}(t)\) of \(\tilde{\mathcal{L}}\), then \(\{\omega(t), \nabla(\partial/\partial t)\omega(t), \nabla(\partial/\partial t)^2\omega(t)\}\) are linearly dependent in 
cohomology. Concretely, this means that if \(L\) is the corresponding second-order differential linear
operator, a quadratic polynomial in \(\partial/\partial t\), then the image of \(\omega(t)\) under \(L\) is exact. Similarly,
\(\{\tilde{\omega}(t), \nabla(\partial/\partial t)\tilde{\omega}(t), \nabla(\partial/\partial t)^2\tilde{\omega}(t)\}\) are linearly dependent and give a second-order differential operator \(\tilde{L}\) that makes \(\tilde{\omega}(t)\) exact. It follows that the periods, defined as the integral of \(\omega(t)\) and \(\tilde{\omega}(t)\) over any fixed element of \(H_1(C, \mathbb{C})\) are annihilated by \(L\) and \(\tilde{L}\), respectively. (Here
‘fixed’ means that we use the property of being a local system to identify the homology groups
\(H_1(C, \mathbb{C}) = H_1(C_1, \mathbb{Z}) \otimes \mathbb{C}\) with each other locally.) These are the well-known Picard–Fuchs
differential equations satisfied by periods.

Now assume that \(W\) is a Teichmüller curve in \(M_2\) or, more generally, with quadratic trace
field. We show that the periods just described are (twisted) modular forms of weight \((1, 0)\) and
\((0, 1)\), respectively, with respect to a modular embedding \(\varphi\) as defined in §1. More precisely we
have the following correspondence.

**Proposition 5.6.** Suppose that \(W\) is a Teichmüller curve with uniformization \(\mathbb{H}/\Gamma\) as above,
and let \(L\) and \(\tilde{L}\) be the rank-two differential operators associated with sections \(\omega(t)\) and \(\tilde{\omega}(t)\)
of \(\mathcal{L}\) and \(\tilde{\mathcal{L}}\) as above. Then there is a rank-one submodule (in the rank-two \(\mathcal{O}_\Gamma(W)\)-module
of solutions of \(L\)) consisting of holomorphic modular forms of weight \((1, 0)\), and a rank-one
submodule (in the rank-two \(\mathcal{O}_\Gamma(W)\)-module of solutions of \(\tilde{L}\)) consisting of twisted holomorphic
modular forms of weight \((0, 1)\). If \(\omega(t)\) (respectively \(\tilde{\omega}(t)\)) extends to a stable form over a cusp
of \(W\), then the corresponding (twisted) modular form is holomorphic at this cusp.

This defines a 1:1 correspondence between holomorphic sections of \(\mathcal{L}\) over \(\mathbb{W}\) and holomorphic
twisted modular forms on \(\Gamma\) of weight \((1, 0)\), and a 1:1 correspondence between holomorphic
sections of \(\tilde{\mathcal{L}}\) over \(\mathbb{W}\) and holomorphic twisted modular forms on \(\Gamma\) of weight \((0, 1)\).

**Proof.** In [Möller06a] it was shown that there exists an oriented basis \(\beta, \alpha\) of the kernel of \(\tilde{\omega}\)
in \(H_1(C, \mathbb{R})\) such that the monodromy representation of \(\pi_1(W) = \Gamma\) on that subspace is the
identity, and similarly a basis \(\beta^\sigma, \alpha^\sigma\) of the kernel of \(\omega\) with respect to which the monodromy
representation is given by the Galois conjugate group \(\Gamma^\sigma\).

Consequently, the period map \(z \mapsto \int_{\beta} \omega(z)/\int_{\alpha} \omega(z)\) is equivariant with \(\Gamma\) acting on domain
and range, hence the identity after an appropriate conjugation by a Möbius transformation.
Moreover, the period map \(z \mapsto \int_{\beta^\sigma} \tilde{\omega}(z)/\int_{\alpha^\sigma} \tilde{\omega}(z)\) is equivariant with \(\Gamma\) acting on the domain
and $\Gamma^\sigma$ on the range. Hence, this map agrees with $\varphi$ in the definition of the modular embedding by the uniqueness of modular embeddings.

As said above, the periods $f(z) = \int_\alpha \omega(z)$ and $f_1(z) = \int_\beta \omega(z) = z f(z)$ span the space of solutions of $L$ (pulled back to $\mathbb{H}$ via $t$). The statement above about the monodromy implies that for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$

$$
\left( \begin{array}{c} f_1(\gamma z) \\ f(\gamma z) \end{array} \right) = \gamma \left( \begin{array}{c} f_1(z) \\ f(z) \end{array} \right) = \left( \begin{array}{c} af_1(z) + bf(z) \\ cf_1(z) + df(z) \end{array} \right) = \left( \begin{array}{c} (az + b)f(z) \\ (cz + d)f(z) \end{array} \right).
$$

The second row implies that $f$ is a twisted modular form for $\Gamma$ of weight $(1,0)$. Similarly, the periods $\tilde{f}(z) = \int_{\alpha^\sigma} \tilde{\omega}(z)$ and $f_1(z) = \int_{\beta^\sigma} \tilde{\omega}(z) = \varphi(z) \tilde{f}(z)$ span the space of solutions of $\tilde{L}$ and we have

$$
\left( \begin{array}{c} \tilde{f}_1(\gamma z) \\ \tilde{f}(\gamma z) \end{array} \right) = \gamma^\sigma \left( \begin{array}{c} \tilde{f}_1(z) \\ \tilde{f}(z) \end{array} \right) = \left( \begin{array}{c} a^\sigma \tilde{f}_1(z) + b^\sigma \tilde{f}(z) \\ c^\sigma \tilde{f}_1(z) + d^\sigma \tilde{f}(z) \end{array} \right) = \left( \begin{array}{c} (a^\sigma \varphi(z) + b^\sigma) \tilde{f}(z) \\ (c^\sigma \varphi(z) + d^\sigma) \tilde{f}(z) \end{array} \right).
$$

Again, the second row implies that $\tilde{f}$ is a twisted modular form for $\Gamma$ of weight $(0,1)$.

Holomorphicity of $f$ and $\tilde{f}$ in the interior of $\mathbb{H}$ is obvious by the definition of a period. To show that they are holomorphic at the cusps, we may assume without loss of generality that $z_0 = \infty$ and $t_0 = 0$. There, it follows from the definition of the monodromy representation that $\alpha$ (respectively $\alpha^\sigma$) is characterized in $(\tilde{\omega})^\perp$ (respectively in $(\omega)^\perp$) as the elements invariant under the local monodromy group. The period of a stable form along such a cycle is finite.

To establish the last statement of the proposition, we just need to assign to every holomorphic twisted modular form of weight $(1,0)$ (respectively weight $(0,1)$) a section of $L$ (respectively of $\tilde{L}$). This is well known in the untwisted case and was done in both the untwisted and twisted cases in §4 of Part I.

We end with a remark on Galois conjugation and spin. We defined $\Gamma$ to be monodromy group of the local system $L$. Then, of course, the monodromy group of the Galois conjugate $\tilde{L}$ is $\Gamma^\sigma$. We will see in the next section in an example, and at the end of the paper in general, that the solutions $y$ and $\tilde{y}$ also have coefficients in the field $K$. However, the Galois conjugate solution $y^\sigma$ is neither equal to $\tilde{y}$ nor to any other solution of $\tilde{L}$. In fact, $y^\sigma$ is naturally a solution of a differential operator associated with the Galois conjugate Teichmüller curve $W^\sigma$. For $D \equiv 1 \mod 8$ this is the curve with the other spin invariant (see Theorem 5.4(iv)). For $D \not\equiv 1 \mod 8$ this Galois conjugate curve is isomorphic to the original curve by McMullen’s classification recalled in Theorem 5.2. The equation of this curve for $D = 13$ is given explicitly in [BM10a].

6. Example: the curve $W_{17}^1$ and its associated differential equations

Our running example, from now until the end of Part II, will be the Teichmüller curve $W_{17}^1$ on the Hilbert modular surface $X_{17}$. In this section we gather the known results for this curve, computing the Veech group and summarizing the construction from [BM10a] to compute the equation of the universal family and the corresponding Picard–Fuchs differential equations.

6.1 The Veech group for $D = 17$ and spin 1

For small values of $D$ the Veech groups $\Gamma^\delta_D = \Gamma(W^\delta_D)$ can be calculated using the algorithm in [McM03]. This is sufficient for our purposes, but we emphasize that a general algorithm to compute the Veech group for any Veech surface has been developed and implemented by Mukamel [Muk12]. We describe this in detail for the case $D = 17$, $\delta = 1$, i.e. for the Teichmüller...
curve of non-trivial spin. McMullen’s algorithm gives a subgroup of $\text{SL}(2, \mathcal{O}_D)$, so the group that we will get (which we will denote simply by $\Gamma$, or by $\Gamma_{17}$ when needed for clarity, with quotient $W = \mathbb{H}/\Gamma$) is actually the conjugate $\Delta \Gamma_{17} \Delta^{-1}$ of $\Gamma_{17}$ by $\Delta = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{D} \end{pmatrix}$, and for the same reason the function $\varphi$ used to make the modular embedding will go from $\mathbb{H}$ to $\mathbb{H}^{-}$ rather than from $\mathbb{H}$ to $\mathbb{H}$. Later, when we use this modular embedding explicitly to compare the twisted modular forms on $W$ with standard Hilbert modular forms for $\mathbb{Q}(\sqrt{17})$, we will conjugate back to make the comparison easier.

We denote by $\alpha = (1 + \sqrt{17})/2$ the standard generator of $\mathcal{O} = \mathcal{O}_{17}$ over $\mathbb{Z}$, and for ease of reading will sometimes use the abbreviated notation $[m,n]$ for $m + n\alpha \in \mathcal{O}$. The group $\Gamma$ can be embedded into $\text{SL}(2, \mathbb{R})$ by the standard embedding of $\mathcal{O}$ into $\mathbb{R}$ and then acts discretely. (Note that the other embedding of $\mathcal{O}$ into $\mathbb{R}$ would lead to a non-discrete subgroup of $\text{SL}(2, \mathbb{R})$.) A fundamental domain for this action is shown in Figure 1(b), while Figure 1(a) shows the explicit ‘L-shaped region’ needed to apply the algorithm and obtain the fundamental domain. The group $\Gamma$ has three cusps, at $z = \infty$, 1, and $\alpha/2$, and an elliptic fixed point of order two at $z = i$, where $z$ is the coordinate in $\mathbb{H}$. The stabilizers of the cusps are the infinite cyclic groups generated by the parabolic elements

$$M_\infty = \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}, \quad M_1 = \begin{pmatrix} -2\alpha - 2 & 2\alpha + 3 \\ -2\alpha - 3 & 2\alpha + 4 \end{pmatrix}, \quad M_{\alpha/2} = \begin{pmatrix} -2\alpha - 3 & 3\alpha + 4 \\ -2\alpha - 2 & 2\alpha + 5 \end{pmatrix},$$

and the stabilizer of $i$ is generated by the element $M_i = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. The presentation of the group $\Gamma$ given by McMullen’s algorithm is then

$$\Gamma = \langle M_\infty, M_i, M_1, M_{\alpha/2} \mid M_i^2 = -1, M_\infty M_i M_1 M_{\alpha/2} = 1 \rangle.$$

It will be useful in the following to deal not only with the Fuchsian group $\Gamma$ but also with a certain index-four subgroup $\Pi$ of it, already mentioned in §3. This group is more convenient for purposes of calculation because it is free and also because the universal genus two curve over $\mathbb{H}/\Pi$ has a stable model. We want that $\tilde{\Pi} = \Pi \times \{ \pm 1 \}$ has index two in $\Gamma$. (This already implies that $\Pi$ has no torsion and is thus free.) This group is not unique. We fix the choice

$$\Pi = \langle M_\infty, M_{\alpha/2}, M_1^2, -M_1^{-1} M_\infty M_1 \rangle.$$
The group \( \tilde{\Gamma} = \Gamma \times \{ \pm 1 \} \) is the kernel of the homomorphism from \( \Gamma \to \{ \pm 1 \} \) sending \( M_\infty \) and \( M_{0/2} \) to 1 and \( M_i \) and \( M_1 \) to \(-1\).

Both curves \( \mathbb{H}/\Gamma \) and \( \mathbb{H}/\Pi \) have genus zero, so there are modular functions \( s(z) \) and \( t(z) \) on \( \Gamma \) and \( \Pi \) giving isomorphism between their compactifications and \( \mathbb{P}^1(\mathbb{C}) \) (hauptmoduls). We can normalize them so that the involution induced by \( M_i : z \mapsto -1/z \) corresponds to \( t \mapsto 1/t \) and that the three cusps of \( \mathbb{H}/\Pi \) are at 0, 1, and \( \infty \). Then the quotient map \( t \mapsto s \) is given by

\[
s = -\frac{4 \kappa_0 t}{(t-1)^2} \quad \text{where} \quad \kappa_0 = \frac{-895 + 217\sqrt{17}}{256}
\]

and the values of \( s \) and \( t \) at the cusps and elliptic fixed points are given, according to the calculations in [BM10a] (where a different parameter on \( \mathbb{H}/\Pi \) was used), by the table

<table>
<thead>
<tr>
<th></th>
<th>( z )</th>
<th>( \infty )</th>
<th>0</th>
<th>( \alpha/2 )</th>
<th>(-2/\alpha)</th>
<th>( i )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \Pi )</td>
<td>( t = t(z) )</td>
<td>0</td>
<td>( \infty )</td>
<td>1</td>
<td>( \lambda^{-1} )</td>
<td>( \lambda )</td>
</tr>
<tr>
<td>( \Gamma )</td>
<td>( s = s(z) )</td>
<td>0</td>
<td>0</td>
<td>( \infty )</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

where

\[
\lambda = t \left( -\frac{2}{\alpha} \right) = \frac{31 - 7\sqrt{17}}{2}, \quad \kappa_0 = s(i) = -\frac{(\lambda - 1)^2}{4\lambda}.
\]

For later use we emphasize that both \( t \) and \( s \) are local parameters of the Teichmüller curve at the cusp \( z = \infty \). The whole situation is summarized by the following diagram.

\[
\begin{array}{ccc}
W_{\Pi} = \mathbb{H}/\Pi & \xrightarrow{t} & \mathbb{P}^1(\mathbb{C}) \\
2:1 \downarrow & & \downarrow \text{inclusion} \\
W = \mathbb{H}/\Gamma & \xrightarrow{s} & \mathbb{P}^1(\mathbb{C})
\end{array}
\]

### 6.2 The universal family over \( W \)

The modular curve \( W = \mathbb{H}/\Gamma \) parametrizes a family of genus two curves with real multiplication by \( \mathcal{O} = \mathbb{Z} + \mathbb{Z} \alpha \) on their Jacobians. This family, and its associated Picard–Fuchs differential equations, was determined explicitly in [BM10a]. In this subsection we review these results, and also give somewhat simpler equations by making suitable changes of coordinates. We remark that meanwhile equations of more Teichmüller curves have been computed by a different method by Kumar and Mukamel [KM14].

The explicit equation in [BM10a] was actually given for the family over the double cover \( \mathbb{H}/\Pi \) of \( W \), with the parameter \( t \), and has the form

\[
Y^2 = P_0(X,t) = (X + (At + B)) \times (X + (Bt + A)) \\
\times (X^3 + C(t+1)X^2 + (D(t+1)^2 + Et)X + F(t+1)^3 + Gt(t+1)),
\]

with coefficients \( A, \ldots, G \) given (with our above notation \( [m,n] = m + na \)) by

\[
A = 5 [2,1], \quad B = -2 [5,3], \quad C = [3,1], \quad D = -\frac{1}{4} [827,529], \\
E = 2^4 \cdot 17 [3,2], \quad F = -\frac{1}{2} [4597,2943], \quad G = 2 \cdot 17 \cdot [271,173].
\]

(These coefficients are not quite as bad as they look since they all factor into small prime factors, e.g. \( D = -\pi_2^{11}/4\varepsilon^3 \) and \( F = -\pi_2^{14}\varepsilon^4/2 \) where \( \varepsilon = 4 + \sqrt{17} \) is the fundamental unit of \( \mathbb{Q}(\sqrt{17}) \).)
6.3 The Picard–Fuchs equations for \( W \) and their solutions

As already discussed in \( \S 3 \), even though we are considering only the single curve \( W = \mathbb{H}/\Gamma \), there are two Picard–Fuchs differential equations, corresponding to the variation of the periods of the two eigendifferentials \( \omega \) and \( \tilde{\omega} \) for the action of \( \mathcal{O} \) on the space of holomorphic differentials
Theta derivatives

of the fibers. It will be crucial for our calculations to have both of them, since together they will
tell us explicitly how the Teichmüller curve \( W \) is embedded in the Hilbert modular surface \( X_{17} \).

Obtaining the Picard–Fuchs differential equations satisfied by the periods of the two
eigendifferentials \( \omega \) and \( \tilde{\omega} \) is straightforward once the equation of the family of curves has been obtained.
One has to find differential operators \( L \) and \( \tilde{L} \) mapping the one-forms \( \omega \) and \( \tilde{\omega} \) to
exact forms. The result, given in [BM10a], is a pair of differential operators of the same form as
in (20), namely

\[
L = \frac{d}{dt} A(t) \frac{d}{dt} + B(t), \quad \tilde{L} = \frac{d}{dt} \tilde{A}(t) \frac{d}{dt} + \tilde{B}(t),
\]

where \( A(t) \) and \( B(t) \) are the polynomials given by

\[
A(t) = t (t - 1) (t - \lambda)(t - \lambda^{-1}) = t^4 - \beta \lambda^2 + \beta t^2 - t,
\]

\[
B(t) = \frac{3}{4} (3t^2 - (\beta + \gamma) t + \gamma),
\]

with \( \lambda = (31 - 7\sqrt{17})/2 \) as in (37) and \( \beta \) and \( \gamma \) defined by

\[
\beta = \lambda + \lambda^{-1} + 1 = \frac{1087 - 217\sqrt{17}}{64}, \quad \gamma = \frac{27 - 5\sqrt{17}}{4},
\]

and where \( \tilde{A}(t) \) and \( \tilde{B}(t) \) are the rational functions

\[
\tilde{A}(t) = A(t) \left( t^2 + \frac{137 - 95\sqrt{17}}{128} t + 1 \right),
\]

\[
\tilde{B}(t) = \left( \frac{1}{4} t^4 + \frac{1113 - 399\sqrt{17}}{512} t^3 - \frac{260375 - 69633\sqrt{17}}{16384} t^2 - \frac{1387 - 301\sqrt{17}}{128} t + \frac{23 - 5\sqrt{17}}{8} \right) \left( t^2 + \frac{137 - 95\sqrt{17}}{128} t + 1 \right)^2.
\]

The differential operator \( L \) has five singularities, at infinity and at the roots of \( A(t) \). The
differential operator \( \tilde{L} \) has seven singularities, these five and two more at the poles of \( \tilde{A} \), but
these last two are only apparent singularities of the differential equation, i.e. all solutions of the
equation are holomorphic at these points.

The unique solutions in \( 1 + t \mathbb{C}[[t]] \) of the differential equations \( Ly = 0 \) and \( \tilde{L}y = 0 \) can easily
be calculated recursively. The first few terms are given by

\[
y = 1 + \frac{81 - 15\sqrt{17}}{16} t + \frac{4845 - 1155\sqrt{17}}{64} t^2 + \frac{3200225 - 775495\sqrt{17}}{2048} t^3 + \ldots
\]

\[
\approx 1 + 1.197t + 1.294t^2 + 1.356t^3 + 1.402t^4 + 1.439t^5 + \ldots
\]

\[
\tilde{y} = 1 + \frac{23 - 5\sqrt{17}}{8} t + \frac{5561 - 1343\sqrt{17}}{128} t^2 + \frac{452759 - 109793\sqrt{17}}{512} t^3 + \ldots
\]

\[
\approx 1 + 0.2981t + 0.1849t^2 + 0.1384t^3 + 0.1131t^4 + 0.0973t^5 + \ldots.
\]

There are also unique power series \( y_1 \) and \( \tilde{y}_1 \) without constant term such that \( y \log(t) + y_1 \) and
\( \tilde{y} \log(t) + \tilde{y}_1 \) are solutions of the same differential equations as \( y \) and \( \tilde{y} \), respectively. These series begin

\[
y_1 = \frac{439 - 97\sqrt{17}}{64} t + \frac{563089 - 135575\sqrt{17}}{4096} t^2 + \frac{200641639 - 48642353\sqrt{17}}{65536} t^3 + \ldots
\]

\[
\approx 0.6103t + 1.001t^2 + 1.283t^3 + 1.504t^4 + 1.687t^5 + \ldots
\]

\[
\tilde{y}_1 = \frac{1575 - 369\sqrt{17}}{128} t + \frac{1749337 - 423695\sqrt{17}}{8192} t^2 + \frac{1764480419 - 427927381\sqrt{17}}{393216} t^3 + \ldots
\]

\[
\approx 0.4185t + 0.2927t^2 + 0.2305t^3 + 0.1958t^4 + 0.1748t^5 + \ldots.
\]
We have given the numerical values of the first coefficients of each of these four power series to emphasize that they are quite small (and the same is true of the first few hundred, which we have computed). In fact, the coefficients in each case grow like \( \lambda^n \), where \( \lambda = 1.06913 \ldots \) is the number defined by (37), since the radius of convergence is the absolute value of the nearest singularity \( t \neq 0 \), and the singularities are at \( t = 0, \lambda^{-1}, 1, \lambda \) and \( \infty \). The growth in each case is quite regular, with the coefficient of \( t^n \) being asymptotic to a constant times \( \lambda^n / n \). It is perhaps worth mentioning that if we took the Galois conjugates \( L^\sigma \) and \( \tilde{L}^\sigma \) of the differential operators \( L \) and \( \tilde{L} \), which give the Picard–Fuchs equations for the other Teichmüller curve \( W_{17}^1 \) (cf. [BM10a] or §9), then the power series \( y, \ldots, \tilde{y} \) would also be replaced by their Galois conjugates and would look algebraically very similar to those above, but would have completely different real coefficients and growth, e.g. the expansion of \( y^\sigma \) begins

\[
y^\sigma = 1 + \frac{81 + 15\sqrt{7}}{16} t + \frac{4845 + 1155\sqrt{7}}{64} t^2 + \frac{3200225 + 775495\sqrt{7}}{2048} t^3 + \ldots
\]

\[
\approx 1 + 8.928 t + 150.11 t^2 + 3123.9 t^3 + 71667 t^4 + 1738907 t^5 + \ldots,
\]

now with coefficients growing like \( (\lambda^\sigma)^n / n \) with \( \lambda^\sigma = 29.93086 \ldots \). The corresponding Fuchsian group, although isomorphic to \( \Pi \) as an abstract group, is not conjugate to it in \( \text{SL}(2, \mathbb{R}) \), and the quotients of the upper half-plane by these two groups, which are the curves \( W_D^0 \) and \( W_D^1 \), represent different points of the moduli space \( \mathcal{M}_{0,5} \).

Another very striking property of the expansions of \( y \) and \( \tilde{y} \) given above (and then of course also of their conjugates \( y^\sigma \) and \( \tilde{y}^\sigma \)) is that the only denominators one sees are powers of two, i.e. the first few coefficients of these power series all belong to the ring \( \mathcal{O} \left[ \frac{1}{2} \right] \). A calculation to higher accuracy shows that the same holds for the first few hundred coefficients, and in fact it is a theorem, proved in [BM10a], that it holds for all coefficients. We will return to this question at the end of the next section because it is has a very interesting aspect that was in fact the point of departure for our whole investigation.

### 7. Arithmetic properties of modular forms for \( W_{17}^1 \)

With the preparations in the preceding sections we can now compute in §7.1 the modular embedding \( \varphi \) in the example \( D = 17 \). In the process we compute the Fourier expansions of some modular forms and later, in §7.3, we completely determine the ring of twisted modular forms in this specific example. This arithmetic of the coefficients reveals two surprising phenomena, a transcendental constant needed for the correct choice of the \( q \)-parameter and the integrality statement mentioned above and proved in [BM10a], which cannot be explained using one modular \( q \)-variable. We discuss these in §7.2, and provide the explanations in §8.

#### 7.1 Modular parametrization of the differential equations

We have already mentioned that the differential equations \( Ly = 0 \) and \( \tilde{L}\tilde{y} = 0 \) have the same form as the differential equation (20) satisfied by ordinary or twisted modular forms with respect to a hauptmodule. This is of course not a coincidence: we have

\[
y(t(z)) = f(z), \quad \tilde{y}(t(z)) = \tilde{f}(z) \quad \text{for } \Im(z) \text{ large, } |t(z)| \text{ small}
\]

where \( t : \mathbb{H}/\Pi \to \mathbb{C} \) is the map defined in §6.1 and \( f(z) \) and \( \tilde{f}(z) \) are a modular form of weight one and a twisted modular form of weight \( (0,1) \), respectively, on the same group \( \Pi \). In this subsection we will work out this statement in more detail, obtaining in particular a way to calculate the expansion (5) of the function \( \varphi : \mathbb{H} \to \mathbb{H}^- \) whose graph gives the embedding of
### Theta derivatives

\( \mathbb{H}/\Pi \) into \( \mathbb{H} \times \mathbb{H}^-/\text{SL}(2, \mathcal{O}_1) \) as discussed in §1. (Here we need \( \mathbb{H}^- \), rather than \( \mathbb{H} \) as in §1, because we have conjugated the original Veech group by \( \Delta = (1/0, 1/\sqrt{17}) \) to embed it into \( \text{SL}(2, \mathcal{O}_D) \) and the Galois conjugate of \( \Delta \) has negative determinant.) We will also calculate the \( q \)-expansion of \( \tilde{f}(z) \), obtaining our first explicit example of a twisted modular form. In §8 we will use this information to determine completely the rings of twisted modular forms for \( \Gamma \) and \( \Pi \) and the algebraic description of the Teichmüller curve \( W \) inside the Hilbert modular surface \( X_{17} \).

We begin with the functions \( y \) and \( f \). As stated in §6.1, the cusp at infinity for either \( \Gamma \) or \( \Pi \) has width \( \alpha \), i.e. its stabilizer is generated by the transformation \( z \mapsto z + \alpha \), where \( \alpha = (1+\sqrt{17})/2 \) is our standard generator of \( \mathcal{O}_{17} \), so any modular function or modular form on either group can be written as a power series in the variable \( q = e^{2\pi i z/\alpha} \). On the other hand, we know that the space of solutions of the differential equation satisfied by any weight-one modular form \( f(z) \) with respect to any modular function on the same group is spanned by \( f(z) \) and \( z f(z) \). Our first thought is thus that \( q \) coincides with the ‘mirror parameter’

\[
Q = Q(t) = t \exp(y_1/y),
\]

where \( y = y(t) \) and \( y_1 = y_1(t) \) are the two power series in \( t \) defined in §6.3. This is indeed what happens in the case of the Apéry or Apéry-like differential equations (see [Zag09]), at least if one normalizes the hauptmodule correctly. Here, however, it is not quite true. We can see this numerically as follows. The function \( Q(t) \) has a Taylor expansion beginning

\[
Q(t) = t + \frac{439 - 97\sqrt{17}}{64} t^2 + \frac{249125 - 60195\sqrt{17}}{2048} t^3 + \cdots.
\]

We can invert this power series to obtain

\[
t = t(Q) = Q - \frac{439 - 97\sqrt{17}}{64} Q^2 + \frac{103549 - 24971\sqrt{17}}{2048} Q^3 + \cdots
\]

and then substitute this into the expansion of \( f(z) = y(t) \) to express \( f(z) \) as a power series

\[
f(z) = 1 + \frac{81 - 15\sqrt{17}}{16} Q + \frac{8613 - 2019\sqrt{17}}{512} Q^2 + \frac{726937 - 175823\sqrt{17}}{16384} Q^3 + \cdots
\]

\[
\approx 1 + 1.197 Q + 0.563 Q^2 + 0.122 Q^3 + 0.0082 Q^4 - 0.0011 Q^5 - \cdots
\]

in the new local parameter \( Q \) at infinity. Looking at the first few numerical coefficients in this expansion, we see that they seem to be tending to zero rapidly, suggesting that the radius of convergence of this power series is larger than one, which is the value it would have to have if we were expanding with respect to \( q \). The point is that, although the function \( \log Q = \log t + y_1/y \) has the same behavior at infinity as \( 2\pi i z/\alpha \), namely, that it is well defined up to an integer multiple of \( 2\pi i \), this property determines it only up to an additive constant. Therefore, \( q \) and \( Q \) are related by

\[
Q = A q = A e^{2\pi i z/\alpha}
\]

for some constant \( A \neq 0 \) that has no reason to be equal to one. The radius of convergence of the series in (47) is then equal to the absolute value of this constant.

We can use this idea, or a modification of it, to calculate \( A \) numerically. First, by computing a few hundred coefficients of the series in (47) and calculating its radius of convergence by the standard formula \( R = \lim \inf |a_n|^{-1/m} \), where \( a_n \) denotes the \( n \)th coefficient, we find that \( |A| \) is roughly equal to 7.5. However, this direct approach has very poor convergence (because the coefficients of the expansion of \( f(z) \) in \( Q \), unlike those of the same function when written as
gives the numerical value \(\varphi\) where \(z\) is related to the variable \(t\). From the data in the table (36), we know that the value of the scaling constant \(A\) of the uniformizing map \(H \mapsto z\) and we will see later that this guessed value is indeed the correct one.

To find the actual value to high precision, we apply a simple trick. From the data in the table (36), we know that the value of \(1/(t(z) + 1)\) has a simple pole at \(z = i\) and that if we express this function as a power series in \(q\) (respectively \(Q\)), then its singularity nearest the origin is a simple pole at \(q_0 = e^{-2\pi/\alpha}\) (respectively \(Q_0 = Aq_0\)). In other words, \(1/(1 + t)\) is the sum of \(c/(1 − Q/Q_0)\) for some non-zero constant \(c\) and a function holomorphic in a disk of radius strictly larger than \(|Q_0|\). This implies that if we expand \(1/(1 + t) = \sum b_nQ^n\), then the coefficients \(b_n\) are given by \(b_n = cQ_0^n(1 + O(a^{-n}))\) for some \(a > 1\), and hence that the quotients \(b_{n+1}/b_n\) tend to \(Q_0\) with exponential rapidity. Calculating a few hundred of the coefficients \(b_n\) numerically, we find from this the value

\[
A \approx −7.48370872991173536914114556623211
\]

to very high precision. After some trial and error we can recognize this number ‘in closed form’ as

\[
A \approx −2(3 + \sqrt{17})\left(\frac{5 - \sqrt{17}}{2}\right)^{(\sqrt{17} − 1)/4},
\]

and we will see later that this guessed value is indeed the correct one.

Equations (46)–(49) now give as many terms as desired of the \(q\)-expansions of the modular function \(t(z)\) and modular form \(f(z)\). We can (and of course did) then use this to check the correctness of these equations numerically to high accuracy by verifying the invariance of \(t(z)\) and the invariance of \(f(z)\) up to an automorphy factor \(cz + d\), under modular transformations \(z \mapsto (az + b)/(cz + d)\) in the group \(\Pi\). Similarly, by inverting (45) we can also give the inverse of the uniformizing map \(\mathbb{H} \to W_{\Pi}\) explicitly as

\[
z(t) = \frac{\alpha}{2\pi i} \log \frac{Q}{A} = \frac{\alpha}{2\pi i} \left(\log t + \frac{y_1(t)}{y(t)} − \log A\right)
= \frac{\alpha}{2\pi i} \left(\log \frac{t}{A} + \frac{439 − 97\sqrt{17}}{64} t + \frac{321913 − 77807\sqrt{17}}{4096} t^2 + \cdots\right).
\]

Exactly the same considerations apply to the second differential operator \(\tilde{L}\), with the difference that here the mirror parameter

\[
\tilde{Q} = t e^{\tilde{y}_1/\tilde{y}} = t + \frac{1575 − 369\sqrt{17}}{128} t^2 + \frac{4814915 − 1166773\sqrt{17}}{16384} t^3 + \cdots
\]

is related to the variable \(z\) in the upper half-plane by

\[
\tilde{Q} = \tilde{A} \tilde{q} \quad \text{with} \quad \tilde{q} = e^{2\pi i \varphi(z)/\alpha},
\]

where \(\varphi : \mathbb{H} \to \mathbb{H}^−\) is the twisting map and \(\tilde{A}\) is some constant. A calculation like the one for \(A\) gives the numerical value

\[
\tilde{A} \approx −40.9565407890298922716044572957685,
\]
Theta derivatives

which we can recognize as the ‘conjugate-in-the-exponent’ of the value in (49):

\[ \bar{A} = -2 (3 + \sqrt{17}) \left( \frac{5 - \sqrt{17}}{2} \right)^{(-\sqrt{17} - 1)/4}. \]  

(51)

We will show later that also this formula is indeed correct.

We can now calculate the Fourier expansions of both the twisted modular form \( \tilde{f}(z) = y_1(t) \) and the twisting map \( \varphi : \mathbb{H} \to \mathbb{H}^{-} \) as

\[ \tilde{f}(z) = 1 + \frac{23 - \sqrt{17}}{8} A q + \frac{1951 - 473 \sqrt{17}}{256} A^2 q^2 + \frac{184453 - 44739 \sqrt{17}}{8192} A^3 q^3 + \cdots \]

and

\[
\varphi(z) = \frac{a^\sigma}{2\pi i} \log \frac{\bar{Q}}{A} = \frac{a^\sigma}{2\pi i} \left( \log t + \frac{\bar{y}_1(t)}{\bar{y}(t) - \log A} \right) \\
= \frac{a^\sigma}{2\pi i} \left( \log \frac{t}{A} + \frac{1575 - 369 \sqrt{17}}{128} t + \frac{1208617 - 292799 \sqrt{17}}{8192} t^2 + \cdots \right) \\
= -9 + \sqrt{17} z + \frac{1 - \sqrt{17}}{4\pi i} A q + \frac{29767 + 7249 \sqrt{17}}{8192} A^2 q^2 + \frac{3091637 - 749587 \sqrt{17}}{393216} A^3 q^3 + \cdots \].

(52)

Again these Fourier expansions, unlike the expansions of the same functions as power series in \( t(z) \), converge exponentially rapidly for all \( z \) in the upper half-plane and can be used to compute the functions \( \tilde{f}(z) \) and \( \varphi(z) \) numerically and to verify the modular transformation properties (1) and \( \tilde{f}((az + b)/(cz + d)) = (c^\sigma \varphi(z) + d^\sigma) \tilde{f}(z) \) numerically to high accuracy, giving us the first explicit example of a non-classical twisted modular form on a Teichmüller curve.

7.2 Modularity and integrality

At the end of §6.3 we stated that all of the coefficients of the expansions of \( y \) and \( \bar{y} \) as power series in \( t \) belong to the ring \( \mathcal{O} \left( \frac{1}{2} \right) \). This integrality has a rather puzzling aspect, which we discuss here and resolve in §8.3.

If we write \( y = \sum c_n t^n \), then the differential equation \( L y = 0 \) translates into the recursion

\[ (n+1)^2 c_{n+1} = (\beta(n^2 + n) + \frac{3}{4} \gamma) c_n - (\beta(n^2 - \frac{1}{4}) + \frac{3}{4} \gamma) c_{n-1} + (n - \frac{1}{2})^2 c_{n-2} \]

for the coefficients \( c_n \), where \( \beta \) and \( \gamma \) are given by (42). The integrality (away from two) of the \( c_n \) is far from automatic from this recursion, because at each stage one has to divide a linear combination of previous coefficients by \( (n+1)^2 \), so that \( a \text{ priori} \) one would only expect \( n!^2 c_n \) to be \( 2 \)-integral. Divisibility properties of this type are familiar from well-known recursions such as the recursion

\[ (n+1)^2 A_{n+1} = (11n^2 + n + 3) A_n - n^2 A_{n-1} \]  

(53)

used by Apéry in his famous proof of the irrationality of \( \zeta(2) \), or the similar one he used in his even more famous proof of the irrationality of \( \zeta(3) \). However, they are extremely rare. For instance, in [Zag09] it was found that of the first 100 000 000 members of the three-parameter family of recursions obtained by varying the coefficients ‘11’, ‘3’, and ‘-1’ in (53), only seven (if one excluded certain degenerate families, and up to scaling) had integral solutions.
Apéry proved the integrality of the solution of his recursion (53) by giving the explicit closed formula \( A_n = \sum_{k=0}^n \left( \frac{1}{3} \right)^k \binom{n+k}{k} \). We do not know a corresponding expression in our case. However, soon after Apéry’s original proof, a more conceptual explanation was found by Beukers [Beu87], who saw that the differential equation corresponding to Apéry’s recursion has a modular parametrization \( y = f(z) \in M_1(\Gamma) \), \( t = t(z) \in M_0^{\text{per}}(\Gamma) \) of the type discussed in § 4, the group \( \Gamma \) in this case being \( \Gamma_1(5) \), and this implies the integrality because we have \( f(z) \in \mathbb{Z}[[q]] \), \( t(z) \in q + q^2 \mathbb{Z}[[q]] \) and hence \( y \in \mathbb{Z}[[t]] \). Similar statements hold for all seven of the ‘Apéry-like’ equations mentioned above, leading to the conjecture (which was made explicitly in [Zag09]) that the integrality property for recursions of this type occurs precisely when the corresponding differential equation has a modular parametrization.

The surprise is now this. In our case, just as in the seven ‘Apéry-like’ ones, the differential equation (at least for \( y \)) is modular, and \textit{does} have power series solutions with integral coefficients (away from two), but now the modularity does not explain the integrality in the same way as above, because here the relevant Fuchsian group is not arithmetic and the \( q \)-expansions of \( t \) and \( y \) are not integral. Indeed, as we saw in § 7.1, the coefficients of these \( q \)-expansions are not even algebraic numbers, since they involve powers of the scaling constant \( A \), which according to (49) and Gelfond’s theorem is a transcendental number. But even if we rescale by replacing \( q \) by \( Q = Aq \), then, although the first few coefficients as listed in equations (46) and (47) have denominators that are powers of two, this property fails if we compute more coefficients. For example, the coefficient of \( Q^{11} \) in \( f(z) \) equals

\[
\frac{16063132006911958155776129 - 3895881761337356780171815\sqrt{17}}{2^{53} \cdot 3^3 \cdot 5 \cdot 7},
\]

and calculating further we find that the first 100 coefficients contain in their denominators all primes less than 100 that do not split in \( \mathbb{Q}(\sqrt{17}) \), and similarly for \( t(z) \). Thus, although our differential equation (40) does not actually contradict the hypothetical statement

‘integrality occurs only when the differential equation is modular’

mentioned above, the mechanism

\[
y(t) \text{ modular } \Rightarrow y \text{ and } t \text{ both have integral } q\text{-expansions} \\
\Rightarrow y \text{ has an integral } t\text{-expansion}
\]

which previously explained that statement now breaks down completely. This puzzle, which was in fact the original motivation for the investigation described in this paper, will be solved in § 8, where we will provide a purely modular explanation of the integrality property by expanding \( y \) and \( \tilde{y} \) with respect to both \( q \) and \( \tilde{q} \), using Hilbert modular forms rather than modular forms in one variable.

7.3 The ring of twisted modular forms for \( W \) and \( W_\Pi \)

We can now calculate the rings \( M_{*,\psi}(\Gamma, \phi) \) and \( M_{*,\psi}(\Pi, \phi) \) of twisted modular forms on the Teichmüller curve \( W = \mathbb{H}/\Gamma \) and its double cover \( W_\Pi = \mathbb{H}/\Pi \). This information will be used in the following section to embed the curve \( W_{17} \) into the Hilbert modular surface \( X_{17} \).

We already know two twisted modular forms on \( \Pi \), namely \( f(z) = y(t(z)) \) in \( M_{1,0}(\Pi) \) and \( \tilde{f}(z) = \tilde{y}(t(z)) \) in \( M_{0,1}(\Pi) \). (From now on we omit the ‘\( \phi \)’.) Any holomorphic or meromorphic twisted modular form of weight \( (k, \ell) \) on \( \Pi \) is then equal to \( f^k \tilde{f}^\ell \) times a rational function of \( t = t(z) \). The next proposition tells us which ones are holomorphic.
Theta derivatives

Proposition 7.1. For $k, \ell \geq 0$ the vector space of twisted modular forms of weight $(k, \ell)$ on $\Pi$ is given by

\[ M_{(k, \ell)}(\Pi) = \left\langle f(z)^k \tilde{f}(z)^\ell t(z)^c \mid 3k + \ell \geq 2c \geq 0 \right\rangle_{\zeta}. \]  

(54)

Proof. The group $\Pi$ has no elliptic fixed points and five cusps, one of which is irregular, so Theorem 5.5 implies the dimension formula

\[ \dim M_{(k, \ell)}(\Pi) = 1 + \left\lfloor \frac{3k + \ell}{2} \right\rfloor. \]  

(55)

Since the right-hand side of this equals the number of monomials $f^k \tilde{f}^\ell t^c$ in (54), it suffices to prove that each of these monomials is holomorphic or equivalent, that $f(z)$ and $\tilde{f}(z)$ are holomorphic everywhere (including at the cusps) and vanish to orders $\frac{1}{2}$ and $\frac{3}{2}$, respectively, at the cusp $t = \infty$, where the order is measured with respect to the local parameter $1/t$. The holomorphy at the cusps is a special case of Proposition 5.6, since the construction of the defining (38) of $W$ given in § 6.2 was based on choosing the coefficients in such a way as to make the two differential forms $\omega = dX/Y$ and $\tilde{\omega} = X dX/Y$ stable at all of the cusps of $W$. We therefore only need to check that the order of vanishing of $f$ and $\tilde{f}$ at $z = 0$ (corresponding to $t = \infty$) are at most, and hence exactly, equal to $\frac{3}{2}$ and $\frac{1}{2}$, respectively. We will give two arguments to see this.

The first way is to use the action of the element $S = M_1$ of $\Gamma$, which corresponds to the involution $t \mapsto 1/t$ on the base of the family (38). This involution extends via $X \mapsto X/t$ and $Y \mapsto Y/t^{5/2}$ to an involution $\iota$ of the whole family, with $\iota^* \omega(t) = t^{3/2} \omega(t)$ and $\iota^* \tilde{\omega}(t) = t^{1/2} \tilde{\omega}(t)$. Near $t = 0$, the section $f$ is the period of $\omega$ along the unique cycle (up to scale) $\beta_0$ that is orthogonal to $\tilde{\omega}(t)$ and extends across $t = 0$. Near $t = \infty$, it is the period of $\omega$ along the unique cycle (up to scale) $\beta_\infty$ that is orthogonal to $\tilde{\omega}(t)$ and extends across $t = \infty$. From these defining properties it follows that $\iota^* \beta_0$ is proportional to $\beta_\infty$. The same argument applies for $\tilde{f}$. In modular terms, this translates into the statement that the function $f(z)$ transforms via

\[ \frac{1}{z} f\left(-\frac{1}{z}\right) = f(z) t(z)^{3/2} \]  

(56)

for some appropriate choice of the square root of $t(z)^{3/2}$ (which we need only make at one point since this function has no zeros or poles in $\mathbb{H}$), and similarly

\[ \frac{1}{\varphi(z)} \tilde{f}\left(-\frac{1}{z}\right) = \tilde{f}(z) t(z)^{1/2}. \]  

(57)

Equations (56) and (57) clearly imply the statement that the modular forms $f^2 t^2$ and $\tilde{f}^2 t$ at $\infty$ are holomorphic everywhere, as claimed.

The other approach, not using the accidental fact that the cusps 0 and $\infty$ of $\mathbb{H}/\Pi$ happen to be interchanged by an element in the normalizer of $\Pi$ and therefore applicable in other situations, is based on the equation

\[ \frac{\alpha}{2\pi i} t'(z) = f(z)^2 A(t(z)), \]  

(58)

where $A(t)$ is the fourth-degree polynomial given in (41), see (19). Since the polynomial $A(t)$ is divisible by $t$, we find that $f(z)^2$ multiplied by a cubic polynomial in $t(z)$ is equal to the logarithmic derivative of the modular function $t(z)$, and hence is holomorphic at the cusp $t = \infty$, so the order of $f$ at $t = \infty$ is $\leq \frac{3}{2}$, as desired. A similar argument, this time using (21), applies also to $\tilde{f}$, with the polynomial $A(t)$ replaced by the rational function $\tilde{A}(t)$, which grows like $t^2$ as $t \to \infty$.  

\[ \square \]
We consider the cusp degenerations of the differentials \( \omega \) de\ngebenerations of the differentials \( F \) (twisted or not) of even weight \( k \) on \( \Pi \). But the subgroup \( \Pi^\prime \) of \( \Pi \) is not normal and is not normalized by \( S \), so the intersection of \( \Pi \) with \( \Pi^\prime = S\Pi S^{-1} \) is a proper subgroup (of index two) in \( \Pi \). The space of holomorphic modular forms of weight one on this group is four-dimensional, spanned by the functions \( ft^{j/2} \) with 0 \( \leq j \leq 3 \), with the spaces of modular forms of weight one on \( \Pi \) and \( \Pi^\prime \) separately being spanned by \( (f, ft) \) and \( (ft^{1/2}, ft^{3/2}) \), respectively. Note that the group \( \Pi^\prime \) does not contain the stabilizer \( M_{\infty} \) of \( \infty \) in \( \Pi \) (or \( \Pi^\prime \)), but only its square, so that it has width two and hence a local uniformizer \( q^{1/2} \) at \( \infty \). This group has genus zero, with \( t(z)^{1/2} \) as a hauptmodule.

(ii) In the above proof we gave an implicit estimate of the period integrals that define \( f \) and \( f \) in the neighborhood of any cusp. These periods are given by integrating \( \omega = dX/Y \) and \( \tilde{\omega} = XdX/Y \) over a linear combination of paths that are invariant under the local monodromy around the given cusp. It is perhaps worthwhile giving a more explicit proof in the special case at hand, since this makes the argument clearer and also shows how to find the full expansion, and not just the order of vanishing, of \( f \) and \( f \) at every cusp. We will just give the main formulas, without complete details. At each cusp we choose a local parameter \( \varepsilon = \varepsilon_j \). Then we can find the expansions of \( y(t) = f(z) \) and \( \tilde{y}(t) = \tilde{f}(z) \) near \( t = t_j \) by looking at the explicit form of the degenerations of the differentials \( \omega \) and \( \tilde{\omega} \) there, as explained at the end of the discussion in §6.2. We consider the cusp \( t = \infty \) here and the other cusps even more briefly in remark (iii) below. Near \( t = \infty \) we make the substitutions \((X, t) = ((cT^2 - A)/\varepsilon, 1/\varepsilon)\), with \( c = (17 - 3\sqrt{17})/2 \) and \( A = 5(5 + \sqrt{17})/2 \) as in (39), and where \( \varepsilon \) tends to zero. Then by direct computation we find

\[
P_b(X, t) = (c/\varepsilon)^5[TP^2 - \lambda_2^2T^2 - \lambda_1^2] + O(\varepsilon^{-4}),
\]

with \( \lambda_1 = (5 + \sqrt{17})/2 \) and \( \lambda_2 = (3 + \sqrt{17})/4 \). The fact that the leading coefficient of the right-hand side as a Laurent series in \( \varepsilon \) is a square corresponds to the degeneration of the fiber over the cusp to a rational curve, and lets us compute the differential form \( \omega = dX/\sqrt{P_b(X, T)} \) as

\[
\omega = 2\varepsilon^{3/2}c^{3/2} \left[ \frac{1}{(T^2 - \lambda_1^2)(T^2 - \lambda_2^2)} \right] dT,
\]

where the omitted terms contain higher powers of \( \varepsilon \) with coefficients that are rational functions of \( T \) having poles only at \( \pm \lambda_1, \pm \lambda_2 \) that can easily be found explicitly with a suitable mathematical software program. The factor \( \varepsilon^{3/2} \) gives the vanishing order we claimed, and the rest of the expansion gives us the complete expansion of \( f(z) \) near \( z = 0 \). Specifically, the homology of \( \mathbb{P}^1\setminus\{\pm \lambda_1, \pm \lambda_2, \infty\} \) is spanned by the four small loops \( \gamma_i \) around the four poles \( \pm \lambda_1 \) and \( \pm \lambda_2 \), and the integral of the above form around each such loop is given simply by the residue of the form at that pole, so that we can easily calculate the periods around each \( \gamma_i \) to any order in \( \varepsilon \). When one does this calculation, one finds that these integrals are given, up to a constant, by

\[
\int_{\gamma_1, \gamma_2, \gamma_3, \gamma_4} \omega(\varepsilon) = \left( 1, -1, \frac{1 + \sqrt{17}}{2}, -\frac{1 + \sqrt{17}}{2} \right) \left( 1 + \frac{81 - 15\sqrt{17}}{16} + \frac{4845 - 1155\sqrt{17}}{64} \varepsilon^2 + \ldots \right).
\]

The surprising observation that they are all proportional is explained by the fact the loops \( \gamma_i \) correspond to the core curves (in both directions) of parallel cylinders in the generating flat surface, here concretely the vertical cylinders in Figure 1(a). More precisely, these curves stay parallel in a neighborhood of the cusp of the Teichmüller curve by definition of these curves, and hence the periods remain proportional. Another striking property of the above expansion,
namely the integrality (up to powers of two) of its coefficients as a power series in the local parameter \( \varepsilon = 1/t \) at the cusp \( t = \infty \), is obvious both from the proof in [BM10a] and from the one that we will give in § 8.

(iii) We now also indicate briefly how to find the Fourier expansions of \( f \) and \( \tilde{f} \) at a cusp \( t_j \) other than \( t = 0 \). To define them, we must first choose a matrix \( M_j \) mapping the point \( z_j \) with \( t(z_j) = t_j \to \infty \). For definiteness’s sake we choose \( M_1 = \text{Id} \) and \( M_j = \begin{pmatrix} 0 & -1 \\ 1 & z_j \end{pmatrix} \) for \( j \neq 1 \). The \textit{width} \( w_j \) of the cusp is defined as the smallest positive number \( w \) with \( M_j^{-1}(0, 1) M_j \subset \Gamma \), and we define \( q_j = e^{2\pi i M_j(z)/w_j} \) and \( Q_j = A_j q_j \), where \( A_j \) is chosen so that \( Q_j = \varepsilon + O(\varepsilon^2) \) as \( \varepsilon \to 0 \). Note that both the width \( w_j \) and the value of the scaling constant \( A_j \) depend on our choices of \( \varepsilon_j \) and \( M_j \); with the choices given above, they are given as in Table 1 for representatives of \( \Gamma \)-equivalence classes of cusps. At each cusp we expand the integral as a power series in the local parameter, observing that each coefficient is the integral of a rational function on a punctured Riemann sphere, and proceed just as we did above for \( t \to \infty \). We omit the calculations and give only the results (for \( f \); those for \( \tilde{f} \) can be obtained in the same way):

\[
\begin{align*}
\left. f \right|_1 M_2 &= \frac{1}{8} (3 + \sqrt{17})^{5/2} (4 + \sqrt{17})^{1/2} \left( 1 - \frac{3}{4} Q_3 + \frac{3807 + 915 \sqrt{17}}{128} Q_3^2 + \cdots \right), \\
\left. f \right|_1 M_3 &= \frac{-i}{2^{29/2}} (3 + \sqrt{17})^{11/4} (4 + \sqrt{17})^{-1/4} \left( 1 + \frac{-255 + 1959 \sqrt{17}}{1024} Q_4 + \cdots \right).
\end{align*}
\]

This concludes our discussion of the modular forms and twisted modular forms on the group \( \Pi \). Proposition 7.1 also allows us to describe the twisted modular forms on the Teichmüller curve itself, i.e. for the group \( \Gamma \). Since this group contains \( -I \), there are no twisted modular forms of weight \((k, \ell)\) for \( k + \ell \) odd. Theorem 3.2 and the arguments in the preceding proof immediately imply the following statement.

**Proposition 7.2.** The ring of weight modular forms of even total weight \( k + \ell \) for \( \Gamma \) has the graded dimensions

\[
\dim M_{k, \ell}(\Gamma) = 1 + \left\lfloor \frac{3k + \ell}{4} \right\rfloor.
\]

It consists of the invariants of \( M_{k, \ell}(\Pi) \) as given in (54) under the involution

\[
t \mapsto t^{-1}, \quad f \mapsto ft^{3/2}, \quad \tilde{f} \mapsto \tilde{f}t^{1/2}.
\]

As a corollary, we see that the ring of \textit{parallel weight} twisted modular forms on \( \Gamma \) is freely generated by the two forms

\[
\xi = (1 - t)^2 f \tilde{f}, \quad \eta = \frac{45 - 11 \sqrt{17}}{8} t f \tilde{f} = -\frac{19 + 5 \sqrt{17}}{4} s \xi,
\]

\[\text{(59)}\]
with \( s \) as in (35). (The numerical factors in the definition of \( \eta \) have been included for later convenience.) In particular, when we embed the Teichmüller curve into the Hilbert modular surface, then the restriction of a Hilbert modular form of weight \((k,k)\) is a homogeneous polynomial of degree \( k \) in \( \xi \) and \( \eta \), or equivalently equals \( \xi^k \) times a polynomial in \( s \) of degree at most \( k \). We will use this in the next section to find an explicit description of this embedding.

8. The Hilbert modular embedding of \( W_{17}^1 \)

We continue to study the example of the special Teichmüller curve \( W = W_{17}^1 \), using the information given in the previous section to give a complete description of the embedding of \( W \) into the Hilbert modular surface \( X_{17} \).

8.1 Modular embedding via Eisenstein series

We can use our explicit knowledge of the twisting map \( \varphi : \mathbb{H} \to \mathbb{H}^- \) and of the twisted modular forms on \( \Pi \) and on \( \Gamma \) to embed \( \mathbb{H}/\Pi \) and \( \mathbb{H}/\Gamma \) into \( X_{17} \) by expressing the restrictions of Hilbert modular forms as polynomials in \( f, \tilde{f}, \) and \( t \) (or simply in \( \xi \) and \( \eta \) if we restrict our attention to forms of parallel weight). We begin by recalling the main points of the theory of Hilbert modular forms, using \( \text{SL}(2, \mathbb{O}) \) rather than \( \text{SL}(\mathbb{O}^\vee \oplus \mathbb{O}) \) since it is more familiar, and then later transform our results back to \( \text{SL}(\mathbb{O}^\vee \oplus \mathbb{O}) \) for \( D = 17 \), using the isomorphism of the two groups in this case.

A textbook reference for the notions and claims in this section is e.g. [vdG87].

A Hilbert modular form of weight \((k, \ell)\) on the full modular group \( \text{SL}(2, \mathbb{O}_D) \) is a holomorphic function \( F : \mathbb{H}^2 \to \mathbb{C} \) satisfying the transformation law

\[
F(\gamma z_1, \gamma^T z_2) = (cz_1 + d)^k (e^a z_2 + d^a)^\ell F(z_1, z_2)
\]

for \( g = (a \ b \\
\ c \ d) \in \text{SL}(2, \mathbb{O}_D) \). If \( k = \ell \) (‘parallel weight’), we call the weight simply \( k \). We denote the space of such forms by \( M_k,\ell(\text{SL}(2, \mathbb{O}_D)) \), or simply \( M_k(\text{SL}(2, \mathbb{O}_D)) \) if \( k = \ell \), and more generally write \( M_{k,\ell}(\Gamma, \chi) \) for the corresponding space for forms with respect to a subgroup \( \Gamma \) of \( \text{SL}(2, \mathbb{O}_D) \) and character \( \chi : \text{SL}(2, \mathbb{O}) \to \mathbb{C}^* \). On \( M_k(\text{SL}(2, \mathbb{O}_D)) \) we have the involution induced by the symmetry \( \iota : (z_1, z_2) \mapsto (z_2, z_1) \) of \( \mathbb{H}^2 \), so we can split this space into the direct sum of the subspaces \( M_k^\pm(\text{SL}(2, \mathbb{O}_D)) \) of symmetric and antisymmetric Hilbert modular forms.

A Hilbert modular form \( F \) on \( \text{SL}(2, \mathbb{O}_D) \) has a Fourier expansion of the form

\[
F(z) = F(z_1, z_2) = b_0 + \sum_{\nu \in \mathbb{O}_D^\vee, \nu \neq 0} b_\nu \ e(\text{tr}(\nu z))
\]

where \( \mathbb{O}_D^\vee \) is the inverse discriminant and \( \text{tr}(\nu z) \) for \( z = (z_1, z_2) \in \mathbb{H}^2 \) means \( \nu z_1 + \nu^T z_2 \). If we choose a \( \mathbb{Z}_D \)-basis for \( \mathbb{O}_D^\vee \) and write \( X \) and \( Y \) for the corresponding exponential functions \( e(\text{tr}(\nu z)) \), then the right-hand side of (60) becomes a Laurent series in \( X \) and \( Y \) (or even a power series if the basis is chosen appropriately). In practice, it is sometimes more convenient to choose only a \( \mathbb{Q} \)-basis for \( \mathbb{O}_D^\vee \), in which case we work with power series with congruence conditions on the exponents of \( X \) and \( Y \). The simplest choice is

\[
X = X(z) = e \left( \frac{z_1 + z_2}{2} \right), \quad Y = Y(z) = e \left( \frac{z_1 - z_2}{2\sqrt{D}} \right),
\]

in which case \( \{e(\text{tr}(\nu z))\} = \{X^m Y^n \mid m \equiv n \pmod{2}\} \) and (60) becomes

\[
F(z) = \sum_{m=0}^{\infty} \left( \sum_{\substack{n \leq |m| \sqrt{D} \\
\ n \equiv m \pmod{2}} b_{m,n} Y^n \right) X^m \in \mathbb{C}[Y, Y^{-1}][[X]].
\]

For Hilbert modular forms the same remarks as in §5.1 for Hilbert modular surfaces apply concerning the (less standard) Hilbert modular groups \( \text{SL}(\mathbb{O}^\vee \oplus \mathbb{O}) \) or functions on \( \mathbb{H} \times \mathbb{H}^- \),
The only difference in the latter case being that the condition $\nu \gg 0$ in (60) must be replaced by $\nu > 0 > \nu^\sigma$.

The simplest Hilbert modular forms to construct are the Eisenstein series of weight $(k, k) (k = 2, 4, 6, \ldots)$, with Fourier expansions given by

$$\mathcal{E}_k^D(z) = \frac{\zeta_k(1-k)}{4} + \sum_{\nu \in \mathcal{O}_D^\vee, \nu > 0} \sigma_{k-1}(\nu \sqrt{D}) e(\operatorname{tr}(\nu z)).$$

(63)

Here $\zeta_k(s)$ denotes the Dedekind zeta function of $K = \mathbb{Q}(\sqrt{D})$ and $\sigma_{k-1}(\nu \sqrt{D})$ for $\nu \in \mathcal{O}_D^\vee$ with $\nu \gg 0$ is given by

$$\sigma_{k-1}(\nu \sqrt{D}) := \sum_{b | \nu \sqrt{D}} N(b)^{k-1} = \sum_{d | \nu \sqrt{D}} d^{k-1} \sigma_{k-1} \left( \frac{D \nu / d^2}{d} \right),$$

where the first sum runs over integral ideals $b$ of $K$ dividing the integral ideal $\nu \sqrt{D}$ and the second sum (whose equality with the first is proved in [Zag76, Lemma on p. 66]) runs over positive integers $d$ such that $d^{-1} \nu \in \mathcal{O}_D^\vee$, and where $\sigma_{k-1}(m)$ for $m \in \mathbb{N}$ has its usual meaning as the sum of the $(k-1)$st powers of the (positive) divisors of $m$. In particular, for $D = 17$ the first three Eisenstein series $\mathcal{E}_k = \mathcal{E}_k^{17}$ have Fourier expansions beginning

$$\mathcal{E}_2(z) = \frac{1}{12} + (3Y^3 + 7Y + 7Y^{-1} + 3Y^{-3})X + (Y^8 + 21Y^6 + 14Y^4$$
$$+ 45Y^2 + 18 + 45Y^{-2} + 14Y^{-4} + 21Y^{-6} + Y^{-8})X^2 + \cdots,$n
$$\mathcal{E}_4(z) = \frac{41}{120} + (9Y^3 + 73Y + 73Y^{-1} + 9Y^{-3})X + (Y^8 + 657Y^6$$
$$+ 2198Y^4 + 5265Y^2 + 4914 + 5265Y^{-2} + \cdots + Y^{-8})X^2 + \cdots,$n
$$\mathcal{E}_6(z) = \frac{5701}{252} + (33Y^3 + 1057Y + 1057Y^{-1} + 33Y^{-3})X + (Y^8 + 34881Y^6$$
$$+ 371294Y^4 + 1116225Y^2 + 1419858 + \cdots + Y^{-8})X^2 + \cdots,$n

with $X$ and $Y$ as in (61). As a check, we can verify that if we set $X = q$ and $Y = 1$, corresponding to the natural embedding $z \mapsto (z, z)$ of $\mathbb{H}/\SL(2, \mathbb{Z})$ into $\mathbb{H}^2/\SL(2, \mathbb{O})$, then these Fourier expansions agree to the accuracy computed (several hundred terms) with those of the classical $\SL(2, \mathbb{Z})$ modular forms $\frac{1}{12} E_4$, $\frac{41}{120} E_4^2$, and $\frac{44}{3} E_4^3 + \frac{2099}{252} E_2^2$, respectively.

We now compute the restrictions of these Eisenstein series to the Teichmüller curve $W = \mathbb{H}/\Gamma$ (or, rather, of its double covering $\mathbb{H}/\Pi$) that we studied in §6. As explained there, the algorithm used produced $\Gamma$ as a subgroup of $\SL(2, \mathbb{O})$, rather than $\SL(\mathbb{O}^\vee \oplus \mathbb{O})$, so that the twisting function $\varphi(z)$ maps $\mathbb{H}$ to $\mathbb{H}^{-}$ (cf. (52)). Hence, we must use the embedding of $W$ into $\mathbb{H}^2/\SL(2, \mathbb{O})$ by

$$z \mapsto (z_1, z_2) = (\varepsilon z, \varepsilon^0 \varphi(z)) \quad (z \in \mathbb{H}),$$

(64)

where $\varepsilon = 4 + \sqrt{17}$ is the fundamental unit of $\mathbb{Q}(\sqrt{17})$. Using the expansions of $Q$ and $\tilde{Q}$ as power series in $t$ that were given in §7.1, we find that the monomial $X^m Y^n = e(\operatorname{tr}(\nu z))$ for $\nu = m/2 + n/2 \sqrt{17} \in \mathbb{O}^\vee$ has the $t$-expansion

$$X^m Y^n = e(\varepsilon \nu z + \varepsilon^0 \nu^0 \varphi(z)) = (Q/A)^{\varepsilon \nu z} \left( \frac{\tilde{Q}}{A} \right)^{\varepsilon^0 \nu^0 z}$$

$$= (-2(3 + \sqrt{17}))^{-(21m+5n)/2} \left( \frac{5 - \sqrt{17}}{2} \right)^{-8m-2n} t^{(21m+5n)/2}$$

$$\times \left( 1 + \frac{16597m + 3985n - (3827m + 919n) \sqrt{17}}{128} \right).$$

(65)

3 For general discriminants there are several Hilbert–Eisenstein series for each value of $k$, with sum $\mathcal{E}_k^D$. If the class number of $D$ is one, as is the case for $D = 17$, there is only one.
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(More precisely, this is true under the assumption that the constants \( A \) and \( \tilde{A} \) relating \( q = e(z/\alpha) \) and \( \tilde{q} = e(\varphi(z)/\alpha^2) \) to \( Q = t + \cdots \) and \( \tilde{Q} = t + \cdots \) indeed have the values given in (49) and (51); we will return to this point below.) Inserting the expansion (65) into the Fourier development by linear algebra. The result of the computation for the first three Eisenstein series is in (59), whose expansions in \( t \) series in \( \tilde{t} \) and \( \eta \), precisely on the Teichmüller curve and its image under the involution \( \iota \) [Bai07] that there must be a symmetric Hilbert modular form of weight 12 which vanishes reducible and \( W \) on the Hilbert modular surface \( W \) vanishes. We have thus obtained an explicit algebraic equation cutting out the Teichmüller curve \( W \) (in which we have omitted 17 equally gigantic intermediate terms) whose restriction to \( s \) to \( \Gamma \), so that the constant terms at the two cusps \( s(0) = 0 \) and \( s(1) = \infty \) of the restriction of any Hilbert modular form to \( W \) are the same up to scaling.) Now by elimination we find a polynomial

\[
P(e_2, e_4, e_6) = (3465994203567 - 840620808790 \sqrt{17} e_6^4 + \cdots + (7395484320944244318526129490625 - 1711627845603248913114298550625 \sqrt{17}) e_2^{12}
\]

(in which we have omitted 17 equally gigantic intermediate terms) whose restriction to \( W \) vanishes. We have thus obtained an explicit algebraic equation cutting out the Teichmüller curve \( W \) on the Hilbert modular surface \( X_{17} \), but it is too big in the sense that its vanishing locus is reducible and \( W \) is only one of its components. Indeed, we know from the results of Bainbridge [Bai07] that there must be a symmetric Hilbert modular form of weight 12 which vanishes precisely on the Teichmüller curve and its image under the involution \( \iota : (z_1, z_2) \mapsto (z_2, z_1) \), whereas the above equation has weight 24. The reason for this is twofold:

- the Eisenstein series generate only a subring of the full ring of symmetric Hilbert modular forms, and there is no reason that the minimal defining equation of \( W \) should belong to this subring; and
- we did not even use all of the Eisenstein series, but only \( e_2, e_4, \) and \( e_6 \).

The second point can be dealt with by expressing the restriction of each \( e_k \) to \( W \) as a polynomial in \( \xi \) and \( \eta \) and looking for the first weight in which some linear combination of monomials in these restricted Eisenstein series vanishes. This weight, however, still turns out to be 14 rather than 12, and the answer is not even unique: there is a two-dimensional space of linear combinations of \( e_7, \ldots, e_{14} \) that vanish on \( W \), all having huge coefficients so that we do not reproduce them here. (As a side remark, it is actually surprising that there should be even
Theta derivatives

one relation in such a low weight, let alone two, since there are 15 monomials of weight 14 in $\Gamma_2, \ldots, \Gamma_{14}$ and also 15 monomials of weight 14 in $\xi$ and $\eta$, so that one would not expect the former to lie in a non-trivial subspace, let alone a subspace of codimension two. The first weight in which there has to be a relation for dimensional reasons is 16. Similarly, it is surprising that we found a relation among $\xi_2, \xi_4,$ and $\xi_6$ in as low a weight as 24, since $a$ priori the first weight in which the number of monomials in these three forms is larger than the number of monomials in $\xi$ and $\eta$ is 38. This suggests that the restrictions of the Eisenstein series to $W$ have some non-generic property, but we do not know what it is.) To address the first point, we need to have a full set of generators of the ring of Hilbert modular forms. Finding such a set of generators in general is a comparatively difficult (though always algorithmically solvable) problem, but in the case $D = 17$ the result has been obtained by Hermann [Her81]. We will describe his results in the next subsection and use them to determine the symmetric Hilbert modular form of minimal weight 12 vanishing on $W$.

Another pertinent remark is that, although we have so far only used Hilbert modular forms of ‘parallel’ weight $(k, k)$, whose restriction to $W$ is a polynomial in $\xi$ and $\eta$, one can equally well consider Hilbert modular forms of mixed weight $(k, \ell)$ with $k \neq \ell$, in which case the restrictions become polynomials in $f, \tilde{f}$ and $t^{\pm 1}$. For example, the Rankin–Cohen bracket $2E_2(z)E_4(z) - E_2(z)E_4'(z)$, where the prime denotes differentiation with respect to the first variable $z_1$, is a Hilbert modular form of weight $(8, 6)$, and its restriction to $W$ could be computed explicitly as $f(z)^8 \tilde{f}(z)^6$ times a Laurent polynomial in $t(z)$. Such mixed weight forms will play a role in Part III of this paper, e.g. in §9, where we will use a different construction to find explicitly a non-symmetric Hilbert modular form of non-parallel weight $(3, 9)$ which vanishes precisely on $W$, again in accordance with the general results of Bainbridge.

8.2 Hilbert modular forms for the discriminant 17

We continue to work with the Hilbert modular group $\Gamma_D = \text{SL}(2, \mathcal{O})$ for the case $D = 17$, i.e. for $\mathcal{O} = \mathbb{Z}[\alpha]$ with $\alpha = (1 + \sqrt{17})/2$. We will describe the structure of the ring of symmetric Hilbert modular forms of parallel even weight, following Hermann [Her81], who obtains these modular forms by restriction of Siegel modular forms of genus two. Later we will look also at non-symmetric Hilbert modular forms and Hilbert modular forms of odd or non-parallel weight on $\Gamma_1$.

Hermann begins with the 16 genus two Siegel theta series

$$\Theta_{m,m'}(Z,v) = \sum_{x \in \mathbb{Z}^2} \mathbf{e} \left( \frac{1}{2} xZx^T + x(v + m')^T \right). \quad (66)$$

Here $m, m' \in \{0, \frac{1}{2}\}^2$ and the independent variables $Z$ and $v$ are in the Siegel half-space $\mathbb{H}_2$ and in $\mathbb{C}^2$, respectively. Ten of these (those for which the theta characteristic $(m, m')$ is even, i.e. $4m \cdot m' \equiv 0 \pmod{2}$) are even functions of $v$ and hence give Siegel modular forms of weight $\frac{1}{2}$ after restricting to $v = 0$. (The other six are odd and hence give zero on restriction, but their derivatives with respect to $v$ give non-trivial restrictions that will play a crucial role in the constructions of Part III of this paper.) Using a modular embedding from $\mathbb{H}^2$ to $\mathbb{H}_2$ such as that described in the previous subsection, we get 10 Hilbert theta series, all of weight $\frac{1}{2}$ with respect to a suitable subgroup of $\Gamma_D$.

It is convenient to re-index the 16 theta characteristics in a way that makes the action of $\Gamma_D$ more transparent. This part works for any $D \equiv 1 \pmod{8}$, i.e. for $D$ such that the prime 2 splits as $\pi_2 \pi_2^\prime$ in $K = \mathbb{Q}(\sqrt{D})$ for some prime ideal $\pi_2 \neq \pi_2^\prime$ (in our case, the principal ideal generated
by $1 + \alpha$). Define sets $S$ and $\hat{S}$ by

$$S = \{0, 1, \infty\}, \quad \hat{S} = S \cup \{X\}$$

where $X$ is a symbol, and let $\Gamma_D$ act on $\hat{S} \times \hat{S}$ by fixing $X$ and identifying $S \times S$ with $\mathbb{P}^1(\mathcal{O}/\pi_2) \times \mathbb{P}^1(\mathcal{O}/\sigma(\pi_2))$. We match the usual indexing by characteristic with these symbols by Table 2, in which the even characteristics correspond to the pairs $(a, b) \in \hat{S}^2$ in which $a$ and $b$ are either both equal to or both different from $X$, and the odd characteristics to the pairs $(a, b)$ in which exactly one of $a$ and $b$ is equal to $X$. This gives us by restriction 10 Hilbert modular forms $\theta_{X,X}$ and $\theta_{a,b}$ $(a, b \in S)$ of weight $\frac{1}{2}$.

Set $\Theta = \theta_{X,X}^2$ and for each permutation $\pi$ of the set $S$ set $\eta_\pi = \pm \prod_{s \in S} \theta_s, \pi(s)$, where the sign is +1 if $\pi$ is the identity and $-1$ otherwise. Up to powers of $i$, the Hilbert modular group preserves $\Theta$ and permutes the $\eta_\pi$, preserving the parity of $\pi$, so that if we set

$$\eta_\pm = \sum_{\pi \text{ even}} \eta_\pi \pm \sum_{\pi \text{ odd}} \eta_\pi$$

then $\Theta$, $\eta_+$ and $\eta_-$ are Hilbert modular forms, with multiplier systems, for the full Hilbert modular group. More precisely, we have

$$\Theta \in M_1(\Gamma_D, v_0), \quad \eta_+^2 \in M_3(\Gamma_D, v_0), \quad \eta_+ \eta_- \in M_3(\Gamma_D, v_0^{-1}), \quad (67)$$

where

$$v_0 : \Gamma_D \to \mu_4, \quad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mapsto -1, \quad \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \mapsto i^{\text{tr}(x)} \quad \text{for } x \in \Theta$$

is a character of order four. The form $\theta_{X,X}$ is antisymmetric with respect to the involution $\iota$ and hence vanishes on the diagonal $\mathbb{H}/\text{SL}(2, \mathbb{Z}) \subset \mathbb{H}^2/\text{SL}(2, \mathcal{O}_D)$. Moreover, in the case $D = 17$ this is its full vanishing locus, so that any Hilbert modular form vanishing on the diagonal is divisible by $\theta_{X,X}$ and any symmetric Hilbert modular form vanishing on the diagonal is divisible by $\Theta$. For example, since the restrictions of both $\eta_+^2$ and $\eta_-^2$ are proportional to $\sqrt{\Delta}$ (where we use $E_4$, $E_6$ and $\Delta = (E_4^3 - E_6^2)/1728$ to denote the standard generators of $M_*(\text{SL}(2, \mathbb{Z}))$), some linear combination of them, which turns out to be $\eta_+^2 - 4\eta_-^2$, vanishes on the diagonal and hence is divisible by $\Theta$. This gives us the following five symmetric Hilbert modular forms of even weight and trivial character:

$$G_2 = \frac{\eta_+^2 - 4\eta_-^2}{\Theta}, \quad G_4 = \eta_+ \eta_- \Theta, \quad H_4 = \Theta^4, \quad G_6 = \eta_-^2 \Theta^3, \quad H_6 = \eta_-^3 \eta_+,$$

where the index of each form indicates its weight.
The Fourier expansions of combination of 1, $C$ final iteration gives a fourth polynomial yet a third polynomial $P$ so $\eta$ has the same restriction to the diagonal as $\eta$. Combining these successive identities we have written $F$ as $P_0 + P_1 + P_2 + P_3 + H_4$ where each $P_i$ belongs to $\mathbb{C}[G_2, H_6]$ and $P_4 \in \mathbb{M}_{-4}(\Gamma_1)$, and now iterating the whole argument we see that $F$ is a linear combination of $1, G_4, G_2^2$, and $H_6$ with coefficients in $\mathbb{C}[G_2, H_4, H_6]$ as claimed. \hfill \Box

**Example.** The Fourier expansions of $\eta_+, \eta_-$, and $\Theta$ begin

$$
\eta_+ = -4X^{1/4}(Y + 1 - Y^{-1}) + (13Y^4 - 19Y^2 - 19Y^{-2} + 13Y^{-4}) X + \cdots,
$$

$$
\eta_- = 16X^{1/4}(1 - (Y^5 + 3Y^3 - Y - Y^{-1} + 3Y^{-3} + Y^{-5}) X + \cdots),
$$

$$
\Theta = 4X^{1/2}((Y^2 - 2)^2 - 2(Y^5 - Y - Y^{-1} + Y^{-5}) X + \cdots).
$$

(As a check, if we set $Y = 1$ and $X = q$ then these reduce to $-8\eta^6, 16\eta^6$, and $0$.) Comparing with the expansions of the first three Eisenstein series $\mathcal{E}_k$ given above, we find that these forms are given in terms of Hermann’s generators of $\mathbb{M}^+(\Gamma_1)$ by

$$
-192 \mathcal{E}_2 = G_2, \quad 640 \mathcal{E}_4 = \frac{41}{18} G_2^3 - 39 G_4 - 57 H_4,
$$

$$
14336 \mathcal{E}_6 = -\frac{5791}{72} G_2^3 + 8571 G_4 G_2 + 11463 H_4 G_2 - \frac{32865}{4} G_6 - 6285 H_6.
$$

**Corollary 8.2.** The function field of the symmetric Hilbert modular surface with $D = 17$ is the rational function field is generated by the two functions

$$
U = \frac{H_4}{G_4} \left( = \frac{\Theta}{\eta_+ \eta_-} \right), \quad V = \frac{H_6}{H_6 - G_2 G_4} \left( = \frac{\eta_+^2}{4 \eta_-^2} \right).
$$

**Proof.** The relations among Hermann’s generators imply

$$
\begin{pmatrix}
G_4 & H_4 & G_6 & H_6 \\
G_2^2 & G_2^2 & G_2^2 & G_2^2
\end{pmatrix}
= \begin{pmatrix}
UV & U^2 V & U^2 V^2 & U^2 V^2 \\
4(V - 1)^2 & 4(V - 1)^2 & 4(V - 1)^3 & 4(V - 1)^3
\end{pmatrix}.
$$

So the corollary follows immediately from the theorem. \hfill \Box
### 8.3 The equation of the Teichmüller curve

The corollary just given tells us that the Hilbert modular surface for $D = 17$ is rational, with coordinates $U$ and $V$. In particular, the image of the Teichmüller curves $W_{17}^\pm$ on this surface must be given by polynomial equations in these coordinates. In this subsection we will give these equations, which turn out to be several orders of magnitude simpler than the previously obtained equation $P(\varepsilon_2, \varepsilon_4, \varepsilon_6) = 0$. We will also describe better systems of Fourier coordinates and will resolve two questions that we raised earlier by showing that the values of $A$ and $\tilde{A}$ given in (49) and (51) on the basis of numerical computations are indeed correct and by giving a purely modular proof of the integrality (away from the prime two) of the Taylor expansions of $y$ and $\tilde{y}$ as power series in $t$.

**Theorem 8.3.** On the (rational) symmetric Hilbert modular surface with coordinates $U$ and $V$, the Teichmüller curves $W_{17}^1$ and $W_{17}^0$ are given by the equations

$$W_{17}^1 : V + \frac{5 + \sqrt{17}}{2} U^2 + 3 \frac{7 + \sqrt{17}}{8} U + \frac{1 - \sqrt{17}}{8} = 0$$

and

$$W_{17}^0 : V + \frac{5 - \sqrt{17}}{2} U^2 + 3 \frac{7 - \sqrt{17}}{8} U + \frac{1 + \sqrt{17}}{8} = 0.$$  

**Proof.** Since we gave the expressions for the Hilbert–Eisenstein series $\varepsilon_2$, $\varepsilon_4$ and $\varepsilon_6$ in the Hermann generators in the last subsection, we could derive (70) from the results of §8.1 giving the restrictions to the curve $W_{17}^1$ of these Eisenstein series. However, it is much simpler to work directly with Hermann’s generators, obtaining their Fourier expansions from those of $\eta_{\pm}$ and $\Theta$ as given above and then using (65) to obtain the $t$-expansions of their restrictions to $W$. The results of the computations are given in the following table, in which we have used the results from the end of §7.3 to write the expansion of the restriction of each of $G_1$, $H_1$, $G_6$, and $H_6$ to $W$ (that of $G_2 = -192\varepsilon_2$ was already given above) as a power of $\xi$ times a polynomial in $s$:

<table>
<thead>
<tr>
<th>$F$</th>
<th>$\varepsilon G_1/2^4 \pi_2^8$</th>
<th>$\varepsilon H_4/2^4 \pi_2^6$</th>
<th>$\varepsilon^2 G_6/2^6 \pi_2^{12}$</th>
<th>$\varepsilon^2 H_6/2^6 \pi_2^{14}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\xi^{-k} F</td>
<td>_W$</td>
<td>$(s-1)(s-\kappa_1)$</td>
<td>$s(s-1)$</td>
<td>$s^2 (s-1)^2$</td>
</tr>
</tbody>
</table>

Here $\kappa_1 = \pi_2^{-2} = (13 - 3\sqrt{17})/8$. From the definitions of the Hilbert modular functions $U$ and $V$ it then follows that their restrictions to $W$ are given by

$$U|_W = \frac{\kappa_1}{s-\kappa_1}, \quad V|_W = \frac{-1 + \sqrt{17}}{8} \frac{s(s-1)}{(s-\kappa_1)^2},$$

and (70) follows immediately. Equation (71) is proved in a similar way using the Galois conjugate differential equation, as discussed in the remark at the end of §5.5; the resulting expansions are the Galois conjugates of those for $W_{17}^1$ and since the Hermann generators have rational Fourier coefficients the final equation is necessarily also the Galois conjugate of that of $W_{17}^1$. In fact, this Galois conjugation property holds for all $D$, as was already recalled in Theorem 5.4(iv) of §5.4. 

\[\Box\]

We end this section by discussing four points related to the equations given in Theorem 8.3.

1. Equation (70) describes a Hilbert modular function that vanishes precisely on the curve $W_{17}^1$. We can also ask for the holomorphic Hilbert modular form of smallest weight with
The same property. If we multiply the left-hand side of (70) through by \( \eta_1^2 \eta_2^2 \), then by (69) the result is
\[
F_{17}^1 = \frac{1}{4} \eta^4 + \frac{5 + \sqrt{17}}{2} \Theta^6 + 3 \frac{7 + \sqrt{17}}{8} \Theta^3 \eta_1 \eta_2 + \frac{1 - \sqrt{17}}{8} \eta_2 \eta_2^2, \tag{73}
\]
and by Theorem 8.3 this vanishes precisely on \( W_{17}^1 \). According to (67), \( F_{17}^1 \) is a holomorphic Hilbert modular form of weight six on the full Hilbert modular group, but with quadratic character \( v_0 \), where \( v_0 \) is defined as in (68). If we further multiply it by \( \Theta^2 \), then we get a Hilbert modular form of weight eight on the full Hilbert modular group and with trivial character, given in terms of the Hermann generators by
\[
\Theta^2 F_{17}^1 = \frac{1}{4} G_2 G_6 + \frac{5 + \sqrt{17}}{2} H_4^2 + 3 \frac{7 + \sqrt{17}}{8} G_4 H_4 + \frac{9 - \sqrt{17}}{8} G_2^4,
\]
but this form now vanishes not only on \( W_{17}^1 \), but also (doubly) on the diagonally embedded modular curve \( \mathbb{H}/\text{SL}(2, \mathbb{Z}) \subset \mathbb{H}^2/\text{SL}(2, \mathbb{R}) \). On the other hand, if we multiply \( F_{17}^1 \) by its Galois conjugate \( F_{17}^0 = (F_{17}^1)^\sigma \), then the product \( F_{17} \) vanishes precisely on the full Teichmüller locus \( W_{17} = W_{17}^0 \cup W_{17}^1 \), and this is now a holomorphic Hilbert modular form on the full modular group \( \text{SL}(2, \mathbb{O}) \) and with trivial character, given in terms of the basis of \( M_{12}(\text{SL}(2, \mathbb{O})) \) from Theorem 8.1 by
\[
F_{17} = \text{(explicit polynomial in } G_2, G_4, H_4, G_6, H_6 \text{ with rational coefficients}).
\]
We do not write out the polynomial, since it is a bit complicated, but observe that it involves only 11 of the 16 generators of \( M_{12}(\text{SL}(2, \mathbb{O})) \). The fact that there is a single Hilbert modular form of weight 12 whose vanishing locus is precisely the union of the Teichmüller curves on \( X_D \) is a special case of the theorem of Bainbridge, already mentioned in § 8.1, stating that such a form \( F_D \) exists for every \( D \). We will give a different proof of this in Part III by constructing \( F_D \in M_{12}(\text{SL}(\mathbb{O}^\vee \oplus \mathbb{O})) \) in general as a product of 12 derivatives of theta series of weight \( (\frac{1}{2}, \frac{3}{2}) \) or \( (\frac{3}{2}, \frac{1}{2}) \).

(2) The next point concerns the choice of coordinates for our Fourier expansions. We replace the previously used coordinates \( X \) and \( Y \) from (61) by the new Fourier variables
\[
X_1 = X^{-3} Y^{13}, \quad Y_1 = X^5 Y^{-21}.
\]
This has several advantages. First of all they form a \( \mathbb{Z} \)-basis of the group of Fourier monomials \( e(\text{tr}(xz)) \), whereas \( X \) and \( Y \) generated a subgroup of index two. Second, symmetric Hilbert modular forms of even weight are symmetric in \( X_1 \) and \( Y_1 \), as one sees using the action of \( \varepsilon \). For instance the \((X_1, Y_1)\)-expansions of the first two Hermann generators begin
\[
-\frac{1}{192} G_2 = \frac{1}{12} + Y_1 X_1 + (9Y_1^2 + 3Y_1^3) X_1^2 + (3Y_1^2 + 10Y_1^3 + 15Y_1^4) X_1^3 + \cdots,
\]
\[
\frac{1}{256} G_4 = -Y_1 X_1 + (-9Y_1^2 + Y_1^3) X_1^2 + (Y_1^2 - 14Y_1^3 + Y_1^4) X_1^3 + \cdots,
\]
in which the coefficients of \( Y_1^3 X_1^2 \) and \( Y_1^2 X_1^3 \) are the same. Third, and most important, both are holomorphic near the cusp of \( W \) and hence have power series expansions in \( t \), with valuations one and zero there rather than \( \frac{21}{12} \) and \( \frac{5}{2} \) as for \( X \) and \( Y \). This has to do with the Hirzebruch resolution of the cusp singularities, according to which different \( \mathbb{Z} \)-bases of the group just mentioned are good coordinates in different parts of the resolution cycle. (We will discuss this in much more detail in § 11.) Here we have to choose the coordinates that are adapted to the point of the cusp resolution through which \( W \) passes. Explicitly, these expansions are
\[
X_1 = X^{-3} Y^{13} = \frac{-11 + 3\sqrt{17}}{64} t + \frac{403 - 229\sqrt{17}}{2048} t^2 + \cdots,
\]
\[
Y_1 = X^5 Y^{-21} = \frac{21 - 5\sqrt{17}}{2} - \frac{895 - 217\sqrt{17}}{8} t + \cdots. \tag{74}
\]
(For comparison, the leading terms of $X$ and $Y$ are $\sqrt{(524445 - 220267\sqrt{17})/2^80}t^{21/2}$ and $\sqrt{-(3 + 11\sqrt{17})/2^{20}}t^{5/2}$, respectively.) It is then very easy to restrict a Hilbert modular form $F$ given in its $(X_1, Y_1)$-expansion to $W$: to get the expansion up to order $t^6$ it suffices to expand $F$ up to $X_1^6$ in $\mathbb{C}[Y_1][[X_1]]$.

(3) The third point concerns the correctness of the values of $A$ and $\tilde{A}$ in (49) and (51). We have been assuming throughout (for instance in (74)) that these guesses were correct, and now finally prove it. We write

$$X_1 = -\frac{11 + 3\sqrt{17}}{64} c_1 t \exp\left(\varepsilon \alpha \lambda \frac{y_1}{y} + (\varepsilon \alpha \lambda)^\sigma \frac{\tilde{y}_1}{y}\right),$$

$$Y_1 = \frac{21 - 5\sqrt{17}}{2} c_2 t \exp\left(\varepsilon \alpha \mu \frac{y_1}{y} + (\varepsilon \alpha \mu)^\sigma \frac{\tilde{y}_1}{y}\right), \quad (75)$$

where $\lambda = (13 - 3\sqrt{17})/2\sqrt{17}$ and $\mu = (-21 + 5\sqrt{17})/2\sqrt{17}$ (the factor $\varepsilon$ in the exponent takes care of the identification of $\mathbb{H} \times \mathbb{H}$ and $\mathbb{H} \times \mathbb{H}^-$, the factor $\alpha$ is the cusp width, and the factors $\lambda$ and $\mu$ come from the passage from $(X, Y)$ to $(X_1, Y_1)$) and where $c_1$ and $c_2$ are numerical factors that are both equal to one if (and only if) the formulas for $A$ and $\tilde{A}$ in (49) and (51) are correct. (One could, and we originally did, also do the whole calculation without including the prefactors $-(11 + 3\sqrt{17})/64$ and $(21 - 5\sqrt{17})/2$ in (75), but then the numbers in the calculation would be even worse.) We substitute these expressions into $G_2$, divide by $\xi^2$, with $\xi$ as in (59), and write the quotient as a power series $\sum_i C_i s^i$, with $s = 4t/(1 - t)^2$ as usual. By Proposition 7.2, we know that this power has to reduce to a quadratic polynomial if $c_1$ and $c_2$ have the correct values. If we instead treat $c_1$ and $c_2$ as unknowns, then the coefficients of this power series are polynomials in $c_1$ and $c_2$ (with huge coefficients), and we have to show that the infinite system of polynomial equations $C_i(c_1, c_2) = 0$ for all $i > 2$ has the unique solution $c_1 = c_2 = 1$. By computer calculation we find that the greatest common divisor (gcd) of the resultants $\text{Res}_{c_1}(C_3, C_4)$ and $\text{Res}_{c_1}(C_4, C_5)$ is equal to $-2^{-111} c_2^5 (c_2 - 1)$, and since $c_2$ cannot be zero, it must be one. Then substituting $c_2 = 1$ into the g.c.d. of $C_2$ and $C_4$ gives $c_1 = 1$.

(4) The final point concerns the integrality (away from the prime two) of our Fourier expansions. The coefficients of the expansion of any Hilbert modular form in $X_1$ and $Y_1$ with rational coefficients always has bounded denominators, like in the examples for $G_2$ and $G_4$ above. The same is true also for Hilbert modular functions, e.g.

$$U + 1 = (Y_1 - Y_1^2) X_1 + (-Y_1 - 15Y_1^2 + 17Y_1^3 - Y_1^4) X_1^2 + \cdots,$$

$$\frac{1}{2} V = Y_1 X_1 - (22Y_1^2 + 2Y_1^3) X_2 + (-2Y_2^2 + 289Y_1^3 + 12Y_1^4 + Y_1^5) X_1^3 + \cdots$$

for the generators of the field of symmetric Hilbert modular functions given in Corollary 8.2 above. We introduce new generators of this function field, namely $U_1 = 1 - (4U + 1)/V$ and $V_1 = \frac{1}{2} V$. (Other choices would be equally good.)

**Proposition 8.4.** In the new set of generators the equality

$$\mathbb{Z}[[X_1, Y_1]]^\text{sym} = \mathbb{Z}[[U_1, V_1]]$$

of power series rings holds.

**Proof.** We have $\mathbb{Z}[[X_1, Y_1]]^\text{sym} = \mathbb{Z}[[S, P]]$, where $S = X_1 + Y_1$ and $P = X_1 Y_1$. We have already seen that $U$ and $V$, and hence also $U_1$ and $V_1$, are symmetric in $X_1$ and $Y_1$. We can thus express $U_1$ and $V_1$ as a power series in $S$ and $P$. Concretely, these expansions start

$$U_1 = S + (-7 + 3S + 3S^2) P + (13 - 65S + 37S^2 + 6S^3) P^2 + \cdots,$$

$$V_1 = P + (-22 - 2S) P^2 + (289 + 12S + S^2) P^3 + \cdots.$$
They have integral expansions, so \( \mathbb{Z}[[U_1, V_1]] \subseteq \mathbb{Z}[[S, P]] \). Conversely, since the expansions begin \( S + O(P) \) and \( P + O(P^2) \), we can recursively compute \( S \) and \( P \) as power series in \( U_1 \) and \( V_1 \), and these again have integral coefficients:

\[
S = U_1 + (7 + 3U_1 + 3U_1^2) V_1 + (120 - 20U_1 - 82U_1^2 + 6U_1^3) V_1^2 + \cdots,
\]

\[
P = V_1 + (22 + 2U_1) V_1^2 + (693 + 158U_1 + U_1^2) V_1^3 + \cdots.
\]

It follows that \( \mathbb{Z}[[U_1, V_1]] \supseteq \mathbb{Z}[[S, P]] \), as desired. \( \square \)

**Corollary 8.5.** The restriction to \( W_H \) of any symmetric Hilbert modular form of even weight with integral Fourier coefficients belongs to \( R[[t]] \), where \( R = O_{17}[\frac{1}{2}] \).

**Proof.** Since \( \kappa_0 \in R^\times \), equation (35) implies that \( R[[t]] = R[[s]] \). Since \( \kappa_1 \in R^\times \), equation (72) implies that the restrictions of \( U_1 \) and \( W_1 \) to \( W \) belong to this ring. Since any Hilbert modular form has a Fourier expansion with exponents in a cone strictly contained in the positive quadrant (explicitly, if \( F = \sum_{r,s} a_{r,s} X_1^r Y_1^s \), then \( a_{r,s} = 0 \) unless \( (9 - \sqrt{17})/8r \leq s \leq (9 + \sqrt{17})/8r \)), it contains only finitely many monomials contributing to any fixed power of \( t \) in the \( t \)-expansion of its restriction to \( W \). (Explicitly, \( X_1^r Y_1^s + X_1^r Y_1^s \) is divisible by \( P_{\text{min}}(r,s) \), and \( P|_W = O(t) \).) The corollary follows. \( \square \)

**Proposition 8.6.** The power series \( y(t) \) and \( \tilde{y}(t) \) have \( O_{17} \)-integral expansions up to denominator two, i.e. \( y(t), \tilde{y}(t) \in R[t] \), where \( R = O_{17}[\frac{1}{2}] \).

**Proof.** The differential operators \( D_1 = (\sqrt{17}/2\pi i)(\partial/\partial z_1) \) and \( D_2 = (\sqrt{17}/2\pi i)(\partial/\partial z_2) \) can be written as

\[
D_1 = \lambda X_1 \frac{\partial}{\partial X_1} + \mu Y_1 \frac{\partial}{\partial Y_1}, \quad D_2 = \lambda^\sigma X_1 \frac{\partial}{\partial X_1} + \mu^\sigma Y_1 \frac{\partial}{\partial Y_1},
\]

where \( \lambda = (13 - 3\sqrt{17})/2 \) and \( \mu = (-21 + 5\sqrt{17})/2 \), and hence map \( R[[U_1, V_1]] \) to itself. On the other hand, they send Hilbert modular functions to meromorphic Hilbert modular forms of weight \((2,0)\) and \((0,2)\), respectively. By Proposition 7.2, the quotients of the restrictions to \( W_H \) of these derivatives by \( y^2 \) (respectively \( \tilde{y}^2 \)) must be rational functions of \( t \). We have to make the right choices of these functions in order not to introduce unwanted denominators. Since \( 1/U \) and \( V/U^2 \) restrict to polynomials in \( s \) (cf. (72)), we choose these as the Hilbert modular functions to be differentiated, finding

\[
y^2 = c \frac{(1-t)^4}{P(t)} \cdot D_1 \left( \frac{V}{U^2} \right) \bigg|_{W_H}, \quad \tilde{y}^2 = \tilde{c} \frac{(1-t)^2}{t(1+t)} \cdot D_2 \left( \frac{1}{U} \right) \bigg|_{W_H},
\]

where \( c = (29 + 7\sqrt{17})/2 \), \( \tilde{c} = (7 + \sqrt{17})/2 \) and \( P(t) \) is the polynomial

\[
P(t) = t (1+t) \left( 1 - \frac{31 - 7\sqrt{17}}{2} t \right) \left( 1 - \frac{31 + 7\sqrt{17}}{64} t \right) \left( 1 - \frac{647 - 153\sqrt{17}}{8} t + t^2 \right).
\]

Since \( c \) and \( \tilde{c} \) belong to \( R \) and the polynomials in the denominators are in \( 1 + tR[t] \), it follows that \( y^2 \) and \( \tilde{y}^2 \), and hence also \( y \) and \( \tilde{y} \), are in \( R[[t]] \). \( \square \)

Although we have been working with \( D = 17 \) all the time and using concrete generators of the field of Hilbert modular functions, it is clear from the proof that the basic principle, the use of integral coefficients for Hilbert modular forms and the base change in two variables, can be applied for any \( D \), giving the \( O_D \)-integrality (up to finitely many primes in the denominator) of solutions \( y(t) \) and \( \tilde{y}(t) \) of the corresponding Picard–Fuchs equations for any \( D \).
In Part II we deduced the equation of the Teichmüller curve in the Hilbert modular surface for \( D = 17 \) and of a Hilbert modular form cutting out the Teichmüller curve starting from the differential operators \( L \) and \( \tilde{L} \), which in turn were deduced from the explicit algebraic model of the family as given in (38). In Part III we will show that there is a general construction of a Hilbert modular form of mixed weight that cuts out the Teichmüller curve. The construction, given in \( \S 9 \), uses derivatives of theta functions. The short proof depends on the description of Teichmüller curves using eigenforms for real multiplication with a double zero (see Theorem 5.2). We also verify that in the case \( D = 17 \) we get the same equation for the Teichmüller curve as already obtained in \( \S 8 \).

Next, in \( \S \S 10 \) and 12, we develop the theory of Teichmüller curves in genus two ‘from scratch’ starting from the new definition as vanishing loci of theta derivatives. In particular, we give new proofs from this point of view of the cusp classification and of the facts that these curves are Kobayashi geodesics and are disjoint from the reducible locus and hence are Teichmüller curves. We do not know how to reprove the irreducibility from the viewpoint of theta functions.

Along the way, in \( \S 11 \) we show that Bainbridge’s compactification of Hilbert modular surfaces using the moduli space of curves is indeed a toroidal compactification. Recall that this property of Hirzebruch’s compactification was the model on which the notion of toroidal compactifications was developed.

9. Teichmüller curves are given by theta derivatives

Bainbridge has shown in [Bai07, Theorem 10.2], that the Teichmüller curves \( W_D \) defined in \( \S 5.3 \) are given as the vanishing locus of a modular form (3, 9) for all \( D \). We determine this form explicitly. It turns out to be a product of derivatives of theta series restricted from the Siegel half-space to Hilbert modular varieties.

9.1 Theta functions and their restrictions to Hilbert modular varieties

We recall the definition of the classical theta functions and properties of their derivatives. Although we are ultimately interested in \( g = 2 \) only, we can keep \( g \) general without effort when setting up the definitions.

For \( m, m' \in (\frac{1}{2}\mathbb{Z})^g \) (considered as row vectors) we define the (Siegel) theta function

\[
\Theta_{(m,m')} : \left\{ \begin{array}{ccc}
\mathbb{C}^g \times \mathbb{H}_g & \to & \mathbb{C}, \\
(v, Z) & \to & \sum_{x \in \mathbb{Z}^g + m} e^{\left(\frac{1}{2}xZx^T + x(v + m')^T\right)},
\end{array} \right.
\]

with characteristic \((m, m')\). The evaluation of a theta function at \( v = 0 \) is called a theta constant. The theta function (and the characteristic \((m, m')\)) is called odd if \( 4m(m')^T \) is odd and even otherwise. Odd theta constants vanish identically as functions of \( Z \). Up to sign, \( \Theta_{(m,m')} \) depends only on \( m \) and \( m' \) modulo \( \mathbb{Z}^g \).

The theta constants \( \Theta_{(m,m')}(0, Z) \) are modular forms of weight \( \frac{1}{2} \) for some subgroup (in fact \( \Gamma(4, 8) \), see e.g. [Igu72]) of \( \text{Sp}(2g, \mathbb{Z}) \). The partial derivatives with respect to any \( v_i \) are not modular, but if we restrict to \( v = 0 \) and consider the gradient (as column vector)

\[
\nabla(\Theta_{(m,m')}(0, Z)) = \left( \frac{\partial}{\partial v_i} \Theta_{(m,m')}(v, Z) \big|_{v=0} \right)_{i=1, \ldots, g},
\]

\(2318\)
then we get a vector-valued modular form. That is, if \( \Theta_{(m,m')} (0, Z) = 0 \), one calculates that for any \( M = (A \, B) \in \Gamma(4,8) \) the gradient transforms as

\[
\nabla (\Theta_{(m,m')})(0, M \cdot Z) = \zeta_8 \det (CZ + D)^{1/2} (CZ + D) \nabla (\Theta_{(m,m')})(0, Z),
\]

where \( \zeta_8 \) is an eighth root of unity depending on \( M \).

If \( K \) is a totally real number field of degree \( g \) over \( \mathbb{Q} \) with ring of integers \( \mathcal{O} \), then just as in the special case \( g = 2 \) we can define a \( g \)-dimensional Hilbert modular variety \( X_K = \mathbb{H}^g / \text{SL}(O^V \oplus \mathcal{O}) \) and a Siegel modular embedding \( (\Psi, \psi) \) of \( X_K \) into \( \mathbb{H}_g / \text{Sp}(2g, \mathbb{Z}) \), given by a matrix \( B \in \text{GL}(g, \mathbb{R}) \) as in (25). Recall that this means that \( \Psi : \text{SL}(O^V \oplus \mathcal{O}) \to \text{Sp}(2g, \mathbb{Z}) \) is a homomorphism and that \( \psi(z) = B^T \text{diag}(z_1, \ldots, z_g) B \) is a map that is equivariant with respect to \( \Psi \). We then denote by \( \theta_{(m,m')} (z) = \Theta_{(m,m')} (0, \psi(z)) \) and \( \nabla \theta_{(m,m')} (z) = \nabla \Theta_{(m,m')} (0, \psi(z)) \) the restriction of the theta functions and their gradients to \( \mathbb{H}_g \). We also write \( \theta_{(m,m'), \Psi} \) if we want to emphasize the dependence on the modular embedding. The modularity of the Siegel theta functions imply that the (Hilbert) theta constants \( \theta_{(m,m')} (z) \) are modular forms with a character of order eight of weight \( \left( \frac{1}{2}, \ldots, \frac{1}{2} \right) \) for a subgroup of finite index of the Hilbert modular group \( \text{SL}(O^V \oplus \mathcal{O}) \). The theta constants are non-zero if and only if \( (m, m') \) is even, while the theta gradients are modular if and only if \( (m, m') \) is odd.

The modular transformation of the derivative of theta constants for \( (m, m') \) odd now reads

\[
\nabla \theta_{(m,m')} (\gamma \cdot z) = \zeta_8 \det (c \psi(z) + d)^{1/2} B^{-1} J(z, \gamma) B \nabla \theta_{(m,m')} (z)
\]

for \( \gamma = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \) in the subgroup \( \Psi^{-1} (\Gamma(4,8)) \subset \text{SL}(O^V \oplus \mathcal{O}) \), where \( \hat{c} \) for \( c \in \mathbb{K} \) denotes the diagonal matrix with entries \( \sigma_j (c) \) given by the different real embeddings \( \sigma_j \) of \( \mathbb{K} \) and \( J = J(z, \gamma) \) is the diagonal matrix with entries \( \sigma_j (c) z_j + \sigma_j (d) \). Consequently, the vector-valued modular form \( B \nabla \theta_{(m,m')} \) transforms with the automorphy factor \( \zeta_8 \det (c \psi(z) + d)^{1/2} J(z, \gamma) \), which is also a diagonal matrix. We will calculate the root of unity \( \zeta_8 \) for \( K = \mathbb{Q}(\sqrt{D}) \) with \( D \equiv 1 \pmod{8} \) in detail below.

To summarize, the \( i \)th entry \( D_i \theta_{(m,m')} (z) \) of the column vector \( B \nabla \theta_{\psi} \) is a Hilbert modular form of multiweight \( \left( \frac{1}{2}, \ldots, \frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2} \right) \) and a character stemming from the eighth root of unity.

We can also express the functions \( D_i \theta_{(m,m')} \) as derivatives in certain eigendirections. For this purpose replace the original coordinates \( u = (v_1, \ldots, v_g) \in \mathbb{C}^g \) by the ‘eigendirection coordinates’ \( u = BV = (u_1, \ldots, u_g) \). Then we may write

\[
D_i \theta_{(m,m')} (z) = \frac{\partial}{\partial u_i} \Theta_{(m,m')} (v, \psi(z))|_{u=0}.
\]

9.2 The curve \( W_D \) is the vanishing locus of the theta derivatives

With this preparation we can now determine Bainbridge’s modular form for general \( D \).

**Theorem 9.1.** The function

\[
\mathcal{D} \Theta(z) = \prod_{(m,m') \text{ odd}} D_2 \theta_{(m,m')} (z)
\]

is a modular form of weight \( (3,9) \) (with character) for the full Hilbert modular group \( \text{SL}(O_D^V \oplus O_D) \). Its vanishing locus is precisely the Teichmüller curve \( W_D \).
We will give additional properties of the modular form $\mathcal{D}\theta$ if the prime (2) splits, i.e. if $D \equiv 1 \pmod{8}$. In that case, as already mentioned in §8.2, the discussion of the numbering of theta characteristics given there for $D = 17$ always holds. In particular, one has a quartic character $\nu_0$ given by (68). Here, $\mathcal{D}\theta$ is a product of two modular forms and its character is $\nu_0^2$. Indeed, using the shorthand notation introduced in §8.2 we calculate the action of generators of $\text{SL}(O^\vee \oplus O)$ on the functions $D_2\theta_{(m,m')}$. From Table 2 in §8.2, we deduce that the products of theta derivatives

$$D_L = D_2\theta_{0X} \cdot D_2\theta_{1X} \cdot D_2\theta_{\infty X} \quad \text{and} \quad D_R = D_2\theta_{X0} \cdot D_2\theta_{X1} \cdot D_2\theta_{X\infty} \quad (77)$$

are modular forms for the full group $\text{SL}(O^\vee \oplus O)$ of weight $(\frac{3}{2}, \frac{9}{2})$ with a character of order eight according to Table 3 below, in which $\alpha = (1 + \sqrt{D})/2$ as usual. The function $\mathcal{D}\theta$ then equals $D_LD_R$ and has character $\nu_0^2$ of order two. This computation also shows that if $D \equiv 1 \pmod{8}$, then the Teichmüller curve $W_D$ has two components, the vanishing loci of $D_L$ and of $D_R$. This is our equivalent of McMullen’s spin invariant that we referred to after Theorem 5.2.

Before proceeding to the proof of Theorem 9.1, we write out the Fourier expansions of the theta derivatives explicitly. Fix a basis $(\omega_1, \omega_2)$ of $\mathfrak{O}_D$ as we did in (25) for the construction of a Siegel modular embedding, say $(\omega_1, \omega_2) = (1, \gamma_0 = (D + \sqrt{D})/2)$. Then the first row the matrix $B$ is the basis of $\mathfrak{O}$ and the second row consists of the Galois conjugates. Given $\omega \in \mathbb{Z}^2$, we let

$$\rho(x) = (x + m) \cdot (1, \gamma_0)^T$$

for $x \in \mathbb{Z}^2$. Then for $(m, m')$ odd the theta derivative is given by

$$D_2\theta_{(m,m')}((z_1, z_2) = i^{4m(m')^T} \sum_{x \in \mathbb{Z}^2} (-1)^{2x \cdot (m')} \sigma(\rho(x))^2/2, \quad (78)$$

where $q_j = e(z_j)$ as usual.

**Proof of Theorem 9.1.** The modularity of $\mathcal{D}\theta$ for some level subgroup follows from the corresponding property of the Siegel theta functions, as shown in §9.1. It follows from the calculations above for $D \equiv 1 \pmod{8}$ that the factor group permutes the set of theta characteristics, preserving their parity. Moreover, we read off from the table that $\mathcal{D}\theta$ has character $\nu_0^2$ in this case. In other cases $D \not\equiv 1 \pmod{8}$, one checks similarly that the factor group still permutes theta characteristics, preserving their parity. In fact, classical calculations of Igusa show the full symplectic group $\text{Sp}(4, \mathbb{Z})$ preserves the parity. Consequently, $\mathcal{D}\theta$ is still a modular form, with some character, for the full Hilbert modular group.

For the second statement, let $C$ be a curve of genus two with period matrix $Z$. It is cut out in its Jacobian as the vanishing locus of $\Theta(0,0)(v, Z)$. See e.g. [FK80, chs VI and VII.1] for standard properties of theta functions and Weierstrass points. The two-torsion points in $\text{Jac}(C)$ are $Zm^T + (m')^T$ for $m, m' \in (\frac{1}{2}\mathbb{Z})^2$. The Weierstrass points of $C$ are precisely the six points $Zm^T + (m')^T$ for $(m, m')$ odd. Moreover, a holomorphic differential form $\omega$ has a double zero on $C$ if and only if $\omega$ vanishes (automatically doubly) at a Weierstrass point.

---

**Table 3.** The action of $\text{SL}(2, \mathbb{O})$ on the $D_2\theta_L$ and $D_2\theta_R$. 

<table>
<thead>
<tr>
<th>$f(z_1, z_2)$</th>
<th>$f(z_1 + 1, z_2 + 1)$</th>
<th>$f(z_1 + \alpha, z_2 + \sigma(\alpha))$</th>
<th>$z_1^{-3/2}z_2^{-9/2}f(1/z_1, 1/z_2)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$D_L$</td>
<td>$-D_L$</td>
<td>$-\zeta_DL$</td>
<td>$-DL$</td>
</tr>
<tr>
<td>$D_R$</td>
<td>$-D_R$</td>
<td>$\zeta_DL$</td>
<td>$-D_R$</td>
</tr>
</tbody>
</table>
Suppose that the point $[C] \in M_2$ lies on $W_D$. Then $C$ has real multiplication, so $Z = \psi(z)$ for some fixed Siegel modular embedding $\psi$. We identify the universal covering $V$ of the Jacobian $J(C)$ with the dual to $H^0(J(C), \Omega^1_{J(C)})$. We claim that the characterizing condition on the existence of an eigenform with a double zero (see Theorem 5.2) is equivalent to the vanishing of the derivative of the Riemann theta function in the second eigendirection $\omega_2$ at a Weierstrass point. In fact, the definition of the Abel–Jacobi map $C \rightarrow J(C)$ implies that its projectivized tangent map is given by $p \mapsto (\omega_1(p) : \omega_2(p))$ in the chosen eigenbasis of $H^0(J(C), \Omega^1_{J(C)})$ and so $\omega_1$ vanishes at $p$ if and only if the $\omega_2$-derivative of the defining equation of $C$ in $J(C)$ vanishes at $p$. Since we defined $u = Bv$ this just means

$$\sum_{j=1}^{2} \omega_j^2 \left( \frac{\partial}{\partial v_j} \Theta_{(0,0),D}(v, \psi(z)) \right) \bigg|_{v=\psi(z)m^T+(m')^T} = 0 \quad (79)$$

for some even $(m, m')$.

By the product rule, the derivative with respect to $v_i$ of the defining equation

$$\Theta_{(0,0)}(v + Zm^T + (m')^T, Z) = e^{-\pi imZm^T - 2\pi im(m + m')^T} \cdot \Theta_{(m,m')}(v, Z)$$

for $Z = \psi(z)$ has two summands. The factor given by differentiating the exponential term vanishes by definition for every pair of a point $z \in \mathbb{H}^2$ and $v \in \text{Jac}(C_z)$. Since $e(\cdot)$ is never zero, the vanishing of the derivative (79) is equivalent to $z$ being in the vanishing locus of $\mathcal{D}\theta$.

To show that the modular form $M_{W_D}$ vanishes nowhere else there are several options. The first is to remark that the above argument can be inverted for Jacobians of smooth curves. So one has just to show that the vanishing locus of $\mathcal{D}\theta$ is disjoint from the reducible locus. We give two proofs of this fact that do not rely on any Teichmüller theory in § 12. Yet another way to conclude is to compare the weight of $\mathcal{D}\theta$ with the modular form that cuts out $W_D$ in Bainbridge’s theorem from [Bai07, Theorem 10.2]. They are both of weight $(3, 9)$. \hfill \Box

The example $D = 17$ revisited. If we calculate the theta series and their derivatives for $D = 17$ as was done in § 8.2, then we can verify that the product of $D_R(z_1, z_2)$ with $D_R(z_2, z_1)$ is indeed proportional to the function $F_{17}$ given in (73), and similarly that the product of $D_L(z_1, z_2)$ with $D_L(z_2, z_1)$ is proportional to the Galois conjugate function $F_{17}^0$.

10. Cusps and multiminimizers

Fix an invertible $\mathcal{O}_D$-ideal $a$. Our aim in this section is to list the branches of the theta derivative vanishing locus $W_D$ through the cusp of the Hilbert modular surface $X_D$ determined by the class of $a$. Our Ansatz is to describe the branch of $W_D$ defined by a Hilbert modular form in $(q_1, q_2)$ by $q_1 = q^a$ and $q_2 = q^{\sigma(a)}(1 + P)$, where $q$ a suitable local parameter and $P$ a power series with positive valuation in $q$. This will lead us to consider a minimization problem on quadratic forms. The solutions are given ‘multiminimizers’.

The main result of this section will be the following characterization of cusps of $W_D$, derived from Fourier expansions only. We call an indefinite quadratic form $Q = [a, b, c]$ \textit{standard} \footnote{There seems to be no standard terminology for these quadratic forms and quadratic irrationalities. They appear implicitly in [CZ93] and [Bai07].} if $a > 0 > c$ and $a + b + c < 0$. To a quadratic form $Q$ we associate the quadratic irrationality $\lambda_Q = (-b + \sqrt{D})/2a$. The quadratic form is standard if and only if $\lambda = \lambda_Q$ satisfies

$$\lambda > 1 > 0 > \lambda^2. \quad (80)$$

A quadratic irrationality $\lambda$ satisfying (80) will also be called \textit{standard}. \hfill 2321
Theorem 10.1. Suppose that $D$ is a fundamental discriminant or, more generally, that $a$ is an invertible $\mathcal{O}_D$-ideal. Then there is a bijection between the cusps of $W_D$ mapping to the cusp $a$ of $X_D$ and the set of pairs $(Q, \tau)$ consisting of a standard quadratic form $Q = [a, b, c]$ of discriminant $D$ such that $[\langle 1, \lambda_Q \rangle ] = [a]$ together with a residue class $\tau \in \mathbb{Z}/(a, c)\mathbb{Z}$.

This will be proved in $\S$ 10.2. We will also comment on the case of non-invertible ideals after a discussion of cusps of Hilbert modular surfaces in the same subsection.

Given that $W_D$ is a Teichmüller curve, as we showed in Theorem 9.1 using the eigenform definition and will show again in the next section using the theta viewpoint only, this characterization reproves the list of cusps of [McM05]. (See also [Bai07, Theorem 8.7(5)].) For a comparison, we will briefly sketch the approach based on flat surfaces in $\S$ 10.3.

The proof of Theorem 10.1 can be applied verbatim to prove the following result on the reducible locus $P_D$ (see $\S$ 5.2) from the theta function viewpoint. This result was also proven by Bainbridge using the flat surface viewpoint on cusps.

Theorem 10.2. Let $D$ and $a$ be as above. Then there is a bijection between the cusps of $P_D$ mapping to the cusp $a$ of $X_D$ and the set of pairs $(Q, \tau)$ as in Theorem 10.1. In particular, for every given $a$ the numbers of cusps of $P_D$ and $W_D$ mapping to the cusp $a$ of $X_D$ coincide.

We will end the section by giving an algorithm in $\S$ 10.4 to compute multiminimizers using continued fractions.

10.1 Multiminimizers

Let $F(x, y) = Ax^2 + Bxy + Cy^2$ be a positive-definite binary quadratic form with real coefficients. On each of the cosets of $2\mathbb{Z}^2$ in $\mathbb{Z}^2$ this form assumes its minimum a finite number of times. Of course on the trivial coset the minimum is zero and is assumed exactly once, while on each of the other cosets the minimum is generically attained exactly twice, by some non-zero vector and its negative. We call the form $[A, B, C]$ multiminimizing if on at least one of the three non-trivial cosets the minimum is attained more than twice. These forms are classified by the following proposition.

Proposition 10.3. A positive-definite binary quadratic form is multiminimizing if and only if it is diagonalizable over $\mathbb{Z}$. In this case, there is exactly one coset of $2\mathbb{Z}^2$ in $\mathbb{Z}^2$ on which the form has a multiple minimum; this minimum is assumed exactly twice (up to sign) and is the sum of the minima in the other two cosets.

Proof. This is proved using reduction theory. Suppose that the form $F$ is multiminimizing. Since this property is obviously $\text{SL}(2, \mathbb{Z})$-invariant, we can assume that $F$ is reduced, i.e. $F = [A, B, C]$ with $C \geq A \geq |B|$. By Cauchy’s inequality we have $|Bxy| \leq A(x^2 + y^2)/2$, and hence

$$F(x, y) \geq \frac{A}{2} x^2 + \left( C - \frac{A}{2} \right) y^2,$$

for all $(x, y) \in \mathbb{R}^2$. In particular,

$|x| \geq 1, \ |y| \geq 2 \ \Rightarrow \ F(x, y) \geq 4C - 3A/2 > A,$
$|x| \geq 2, \ |y| \geq 1 \ \Rightarrow \ F(x, y) \geq C + 3A/2 > C,$
$|x| \geq 3, \ |y| \geq 1 \ \Rightarrow \ F(x, y) \geq C + 3A > A + |B| + C,$
$|x| \geq 1, \ |y| \geq 3 \ \Rightarrow \ F(x, y) \geq 9C - 4A > A + |B| + C.$

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These equations show that the smallest value of $F$ on the coset ‘(odd, even)’ is attained only at $(±1,0)$ and equals $A$, that the smallest value of $F$ on the coset ‘(even, odd)’ is attained only at $(0,±1)$ and equals $C$, and that the two smallest values of $F$ on the coset ‘(odd, odd)’ are attained only at $(±1,±1)$ and are equal to $A−B+C$ and $A+B+C$. In particular, $F$ is multiminimizing if and only if $B$ vanishes, in which case the only coset on which it attains its minimum more than once is ‘(odd, odd)’ and the minimum there is attained exactly twice (up to sign) and is the sum of the unique minima in the other two cosets. □

Let $a$ be a fractional $O_D$-ideal in a real quadratic field $K$ and $ξ$ a non-zero coset of $a$ in $\frac{1}{2}a$. (Thus there are three possibilities for $ξ$ given $a$.) We denote by $\widetilde{\text{MM}}(a,ξ)$ the set of non-zero $α ∈ K$ such that the quadratic form $F_α(x) = \text{tr}(αx^2)$ is positive-definite and assumes its minimum value on the coset $ξ$ more than twice (up to sign), and call the elements of this set multiminimizers for $ξ$. We denote by $\text{MM}(a)$ the set of all multiminimizers for $a$, i.e. the union of the sets $\widetilde{\text{MM}}(a,ξ)$ for all three cosets $ξ$. Clearly, $\text{MM}(a)$ is invariant under multiplication by positive rational numbers and by the squares of elements of the group $U_D$ of units $ε$ of $O_D$. We set

$$\text{MM}(a) = \widetilde{\text{MM}}(a)/(Q_+^* \cdot U_D^2).$$ (81)

Later we will often use the representatives of $α ∈ \text{MM}(a)$ that are primitive in $(a^2)^\land$. They are unique up to multiplication by $U_D^2$.

**PROPOSITION 10.4.** Let $a$ be a fixed fractional $O_D$-ideal of $K$. Then there is a bijection between $\text{MM}(a)$ and the set of standard quadratic forms $Q = [a,b,c]$ with $b^2 − 4ac = D$ in the wide ideal class of $a$.

Moreover, given $α ∈ \widetilde{\text{MM}}(a)$, there is a unique basis $(ω_1,ω_2)$ such that $ω_2 > ω_1 > 0$ of $a$ with respect to which the form $F_α$ is diagonal, and the coset $ξ$ is then $\frac{1}{2}(ω_1 + ω_2) + a$.

**Proof.** Given $Q = [a,b,c]$ choose $μ ∈ K^\times$ positive such that $μ⟨1,λQ⟩ = a$ and take

$$α = \frac{-ac}{μ^2λQ\sqrt{D}} = \frac{a}{μ^2} \cdot \frac{b + \sqrt{D}}{2\sqrt{D}}.$$ (82)

This is positive-definite if $Q$ is standard. Since $Q$ is simple the basis $ω_1 = μ$, $ω_2 = μλQ$ satisfies the conditions stated. Moreover, in this basis the quadratic form $F_α$ is

$$F_α = [A,0,C], \quad A = \text{tr}(αω_1^2) = a, \quad C = \text{tr}(αω_2^2) = −c$$

since $\text{tr}(αω_1ω_2) = \text{tr}(−ac/\sqrt{D}) = 0$. Thus, $α$ is a multiminimizer.

Conversely, if $α$ is a representative of a class in $\text{MM}(a)$, then it follows from Proposition 10.3 that there exists a unique basis $(ω_1,ω_2)$ of $a$ (up to interchanging the $ω_i$ and changing their signs) of $a$ with respect to which $F_α(x)$ is diagonal, and the coset $ξ$ is then $\frac{1}{2}(ω_1 + ω_2) + a$. This also proves the last statement. In this basis $F_α = [A,0,C]$ with $A = \text{tr}(ω_1^2)$ and $C = \text{tr}(ω_2^2)$. We choose signs and order the basis such that $λ = ω_2/ω_1 > 1$ and claim that then $σ(λ) < 0$, so that $λ$ is the root of a standard form. In fact, we have $α = q/(ω_1ω_2\sqrt{D})$ for some $q ∈ Q_+^\times$. Thus $C = \text{tr}(qλ/\sqrt{D})$ and

$$A = \text{tr}(q/(λ\sqrt{D})) = −N(λ)C.$$

Since $F_α$ is positive-definite (by definition of a multiminimizer), the numbers $A$ and $C$ are positive, so this implies that $N(λ) < 0$ as claimed. One checks immediately that $λ$ does not depend on the representative of the multiminimizer in $\text{MM}(a,ξ)$ we have chosen. We take $Q = [a,b,c]$
so that $\lambda$ satisfies $a\lambda^2 + b\lambda + c = 0$ with $a > 0$ and $a, b, c$ coprime integers, and since $\langle 1, \lambda \rangle$ is an invertible $\mathcal{O}_D$ module, we then have $b^2 - 4ac = D$.

Obviously, the composition $\lambda_Q \rightarrow \alpha(\lambda_Q) \rightarrow \lambda(\alpha)$ is the identity. In the other direction, note that $\alpha$ is determined by $(\omega_1, \omega_2)$ up to a positive rational number and that $\mu$ with $\mu(1, \lambda_Q) = a$ is determined up to a unit. Consequently, each of the distinguished basis elements $\omega_i$ is determined up to a unit and $\alpha$ is determined up to a square of this unit. Since multiminimizers were defined in (81) by these two equivalence relations, this shows the bijection we claimed.

\[\square\]

### 10.2 Cusps of Hilbert modular surfaces

Classically, cusps of the Hilbert modular surface $X_D$ are defined to be the equivalence classes of points in $\mathbb{P}^1(K)$ under the action of $\text{SL}(\mathcal{O}^\vee \oplus \mathcal{O})$. Equivalently, we may define a cusp as an exact sequence

$$0 \to a^\vee \to \mathcal{O}_D^\vee \oplus \mathcal{O}_D \to a \to 0$$

of torsion-free $\mathcal{O}$-modules up to the action of $\text{SL}(\mathcal{O}^\vee \oplus \mathcal{O})$ on $\mathcal{O}^\vee \oplus \mathcal{O}$ and its sub-$\mathcal{O}$-modules. The modules $a$ arising in this way are quasi-invertible, i.e. invertible $\mathcal{O}_E$-module for some order $\mathcal{O}_E \supset \mathcal{O}_D = \mathcal{O}$. Yet another equivalent viewpoint to define a cusp is by the class of an invertible $\mathcal{O}_E$-ideal $a$ together with an element $r \in \mathbb{Z}/(\sqrt{D/E})\mathbb{Z}$.

We briefly recall how to see the equivalence of these definitions. For the equivalence of the first two definitions, intersect the line $L \subset K^2$ determined by a point in $\mathbb{P}^1(K)$ with a fixed embedding of $\mathcal{O}^\vee \oplus \mathcal{O}$ in $K^2$ to get the exact sequence and conversely tensor the exact sequence.

As a preparation for the proof of Theorem 10.1 we determine the Fourier series of $\mathcal{D}\theta$ at a given cusp $a$ of $X_D$. For a basis $\omega = (\omega_1, \omega_2)$ of the ideal $a$ let $\rho_\omega(x) = (x + m) \cdot \omega^T$ for $x \in \mathbb{Z}^2$, with the dependence on $m$ suppressed in the notation. Then for $(m, m') \in (\frac{1}{2}\mathbb{Z})^2$ odd we define

$$D_2\theta_{(m, m'), \omega}(z_1, z_2) = \sum_{x \in \mathbb{Z}^2} (-1)^{2x \cdot (m')^T} \sigma(\rho_\omega(x)) q_1^{\rho_\omega(x)^2/2} q_2^{\sigma(\rho_\omega(x))^2/2}, \quad (83)$$

where $q_i = e(z_i)$. Note that $D_2\theta_{(m, m'), \omega}$ depends on the chosen basis $\omega$, but a base change can be compensated for by letting the base change matrix also act on the characteristic $(m, m')$. Consequently, the product

$$\mathcal{D}\theta_a = \prod_{(m, m') \text{ odd}} D_2\theta_{(m, m'), \omega} \quad (84)$$

is of the form $\sum_{y \in \mathcal{A}} c(y) \exp(\text{tr}(y^2 z))$ and thus invariant under an upper triangular matrix in $\text{SL}(a^\vee \oplus a)$.

**Lemma 10.5.** The Fourier expansion of $\mathcal{D}\theta$ at the cusp $a$ is proportional to $\mathcal{D}\theta_a$.

**Proof.** To avoid the generally hard problem of finding the Fourier development of a modular form at a different cusp we use the fact that the vanishing locus of $\mathcal{D}\theta$ has an intrinsic formulation in terms of eigenforms. In Theorem 9.1 we proved that this vanishing locus corresponds to the set of principally polarized abelian varieties with real multiplication such that the first eigenform has a double zero. This proof works for any Siegel modular embedding, for example the one given at the end of § 5.1, where the locus of real multiplication is $X_{D,a}$ and $\psi$ is constructed with the help of the matrix $B$ as in (25) having $\omega = (\omega_1, \omega_2)$ as its first column. The restriction of the
Siegel theta function with characteristic \((m, m')\) via this modular embedding is just \(D_2\theta_{(m, m')}\). To complete the proof, we note that the cusp at infinity of \(X_{D, a}\) is just the cusp \(a\) of \(X_D\). To see this, take a matrix \(\left(\frac{a\gamma}{a'} \ a'\right) \cap SL(2, K)\). It conjugates \(SL(0^\vee \oplus 0)\) into \(SL(a^\vee \oplus a)\), since \(a^\vee = a^{-1} 0^\vee\) and the line at infinity in \(\mathbb{P}^1_K\) intersects \(a^\vee \oplus a\) in the submodule \(a^\vee\), as required.

\[\text{Proof of Theorem 10.1.}\] We first determine the cusps of the vanishing locus of \(\mathbb{D}a\), as defined in (84), that map to the cusp \(\infty\) of the Hilbert modular surface \(X_{D, a}\). Because functions in a neighborhood of the cusp \(a\) have the form \(\sum_{\nu} c_\nu \chi(\nu z)\) with \(\nu \in (a^\vee a^{-1})^\vee = a^2\), we can choose a local parameter \(\nu\) at a branch of this locus, with \(q = 0\) at the cusp, of the form \(q = e(z/\nu)\) with \(\alpha \in (a^2)^\vee\) primitive. A lift of the vanishing locus to \(\mathbb{H} \times \mathbb{H}\) looks locally like \(z_2 = \varphi(z_1)\), where \(\varphi(z) = \alpha^2/z + C + \varepsilon(q)/2\pi i\) as \(\Im(z) \to \infty\) for some \(C \in \mathbb{C}\) and some power series \(\varepsilon\) in \(q\) with no constant term. Then we have

\[e(\nu z_1 + \nu^2 z_2)\big|_{\text{locus}} = e(C\nu^2) q^{\nu^2(\nu)\varepsilon(q)} \quad \text{for all } \nu \in a^2, \quad (85)\]

so that the restriction of any Hilbert modular form becomes a power series in \(q\). Making a different choice of the lifting would change \(C\) by an integral multiple of \(N(a)^2/\sqrt{D}/\alpha\), so that the quantity

\[S = e\left(\frac{C\alpha}{N(a)^2\sqrt{D}}\right) \in \mathbb{C}^* \quad (86)\]

is independent of the choice of the lifting. We will show below that \(S\) is in fact a rational power of an element of \(K^*\).

The resulting \(q\)-exponents after plugging (85) into (83) are of the form \(\nu = \nu^{tr(\nu)}\). In order for the theta derivative to vanish, the smallest exponent of \(q\) must occur twice, so this quadratic form has to take its minimum twice (with \(x\) and \(-x\) not distinguished). Hence, \(\alpha\) is a multiminimizer for \(a\). Recall that this specifies \(\alpha\) only up to \(Q^*_a U^2_D\), but here the fact that \(q = e(z/\nu)\) is a local parameter (or, equivalently, that \(\alpha\) is primitive) eliminates the \(Q^*_a\)-ambiguity. The \(U^2_D\)-ambiguity corresponds to the fact that this group (considered as diagonal matrices in the Hilbert modular group) stabilizes the given cusp \(\alpha\) of the Hilbert modular surface. Since \(U^2_D\) acts transitively on the three non-trivial cosets \(\zeta\) of \(a/a^\vee\) and since a multiminimizer has the multiple minimum property on exactly one of the three cosets by Proposition 10.3, we may suppose from now on that the branch is chosen such that \(D_2\theta_{(m, m')}\) for \(m = \left(\frac{1}{2}, \frac{1}{2}\right)\) vanishes on that branch.

We next determine \(\alpha\) and exhibit for this purpose a convenient basis of \(a^2\). We suppose that \(\omega_1, \omega_2\) was from the beginning of the discussion the distinguished basis of \(a\) associated with a multiminimizer in Proposition 10.4. We let \(g = \gcd(a, c)\), where \(ax^2 + bx + c = 0\) is the minimal polynomial of \(\lambda = \omega_2/\omega_1\). Choose \(s, t \in \mathbb{Z}\) such that \(sa + tc = g\). It is easily verified that

\[a^2 = \langle \omega_1^2, \omega_1 \omega_2, \omega_2^2 \rangle = \langle \alpha^*, \beta^* \rangle,\]

where

\[\alpha^* = g/\omega_1^2 + tb/a \omega_1 \omega_2 \quad \text{and} \quad \beta^* = 1/\omega_1 \omega_2\]

since \(\gcd(a, b, c) = 1\). (In fact, \(a^2\) contains \((c/a)\omega_1^2 + (b/a)\omega_1 \omega_2\) using the minimal polynomial, so \((b/g)\omega_1 \omega_2\) and finally \((1/g)\omega_1 \omega_2\) by the gcd condition. The ideal also contains \((ct/a)\omega_1^2 + (tb/a)\omega_1 \omega_2\) and, since \(\gcd(a, tc) = g\), also \(\alpha^*\). The converse inclusion follows from the line below (89).) This basis is chosen such that the dual basis is \(\langle \alpha, \beta \rangle\) where

\[\alpha = \frac{-ac}{g\omega_1^2} \cdot \frac{a}{g\omega_1^2} \cdot \frac{b + \sqrt{D}}{2\sqrt{D}}. \quad (87)\]
As local coordinates on the Hilbert modular surface we now pick $X = q_1^{\alpha} q_2^{\sigma(\alpha^*)}$ and $Y = q_1^{\theta} q_2^{\sigma(\beta^*)}$, which are power series in the local coordinate $q$ beginning with $q$ and $S$, respectively. In these coordinates the factors of $D\theta_\alpha$ have the expansion

$$D_2\theta_{(m,m'),\omega}(z_1, z_2) = \sum_{x \in \mathbb{Z}^2} (-1)^{2x - (m')^T} \rho_\omega(x)^\sigma X^{F(\tilde{x}_1, \tilde{x}_2)/2} Y^{G(\tilde{x}_1, \tilde{x}_2)/2},$$

(88)

where $\tilde{x}_i = x_i + \frac{1}{2}$ and where

$$F = \left[ \frac{a}{g}, 0, -c \right], \quad G = [-bt, 2g, bs].$$

(89)

(This follows from $(a/g)\alpha^* - bt\beta^* = \omega_1^2$, $2g\beta^* = 2\omega_1\omega_2$ and $-(c/g)\alpha^* - bs\beta^* = \omega_2^2$.) Then $D_2\theta_{(m,m'),\omega}/X^{1/4}Y^{1/8}$ has an expansion in integral powers of $X$ and $Y^{1/2}$, but we actually need the product $\mathcal{F} = D_2\theta_{(1/2,1/2),(1/2,0),\omega} D_2\theta_{(1/2,1/2),(0,1/2),\omega}$, and this product, divided by $X^{1/4}Y^{1/4}$, has integral powers of $Y$ as well.

Finally we need to show that map from cusps to multiminimizers is onto and that the fibers have cardinality $g$. Analyzing the lowest-order coefficient in $q$ of $\mathcal{F}$, to which precisely the summands $\tilde{x}_1 = \pm \frac{1}{2}$ and $\tilde{x}_2 = \pm \frac{1}{2}$ contribute, and noting that $G(\frac{1}{2}, \frac{1}{2}) - G(\frac{1}{2}, -\frac{1}{2}) = g$, we find that the $S$ of (86) has to be a solution of

$$S^g = \left( \frac{\rho_\omega(\frac{1}{2}, -\frac{1}{2})}{\rho_\omega(\frac{1}{2}, \frac{1}{2})} \right)^\sigma.$$

(90)

This equation has precisely $g$ solutions, differing by $g$th roots of unity. For each such solution of the lowest-order term in $q$ there is a unique power series $\varepsilon(q)$ such that (85) is in the vanishing locus of $D\theta_\alpha$, since the coefficients of $\varepsilon(q)$ are recursively determined by a triangular system of equations. (We will discuss the arithmetic properties of $\varepsilon(q)$ in §13.1.)

This completes the proof for the vanishing locus of $D\theta_\alpha$ at the cusp $\infty$ and by Lemma 10.5 also the proof of Theorem 10.1. □

A statement such as Theorem 10.1 can certainly be proven along the same lines also if $\alpha$ is not an invertible $\mathcal{O}_D$-ideal (which can of course only happen for non-fundamental $D$). This has an effect in Lemma 10.5, and the fact that $\gcd(a, b, c) > 1$ changes the computation of the basis $\{\alpha^*, \beta^*\}$ that was used in the proof. Since our aim is just to demonstrate the method of reproving the properties of genus two Teichmüller curves using theta functions, we do not carry this out in detail.

**Proof of Theorem 10.2.** The reducible locus is the vanishing locus of the product of all $10$ even theta functions. Thus branches of the vanishing locus have to be parametrized as in (85) with $\alpha$ a multiminimizer. For $m = (0, 0)$ the forms are never multiminimizing and in all other cases the proof proceeds as the proof of Theorem 10.1. Only the coefficients of (90) change, but not the exponents. This does not affect the number of cusps for each multiminimizer. □

### 10.3 Cusps of $W_D$ via flat surfaces and prototypes

The cusps of the Teichmüller curve $W_D$ were first determined in [McM05] based on the following observations. It was discovered by Veech along with the definition of Teichmüller curves that cusps correspond to directions (considered as elements of $\mathbb{R}^2/\mathbb{R}^*$) of *saddle connections* (i.e. geodesics for the flat metric $|\omega|$ starting and ending at a zero of $\omega$). Applying a rotation
to the flat surface, we may suppose that the direction is horizontal, so that the subgroup of
$GL(2, \mathbb{R})$ stabilizing the direction consists of upper triangular matrices. McMullen discovered
that flat surfaces parametrized by $W_D$ always decompose in saddle connection directions into
two cylinders as indicated in Figure 2. The action of the upper triangular group allows us to
assume the upper cylinder to be a square, while still having the freedom to be normalized by the
action of $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. McMullen further deduces from real multiplication that the two cylinders
must be isogenous with a homothety in a real quadratic field. We thus may suppose that $(a, c, q)$
are integral and $\lambda$ is real quadratic. A more careful analysis of the isogeny (see [McM05, (2.1)]
or [Bai07, Proposition 7.20]), imposing moreover that $O_D$ is the exact endomorphism ring of a
generic abelian surface parametrized by the Teichmüller curve, implies that $\lambda = [a, b, c]$ with $a$
and $c$ as in Figure 2 and such that $(a, b, c, r) = 1$.

Obviously $a > 0$, $c < 0$ and the side length inequality $-a\lambda^\sigma < -c$ is equivalent to $a + b + c < 0$. By the action of $T$ we may reduce mod $\gcd(a, c)$. Altogether this implies that cusps
of $W_D$ correspond bijectively to pairs $(\lambda = [a, b, c], q)$ with $\lambda$ standard and $q \in \mathbb{Z}/\gcd(a, c)$, as in
Theorem 10.1.

10.4 Computing multiminimizers

There are many variants of how to expand real numbers into continued fractions. One may
subtract (slowly) one at each step or group these steps together, one may subtract one while
being greater than one, greater than zero, or until becoming negative. We will need two of them,
one here and one in §11.2.

The slow plus greater-than-one continued fraction expansion of $x \in \mathbb{R}_{>1}$ is defined by $x_0 = x$
and then inductively by

$$x_{n+1} = \begin{cases} 
x_n - 1 & \text{if } x_n > 2, \\
1/(x_n - 1) & \text{if } 2 > x_n > 1,
\end{cases}$$

so that

$$x = 1 + \cdots + 1 + \left(1 + \frac{1}{1 + \cdots + 1 + \left(1 + \frac{1}{\ddots}\right)}\right).$$
The class of the ideal \([a] = \langle 1, x_n \rangle\) is unchanged under this continued fraction algorithm, since \(\langle 1, x_n \rangle = \langle 1, x_n - 1 \rangle = (x_n - 1)/1 = 1/(x_n - 1)\).

**Lemma 10.6.** The slow plus greater-than-one continued fraction expansion of any quadratic irrational \(\lambda\) is periodic, and is pure periodic if and only if \(\lambda\) is standard.

**Proof.** The classical proof of Lagrange (for the more usual fast continued fraction algorithm with plus sign, where \(\lambda\) is sent to \(\lambda - [\lambda]\) if \(\lambda > 1\) and to \(1/\lambda\) if \(0 < \lambda < 1\)) applies here as well. The slow algorithm here just introduces intermediate steps and combines two steps when \(1 < \lambda < 2\).

This observation gives us an algorithm to compute all multiminimizers for a given ideal \(a\). Write \(a = \mu_0(\lambda_0, 1)\) and apply the continued fraction algorithm to \(\lambda_0\) until it becomes pure periodic, say at a standard quadratic irrational \(\lambda_1\) with \(a = \mu_1(\lambda_1, 1)\). The first multiminimizer is then the \((U_2D \cdot \mathbb{Q}^*)\)-class of \(\alpha_1 = -1/(\mu_1^2(\lambda_1, \sqrt{D}))\) as in (82). Proceeding in this way with \(\alpha_k = -1/(\mu_k^2(\lambda_k, \sqrt{D}))\) where \(a = \mu_k(\lambda_k, 1)\) for \(\lambda_2, \ldots, \lambda_n\) along the period of \(\lambda_1\) gives all of the multiminimizers for \(a\).

In each of the steps we may choose \(\alpha\) in its \(\mathbb{Q}^*\)-class to be primitive in some fixed \(\mathcal{O}_D\)-module. We will use this normalization for the Bainbridge compactification in the next section.

11. The Hirzebruch and the Bainbridge compactification

Hirzebruch’s minimal smooth compactification is a toroidal compactification. In this section we reinterpret the Bainbridge compactification of Hilbert modular surfaces as a toroidal compactification, see Theorem 11.5. This compactification was defined originally via the Deligne–Mumford compactification of the moduli space of one-forms on Riemann surfaces, We will see that both Hirzebruch’s and Bainbridge’s compactification can be computed using continued fraction algorithms.

We start with a review of toroidal compactifications from the viewpoint of curve degenerations and we recall Hirzebruch’s continued fraction algorithm. The procedure presented in §10.4 to compute multiminimizers is very similar to Hirzebruch’s, but the continued fraction algorithms are different. The geometry of Bainbridge’s compactification has been described in [Bai07]. We recall this material in §11.4, since it is part of the main Theorem 11.5. Once this is proven, all geometric properties will follow from general facts about toroidal compactifications. (See Proposition 11.2.) The proof of the main theorem will be given in §11.5.

The description of toroidal compactifications will make it easy to determine (in §11.6) where the cusps of modular curves \(F_N\) or Teichmüller curves \(W_D\) intersect the cusp resolution cycle both of the Hirzebruch compactification and the Bainbridge compactification. Finally, in §11.8 we compare the two continued fraction algorithms governing the two compactifications and give formulas for the total lengths or the cusp resolution cycles in both cases.

11.1 Toroidal compactifications

The reader may consult the textbook by Ash et al. [AMRT75] as a general reference on toroidal compactifications. We give a self-contained treatment for the case of Hilbert modular surfaces. Locally near \((\infty, \infty) \in \mathbb{H}^2\) a Hilbert modular surfaces is \(\mathbb{H}^2/G(M, V)\), where \(M\) is a complete submodule of \(K\) (i.e. an additive subgroup of \(K\) that is free abelian of rank two) and \(G(M, V) \subset \text{SL}(\mathcal{O}_D^* \oplus \mathcal{O}_D)\) is the semidirect product of \(M\) and some subgroup \(V\) of totally positive units (see e.g. [vdG87]).
We study the limiting behavior of a complex curve $C$ in $\mathbb{H}^2$ parametrized by $\tau \in \mathbb{H}$ and having the asymptotic form
\[
\begin{align*}
 z_1 &= \gamma \tau + A_0 + A_1 q + A_2 q^2 + \cdots \\
 z_2 &= \gamma' \tau + B_0 + B_1 q + B_2 q^2 + \cdots
\end{align*}
\] (91)
for $\Im(\tau) \to \infty$, where $\gamma \in K$ is totally positive, $q = e(\tau)$ and $A_i, B_i \in \mathbb{C}$. If this curve descends to an algebraic curve in the Hilbert modular surface $X_D$, so that the intersection with the subgroup $M$ of the cusp stabilizer is not trivial, we can always assume that the leading coefficients are of that form. In fact, such a curve is always given as (part of) the vanishing locus of some Hilbert modular form. Looking at the $q$-expansion of this Hilbert modular form, we deduce (91) by the same argument as given at the beginning of the proof of Theorem 10.1.

To each totally positive $\alpha \in M$ we associate a copy $\mathbb{P}^1_\alpha$ of $\mathbb{P}^1(\mathbb{C})$. We have a map
\[
X_\alpha : \left\{ \begin{array}{l}
\mathbb{H}^2/M \to \mathbb{C}^*_\alpha \subset \mathbb{P}^1_\alpha \\
(z_1, z_2) \mapsto \mathfrak{e}(\alpha \gamma z) = \mathfrak{e}\left(\frac{\alpha^\sigma z_1 - \alpha z_2}{N(M)\sqrt{D}}\right),
\end{array} \right.
\]
where $N(M)$ denotes the norm of $M$. Using this identification, we glue $\mathbb{P}^1_\alpha$ at the cusp $(\infty, \infty)$ of $\mathbb{H}^2$ topologically as follows. A sequence of points $z = (z_1, z_2) \in \mathbb{H}^2$ with imaginary parts $y_1, y_2$ both tending to $\infty$ with limiting value of $y_1/y_2$ being equal to $t \in \mathbb{R}_+$ converges to the point $0 \in \mathbb{P}^1_\alpha$ if $t < \alpha/\alpha^\sigma$, to the point $\infty \in \mathbb{P}^1_\alpha$ if $t > \alpha/\alpha^\sigma$ and to a finite point $X = \mathfrak{e}(\theta/N(M)\sqrt{D})$ for some $\theta \in \mathbb{C}$ if $t = \alpha/\alpha^\sigma$ and if $\alpha^\sigma z_1 - \alpha z_2 = \theta + o(1)$.

Consequently, the curve $C$ meets $\mathbb{P}^1_\alpha$ at $\infty$ if $\text{tr}(\alpha \gamma) < 0$, at $0$ if $\text{tr}(\alpha \gamma) > 0$, and at a finite, non-zero, point if $\gamma$ is a rational multiple of $\alpha$.

To any oriented $\mathbb{Q}$-basis $(\alpha, \beta)$, i.e. with $\alpha/\alpha^\sigma < \beta/\beta^\sigma$, corresponds a pair of projective lines $\mathbb{P}^1_\alpha$ and $\mathbb{P}^1_\beta$ meeting at one point $(\infty, 0) \in \mathbb{P}^1_\alpha \times \mathbb{P}^1_\beta$. Suppose that $\gamma \in K$ has the property
\[
\frac{\alpha}{\alpha^\sigma} < \frac{\gamma}{\gamma^\sigma} < \frac{\beta}{\beta^\sigma}
\]
holds, so that the curve $C$ passes through the point $(\infty, 0)$. If we write $\gamma = p\alpha + q\beta$ with $p, q \in \mathbb{Q}_+$ then, near the point $(\infty, 0)$ the curve (91) looks like
\[
(X_\alpha^{-1})^p = (X_\beta)^q.
\]

We will, of course, apply this in particular to the curves $F_N$ and to Teichmüller curves.

A $V$-invariant partial compactification $\overline{\mathbb{H}^2}/M$ of a Hilbert modular surface is defined by adding not just one $\mathbb{P}^1_\alpha$, but an appropriate sequence of $\mathbb{P}^1_\alpha$.

**Definition 11.1.** A sequence of numbers $(\alpha_n)_{n \in \mathbb{N}}$ in $M$ forms a fan if (i) the $\alpha_n$ are all totally positive, (ii) $V \cdot \{\alpha_n\}_{n \in \mathbb{N}} = \{\alpha_n\}_{n \in \mathbb{N}}$ and (iii) the ratios $\alpha_n^\sigma/\alpha_n$ are a strictly decreasing sequence.

Our definition includes that the fan is $V$-invariant. If $V = \langle \varepsilon \rangle$, then (ii) is equivalent to the existence of some $k$ such that $\varepsilon \alpha_n = \alpha_{n+k}$. The minimal positive $k$ with that property will be called the length of the fan.

**Proposition 11.2.** A fan $(\alpha_n)_{n \in \mathbb{N}}$ determines a partial compactification $\overline{\mathbb{H}^2}/M$ of $\mathbb{H}^2/M$ with the following properties. For each $n$ there is an irreducible curve $\mathbb{P}^1_{\alpha_n}$ in $\overline{\mathbb{H}^2}/M \setminus \mathbb{H}^2/M$. The curves $\mathbb{P}^1_{\alpha_n}$ and $\mathbb{P}^1_{\alpha_{n+1}}$ intersect in one point. For $k > 1$ the curves $\mathbb{P}^1_{\alpha_n}$ and $\mathbb{P}^1_{\alpha_{n+k}}$ are disjoint in $\mathbb{H}^2/M$. 

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The action of $V$ on $\mathbb{H}^2/M$ extends to an action on $\overline{\mathbb{H}^2}/M$. Hence, a fan determines a partial compactification $\overline{\mathbb{H}^2}/G(M,V)$ of $\mathbb{H}^2/G(M,V)$.

The partial compactification is always smooth along $\mathbb{P}^1_{\alpha_n}$ minus the intersection points with the $\mathbb{P}^1_{\alpha_{n+1}}$. At the intersection point of $\mathbb{P}^1_{\alpha_n}$ and $\mathbb{P}^1_{\alpha_{n+1}}$ the compactification is smooth if and only $M = Z\alpha_n + Z\alpha_{n+1}$. More generally, if $Z\alpha_n + Z\alpha_{n+1}$ has index $k$ in $M$, then $\overline{\mathbb{H}^2}/M$ has a cyclic quotient singularity of order $k$ at that point. In particular, the compactification is normal.

**Proof.** We first recall the general setup of two-dimensional toric varieties, e.g. the book of Fulton [Ful93] provides an introduction to this topic. For each two-dimensional cone $\sigma$ given, say, as the span of $\alpha_1$ and $\alpha_{n+1}$ we let $U_\sigma$ be the variety with coordinate ring $\mathbb{C}[\sigma^\vee \cap M^\vee]$. For each one-dimensional cone $\tau$ at the boundary of $\sigma_2$, say spanned by $\alpha_n$ we let $V_\tau \subset U_\sigma$ be the variety with coordinate ring $\mathbb{C}[\tau^\vee \cap M^\vee]$. If $\tau$ is contained both in $\sigma_1$ and $\sigma_2$ we may glue $U_{\sigma_1}$ and $U_{\sigma_2}$ along the open set $V_\tau$. In particular $a_n$ (or $\tau = \langle a_n \rangle$) determines a rational curve $\mathbb{P}^1_{\alpha_n} = (U_{\sigma_1} \cup U_{\sigma_2}) \setminus V_\tau$. The zero-dimensional cone $0$ corresponds to the variety $W$ with coordinate ring $\mathbb{C}[M^\vee]$ and sits as open part in all the $V_\tau$ and $U_\sigma$. Here $M^\vee$ is the dual $\mathbb{Z}$-module, which we will identify from now on as a submodule of $K$ using the trace pairing.

In our situation we want to identify $W$ with $\mathbb{C}^2/M$. On the level of local coordinate rings this is done by assigning to $b \in M^\vee$ the coordinate function

$$X_b^\vee(z_1, z_2) = e(bz_1 + b^\sigma z_2).$$

(This is the same coordinate as $X_{\alpha_1}^\vee$ associated above with $\alpha \in M$, using the identification of $M$ with $M^\vee$ via $\alpha \mapsto \alpha^\vee = \alpha^\sigma/(N(M)\sqrt{D})$. We refer to $\alpha^\vee$ as the *trace dual* of $\alpha$.) Since $\mathbb{H}^2/M$ sits inside $\mathbb{C}^2/M$, the partial compactification of $W$ by the $U_\sigma$ defines the desired partial compactification of $\mathbb{H}^2/M$.

Given a cone $\tau$ generated by $\alpha_n$, the element $\alpha_n^\vee$ is the unique (up to sign) primitive element of $M^\vee$ with $\text{tr}(\alpha_n \alpha_n^\vee) = 0$. If we complete $\alpha_n^\vee$ to a basis of $M^\vee$ using $\beta_n$, whose sign we may choose such that $\text{tr}(\alpha_n^\vee \beta_n) > 0$, then the curve $\{X_{\beta_n}^\vee = 0\}$ is independent of the choice of $\beta_n$ and coincides with the curve $\mathbb{P}^1_{\alpha_n}$ defined in the text preceding the proposition.

Given two consecutive elements $\alpha_n$ and $\alpha_{n+1}$ of the fan, the monotonicity of the ratios implies that

$$\text{tr}(\alpha_n^\vee \alpha_{n+1}) < 0 \quad \text{and} \quad \text{tr}(\alpha_n^\vee \alpha_n) > 0.$$ 

Hence, $\alpha_n^\vee$ and $-\alpha_{n+1}^\vee$ can play the role of $\beta_n$ above. Consequently, the intersection point of $\mathbb{P}^1_{\alpha_n}$ and $\mathbb{P}^1_{\alpha_{n+1}}$ is the point $\{X_{\alpha_n}^\vee = 0, (X_{\alpha_{n+1}}^\vee)^{-1} = 0\}$. Since $X_{\alpha_j}$ is a coordinate on $\mathbb{P}^1_{\alpha_j}$, we retrieve that the intersection point of $\mathbb{P}^1_{\alpha_n}$ and $\mathbb{P}^1_{\alpha_{n+1}}$ is $(\infty, 0)$ in that coordinate system.

The disjointness of $\mathbb{P}^1_{\alpha_n}$ and $\mathbb{P}^1_{\alpha_{n+k}}$ is obvious from the gluing procedure. The statement on the $V$-action is an obvious consequence of the $V$-invariance of a fan.

The singularity statements are described in detail in [Ful93, §2.2].

Replacing the $\mathbb{Q}$-basis $(\alpha, \beta)$ by $(\alpha + \beta, \beta)$ respectively by $(\alpha, \alpha + \beta)$ corresponds to performing a sigma-transformation (or blowup) defined by the coordinate changes

$$(X^{-1}_\alpha, X_{\beta}) \mapsto (X^{-1}_\alpha X^{-1}_\beta, X_{\beta})$$

respectively

$$(X^{-1}_\alpha, X_{\beta}) \mapsto (X^{-1}_{\alpha}, X_{\alpha} X_{\beta})$$

at that point. It follows that the partial compactifications defined by any two fans are related by repeated blowing up and blowing down.

\[2330\]
11.2 Hirzebruch’s compactification

We describe the fan of Hirzebruch’s compactification. There are many detailed expositions of this, in particular [Hir73] and [vdG87, ch. II]. The aim is to compare the fans of Hirzebruch and Bainbridge below. We will use the letters $A_k$ for Hirzebruch’s fan (which is consistent with [vdG87]) and subscripts $k$ for the indexing that produces an increasing sequences of slopes. (Thus, if we replace $A_0$ by $A_{k_0-1}$ for some $k_0$ we fit exactly the Definition 11.1.)

Suppose we want to compactify the Hilbert modular surface $X_D$ at the cusp $a$ or equivalently the cusp at $\infty$ in $X_{D,a}$. Then, in the notation of §11.1 the module $M = \mathfrak{a}^\vee (a^{-1}) = (a^2)^\vee$. We choose the $A_k$ to be the set of extreme points in $M^+ = M \cap (\mathbb{R}_+)^2$, i.e. the points lying on the convex hull conv$(M^+)$ of $M^+$ in $(\mathbb{R}_+)^2$, indexed by increasing slope then the $A_k$ form a fan since conv$(M^+)$ is $V$-invariant. This compactification is smooth, because any two adjacent points on the boundary of conv$(M^+)$ form a $\mathbb{Z}$-basis of $M$ ([Hir73] or [vdG87], (II) Lemma 2.1). Since the $A_k$ lie on the boundary of the convex hull, we can write

$$A_{k-1} + A_{k+1} = p_k A_k \quad \text{with } p_k \in \mathbb{Z}, \ p \geq 2. \quad (92)$$

On this smooth compactification the self-intersection number of $\mathbb{P}^1_{A_k}$ is $-p_k$ (see [vdG87, §II.2]). Consequently, the compactification using at a cusp with stabilizer $G(M,V)$ the fan given by the boundary points of conv$(M^+)$ is the minimal smooth compactification.

We call $x$ reduced (or a reduced quadratic irrationality) (see [vdG87, §2.5] or [Zag81]) if $x$ is real quadratic and

$$x \geq 1 > x^2 > 0. \quad (93)$$

The fast-minus continued fraction algorithm of $x \in \mathbb{R} > 1$ is defined by $x_0 = x$ and then inductively by

$$x_{k+1} = 1/(p_k - x_k) \quad \text{where } p_k = \lfloor x_k \rfloor, \quad (94)$$

so that

$$x_k = p_k - \frac{1}{p_{k+1} - \frac{1}{\ddots}}. \quad (94)$$

Note that the narrow class of the ideal $[M] = \langle 1, x \rangle$ is unchanged under this continued fraction algorithm, since $\langle 1, x \rangle = \langle p-x \rangle \langle 1, 1/(p-x) \rangle$. The following lemma is the analogue of Lemma 10.6 above.

**Lemma 11.3.** The fast minus continued fraction expansion of any quadratic irrational $x$ is periodic, and is pure periodic if and only if $x$ is reduced.

**Proof.** This is well known and is stated, for instance, in [Hir73, §2.5], where the relation of the fast plus continued fraction expansion and the fast minus continued fraction expansion is also given. \qed

This observation gives us an algorithm to compute the convex hull. Write $M = \mu_0 \langle 1, x_0 \rangle$ and apply the continued fraction algorithm until it becomes pure periodic, say at a reduced quadratic irrational $x_1$ with $M = \mu_1 \langle 1, x_1 \rangle$. Take $A_0 = \mu_1$ and $A_1 = \mu_1 x_1$ and then $A_k = A_{k-1}/x_k$ for $k \geq 2$, where $x_1, x_2, \ldots, x_n$ form the cycle of the continued fraction algorithm. The recursive definition of the $x_k$ in (94) then is equivalent to (92).

**Conclusion.** The Hirzebruch compactification is given by the fan stemming from the lower convex hull or, equivalently, triggered by the fast minus continued fraction algorithm.
11.3 Bainbridge’s compactification

Bainbridge’s compactification of a Hilbert modular surface is defined using the Deligne–Mumford compactification of the moduli space of curves of genus two. For the details we recall facts about various bundles of one-forms over moduli spaces.

In §5.3 we introduced together with Teichmüller curves the vector bundle $\Omega M_g$ of holomorphic one-forms over the moduli space $M_g$ of curves of genus $g$. The moduli space of curves comes with the Deligne–Mumford compactification by stable curves $\overline{M}_g$ and the vector bundle $\Omega M_g$ extends to a vector bundle $\Omega \overline{M}_g$, whose sections are stable forms. A stable form is a differential form on the normalization of the stable curves, holomorphic except for at most simple poles at the preimages of nodes and such that the residues at the two branches of a node add up to zero. We refer to $\Omega \overline{M}_g$ and to the corresponding projective bundle $P\Omega \overline{M}_g$ as the Deligne–Mumford compactification of $\Omega M_g$ (respectively of $P\Omega M_g$).

Strictly contained between $M_g$ and $\overline{M}_g$ is the partial compactification $\tilde{M}_g$ of stable curves of compact type, i.e. stable curves whose Jacobian is compact or equivalently of arithmetic genus $g$.

For $g = 2$ the Torelli map $t : \tilde{M}_2 \to A_2$ is an isomorphism. It extends to an isomorphism of the bundles of stable one-forms $t : \Omega \tilde{M}_2 \to \Omega A_2$ and also to the projectivized bundles $t : P\Omega \tilde{M}_2 \to P\Omega A_2$.

On the other hand, we have seen in §5.1 that a Hilbert modular surface $X_D$ parametrizes principally polarized abelian varieties with real multiplication. There is a unique holomorphic one-form on such an abelian variety that is an eigenform for the action of real multiplication (with the embedding $K \to \mathbb{R}$ that we fixed throughout). The quotient map of a Siegel modular embedding by the action of $SL(2,\mathcal{O}_D)$ defines a map $X_D \to A_2$ and the choice of an eigenform lifts this map to an injection $\psi : X_D \to P\Omega A_2$.

We thus use the same letter for this map as for the modular embedding. The image $t^{-1}(\psi(X_D))$ is called the eigenform locus (maybe the projectivized eigenform locus would be more precise). It parametrizes stable curves of genus two of compact type with real multiplication by $\mathcal{O}_D$. We denote by $\overline{X_D}^{DM}$ the closure of the eigenform locus in the Deligne–Mumford compactification $P\Omega \overline{M}_2$.

**Definition 11.4.** The Bainbridge compactification (called the geometric compactification in [Bai07]) $\overline{X_D}^B$ of a Hilbert modular surface $X_D$ is the normalization of $\overline{X_D}^{DM}$.

We now identify this compactification as a toroidal compactification. We restrict to the case $D$ fundamental for simplicity and since the preparations in §10 have been carried out for this case only. In §11.5 we will prove the following theorem.

**Theorem 11.5.** Suppose that $D$ is a fundamental discriminant. For each cusp given by the ideal class $a$ the sequence of multiminimizers for $a$, as derived from the continued fraction algorithm in §10.4, forms a fan. The Bainbridge compactification is the toroidal compactification of the Hilbert modular surface obtained by using this sequence of multiminimizers.

Comparing with §11.2, we can summarize this theorem and the algorithm in §10.4 as follows.

**Conclusion.** The Bainbridge compactification is given by the fan stemming from the multiminimizers or, equivalently, triggered by the slow plus greater-than-one continued fraction algorithm.
In §11.4 we review the properties of the compactification by Bainbridge. For the proof of Theorem 11.5 we will need only part of these properties. The local structure at the intersection points of the compactification curves is forced by normality and could also be derived from Proposition 11.2.

11.4 Period coordinates and properties of the Bainbridge compactification

In order to identify the Bainbridge compactification and to prove Theorem 11.5 we use the part of the work of Bainbridge, where he gives a coordinate system of $\Omega M_2$ and describes $\overline{X_D}^{DM}$ in there. He uses a lift by choosing a scaling of the one-form $\omega$ by fixing locally a loop $\alpha_1$ and by imposing that $\int_{\alpha_1} \omega = r_1$ for some $r_1 \in K$. We will then compare the coordinates introduced for toroidal compactifications to this coordinate system and thereby prove the claimed isomorphism.

There are two relevant types of coordinate systems, both called period coordinates depending on the type of the stable curves. We follow [Bai07, §6.6]. If $(X, \omega)$ is in the boundary of $\overline{X_D}^{DM}$, then $g(X)$ is zero. The first type of coordinate system is around a stable curve $X$ with two non-separating nodes. We moreover suppose that $\omega$ has two simple zeros. We let $\pm r_1$ and $\pm r_2$ be the residues of $\omega$. Choose loops $\alpha_1, \alpha_2$ around the punctures such that $\int_{\alpha_i} \omega = r_i$. For a smooth surfaces in a neighborhood choose loops $\beta_1, \beta_2$ that complete the $\alpha_i$ curves to a symplectic basis. Finally, choose a path $I$ joining the two zeros of $\omega$. Then on a neighborhood of $(X, \omega)$ in $\Omega M_2$ the functions

$$
v^{(E)} = \int_{\alpha_1} \omega, \quad w^{(E)} = \int_{\alpha_2} \omega, \quad y^{(E)} = e\left(\int_{\beta_1} \omega/v\right), \quad z^{(E)} = e\left(\int_{\beta_2} \omega/w\right), \quad x^{(E)} = \int_I \omega
$$

are well defined, i.e. independent of ambiguity in the choice of $\beta_i$ given by Dehn twists around the corresponding $\alpha_i$. These five functions form a system of coordinates on $\Omega M_2$. We provide them with a superscript $E$ (edge) to distinguish them. They will correspond to edges of the boundary of $\overline{X_D}^{DM}$. Note that $x = x^{(E)}$ is only well defined up to an additive constant depending on the path of integration and that its sign depends on the choice of ordering of the two zeros.

From the geometry of the Deligne–Mumford compactification Bainbridge derives the following proposition that we use to prove Theorem 11.5.

**Proposition 11.6** [Bai07, Proposition 7.18, Theorems 7.17 and 7.22]. A pair $(r_1, r_2)$ appears as residues of an eigenform of $X_D$ if and only if $\lambda = r_2/r_1 \in \mathbb{Q}(\sqrt{D})$ and $N(\lambda) < 0$. More precisely, the irreducible components of the boundary of $\overline{X_D}^{DM}$ are in bijection with the unordered projective tuples $(r_1 : r_2)$, or equivalently to standard quadratic irrationals $\lambda$ in $\mathbb{Q}_D$.

Near a boundary component labeled by $\lambda$, the Hilbert modular surface $\overline{X_D}^{DM}$ is cut out in $\Omega M_2$ by the equations

$$
v^{(E)} = r_1, \quad w^{(E)} = r_2 \quad \text{and} \quad (y^{(E)})^a = (z^{(E)})^{-c},
$$

and the boundary curve is given in these coordinates as $\{y^{(E)} = z^{(E)} = 0\}$.

The boundary of $\overline{X_D}^B$ is a union of rational curves $C_\lambda$ where $\lambda = r_1/r_2$.

We explain the last statement. If $\gcd(a, c) > 1$, then (96) shows that the compactification is not normal near $\{y^{(E)} = z^{(E)} = 0\}$; the normalization has $\gcd(a, c)$ local branches. Nevertheless, the preimage of $\{y^{(E)} = z^{(E)} = 0\}$ in the normalization $\overline{X_D}^B$ is a connected curve $C_\lambda$. 

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The normalization map is a cyclic covering of order $\gcd(a, c)$ ramified precisely over the intersection points of $C_\lambda$ with its two adjacent curves in the cusp resolution.

We include the following proposition for a complete description of the Bainbridge compactification. It is not needed for the proof of Theorem 11.5.

The second type of coordinate system is around a stable curve $X$ consisting of two irreducible components joined at three non-separating nodes. Here we choose $\alpha_1, \alpha_2, \alpha_3$ to be the loops around the punctures and let $r_i = \int_{\alpha_i} \omega$ be the residue. We may orient the $\alpha_i$ so that $r_1 - r_2 + r_3 = 0$. The one-form $\omega$ necessarily has one zero on each of the irreducible components. We let $\gamma_i$ be a path joining these two zeros that crosses $\alpha_i$ once with positive intersection and no other $\alpha_j$. Then the five functions

$$v^{(V)} = \int_{\alpha_1} \omega, \quad w^{(V)} = \int_{\alpha_2} \omega, \quad x^{(V)} = e^{\int_{\gamma_1} \omega / (w^{(V)} - v^{(V)})}, \quad y^{(V)} = e^{\int_{\gamma_2} \omega / (w^{(V)})}, \quad z^{(V)} = e^{\int_{\gamma_3} \omega / v^{(V)}}$$

form a system of coordinates on $\Omega M_2$ near $(X, \omega)$. We provide them with a superscript $V$ (vertex of the compactification cycle) to distinguish them. We also have to say how the coordinates $v^{(V)}$, $z^{(V)}$ and $v^{(E)}$, $z^{(E)}$ are related near a vertex. Suppose we unpinch the node corresponding to $\alpha_3$. Then on this nearby surface

$$\beta_1 = \gamma_1 - \gamma_3, \quad \beta_2 = \gamma_2 + \gamma_3, \quad I = \gamma_3.$$

Consequently, we have

$$v^{(E)} = v^{(V)} = v, \quad w^{(E)} = w^{(V)} = w$$

and

$$x^{(V)} = e^{(x^{(E)}) / (w - v)}, \quad y^{(V)} = y^{(E)} \cdot e^{(x^{(E)}/v)}, \quad z^{(V)} = z^{(E)} \cdot e^{(x^{(E)}/w)}.$$

The following proposition describes the intersection points of boundary curves of $X_D$. It is proven in [Bai07, Theorem 7.27] but it is also an immediate consequence of Theorem 11.5 and Proposition 11.2, in particular its last statement.

**Proposition 11.7.** If $\lambda^+$ denotes the successor of $\lambda$ for the slow plus greater-than-one continued fraction, then the curves $C_\lambda$ and $C_{\lambda^+}$ have exactly one point $c_\lambda$ in common. Near this intersection point the Hilbert modular surface $X_D$ is cut out in $\Omega M_{c}$ by the equations

$$v^{(V)} = r_1, \quad w^{(V)} = r_2 \quad \text{and} \quad (y^{(V)})^a = (z^{(V)})^{-c} (x^{(V)})^{-a - b - c}.$$

The point $c_\lambda$ is a cyclic quotient singularity of order

$$m_\lambda = \frac{a}{\gcd(a, c) \gcd(a, a + b + c)}.$$

**11.5 The proof of Theorem 11.5**

The first step is to show that the multimimimizers form a fan. For the cusp $a$ we may choose the normalization as in (87) and suppose that the multimimimizers are primitive elements in $(a^2)^\vee$. We next examine the slopes. Suppose that at some step of the slow plus greater-than-one continued fraction we have the representation $a = \mu_k (\lambda_k, 1)$. Then

$$\frac{\sigma(\alpha_k)}{\alpha_k} = \frac{\lambda_k \mu_k^2}{\sigma(\lambda_k \mu_k^2)} \quad \text{and} \quad \frac{\sigma(\alpha_{k+1})}{\alpha_{k+1}} = \frac{(\lambda_k - 1) \mu_k^2}{\sigma((\lambda_k - 1) \mu_k^2)}$$
in both cases of the continued fraction algorithm. The ratio of these two fractions is

\[
\frac{\sigma(\alpha_k)}{\alpha_k} \bigg/ \frac{\sigma(\alpha_{k+1})}{\alpha_{k+1}} = \left(1 + \frac{1}{\lambda_k} \right) \left(1 - \frac{1}{\sigma(\lambda_k)} \right) > 1
\]

since \( \lambda_k \) is standard. Consequently the slopes of multiminimizers are decreasing. Since the continued fraction algorithm is periodic this shows that the sequence multiminimizers in continued fraction order forms a fan. We denote the compactification of the Hilbert modular surface using these fans of multiminimizers by \( \overline{X}_{D}^{\text{MM}} \).

We let \( \overline{X}_{D}^{\text{MM},*}, \overline{X}_{D}^{\text{B},*} \), respectively \( \overline{X}_{D}^{\text{DM},*} \) be the complement of the intersection points of the cusps resolution curves in the three compactifications, i.e. obtained by removing the codimension two boundary strata. Our aim to show that there is a map \( m : \overline{X}_{D}^{\text{MM},*} \to \overline{X}_{D}^{\text{DM},*} \)

which is an isomorphism over \( X_D \) and which maps the boundary components labeled by a standard quadratic irrational \( \lambda \) onto the component with the same label by an unramified cyclic covering of degree \( g = \gcd(a,c) \). Since the Bainbridge compactification is normal, the map \( m \) factors through a map \( \hat{m} : \overline{X}_{D}^{\text{MM},*} \to \overline{X}_{D}^{\text{B},*} \), which is an isomorphism by the local description of the map \( \overline{X}_{D}^{\text{B},*} \to \overline{X}_{D}^{\text{DM},*} \). Since the multiminimizer compactification and the Bainbridge compactification are normal, the codimension two indeterminacy of \( \hat{m} \) (on domain an range) can be resolved to a global isomorphism.

We want to define \( m \) in a neighborhood of a point on the component of the boundary of \( \overline{X}_{D}^{\text{MM},*} \) given by the multiminimizer \( \alpha \) with corresponding standard quadratic irrational \( \lambda \). Since \( \alpha^\vee \) is the unique (up to sign) primitive element in \( (a^2)^\vee \) with \( \text{tr}(a\alpha^\vee) = 0 \), local coordinates near this point as defined in the proof of Proposition 11.2 are just the coordinates \( X = q_1^\alpha q_2^\sigma(\alpha^*) \) and \( Y = q_1^\beta q_2^\sigma(\beta^*) \) used in the proof of Theorem 10.1.

The map \( m \) is given by assigning to a point in \( X_D \) the vanishing locus of the Siegel theta function restricted to \( X_D \) together with the first eigenform. In order to understand the local behavior of this map near the boundary we may choose any convenient translate of the theta function. As in the previous proofs we take the characteristic \( (\frac{1}{2}, \frac{1}{2}), (\frac{1}{2}, 0) \) and the basis \( \omega = (\omega_1, \omega_2) \) of \( a \) that is distinguished by the multiminimizer. Moreover, we define elliptic coordinates

\[
S = e(\text{tr}(\omega^\vee u)), \quad T = e(\text{tr}(\omega^\vee u)).
\]

In the coordinates \( X, Y, S, T \), the theta function is

\[
\theta_{(m,m'),\omega}(z_1, z_2) = \sum_{x \in \mathbb{Z}^2} (-1)^{2x(m')^T} X^{F(\bar{x}_1, \bar{x}_2)/2} Y^{G(\bar{x}_1, \bar{x}_2)/2} S^{x_1 T x_2}
\]

\[
= 2 X^{(a+c)/2} S^{-1/2} T^{-1/2} (Y^{G(1/2,1/2)/2} (ST - 1) + Y^{G(1/2,-1/2)/2} (S - T) + O(X)),
\]

where \( \bar{x}_i = x_i + \frac{1}{2} \) and where

\[
F = [a/g, 0, -c/g], \quad G = [-bt, 2g, -bs].
\]

Consequently, since the boundary is given by \( X = 0 \) the vanishing locus of \( \theta \) degenerates there. As in § 6.2 the limiting curve is a rational curve. We abbreviate \( Z = Y^{G(1/2,1/2)/2}/Y^{G(1/2,-1/2)/2} = Y^{g/2}, \) and obtain (for \( Z \) fixed) the equation

\[
Z^{-1}(ST - 1) + (S - T) = 0 \quad \text{or, equivalently,} \quad (S - Z)(T + Z) = 1 - Z^2
\]

for the rational curve. A parametrization with coordinate \( t \) is given by \( S = t + Z, \quad T = (1 - Z^2)/t - Z \).
On a boundary component of the genus two Deligne–Mumford compactification parametrizing rational curves with two nodes, the cross-ratio of the four points is a coordinate. This is also the relative period \( x_E \) that was used by Bainbridge as coordinate (cf. Proposition 11.6), up to a Möbius transformation depending on \( r_1 \) and \( r_2 \), as one checks easily by integrating the limiting stable form.

Our last task is therefore to express the cross-ratio in terms of the coordinate \( X \). The first eigenform is

\[
\omega = du_1 = (\omega_2^\vee)^\sigma \frac{dS}{S} - (\omega_1^\vee)^\sigma \frac{dT}{T} = \left( \frac{(\omega_2^\vee)^\sigma}{t + Z} + \frac{(\omega_1^\vee)^\sigma}{t(1 - (Z/(1 - Z^2))t)} \right) dt, \tag{100}
\]

in the given rational parametrization. The cross-ratio of the four poles 0, \( \infty \), \( -Z \), and \( (1 - Z^2)/Z \) is given by a Möbius transformation of \( Z - 2 = \gcd(a,c) \). This completes the proof of the properties claimed about the map \( m \), and of Theorem 11.5.

11.6 Intersection of \( F_N \) with Bainbridge’s and Hirzebruch’s boundary

We can now answer the question about the intersection of the modular and Teichmüller curves with the two compactifications. For the Hirzebruch compactification the statement in Theorem 11.8 is of course already in the literature.

Fix the cusp to be infinity and let \( F_N \) be one of the modular curves, as defined in § 5.2, passing through the cusp at infinity. Then \( F_N \) can be given by an equation of the form

\[
\lambda z_1 + \lambda^\sigma z_2 + B = 0 \quad (B \in \mathbb{Z}, \ \lambda \in \mathbb{O}, \ \lambda \lambda^\sigma = -N). \tag{101}
\]

The following result now follows immediately from Proposition 11.2 and the descriptions of the Hirzebruch and Bainbridge compactifications given in this section.

**Theorem 11.8.** The curve \( F_N \) passes through an interior point of the cusp resolution cycle of the Bainbridge compactification (respectively of the Hirzebruch compactification) if there is a multiminimizer \( \alpha_n \) for this cusp such that \( \alpha_n/\lambda \in \mathbb{Q} \) (respectively if there is an element \( A_n \) of Hirzebruch’s lower convex hull fan such that \( A_n/\lambda \in \mathbb{Q} \)).

Otherwise, if

\[
\frac{\alpha_n}{\lambda^\sigma} > \frac{\lambda}{\lambda^\sigma} > \frac{\alpha_{n+1}}{\lambda^\sigma} \quad \text{respectively} \quad \frac{A_n}{A^\sigma_n} > \frac{\lambda}{\lambda^\sigma} > \frac{A_{n+1}}{A^\sigma_{n+1}},
\]

then the curve \( F_{N_n} \) passes through the node corresponding to the intersection of the curves associated with \( \alpha_n \) and \( \alpha_{n+1} \) (respectively with \( A_n \) and \( A_{n+1} \)).

This result together with Theorems 10.1 and 10.2 reproves from the theta viewpoint another result of Bainbridge.

**Corollary 11.9.** The curves \( W_D \) and the components of the reducible locus \( P_D \) intersect the boundary of the Bainbridge compactification only in interior points of the boundary curves.

Note that the component \( P_{D,\nu} \) of \( P_D \) as defined in Proposition 5.1 is given by (101) with \( \lambda = \nu \sqrt{D}, \ B = 0 \).

11.7 Examples

**Case** \( D = 17 \). We consider the cusp at \( \infty \) for \( \text{SL}(\mathbb{O}^\vee \oplus \mathbb{O}) \). In this case \( M = \mathbb{O}_D^\vee \) and the standard quadratic forms are \([2, -3, -1], [2, -1, -2], [1, -3, -2], [1, -1, -4], \) and \([1, 1, -4] \). The following table contains the corresponding multiminimizers \( \alpha \in \mathbb{O}_D^\vee \), scaled by the factor \( \sqrt{17} \).
### Theta derivatives

<table>
<thead>
<tr>
<th>Standard $n$</th>
<th>[1, −3, −2]</th>
<th>[1, −1, −4]</th>
<th>[1, 1, −4]</th>
<th>[2, −3, −1]</th>
<th>[2, −1, −2]</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda$</td>
<td>$\frac{1}{2} \cdot \sqrt[4]{17}$</td>
<td>$\frac{2}{2} \cdot \sqrt[4]{17}$</td>
<td>$\frac{-1}{2} \cdot \sqrt[4]{17}$</td>
<td>$\frac{3}{2} \cdot \sqrt[4]{17}$</td>
<td>$\frac{5}{4} \cdot \sqrt[4]{17}$</td>
</tr>
<tr>
<td>$\sqrt[4]{17} \alpha_n$</td>
<td>$\frac{3 + \sqrt[4]{17}}{2}$</td>
<td>$\frac{1 + \sqrt[4]{17}}{2}$</td>
<td>$\frac{-1 + \sqrt[4]{17}}{2}$</td>
<td>$\frac{3 + \sqrt[4]{17}}{2}$</td>
<td>$\frac{1 + \sqrt[4]{17}}{2}$</td>
</tr>
<tr>
<td>$\sqrt[4]{17} A_k$</td>
<td>$\frac{3 + \sqrt[4]{17}}{2}$</td>
<td>$\frac{2}{2} \cdot \sqrt[4]{17}$</td>
<td>$\frac{3 + \sqrt[4]{17}}{2}$</td>
<td>$\frac{3 + \sqrt[4]{17}}{2}$</td>
<td>$\frac{4 + \sqrt[4]{17}}{2}$</td>
</tr>
<tr>
<td>$x_k$</td>
<td>$\frac{7 + \sqrt[4]{17}}{2}$</td>
<td>$\frac{9 + \sqrt[4]{17}}{2}$</td>
<td>$\frac{7 + \sqrt[4]{17}}{2}$</td>
<td>$\frac{5 + \sqrt[4]{17}}{2}$</td>
<td>$\frac{5 + \sqrt[4]{17}}{2}$</td>
</tr>
<tr>
<td>$k$</td>
<td>$\frac{4}{5}$</td>
<td>$\frac{4}{3}$</td>
<td>$\frac{4}{2}$</td>
<td>$\frac{2}{1}$</td>
<td>$\frac{2}{1}$</td>
</tr>
</tbody>
</table>

By the singularity criterion Proposition 11.2 this compactification is smooth.

A reduced quadratic irrationality is $x_1 = (5 + \sqrt[4]{17})/2$. Its fast minus continued fraction is listed in the lower part of that table. The point $A_1 = (\sqrt[4]{17})(4 + \sqrt[4]{17})$ lies on the lower convex hull of $\mathcal{O}_n^\vee \cap (\mathbb{R}_+)^2$. By the algorithm for Hirzebruch’s compactification the subsequent points on the lower convex hull are defined by $A_{k+1} = A_k/x_k$, as listed in the table.

The Bainbridge compactification has no singularities at the points $c_\lambda$ and the number of reduced quadratic forms equals the number of standard quadratic forms. Hence, the Hirzebruch and the Bainbridge compactification coincide in this case. This is consistent with the table listing the same values for $\sqrt[4]{17} \alpha_n$ and for $\sqrt[4]{17} A_k$.

**Case $D = 41$.** There are 11 standard and also 11 reduced quadratic forms. But here the Hirzebruch and the Bainbridge compactification do not coincide.

<table>
<thead>
<tr>
<th>Standard $n$</th>
<th>[1, −5, −4]</th>
<th>⋯</th>
<th>[4, −5, −1]</th>
<th>[2, −3, −4]</th>
<th>[2, 1, −5]</th>
<th>⋯</th>
<th>[4, −3, −2]</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda_n$</td>
<td>$\frac{1}{2} \cdot \sqrt[4]{41}$</td>
<td>⋯</td>
<td>$\frac{5 + \sqrt[4]{41}}{2}$</td>
<td>$\frac{3 + \sqrt[4]{41}}{2}$</td>
<td>$\frac{-1 + \sqrt[4]{41}}{2}$</td>
<td>⋯</td>
<td>$\frac{3 + \sqrt[4]{41}}{2}$</td>
</tr>
<tr>
<td>$\sqrt[4]{41} \alpha_n$</td>
<td>$\frac{-5 + \sqrt[4]{41}}{2}$</td>
<td>⋯</td>
<td>$\frac{5 + \sqrt[4]{41}}{2}$</td>
<td>$\frac{19 + 3\sqrt[4]{41}}{2}$</td>
<td>$\frac{83 + 13\sqrt[4]{41}}{2}$</td>
<td>⋯</td>
<td>$\frac{429 + 67\sqrt[4]{41}}{2}$</td>
</tr>
<tr>
<td>$\sqrt[4]{41} A_k$</td>
<td>$\frac{-5 + \sqrt[4]{41}}{2}$</td>
<td>⋯</td>
<td>$\frac{5 + \sqrt[4]{41}}{2}$</td>
<td>$\frac{19 + 3\sqrt[4]{41}}{2}$</td>
<td>$\frac{826 + 129\sqrt[4]{41}}{2}$</td>
<td>⋯</td>
<td>$\frac{826 + 129\sqrt[4]{41}}{2}$</td>
</tr>
<tr>
<td>$x_k$</td>
<td>$\frac{13 + \sqrt[4]{41}}{2}$</td>
<td>⋯</td>
<td>$\frac{11 + \sqrt[41]}{2}$</td>
<td>$\frac{9 + \sqrt[41]}{2}$</td>
<td>$\frac{7 + \sqrt[41]}{2}$</td>
<td>⋯</td>
<td>$\frac{11 + \sqrt[41]}{2}$</td>
</tr>
<tr>
<td>$k$</td>
<td>$\frac{8}{11}$</td>
<td>⋯</td>
<td>$\frac{6}{8}$</td>
<td>$\frac{5}{10}$</td>
<td>$\frac{4}{10}$</td>
<td>⋯</td>
<td>$\frac{1}{1}$</td>
</tr>
</tbody>
</table>

At the intersection points of the curves $C_{\lambda_7}$ and $C_{\lambda_8}$ and also at the intersection points of $C_{\lambda_8}$ and $C_{\lambda_9}$ the Bainbridge compactification is smooth, but

$$\alpha_7 + \alpha_9 = \alpha_8 \quad \text{and also} \quad \alpha_9 + \alpha_{11} = \alpha_{10}$$

and hence this compactification is not minimal. In fact, $C_{\lambda_8}$ (and also $C_{\lambda_{10}}$) is a $\langle -1 \rangle$-curve and the corresponding values $\alpha_8$ and $\alpha_{10}$ do not show up in the list of $A_k$. On the other hand, at the intersection point of $C_{\lambda_6}$ and $C_{\lambda_7}$ the Bainbridge compactification has a quotient singularity of order two, since $\alpha_6$ and $\alpha_7$ generate an index two subgroup of $\mathcal{O}_n^\vee$. It can be resolved by blowing up, adding a $\langle -2 \rangle$-curve, corresponding to the value $\sqrt[4]{41} A_5 = 6 + \sqrt[4]{41}$ that does not show up in the list of $\sqrt[4]{41} \alpha_n$. The singularity can also be read off from the quadratic form $[4, 5, −1]$ and Proposition 11.6. In terms of the convex hull of $\mathcal{O}_n^\vee$, the multiminimizer fan has two interior points and skips two boundary points, as is shown in Figure 3 below.

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Figure 3. The multiminimizer fan (thick black points) and the Hirzebruch fan (circles) for $D = 41$, connected by red respectively by black lines. The rightmost thick black point is an interior point of the convex hull. Another point of the Hirzebruch fan skipped by the multiminimizer fan is not drawn (far to the right, close to the horizontal axis).

11.8 Relating the two compactifications and two continued fraction algorithms

The preceding examples show that the Bainbridge and the Hirzebruch compactification may sometimes agree, but that they are different in general. They also illustrate the general algorithm how to go from Bainbridge’s compactification to Hirzebruch’s: blow down curves corresponding to interior points and blow up points where boundary points of the lower convex hull have been omitted by the multiminimizers.

At first glance the previous examples suggest that at least the length of the boundary cycles in the Bainbridge and the Hirzebruch compactification agree. This is true for class number one, but otherwise the truth is more subtle, as we now explain.

To determine the geometry of the cusp $a$ in the Bainbridge compactification, we need to run the multiminimizer algorithm given in §10.4 for $\lambda$ such that $a = \langle 1, \lambda \rangle$ and get as an output $\alpha \in (a^2)^\vee$. The geometry of Hirzebruch’s minimal smooth resolution, however, depends only on the square of $a^2$, since we need to determine by the algorithm in §11.2 the lower convex hull of $(a^2)^\vee \cap \mathbb{R}_2^\geq$.

Consequently, whenever the squaring map is not an isomorphism on the ideal class group of $K = \mathbb{Q}(\sqrt{D})$, only the reduced quadratic irrationalities $x$ such that $((1, x))^\vee$ is a square appear as labels in the cusp resolution of the Hilbert modular surface $X_D$ in the focus of our interest. However, all the curves labeled by standard quadratic forms appear on $X_D$. An example to
Theta derivatives

illustrate this is \( D = 65 \), where the class group (both in the narrow and wide sense) is of order two, so the multiminimizers for both cusps lie in an ideal equivalent to \( \mathcal{O}_{65} \). Representatives of the multiminimizers are

\[
\text{MM}(\mathcal{O}_{65}) = \left\{ \frac{5 + \sqrt{65}}{2}, \frac{7 + \sqrt{65}}{2}, \frac{23 + 3\sqrt{65}}{2}, \frac{8 + \sqrt{65}}{2}, \frac{153 + 19\sqrt{65}}{2}, \frac{137 + 17\sqrt{65}}{2}, \frac{395 + 49\sqrt{65}}{2}, \frac{1685 + 209\sqrt{65}}{2}, \frac{1943 + 241\sqrt{65}}{2} \right\}
\]

while the elements on the convex hull, up to multiplication by \( U^2_{65} \), are given by

\[
\left\{ \frac{5 + \sqrt{65}}{2}, \frac{7 + \sqrt{65}}{2}, \frac{23 + 3\sqrt{65}}{2}, \frac{8 + \sqrt{65}}{2}, \frac{153 + 19\sqrt{65}}{2}, \frac{137 + 17\sqrt{65}}{2}, \frac{395 + 49\sqrt{65}}{2}, \frac{1685 + 209\sqrt{65}}{2}, \frac{1943 + 241\sqrt{65}}{2} \right\}
\]

There is however an equality of total lengths of cusp resolution cycles, if one takes all Hilbert modular surfaces \( \mathbb{H}/\text{SL}(b \oplus \mathcal{O}_D) \) into account. We start our considerations on the level of quadratic irrationalities.

Reduced quadratic irrationalities as defined in (93) are well known and have been used to label the boundary curves of Hirzebruch’s compactification. Standard quadratic irrationalities as defined in (80) have been used to label the boundary curves of Bainbridge’s compactification. There is an obvious bijection between these two classes, given by

\[
\begin{array}{ccc}
\text{reduced} & \xrightarrow{x > 1 > x^\sigma > 0} & \text{standard} \\
\text{conjugation:} & \lambda = \frac{x}{x-1} & \lambda > 1 > 0 > \lambda^\sigma
\end{array}
\]

In order to pass from this correspondence of quadratic irrationalities to cusp resolutions, we define

\[
C^\text{std}(a) = \{ (\mu, \lambda) \in K^+/U^2_D \times K : a = \mu(1, \lambda), \lambda \text{ standard} \}
\]

and

\[
C^\text{red}(a) = \{ (\rho, x) \in K^+/U^2_D \times K : a = \rho(1, x), x \text{ reduced} \}.
\]

Then the map

\[
\phi : C^\text{std}(a) \rightarrow C^\text{red}(a), \quad (\mu, \lambda) \mapsto (\rho, x) = \left( \mu(\lambda - 1), \frac{\lambda}{\lambda - 1} \right)
\]

is obviously a bijection.

**Proposition 11.10.** For any \( a \), the length of the cycle of the Bainbridge compactification for the cusp corresponding to \( a^2 \) is equal to the length of the cycle of the Hirzebruch compactification for the cusp corresponding to \( a \).
Proof. Curves in the Bainbridge compactification for \( a^2 \) are in bijection to \( C^{\text{std}}(a^2) \) by Theorem 11.5 together with Proposition 10.4. Curves in the Hirzebruch compactification for \( a \) are in bijection to \( C^{\text{red}}(a^2) \) by the algorithm in §11.2. We now can use the identification \( \phi \). This explains the initial observations for \( D = 17 \) and \( D = 41 \) and in general for \( a = a^2 = O_D \).

For a more symmetric formulation we recall that the isomorphism class of the Hilbert modular surface \( \mathcal{H}_2 / \text{SL}(b \oplus \mathcal{O}_D) \) depends on \( b \) only up to squares of ideals and multiplication by a totally positive element in \( K \). Consequently, there are \( 2^{t-1} = |\text{Ker}(\text{Sq} : \mathfrak{Cl}^+(D) \to \mathfrak{Cl}^+(D))| \) Hilbert modular surfaces for a given \( D \), where \( t \) is the number of distinct prime factors of \( D \). This also implies that every ideal class \( a \) (in the wide sense) appears \( 2^{t-1} \) times as the module \( M \) associated with a cusp \( \mathbb{H}^2 / G(M, V) \) on the total collection of Hilbert modular surfaces for a given \( D \). Altogether, this implies that the number

\[
\ell(D) = \sum_{[a,b,c]\text{reduced}} 1 = \sum_{[a,b,c]\text{standard}} 1
\]

appears in two incarnations.

**Proposition 11.11.** The total number of boundary curves of the Bainbridge compactification of \( X_D \) is equal to \( \ell(D) \).

The total number of boundary curves of the Hirzebruch compactifications of the Hilbert modular surfaces \( \mathbb{H}^2 / \text{SL}(b_i \oplus \mathcal{O}_D) \) for \( b_i \) in a set of representatives of \( \mathfrak{Cl}^+(D) / \text{Sq}(\mathfrak{Cl}^+(D)) \) is equal to \( 2^{t-1} \ell(D) \).

We could presumably make this statement even more symmetrical if we compactified all Hilbert modular surfaces \( \mathbb{H}^2 / \text{SL}(b \oplus \mathcal{O}_D) \) using multimimimizers defined by theta functions, but we leave it to the reader to explore this.

**12. Uniformization and disjointness from the reducible locus**

The aim of this section is to give an independent proof of ‘\( W_D \) is a Teichmüller curve’ using the definition via theta functions and applying Theorem 5.3. We emphasize that we derive all properties of \( W_D \) ab ovo, i.e. using just the definition (76) as the vanishing locus of \( \mathcal{D}\theta \) and without using anything that follows from the ‘geodesic’ definition. Here again we simplify our counting task by restricting to fundamental discriminants.

**12.1 Transversality of \( W_D \) to the foliation \( \mathcal{F}_1 \)**

Recall that in §5.3 we defined the ‘first Hilbert modular foliation’ \( \mathcal{F}_1 \) of the Hilbert modular surface \( X_D \) to be the foliation defined by the constancy of the first coordinate in the uniformization (i.e. by the equation \( d_{z1} = 0 \), which is invariant under the action of the Hilbert modular group). Let \( \varphi : \mathbb{H} \to \mathbb{H} \) be a holomorphic map such that \( z \mapsto (z, \varphi(z)) \) defines a branch of the vanishing locus of \( \mathcal{D}\theta \), and suppose that the corresponding component \( W_D \) of the vanishing locus has the uniformization \( W_D = \mathbb{H} / \Gamma \).

**Theorem 12.1.** Suppose that \( D \) is a fundamental discriminant. Then the restriction to \( W_D \) of the derivative \( \mathcal{D}^3\theta = (\partial / \partial z_2)\mathcal{D}\theta \) is a \( \varphi \)-twisted modular form of weight \((3, 11)\) for \( \Gamma \) that vanishes only at the cusps of \( W_D \).
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Since the vanishing locus of a holomorphic function \( F \) is transversal to \( \mathcal{T}_1 \) at a point \( p \) if and only if \( (\partial/\partial z_2)F(p) \neq 0 \), the theorem immediately implies the transversality we want to prove. For all \( (m, m') \in (\frac{1}{2}\mathbb{Z})^2 \) and for any basis \( \omega = (\omega_1, \omega_2) \) of an \( \mathcal{O} \)-ideal \( a \), define

\[
D_2^2\theta_{(m, m')}(z, \omega)(z) = \frac{\partial^3}{\partial z_2^3} (\Theta_{(m, m')}(u, \psi_\omega(z)))|_{u=0},
\]

where the Siegel modular embedding \( \psi_\omega \) is defined using \( \omega \). If we drop the index \( \omega \) we tacitly assume that we have chosen some basis of \( \mathcal{O} \), as we did in (78). It follows that

\[
D_2^2\theta_{(m, m')}(z_1, z_2) = \frac{\partial}{\partial z_2} (D_2\theta_{(m, m')}(z_1, z_2))
\]

(by the heat equation or by direct computation). Consequently,

\[
D^3\theta(z_1, z_2) = \sum_{(m_0, m'_0) \text{ odd}} D_2^3\theta_{(m_0, m'_0)}(z_1, z_2) \prod_{(m, m') \text{ odd}} D_2\theta_{(m, m')}(z_1, z_2),
\]

explaining also the name given to this twisted modular form.

**Proof.** By applying the chain rule one sees that the restriction of the \( z_i \)-derivative of a Hilbert modular form to the vanishing locus of the form satisfies a modular transformation property with respect to the subgroup stabilizing this vanishing locus. This calculation also shows that the \( z_i \)-derivative increases the weight in the \( i \)th component by two, proving the first claim.

To prove the second claim we will show that the vanishing orders of \( D^3\theta \) at the cusps sum up to the total vanishing order of a twisted modular form of this bi-weight. Since \( D^3\theta \) is holomorphic, it cannot then have any zeros at finite points. On a minimal compactification of the Hilbert modular surface \( X_D \) the number of intersection points of two modular forms of bi-weights \( (k_1, \ell_1) \) and \( (k_2, \ell_2) \) is \( \frac{1}{4}(k_1\ell_2 + k_2\ell_1)|\chi(X_D)| \). This follows e.g. from [vdG87, §IV.2]. This calculation is still valid when intersecting the vanishing locus \( W_D \) of a modular form with a section of the bundle of modular forms of bi-weight \( (k_2, \ell_2) \) to \( W_D \). Since here \( (k_1, \ell_1) = (3, 9) \) and \( (k_2, \ell_2) = (3, 11) \), the function \( D^3\theta_\alpha \) has \( 15|\chi(X_D)| \) zeros on the closure of \( W_D \). We have to show that they all lie at the cusps.

Note that precisely one of the two quadratic forms \([a, b, c]\) and \([-c, -b, -a]\) of discriminant \( D \) with \( a > 0 \) and \( c < 0 \) satisfies the additional condition \( a + b + c < 0 \) required to make it a standard quadratic form, since \( a + b + c = 0 \) would imply that \( D \) is a square. From (34) we consequently deduce that

\[
15|\chi(X_D)| = \frac{1}{2} \sum_{D=b^2-4ac, a>0, c<0} a = \frac{1}{2} \sum_{[a,b,c]\text{ standard}} \sum_{D=b^2-4ac} (a + |c|).
\]

To complete the proof it thus suffices to show that at each of the \( g \) cusps of \( W_D \) corresponding to \( \lambda \), the vanishing order of \( D^3\theta \) is at least (and, hence, precisely) equal to \( (a + |c|)/2g \). Here \( \lambda \) is a zero of the standard form \([a, b, c]\) and \( g = \gcd(a, c) \). (The order of zero may indeed be half-integral, in accordance with the fact that \( D\theta \) is a modular form with a quadratic character.)

By Lemma 10.5 the vanishing orders of \( D^3\theta \) at the cusps of \( W_D \) mapping to \( \alpha \) can be computed as the vanishing orders of

\[
D^3\theta_\alpha(z_1, z_2) = \sum_{(m_0, m'_0) \text{ odd}} D_2^3\theta_{(m_0, m'_0)}(z_1, z_2) \prod_{(m, m') \text{ odd}} D_2\theta_{(m, m')}(z_1, z_2),
\]

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on the Hilbert modular surface \( X_{D,a} \) at the cusps mapping to infinity. By Theorem 10.1 we associate a multiminimizer \( \alpha \) to such a cusp and we may assume that \( \omega = (\omega_1, \omega_2) \) has been chosen to be the distinguished bases for this multiminimizer. If \( t \) is a local parameter of cusps of \( W_D \), as in the proof of Theorem 10.1, then the terms appearing in the expansion of \( D_2\theta_{(m_0, m_0')}^2 \), \( \omega \) and \( D_2^3\theta_{(m_0, m_0')}^2 \), \( \omega \) are \( t^{F(\bar{x}_1, \bar{x}_2)/2} \), where \( \bar{x}_i = x_i + m_i \) with \( x_i \in \mathbb{Z} \) and where

\[
F = \left[ \frac{a}{g}, 0, -\frac{c}{g} \right]
\]

as in (89). The minimal \( t \)-exponents are greater than or equal to \((a + |c|)/8g, a/8g, \) and \(|c|/8g\) in case \( m \) is equal to \((1/2, 1/2), (1/2, 0), \) and \((0, 1/2)\), respectively, both for \( D_2\theta \) and \( D_2^2\theta \). Since for each of these \( m \) there are precisely two \( m' \) such that \((m, m')\) is odd, the total vanishing order is at least \((a + |c|)/2g\), which is what we wanted to show.

\[
\Box
\]

12.2 Disjointness of \( W_D \) from the reducible locus (by counting zeros)

In this and the following subsection we give two completely different proofs of the following result, which is the second half of what we need to apply the criteria of Theorem 5.3 and show that the vanishing locus of \( \mathcal{D}\theta \) is a Teichmüller curve. The first proof is similar to that used for Theorem 12.1, by comparing the number of known zeros of a twisted modular form with its total number of zeros.

**Theorem 12.2.** The vanishing locus \( W_D \) of \( \mathcal{D}\theta \) is disjoint in \( X_D \) from the reducible locus \( P_D \).

**Proof.** Recall that the reducible locus is the vanishing locus of the product of all 10 even theta functions. This product is a Hilbert modular form of weight \((5, 5)\), so its restriction to \( W_D \) is a modular form for \( W_D \) of bi-weight \((5, 5)\). As in the preceding proof we deduce that the degree of its divisor (on a compactification of \( W_D \)) is \( \frac{1}{2}(5 \cdot 3 + 5 \cdot 9)|\chi(X_D)| = 15|\chi(X_P)| \).

As in the preceding proof it suffices to show that the restriction of the product \( \prod_{(m,m') \text{even}} \theta_{(m,m')} \) to \( W_D \) vanishes at each of the \( g \) cusps of \( W_D \) corresponding to \( \lambda \) to the order at least \((a + |c|)/2g\). Here again we can work at the cusp \( \infty \) of \( X_{D,a} \). There, by the same argument as in Lemma 10.5, the product of the 10 even theta functions is given by \( \prod_{(m,m') \text{even}} \theta_{(m,m')}(\omega) \), where \( \omega \) is some basis of \( \mathfrak{a} \) and where \( \theta_{(m,m')}(\omega)(z) = \Theta_{(m,m'),\omega}(0, \psi_\omega(z)) \) with the modular embedding \( \psi_\omega \) defined using \( \omega \). The rest of the proof proceeds as above. To each cusp we associate its multiminimizer and take \( \omega \) to be the distinguished basis. The terms appearing in the expansion of \( \theta_{(m,m')}(\omega) \) at such a cusp are \( t^{F(\bar{x}_1, \bar{x}_2)/2} \), where \( \bar{x}_i = x_i + m_i \) with \( x_i \in \mathbb{Z} \) and where \( F = [a/g, 0, -c/g] \). The minimal \( t \)-exponents are greater than or equal to \((a + |c|)/8g, a/8g, |c|/8g\), and \( 0 \) in case \( m \) is equal to \((1/2, 1/2), (1/2, 0), (0, 1/2)\), and \((0, 0)\), respectively. Each of the first three cases occurs twice among the ten even theta characteristics (and the irrelevant last case four times). Summing up these contributions gives again the vanishing order at least \((a + |c|)/2g\) that we claimed.

\[
\Box
\]

12.3 Disjointness of \( W_D \) from the reducible locus (via theta products)

In this subsection we give a proof of Theorem 12.2 based on a completely different idea, by establishing a formula for the restriction of theta derivatives to the reducible locus. Let

\[
\begin{align*}
\theta_{00}(z) &= \sum_{n \in \mathbb{Z}} q^{n^2/2}, & \theta_{01}(z) &= \sum_{n \in \mathbb{Z}+1/2} q^{n^2/2}, \\
\theta_{0\frac{1}{2}}(z) &= \sum_{n \in \mathbb{Z}} (-1)^n q^{n^2/2}, & \theta_{(1/2)(1/2)}(z) &= \sum_{n \in \mathbb{Z}+1/2} (-1)^{n-1/2} n q^{n^2/2}.
\end{align*}
\]
Theta derivatives

(Here \(\theta_{(1/2)(1/2)}\) should perhaps be called \(\theta'_{(1/2)(1/2)}\), since the corresponding Jacobi theta constant vanishes identically, but with these notations it will be easier to write a closed expression.) The product formulas found by Jacobi for these four functions, which in a modern notation say that they are equal to \(\eta(2z)/\eta(z)^2\), \(2\eta(4z)^2/\eta(2z)\), \(\eta(z)^2/\eta(2z)\), and \(\eta(z)^3\), respectively, show that none of these functions vanish anywhere in the upper half-plane. Recall from Proposition 5.1 that the reducible locus is the union of irreducible curves \(P_{D,\nu} = F_N(\nu)\) for \(\nu = (r + \sqrt{D})/2\sqrt{D}\), where \(D = r^2 + 4N\) with \(N \in \mathbb{N}\). Therefore, Theorem 12.2 follows from the following theorem, which for simplicity we formulate only for \(D\) odd, the case of even \(D\) being similar.

**Theorem 12.3.** Let \(D \equiv 1 \pmod{4}\) be a fundamental discriminant. Then for any odd theta characteristic \((m, m')\) the restriction of the modular form \(D_2\theta_{(m, m')}\) to the curve \(P_{D, \nu}\) for \(\nu\) as above has the factorization

\[
D_2\theta_{(m, m')} (\nu z, \nu^\sigma z) = \begin{cases} 
-\theta_{\hat{m}_1 m'_1} (z) \theta_{m_2 m'_2} (N z) & \text{if } m_1 = m'_1 = 1/2, \\
-\sqrt{D} \nu^\sigma \theta_{\hat{m}_1 m'_1} (z) \theta_{m_2 m'_2} (N z) & \text{if } m_2 = m'_2 = 1/2,
\end{cases}
\]

as a product of Jacobi theta functions, where \(\hat{m}_1\) and \(\hat{m}_2\) are defined by

\[
\hat{m}_1 = m_1 + m_2 \pmod{1}, \quad \hat{m}_2 = m'_1 + m'_2 \pmod{1} \quad \text{if } r \equiv 1 \pmod{4},
\]

\[
\hat{m}_1 = m_1, \quad \hat{m}_2 = m_2, \quad \text{if } r \equiv 3 \pmod{4},
\]

for \(m = (m_1, m_2), \quad m' = (m'_1, m'_2)\). In particular, this restriction vanishes only at cusps.

**Proof.** As above we use a tilde to denote elements of the shifted lattice, i.e. \(\tilde{x}_i = x_i + m_i\). The restriction of the theta derivative to \(P_{D, \nu}\) is

\[
D_2\theta_{(m, m')} (\nu z, \nu^\sigma z) = \sum_{(x_1, x_2) \in \mathbb{Z}^2} (-1)^{2(x_1, x_2)(m')^T} \rho(\tilde{x}_1, \tilde{x}_2)^\sigma q^{\text{tr}(\nu(\tilde{x}_1, \tilde{x}_2)^2)/2},
\]

where \(\rho(x_1, x_2) = x_1 + ((1 + \sqrt{D})/2) x_2\). With the \(\mathbb{Z}\)-linear transformation \(\tilde{y}_1 = \tilde{x}_1 + ((r + 1)/2) \tilde{x}_2\) \(\tilde{y}_2 = -\tilde{x}_2\) we obtain

\[
\rho(\tilde{x}_1, \tilde{x}_2) = \tilde{y}_1 + \frac{r - \sqrt{D}}{2} \tilde{y}_2 = \tilde{y}_1 + \sqrt{D} \nu^\sigma \tilde{y}_2 =: \tau(\tilde{y}_1, \tilde{y}_2)
\]

and

\[
D_2\theta_{(m, m')} (\nu z, \nu^\sigma z) = \sum_{(\tilde{y}_1, \tilde{y}_2) \in \mathbb{Z}^2 + (\hat{m}_1, m_2)} \epsilon(\tilde{y}_1, \tilde{y}_2) \tau(\tilde{y}_1, \tilde{y}_2) q^{(\tilde{y}_1^2 + \hat{m}_1 \tilde{y}_1 + \hat{m}_2 \tilde{y}_2)/2},
\]

where

\[
\epsilon(\tilde{y}_1, \tilde{y}_2) = (-1)^{2(\hat{m}_1 \tilde{y}_1 - \hat{m}_2 \tilde{y}_2)} + 2(\hat{m}_2 - \hat{m}_1) \hat{m}_1 \hat{m}_2
\]

and where we have used \(\text{tr}(\nu \tau(\tilde{y}_1, \tilde{y}_2)^2) = \tilde{y}_1^2 + \hat{m}_2 \tilde{y}_2\). The \(q\)-exponent is invariant under both \(\tilde{y}_1 \mapsto -\tilde{y}_1\) and \(\tilde{y}_2 \mapsto -\tilde{y}_2\). Under, say, \(\tilde{y}_2 \mapsto -\tilde{y}_2\) the sign of \(\epsilon(\tilde{y}_1, \tilde{y}_2)\) is unchanged unless \(m_2 = \hat{m}_2 = 1/2\), hence, unless \(m_2 = \hat{m}_2 = 1/2\), the \(\sqrt{D} \nu^\sigma \tilde{y}_2\) contribution of \(\tau(\tilde{y}_1, \tilde{y}_2)\) cancels and one checks the formula by multiplying the unary theta functions. For an odd theta constant \((m, m')\) precisely one of the cases \(m_1 = \hat{m}_1 = 1/2\) or \(\hat{m}_2 = m_2 = 1/2\) happens and a similar cancelation gives the formula in the second case, too. Note that in each case of the theorem, one of the theta series in the decomposition is the function \(\theta_{(1/2)(1/2)}\) of weight \(3/2\), so that the total weight is always two.

**Open problem.** Can one reprove the irreducibility, stated in Theorem 5.2 and proved by McMullen using combinatorial number theory of the set of cusps, exclusively with techniques of (Hilbert) modular forms? 

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13. Applications

13.1 The modular embedding via theta functions

Using the description of $W_D$ as the vanishing locus of $D\theta$, we can now give the modular embedding $\varphi$ as in the 'Fourier expansion' as defined in (2) for any cusp of $W_D$. In fact, most of this was already achieved in the proof of Theorem 10.1. There we identified a cusp of $W_D$ with a standard quadratic form $[a, b, c]$ (and, hence, a multiminimizer $\alpha$) together with an element in $r \in \mathbb{Z}/g\mathbb{Z}$ where $g = \gcd(a, c)$ (equivalently, a solution $S$ of (90)). Recall from this proof that for each such solution there is a unique branch, given by (85), of the locus $D_2\theta_{(m,m')}((z_1, z_2)) = 0$. The map $z \mapsto (z, \varphi(z))$ describing this branch was given by

$$\varphi(z) = \frac{\alpha^*}{\alpha}z + C + \frac{\varepsilon(q)}{2\pi i}.$$ (102)

We can now describe the arithmetic properties of this expansion.

**Theorem 13.1.** The coefficients of the modular embedding describing the branch determined by the quadratic form $[a, b, c]$ and number $S$ as in (90) through the cusp $a$ of $X_D$ have the following properties.

(i) The constant $C$ in (102) belongs to $(1/2\pi i)K^\times \log K^\times$. In fact, $e(ga/C/(N(a)^2\sqrt{D})) \in K\backslash \mathbb{Q}$.

(ii) The number $S$ lies in $K^{1/g}$.

(iii) For each $\beta$ completing the multiminimizer $\alpha$ determined by $[a, b, c]$ to a basis of $(a^2)^\vee$ there exists $A \in \mathbb{C}^*$ such that in the local parameter $Q = Aq$ of the cusp $e(\nu z + \nu^\sigma \varphi(z)) = S^{\tr(\beta)}Q^{\tr(\alpha^\vee)}e^{\sigma(\nu)e(q)}$ for all $\nu \in a^2$.

The scalar $A$ is transcendental of Gelfond–Schneider type, more precisely of the form $x^y$ with $x$ and $y$ in $K\backslash \mathbb{Q}$.

(iv) The coefficients $a_n$ of the power series $\varepsilon(q) = \sum_{n \geq 1} a_nQ^n$ expanded in a local parameter $Q$ as in property (ii) lie in the number field $K(S)$.

**Proof.** Statement (i) obviously follows from (90) and (86). For property (ii) it suffices to test for $\nu$ the dual basis $\{\alpha^*, \beta^*\} \subset a^2$ of $\{\alpha, \beta\}$. Plugging in $\beta^*$ confirms that $S$ here is the same as in (86), since $\beta^* = \alpha^*/N(a)^2\sqrt{D}$. Plugging in $\alpha^* = \beta^*/N(a)^2\sqrt{D}$ implies that

$$A = e\left(\frac{\beta C}{N(a)^2\sqrt{D}}\right) = S^{\beta^*/\alpha} \in K^{\beta^*/\alpha}.$$  

The last statement (iii) follows since the coefficients of Fourier expansion of the theta function lie in $K(S)$ and solving recursively for the $a_n$ involves only these coefficients and integral powers of $S$. For concreteness, we perform the first step in this procedure in the case $|c| < a$. The summands $(\tilde{x}_1, \tilde{x}_2) = (\pm \frac{1}{2}, \pm \frac{1}{2})$ contribute to the lowest-order term (in $Q$) of the restriction of the theta derivative to the branch $(z, \varphi(z))$. The next lowest-term is determined by $(\tilde{x}_1, \tilde{x}_2) = (\pm \frac{3}{2}, \pm \frac{3}{2})$ and we can solve for

$$a_1 = S^{\sigma + 2b} \rho_{\omega}(\frac{1}{2}, \frac{3}{2})(\sigma(\alpha^*)F(\frac{1}{2}, \frac{3}{2}) + \sigma(\beta^*)G(\frac{1}{2}, \frac{3}{2}))$$

$$\rho_{\omega}(\frac{1}{2}, \frac{1}{2})(\sigma(\alpha^*)F(\frac{1}{2}, \frac{1}{2}) + \sigma(\beta^*)G(\frac{1}{2}, \frac{1}{2}))$$

using the notation $\rho_{\omega}$ introduced in the lines before (83), where $F$ and $G$ given in (89) and $g = sa + tc$. □
We remark that the ‘$A$’ of §7.1 is not quite the same as the one above, and that the numerical value given in (49) is not of the form $x^y$ with $x, y \in K$, but an algebraic multiple of this. This is because we normalized our $Q$ in §7.1 so that the expansion of $t = Q + \cdots$ had leading coefficient one. If we changed $Q$ by the algebraic factor, then $t$ and $y$ would still have $Q$-expansions with coefficients in $K$, so that this would be an equally good choice in the special case $D = 17$, but for the general statement it seems best to normalize $Q$ as above.

13.2 Fourier coefficients of twisted modular forms

Fix $D$, a fractional $\mathcal{O}_D$-ideal $a$, and a branch of the vanishing locus of $\mathcal{O}$ through the cusp at $\infty$ of the Hilbert modular surface $X_{D,a}$ given by the quadratic form $[a, b, c]$ and $S$ as above. Let $\Gamma \subset \text{SL}(a) \oplus a$ be the subgroup stabilizing this branch. It is the Fuchsian group uniformizing the curve $W_D$, normalized so that the cusp labeled with $([a, b, c], S)$ is the cusp at $\infty$.

The first twisted modular form for $\Gamma$ we encountered was the form $\varphi'(z)$ of bi-weight $(2, -2)$ in the case $D = \sqrt{17}$. By the preceding theorem we know that this modular form has a Fourier expansion of the form

$$\varphi'(z) = \sum_{n \geq 0} b_n(Aq)^n$$

with $A$ transcendental of Gelfond–Schneider type and $b_n$ in the field $K(S)$ (which is equal to $K$ in this case, since $g = 1$). We will show that such a statement holds for all twisted modular forms.

This section is inspired by work of Wolfart in the case of non-compact Fuchsian triangle groups. Let $\Delta(\infty, q, r)$ be such a group. We normalize it so that $\Delta(\infty, q, r) \subset \text{SL}(2, \mathbb{Q})$ and that $\infty$ is a cusp of $\Delta(\infty, q, r)$. In this situation Wolfart shows in [Wol83] (see also [Wol85]) that there exists some $A \in \mathbb{C}$ such that the space of (ordinary, i.e. of bi-weight $(k, 0)$) modular forms for $\Delta(\infty, q, r)$ admits a basis given by forms $f_m$ with Fourier expansions

$$f_m(z) = \sum_{n \geq 0} r_{m,n}(Aq)^n \quad \text{with } r_{m,n} \text{ rational},$$

where $q = e(z/a_0)$. The constant $A$ is transcendental of Gelfond–Schneider type if the Fuchsian group is non-arithmetic and is algebraic in the finitely many other cases.

The uniformizing group of the curves $W_D$ is non-arithmetic and the following result extends Wolfart’s non-algebraicity result to this class of curves.

**Theorem 13.2.** The space of twisted modular form of all bi-weights for the uniformizing group $\Gamma$ of $W_D$ has a basis of forms with Fourier expansions $\sum_{n \geq 0} a_n Q^n$ with $a_n \in K(S)$ with $Q = Ae(z/\alpha)$ and $S$ as in Theorem 13.1(ii). The number $A$, and also the radius of convergence $|A|$ of this series, is transcendental of Gelfond–Schneider type.

Wolfart’s proof of $A$ being of Gelfond–Schneider type relies on properties of $\Gamma$-functions and trigonometric calculations. It overlaps with our result in the few cases ($D = 5, 8, 12$) where the uniformizing group of $W_D$ is a triangle group. The rationality of $r_n$ in Wolfart’s result follows easily from the rationality of the coefficients of the Picard–Fuchs differential operators. This rationality does not hold for general $D$, as can be seen from our example (41). The statement of the theorem can indeed not be strengthened to rationality of the coefficients, even in the case $k = \ell = 0$, since the modular function $t$ in (46) admits no rescaling that has rational coefficients.
whose Fourier expansions have rational coefficients. Suppose that $f = \sum_{\nu \in \mathbb{Z}^2} c_\nu e^{i(\nu z_1 + \nu^\prime z_2)}$ is such a basis element. Then the restriction to $(z_1, z_2) = (z, \varphi(z))$ with $\varphi$ as in (102) is

$$f(q) = \sum_{\nu \in \mathbb{Z}^2} c_\nu \text{tr}(\beta\nu) \, Q^{\text{tr}(\alpha\nu)} e^{\sigma(\nu)\varepsilon(q)}$$

and so for this twisted modular form the claim directly follows from Theorem 13.1.

For $k$ and $\ell$ sufficiently large the restriction map is surjective since $L_1 \otimes L_2$ is ample on $X_D$. Here $L_1$ and $L_2$ are the natural line bundles on $X_D$ such that Hilbert modular forms of weight $(k, \ell)$ are sections of $L_1^k \otimes L_2^\ell$. (The surjectivity in fact holds already for $k \geq 4$ and $\ell \geq 10$. To see this, we tensor the structure sequence for $W_D \subset X_D$ with $L_1^k \otimes L_2^\ell$ and note that the cokernel of the restriction map lies in $H^1(X_D, J_{W_D} \otimes L_1^k \otimes L_2^\ell)$. Since $J_{W_D} \cong \mathcal{O}_{X_D}(-W_D) \cong L_1^{-9}L_{-9}$ and since $L_1 \otimes L_2$ is ample on $X_D$, Kodaira’s vanishing theorem implies the claim.)

For the remaining cases, note that with $f$ and $f$ also $1/f$ and $fg$ have Fourier expansions as claimed in the theorem. It thus suffices to consider the products of modular forms of bi-weight $(k, \ell)$ with a given modular form of large enough weight and then to apply the restriction argument. \qed

### 13.3 The foliation by constant absolute periods

As quotients of $\mathbb{H} \times \mathbb{H}$, Hilbert modular surfaces come with two natural holomorphic foliations, which we called $\mathcal{F}_1$ and $\mathcal{F}_2$. They also admit an interesting foliation defined using the $\text{SL}(2, \mathbb{R})$ action on the space $\Omega M_2$. The leaves of this foliation are upper half-planes, but the foliation is not holomorphic. It is studied in detail in [McM07]. In the context of studying the $\text{SL}(2, \mathbb{R})$-action a natural local coordinate system on $\Omega M_2$ is given by period coordinates, i.e. by integrating the holomorphic one-form $\omega$ along a chosen basis of the first homology relative to the zeros of $\omega$. In the special case of genus two, we have used this coordinate system in (95).

We may identify the (Torelli preimage of a) Hilbert modular surface as a subset of $\mathbb{P}\Omega M_2 = (\Omega M_2)/\mathbb{C}^*$ by mapping $X$ to the class of $(X, \omega)$, where $\omega$ is the eigenform for the first embedding of $K$ (in the order that we have chosen once and for all). It follows from [Bai07] or [McM07] that the foliation by constant absolute periods, where only $x_E$ defined in (95) is allowed to vary, is the first foliation $\mathcal{F}_1$. The function $x_E$ is not globally well defined: its sign depends on the choice of an orientation and it may also be altered by a constant by adding the period of a closed loop. However, $q = (dx_E)^2$ is a well-defined quadratic differential on each leaf of $\mathcal{F}_1$, independent of these choices.

The horizontal trajectories of this quadratic differential exhibit beautiful structures on the leaves of $\mathcal{F}_1$. They have been determined by McMullen in [McM12], using the following theorem.

**Theorem 13.3.** The quadratic differential $q$ is proportional to the restriction of the meromorphic modular form

$$Q(z_1, z_2) = \left( \prod_{(m, m') \text{ odd}} D_2 \theta_{(m, m')}(z_1, z_2) \right) \bigg/ \left( \prod_{(m, m') \text{ even}} \theta_{(m, m')}(z_1, z_2) \right)$$

of weight $(-2, 4)$ to the leaf where $z_1$ is constant.

**Proof.** This follows directly from [Bai07, Theorem 10.2] and Theorem 9.1. More precisely, Bainbridge has determined the dependence on $z_1$. Since the quadratic differential depends on the choice of a holomorphic one-form on each Riemann surface, there exists a linear map $Q_1$
from the first eigenform bundle to quadratic differentials on the leaves of $\mathcal{F}_1$ that is locally defined by $q = (dx_E)^2$. Such a map is the same object as a meromorphic modular form of weight $(-2, 4)$. This modular form $Q_1$ vanishes at $W_D$, where the zeros collide, and acquires a pole at the reducible locus $P_D$, where the zeros are infinitely far apart. It is shown moreover in [Bai07, Theorem 10.2] that both the vanishing order and the pole order are equal to one. Consequently, by Theorem 9.1 and the fact that the even theta characteristics vanish precisely at the reducible locus, $Q/Q_1$ is a holomorphic function on $X_D$. It then extends holomorphically to the Baily–Borel compactification by Hartog’s theorem, since the boundary has codimension two. We deduce that $Q/Q_1$ has to be constant, which proves the claim.

Note added in proof. We have just learned that our notion of twisted modular form is not new, but has already occurred in the work of Min Ho Lee under the name of mixed modular form and is described in detail in the first chapter of his 2004 book Mixed automorphic forms, torus bundles, and Jacobi forms [Lee04]. In fact, his notion is somewhat more general than ours in that the underlying map $\varphi$ can be equivariant with respect to any homomorphism $\Gamma \to \text{SL}(2, \mathbb{R})$, not specifically the Galois conjugation for a subgroup of a Hilbert modular group. So his specific results do not apply to Teichmüller curves or to modular embeddings in Hilbert modular surfaces, but some parts of the theory, in particular the connection to differential equations and to periods (in his case of an elliptically fibered surface rather than of a Teichmüller curve) are the same. We would like to thank YoungJu Choie for bringing Lee’s work to our attention.

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References
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Theta derivatives


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