RATIONAL PERIOD FUNCTIONS FOR $PSL(2,\mathbb{Z})$

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A *rational period function* (RPF) of weight $2k$ ($k \geq 0$) for $PSL(2,\mathbb{Z})$ is a rational function $q(z)$ satisfying the two identities

$$q(z) + z^{-2k} q\big(\frac{-1}{z}\big) = 0, \qquad q(z) + z^{-2k} q\big(1 - \frac{1}{z}\big) + (z-1)^{-2k} q\big(\frac{-1}{z-1}\big) = 0.$$

The definition is motivated by the fact that the corresponding identities in negative weight $2-2k$ describe the so-called *period polynomials*, a space of polynomials closely related to the space of cusp forms on $PSL(2,\mathbb{Z})$ of weight $2k$. Rational period functions were first considered by Knopp [6], who gave non-trivial examples (e.g. $(z^2-z-1)^{-k} + (z^2+z-1)^{-k}$ if k is odd) and showed that an arbitrary RPF can have poles only at 0 or at real quadratic irrationalities. Further work was done by several authors (cf. references in §2.4). Finally, Ash [1] described the set of RPF's completely, not only for the full modular group, but also for subgroups of finite index. In this paper we will give a very explicit construction of all RPF's for $PSL(2,\mathbb{Z})$, in particular recovering Ash's result in this case in a more concrete form than the one he gives, and also give an explicit description of the action of Hecke operators on RPF's. The final result on the classification of RPF's for $PSL(2,\mathbb{Z})$is stated in §2.4, while the discussion of Hecke operators is in Section 3. Some numerical examples illustrating the theory are given in §4.

1. Review of reduction theory and of the theory of periods

In §1.1 we review the reduction theory of indefinite binary quadratic forms and prove a lemma which will be used for the classification of rational period functions. In §1.2 we recall the Eichler-Shimura theory of periods of modular forms for $\Gamma = PSL(2,\mathbb{Z})$ (as presented, say, in [10]) and also describe certain complements to this theory proved in [8].

1.1. We consider binary quadratic forms $Q(x,y) = Ax^2 + Bxy + Cy^2$ with A, B, $C \in \mathbb{Z}$; we will denote such a form by $Q = [A,B,C]$ and its discriminant $B^2 - 4AC$ by $D = D(Q)$. We will consider only forms of positive non-square discriminant. The group Γ acts on the set of forms of discriminant D by $(Q|\gamma)(x,y) = Q((x,y)\gamma^t)$ or more explicitly by $[A,B,C] \,|\! \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = [Aa^2{+}Bac{+}Cc^2, 2Aab{+}B(ad{+}bc){+}2Ccd, Ab^2{+}Bbd{+}Cd^2]$. The action can be extended to $PGL(2,\mathbb{Z})$ by letting $(Q|\gamma)(x,y)$ denote $-Q((x,y)\gamma^t)$ if $\det(\gamma) = -1$. We will use the letter $\mathcal{A}$ to denote a Γ-equivalence class of forms Q and $\mathcal{A}^{-1}$, $\Theta\mathcal{A}$, and $\Theta\mathcal{A}^{-1}$ to denote the classes of the quadratic forms $[A,-B,C]$, $[-A,B,-C]$ and $[-A,-B,-C]$. The notation comes from the fact that the classes of primitive Q with given discriminant D form a group $\mathrm{Cl}(D)$, the strict ideal class group of the order $\mathcal{O}_D = \mathbb{Z} + \mathbb{Z}\frac{D+\sqrt{D}}{2}$, and that $\mathcal{A}^{-1}$ and $\Theta\mathcal{A}$ correspond to the inverse of $\mathcal{A}$ and to the product of $\mathcal{A}$ with Θ in this group, Θ being the class of principal ideals having a generator of negative norm. The element $\Theta \in \mathrm{Cl}(D)$ has order 1 or 2 depending whether Pell's equation $t^2 - Du^2 = -4$ has an integral solution or not. The smallest D for which there exists an $\mathcal{A}$ with $\mathcal{A}$, $\mathcal{A}^{-1}$, $\Theta\mathcal{A}$ and $\Theta\mathcal{A}^{-1}$ all distinct is 316.

We call the form $Q=[A,B,C]$ *simple* if $A>0>C$. Clearly there are only finitely many simple forms of a given discriminant, since the equation $D=B^2+4|AC|$ bounds A, B, and C. We call Q *reduced* if $A>0$, $C>0$, $B>A+C$. The map $[A,B,C]\mapsto[A,B-2A,C-B+A]$ is a bijection between reduced forms and simple forms $[A',B',C']$ with $A'+B'+C'>0$; in particular, there are exactly half as many reduced forms as simple forms of a given discriminant. To each form $Q=[A,B,C]$ we associate the real quadratic irrationality $\alpha=\alpha_Q=\frac{B+\sqrt{D}}{2A}$, one of the roots of $Q(x,-1)=0$. The correspondence $Q\leftrightarrow\alpha_Q$ is bijective (of course $Q(X,-1)=0$ has a second root $\alpha_Q^\sigma=\frac{B-\sqrt{D}}{2A}$, the conjugate of α, but this corresponds under the bijection to $-Q$, not Q), so that we can think of classes $\mathcal{A}$ as Γ-equivalence classes of quadratic irrationalities and speak of simple and reduced quadratic irrationalities (denoted x and w, respectively) in the obvious way. We have

$$w \text{ reduced } \iff w>1>w^\sigma>0, \qquad x \text{ simple } \iff x>0>x^\sigma,$$

and the above bijection between reduced forms and half of all simple forms is given by $x=w-1$. (The condition $A+B+C>0$ for a simple form corresponds to $x^\sigma>-1$; the involution $[A,B,C]\mapsto[-C,-B,-A]$ or $x\mapsto 1/x$ interchanges this half of the set of simple forms or simple irrationalities with the other half.)

The reduced forms in a given class $\mathcal{A}$ can be put into a cycle $\{Q_\nu \mid \nu\in\mathbb{Z}/r\mathbb{Z}\}$ (r=number of reduced forms) with $Q_\nu=Q_{\nu+1}|\left(\begin{smallmatrix}0&-1\\1&n_\nu\end{smallmatrix}\right)$ for some integers $n_\nu\ge 2$. Then $w_\nu=1/(n_\nu-w_{\nu+1})$, so that each of the finitely many w_ν has a pure periodic continued fraction expansion $n_\nu-1/(n_{\nu+1}-1/(n_{\nu+2}-\dots))=((\overline{n_\nu,n_{\nu+1},\dots,n_{\nu+r-1}}))=$ of length r; conversely, any pure periodic continued fraction of this form except $((\overline{2}))=1$ is a reduced quadratic irrationality. The simple irrationalities in the class $\mathcal{A}$ are the numbers $w_\nu-l$, $\nu\in\mathbb{Z}/r\mathbb{Z}$, $1\le l\le n_\nu-1$. They also form a cycle; more precisely, one has:

Lemma. *Define a map* $\Phi:\mathbb{R}_+\to\mathbb{R}_+$ *by*

$$\Phi(x)=\begin{cases} x-1 & \text{if } \ x\ge 1,\\ \dfrac{x}{1-x} & \text{if } \ 0\le x<1.\end{cases}$$

Then the finite orbits of Φ *are the set* $\{0\}$ *and the sets* $Z_{\mathcal{A}}=\{\alpha_Q,\ Q\in\mathcal{A} \text{ simple}\}$, *where* $\mathcal{A}$ *runs over all* Γ*-equivalence classes of primitive indefinite binary quadratic forms with integral coefficients and non-square discriminant.*

Proof. Clearly any cycle except $\{0\}$ has the form $\{x_1,x_1-1,\dots,x_1-m_1,x_2,x_2-1,\dots,x_2-m_2,\dots,x_r,x_r-1,\dots,x_r-m_r\}$ for some real numbers $x_1,\dots,x_r>0$, where $m_\nu=[x_\nu]\ge 0$ (greatest integer $\le x_\nu$) and $x_{\nu+1}=(x_\nu-m_\nu)/(1-x_\nu+m_\nu)$ (here ν is to be taken modulo r). Set $w_\nu=x_\nu+1$ and $n_\nu=m_\nu+2\ge 2$; then $n_\nu=[w_\nu]+1$ and $w_\nu=n_\nu-\frac{1}{w_{\nu+1}}$, so the w_ν are a cycle of pure periodic continued fractions and are precisely the cycle of irreduced irrationalities in some $PSL(2,\mathbb{Z})$-equivalence class $\mathcal{A}$ as described above. $\square$

1.2. Fix an integer $k>1$ and denote by $\mathbf{S}=\mathbf{S}_{2k}$ the space of cusp forms of weight $2k$ on Γ and by $\mathbf{V}=\mathbf{V}_{2k-2}$ the space of polynomials of degree $\le w:=2k-2$. To each $f\in\mathbf{S}$ is associated the *period polynomial* $r(f)\in\mathbf{V}$ defined by

$$r(f)(X)=\int_0^\infty f(z)(z-X)^w\,dz=\sum_{n=0}^{w}(-1)^n\binom{w}{n}r_n(f)X^{w-n},\quad r_n(f)=\int_0^\infty f(z)z^n\,dz.$$

Denote by $| = |_{2-2k}$ the action of Γ on $\mathbf{V}$ defined by $\big(h|\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)\big)(X) = (cX+d)^w h\big(\frac{aX+b}{cX+d}\big)$. An easy calculation shows that $(r(f)|\gamma)(X)$ for $f \in \mathbf{S}$ and $\gamma \in \Gamma$ is just the integral of $f(z)(z-X)^w$ from $\gamma^{-1}(0)$ to $\gamma^{-1}(\infty)$. It follows immediately that $r(f)$ belongs to the subspace $\mathbf{W} = \mathbf{W}_{2k-2} = \ker(|(1+T)) \cap \ker(|(1+U+U^2))$ of $\mathbf{V}$, where T and U denote the two generators $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1 & -1 \\ 1 & 0\end{smallmatrix}\right)$ of Γ of order 2 and 3, and the definition of "$|$" has been extended from Γ to the group ring $\mathbb{Z}[\Gamma]$ by linearity. The space $\mathbf{V}$ splits into the direct sum of the subspaces $\mathbf{V}^+$, $\mathbf{V}^-$ of even and odd polynomials, and $\mathbf{W}$ is the direct sum of the subspaces $\mathbf{W}^\pm = \mathbf{W} \cap \mathbf{V}^\pm$, since the matrix $\epsilon = \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1\end{smallmatrix}\right) \in GL(2,\mathbb{Z})$ commutes with T and conjugates U into TU^2T. Write $r(f)$ for $f \in \mathbf{S}$ as $r^+(f) + r^-(f)$ with $r^\pm \in \mathbf{W}^\pm$. If $f \in \mathbf{S}$ is a normalized Hecke eigenform, then there are non-zero numbers $\omega^+(f) \in i\mathbb{R}$ and $\omega^-(f) \in \mathbb{R}$ with product $i(f,f)$ ($(\ , \ )$ = Petersson scalar product in $\mathbf{S}$) such that the polynomials $\omega^\pm(f)^{-1} r^\pm(f)$ both have coefficients in the algebraic number field generated by the Fourier coefficients of f. The map r^- is an isomorphism from $\mathbf{S}^-$ to $\mathbf{W}^-$, while r^+ is an isomorphism from $\mathbf{S}^-$ onto a codimension 1 subspace of $\mathbf{W}^+$ not containing the element $p_{k,0}(X) = X^w - 1$ of $\mathbf{W}^+$. In other words, there is an exact sequence

$$0 \longrightarrow \mathbf{S}_{2k} \oplus \mathbf{S}_{2k} \xrightarrow{(r^+,r^-)} \mathbf{W}_{2k-2} \xrightarrow{\lambda} \mathbb{C} \longrightarrow 0\,,$$

where $\lambda(p_{k,0}) \neq 0$. The map λ is given explicitly by the formula

$$\lambda\Big(\sum_{n=0}^{w} \binom{w}{n} r_n X^{w-n}\Big) = \sum_{r=1}^{k} \beta_r \beta_{k-r} \sum_{\substack{n=0 \\ n \text{ even}}}^{w} \binom{2r-1}{n} r_n,$$

where $\beta_m = \dfrac{B_{2m}}{(2m)!}$ ($B_{2m} = (2m)^{\text{th}}$ Bernoulli number) for $m > 0$, $\beta_0 = 2$ ([8]; the formula is both normalized and stated slightly differently there).

The space $\mathbf{W}_{2k-2}/\langle p_{k,0}\rangle$ can be identified with $H^1_{\text{par}}(\Gamma, \mathbf{V}_{2k-2})$, the subspace of the first cohomology group of Γ with coefficients in the representation $\mathbf{V}$ given by cocycles which vanish on the parabolic element $S = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1\end{smallmatrix}\right) = UT$. The isomorphism is given by mapping $q \in \mathbf{W}$ to the class of the cocycle ϕ with $\phi(T) = q$ and $\phi(S) = 0$ (these properties determine ϕ because Γ is generated by S and T, and ϕ is a cocycle exactly for $q \in \mathbf{W}$). It follows that we can also canonically identify the space $\mathbf{W}^0_{2k-2} = \text{Ker}\,(\lambda) = r^+(\mathbf{S}_{2k}) \oplus r^-(\mathbf{S}_{2k}) \subset \mathbf{W}_{2k-2}$ with $H^1_{\text{par}}(\Gamma, \mathbf{V}_{2k-2})$.

The relation of the period theory with the theory of reduced quadratic forms as reviewed in §1.1 is as follows. For each Γ-equivalence class $\mathcal{A}$ of binary quadratic forms, define elements $P_{k,\mathcal{A}}$, $Q_{k,\mathcal{A}}$, and $R_{\mathcal{A}}$ of $\mathbf{V}$ by

$$P_{k,\mathcal{A}}(X) = \sum_{\substack{Q \in \mathcal{A} \\ Q \text{ simple}}} Q(X,-1)^{k-1}, \qquad Q_{k,\mathcal{A}}(X) = \sum_{\substack{Q \in \mathcal{A} \\ Q \text{ reduced}}} Q(X,-1)^{k-1}$$

and $R_{k,\mathcal{A}} = P_{k,\mathcal{A}} + (-1)^k P_{k,\Theta\mathcal{A}^{-1}}$ (recall that $\Theta\mathcal{A}^{-1}$ denotes the class consisting of all forms $-Q$ with $Q \in \mathcal{A}$). Then the relationship between simple and reduced forms as described in §1.1 implies the equalities

$$R_{k,\mathcal{A}} = P_{k,\mathcal{A}}|(1-T) = (Q_{k,\mathcal{A}} + (-1)^k Q_{k,\Theta\mathcal{A}^{-1}})|(-U+U^2).$$

From these and the equations $T^2 = 1$, $U^3 = 1$ it follows that $R_{k,\mathcal{A}}$ belongs to the subspace $\mathbf{W}$ of $\mathbf{V}$. Therefore by the decomposition $\mathbf{W}_{2k-2} = \mathbf{W}^0_{2k-2} \bigoplus \mathbb{C}p_{k,0}$ we must have

$$R_{k,\mathcal{A}} = r^+(f^+_{k,\mathcal{A}}) + r^-(f^-_{k,\mathcal{A}}) + \rho_k(\mathcal{A})\, p_{k,0}$$

for some cusp forms $f^\pm_{k,\mathcal{A}} \in \mathbf{S}$ and number $\rho_k(A)$; in fact,

$$f^\pm_{k,\mathcal{A}}(z) = \frac{D^{k-1/2}}{2\pi i\binom{2k-2}{k-1}}\Bigl(\sum_{Q\in\mathcal{A}} \frac{1}{Q(z,1)^k} \pm \sum_{Q\in\mathcal{A}} \frac{1}{Q(z,-1)^k}\Bigr), \qquad \rho_k(A) = \frac{\zeta_{\mathcal{A}}(1-k)}{\zeta(1-2k)},$$

where $\zeta(s)$ is the Riemann zeta function and $\zeta_{\mathcal{A}}(s)$ the partial zeta function of the ideal class $\mathcal{A}$. For all of this, see sections 2.1-2.3 of [8], esp. Theorem 5 (where, however, the normalizations are slightly different and the factor $(-1)^k$ before $Q_{\Theta\mathcal{A}^{-1}}$ was omitted). We denote by $R^0_{k,\mathcal{A}} = R_{k,\mathcal{A}} - \rho_k(\mathcal{A})\, p_{k,0}$ the component of $R_{k,\mathcal{A}}$ in $\mathbf{W}^0_{2k-2}$.

2. Classification of rational period functions for $PSL(2,\mathbb{Z})$

We now turn from negative weight $2-2k$ to positive weight $2k$, so that we consider the action $\bigl(h|_{2k}\bigl(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\bigr)\bigr)(X) = (cX+d)^{-2k} h\bigl(\frac{aX+b}{cX+d}\bigr)$ on rational functions (here we must consider rational functions rather than polynomials because the action $|_{2k}$, unlike $|_{2-2k}$, never maps polynomials to polynomials). We define the space of rational period polynomials of weight $2k$ in analogy to $\mathbf{W}_{2k-2}$ by

$$\mathbf{RPF} = \mathbf{RPF}_{2k} = \bigl\{\text{rational functions } q(z) \bigm| q|_{2k}(1+T) = q|_{2k}(1+U+U^2) = 0\bigr\}.$$

(This is the same as the formula in the introduction.) Our object is to obtain a complete description of this space.

2.1. Suppose $q \in \mathbf{RPF}$, and denote by $\mathcal{S} \subset \mathbb{C}$ the pole set of q. The definition of $\mathbf{RPF}$ clearly implies

$$\alpha \in \mathcal{S}, \quad \alpha \neq 0 \implies T\alpha\ \bigl(= \frac{-1}{\alpha}\bigr) \in \mathcal{S},$$
$$\alpha \in \mathcal{S}, \quad \alpha \neq 0,1 \implies U\alpha\ \bigl(= 1 - \frac{1}{\alpha}\bigr) \text{ or } U^2\alpha\ \bigl(= \frac{1}{1-\alpha}\bigr) \in \mathcal{S}.$$

In this subsection we will deduce from this the possible structure of the set $\mathcal{S}$.

Lemma 1. $\mathcal{S} \subset \mathbb{R}$.

Proof. Suppose $\alpha \in \mathcal{S}$, $\alpha \notin \mathbb{R}$. By what we have just said, at least one of $UT\alpha = \alpha+1$ and $U^2T\alpha = \alpha/(\alpha+1)$ belongs to $\mathcal{S}$, so we get a sequence $\alpha_0 = \alpha, \alpha_1, \ldots$ of elements $\alpha_n \in \mathcal{S}$ with α_{n+1} equal to $\alpha_n + 1$ or $\alpha_n/(\alpha_n+1)$. But it is obvious that the argument (taken in the interval $(-\pi,\pi)$) of a non-real complex number α decreases strictly in absolute value when α is replaced by either $\alpha+1$ or $\alpha/(\alpha+1)$. It follows that $|\arg(\alpha_0)| > |\arg(\alpha_1)| > |\arg(\alpha_2)| > \ldots > 0$. This is impossible since $\mathcal{S}$ is a finite set. $\square$

Lemma 2. *The set $\mathcal{S} \smallsetminus \{0\}$ is the (disjoint) union of sets $\mathcal{Z}_{\mathcal{A}} \cup T\mathcal{Z}_{\mathcal{A}}$, where $\mathcal{Z}_{\mathcal{A}}$ is the cycle associated to a Γ-equivalence class of binary quadratic forms $\mathcal{A}$ as in §1.1.*

Proof. The proof is similar to that of Lemma 1. We may assume (by replacing α by $T\alpha$ if necessary) that $\alpha > 0$; then $T\alpha \neq 0, 1$ and therefore $\mathcal{S}$ contains either $UT\alpha = \alpha + 1$ or $U^2T\alpha = \alpha/(\alpha+1)$, which is again positive. This again gives a sequence $\alpha_0 = \alpha$, $\alpha_1, \ldots$ of elements $\alpha_n \in \mathcal{S}$ where for each n either $\alpha_{n+1} = \alpha_n + 1$ or $\alpha_{n+1} = \alpha_n/(\alpha_n+1)$. In the first case $\alpha_{n+1} > 1$ and in the second case, $0 < \alpha_{n+1} < 1$, so in both cases one finds $\Phi(\alpha_{n+1}) = \alpha_n$, where $\Phi : \mathbb{R} \to \mathbb{R}$ is the map defined in §1.1. But $\alpha_m = \alpha_n$ for some $m > n \geq 0$ by the finiteness of $\mathcal{S}$, so we get a finite orbit $\{\alpha_m, \alpha_{m-1}, \ldots, \alpha_{n+1}\}$ of Φ. By the Lemma of §1.1, it follows that α_n, and hence $\alpha = \alpha_0 = \Phi^n(\alpha_n)$, belongs to a cycle $\mathcal{Z}_{\mathcal{A}}$ of simple real quadratic irrationalities, and that all of $\mathcal{Z}_{\mathcal{A}}$ is contained in $\mathcal{S}$. □

Lemma 3. *If $\alpha \neq 0 \in \mathcal{S}$, then* exactly *one of $U\alpha$ and $U^2\alpha$ belongs to $\mathcal{S}$.*

Proof. By what we have just proved, any non-zero element of $\mathcal{S}$ is a quadratic irrationality of negative norm. But it is easily checked that if $1 > \alpha > 0 > \alpha^\sigma$ or $1 > \alpha^\sigma > 0 > \alpha$, then $U^2\alpha$ and $U^2\alpha^\sigma$ are both positive, while if $\alpha > 1 > 0 > \alpha^\sigma$ or $\alpha^\sigma > 1 > 0 > \alpha$, then $U\alpha$ and $U\alpha^\sigma$ are both positive. Thus in all cases one of $U\alpha$ and $U^2\alpha$ has positive norm and cannot belong to $\mathcal{S}$. □

2.2 The results of §2.1 describe the pole set $\mathcal{S}$ of an RPF q. We now consider the principal part of q at its pole. We use the notation $\mathrm{PP}_\alpha[f]$ to denote the principal part of a rational function $f(z)$ near a point $z = \alpha \in \mathbb{C}$, i.e., the unique polynomial in $(z-\alpha)^{-1}$ without constant term such that $f - \mathrm{PP}_\alpha[f]$ is regular at α. We first show:

Lemma 4. *Suppose that $\alpha \neq 0$ is a pole of $q \in$ **RPF**. Then the order of the pole of q at α is exactly k.*

Proof. The equation $q|_{2k}(1+T) = 0$ implies that $\mathrm{PP}_{T\alpha}[q] = -\mathrm{PP}_\alpha[q]|_{2k}T$, and the equation $q|_{2k}(1+U+U^2) = 0$ together with Lemma 3 implies that $\mathrm{PP}_{V\alpha}[q] = -\mathrm{PP}_\alpha[q]|_{2k}V^{-1}$, where V is that one of U and U^2 for which $V\alpha \in \mathcal{S}$. We also know that α belongs to a cycle $\{\alpha_0 = \alpha = \alpha_r, \alpha_1, \ldots, \alpha_{r-1}\}$ with $\alpha_{n+1} = UT\alpha_n$ or $U^2T\alpha_n$ for each n, i.e. with $\alpha_n = \gamma_n\alpha$ for some $\gamma_n \in \Gamma$, $\gamma_0 = 1$, $\gamma_{n+1} = U^{\pm 1}T\gamma_n$ $(n \geq 0)$. By induction on n we obtain $\mathrm{PP}_{\alpha_n}[q] = \mathrm{PP}_\alpha[q]|_{2k}\gamma_n^{-1}$ and in particular $\mathrm{PP}_\alpha[q] = \mathrm{PP}_\alpha[q]|_{2k}\gamma_r^{-1}$. Write $\gamma_r = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ and suppose that $\mathrm{PP}_\alpha[q](z) = (z-\alpha)^{-m} + \cdots$ where "$\cdots$" denotes an expression with a pole of order $< m$ at α. Then

$$\begin{aligned}
\left(\mathrm{PP}_\alpha[q]|_{2k}\gamma^{-1}\right)(z) &= (cz-a)^{-2k}\,\mathrm{PP}_\alpha[q]\Big(\frac{dz-b}{-cz+a}\Big) \\
&= (cz-a)^{-2k}\,\Big(\frac{dz-b}{-cz+a} - \alpha\Big)^{-m} + \cdots \\
&= \frac{(-cz+a)^{-2k+m}}{(c\alpha+d)^m}\,(z-\alpha)^{-m} + \cdots \\
&= (c\alpha+d)^{2k-2m}(z-\alpha)^{-m} + \cdots
\end{aligned}$$

since $c\alpha + d = (-c\alpha+a)^{-1}$. But $c\alpha + d$ is clearly irrational (it is in fact a fundamental unit of the quadratic order associated to α), so we must have $k = m$. □

Lemma 4 implies that for each $\alpha \neq 0$ which can occur as a pole of a rational period function q of weight $2k$ there is a (unique) function of the form

$$q_{k,\alpha}(z) = \frac{1}{(z-\alpha)^k} + \frac{a_1(\alpha)}{(z-\alpha)^{k-1}} + \cdots + \frac{a_k}{z-\alpha}$$

such that the $\mathrm{PP}_\alpha[q]$ is a multiple of $q_{k,\alpha}$ for any $q \in \mathbf{RPF}_{2k}$.

Lemma 5. *Let α be a simple quadratic irrationality, α^σ its conjugate. Then*

$$q_{k,\alpha}(z) = PP_\alpha\left[\frac{(\alpha-\alpha^\sigma)^k}{(z-\alpha)^k(z-\alpha^\sigma)^k}\right] = PP_\alpha\left[\frac{D^{k/2}}{Q(z,-1)^k}\right],$$

where Q is the quadratic form associated to α and D the discriminant of Q.

Proof. It is easily checked that $Q(z,-1)^{-k}|_{2k}\gamma = Q(z,-1)^{-k}$ for any $\gamma \in \Gamma$ with $\gamma\alpha = \alpha$. But the proof of Lemma 4 shows that this property characterizes $q_{k,\alpha}(z)$ up to a constant (since any function satisfying it has a pole of order *exactly* k at α). Since the leading term of $Q(z,-1)^{-k}$ is $D^{-k/2}$, the proportionality constant is as given in the lemma. □

It follows from Lemma 5 that any function in **RPF** has the form

$$q(z) = \sum_\alpha C_\alpha q_{k,\alpha}(z) \ + \ q_0'(z)$$

for some constants $C_\alpha \in \mathbb{C}$ and some function $q_0'(z) \in \mathbb{C}[z, z^{-1}]$, where the sum ranges over finitely many cycles of real quadratic irrationalities α with $\alpha\alpha' < 0$. The first sentence of the proof of Lemma 4 makes it clear that the coefficients C_α alternate as we go around any cycle α, $T\alpha$, $U^{\pm1}T\alpha$, $TU^{\pm1}T\alpha$, $\cdots$, i.e., that C_α equals $\mathrm{sgn}(\alpha)\, C_{\mathcal{A}}$ where $C_{\mathcal{A}}$ depends only on the class $\mathcal{A}$ corresponding to α. Finally, we apply the operator $|_{2k}(1-T)$ to the decomposition of q, observing that $q|(1-T) = 2q$ and that the function $q_0 = q_0'\,|(1-T)$ still belongs to $\mathbb{C}[z,z^{-1}]$. This gives a representation

$$q(z) = \sum_{\mathcal{A}} C_{\mathcal{A}} q_{k,\mathcal{A}}(z) \ + \ q_0(z), \qquad q_0(z) \in \mathbb{C}[z,z^{-1}]|_{2k}(1-T)$$

where

$$q_{k,\mathcal{A}}(z) = \sum_{\alpha \in \mathcal{Z}_{\mathcal{A}}} \big(q_{k,\alpha}(z) - q_{k,T\alpha}(z)\big)|_{2k}(1-T)$$

and the coefficients $C_{\mathcal{A}}$ are non-zero for only finitely many classes $\mathcal{A}$.

2.3. It remains to determine which linear combinations of the functions $q_{k,\mathcal{A}}$ can be corrected by a function q_0 having a pole only at $z=0$ in such a way as to become rational period functions. Since $q_{k,\mathcal{A}}|(1+T)$ vanishes by construction, the condition $q|(1+T) = 0$ is automatic if q_0 is chosen to be anti-invariant under the involution T. We must therefore compute the image of $q_{k,\mathcal{A}}$ under $1+U+U^2$. The poles of $q_{k,\mathcal{A}}$ at quadratic irrationalities in $\mathcal{Z}_{\mathcal{A}} \cup T\mathcal{Z}_{\mathcal{A}}$ cancel out when we slash with $1+U+U^2$ by construction, but there are new poles at $z=0$ and $z=1$ coming from the automorphy factors z^{-2k} and $(z-1)^{-2k}$ which arise from the operations $|_{2k}U$ and $|_{2k}U^2$. These poles have order at most $2k$. By subtracting from q a multiple of $1-z^{-2k}$, which can easily be checked to be an RPF, we can assume that the pole at 0 has order $\le 2k-1$. The pole at 1 then also has order $\le 2k-1$ because $q_{k,\mathcal{A}}|(1+U+U^2)$ is annihilated by $1-U$ (since $(1+U+U^2)(1-U) = 1-U^3 = 0$ in the group ring of Γ). Thus $q_{k,\mathcal{A}}|(1+U+U^2)$ belongs to the space

$$\mathbf{H} = \mathbf{H}_{2k} = \big\{\phi \,\in\, z^{-2k+1}(z-1)^{1-2k}\,\mathbb{C}[z] \;\big|\; \phi|_{2k}(1-U) = 0\big\}.$$

Our first goal is to determine the structure of $\mathbf{H}$.

Lemma 6. *There is an isomorphism $\mu : \mathbf{H}_{2k} \to Ker\big(1+U+U^2, \mathbf{V}_{2k-2}\big)$ given as follows: if*

$$h(z) = \sum_{n=0}^{w} \frac{a_n}{z^{n+1}} + \sum_{n=0}^{w} \frac{b_n}{(z-1)^{n+1}} \qquad\qquad (w = 2k-2)$$

is an element of $\mathbf{H}$, and $A(X) = \sum_{n=0}^{w} \binom{w}{n} a_n X^n$, $B(X) = \sum_{n=0}^{w} \binom{w}{n} b_n X^n \in \mathbf{V}_{2k-2}$, then $\mu(h) = A|_{2-2k}TU = B|_{2-2k}U^2$.

Proof. If $h(z) \in \mathbf{H}$, the property $h(z) = z^{-2k}h(1-\frac{1}{z})$ shows that h tends to 0 as z tends to infinity, so h has the form given in the lemma. Then by partial fractions

$$\begin{aligned}(h|U)(z) &= \sum a_n z^{-\tilde{n}-1}(z-1)^{-n-1} - \sum (-1)^n b_n z^{-\tilde{n}-1} \\ &= \sum z^{-n-1}\Big((-1)^{n+1} b_{\tilde{n}} - \sum_{\tilde{m}\ge n} (-1)^m \tbinom{\tilde{n}}{\tilde{m}-n} a_m\Big) \\ &\quad + \sum (z-1)^{-n-1}\Big(\sum_{m\ge n} (-1)^{m-n} \tbinom{\tilde{n}}{m-n} a_m\Big),\end{aligned}$$

where n in the summations goes from 0 to w and $\tilde{n}$, $\tilde{m}$ denote $w-n$, $w-m$. The condition $h = h|U$ is thus equivalent to the properties

$$a_n = (-1)^{n+1} b_{\tilde{n}} - \sum_{\tilde{m}\ge n} (-1)^m \tbinom{\tilde{n}}{\tilde{m}-n} a_m, \qquad b_n = \sum_{m\ge n} (-1)^{m-n} \tbinom{\tilde{n}}{m-n} a_m$$

or

$$\tbinom{w}{n} a_n = (-1)^{n+1} \tbinom{w}{n} b_{\tilde{n}} - \sum_m (-1)^m \tbinom{w}{m} a_m \tbinom{\tilde{m}}{n}, \qquad \tbinom{w}{n} b_n = \sum_m (-1)^{n-m} \tbinom{w}{m} a_m \tbinom{m}{n}.$$

Multiplying by X^n and summing from 0 to w, we find

$$h = h|U \iff A(X) = -X^w B\big(-\frac{1}{X}\big) - (X+1)^w A\big(\frac{-1}{X+1}\big), \;\; B(X) = A(X-1)$$

or $A = B|UT$, $B|(1+U+U^2) = 0$. Thus $\mathbf{H}$ is isomorphic to $\mathrm{Ker}\,(1+U+U^2, \mathbf{V})$ via $h \mapsto B$ or alternatively via $h \mapsto B|U^2 = \mu(h)$. $\square$

The following result is the key to the solution of our problem.

Lemma 7. *Let $\mathcal{A}$ be a Γ-equivalence class of forms as usual. Then*

$$\mu\big(q_{k,\mathcal{A}}|_{2k}(1+U+U^2)\big) = (-1)^{k-1} \binom{2k-2}{k-1} D^{\frac{1-k}{2}} R_{k,\mathcal{A}},$$

where $R_{k,\mathcal{A}}$ is the element of $\mathbf{W}_{2k-2}$ defined in §1.2.

Proof. Write ψ for the map $\bigoplus_{0\le n\le w} \mathbb{C}z^{-n-1} \to \mathbf{V}$ defined by $z^{-n-1} \mapsto \binom{w}{n} X^n$, so that the μ of Lemma 6 is given by $\mu(h) = \psi(\mathrm{PP}_0[h])|TU$. (We use $|$ to denote $|_{2k}$ when applied to rational functions of z and $|_{2-2k}$ when applied to polynomials in X.) We begin by calculating $\psi(\mathrm{PP}_0[q_{k,\alpha}|T])$. By Lemma 5 and the binomial theorem,

$$q_{k,\alpha}(z) = \mathrm{PP}_\alpha\big[(z-\alpha)^{-k}\big(1-\frac{z-\alpha}{\alpha'-\alpha}\big)^{-k}\big] = \sum_{n=0}^{k-1} \big(\frac{1}{\alpha'-\alpha}\big)^{k-1-n} \binom{\tilde{n}}{k-1} (z-\alpha)^{-n-1},$$

where $\tilde{n} = w - n$ as before. On the other hand,

$$\mathrm{PP}_0\big[\,(z-\alpha)^{-n-1}|T\,\big] = \mathrm{PP}_0\big[\frac{(-1)^{n+1}}{z^{\tilde{n}+1}(1+\alpha z)^{n+1}}\big] = \sum_{m=0}^{\tilde{n}}(-1)^{m+1}\alpha^{\tilde{n}-m}\binom{\tilde{m}}{n}z^{-m-1}$$

(binomial theorem again), so

$$\psi\big(\mathrm{PP}_0\big[\,(z-\alpha)^{-n-1}|T\,\big]\big) = \sum_{m=0}^{\tilde{n}}(-1)^{m+1}\alpha^{\tilde{n}-m}\binom{\tilde{m}}{n}\binom{w}{m}X^m = -\binom{w}{n}(\alpha-X)^{\tilde{n}}.$$

Therefore

$$\begin{aligned}\psi\big(\mathrm{PP}_0\big[\,q_{k,\alpha}\,|T\big]\big) &= -\sum_{n=0}^{k-1}\big(\frac{-1}{\alpha-\alpha'}\big)^{k-1-n}\binom{\tilde{n}}{k-1}\binom{w}{n}(\alpha-X)^{\tilde{n}}\\ &= -\binom{w}{k-1}\frac{(\alpha-X)^{k-1}}{(\alpha-\alpha')^{k-1}}\sum_{n=0}^{k-1}\binom{k-1}{n}(X-\alpha)^{k-1-n}(\alpha-\alpha')^n\\ &= (-1)^k\binom{w}{k-1}\left(\frac{(\alpha-X)(\alpha'-X)}{\alpha-\alpha'}\right)^{k-1}\\ &= (-1)^k\binom{w}{k-1}D^{(1-k)/2}\,Q_\alpha(X,-1)^{k-1},\end{aligned}$$

where $Q_\alpha(X,-1)$ is the quadratic form associated to α.

On the other hand, we have

$$\mathrm{PP}_0\big[\,q_{k,\alpha}|(1-T)(1+U+U^2)\,\big] = \mathrm{PP}_0\big[\,q_{k,\alpha}|(-T+U)\,\big] = \mathrm{PP}_0\big[\,\big(-q_{k,\alpha}+q_{k,\alpha-1}\big)|T\,\big]$$

because the function $q_{k,\alpha}|\gamma$ has a pole at 0 only for those γ with $\gamma(0)=\infty$ and $q_{k,\alpha}|S = q_{k,\alpha-1}$. Consequently

$$\mathrm{PP}_0\big[\,q_{k,\mathcal{A}}|(1+U+U^2)\,\big] = \sum_{\alpha\in\mathcal{Z}_\mathcal{A}}\mathrm{PP}_0\big[\,\big(-q_{k,\alpha}+q_{k,\alpha-1}+q_{k,T\alpha}-q_{k,T\alpha-1}\big)|T\,\big]$$

and hence by the computation above

$$\mu\big(q_{k,\mathcal{A}}|(1+U+U^2)\,\big) = (-1)^k\binom{w}{k-1}\,D^{(1-k)/2}\,P_{k,\mathcal{A}}|(-1+S+T-TS)|TU$$

with $P_{k,\mathcal{A}} = \sum_{\alpha\in\mathcal{Z}_\mathcal{A}}Q_\alpha(z,-1)^{k-1}$ as in §1.2. For the last equality we used the easily checked fact that $Q_{\gamma\alpha}(X,-1)^{k-1} = Q_\alpha(X,-1)^{k-1}|\gamma^{-1}$ for any $\gamma\in\Gamma$. The lemma now follows since

$$P_{k,\mathcal{A}}|(-1+S+T-TS)|TU = R_{k,\mathcal{A}}|(U^2-TU) = R_{k,\mathcal{A}}|(U^2+U) = -R_{k,\mathcal{A}}$$

because $R_{k,\mathcal{A}}\in\mathbf{W}_{2k-2}$. $\square$

Lemma 8. *For $0 \le n \le w$ we have*

$$\mu\big(z^{-n-1}|_{2k}(1-T)(1+U+U^2)\big) = \binom{w}{n} X^n|_{2-2k}(1+T)(U-U^2).$$

Proof. Let $h(z) = z^{-n-1}|_{2k}(1-T)(1+U+U^2)$. By partial fractions, we find

$$\begin{aligned} h(z) &= z^{-n-1} + (-1)^n z^{-\tilde{n}-1} + z^{-\tilde{n}-1}(z-1)^{-n-1} + (-1)^n z^{-n-1}(z-1)^{-\tilde{n}-1} + \cdots \\ &= z^{-n-1} + (-1)^n z^{-\tilde{n}-1} - (-1)^n \sum_{m=0}^{\tilde{n}} \tbinom{\tilde{m}}{n} z^{-m-1} - \sum_{m=0}^{n} \tbinom{\tilde{m}}{\tilde{n}} z^{-m-1} + \cdots, \end{aligned}$$

where $\cdots$ denotes terms regular at $z=0$. A short calculation now shows that the "$A(X)$" of Lemma 6 is $\binom{w}{n}X^n|(1+T-UT-TUT)$, so $\mu(h) = \binom{w}{n}X^n|_{2-2k}(1+T)(U-U^2)$. $\square$

The proof of the following result, though simple, cost us a considerable effort to find.

Lemma 9. *The intersection of $\mathbf{V}|_{2-2k}(1+T)(U-U^2)$ with $\mathbf{W}$ is one-dimensional, spanned by X^w-1.*

Proof. Suppose $A \in \mathbf{V}$ with $B = A|(1+T)(U-U^2) \in \mathbf{W}$ (where $|$ denotes $|_{2-2k}$). Since B is automatically in $\operatorname{Ker}(1+U+U^2)$, this is equivalent to $B|(1+T)=0$. But $B|(1+T) = A|(1+T)(1-U+U^2)(S-1)$ since $(1+T)(U-U^2)(1+T) = (1+T)(1-U+U^2)(S-1)$ in $\mathbb{Z}[\Gamma]$ $(S=UT)$. Since the kernel of $S-1$ in $\mathbf{V}$ is the one-dimensional space spanned by the constant function 1 (a periodic polynomial is constant!), and since $(1-U+U^2) : \mathbf{V} \to \mathbf{V}$ is an isomorphism with inverse $\frac{1}{2}(1+U)$, this says that $A|(1+T)$ belongs to the one-dimensional space spanned by $1|(1+U) = 1+X^w$, and consequently that B belongs to the one-dimensional space spanned by $1|(1+U)(U-U^2) = X^w-1$. $\square$

2.4. We can now give a complete description of RPF's of arbitrary positive weight:

Theorem 1. *For each Γ-equivalence class $\mathcal{A}$ of indefinite binary quadratic forms of non-square discriminant D and each integer $k>0$ define*

$$r^0_{k,\mathcal{A}}(z) = \frac{(-\sqrt{D})^{k-1}}{\binom{2k-2}{k-1}} \sum_{\alpha\in\mathcal{Z}_{\mathcal{A}}} \big(q_{k,\alpha}(z) - q_{k,T\alpha}(z)\big)|_{2k}(1-T) \;-\; \rho_k(\mathcal{A})\big(z^{-1}+z^{-2k+1}\big),$$

where the summation runs over the cycle of simple quadratic irrationalities α corresponding to $\mathcal{A}$, $q_{k,\alpha}(z)$ denotes the principal part at α of $D^{k/2}Q_\alpha(z)^{-k}$, $Q_\alpha(z)$ being the quadratic polynomial of discriminant D with root α, and $\rho_k(\mathcal{A}) = \zeta_{\mathcal{A}}(1-k)/\zeta(1-2k)$ as in §1.2. Also define $r_{k,0}(z) = 1-z^{-2k}$. Then

i) Any rational period function of weight $2k$ has the form

$$q(z) = \sum_{\mathcal{A}} c_{\mathcal{A}}\, r^0_{k,\mathcal{A}}(z) \;+\; c_0\, r_{k,0}(z)$$

for some complex constants $c_{\mathcal{A}}$ (almost all zero) and c_0.

ii) Conversely, a sum as in i) is an RPF if and only if $\sum c_{\mathcal{A}} R^0_{k,\mathcal{A}} = 0$, where

$$R^0_{k,\mathcal{A}}(X) = \sum_{\substack{[A,B,C]\in\mathcal{A}\\ AC<0}} \mathit{sign}(A)\,\big(AX^2-BX+C\big)^{k-1} - \;\rho_k(\mathcal{A})\big(X^{2k-2}-1\big) \in \mathbf{W}^0_{2k-2}$$

is the rational period polynomial defined at the end of §*1.*

Proof. We already know that any RPF q has the form $\sum C_{\mathcal{A}} q_{k,\mathcal{A}} + q_0$ where q_0 is a polynomial of degree $\leq 2k$ in z^{-1} satisfying $q|T = -q$. Equivalently, we can write q as $\sum c_{\mathcal{A}} r^0_{k,\mathcal{A}} + c_0 r_{k,0} - r_0|(1-T)$ where $r_0 \in \bigoplus_{0 \leq n \leq w} \mathbb{C} z^{-n-1}$. This expression is automatically annihilated by $1+T$, while its image under $1+U+U^2$ belongs to $\mathbf{H}_{2k}$, so Lemma 6 implies that q is an RPF if and only if $\sum c_{\mathcal{A}} \mu\bigl(r^0_{k,\mathcal{A}}|_{2k}(1+U+U^2)\bigr)$ equals $\mu\bigl(r_0|_{2k}(1-T)(1+U+U^2)\bigr)$. But $\mu\bigl(r^0_{k,\mathcal{A}}|_{2k}(1+U+U^2)\bigr)$ equals $R^0_{k,\mathcal{A}} \in \mathbf{W}^0_{2k-2}$ by Lemma 7 and the special case $n=0$ of Lemma 8, while $\mu\bigl(r_0|_{2k}(1-T)(1+U+U^2)\bigr) \in \mathbf{V}|_{2-2k}(1+T)(U-U^2)$ by Lemma 8. Lemma 9 now implies that q is an RPF iff $\sum c_{\mathcal{A}} R^0_{k,\mathcal{A}} = 0$ and $\mu\bigl(r_0|_{2k}(1-T)(1+U+U^2)\bigr) = 0$. But the latter equation implies that already $r_0|_{2k}(1-T) = 0$. [Proof: by Lemma 8, this statement is equivalent to the assertion that if $R|_{2-2k}(1+T)(U-U^2) = 0$ for some $R \in \mathbf{V}$, then already $R|_{2-2k}(1+T) = 0$, and this is true because $R|_{2-2k}(1+T)$ is invariant under both T and U and hence under the whole group $PSL(2,\mathbb{Z})$.] Hence q has the form given in the theorem. $\square$

An equivalent formulation of Theorem 1 is that there is an exact sequence

$$0 \longrightarrow \mathbf{RPF}_{2k} \longrightarrow \bigoplus_{\mathcal{A}} \mathbb{C} \oplus \mathbb{C} \longrightarrow \mathbf{W}^0_{2k-2} \longrightarrow 0\,,$$

where the maps are defined by sending $\sum c_{\mathcal{A}}\, r^0_{k,\mathcal{A}} + c_0 r_{k,0}$ to $\bigl(\{c_{\mathcal{A}}\}_{\mathcal{A}},\, c_0\bigr)$ and $\bigl(\{c_{\mathcal{A}}\}_{\mathcal{A}},\, c_0\bigr)$ to $\sum c_{\mathcal{A}}\, R^0_{k,\mathcal{A}}$, the latter map being surjective because the functions $R^0_{k,\mathcal{A}}$ span W^0_{2k-2} [9]. We also have the following

Corollary. *For every class of forms $\mathcal{A}$ and integer $k > 0$ the function $q_{k,\mathcal{A}} - (-1)^k q_{k,\Theta\mathcal{A}^{-1}}$ belongs to $\mathbf{RPF}_{2k}$.*

Proof. This follows immediately from the theorem since $R_{k,\mathcal{A}} = (-1)^k R_{k,\Theta\mathcal{A}^{-1}}$ and $\rho_k(\mathcal{A}) = (-1)^k \rho_k(\Theta\mathcal{A}^{-1})$. However, a direct proof is easier. The function $D^{k/2} Q_\alpha(z)^{-k}$ has poles only at α and α' and is small at infinity, so it is the sum of its principal parts $q_{k,\alpha}(z)$ and $(-1)^k q_{k,\alpha'}(z)$. On the other hand,

$$q_{k,\mathcal{A}}(z) = \sum_{\substack{\alpha \leftrightarrow Q \in \mathcal{A} \\ \alpha\alpha' < 0}} \operatorname{sign}(\alpha)\, q_{k,\alpha}(z)|_{2k}(1-T),$$

and under the correspondence we are using, if $\alpha \leftrightarrow Q \in \mathcal{A}$, then $\alpha' \leftrightarrow -Q \in \Theta\mathcal{A}^{-1}$. Hence

$$q_{k,\mathcal{A}}(z) - (-1)^k q_{k,\Theta\mathcal{A}^{-1}}(z) = \sum_{\substack{[A,B,C] \in \mathcal{A} \\ AC < 0}} \operatorname{sign}(A)\, (Az^2 - Bz + C)^{-k},$$

and this is automatically a true RPF because it is a sum of terms Q^{-k} and $Q^{-k}|_{2-2k}\gamma = (Q \circ \gamma)^{-k}$ for any $\gamma \in \Gamma$: the problems of §2.3 were caused entirely by the fact that this identity breaks down when Q^{-k} is separated into its two principal parts. $\square$

Finally, we should comment on the overlap between the presentation given in this section and previous work. As already mentioned in the introduction, the notion of rational period function was introduced by M. Knopp in [6]. Lemma 1 and part of Lemma 2 (namely, that any non-zero pole of an RPF is a real quadratic irrationality) were proved by him in the later paper [7]. The existence of RPF's with poles in an arbitrary real quadratic field was shown in [2]. Lemma 4 is due to J. Hawkins [4], who also found the cycle structure of the poles of an RPF and observed that there is a relation between the obstructions to the RPF property for sums over cycles and the existence of cusp forms of weight $2k$. The construction of special RPF's given in the above corollary was obtained in [3]. We have preferred to give our own proofs of these results because they are shorter and in order to keep our treatment self-contained. The general result of Ash mentioned in the introduction is the analogue of the exact sequence given above for $\Gamma' \subset \Gamma$: the space $\mathbf{W}^0_{2k-2}$ replaced by $H^1_0(\Gamma', \mathbf{V}_{2k-2})$, the space $\mathbf{RPF}_{2k}$ replaced by a similarly defined first cohomology group of Γ' with coefficients in the Γ'-module of rational functions, and the summation ranging over Γ'-equivalence classes of indefinite quadratic forms of non-square discriminant. However, this more general result, whose proof relies on cohomological methods, does not give an explicit description of the generalized rational period functions.

3. Hecke operators

A definition of Hecke operators on rational period functions was given by Knopp [6], based on the modular integrals (Eichler integrals) which he had introduced earlier [5]. In §3.1. we recall the definition of modular integrals and use it to motivate a purely algebraic definition whose main properties we then investigate.

It will be convenient to introduce the following notation. For each integer $n > 0$ we set $\mathcal{M}_n = \{M \in M_2(\mathbb{Z})/\{\pm 1\}\} \mid \det(M) = n\}$, so that $\mathcal{M}_1 = \Gamma$. We write $\mathcal{M}_+$ for $\bigcup \mathcal{M}_n = \{M \in M_2(\mathbb{Z})/\{\pm 1\}\} \mid \det(M) > 0\}$ and $\mathcal{R}_n = \mathbb{Z}[\mathcal{M}_n]$, $\mathcal{R}_+ = \mathbb{Z}[\mathcal{M}_+] = \bigoplus_n \mathcal{M}_n$ for the sets of finite integral linear combinations of elements of $\mathcal{M}_n$ and $\mathcal{M}_+$, respectively. Thus $\mathcal{R}_+$ is a (non-commutative) ring with unit and is "multiplicatively graded" in the sense that $\mathcal{R}_n\mathcal{R}_m \subseteq \mathcal{R}_{mn}$ for all m, $n > 0$; in particular, each $\mathcal{R}_n$ is a left and right module over the group ring $\mathcal{R}_1 = \mathbb{Z}[\Gamma]$ of Γ. We denote by $\mathcal{J}$ the right ideal $(1+T)\mathcal{R}_1 + (1+U+U^2)\mathcal{R}_1$ of $\mathcal{R}_1$. Finally, the slash operator $|$ will always mean $|_{2k}$ unless otherwise specified, the action being extended from Γ to $\mathcal{M}_+$ by $(h|M)(z) = n^{2k}(cz+d)^{-2k}h\big(\frac{az+b}{cz+d}\big)$ for $M = \big(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\big) \in \mathcal{M}_n$. Thus $\mathcal{R}_n$ and $\mathcal{R}_+$ act on the spaces of rational or meromorphic functions in the upper half plane, and $\mathbf{RPF} = \mathbf{RPF}_{2k}$ is simply the space of rational functions annihilated by the ideal $\mathcal{J}$.

3.1. A *modular integral* of weight $2k$ is by definition a meromorphic function F on the upper half plane $\mathcal{H}$, small at infinity, satisfying

$$F|S = F \qquad \text{and} \qquad F|T = F + q$$

for some rational function q. In other words, F is periodic of period 1 and the difference $z^{-2k}F(-1/z) - F(z)$ is a rational function of z; the condition "small at infinity" means that $F(z)$ has a Fourier expansion $\sum_{n>0} a(n)\, e^{2\pi i n z}$ for $\Im(z) \gg 0$. The definition implies immediately that q satisfies $q|(1+T) = q|(1+U+U^2) = 0$, i.e., that q is an RPF. Conversely Knopp proved that every $q \in \mathbf{RPF}$ comes in this way from a modular integral F. This F is obviously not unique: we can add to F any meromorphic cusp form of weight $2k$ without changing q, and even if we specify the principal parts of F at every point of $\mathcal{H}$, then F is still only well-defined up to the action of a cusp form $f \in \mathbf{S}$. The terminology "modular integral," by the way, comes from the analogous definition for the action of Γ by $|_{2-2k}$, where

q is now a polynomial; here F can be obtained as an integral of a cusp form with period polynomial q.

On $\mathbf{S}$ we have Hecke operators defined by $f|T_n = \sum_{M\in\Gamma\backslash\mathcal{M}_n} f|M$. This definition cannot be applied to F because F is not Γ-invariant (with respect to the action $|$), but F is invariant under the action of the stabilizer $\Gamma_\infty = \langle S\rangle$ of infinity, so the operator

$$T_n^\infty \;=\; \sum_{\substack{M\in\Gamma_\infty\backslash\mathcal{M}_n\\ M(\infty)=\infty}} M = \sum_{\substack{ad=n,\,a,d>0\\ b \pmod d}} \begin{pmatrix} a & b\\ 0 & d\end{pmatrix} \;\in\; \big((S-1)\mathcal{R}_n\big)\backslash\mathcal{R}_n$$

has a well-defined effect on F. It is easy to see that $F|T_n^\infty$ is again periodic with period 1, and in fact it is a new modular integral whose associated rational function depends only on q (and n), and this defines Knopp's Hecke operator. To phrase this more algebraically, suppose that there are elements X_n, $\tilde{T}_n$ and Y_n in $\mathcal{M}_n$ satisfying

$$T_n^\infty(S-1) = (S-1)\,X_n, \qquad T_n^\infty(T-1) = (T-1)\,\tilde{T}_n + (S-1)\,Y_n;$$

then

$$\begin{aligned}\big(F|T_n^\infty\big) \mid (S-1) &= \big(F|(S-1)\big) \mid X_n = 0,\\ \big(F|T_n^\infty\big) \mid (T-1) &= \big(F|(T-1)\big) \mid \tilde{T}_n + \big(F|(S-1)\big) \mid Y_n = q \mid \tilde{T}_n,\end{aligned}$$

which shows that $F|T_n^\infty$ is a modular integral with associated rational function $q|\tilde{T}_n$. We now show that X_n, $\tilde{T}_n$ and Y_n exist and that the image of q under $\tilde{T}_n$ is independent of the choice of these three operators and is again an RPF, and moreover that the various $\tilde{T}_n : \mathbf{RPF} \to \mathbf{RPF}$ commute. This gives a purely algebraic definition of the Hecke operators, independent of the existence of modular integrals for arbitrary RPF's, which is a non-trivial analytic fact.

Theorem 2. *For each integer $n \ge 1$,*

$$T_n^\infty(S-1) \equiv 0\,, \qquad T_n^\infty(T-1) \equiv (T-1)\tilde{T}_n \pmod{(S-1)\mathcal{R}_n}$$

for a certain element $\tilde{T}_n \in \mathcal{R}_n$. This element is unique modulo $\mathcal{J}\mathcal{R}_n$ and satisfies $\tilde{T}_n\mathcal{J} \subseteq \mathcal{J}\mathcal{R}_n$. If m is a second positive integer, then the elements $\tilde{T}_m$, $\tilde{T}_n \in \mathcal{R}_+$ satisfy the product formula

$$\tilde{T}_m\tilde{T}_n = \sum_{d|(m,n)} d^{-1}\begin{pmatrix} d & 0\\ 0 & d\end{pmatrix}\tilde{T}_{mn/d^2} \pmod{\mathcal{J}\mathcal{R}_{mn}}.$$

Corollary. *If $q \in \mathbf{RPF}$, then $q|\tilde{T}_n$ is well-defined (independent of the choice of $\tilde{T}_n$) and again belongs to $\mathbf{RPF}$. The operators $\tilde{T}_n : \mathbf{RPF} \to \mathbf{RPF}$ satisfy $\tilde{T}_m\tilde{T}_n = \tilde{T}_n\tilde{T}_m = \sum_{d|(m,n)} d^{2k-1}\tilde{T}_{mn/d^2}$.*

Proof. For the assertion $T_n^\infty(S-1) \in (S-1)\mathcal{R}_n$ we compute (for given a, $d > 0$ with $ad = n$)

$$\begin{aligned}\sum_{0\le b<d}\begin{pmatrix} a & b\\ 0 & d\end{pmatrix}(S-1) &= \sum_{0\le b<d}\left(\begin{pmatrix} a & a+b\\ 0 & d\end{pmatrix} - \begin{pmatrix} a & b\\ 0 & d\end{pmatrix}\right)\\ &= \sum_{0\le b<a+d}\begin{pmatrix} a & b\\ 0 & d\end{pmatrix} - \sum_{0\le b<a}\begin{pmatrix} a & b\\ 0 & d\end{pmatrix} - \sum_{0\le b<d}\begin{pmatrix} a & b\\ 0 & d\end{pmatrix}\\ &= \sum_{0\le b<a}\left(\begin{pmatrix} a & b+d\\ 0 & d\end{pmatrix} - \begin{pmatrix} a & b\\ 0 & d\end{pmatrix}\right) = \sum_{0\le b<d}(S-1)\begin{pmatrix} a & b\\ 0 & d\end{pmatrix}.\end{aligned}$$

For the assertion $T_n^\infty(T-1) \equiv (T-1)\tilde{T}_n \pmod{(S-1)\mathcal{R}_n}$ we need the following lemma.

Lemma 1. *$\gamma - 1 \in (T-1)\mathcal{R}_1 + (S-1)\mathcal{R}_1$ for any element $\gamma \in \Gamma$.*

Proof. Suppose that this property holds for some $\gamma \in \Gamma$. Then the elements $T\gamma - 1 = (T-1)\gamma + (\gamma - 1)$, $S\gamma - 1 = (S-1)\gamma + (\gamma-1)$ and $S^{-1}\gamma - 1 = (S-1)(-S^{-1}\gamma) + (\gamma - 1)$ also belong to $(T-1)\mathcal{R}_1 + (S-1)\mathcal{R}_1$. Since T $(=T^{-1})$ and S generate Γ, the lemma now follows by induction on the word length. □

Now write T_n^∞ as $\sum M_i$. For each index i there is a unique index i' such that $M_iT = \gamma_i M_{i'}$ with $\gamma_i \in \Gamma$. Then $T_n^\infty(T-1) = \sum_i \bigl(\gamma_i M_{i'} - M_i\bigr) = \sum_i (\gamma_i - 1)M_{i'}$, and this belongs to $(T-1)\mathcal{R}_n + (S-1)\mathcal{R}_n$ by Lemma 1. □

To prove the other assertions of the theorem, we need a characterization of the elements of $\mathcal{J}\mathcal{M}_n$. To get this, we introduce the following concept. Suppose V is an abelian group on which Γ acts on the left. Then V is a left $\mathcal{R}_1$-module. For $X \in \mathcal{R}_1$ we write $\operatorname{Ker}(X)$ for $\{v \in V \mid Xv = 0\}$ and $\operatorname{Im}(X)$ for $XV = \{Xv \mid v \in V\}$. The intersection of $\operatorname{Ker}(1+T)$ and $\operatorname{Ker}(1+U+U^2)$ has an interpretation in terms of the cohomology of Γ with coefficients in the module V; we call V *acyclic* if this intersection is zero and if, furthermore $\operatorname{Ker}(1-T) = \operatorname{Im}(1+T)$ and $\operatorname{Ker}(1-U) = \operatorname{Im}(1+U+U^2)$. (The second hypothesis can be omitted if we work over $\mathbb{Q}$, since $(1-T)v = 0 \Rightarrow v = (1+T)(\frac{1}{2}v)$ and $(1-U)v = 0 \Rightarrow v = (1+U+U^2)(\frac{1}{3}v)$.)

Lemma 2. *$\mathcal{R}_n$ is an acyclic $\mathcal{R}_1$-module for all n.*

Proof. Suppose $X = \sum n_M\, M$ $(n_M \in \mathbb{Z},\ M \in \mathcal{M}_n)$ is an element of $\mathcal{R}_n$. If $X \in \operatorname{Ker}(1+T) \cap \operatorname{Ker}(1+U+U^2)$, then the function $q = r|X$ is an RPF for any rational function $r(z)$. Choose $r(z) = (z-\lambda)^{-1}$ where $\lambda \in \mathbb{C}$ is not rational or quadratic; then $q(z)$ has a pole at each point $M^{-1}\lambda$ with $n_M \neq 0$ (these cannot cancel since $M_1^{-1}\lambda = M_2^{-1}\lambda$ would lead to a quadratic equation for λ), and this contradicts the fact proved in §2 that an RPF can have poles only at rational or quadratic irrational points. On the other hand, if X is left invariant under T, then $n_{TM} = n_M$ for all M, and since $M \neq TM$ this means that X can be written as an integral linear combination of elements $M + TM = (1+T)M \in \mathcal{R}_n$. Similarly, $X = UX$ implies $n_M = n_{UM} = n_{U^2M}$ for all M and hence $X \in (1+U+U^2)\mathcal{R}_n$. This proves the second hypothesis in the definition of acyclicity. □

Lemma 3. *If V is an acyclic Γ-module and $v \in V$, then*

$$(1-T)v \in (1-S)V \quad\Longleftrightarrow\quad v \in (1+T)V + (1+U+U^2)V = \mathcal{J}V.$$

Proof. The direction "$\Leftarrow$" is true for any Γ-module, since $v = (1+T)x + (1+U+U^2)y$ implies

$$(1-T)v = (1-S^{-1}U)(1+U+U^2)y = (S-1)S^{-1}(1+U+U^2)y.$$

Conversely, assume that V is acyclic and $(1-T)v = (1-S)w$ for some $w \in V$. Then

$$(1-T)(v-w) = (T-S)w = (1-U)Tw.$$

This element must vanish since $\operatorname{Im}(1-T) \cap \operatorname{Im}(1-U) \subseteq \operatorname{Ker}(1+T) \cap \operatorname{Ker}(1+U+U^2) = \{0\}$. But then $v - w \in \operatorname{Im}(1+T)$ and $Tw \in \operatorname{Im}(1+U+U^2)$ by the second hypothesis in the definition of acyclicity, so $v = (v-w) + (1+T)w - Tw \in (1+T)V + (1+U+U^2)V$. □

Lemmas 2 and 3 give a characterization of $\mathcal{JR}_n$ as $\{X \in \mathcal{R}_n \mid (1-T)X \in (1-S)\mathcal{R}_n\}$. The uniqueness of $\tilde{T}_n$ modulo $\mathcal{JR}_n$ (and consequently of the image $q|\tilde{T}_n$ for an RPF q) follows immediately from this characterization and the definition of $\tilde{T}_n$. The fact that $\tilde{T}_n\mathcal{J} \subseteq \mathcal{JR}_n$ (and consequently that $q|\tilde{T}_n$ is again an RPF) also follows easily, since the equations

$$T_n^\infty(S-1) \equiv 0\,, \qquad T_n^\infty(T-1) \equiv (T-1)\tilde{T}_n \pmod{(S-1)\mathcal{R}_n}$$

imply $(1-T)\tilde{T}_n(1+T) \equiv 0$ and

$$(1-T)\tilde{T}_n(1+U+U^2) \equiv T_n^\infty(1-S^{-1}U)(1+U+U^2) \equiv T_n^\infty(S-1)S^{-1}(1+U+U^2) \equiv 0$$

modulo $(S-1)\mathcal{R}_n$. Finally,

$$\begin{aligned}(T-1)\Bigl(\tilde{T}_m\tilde{T}_n - \sum_{d|(m,n)} d^{-1}\left(\begin{smallmatrix} d & 0 \\ 0 & d\end{smallmatrix}\right)\tilde{T}_{mn/d^2}\Bigr)&\\ \equiv T_m^\infty(T-1)\tilde{T}_n - \sum d^{-1}\left(\begin{smallmatrix} d & 0 \\ 0 & d\end{smallmatrix}\right)(T-1)\tilde{T}_{mn/d^2}&\\ \equiv T_m^\infty\bigl[T_n^\infty(T-1)-(S-1)Y_n\bigr] - \sum d^{-1}\left(\begin{smallmatrix} d & 0 \\ 0 & d\end{smallmatrix}\right)T_{mn/d^2}^\infty(T-1)&\\ \equiv \Bigl(T_m^\infty T_n^\infty - \sum d^{-1}\left(\begin{smallmatrix} d & 0 \\ 0 & d\end{smallmatrix}\right)T_{mn/d^2}^\infty\Bigr)(T-1) \qquad &\pmod{(S-1)\mathcal{R}_{mn}},\end{aligned}$$

and

$$T_m^\infty T_n^\infty - \sum_{d|(m,n)} d^{-1}\left(\begin{smallmatrix} d & 0 \\ 0 & d\end{smallmatrix}\right)T_{mn/d^2}^\infty \equiv 0 \pmod{(S-1)\mathcal{R}_{mn}}$$

by the usual calculation for the commutation properties of Hecke operators. This completes the proof of Theorem 2. □

3.2. In this section we give an explicit combination $\tilde{T}_n$ of matrices of determinant n which has the property used in §3.1. to define the Hecke operators on rational period functions. This operator was given in [12], where it was shown that it preserves the space $\mathbf{W}_{2k-2}$ of period polynomials of degree $2k-2$ and corresponds under the isomorphism $\mathbf{W}_{2k-2} \cong \mathbf{S}_{2k} \oplus \mathbf{S}_{2k} \oplus \mathbb{C}$ to the usual action of Hecke operators on cusp forms, generalizing the description given by Manin [11] of the action of Hecke operators on the constant terms of the period polynomials of cusp forms. However, the proof in [12] relied on an explicit knowledge of a generating function for the period polynomials of Hecke eigenforms of weight $2k$, while the construction here is purely algebraic (and thus shows that the $\tilde{T}_n$ we construct preserves the period subspace $\mathrm{Ker}\,(1+T) \cap \mathrm{Im}\,(1+U+U^2)$ of any Γ-module).

Theorem 3. *Let n be a positive integer. Then the element $\tilde{T}_n \in R_n$ defined by*

$$\tilde{T}_n = \sum_{\substack{ad-bc=n \\ a>c>0 \\ d>-b>0}} \left[\begin{pmatrix} a & b \\ c & d\end{pmatrix} + \begin{pmatrix} a & -b \\ -c & d\end{pmatrix}\right] + \sum_{\substack{ad=n \\ -\frac12 d<b\le\frac12 d}} \begin{pmatrix} a & b \\ 0 & d\end{pmatrix} + \sum_{\substack{ad=n \\ -\frac12 a<c\le\frac12 a \\ c\neq 0}} \begin{pmatrix} a & 0 \\ c & d\end{pmatrix}\,,$$

has the properties given in Theorem 2.

Proof. Define $\tilde{T}_n$ by the formula given. We must show that $(T-1)\tilde{T}_n \equiv T_n^\infty(T-1)$, where (and throughout the proof) $\equiv$ denotes congruence modulo $(S-1)\mathcal{R}_n$.)

The maps

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto TS^{-[a/c]}A = \begin{pmatrix} c & d \\ -(a-c[a/c]) & -b+d[a/c] \end{pmatrix},$$
$$B = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \quad S^{[d/b]}TB = \begin{pmatrix} -c+a[d/b] & -(d-b[d/b]) \\ a & b \end{pmatrix},$$

where [ ] denotes integral part, are easily checked to give inverse bijections between the sets

$$\mathcal{A}_n = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{M}_n \;\middle|\; a > c > 0,\ d > -b \geq 0,\ b = 0 \Rightarrow a \geq 2c \right\}$$

and

$$\mathcal{B}_n = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{M}_n \;\middle|\; a > -c \geq 0,\ d > b > 0,\ c = 0 \Rightarrow d \geq 2b \right\}.$$

Since $S^r A \equiv A$ for any integer r, this gives $\sum_{A\in\mathcal{A}_n} A \equiv T \sum_{B\in\mathcal{B}_n} B$ or

$$\sum_{\substack{a>c>0 \\ d>-b>0 \\ ad-bc=n}} \left[\begin{pmatrix} a & b \\ c & d \end{pmatrix} - T \begin{pmatrix} a & -b \\ -c & d \end{pmatrix} \right] \equiv \sum_{\substack{\frac{1}{2}d\geq b>0 \\ ad=n}} \begin{pmatrix} 0 & -d \\ a & b \end{pmatrix} - \sum_{\substack{\frac{1}{2}a\geq c>0 \\ ad=n}} \begin{pmatrix} a & 0 \\ c & d \end{pmatrix}.$$

Conjugating this equation by $\left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ changes the sign of all the off-diagonal coefficients and preserves the property "$\equiv$". Adding the result to the original equation, we get

$$(1-T) \sum_{\substack{a>c>0 \\ d>-b>0 \\ ad-bc=n}} \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} a & -b \\ -c & d \end{pmatrix} \right) \equiv \sum_{\substack{0<|b|\leq\frac{1}{2}d \\ ad=n}} \left(\begin{pmatrix} 0 & -d \\ a & b \end{pmatrix} - \begin{pmatrix} d & 0 \\ b & a \end{pmatrix} \right).$$

(Recall that we are working with matrices modulo ±1.) Hence

$$(1-T)\,\tilde{T}_n \equiv \sum_{\substack{-\frac{1}{2}d<b\leq\frac{1}{2}d \\ ad=n}} \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} (1-T) + \sum_{\substack{a,d>0 \\ ad=n \\ d \text{ even}}} \left(\begin{pmatrix} 0 & -d \\ a & -\frac{1}{2}d \end{pmatrix} - \begin{pmatrix} d & 0 \\ -\frac{1}{2}d & a \end{pmatrix} \right),$$

as we see after a short calculation. The first sum on the right is $\equiv T_n^\infty(1-T)$, while the second equals

$$\sum_{\substack{xy=n/2 \\ x,y>0}} \left(\begin{pmatrix} 0 & 2y \\ -x & y \end{pmatrix} - \begin{pmatrix} 2x & 0 \\ -x & y \end{pmatrix} \right) = \sum_{\substack{xy=n/2 \\ x,y>0}} (S^2-1) \begin{pmatrix} 2x & 0 \\ -x & y \end{pmatrix} \equiv 0. \qquad \square$$

3.3 Finally, we mention a problem which we have not been able to solve and which seems to be of interest, namely that of writing down explicit modular integrals. Specifically, we would like to find functions $F_{k,\mathcal{A}}$, holomorphic in the upper half-plane and small at infinity, such that $\sum_{\mathcal{A}} c_{\mathcal{A}} F_{k,\mathcal{A}}(z)$ is a modular integral for $q(z) = \sum_{\mathcal{A}} c_{\mathcal{A}} r^0_{k,\mathcal{A}}(z)$ whenever the latter is an RPF, i.e. (according to Theorem 1), whenever $\sum_{\mathcal{A}} c_{\mathcal{A}} R^0_{k,\mathcal{A}}(z) = 0$ in $\mathbf{W}^0_{2k-2}$. For the special case of the RPF $q_{k,\mathcal{A}} - (-1)^k q_{k,\Theta\mathcal{A}^{-1}}$ of the Corollary to Theorem 1, the answer was found by A. Parson: an explicit modular integral is given by (a multiple of) $\sum Q(z,-1)^{-k}$, where the sum ranges over quadratic forms $Q = [a,b,c] \in \mathcal{A}$ with $a > 0$. This can be checked easily: the sum is S-invariant and when one applies $(1-T)$ all the terms with $ac > 0$ drop out, so that one is left with a sum over simple forms. A similar calculation shows that the sum of the principal parts $q_{k,\alpha}$, where α ranges over the roots of forms in $\mathcal{A}$ with $\alpha > \alpha'$ (corresponding to Parson's condition $a > 0$) formally is a modular integral for $q_{k,\mathcal{A}}$. However, this sum diverges because $q_{k,\alpha}$ grows like a^{k-1} as $[a,b,c]$ runs over the class $\mathcal{A}$ (cf. the formula for $q_{k,\alpha}$ given in the proof of Lemma 7), and it is not clear how to write down an expression which makes sense.

4. Numerical examples

For the discriminant $D = 5$ there is only one class of forms I, represented by the simple form $[1,1,-1]$, with $\mathcal{Z}_I = \{\frac{\sqrt{5}+1}{2}, \frac{\sqrt{5}-1}{2}\}$. For k odd, both $q_{k,I}(z)$ and $r^0_{k,I}(z)$ are multiples of

$$\sum_{\alpha\in\mathcal{Z}_I} \big(q_{k,\alpha}(z) - q_{k,T\alpha}(z)\big) = \left(\frac{\sqrt{5}}{z^2-z-1}\right)^k + \left(\frac{\sqrt{5}}{z^2+z-1}\right)^k$$

and belong to $\mathbf{RPF}_{2k}$. (Note that $R_{k,I} = (1+(-1)^k)P_{k,I} = 0$ in this case.) For $k = 2$ we have

$$\sum_{\alpha\in\mathcal{Z}_I} \big(q_{2,\alpha}(z) - q_{2,T\alpha}(z)\big) = -\frac{1}{\sqrt{5}}\left(\frac{4z^3-6z^2-12z+7}{(z^2-z-1)^2} + \frac{4z^3+6z^2-12z-7}{(z^2+z-1)^2}\right).$$

Applying $|_4(1-T)$ gives

$$q_{2,I}(z) = -\frac{2}{\sqrt{5}}\left(\frac{4z^3-6z^2-12z+7}{(z^2-z-1)^2} + \frac{4z^3+6z^2-12z-7}{(z^2+z-1)^2} - \frac{4}{x} + \frac{4}{x^3}\right);$$

and since $\rho_2(I) = 4$ we find

$$r^0_{2,I}(z) = \frac{4z^3-6z^2-12z+7}{(z^2-z-1)^2} + \frac{4z^3+6z^2-12z-7}{(z^2+z-1)^2} - \frac{8}{x} \in \mathbf{RPF}_4\,,$$

corresponding to the fact that the polynomial

$$R_{2,I}(X) = 2P_{2,I}(X) - \rho_2(I)\,p_{2,0}(X) = 2\big[(X^2-X-1)+(X^2+X-1)\big] - 4(X^2-1)$$

vanishes (as of course it must, since $W^0_2 = \{0\}$). Similarly, for $k = 4$ we find that

$$r^0_{4,I}(z) = \frac{4z^7-14z^6+35z^4-42z^2-28z+\frac{45}{2}}{(z^2-z-1)^4} + \frac{4z^7+14z^6-35z^4+42z^2-28z-\frac{45}{2}}{(z^2+z-1)^4} - \frac{8}{x}$$

is again an RPF, corresponding to the identity

$$R_{4,I} = 2\big[(X^2-X-1)^3+(X^2+X-1)^3\big] - 4(X^6-1) = 0 \ \in W^0_6 = \{0\}\,.$$

For $k=6$, however, $r^0_{k,I}$ no longer has to be, and no longer is, an RPF, because of the existence of cusp forms of weight $2k$ on $PSL(2,\mathbb{Z})$. Here we find

$$r^0_{6,I}(z) = \frac{44h(z)}{(z^2-z-1)^6} - \frac{44h(-z)}{(z^2+z-1)^6} - 4\left(\frac{1}{x}+\frac{1}{9x^3}-\frac{1}{7x^5}+\frac{1}{7x^7}-\frac{1}{9x^9}-\frac{1}{x^{11}}\right) - \rho\left(\frac{1}{x}+\frac{1}{x^{11}}\right),$$

where $\rho=\rho_5(I)=\dfrac{3484}{691}$ and $h(z)$ denotes the polynomial

$$h(z) = \frac{z^{11}}{11} - \frac{z^{10}}{2} + \frac{5z^9}{2} + \frac{5z^8}{4} - \frac{15z^7}{7} - \frac{11z^6}{6} + 3z^5 + \frac{5z^4}{2} - \frac{5z^3}{3} - \frac{5z^2}{2} - z + \frac{6227}{5544}.$$

We have $r^0_{6,I}|_{12}(1+T)=0$ but

$$r^0_{6,I}|_{12}(1+U+U^2) = -4k\left(\frac{1}{z}\right) - 4k\left(\frac{1}{1-z}\right) + \rho\sum_{n=1}^{10}\left(\frac{1}{z^n}+\frac{1}{(1-z)^n}\right)$$

with $k(z)=z^{10}+z^9+\frac{4}{3}z^8+\frac{5}{3}z^7+\frac{221}{126}z^6+\frac{67}{42}z^5+\frac{4}{3}z^4+\frac{10}{9}z^3+z^2+z$. This belongs to $\mathbf{H}_{12}$ and is mapped by the isomorphism μ of Lemma 6 to the element

$$R^0_{6,I} = 2\big[(X^2-X-1)^5+(X^2+X-1)^5\big]-\rho\left(X^{10}-1\right) = 20\Big(X^8-3X^6+3X^4-X^2-\frac{36}{691}\,(X^{10}-1)\Big)$$

of the 2-dimensional space W^0_{10}.

Other discriminants work similarly. For $D=12$, for example, there are two classes I and Θ, represented by the simple forms $[1,0,-3]$ and $[3,0,-1]$, respectively, with cycles $\mathcal{Z}_I=\{\sqrt{3},\ \sqrt{3}-1,\ \sqrt{3}+1\}$ and $\mathcal{Z}_\Theta=\{\frac{1}{\sqrt{3}},\ \frac{\sqrt{3}+1}{2},\ \frac{\sqrt{3}-1}{2}\}$. Here for each k the rational function $r^0_{k,I}(z)-(-1)^k\,r^0_{k,\Theta}(z)$ is a multiple of

$$\begin{aligned}&\frac{1}{(z^2-3)^k}+\frac{1}{(z^2-2z-2)^k}+\frac{1}{(z^2+2z-2)^k}\\ -&\frac{1}{(1-3z^2)^k}-\frac{1}{(1-2z-2z^2)^k}-\frac{1}{(1+2z-2z^2)^k}\end{aligned}$$

and is an RPF. For $k=2$, 3, 4, 5 or 7, where $S_{2k}=\{0\}$, the two fuctions $r^0_{k,I}$ and $r^0_{k,\Theta}$ are individually RPF's. For $k=6$ this is no longer true, but subtracting from $r^0_{6,I}$ six times the function $r^0_{6,I}$ for $D=5$ as given above, we do get a function belonging to $\mathbf{RPF}_{12}$, corresponding to the fact that the polynomial

$$\begin{aligned}R^0_{6,I} =&(X^2-3)^5+(X^2+2X-2)^5+(X^2-2X-2)^5+(3X^2-1)^5\\ &+(2X^2+2X-1)^5+(2X^2-2X-1)^5-\frac{218530}{691}\left(X^{10}-1\right).\end{aligned}$$

is 6 times what it was for $D=5$. However, the coefficients of this rational function are already fairly complicated and we do not give them or any further numerical examples.

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