A NOTE ON REYES'S THEOREM ABOUT TRIANGLES

Don Zagier

In the last issue of *Nieuw Archief*, the following elegant theorem was stated by Reyes [1]:

Theorem. *Let $A_1A_2A_3$ be a triangle. We extend the numbering cyclically by $A_{i+3} = A_i$ for $i \in \mathbb{Z}$. Suppose given for each $i \in \mathbb{Z}$ a point X_i on the perpendicular bisector of $A_{i-1}A_i$ such that the three points X_i, A_i, X_{i+1} are collinear for each i. Then the sequence $\{X_i\}$ has period 6.*

(In the notation of [1] the three vertices A_1, A_2, A_3 were denoted by A, B, C and the first four points X_1, X_2, X_3 and X_4 by X, Y, Z and X', and the result was stated in the form that the map $X \mapsto X'$ from the perpendicular bisector of CA to itself is an involution.) The proof in [1], though described as "quite simple," is fairly intricate and uses arguments from both vector algebra and trigonometry. Here is a much shorter geometric proof. Let O be the meeting point of the perpendicular bisectors of the sides (= center of the circumscribed circle) of the triangle $A_1A_2A_3$. The point X_i determines, and is determined by, the angle $\theta_i = \angle OA_{i-1}X_i$. Since X_i is on the bisector of the angle $A_{i-1}OA_i$, we have $\angle OA_iX_i = \theta_i$, and since X_i, A_i and X_{i+1} are collinear we have $\angle OA_iX_{i+1} = \pi - \angle OA_iX_i$ or $\theta_{i+1} = \pi - \theta_i$. Thus the angles θ_i alternate between two values θ_1 and $\pi - \theta_1$, so $\theta_{i+6} = \theta_i$ and $X_{i+6} = X_i$.

[1] W. Reyes, 1996, On a theorem in circle geometry, *Nieuw-Arch.-Wisk.* **14**, no. **2**, pp. 231–233.