NEW POINTS OF VIEW ON THE SELBERG ZETA FUNCTION

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Abstract. The classical Selberg zeta function for the modular group $\Gamma_1 = \mathrm{SL}_2(\mathbb{Z})$ relates the set of traces of hyperbolic elements of Γ_1 and the set of eigenvalues of Maass wave forms for Γ_1. In recent years new connections have emerged relating these two objects to two new objects: the spectrum (and in particular the eigenvalues ± 1) of a certain "transfer operator" $\mathfrak{L}_s$ defined by D. Mayer and the analytic solutions of the "three-term functional equation" $\psi(x) = \psi(x+1) \pm x^{-2s}\psi(x^{-1}+1)$ introduced by J. Lewis. We will give a short survey of these connections and also briefly discuss numerical aspects and extensions to other groups.

§1. The Selberg trace formula and the Selberg zeta function

If Γ is a discrete subgroup of cofinite volume of $G = \mathrm{SL}_2(\mathbb{R})$, acting in the usual way on the upper half-plane $\mathcal{H}$, then we have two interesting associated sequences of positive real numbers. The first is the "length spectrum" $0 < l_1 \le l_2 \le \cdots$, which can be defined for Γ torsion-free and cocompact as the set of lengths of all closed geodesics in $\Gamma\backslash\mathcal{H}$ (counting multiplicities), and in the general case as the set of numbers $\log N(\gamma)$ for γ ranging over all conjugacy classes of hyperbolic elements of Γ, where $N(\gamma)$ is defined as ε^2 for γ conjugate to $\pm\left(\begin{smallmatrix}\varepsilon & 0\\ 0 & 1/\varepsilon\end{smallmatrix}\right)$ with $\varepsilon > 1$. The second is the set $0 = \lambda_0 < \lambda_1 \le \lambda_2 \le \cdots$ of eigenvalues of the hyperbolic Laplace operator $\Delta = -y^2(\partial^2/\partial x^2 + \partial^2/\partial y^2)$ on $\Gamma\backslash\mathcal{H}$. (Here x and y denote the real and imaginary part of the coordinate $z \in \mathcal{H}$ as usual.) One has information about the asymptotics of the set $\{\lambda_j\}$, but the eigenvalues themselves are in general quite mysterious. For the full modular group $\Gamma_1 = \mathrm{SL}_2(\mathbb{Z})$, for instance, many eigenvalues have been computed numerically (the first and third have the values $\lambda_1 = 91.14134\cdots$ and $\lambda_3 = 190.13154\cdots$), but not a single eigenvalue is known "in closed form." The length spectrum, on the other hand, is fairly easily computable and in the case of $\Gamma = \Gamma_1$ consists essentially of the logarithms of units of all real quadratic fields, with multiplicities equal to the class numbers of these fields.

The Selberg trace formula relates these two invariants of the action of Γ. Roughly speaking, it has the form

$$\sum_{j=0}^{\infty} H(\lambda_j) \;=\; \sum_{j=1}^{\infty} \widetilde{H}(l_j)\,, \qquad (1)$$

where H is a sufficiently nice "test function" (analytic and of suitable decay in a suitable neighbourhood of $\mathbb{R}_+$, but otherwise arbitrary) and $\widetilde{H}$ an explicitly given integral transform of H. (Actually, the formula is a little more complicated: the term $\widetilde{H}(\lambda_j)$ corresponding to the conjugacy class of a hyperbolic element which is the nth power of a primitive element must be divided by n, and one has to add to the right-hand side further contributions coming from the parabolic elements and the elements of finite order of the group Γ.)

By choosing for H a suitable function depending on a complex parameter s and exponentiating both sides of (1), one obtains the *Selberg zeta function*, $Z_\Gamma(s)$. The expression for $Z_\Gamma(s)$ as a product over hyperbolic elements takes the form

$$Z_\Gamma(s) \;=\; \prod_\gamma \prod_{m=0}^{\infty} \bigl(1 \;-\; N(\gamma)^{-s-m}\bigr) \qquad (\Re(s) \gg 0), \tag{2}$$

where γ in the first product ranges over all primitive hyperbolic elements of Γ (or equivalently, all primitive geodesics of $\Gamma\backslash\mathcal{H}$), while the expression for $Z_\Gamma(s)$ as a product over eigenvalues of the Laplace operator shows that $Z_\Gamma(s)$ has a meromorphic continuation in s and has zeros (with the appropriate multiplicities) at all the *spectral parameters* of Γ, defined as the numbers s_j with $s_j(1-s_j)=\lambda_j$. (Notice that there are two s_j for each eigenvalue λ_j, the values corresponding to the two eigenvalues given above for $\Gamma=\Gamma_1$ being $\frac12 \pm 9.53369526\cdots i$ and $\frac12 \pm 13.779751\cdots i$.)

In this note we will concentrate mostly on the case of the full modular group $\Gamma_1 = \mathrm{SL}_2(\mathbb{Z})$ and its order 2 extension $\Gamma_+ = \mathrm{GL}_2(\mathbb{Z})$, which can be made to operate on $\mathcal{H}$ by letting $\gamma = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)$ act by $z \mapsto (a\bar z + b)/(c\bar z + d)$ if $\det(\gamma) = -1$. The eigenvalues of the Laplace operator for Γ_+ are just the *even* eigenvalues for Γ_1, i.e., those corresponding to eigenfunctions which are symmetric under $z \mapsto -\bar z$ (of the two eigenvalues for Γ_1 given above, the first corresponds to an odd eigenfunction and the second to an even one), and the corresponding spectral parameters are zeros of the Selberg zeta function $Z_{\Gamma_+}(s)$, which is defined by a formula like (2) but with γ now ranging over the primitive hyperbolic elements of Γ_+ and with the factor $1-N(\gamma)^{-s-m}$ replaced by $1-(-1)^m N(\gamma)^{-s-m}$ whenever γ has determinant -1.

§2. The Mayer transfer operator $\mathfrak{L}_s$

A surprising new interpretation of the Selberg zeta function was found by D. Mayer [9]. Let $\mathbf{D}$ denote the open disc of radius $3/2$ and center 1 and $\mathbf{V}$ the space of functions which are holomorphic in $\mathbf{D}$ and extend continuously to its boundary. Then Mayer defines an operator $\mathfrak{L}_s : \mathbf{V} \to \mathbf{V}$ by the formula

$$\mathfrak{L}_s\phi(z) \;=\; \sum_{n=1}^{\infty} \frac{1}{(z+n)^{2s}}\,\phi\Bigl(\frac{1}{z+n}\Bigr) \qquad (\phi \in \mathbf{V},\;\; z \in \mathbf{D})\,. \tag{3}$$

The series converges for $\Re(s) > \frac12$ for any $\phi \in \mathbf{V}$, and can be continued meromorphically to all s. (Write $\phi(z)$ as the sum of a polynomial of degree k and a function which is $\mathrm{O}(z^{k+1})$ as $z \to 0$; then the RHS of (3) is the sum of a finite linear combination of Hurwitz zeta functions, whose meromorphic continuation is well known, and an infinite sum which converges in the half-plane $\Re(s) > -k/2$.) Mayer shows that this operator has eigenvalues converging rapidly to zero ("nuclear"); in particular, $\mathfrak{L}_s$ is of trace class and $1-\mathfrak{L}_s$ has a determinant in the Fredholm sense, and similarly for $\mathfrak{L}_s^2$. Mayer's result is then:

Theorem. *We have the identities*

$$Z_{\Gamma_1}(s) \;=\; \det(1-\mathfrak{L}_s^2), \qquad Z_{\Gamma_+}(s) \;=\; \det(1-\mathfrak{L}_s)\,. \tag{4}$$

(Actually, Mayer gave the first of these statements, for the usual modular group. The second statement, corresponding to the even eigenfunctions, was proved by Efrat [2].)

Mayer's proof came from the theory of dynamical systems, and more specifically from the dynamics of the Gauss map (continued fraction map) $F : [0,1) \to [0,1)$ which assigns to any $x > 0$ the fractional part of $1/x$. For each complex number s with $\Re(s) > \frac{1}{2}$ and each positive integer n we then have the *partition function*

$$Z_n(F,h_s) \;=\; \sum_{x\in[0,1),\; F^n x = x} h_s(x)h_s(Fx)\cdots h_s(F^{n-1}x)\,,$$

where h_s is the weighting function $h_s(x) = x^{2s}$ and the sum is over all periodic points of F of order dividing n. The relationship with Mayer's operator $\mathfrak{L}_s$ ("transfer operator") comes from the fact that the inverse images of $x \in [0,1)$ are the points $(n+x)^{-1}$ ($n = 1, 2, \ldots$), and using this one can show that

$$Z_n(F,h_s) \;=\; \mathrm{Tr}(\mathfrak{L}_s^n) \;-\; (-1)^n\,\mathrm{Tr}(\mathfrak{L}_{s+1}^n)\,.$$

On the other hand, the periodic points of F correspond to the periodic continued fractions of real quadratic irrationalities and hence to closed geodesics in $\mathcal{H}/\Gamma_+$, and using this correspondence one finds that the sub-product of (2) corresponding to $m = 0$ for the group $\Gamma = \Gamma_1$ is equal to $\exp(\sum_{n=1}^\infty Z_{2n}(F,h_s)/n)$. (Here only even indices occur because the map $x \mapsto (n+x)^{-1}$ has determinant -1. For Γ_+ all indices occur.) Putting these facts together, one obtains (4) after a short calculation. A somewhat more elementary version of the proof, based on the reduction theory of elements of Γ_+, is presented in [4].

§3. Period theory and the Lewis correspondence

The final ingredient of our story is the discovery made some years ago by John Lewis [3] that one can associate to an eigenfunction of the Laplace operator for Γ_+ (i.e., an even eigenfunction for Γ_1) with spectral parameter s a holomorphic solution in $\mathbb{C} \smallsetminus (-\infty,0]$ of the *three-term functional equation*

$$\psi(z) \;=\; \psi(z+1) \;+\; z^{-2s}\,\psi(1+1/z)\,. \tag{5}$$

This result was extended in several ways in subsequent joint work [5]; in particular, it was shown that the correspondence is reversible, that it is enough to consider real-analytic solutions of (5) on the positive real axis, and that there is a similar result for odd eigenfunctions of the Laplacian. The final result can be stated as follows.

Theorem. *Let s be a complex number with $\Re(s) = \frac{1}{2}$. Then there is a canonical isomorphism between the space of even or odd eigenfunctions of the Laplace operator for Γ_1 with eigenvalue $s(1-s)$ and the space of real-analytic functions ψ on $(0,\infty)$ satisfying the even or odd three-term functional equation*

$$\psi(x) \;=\; \psi(x+1) \,\pm\, x^{-2s}\,\psi(1+1/x) \tag{6}$$

and the growth condition $\lim_{x\to\infty} \psi(x) = 0$.

Note that the theorem implies in particular that the space of real-analytic solutions of (6) satisfying the growth condition $\psi(x) = \mathrm{o}(1)$ as $x \to \infty$ is finite-dimensional for all s, and non-zero only for countably many s. This is especially surprising since, as is shown in the paper, the space of real-analytic solutions of (6) satisfying the apparently only slightly weaker condition $\psi(x) = \mathrm{O}(1)$ as $x \to \infty$ is infinite-dimensional for every s. We also mention that there is also a uniform version of the bijection of the theorem, including both the even and odd cases, in which the two functional equations (6) are replaced by the "unified" functional equation $\psi(x) = \psi(x+1) + (x+1)^{-2s}\,\psi(x/(1+x))$.

The above theorem is the exact analogue of the classical theory of Eichler, Shimura and Manin of period polynomials of modular forms, according to which one can canonically associate to a classical cusp form $f(z)$ of weight $2k$ on Γ_1 two polynomials, one even and one odd, which satisfy the above even and odd three-term functional equations (in the opposite order) with $s = 1-k$, obtaining in this way an isomorphism between the space of cusp forms and the spaces of polynomial solutions of each of the two functional equations. (More precisely, for the even polynomials one obtains a codimension one subspace of the space of polynomial solutions of the odd functional equation, the full space being obtained only if one includes an Eisenstein series.) The correspondence is obtained by associating to $f(z)$ its *Eichler integral* $f^*(z)$, defined most easily by the formula $f^*(z) = \sum_{n=1}^{\infty} n^{1-k} a_n\, e^{2\pi i n z}$, where $f(z) = \sum_{n=1}^{\infty} a_n e^{2\pi i n z}$ is the Fourier expansion of f; then the function $\psi(z) = f^*(z) - z^{2k-2} f^*(-1/z)$ turns out to be a polynomial satisfying the above-mentioned "unified" three-term functional equation with $s = 1-k$, and its even and odd parts are the desired polynomials satisfying the two equations (6).

The construction here proceeds similarly, although the formulas are somewhat more complicated and the proof is considerably longer. If $f(z)$ is an eigenfunction of the Laplace operator for Γ_1 (a so-called *Maass wave form*) with spectral parameter s, then, as is well known, f has a Fourier expansion of the form

$$f(z) \;=\; \sqrt{y}\,\sum_{n=1}^{\infty} A_n\, K_{s-\frac{1}{2}}(2\pi n y)\, \cos(2\pi n x) \qquad (z = x+iy \in \mathcal{H}) \tag{7}$$

in the even case and a similar expansion with "sin" instead of "cos" in the odd case, where $K_\nu(t)$ is the usual K-Bessel function and the coefficients A_n are complex numbers which grow at most polynomially in n. Although this function is non-holomorphic, we define its "Eichler integral" to be the *holomorphic* function f^* in the upper half-plane given by

$$f^*(z) \;=\; \sum_{n=1}^{\infty} n^{s-\frac{1}{2}}\, A_n\, e^{2\pi i n z} \qquad (z \in \mathcal{H})\,. \tag{8}$$

We extend this function to $\mathbb{C} \smallsetminus \mathbb{R}$ by setting $f^*(-z) = -f^*(z)$ in the even case (resp. $f^*(-z) = f^*(z)$ in the odd case). Then by relating the Mellin transforms of the two functions $f^*(\pm iy)$ $(y \in \mathbb{R}_+)$ to the L-series of f and using the functional equation of this L-series, one shows that the function $\psi(z)$ defined in $\mathbb{C} \smallsetminus \mathbb{R}$ by the formula $\psi(z) = f^*(z) - z^{-2s} f^*(-1/z)$ (i.e., by exactly the same formula as in the classical case, with k replaced by $1-s$) extends holomorphically from $\mathbb{C} \smallsetminus \mathbb{R}$ to the cut plane $\mathbb{C} \smallsetminus (-\infty, 0]$. The function ψ of the theorem is then simply the restriction of this extended function to the positive real axis. That ψ satisfies the three-term functional equation in $\mathbb{C} \smallsetminus \mathbb{R}$ is a trivial consequence of its definition and the periodicity of f, and would be true for any function in $\mathcal{H}$ with a Fourier expansion of the form (7), i.e., for any eigenfunction of the Laplace operator which is invariant merely under $z \mapsto z+1$ rather than under the whole modular group. The non-trivial property, which holds only if the function defined by (7) is invariant under the full group Γ_1, is the fact that ψ extends analytically across the positive real axis.

For the converse direction, if ψ is a real-analytic solution of (6) satisfying the growth condition in the theorem, one first shows by a "bootstrapping" argument (successive extension of ψ to larger and larger domains) that the function ψ extends holomorphically from $\mathbb{R}_+$ to all of $\mathbb{C} \smallsetminus \mathbb{R}$ and satisfies a suitable growth condition near the cut; then the functional equation of ψ shows that the function $f^*(z)$ defined by the formula $f^*(z) = \psi(z) + z^{-2s}\psi(-1/z)$ is periodic with period 1, and hence has a Fourier expansion of the form (8) for some coefficients $A_n \in \mathbb{C}$, and one then shows that the function $f(z)$ defined by the Fourier expansion (7) with the same coefficients A_n is indeed Γ_1-invariant. The key fact here, which has no analogue in the classical case, is the simple algebraic identity that the two transformations $\psi(z) = f^*(z) - z^{-2s} f^*(-1/z)$ and $f^*(z) = \psi(z) + z^{-2s}\psi(-1/z)$ are inverses of one another up to a non-zero scalar factor for any $s \in \mathbb{C} \smallsetminus \mathbb{Z}$.

§4. Interrelationships and generalizations

We now consider the relationship between the various objects we have described. In §1 we saw how the length spectrum of $\Gamma\backslash\mathcal{H}$ can be encoded in an infinite product, the Selberg zeta function, whose meromorphic continuation has zeros at the spectral parameters s_j of Γ. In §2 we saw how this zeta function, for $\Gamma = \Gamma_+$ or Γ_1, could be written as the Fredholm determinant of $1 - \mathfrak{L}_s$ or $1 - \mathfrak{L}_s^2$, respectively. But this Fredholm determinant is given by a convergent product $\prod_\nu (1 - \beta_\nu(s))$ or $\prod_\nu (1 - \beta_\nu(s)^2)$, where $\beta_\nu(s)$ are the eigenvalues of the operator $\mathfrak{L}_s$, so its vanishing is equivalent to the statement that $\mathfrak{L}_s$ has the eigenvalue 1 or ± 1. We thus obtain as a direct corollary of the theorem in §2 the following statement: *If s is the spectral parameter for an even (resp. odd) eigenfunction of the Laplace operator for $SL_2(\mathbb{Z})$, then there exists a non-zero function $\phi \in \mathbf{V}$ satisfying $\mathfrak{L}_s\phi = \phi$ (resp. $\mathfrak{L}_s\phi = -\phi$).*

The relationship between this statement and the period theory of Maass wave forms which we described in the last section is now very simple: It is obvious from the definition (3) that the operator $\mathfrak{L}_s$ satisfies the functional equation $\mathfrak{L}_s\phi(x) = (x+1)^{-2s}\phi(1/(x+1)) + \mathfrak{L}_s\phi(x+1)$, so if $\phi \in \mathbf{V}$ is an eigenfunction of $\mathfrak{L}_s$ with eigenfunction ± 1 then we have the identity $\phi(x) = \phi(x+1) \pm (x+1)^{-2s}\phi(1/(x+1))$, and after the simple change of variables $\phi(x) = \psi(x+1)$ this is precisely equation (6).

We can therefore summarize the whole story by saying that there are very close relationships between *four* a priori very different objects, as indicated in the following diagram,

$$\begin{tikzcd}
\boxed{\begin{array}{c}\text{A. Set of lengths of}\\ \text{geodesics on } \Gamma_1\backslash\mathcal{H}\end{array}} \arrow[r, leftrightarrow] \arrow[d, leftrightarrow] & \boxed{\begin{array}{c}\text{B. Spectrum of the}\\ \text{Laplace operator } \Delta\end{array}} \arrow[d, leftrightarrow] \\
\boxed{\begin{array}{c}\text{C. Spectrum of the}\\ \text{Mayer operator } \mathfrak{L}_s\end{array}} \arrow[r, leftrightarrow] & \boxed{\begin{array}{c}\text{D. Lewis's three-term}\\ \text{functional equation}\end{array}}
\end{tikzcd}$$

with (A) and (B) being related by the Selberg trace formula, (A) and (C) by the theory of transfer operators applied to the dynamical system associated to continued fractions, (B) and (D) by a generalization to the case of Maass wave forms of the classical theory of periods of modular forms, and (C) and (D) by the fact that an eigenfunction of $\mathfrak{L}_s$ with eigenvalue ± 1 is, up to a shift by 1, a solution of the functional equation (6).

It is also natural to ask what happens when Γ_1 is replaced by some other discrete group $\Gamma \subset \mathrm{SL}_2(\mathbb{R})$. The relationship between the lengths of geodesics and the eigenvalues of the Laplacian for $\Gamma\backslash\mathcal{H}$ given by the Selberg trace formula works equally well, as discussed in §1, but both the Mayer operator and the three-term functional equation depend on the special properties of the group Γ_1, and specifically on the fact that it is generated by the two transformations $S : z \mapsto -1/z$ and $T : z \mapsto z+1$. If Γ is a subgroup of Γ_1 of finite index N, then we can reduce to the case of Γ_1 by replacing the functions on which $\mathfrak{L}_s$ acts and the function ψ by vector-valued functions of length N, with coordinates indexed by the set of cosets $\Gamma\backslash\Gamma_1$, and with the appropriate modifications of the equations (for instance, in the functional equation (6) the coordinates of the two terms on the right must be permuted according to the actions of the two matrices $\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right)$ on $\Gamma\backslash\Gamma_1$), and everything goes through. This is the point of view adopted in the thesis of Martin [8], in which a theory analogous to the one in [5] is developed for the case of holomorphic modular forms of weight 1, a case not covered by the classical period theory of Eichler-Shimura-Manin, and also in the recent work of Manin and Marcolli [7]. However, there is also a more intrinsic approach which works also if the group Γ is not contained in, or even commensurable with, the modular group Γ_1. In the case of the Mayer operator, this requires considering the analogue of the continued fraction expansion appropriate to the group Γ, based on the geometry of a chosen fundamental domain for Γ, and replacing $\mathfrak{L}_s$

by the transfer operator appropriate to this new continued fraction. For example, if we observe that $\mathfrak{L}_s$ is really attached to Γ_+ rather than to Γ_1 (as seen by the fact that the Selberg zeta function for Γ_1 corresponds to the determinant of $1-\mathfrak{L}_s^2$ rather than $1-\mathfrak{L}_s$), then we should already modify $\mathfrak{L}_s$ even for the case of the subgroup Γ_1; the right modification of (3) is the operator $\phi(z) \mapsto \sum_{n=2}^{\infty}(n-z)^{-2s}\phi(1/(n-z))$, whose fixed points correspond via $\phi(z) = (1-z)^{-2s}\psi(1/(1-z))$ to the solutions of the "unified" functional equation mentioned above. For the Lewis correspondence, the most convenient language in which to describe the generalization to arbitrary Γ is that of cohomology of groups. We indicate here very briefly how this works. (The details will be given in [6].)

Define V_s^∞ to be the space of C^∞ functions $\psi : \mathbb{R} \to \mathbb{C}$ such that the function $|x|^{-2s}\psi(1/x)$ on $\mathbb{R} \smallsetminus \{0\}$ extends to a C^∞ function on $\mathbb{R}$ (or equivalently, that the function $\Psi(x,y) := |y|^{-2s}\psi(x/y)$ on $\mathbb{R}^2 \smallsetminus \{(0,0)\}$ is C^∞), and let V_s^ω be the subspace defined the same way but with "C^∞" replaced by "real-analytic." We define an intermediate space $V_s^{\omega/2}$ as the space of functions in V_s^∞ which are real-analytic except at a finite subset of $\mathbb{P}^1(\mathbb{R})$. The group $G = \mathrm{SL}_2(\mathbb{R})$ acts on all three spaces by $(\psi|g)(x) = |cx+d|^{-2s}\psi(g(x))$ for $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G$ (or equivalently, by $\Psi \mapsto \Psi \circ g$). In [5] it is shown that the function ψ occurring in the theorem in §2, if extended from $\mathbb{R}_+$ to $\mathbb{R} \smallsetminus \{(0,0)\}$ by $\psi(x) = \mp\psi(-x)$ (or equivalently by $\psi(x) = -|x|^{-2s}\psi(-1/x)$), belongs to V_s^∞ and hence also to $V_s^{\omega/2}$, and now the three-term functional equation implies exactly that this extended function satisfies the equations $\psi + \psi|S = \psi + \psi|U + \psi|U^2 = 0$, where $U = ST$ is the standard element of Γ_1 of order 3 which together with the involution S generates this group. These equations say that the map $S \mapsto \psi$, $T \mapsto 0$ extends to a cocycle on Γ with values in $V_s^{\omega/2}$, and the fact that T maps to 0 says that this is a parabolic cocycle (one which sends every parabolic element $\gamma \in \Gamma$ to an element in the image of $1-\gamma$). The theorem of §2 can then be reformulated as the statement that the parabolic cohomology group $H^1_{\mathrm{par}}(\Gamma_1, V_s^{\omega/2})$ is isomorphic to the space of Maass wave forms with spectral parameter s for all complex numbers s with real part $1/2$, and the generalization to other Γ is simply the same assertion for $H^1_{\mathrm{par}}(\Gamma, V_s^{\omega/2})$ and the spectral parameters for Γ. (If Γ is contained in Γ_1 and one uses the point of view of vector-valued functions mentioned above, then one would look instead at $H^1_{\mathrm{par}}(\Gamma_1, (V_s^{\omega/2})^{\Gamma\backslash\Gamma_1})$; this is the form in which the corresponding theorem for holomorphic forms of weight 1 occurs in [8].) Finally, it turns out that one can change the above cocycle by a coboundary to obtain a new cocycle, no longer sending T to 0, with values in the smaller space V_s^ω; these new cocycles are no longer parabolic, but satisfy the weaker "semi-parabolic" condition that their value at every parabolic element γ of Γ belongs to $V_s^{\omega/2}|(1-\gamma)$, and the cohomology group $H^1_{\mathrm{par}/2}(\Gamma, V_s^\omega)$ defined by such semi-parabolic real-analytic cocycles modulo coboundaries is again isomorphic to the space of Maass wave forms on Γ with spectral parameter s.

§5. Computational aspects

We end this survey by discussing to what extent the new points of view on the

Selberg zeta function are usable from a practical or computational point of view. The usual Selberg trace formula, both in its general form and in the special case defining the Selberg zeta function, has the drawback that it is very hard to use computationally: both the product (2) and the Weierstrass product expressing $Z_\Gamma(s)$ in terms of its zeros and poles are either divergent or only slowly convergent, and it is not easy to calculate actual values of the Selberg zeta function or to use the Selberg trace formula to get explicit information about the eigenvalues of the Laplace operator from the lengths of geodesics or vice versa. On the other hand, as we mentioned in §4, the eigenvalues $\beta_\nu(s)$ of the operator $\mathfrak{L}_s$ tend to zero rapidly, so that the infinite products $\prod(1-\beta_\nu(s))$ and $\prod(1-\beta_\nu(s)^2)$ defining the Fredholm determinants $\det(1-\mathfrak{L}_s) = Z_{\Gamma_+}(s)$ and $\det(1-\mathfrak{L}_s^2) = Z_{\Gamma_1}(s)$ are rapidly convergent. Hence if we have a good algorithm to compute the eigenvalues $\beta_\nu(s)$ then we also have a good method to compute both the values and the zeros of these Selberg zeta functions.

In fact such an algorithm exists. It was developed by Babenko (cf. [1]) to study the case $s = 1$, corresponding to a famous problem of Gauss (in a letter to Laplace, 1812) and Kuzmin (1928) concerning the distribution of the "tails" of the continued fraction expansions of real numbers, and the method extends in a fairly straightforward way to other values of s. On the one hand, one can obtain a good representation of the operator $\mathfrak{L}_s$ by showing that it is conjugate to (and hence has the same eigenvalues as) the integral operator on an appropriate space of functions on $(0,\infty)$ given by the kernel function

$$K_s(x,y) \;=\; \frac{J_{2s-1}\bigl(2\sqrt{xy}\bigr)}{\sqrt{(e^x-1)(e^y-1)}}\,,$$

where $J_\nu(x)$ is a J-Bessel function. (This is shown in Babenko's paper for $s = 1$ and a modification of the same proof works in general. It also occurs, in a somewhat different formulation, in Lewis's paper [3].) This representation as an integral operator can be used to establish the compactness and various other properties of the operator $\mathfrak{L}_s$. On the other hand, the operator $\mathfrak{L}_s$ sends the function $A_k(x) = x^{k-1}$ ($k = 1, 2, \dots$) to $B_k(x) = \zeta(x+1, k+2s-1)$, where $\zeta(\alpha,\nu) = \sum_{n=1}^\infty (n+\alpha)^{-\nu}$ denotes the Hurwitz zeta function. If one takes m well-chosen points $x_1, \dots, x_m$ in $[0,1]$ (a good choice is the set of points $x_j = \sin^2((2j-1)\pi/4m)$), then it turns out that the eigenvalues of the $m\times m$ matrix $A^{-1}B$, where A is the Vandermonde matrix $\bigl(A_k(x_j)\bigr)_{1\le j,k\le m}$ and B the matrix $\bigl(B_k(x_j)\bigr)_{1\le j,k\le m}$, are good approximations to the desired eigenvalues $\beta_\nu(s)$ of $\mathfrak{L}_s$. (More precisely, the matrix AB^{-1} has some eigenvalues $\beta_{\nu,m}(s)$ which tend exponentially rapidly to $\beta_\nu(s)$ as $m\to\infty$, as well as some "spurious" eigenvalues which tend slowly, like a negative power of m, towards zero as m goes to infinity.) As an example, we list in Table 1 the approximate values of the first 19 eigenvalues $\beta_\nu = \beta_\nu(1)$ of $\mathfrak{L}_1$ obtained in this way using values up to $m = 50$. (The first five of these, based on a calculation with $m = 16$ and correct to about 8 digits, were already given in [1], p. 367, but the table given there also contains one of the spurious eigenvalues which should be discarded.) As a check on the accuracy of the computation, one can compare the sum of the numbers $\beta_\nu(s)$ and the sum of their squares with the values of $\mathrm{Tr}(\mathfrak{L}_s)$ and $\mathrm{Tr}(\mathfrak{L}_s^2)$, both of which can be computed exactly.

$$
\begin{aligned}
\beta_1 &= 1.0000000000000000000000000000, \\
\beta_2 &= -0.3036630028987326585974481219 \\
\beta_3 &= 0.1008845092931040753056376425, \\
\beta_4 &= -0.0354961590216598454088916304\,1, \\
\beta_5 &= 0.01284379036244026481516090344, \\
\beta_6 &= -0.004717777511571031073863594144, \\
\beta_7 &= 0.001748675124305511914510617525, \\
\beta_8 &= -0.0006520208583205029031816912077, \\
\beta_9 &= 0.0002441314655245138906981706362, \\
\beta_{10} &= -0.00009168908376859566896987721814 \\
\beta_{11} &= 0.00003451654616347253214653427704, \\
\beta_{12} &= -0.00001301769787581775595712876501, \\
\beta_{13} &= 0.000004916782323649260641924415470, \\
\beta_{14} &= -0.000001859377799671070019642381811, \\
\beta_{15} &= 0.0000007038108595990962920678071171, \\
\beta_{16} &= -0.0000002666477929249816513412730701, \\
\beta_{17} &= 0.0000001010797864542122092292865860954, \\
\beta_{18} &= -0.00000003831081550679296190236314295, \\
\beta_{19} &= 0.00000001400621008979342385785713695.
\end{aligned}
$$

Table 1. Eigenvalues of $\mathfrak{L}_1$

Applying this method with s of the form $\frac{1}{2} + it$ with t real, we can compute the eigenvalues $\beta_\nu(s)$ to high accuracy for small values of t. If we plot the curve of the largest (in absolute value) eigenvalue $\beta_1(\frac{1}{2}+it)$ in the complex plane, then we indeed find that within the accuracy of the computation it passes through the points -1 and $+1$ for $t \approx 9.53369526135355746$ and $t \approx 13.77975135189074$, respectively, in accordance with Mayer's theorem from §2 and the known numerical values of the first odd and even spectral parameters for Γ_1. An interesting point here would be to find a direct argument showing that the curve really passes exactly through the points -1 and $+1$, not just very near them. It ought to be possible to do this using the functional equation of the Selberg zeta function, as one does for the analogous statement for the zeros of the Riemann zeta function, but I have not seen how to do this.

Finally, we mention that the numerical calculation described above could be used, for instance, to calculate to high accuracy the value of the logarithmic derivative at $s = 1$ of the Selberg zeta function for Γ_1, an invariant which plays a role in the work of Jorgenson and Kramer discussed at this conference.

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