A MODIFIED BERNOULLI NUMBER

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The classical Bernoulli numbers B_n, defined by the generating function

$$\frac{x}{e^x-1}=\sum_{n=0}^{\infty} B_n\,\frac{x^n}{n!}\,, \tag{1}$$

have many famous and beautiful properties, including the following three:

(i) $B_n = 0$ *for odd* $n>1$.

(ii) *The fractional part of* B_n *is given by*

$$B_n \equiv -\sum_{\substack{(p-1)|n\\ p\text{ prime}}} \frac{1}{p} \pmod 1 \qquad (n>0,\ n\text{ even})\,.$$

(iii) B_n *is given asymptotically for large even* n *by the formula*

$$B_n \sim (-1)^{n/2-1}\frac{n!}{2^{n-1}\pi^n} \qquad (n\to\infty,\ n\text{ even}).$$

In this note we show that the rational numbers defined by

$$B_n^* = \sum_{r=0}^{n}\binom{n+r}{2r}\frac{B_r}{n+r} \qquad (n>0) \tag{2}$$

satisfy the following amusing variants of the above three properties:

(I) *The value of* B_n^* *for* n *odd is periodic; more precisely, it is given by*

$n \pmod{12}$	1	3	5	7	9	11
B_n^*	$3/4$	$-1/4$	$-1/4$	$1/4$	$1/4$	$-3/4$

(II) *The fractional part of the number* $\widetilde{B}_n := 2nB_n^* - B_n$ *is given by*

$$\widetilde{B}_n \equiv \sum_{\substack{(p+1)|n\\ p\text{ prime}}} \frac{1}{p} \pmod 1 \qquad (n>0,\ n\text{ even})\,.$$

(III) B_n^* *is asymptotically equal to* $(-1)^{n/2-1}\dfrac{(n-1)!}{(2\pi)^n}$ *for* n *large and even, and is given much more precisely by the approximation*

$$B_n^* \approx (-1)^{n/2}\,\pi\,Y_n(4\pi) \qquad (n\to\infty,\ n\text{ even}),$$

where $Y_n(x)$ *denotes the* n*th Bessel function of the second kind.*

The proofs of (I)–(III) will be given in the next three sections, after which we will give the statement and proof of a fourth property, an exact formula for B_n^* refining the asymptotic formula (III). The proofs, especially those of (I) and (II), are quite fun and the reader is invited to try to find them him/herself before proceeding.

We end the introduction with a small table of the numbers B_n^* and $\widetilde{B}_n$.

n	2	4	6	8	10	12	14	16	18	20
B_n^*	$\frac{1}{24}$	$-\frac{27}{80}$	$-\frac{29}{1260}$	$\frac{451}{1120}$	$-\frac{65}{264}$	$-\frac{6571}{12012}$	$\frac{571}{312}$	$-\frac{181613}{38080}$	$\frac{23663513}{1220940}$	$-\frac{10188203}{83600}$
$\widetilde{B}_n$	0	$-\frac{8}{3}$	$-\frac{3}{10}$	$\frac{136}{21}$	-5	$-\frac{4249}{330}$	$\frac{651}{13}$	$-\frac{3056}{21}$	$\frac{109269}{170}$	$-\frac{247700}{57}$

Proof of (I). Instead of using the familiar generating function (1), we represent the Bernoulli numbers by the generating function

$$F(x) = \sum_{r=1}^{\infty} \frac{B_r}{r} x^r \ \in\ \mathbb{Q}[[x]]\,.$$

This formal power series does not converge anywhere, but occurs in the asymptotic formula

$$\frac{\Gamma'(X)}{\Gamma(X)} \sim \log X - \frac{1}{X} - F\bigl(\frac{1}{X}\bigr) \qquad (X \to \infty)$$

for the logarithmic derivative of the gamma function, and the functional equation $\Gamma(X+1) = X\Gamma(X)$ of the gamma function implies the functional equation

$$F\bigl(\frac{x}{1-x}\bigr) = F(x) + x + \log(1-x) \quad \in\ \mathbb{Q}[[x]] \tag{3}$$

of the power series F. An elementary proof of (3), or of the equivalent but simpler functional equation $G\bigl(\frac{x}{1-x}\bigr) = G(x) - x^2$ for the simpler power series $G(x) = \sum_{r=0}^{\infty} B_r x^{r+1} = x + x^2 F'(x)$, can be obtained by noting that either one of these functional equations is equivalent to the standard recursion formula

$$\sum_{k=0}^{n-1} \binom{n}{k} B_k = 0 \qquad (n > 1) \tag{4}$$

for the Bernoulli numbers, which is in turn an easy restatement of the definition (1).

Now introduce a new power series $F_\lambda(x)$, depending on a parameter λ, by

$$F_\lambda(x) \ = \ F\bigl(\frac{x}{1-\lambda x + x^2}\bigr) - \log(1-\lambda x + x^2) \quad \in\ \mathbb{Q}[[x]]\,.$$

For $\lambda = 2$ this specializes to

$$F_2(x) \ = \ \sum_{r=1}^{\infty} \frac{B_r}{r} \frac{x^r}{(1-x)^{2r}} - 2\log(1-x) = 2\sum_{n=1}^{\infty} B_n^* x^n\,. \tag{5}$$

On the other hand, the functional equation (3) together with the symmetry property $F(-x) = F(x) + x$, which is a restatement of (i), give the functional equations

$$F_{\lambda+1}(x) \;=\; F_\lambda(x) + \frac{x}{1-\lambda x + x^2} \;=\; F_{-\lambda}(-x)$$

for the power series F_λ. We deduce

$$\begin{aligned} F_2(x) - F_2(-x) &= \bigl(F_2(x) - F_1(x)\bigr) \;+\; \bigl(F_1(x) - F_0(x)\bigr) \;+\; \bigl(F_0(x) - F_{-1}(x)\bigr) \\ &= \frac{x}{1-x+x^2} + \frac{x}{1+x^2} + \frac{x}{1+x+x^2} \;=\; \frac{3x - x^3 - x^5 + x^7 + x^9 - 3x^{11}}{1-x^{12}}\,. \end{aligned}$$

Statement (I) follows.

Proof of (II). Rather surprisingly, this property is a consequence of the analogous property (ii) for the usual Bernoulli numbers, the divisibility by $p-1$ being metamorphosed into the divisibility by $p+1$ by the magic of generating functions. We begin by rewriting the definition of $\widetilde{B}_n$ as

$$\widetilde{B}_n = \sum_{r=0}^{n-1} \frac{(n+r)+(n-r)}{n+r} \binom{n+r}{2r} B_r = \beta_n + \beta_{n-1} - B_n\,, \tag{6}$$

where

$$\beta_n = \sum_{r=0}^{n} \binom{n+r}{2r} B_r \qquad\qquad (n \ge 0)\,.$$

Fix a prime p. We want to show that $p\widetilde{B}_n$ is p-integral for all n and is congruent to 1 mod p if $p+1$ divides n and to 0 otherwise. We suppose that $p > 2$. (The case $p = 2$ is similar but easier.) By (ii) we know that pB_r is p-integral and is congruent to -1 mod p if $p-1$ divides $r > 0$ and to 0 otherwise. Equation (6) then immediately gives the p-integrality of $p\widetilde{B}_n$ and the congruence

$$p\widetilde{B}_n \;\equiv\; -\gamma_n - \gamma_{n-1} + \begin{cases} 1 & \text{if } (p-1)|n \\ 0 & \text{otherwise} \end{cases} \tag{7}$$

(here and from now on all congruences are modulo p), where

$$\gamma_n = \sum_{r>0,\ (p-1)|r} \binom{n+r}{2r}\,.$$

As usual, we use generating functions. From the definition of γ_n we have

$$\sum_{n=0}^{\infty} \gamma_n\, x^n = \sum_{r>0,\ (p-1)|r} \frac{x^r}{(1-x)^{2r+1}} = \frac{x^{p-1}}{(1-x)^{2p-1} - x^{p-1}(1-x)}\,.$$

Hence (7) gives

$$\sum_{n>0} p\widetilde{B}_n\, x^n \equiv -(1+x)\sum_{n=0}^{\infty} \gamma_n\, x^n + \frac{x^{p-1}}{1-x^{p-1}}$$
$$\equiv -(1+x)\,\frac{x^{p-1}(1-x)}{(1-x^p)^2 - x^{p-1}(1-x)^2} + \frac{x^{p-1}}{1-x^{p-1}}\,.$$

The expression on the right simplifies to $\dfrac{x^{p+1}}{1-x^{p+1}}$, completing the proof.

Proof of (III). As in the proofs of (I) and (II), we deduce property (III) from its classical Bernoulli analogue. Suppose n is even and large. Then for $r = n-k$ with k fixed (and even) we have

$$\binom{2n+r}{2r}\frac{B_r}{2n+r} = \frac{(-1)^{n/2-1}(n-1)!}{(2\pi)^n}\left(\frac{(-1)^{k/2}(4\pi)^k}{k!} + \mathrm{O}\big(\frac{1}{n}\big)\right),$$

and since $\sum \dfrac{(-1)^{k/2}(4\pi)^k}{k!} = \cos(4\pi) = 1$ this gives the asymptotic formula in (III). The same argument in conjunction with the binomial coefficient identity

$$\binom{2n+r}{2r}\frac{1}{2n+r} = \frac{1}{2r!}\sum_{0\le h\le k/2}\frac{(-1)^h 2^{k-2h}(n-h-1)!}{h!(k-2h)!} \qquad (r = n-k)$$

(which holds because both sides express the coefficient of x^k in $(1-2x+x^2)^{-r}/2r$) lets us replace the asymptotic formula for B_n^* by the full asymptotic expansion

$$(-1)^{n/2-1}B_n^* \sim \sum_{h\ge 0}\frac{(n-h-1)!}{h!}\,(2\pi)^{-n+2h}\,.$$

Consulting standard reference works, we discover that the expression on the right is also the asymptotic development of $-\pi Y_n(4\pi)$, since Y_n is defined by

$$-\pi\, Y_n(2x) = \sum_{h=0}^{n-1}\frac{(n-h-1)!}{h!}x^{-n+2h} + \sum_{h=0}^{\infty}(c_h + c_{n+h} - 2\log x)\,\frac{(-1)^h x^{2h+n}}{h!(n+h)!}$$

with $c_h = 1 + \frac{1}{2} + \cdots + \frac{1}{h} - \gamma = \mathrm{O}(\log h)$. (Cf. [1], 7.2(32).) This proves the second assertion in (III), though without any estimate of the error.

Here are a few numerical values to illustrate the accuracy of the two approximations for B_n^* in (III) :

n	20	30	50
$(n-1)!\,(2\pi)^{-n}$	13.228	10026347.89	$0.75008667460769577\times 10^{23}$
$-\pi\, Y_n(4\pi)$	122.319	40532569.11	$1.69052138468088825\times 10^{23}$
$(-1)^{n/2-1}B_n^*$	121.868	40532573.81	$1.69052138468090709\times 10^{23}$

The poorness of the approximations in the first row is explained by the above asymptotic expansion, which shows that the ratio of $|B_n^*|$ to $(n-1)!(2\pi)^{-n}$ is $1 + Cn^{-1} + \mathrm{O}(n^{-2})$ with $C = 4\pi^2 \approx 40$.

An exact formula. In the case of the usual Bernoulli numbers, the rough asymptotic formula (iii) can be replaced by the exact formula

$$B_n \;=\; \frac{(-1)^{n/2-1}\, n!}{2^{n-1}\,\pi^n}\left(1+\frac{1}{2^n}+\frac{1}{3^n}+\cdots\right) \qquad (n>0,\ n \text{ even}) \tag{8}$$

due to Euler. It is reasonable to look for a corresponding exact formula for B_n^*.

We start with numerical data. From the table above we have $B_{50}^* + \pi Y_{50}(4\pi) \approx 1,884 \times 10^9$, the exact value being $1884415006.56\ldots$. Guessing that this difference might be related to the value $-\pi Y_{50}(8\pi)$, we compute this latter number, which turns out to be $1884414704.76\ \ldots$, suggesting that we are on the right track. Going one step further, we find that the difference $B_{50}^* + \pi Y_{50}(4\pi) + \pi Y_{50}(8\pi)$ equals $301.79\ldots$, rather close to the value $-\pi Y_{50}(12\pi) = 300.89\ldots$. This suggests that B_n^* might be very well approximated by, or even equal to, the number $U_n = \sum_{\ell=1}^{\infty}(-1)^{n/2}\pi Y_n(4\pi\ell)$. However, this sum diverges, since $(-1)^{n/2}\pi Y_n(4\pi\ell)$ behaves like $-\frac{1}{2}\ell^{-1/2}$ for ℓ large, so we must renormalize it, setting

$$U_n := \sum_{\ell=1}^{\infty}\left((-1)^{n/2}\pi Y_n(4\pi\ell) \,+\, \frac{1}{2\sqrt{\ell}}\right) - \frac{1}{2}\,\zeta\bigl(\tfrac{1}{2}\bigr). \tag{9}$$

The series converges only like $\sum \ell^{-3/2}$, but can be replaced by the expression

$$U_n = \sum_{\ell=1}^{\infty}\left((-1)^{n/2}\pi Y_n(4\pi\ell) \,+\, \sum_{k=0}^{K}\frac{c_{n,k}}{\ell^{k+1/2}}\right) - \sum_{k=0}^{K} c_{n,k}\,\zeta\bigl(k+\frac{1}{2}\bigr)\,,$$

$$c_{n,k} = \frac{(-1)^{[\frac{k+1}{2}]}}{2(8\pi)^k\,k!}\,(n+k-\tfrac{1}{2})(n+k-\tfrac{3}{2})\cdots(n-k+\tfrac{1}{2})$$

for any $K \geq 0$, where the ℓth term is $\mathrm{O}(\ell^{-K-3/2})$, so the numerical value of U_n can be computed easily. Comparing it with B_n^*, we find the following table:

n	$B_n^* - U_n$
2	$-0.886968175\ldots$
4	$-1.988273972\ldots$
10	$-4.999969846\ldots$
30	$-15 + 1.292\ldots \times 10^{-13}$
50	$-25 + 5.646\ldots \times 10^{-22}$

suggesting a formula of the form

$$B_n^* + \frac{n}{2} = U_n + \varepsilon_n \qquad (n>0,\ n \text{ even}), \tag{10}$$

where ε_n is positive and goes to 0 rapidly as n tends to infinity. (Notice that the quantity $B_n^* + n/2$ occurring on the left is given by the same expression as in (2), but with B_r replaced by $(-1)^r B_r$.) We now prove this and give an exact formula for the error term ε_n.

(IV) *Define* B_n^* *by* (2) *and* U_n *by* (9). *Then equation* (10) *holds with*

$$\varepsilon_n \;=\; \sum_{k=1}^{\infty}\frac{1}{\sqrt{k(k+4)}}\left(\frac{\sqrt{k}+\sqrt{k+4}}{2}\right)^{-2n} \;\sim\; \frac{1}{\sqrt{5}}\left(\frac{3+\sqrt{5}}{2}\right)^{-n}. \tag{11}$$

Proof of (IV). Using the relation (8) between Bernoulli numbers and zeta values, we find

$$B_n^* = \frac{1}{n} - \frac{n}{4} - \sum_{\substack{2\le r\le n\\ r\text{ even}}} \binom{n+r-1}{2r-1} \frac{(r-1)!}{(2\pi i)^r}\,\zeta(r)$$

and hence

$$B_n^* - U_n = \frac{1}{n} - \frac{n}{4} + \frac{1}{2}\zeta\big(\frac{1}{2}\big) + \sum_{\ell=1}^{\infty} b_{n,\ell}\,, \tag{12}$$

where

$$b_{n,\ell} := (-1)^{n/2-1}\,\pi\, Y_n(4\pi\ell) - \sum_{\substack{2\le r\le n\\ r\text{ even}}} \binom{n+r-1}{2r-1} \frac{(r-1)!}{(2\pi i\ell)^r} - \frac{1}{2\sqrt{\ell}}\,.$$

On the other hand, standard formulas for Bessel functions (cf. [1], 7.2(15) and 7.3(16)) give

$$\begin{aligned}(-1)^{n/2-1}\pi\, Y_n(x) &= 2\,\Re\big(K_n(ix)\big)\\ &= \frac{\sqrt{\pi/2}}{\Gamma(n+\frac{1}{2})} \int_0^\infty \frac{e^{-ix}(t/2+ix)^{n-\frac{1}{2}} + e^{ix}(t/2-ix)^{n-\frac{1}{2}}}{(ix)^n}\, e^{-t}\, t^{n-\frac{1}{2}}\, dt\end{aligned}$$

for n even and positive, and hence, after some simple manipulations, the formula

$$b_{n,\ell} = \frac{2\,\Gamma(\frac{1}{2})}{\Gamma(n+\frac{1}{2})} \int_0^\infty t^{n-1}\, e^{-t} \left(f_n\big(\frac{8\pi\ell}{t}\big) - \frac{1}{4}\sqrt{\frac{t}{\pi\ell}}\right) dt \tag{13}$$

where $f_n(x)$ is defined for $x>0$ by

$$f_n(x) = \frac{(1+ix)^{n-\frac{1}{2}} + (1-ix)^{n-\frac{1}{2}}}{2\,(ix)^n} - \sum_{\substack{2\le r\le n\\ r\text{ even}}} \binom{n-\frac{1}{2}}{n-r} (ix)^{-r}\,.$$

Note that $f(x) = (2x)^{-1/2} + \mathrm{O}(x^{-3/2})$ as $x\to\infty$, so that $b_{n,\ell} = \mathrm{O}(\ell^{-3/2})$, as we already know. Clearly $f_n(x)$ extends to $\mathbb{C}^0 := \mathbb{C} \smallsetminus (-i\infty,-i] \smallsetminus [i,i\infty)$ as an even holomorphic function, and the binomial theorem gives the Taylor expansion

$$f_n(x) = \sum_{r\ge 0} (-1)^r \binom{n-\frac{1}{2}}{n+2r} x^{2r} \qquad (|x|<1)\,.$$

The beta integral identity

$$\binom{n-\frac{1}{2}}{n+2r} = \frac{1}{\pi}\,\frac{\Gamma(n+\frac{1}{2})\Gamma(2r+\frac{1}{2})}{\Gamma(n+2r+1)} = \frac{1}{\pi}\int_0^1 u^{2r-\frac{1}{2}}\,(1-u)^{n-\frac{1}{2}}\,du$$

now gives the integral representation

$$f_n(x) = \frac{1}{\pi} \int_0^1 \frac{u^{-\frac{1}{2}}\,(1-u)^{n-\frac{1}{2}}}{1+x^2u^2}\,du$$

for $|x| < 1$, and by analytic continuation this holds for all $x \in \mathbb{C}^0$. Consider the more general integral

$$f_n(x,s) = \frac{1}{\pi} \int_0^1 \frac{u^{s-1}\,(1-u)^{n-\frac{1}{2}}}{1+x^2u^2}\,du \qquad (x \in \mathbb{C}^0,\ s \in \mathbb{C},\ 0 < \Re(s) < 2)\,,$$

so that $f_n(x) = f_n(x, \frac{1}{2})$. It can be estimated for large x by

$$f_n(x,s) = \frac{1}{\pi} \int_0^\infty \frac{u^{s-1}\,du}{1+x^2u^2} + \mathrm{O}(x^{-s-1}) = \frac{x^{-s}}{2\sin(\pi s/2)} + \mathrm{O}(x^{-s-1})\,.$$

In particular, $\sum_\ell f_n(\ell x, s)$ converges for $\Re(s) > 1$ and all $x > 0$, and using the Poisson summation identity

$$\sum_{\ell=-\infty}^{\infty} \frac{1}{1+\ell^2\,t^2} = \frac{\pi/t}{\tanh \pi/t} = \frac{\pi}{t}\left(1 + 2\sum_{k=1}^{\infty} e^{-2\pi k/t}\right) \qquad (t > 0)$$

we find

$$\sum_{\ell=-\infty}^{\infty} f_n(\ell x, s) = \frac{1}{x}\left(\frac{\Gamma(s-1)\Gamma(n+\frac{1}{2})}{\Gamma(s+n-\frac{1}{2})} + 2\sum_{k=1}^{\infty}\int_0^1 u^{s-2}\,(1-u)^{n-\frac{1}{2}}\,e^{-2\pi k/xu}\,du\right)$$

for $\Re(s) > 1$. Writing the left-hand side of this identity as

$$f_n(0,s) + \sum_{\ell=1}^{\infty}\left(2f_n(\ell x, s) - \frac{(\ell x)^{-s}}{\sin(\pi s/2)}\right) + \frac{\zeta(s)\,x^{-s}}{\sin(\pi s/2)}$$

gives its analytic continuation to $\Re(s) > 0$, and setting $s = \dfrac{1}{2}$, $x = \dfrac{8\pi}{t}$ we obtain

$$\sum_{\ell=1}^{\infty}\left(f_n(\frac{8\pi\ell}{t}) - \frac{1}{4}\sqrt{\frac{t}{\pi\ell}}\right) = -\frac{\Gamma(n+\frac{1}{2})}{8\Gamma(\frac{1}{2})\Gamma(n)}\,t - \frac{\zeta(\frac{1}{2})}{4\sqrt{\pi}}\,t^{\frac{1}{2}} - \frac{\Gamma(n+\frac{1}{2})}{2\Gamma(\frac{1}{2})\Gamma(n+1)}$$
$$+ \frac{t}{8\pi}\sum_{k=1}^{\infty}\int_0^1 u^{-\frac{3}{2}}\,(1-u)^{n-\frac{1}{2}}\,e^{-kt/4u}\,du\,.$$

Combining this with (13) and performing the integrations over t we get

$$\sum_{\ell=1}^{\infty} b_{n,\ell} = -\frac{n}{4} - \frac{\zeta(\frac{1}{2})}{2} - \frac{1}{n} + \frac{\Gamma(n+1)}{4\Gamma(\frac{1}{2})\Gamma(n+\frac{1}{2})}\sum_{k=1}^{\infty}\int_0^1 \frac{u^{-\frac{3}{2}}\,(1-u)^{n-\frac{1}{2}}}{(1+k/4u)^{n+1}}\,du\,.$$

The desired result now follows by inserting this into (12) and using the identity

$$\frac{\Gamma(n+1)}{4\Gamma(\frac{1}{2})\Gamma(n+\frac{1}{2})}\int_0^1 \frac{u^{-\frac{3}{2}}\,(1-u)^{n-\frac{1}{2}}}{(1+k/4u)^{n+1}}\,du = \frac{1}{\sqrt{k(k+4)}}\left(\frac{\sqrt{k}+\sqrt{k+4}}{2}\right)^{-2n},$$

which can be proved either from standard hypergeometric formulas or by expanding both sides as power series in $1/k$ for $k > 4$ and then using analytic continuation.

Remarks. We end this paper with a number of remarks.

1. The formula given in (i), which can be rewritten in the form

$$B_n^* = \frac{1}{2}\left(\frac{-3}{n}\right) + \frac{1}{4}\left(\frac{-4}{n}\right) \qquad (n \text{ odd}),$$

occurred originally in [4] in the context of the Eichler-Selberg trace formula for the traces of the Hecke operator T_ℓ acting on modular forms on $SL_2(\mathbb{Z})$. The method of proof there gave a formula for these traces which had a somewhat different form from the classical one, and in particular involved Bernoulli numbers. The specialization of the formula to the case $\ell = 1$ gave the dimension of $M_k(SL_2(\mathbb{Z}))$ in terms of B_{k-1}^*, and the equality of this expression with the standard dimension formula required the periodicity property (I).

2. The same idea as was used to prove (I) leads to another simple expression for the modified Bernoulli numbers B_n^*. Specifically, from

$$F_2(x) = F_0(x) + \frac{x}{1-x+x^2} + \frac{x}{1+x^2}$$

and the binomial formula we get the identity

$$2B_{2n}^* = \left(\frac{-3}{n}\right) + \sum_{r=0}^{n} (-1)^{n+r} \binom{n+r}{2r} \frac{B_{2r}}{n+r},$$

whose second term has a pleasing similarity to the original formula (2) defining B_n^*.

3. Next, we mention that the definition (2) can also be inverted to express the ordinary Bernoulli numbers in terms of the modified ones, should we for any reason wish to do so. Indeed, from (5) we have

$$\sum_{n=1}^{\infty} \frac{B_n}{n} x^n = 2\log\left(\frac{\sqrt{1+4x}-1}{2x}\right) + 2\sum_{r=1}^{\infty} B_r^* \left(\frac{1+2x-\sqrt{1+4x}}{2x}\right)^r,$$

and comparing coefficients of B_n we get

$$B_n = (-1)^n \binom{2n}{n} + 2\sum_{r=1}^{n} (-1)^{n-r}\, r \binom{2n}{n-r} B_r^*.$$

4. Finally, we mention that there are (at least) two ways of calculating Bernoulli numbers which are faster than the standard recursion (4). The first is due to M. Kaneko [3]. The second I noticed myself, but Kaneko has informed me that it is in fact a classical identity going back to Kronecker. (See [2] for a historical survey.) Nevertheless, these formulas are both pretty and useful, so for the sake of popularization we reproduce them here.

a. One can replace (4) by a recursion of the same type, but with only half as many terms, namely, setting $b_n = (n+1)B_n$,

$$b_{2n} = -\frac{1}{n+1} \sum_{i=0}^{n-1} \binom{n+1}{i} b_{n+i} \tag{14}$$

together with the conditions $b_1 = -1$ and $b_{2n+1} = 0$ for $n > 0$. Equation (14) can be seen as a special case of the following fact: Define an involution * on the set of sequences $\{a_0, a_1, a_2, \dots\}$ by $A^*(x) = e^{-x} A(-x)$, where $A(x) := \sum_{n=0}^{\infty} a_n x^n/(n+1)!$, or more explicitly by $a_n^* = (-1)^n \sum_{i=0}^{n} \binom{n+1}{i+1} a_i$. Then the expression $\sum_{i=0}^{n} \binom{n}{i} a_{n+i-1}$ $(n \geq 1)$ is anti-invariant under * and hence vanishes if $A^* = A$. (Note that $A(x) = x/(e^x - 1) = A^*(x)$ for $a_n = b_n$.)

b. The Bernoulli numbers can be calculated directly, rather than recursively, by the closed formula

$$B_k = \sum_{n=0}^{k} (-1)^n \binom{k+1}{n+1} \frac{0^k + \cdots + n^k}{n+1} \,. \tag{15}$$

To prove this, we apply Bernoulli's famous formula for $1^k + \cdots + n^k$ to get

$$\frac{0^k + \cdots + n^k}{n+1} = \frac{B_{k+1}(n+1) - B_{k+1}}{(k+1)(n+1)} = (\text{polynomial of degree } k \text{ in } n) + \frac{B_k}{n+1} \,,$$

where $B_r(x)$ denotes the rth Bernoulli polynomial; taking the $(k+1)$st difference of both sides kills the polynomial on the right, leaving only an easily computed multiple of B_k. Formula (15) is much more convenient for numerical computations than the recursion formula, at least if one wants to compute individual Bernoulli numbers rather than a table up to some limit, since the number of steps needed to compute B_k is $\mathrm{O}(k)$ rather than $\mathrm{O}(k^2)$ (each term in the sum can be computed from its predecessor in $\mathrm{O}(1)$ steps). Indeed, even for computing a table, (15) is sometimes more useful than the recursion (4), since the time required is about the same but the storage requirements are reduced from $\mathrm{O}(k)$ to $\mathrm{O}(1)$. Here is a one-line PARI program implementing the formula (15) (for $k > 0$):

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B(k)=h=0;s=1;c=k+1;for(n=2,k+1,c=c*(n-k-2)/n;h=h+c*s/n;s=s+n^k);h
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