APPENDIX. THE MELLIN TRANSFORM AND RELATED ANALYTIC TECHNIQUES

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1. The generalized Mellin transformation

The Mellin transformation is a basic tool for analyzing the behavior of many important functions in mathematics and mathematical physics, such as the zeta functions occurring in number theory and in connection with various spectral problems. We describe it first in its simplest form and then explain how this basic definition can be extended to a much wider class of functions, important for many applications.

Let $\varphi(t)$ be a function on the positive real axis $t > 0$ which is reasonably smooth (actually, continuous or even piecewise continuous would be enough) and decays rapidly at both 0 and ∞, i.e., the function $t^A\varphi(t)$ is bounded on $\mathbb{R}_+$ for any $A \in \mathbb{R}$. Then the integral

$$\widetilde{\varphi}(s) \;=\; \int_0^\infty \varphi(t)\, t^{s-1}\, dt \tag{1}$$

converges for any complex value of s and defines a holomorphic function of s called the *Mellin transform* of $\varphi(s)$. The following small table, in which α denotes a complex number and λ a positive real number, shows how $\widetilde{\varphi}(s)$ changes when $\varphi(t)$ is modified in various simple ways:

$$\begin{array}{ccccc} \varphi(\lambda t) & t^\alpha\varphi(t) & \varphi(t^\lambda) & \varphi(t^{-1}) & \varphi'(t) \\ \lambda^{-s}\widetilde{\varphi}(s) & \widetilde{\varphi}(s+\alpha) & \lambda^{-s}\widetilde{\varphi}(\lambda^{-1}s) & \widetilde{\varphi}(-s) & (1-s)\widetilde{\varphi}(s-1) \end{array}\,. \tag{2}$$

We also mention, although we will not use it in the sequel, that the function $\varphi(t)$ can be recovered from its Mellin transform by the *inverse Mellin transformation* formula

$$\varphi(t) \;=\; \frac{1}{2\pi i}\int_{C-i\infty}^{C+i\infty} \widetilde{\varphi}(s)\, t^{-s}\, ds\,,$$

where C is any real number. (That this is independent of C follows from Cauchy's formula.)

However, most functions which we encounter in practise are not very small at both zero and infinity. If we assume that $\varphi(t)$ is of rapid decay at infinity but grows like t^{-A} for some real number A as $t \to 0$, then the integral (1) converges and defines a holomorphic function only in the right half-plane $\Re(s) > A$. Similarly, if $\varphi(t)$ is of rapid decay at zero but grows like t^{-B} at infinity for some real number B, then $\widetilde{\varphi}(s)$ makes sense and is holomorphic only in the left half-plane $\Re(s) < B$, while if $\varphi(t)$ has polynomial growth at both ends, say like t^{-A} at 0 and like t^{-B} at ∞ with $A < B$, then $\widetilde{\varphi}(s)$ is holomorphic only in the strip $A < \Re(s) < B$. But it turns out that in many cases the function $\widetilde{\varphi}(t)$ has a meromorphic extension to a larger half-plane or strip than the one in which the original integral (1) converges, or even to the whole complex plane. Moreover, this extended Mellin transform can sometimes be defined even in cases where $A > B$, in which case the integral (1) does not converge for any value of s at all.

Let us start with the frequently occurring case where $\varphi(t)$ is of rapid decay at infinity and is C^∞ at zero, i.e., it has an asymptotic expansion $\varphi(t) \sim \sum_{n=0}^\infty a_n t^n$ as $t \to 0$. (Recall that this means that the difference $\varphi(t) - \sum_{n=0}^{N-1} a_n t^n$ is $\mathrm{O}(t^N)$ as $t \to 0$ for any integer $N \geq 0$; it is not required that the series $\sum a_n t^n$ be convergent for any positive t.) Then for s with $\Re(s) > 0$ and any positive integer N the integral (1) converges and can be decomposed as follows:

$$\begin{aligned}\widetilde{\varphi}(s) &= \int_0^1 \varphi(t)\, t^{s-1}\, dt + \int_1^\infty \varphi(t)\, t^{s-1}\, dt \\ &= \int_0^1 \Bigl(\varphi(t) - \sum_{n=0}^{N-1} a_n t^n\Bigr) t^{s-1}\, dt + \sum_{n=0}^{N-1} \frac{a_n}{n+s} + \int_1^\infty \varphi(t)\, t^{s-1}\, dt\,.\end{aligned}$$

The first integral on the right converges in the larger half-plane $\Re(s) > -N$ and the second for all $s \in \mathbb{C}$, so we deduce that $\widetilde{\varphi}(s)$ has a meromorphic continuation to $\Re(s) > -N$ with simple poles of residue a_n at $s = -n$ $(n = 0, \ldots, N-1)$ and no other singularities. Since this holds for every n, it follows that the Mellin transform $\widetilde{\varphi}(s)$ in fact has a meromorphic continuation to all of $\mathbb{C}$ with simple poles of residue a_n at $s = -n$ $(n = 0, 1, 2, \ldots)$ and no other poles. The same argument shows that, more generally, if $\varphi(t)$ is of rapid decay at infinity and has an asymptotic expansion

$$\varphi(t) \sim \sum_{j=1}^\infty a_j\, t^{\alpha_j} \qquad (t \to 0) \tag{3}$$

as t tends to zero, where the α_j are real numbers tending to $+\infty$ as $j \to \infty$ or complex numbers with real parts tending to infinity, then the function $\widetilde{\varphi}(s)$ defined by the integral (1) for $\Re(s) > -\min_j \Re(\alpha_j)$ has a meromorphic extension to all of $\mathbb{C}$ with simple poles of residue a_j at $s = -\alpha_j$ $(j = 1, 2, \ldots)$ and no other poles. Yet more generally, we can allow terms of the form $t^\alpha (\log t)^m$ with $\lambda \in \mathbb{C}$ and $m \in \mathbb{Z}_{\geq 0}$ in the asymptotic expansion of $\varphi(t)$ at $t = 0$ and each such term contributes a pole with principal part $(-1)^m m!/(s+\alpha)^{m+1}$ at $s = -\alpha$, because $\int_0^1 t^{\alpha+s-1} (\log t)^m\, dt = (\partial/\partial\alpha)^m \int_0^1 t^{\alpha+s-1}\, dt = (-1)^m m!/(\alpha+s)^{m+1}$ for $\Re(s+\alpha) > 0$.

By exactly the same considerations, or by replacing $\varphi(t)$ by $\varphi(t^{-1})$, we find that if $\varphi(t)$ is of rapid decay (faster than any power of t) as $t \to 0$ but has an asymptotic expansion of the form

$$\varphi(t) \sim \sum_{k=1}^\infty b_k\, t^{\beta_k} \qquad (t \to \infty) \tag{4}$$

at infinity, where now the exponents β_k are complex numbers whose real parts tend to $-\infty$, then the function $\widetilde{\varphi}(s)$, originally defined by (1) in a left half-plane $\Re(s) < -\max_k \Re(\beta_k)$, extends meromorphically to the whole complex s-plane with simple poles of residue $-b_k$ at $s = -\beta_k$ and no other poles. (More generally, again as before, we can allow terms $b_k t^{\beta_k} (\log t)^{n_k}$ in (3) which then produce poles with principal parts $(-1)^{n_k+1} n_k!\, b_k/(s+\beta_k)^{n_k+1}$ at $s = -\beta_k$.)

Now we can use these ideas to define $\widetilde{\varphi}(s)$ for functions which are not small either at 0 or at ∞, even when the integral (1) does not converge for any value of s. We simply assume that $\varphi(t)$ is a smooth (or continuous) function on $(0, \infty)$ which has asymptotic expansions of the forms (3) and (4) at zero and infinity, respectively. (Again, we could allow more general terms with powers of $\log t$ in the expansions, as already explained, but the corresponding modifications are easy and for simplicity of expression we will assume expansions purely in powers of t.) For convenience we assume that the numbering s such that $\Re(\alpha_1) \leq \Re(\alpha_2) \leq \cdots$ and $\Re(\beta_1) \geq \Re(\beta_2) \geq \cdots$. Then, for any $T > 0$ (formerly we took $T = 1$, but the extra freedom of being able to choose any value of T will be very useful later) we define two "half-Mellin transforms" $\widetilde{\varphi}_{\le T}(s)$ and $\widetilde{\varphi}_{\ge T}(s)$ by

$$\widetilde{\varphi}_{\le T}(s) \;=\; \int_0^T \varphi(t)\, t^{s-1}\, dt \qquad \bigl(\Re(s) \;>\; -\Re(\alpha_1)\bigr)\,,$$
$$\widetilde{\varphi}_{\ge T}(s) \;=\; \int_T^\infty \varphi(t)\, t^{s-1}\, dt \qquad \bigl(\Re(s) \;<\; -\Re(\beta_1)\bigr)\,.$$

Just as before, we see that for each integer $J \ge 1$ the function $\widetilde{\varphi}_{\le T}(s)$ extends by the formula

$$\widetilde{\varphi}_{\le T}(s) \;=\; \int_0^T \Bigl(\varphi(t) - \sum_{j=1}^J a_j t^{\alpha_j}\Bigr)\, t^{s-1}\, dt \;+\; \sum_{j=1}^J \frac{a_j}{s+\alpha_j}\, T^{s+\alpha_j}$$

to the half-plane $\Re(s) > -\Re(\alpha_{J+1})$ and hence, letting $J \to \infty$, that $\widetilde{\varphi}_{\le T}(s)$ is a meromorphic function of s with simple poles of residue a_j at $s = -\alpha_j$ $(j = 1, 2, \dots)$ and no other poles. Similarly, $\widetilde{\varphi}_{\ge T}(s)$ extends to a meromorphic function whose only poles are simple ones of residue $-b_k$ at $s = -\beta_k$. We now define

$$\widetilde{\varphi}(s) \;=\; \widetilde{\varphi}_{\le T}(s) \;+\; \widetilde{\varphi}_{\ge T}(s)\,. \tag{$4\frac{1}{2}$}$$

This is a meromorphic function of s and is independent of the choice of T, since the effect of changing T to T' is simply to add the everywhere holomorphic function $\int_T^{T'} \varphi(t)\, t^{s-1}\, dt$ to $\widetilde{\varphi}_{\le T}(s)$ and subtract the same function from $\widetilde{\varphi}_{\ge T}(s)$, not affecting the sum of their analytic continuations.

In summary, if $\varphi(t)$ is a function of t with asymptotic expansions as a sum of powers of t (or of powers of t multiplied by integral powers of $\log t$) at both zero and infinity, then we can define in a canonical way a Mellin transform $\widetilde{\varphi}(s)$ which is meromorphic in the entire s-plane and whose poles reflect directly the coefficients in the asymptotic expansions of $\varphi(t)$. This definition is consistent with and has the same properties (2) as the original definition (1). We end this section by giving two simple examples, while Sections 2 and 3 will give further applications of the method.

Example 1. Let $\varphi(t) = t^\alpha$, where α is a complex number. Then φ has an asymptotic expansion (3) at 0 with a single term $\alpha_1 = \alpha$, $a_1 = 1$, and an asymptotic expansion (4) at ∞ with a single term $\beta_1 = \alpha$, $b_1 = 1$. We immediately find that $\widetilde{\varphi}_{\le T}(s) = T^{s+\alpha}/(s+\alpha)$ for $\Re(s+\alpha) > 0$ and $\widetilde{\varphi}_{\ge T}(s) = -T^{s+\alpha}/(s+\alpha)$ for $\Re(s+\alpha) < 0$, so that, although the original Mellin transform integral (1) does not converge for any value of s, the function $\widetilde{\varphi}(s)$ defined as the sum of the meromorphic continuations of $\widetilde{\varphi}_{\le T}(s)$ and $\widetilde{\varphi}_{\ge T}(s)$ makes sense, is independent of T, and in fact is identically zero. More generally, we find that $\widetilde{\varphi}(s) \equiv 0$ whenever $\varphi(t)$ is a finite linear combination of functions of the form $t^\alpha \log^m t$ with $\alpha \in \mathbb{C}$, $m \in \mathbb{Z}_{\ge 0}$. (These are exactly the functions whose images $\varphi_\lambda(t) = \varphi(\lambda t)$ under the action of the multiplicative group $\mathbb{R}_+$ span a finite-dimensional space.) In particular, we see that the generalized Mellin transformation is no longer injective.

Example 2. Let $\varphi(t) = e^{-t}$. Here the integral (1) converges for $\Re(s) > 0$ and defines Euler's gamma-function $\Gamma(s)$. From the fact that $\varphi(t)$ is of rapid decay at infinity and has the asymptotic (here even convergent) expansion $\sum_{n=0}^\infty (-t)^n/n!$ at zero, we deduce that $\Gamma(s) = \widetilde{\varphi}(s)$ has a meromorphic continuation to all s with a simple pole of residue $(-1)^n/n!$ at $s = -n$ $(n = 0, 1, \dots)$ and no other poles. Of course, in this special case these well-known properties can also be deduced from the functional equation $\Gamma(s+1) = s\Gamma(s)$ (proved for $\Re(s) > 0$ by integration by parts in the integral defining $\Gamma(s)$), N applications of which gives the meromorphic extension $\Gamma(s) = s^{-1}(s+1)^{-1}\cdots(s+N-1)^{-1}\Gamma(s+N)$ of $\Gamma(s)$ to the half-plane $\Re(s) > -N$.

From the first of the properties listed in (2), we find the following formula, which we will use many times:

$$\varphi(t) \;=\; e^{-\lambda t} \quad\Rightarrow\quad \widetilde{\varphi}(s) \;=\; \Gamma(s)\, \lambda^{-s} \qquad (\lambda > 0)\,. \tag{5}$$

2. Dirichlet series and their special values

In this section we look at functions $\varphi(t)$ for which the Mellin transform defined in Section 1 is related to a Dirichlet series. The key formula is (5), because it allows us to convert Dirichlet series into exponential series, which are much simpler.

Example 3. Define $\varphi(t)$ for $t > 0$ by $\varphi(t) = 1/(e^t - 1)$. This function is of rapid decay at infinity and has an asymptotic expansion (actually convergent for $t < 2\pi$)

$$\frac{1}{e^t - 1} \;=\; \frac{1}{t + \dfrac{t^2}{2} + \dfrac{t^3}{6} + \cdots} \;=\; \sum_{r=0}^{\infty} \frac{B_r}{r!}\, t^{r-1} \tag{6}$$

with certain rational coefficients $B_0 = 1$, $B_1 = -\frac{1}{2}$, $B_2 = \frac{1}{6}$, ... called *Bernoulli numbers*. From the results of Section 1 we know that the Mellin transform $\widetilde{\varphi}(s)$, originally defined for $\Re(s) > 1$ by the integral (1), has a meromorphic continuation to all s with simple poles of residue $B_r/r!$ at $s = 1 - r$ ($r = 0, 1, 2, \ldots$). On the other hand, since $e^t > 1$ for $t > 0$, we can expand $\varphi(t)$ as a geometric series $e^{-t} + e^{-2t} + e^{-3t} + \cdots$, so (5) gives (first in the region of convergence) $\widetilde{\varphi}(s) \;=\; \Gamma(s)\zeta(s)$, where

$$\zeta(s) \;=\; \sum_{m=1}^{\infty} \frac{1}{m^s} \qquad (\Re(s) > 1) \tag{7}$$

is the Riemann zeta function. Since $\Gamma(s)$, as we have seen is also meromorphic, with simple poles of residue $(-1)^n/n!$ at non-positive integral arguments $s = -n$ and no other poles, and since $\Gamma(s)$ (as is well-known and easily proved) never vanishes, we deduce that $\zeta(s)$ has a meromorphic continuation to all s with a unique simple pole of residue $1/\Gamma(1) = 1$ at $s = 1$ and that its values at non-positive integral arguments are rational numbers expressible in terms of the Bernoulli numbers:

$$\zeta(-n) \;=\; (-1)^n \frac{B_{n+1}}{n+1} \qquad (n = 0,\, 1,\, 2,\, \ldots)\,. \tag{8}$$

Example 4. To approach $\zeta(s)$ in another way, we choose for $\varphi(t)$ the *theta function*

$$\vartheta(t) \;=\; \sum_{n=-\infty}^{\infty} e^{-\pi n^2 t} \qquad (t > 0)\,. \tag{9}$$

(The factor π in the exponent has been included for later convenience.) We can write this out as

$$\vartheta(t) \;=\; 1 \,+\, 2\,e^{-\pi t} \,+\, 2\,e^{-4\pi t} \,+\, \cdots\,, \tag{10}$$

and since the generalized Mellin transform of the function 1 is identically $\tilde{0}$ by Example 1, we deduce from (5) that $\widetilde{\varphi}(s) = 2\,\zeta^*(2s)$, where

$$\zeta^*(s) \;=\; \pi^{-s/2}\,\Gamma(s/2)\,\zeta(s)\,. \tag{11}$$

To obtain the analytic properties of $\zeta(s)$ from the results of Section 1, we need the asymptotics of $\vartheta(t)$ at zero and infinity. They follow immediately from the following famous result, due to Jacobi:

Proposition 1. *The function $\vartheta(t)$ satisfies the functional equation*

$$\vartheta(t) \;=\; \frac{1}{\sqrt{t}}\,\vartheta\Bigl(\frac{1}{t}\Bigr) \qquad (t>0)\,. \tag{12}$$

Proof. Formula (12) is a special case of the *Poisson summation formula*, which says that

$$\sum_{n\in\mathbb{Z}} f(n) \;=\; \sum_{n\in\mathbb{Z}} \widehat{f}(n) \tag{13}$$

for any sufficiently well-behaved (i.e., smooth and small at infinity) function $f:\mathbb{R}\to\mathbb{C}$, where $\widehat{f}(y) = \int_{-\infty}^{\infty} f(x)\,e^{2\pi ixy}\,dx$ is the Fourier transform of f. (To prove this, note that the function $F(x)=\sum_{n\in\mathbb{Z}} f(n{+}x)$ is periodic with period 1, so has a Fourier expansion $F(x)=\sum_{m\in\mathbb{Z}} c_m e^{2\pi imx}$ with $c_m = \int_0^1 F(x)e^{-2\pi imx}dx = \widehat{f}(-m)$. Now set $x=0$.) Now consider the function $f_t(x) = e^{-\pi tx^2}$. Its Fourier transform is given by

$$\widehat{f_t}(y) \;=\; \int_{-\infty}^{\infty} e^{-\pi tx^2+2\pi ixy}\,dx \;=\; e^{-\pi y^2/t}\int_{-\infty}^{\infty} e^{-\pi t(x+iy/t)^2}\,dx \;=\; \frac{c}{\sqrt{t}}\,f_{1/t}(y)\,,$$

where $c>0$ is the constant $c=\int_{-\infty}^{\infty} e^{-\pi x^2}dx$. Applying (13) with $f=f_t$ therefore gives $\vartheta(t) = ct^{-1/2}\vartheta(1/t)$, and taking $t=1$ in this formula gives $c=1$ and proves equation (12). □

Now we find from (10) that $\vartheta(t)$ has the asymptotic expansions $\vartheta(t)=1+\mathrm{O}(t^{-N})$ as $t\to\infty$ and $\vartheta(t)=t^{-1/2}+\mathrm{O}(t^N)$ as $t\to 0$, where $N>0$ is arbitrary. It follows from the results of Section 1 that its Mellin transform $\widetilde{\vartheta}(s)$ has a meromorphic extension to all s with simple poles of residue 1 and -1 at $s=1/2$ and $s=0$, respectively, and no other poles. From the formula $\zeta^*(s)=\frac{1}{2}\widetilde{\vartheta}(s/2)$ we deduce that the function $\zeta^*(s)$ defined in (11) is meromorphic having simple poles of residue 1 and 0 at $s=1$ and $s=0$ and no other poles and hence (using once again that $\Gamma(s)$ has simple poles at non-positive integers and never vanishes) that $\zeta(s)$ itself is holomorphic except for a single pole of residue 1 at $s=1$ and vanishes at negative even arguments $s=-2,\,-4,\ldots$. This is weaker than (8), which gives a formula for $\zeta(s)$ at all non-positive arguments (and also shows the vanishing at negative even integers because it is an exercise to deduce from the definition (6) that B_r vanishes for odd $r>1$). The advantage of the second approach to $\zeta(s)$ is that from equation (12) and the properties of Mellin transforms listed in (2) we immediately deduce the famous functional equation

$$\zeta^*(1-s) \;=\; \zeta^*(1-s) \tag{14}$$

of the Riemann zeta-function which was discovered (for integer values $\neq 0,\,1$ of s) by Euler in 1749 and proved (for all complex values $\neq 0,\,1$ of s) by Riemann in 1859 by just this argument.

We next generalize the method of Example 3. Consider a generalized Dirichlet series

$$L(s) \;=\; \sum_{m=1}^{\infty} c_m\,\lambda_m^{-s} \tag{15}$$

where the λ_m are real numbers satisfying $0<\lambda_1<\lambda_2<\cdots$ and growing at least as fast as some positive power of m. (This is an ordinary Dirichlet series if $\lambda_m = m$ for all m.) Assume that the series converges for at least one value s_0 of s. Then it automatically converges in a half-plane (for instance, if $\lambda_m=m$ then the fact that $c_m=\mathrm{O}(m^{s_0})$ implies convergence in the half-plane $\Re(s)>\Re(s_0)+1$) and the associated exponential series

$$\varphi(t) \;=\; \sum_{m=1}^{\infty} c_m\,e^{-\lambda_m t} \qquad (t>0), \tag{16}$$

converges for all positive values of t. We then have:

Proposition 2. *Let $L(s)$ be a generalized Dirichlet series as in* (15)*, convergent somewhere, and assume that the function $\varphi(t)$ defined by* (16) *has an asymptotic expansion of the form*

$$\varphi(t) \sim \sum_{n=-1}^{\infty} a_n t^n \qquad (t \to 0)\,. \tag{17}$$

as $t \to 0$. Then $L(s)$ has a meromorphic continuation to all s, with a simple pole of residue a_{-1} at $s = 1$ and no other singularities, and its values at non-positive integers are given by

$$L(-n) = (-1)^n n!\, a_n \qquad (n = 0,\, 1,\, 2,\, \ldots)\,. \tag{18}$$

Proof. The function $\varphi(t)$ is of rapid decay at infinity and has the asymptotic expansion (17) at zero, so by the results of Section 1 we know that its Mellin transform $\widetilde{\varphi}(s)$ extends meromorphically to all s, with simple poles of residue a_n at $s = -n$ ($n = -1, 0, 1, \ldots$). On the other hand, $\widetilde{\varphi}(s)$ is equal to $\Gamma(s)L(s)$ by formula (5), and we know that $\Gamma(s)$ has simple poles of residue $(-1)^n n!$ at $s = -n$ ($n = 0, 1, \ldots$), has no other zeros or poles, and equals 1 at $s = 1$. The result follows. □

Example 5. Consider the Dirichlet series defined by

$$L_4(s) = \frac{1}{1^s} - \frac{1}{3^s} + \frac{1}{5^s} - \cdots \qquad (\Re(s) > 1)\,. \tag{19}$$

Here the function defined by (16) is given by

$$\varphi(t) = e^{-t} - e^{-3t} + e^{-5t} - \cdots = \frac{1}{e^t + e^{-t}} = \frac{1}{2\cosh t}$$

(geometric series). The asymptotic expansion of this function at $t = 0$ has the form

$$\varphi(t) = \frac{1/2}{1 + t^2/2! + t^4/4! + \cdots} = \frac{1}{2}\sum_{n=0}^{\infty} E_n \frac{t^n}{n!}$$

where the coefficients $E_0 = 1,\ E_1 = 0,\ E_2 = -1, \ldots$ are certain integers called the *Euler numbers*. It follows from the proposition that the function $L_4(s)$ has a holomorphic continuation to all s and that $L_4(-n) = E_n/2$ for all $n \ge 0$. (We can omit the factor $(-1)^n$ because $E_n = 0$ for n odd.) The same method works for any Dirichlet series of the form $\sum_{m=1}^{\infty} \chi(m) m^{-s}$ with coefficients $\chi(m)$ which are periodic of some period (here 4), the most important case being that of Dirichlet L-series, where the coefficients $\chi(m)$ also satisfy $\chi(m_1 m_2) = \chi(m_1)\chi(m_2)$ for all m_1 and m_2.

Example 6. As a final example, consider the *Hurwitz zeta function*, defined by

$$\zeta(s,a) = \sum_{n=0}^{\infty} \frac{1}{(n+a)^s} \qquad (a > 0,\ \Re(s) > 0)\,. \tag{20}$$

Here $\varphi(t) = \sum_{n=0}^{\infty} e^{-(n+a)t} = \dfrac{e^{-at}}{1 - e^{-t}}$. But for any x we have the expansion

$$\frac{e^{xt}}{e^t - 1} \sim \sum_{n=0}^{\infty} \frac{B_n(x)}{n!} t^n \qquad (t \to 0)\,, \tag{21}$$

where the $B_n(x)$ are the *Bernoulli polynomials*

$$B_n(x) = \sum_{r=0}^{n} \binom{n}{r} B_r\, x^{n-r} \tag{22}$$

($B_0(x) = 1$, $B_1(x) = x - \frac{1}{2}$, $B_2(x) = x^2 - x + \frac{1}{6}, \ldots$). We deduce that $\zeta(s,a)$ has a meromorphic continuation in s with a simple pole of residue 1 (independent of a) at $s = 1$, and, generalizing (8), that its values at non-positive integers are given by

$$\zeta(s,-n) = -\frac{B_{n+1}(a)}{n+1} \qquad (n = 0,\, 1,\, 2, \ldots)\,. \tag{23}$$

3. Application: the Casimir effect

In the study of the Casimir effect, described in Chapter 6 of this book, one encounters the "function" defined by the series

$$F(\lambda) \;=\; -\,2\pi \sum_{\ell,\,m,\,n\in\mathbb{Z}} \sqrt{\ell^2+m^2+\lambda^2 n^2}\,, \tag{24}$$

where λ is a positive real variable; the factor -2π here has been included for later convenience. Of course the series is divergent. There are several natural questions:

A. How can $F(\lambda)$ be defined rigorously?
B. How can $F(\lambda)$ be computed effectively for a given $\lambda > 0$?
C. How does $F(\lambda)$ behave asymptotically as $\lambda \to 0$ and as $\lambda \to \infty$?

(For the analysis of the Casimir effect, it is the asymptotics at $\lambda \to \infty$ which are important.) Using the ideas explained in the previous two sections, we will show that the answers are as follows:

A. For complex s with $\Re(s) > \frac{3}{2}$, define

$$Z(\lambda, s) \;=\; \sum_{\ell,\,m,\,n\in\mathbb{Z}}{}' \frac{1}{(\ell^2+m^2+\lambda^2 n^2)^s} \qquad (\lambda>0,\; s\in\mathbb{C},\; \Re(s)>\frac{3}{2}) \tag{25}$$

(the prime on the summation sign means that the term $(\ell, m, n) = (0, 0, 0)$ is to be omitted), a so-called Epstein zeta function. Then $Z(\lambda, s)$ has a meromorphic continuation to all s, with a simple pole of residue $2\pi/\lambda$ at $s = 3/2$ and no other poles, and satisfies the functional equation

$$Z^*(\lambda, s) \;=\; \frac{1}{\lambda}\, Z^*\big(\frac{1}{\lambda}, \frac{3}{2}-s\big)\,, \tag{26}$$

where

$$Z^*(\lambda, s) \;:=\; \pi^{-s}\,\Gamma(s)\, Z(\lambda, s)\,. \tag{27}$$

Then one makes sense of (24) by setting $F(\lambda) = Z^*(\lambda, -\frac{1}{2})$. (Note that $\pi^{1/2}\,\Gamma(-\frac{1}{2}) = -2\pi$.)

B. The value of the function $F(\lambda)$ is given for any positive real number T by

$$F(\lambda) \;=\; \frac{1}{\sqrt{T}} \sum_{\ell,\,m,\,n} \gamma_{-\frac{1}{2}}\big(\pi T(\ell^2+m^2+\lambda^2 n^2)\big) \;+\; \frac{1}{\lambda T^2} \sum_{\ell,m,n} \gamma_2\big(\frac{\pi}{T}(\ell^2+m^2+\lambda^{-2} n^2)\big)\,, \tag{28}$$

where the sums are taken over all triples $(\ell, m, n) \in \mathbb{Z}^3$ and the functions $\gamma_{-\frac{1}{2}}(x)$ and $\gamma_2(x)$ are defined by the formulas

$$\gamma_{-\frac{1}{2}}(x) \;=\; \int_1^\infty e^{-xt}\, \frac{dt}{t^{3/2}} \qquad (x \ge 0) \tag{29}$$

(a variant of the error function) and

$$\gamma_2(x) \;=\; \begin{cases} \big(\dfrac{1}{x} + \dfrac{1}{x^2}\big)\, e^{-x} & \text{if } x > 0, \\ -\dfrac{1}{2} & \text{if } x = 0, \end{cases} \tag{30}$$

respectively. Since both $\gamma_{-\frac{1}{2}}(x)$ and $\gamma_2(x)$ are $\mathrm{O}(e^{-x})$ as $x \to \infty$, formula (28) makes $F(\lambda)$ rapidly computable. More precisely, if we choose $T = \lambda^{-2/3}$ then there are (uniformly in λ) only $\mathrm{O}(M^{3/2})$

terms in the two sums in (28) for which the arguments of $\gamma_{-\frac{1}{2}}$ or γ_2 are $\le M$, so that a relatively small number of terms suffices to compute $F(\lambda)$ to high precision. Here are some sample values:

t	$F(t)$
.1	6.21115704963445320831277821363781739171176675371...
.5	1.74490666842235054522002968176940979179901592336...
1	1.67507382139216375677378965854995727774709002078...
2	3.19240228274691182863701405594738286405890947611...
10	220.76228779131783503658735903111328294590024733 7...

(31)

Each of these numbers was computed independently using (28) for several different values of T. The fact that the answers agreed to the precision given, even though the individual terms of the sums are completely different, gives a high degree of confidence in the correctness of the theoretical and numerical calculations.

Of course we could also use the functional equation (26) to give a much simpler convergent series expansion for $F(\lambda)$ than (28), namely

$$F(\lambda) \;=\; \frac{1}{\lambda}\, Z^*\big(\frac{1}{\lambda},2\big) \;=\; \frac{\lambda^3}{\pi^2} \sum_{\ell,\,m,\,n}{}' \frac{1}{(\lambda^2\ell^2+\lambda^2m^2+n^2)^2}\,, \tag{32}$$

but this formula would be useful for practical purposes because of the slow convergence: summing over $|\ell|$, $|m|$, $|n| \le N$ involves $\mathrm{O}(N^3)$ terms and gives an error of the order of $1/N$, so that we would need some 10^{10} terms to achieve even three digits of precision.

C. The value of $F(\lambda)$ is given for small λ by

$$F(\lambda) \;=\; \frac{2C}{3}\,\lambda^{-1} \,+\, \frac{\pi}{3}\,\lambda \,+\, \mathrm{O}\big(\lambda e^{-\pi/\lambda}\big) \qquad (\lambda \to 0) \tag{33}$$

and for large λ by

$$F(\lambda) \;=\; \frac{\pi^2}{45}\,\lambda^3 \,+\, C' \,+\, \mathrm{O}\big(\sqrt{\lambda}\, e^{-\pi\lambda}\big) \qquad (\lambda \to \infty) \tag{34}$$

where $C = 1-\frac{1}{9}+\frac{1}{25}-\cdots = L_4(2)$ (with L_4 as in $\tilde{(}19)$) is Catalan's constant and $C' = \frac{2}{\pi}\zeta(\frac{3}{2})L_4(\frac{3}{2})$. The values obtained for $\lambda = .1$ and $\lambda = 10$ by retaining only the first two terms in equations (33) or (34), respectively, are

$$F(.1) \;\approx\; 6.21115704963445320831277821232521083463423480852\,,$$
$$F(10) \;\approx\; 220.762287791317835036587358989193358212711493834\,,$$

extremely close to the exact values for these two numbers given above.

Let us now prove each of these assertions.

A. From (5) and the fact that the generalized Mellin transform of the constant function 1 vanishes, we deduce that the function $Z^*(\lambda,s)$ defined by (27) equals the Mellin transform $\widetilde{\varphi_\lambda}(s)$ of the function

$$\varphi_\lambda(t) \;=\; \sum_{\ell,\,m,\,n\in\mathbb{Z}} e^{-\pi(\ell^2+m^2+\lambda^2n^2)t} \;=\; \vartheta(t)^2\,\vartheta(\lambda^2 t) \qquad (\lambda,\ t>0)\,,$$

where $\vartheta(t)$ is the theta series defined in (9). The functional equation (12) of $\vartheta(t)$ implies the functional equation $\varphi_\lambda(t) = \lambda^{-1}t^{-3/2}\varphi_{1/\lambda}(t^{-1})$ of $\varphi_\lambda(t)$, and the meromorphic continuation, description of poles and funcctional equation (26) of $Z^*(s)$ then follow as in Example 4 above.

B. Since the function $\varphi(t)=\varphi_\lambda(t)$ equals 1 to all orders in t as $t\to\infty$ and (by virtue of its functional equation) equals $\lambda^{-1}t^{-3/2}$ to all orders in t as $t\to 0$, the two pieces of the decomposition ($r\frac{1}{2}$) of its Mellin transform are given by

$$\widetilde{\varphi}_{\lambda,\geq T}(s) \;=\; \int_T^\infty \bigl(\varphi_\lambda(t)\,-\,1\bigr)\,t^{s-1}\,dt \;-\; \frac{T^s}{s} \tag{35}$$

and

$$\widetilde{\varphi}_{\lambda,\leq T}(s) \;=\; \int_0^T \bigl(\varphi_\lambda(t)\,-\,\lambda^{-1}\,t^{-3/2}\bigr)\,t^{s-1}\,dt \;+\; \frac{\lambda^{-1}\,T^{s-\frac{3}{2}}}{s-\frac{3}{2}}$$

respectively. Using the functional equation of $\varphi_\lambda(t)$, we see that these are exchanged when we replace λ, s and T by λ^{-1}, $\frac{3}{2}-s$ and T^{-1}, respectively (again making the functional equation of $Z^*(\lambda,s)=\widetilde{\varphi_\lambda}(s)$ evident), so we only have to study $\widetilde{\varphi}_{\lambda,\geq T}(s)$. Substituting the definition of $\varphi_\lambda(t)$ into (35), we find

$$\widetilde{\varphi}_{\lambda,\geq T}(s) \;=\; T^s \sum_{\ell,m,n}{}' \gamma_s\bigl(\pi T(\ell^2+m^2+\lambda^2 n^2)\bigr) \;-\; \frac{T^s}{s}\,,$$

where $\gamma_s(x)$, essentially the incomplete gamma function, is defined for $x>0$ by

$$\gamma_s(x) \;=\; \int_1^\infty t^{s-1}\,e^{-xt}\,dt \qquad\qquad (x>0)\,.$$

The extra term $-T^s/s$ in this formula can be omitted if we drop the prime from the summation signs and define the value of $\gamma_s(0)$ as $-1/s$ (which is indeed the limiting value of $\gamma_s(x)$ as $x\to 0$ if $\Re(s)<1$). The final result, with this convention for $\gamma_s(0)$, is therefore

$$Z^*(\lambda,\,s) \;=\; T^s \sum_{\ell,m,n} \gamma_s\bigl(\pi T(\ell^2+m^2+\lambda^2n^2)\bigr) \;+\; \Bigl[s\mapsto \frac{3}{2}-s,\; \lambda\mapsto\lambda^{-1},\; T\mapsto T^{-1}\Bigr]\,.$$

The special case $s=-\frac{1}{2}$ gives the expansion (28).

C. The leading term in the expansion of $F(\lambda)$ when λ is very big or very small can be obtained from equation (32): if λ is small then the dominating terms in (32) are those with $n=0$, so $F(\lambda)$ is asymptotically equal to $\pi^{-2}\lambda^{-1}\sum'_{\ell,m}(\ell^2+m^2)^{-2} = 4\pi^{-2}\zeta(2)L_4(2)\lambda^{-1}$ (here we have used that $\frac{1}{4}\sum'_{\ell,m}(\ell^2+m^2)^{-s}$ is the Dedekind zeta function of the field $\mathbb{Q}(i)$, which factors as $\zeta(s)$ times $L_4(s)$), while for large λ the dominating terms are those with $\ell=m=0$ and we get $F(\lambda)\sim 2\pi^{-2}\zeta(4)\lambda^3$. To get a more precise estimate, we use equation (28).

Consider first the case $\lambda\to 0$, and choose T in (28) so that $T\to\infty$, $\lambda^2T\to 0$. (The best choice will turn out to be $T=\lambda^{-1}$.) Then in view of the exponential decay $\gamma_s(x)=\mathrm{O}_s(x^{-1}e^{-\pi x})$ of $\gamma_s(x)$ when $x\to\infty$, we find that the only terms in (28) which are not exponentially small are those with $(\ell,m)=(0,0)$ in the first sum and those with $n=0$ in the second one. Hence

$$\begin{aligned} F(\lambda) \;&=\; \frac{1}{\sqrt{T}}\biggl[f_1(\lambda^2T)\,+\,\mathrm{O}\Bigl(\frac{1}{T}\sum_{\ell,m}{}' e^{-\pi T(\ell^2+m^2)}\cdot\sum_n e^{-\pi\lambda^2Tn^2}\Bigr)\biggr] \\ &\quad+\;\frac{1}{\lambda T^2}\biggl[f_2\bigl(\frac{1}{T}\bigr)\,+\,\mathrm{O}\Bigl(\sum_{\ell,m} e^{-\pi(\ell^2+m^2)/T}\cdot\lambda^2T\sum_n{}' e^{-\pi n^2/\lambda^2T}\Bigr)\biggr] \\ &=\; \frac{1}{\sqrt{T}}\,f_1(\lambda^2T)\,+\,\frac{1}{\lambda T^2}\,f_2\bigl(\frac{1}{T}\bigr)\,+\,\mathrm{O}\Bigl(\frac{1}{\lambda T^2}\,e^{-\pi T}\,+\,\lambda\,e^{-\pi/\lambda^2T}\Bigr)\,, \end{aligned} \tag{36}$$

where

$$f_1(\varepsilon) \;=\; \sum_{n\in\mathbb{Z}} \gamma_{-1/2}(\pi\varepsilon n^2)\,, \qquad f_2(\varepsilon) \;=\; \sum_{\ell,\,m\in\mathbb{Z}} \gamma_2(\pi\varepsilon(\ell^2+m^2))\,.$$

From the definition (29) and the functional equation of $\vartheta(x)$ we find (for ε small)

$$\begin{aligned} f_1(\varepsilon) \;&=\; \int_1^\infty t^{-3/2}\,\vartheta(\varepsilon t)\,dt \;=\; \sqrt{\varepsilon}\int_\varepsilon^\infty \vartheta(t)\,\frac{dt}{t^{3/2}} = \sqrt{\varepsilon}\int_0^{1/\varepsilon} \vartheta(x)\,dx \\ &=\; \frac{1}{\sqrt{\varepsilon}} \,+\, \sqrt{\varepsilon}\int_0^{1/\varepsilon}\bigl(\vartheta(x)-1\bigr)\,dx \;=\; \frac{1}{\sqrt{\varepsilon}} \,+\, \sqrt{\varepsilon}\left(\frac{\pi}{3} \,+\, \mathrm{O}\bigl(e^{-\pi/\varepsilon}\bigr)\right), \end{aligned}$$

since

$$\int_0^\infty \bigl(\vartheta(x)-1\bigr)\,dx \;=\; 2\sum_{n=1}^\infty \int_0^\infty e^{-\pi n^2 x}\,dx \;=\; \frac{2}{\pi}\sum_{n=1}^\infty \frac{1}{n^2} \;=\; \frac{\pi}{3}\,.$$

Similarly, noting the exceptional case $x=0$ in the definition (30) of $\gamma_2(x)$, we find

$$\begin{aligned} f_2(\varepsilon) \,+\, \frac12 \;&=\; \int_1^\infty t\,\bigl(\vartheta(\varepsilon t)^2-1\bigr)\,dt \;=\; \frac{1}{\varepsilon^2}\int_\varepsilon^\infty x\,(\vartheta(x)^2-1)\,dx \\ &=\; \frac{1}{\varepsilon^2}\left(\int_0^\infty x\,(\vartheta(x)^2-1)\,dx \,-\, \int_0^\varepsilon (1-x+\mathrm{O}(e^{-\pi/x}))\,dx\right) \\ &=\; \frac{2C}{3}\,\varepsilon^{-2} \,-\, \varepsilon^{-1} \,+\, \frac12 \,+\, \mathrm{O}(e^{-\pi/\varepsilon})\,, \end{aligned}$$

where this time we have used

$$\int_0^\infty x\,\bigl(\vartheta(x)^2-1\bigr)\,dx \;=\; {\sum_{\ell,\,m}}' \int_0^\infty x e^{-\pi(\ell^2+m^2)x}\,dx \;=\; \frac{1}{\pi^2}{\sum_{\ell,\,m}}' \frac{1}{(\ell^2+m^2)^2} \;=\; \frac{4}{\pi^2}\,\zeta(2)L_4(2)\,.$$

Inserting these formulas into (36), we find

$$\begin{aligned} F(\lambda) \;&=\; \frac{1}{\sqrt{T}}\left[\frac{1}{\lambda\sqrt{T}} \,+\, \frac{\pi}{3}\,\lambda\sqrt{T} \,+\, \mathrm{O}(\lambda\sqrt{T}\,e^{-\pi/\lambda^2 T})\right] \\ &\quad+\, \frac{1}{\lambda T^2}\left[\frac{2C}{3}\,T^2 \,-\, T \,+\, \mathrm{O}(e^{-\pi T})\right] \,+\, \mathrm{O}\Bigl(\frac{1}{\lambda T^2}\,e^{-\pi T} \,+\, \lambda\,e^{-\pi/\lambda^2 T}\Bigr) \\ &=\; \frac{2C}{3}\,\lambda^{-2} \,+\, \frac{\pi}{3}\,\lambda \,+\, \mathrm{O}\Bigl(\frac{1}{\lambda T^2}\,e^{-\pi T} \,+\, \lambda\,e^{-\pi/\lambda^2 T}\Bigr)\,. \end{aligned}$$

(Note how the two non-exponentially small terms which depend on T cancel, as they have to.) Taking $T = 1/\lambda$ gives (33).

The proof of (34) is completely analogous. From (28) and the exponential decay of $\gamma_s(x)$ we get

$$F(\lambda) \;=\; \frac{1}{\sqrt{T}}\,f_3(T) \,+\, \frac{1}{\lambda T^2}\,f_4\Bigl(\frac{1}{\lambda^2 T}\Bigr) \,+\, \mathrm{O}\Bigl(\frac{1}{\lambda^2 T^{5/2}}\,e^{-\pi\lambda^2 T} \,+\, \frac{1}{T^{1/2}}\,e^{-\pi/T}\Bigr)$$

as $\lambda\to\infty$, where $T\to 0$ with $\lambda^2 T\to\infty$ and where $f_3(\varepsilon)$ are $f_4(\varepsilon)$ are defined by

$$f_3(\varepsilon) \;=\; \sum_{\ell,\,m\in\mathbb{Z}} \gamma_{-1/2}(\pi\varepsilon(\ell^2+m^2))\,, \qquad f_4(\varepsilon) \;=\; \sum_{n\in\mathbb{Z}} \gamma_2(\pi\varepsilon n^2)\,.$$

This time we find the expansions

$$f_3(\varepsilon) \;=\; \frac{2}{3}\,\varepsilon^{-1} \,+\, C'\,\varepsilon^{1/2} \,+\, \mathrm{O}(e^{-\pi/\varepsilon})\,, \qquad f_4(\varepsilon) \;=\; \frac{\pi^2}{45}\,\varepsilon^{-2} \,-\, \frac{2}{3}\varepsilon^{-1/2} \,+\, \mathrm{O}(\sqrt{\varepsilon}\,e^{-\pi/\varepsilon})$$

for ε small, the non-exponentially small terms which depend on T again cancel, and taking $T = 1/\lambda$ to make the error as small as possible we obtain (34). The details are left to the reader.

As a final comment, we mention that the constants C and C' occurring in (33) and (34) can be evaluated rapidly by the same method as in **B**, with $\vartheta(t)^2$ instead of $\vartheta(t)^2\vartheta(\lambda t)$, using the fact that the Dirichlet series $4\zeta(s)L_4(s)$ is the Mellin transform of $\vartheta(t)^2$. Their numeical values are

$$\begin{aligned} C \;&=\; 0.91596559417721901505460351493238411077414937428167213 4\cdots\,, \\ C' \;&=\; 1.43774554488764350693200343638999965018484034037993399 7\cdots\,. \end{aligned}$$

4. Asymptotics of sums of the form $\sum f(nt)$

In this section we describe an extremely useful, and not sufficiently well known, asymptotic formula for functions given by expansions of the form

$$g(t) \;=\; f(t) \,+\, f(2t) \,+\, f(3t) \,+\, \cdots\,, \tag{37}$$

where $f:\,(0,\infty) \to \mathbb{C}$ is a smooth function of sufficiently rapid decay to ensure the convergence of the series (say $f(t) = \mathrm{O}(t^{-1-\varepsilon})$ as $t \to \infty$) and having a known asymptotic expansion at $t = 0$. In the simplest situation, we assume that f has a power series expansion (which may be only asymptotic rather than convergent, i.e. f need only be differentiable rather than analytic at the origin)

$$f(t) \;\sim\; \sum_{n=0}^{\infty} b_n\,t^n \qquad\quad (t \to 0)\,. \tag{38}$$

First, let us try to guess what the answer should be. On the one hand we can argue à la Euler, simply substituting the expansion (38) into (37) and interchanging the order of summation, without worrying about convergence problems. This gives

$$g(t) \sim \sum_{m=1}^{\infty}\sum_{n=0}^{\infty} b_n\,(mt)^n \;=\; \sum_{n=0}^{\infty} b_n \left(\sum_{m=1}^{\infty} m^n\right) t^n \;=\; \sum_{n=0}^{\infty} b_n\,\zeta(-n)\,t^n\,. \tag{39}$$

Of course this calculation is meaningless, since not only is the interchange of summation not permitted, but each of the interior sums $\sum_m m^n$ is divergent. Nevertheless, we know from Section 2, and Euler knew non-rigorously in 1749, that the numbers $\zeta(-n)$ do make sense and are certain rational numbers given by equation (8), so that at least the final expression in (39) makes sense as a formal power series. Alternatively, we can proceed à la Riemann and consider (37) for t small as $1/t$ times an approximation to the integral

$$I_f \;=\; \int_0^{\infty} f(t)\,dt\,. \tag{40}$$

This suggests instead that the correct asymptotic expansion of $g(t)$ near 0 should be given by

$$g(t) \;\sim\; \frac{I_f}{t} \qquad\quad (t \to 0)\,, \tag{41}$$

and indeed this formula is true, by the very definition of Riemann integrals as limits of sums, if we interpret the symbol of asymptotic equality "$\sim$" in its weak sense, as saying simply that the ratio of the expressions on its left and right tends to 1 as $t \to 0$. However, we are using "$\sim$" in its strong sense, where $g(t) \sim \sum_\lambda a_\lambda t^\lambda$ means that the difference between $g(t)$ and the finite sum $\sum_{\lambda<C} a_\lambda t^\lambda$ is $\mathrm{O}(t^C)$ as $t \to 0$ for any value of C, no matter how large. In this stronger sense, neither (39) nor (41) gives the correct asymptotic development of g. Remarkably enough, however, their sum does give the right answer:

Proposition 3. *Let f be a C^∞ function on the positive real line which has the asymptotic development* (38) *at the origin and, together with all its derivatives, is of rapid decay at infinity. Then the function $g(t)$ defined by* (37) *has the asymptotic development*

$$g(t) \;\sim\; \frac{I_f}{t} + \sum_{n=0}^{\infty} b_n \,\frac{B_{n+1}}{n+1}\,(-t)^n \tag{42}$$

as $t \to 0$, where I_f is defined by equation (40).

Remark. The relationship of Proposition 3 to the discussion of the Mellin transform in §§1–3 is very easy to describe. Suppose for simplicity that $f(t)$ is of rapid decay as $t \to \infty$. Then (38) implies that the Mellin transform $\widetilde{f}(s)$, defined for $\Re(s) > 0$ as $\int_0^\infty f(t)t^{s-1}dt$, has a meromorphic continuation to all s with simple poles of reside b_n at $s = -n$ ($n = 0, 1, 2, \dots$) and with no other singularities. But the transformation rules (2) of the Mellin transform imply that the Mellin transform $\widetilde{g}(s)$ of $g(t)$ equals $\zeta(s)\widetilde{f}(s)$, so it has (at most) simple poles of residue $\widetilde{f}(1) = I_f$ at $s = 1$ and residue $\zeta(-n)b_n$ at $s = -n$ ($n = 0, 1, 2, \dots$). Since $g(t)$ is small at infinity, it follows that, if $g(t)$ has an asymptotic expansion as $t \to 0$ of the form $\sum_j a_j t^{\alpha_j}$ (or even $\sum_j a_j t^{\alpha_j}(\log t)^{m_j}$), then this expansion is necessarily given by eq. (42).

Proof. We begin by describing the Euler-Maclaurin summation formula. To state it, we need the Bernoulli polynomials $B_n(x)$, which can be described by the generating function (22), by the explicit formula (23) in terms of Bernoulli numbers, or, most beautiful, by the property that $\int_a^{a+1} B_n(x)dx = a^n$ for every a. (It is easy to see that there is only one polynomial $B_n(x)$ with this property for each n.) From any of these definitions we can deduce without difficulty that $B_n'(x) = nB_{n-1}(x)$ and that $B_n(x+1) - B_n(x) = nx^{n-1}$.

Now let f be a smooth function on the positive reals. Integration by parts and the fact that $B_1(0) = \frac{1}{2} = -B_1(0)$ but $B_{n+1}(1) = B_{n+1}(0) = B_{n+1}$ for $n \geq 2$ gives

$$\int_0^1 f^{(n)}(x)\,\frac{B_n(x)}{n!}\,dx \;=\; -\int_0^1 f^{(n+1)}(x)\,\frac{B_{n+1}(x)}{(n+1)!}\,dx \;+\; \begin{cases} \frac{1}{2}[f(0)+f(1)] & \text{if } n=0,\\ \frac{B_{n+1}}{(n+1)!}[f^{(n)}(1)-f^{(n)}(0)] & \text{if } n\geq 1 \end{cases}$$

and hence, by induction on N,

$$\int_0^1 f(x)\,dx \;=\; \frac{1}{2}[f(0)+f(1)] + \sum_{n=1}^{N-1}\frac{(-1)^n B_{n+1}}{(n+1)!}[f^{(n)}(1)-f^{(n)}(0)] + (-1)^N\int_0^1 f^{(N)}(x)\,\frac{B_N(x)}{N!}\,dx$$

for every integer $N \geq 1$. Replacing $f(x)$ by $f(x+m-1)$ and summing over $m = 1, 2, \dots, M$ gives

$$\int_0^M f(x)\,dx \;=\; \frac{f(0)}{2} + \sum_{m=1}^{M-1} f(m) + \frac{f(M)}{2} + \sum_{n=1}^{N-1}\frac{(-1)^n B_{n+1}}{(n+1)!}[f^{(n)}(M)-f^{(n)}(0)]$$
$$+\;(-1)^N\int_0^M f^{(N)}(x)\,\frac{\bar{B}_N(x)}{N!}\,dx \tag{43}$$

where $\bar{B}_N(x) = B_N(x - [x])$. This is the Euler-Maclaurin summation formula.

Now assume that f and each of its derivatives is of rapid decay at infinity, so that $\int_0^\infty |f^{(N)}(x)|\,dx$ converges. Since $\bar{B}_N(x)$ is periodic and hence bounded, we can let $M \to \infty$ in (43) to get

$$\sum_{m=1}^{\infty} f(m) \;=\; \int_0^\infty f(x)\,dx \,+\, \sum_{n=0}^{N-1} \frac{(-1)^n B_{n+1}}{(n+1)!}\, f^{(n)}(0) \,-\, (-1)^N \int_0^\infty f^{(N)}(x)\,\frac{\bar{B}_N(x)}{N!}\,dx\,.$$

Replacing $f(x)$ by $f(tx)$ and then x by x/t with $t > 0$ changes this formula to

$$\sum_{m=1}^{\infty} f(mt) \;=\; \frac{1}{t}\int_0^\infty f(x)\,dx \,+\, \sum_{n=0}^{N-1} \frac{(-1)^n B_{n+1}}{(n+1)!}\, f^{(n)}(0)\,t^n \,+\, (-t)^{N-1} \int_0^\infty f^{(N)}(x)\,\frac{\bar{B}_N(x/t)}{N!}\,dx\,.$$

The last integral is bounded as $t \to 0$ with N fixed for the same reason as before, so the final term is $\mathrm{O}_N(t^{N-1})$. Substituting $f^{(n)}(0) = n!\,b_n$ from (38), we get the desired asymptotic formula (42). $\square$

Before giving examples of Proposition 3, we mention three extensions to more general sums.

1. First, instead of (37) we can look at shifted sums of the form $g(t) = \sum_{m=0}^{\infty} f((m+a)t)$, where $a > 0$. Here the "Riemann Ansatz" and the "Euler Ansatz" predict $I_f t^{-1}$ and $\sum_{n=0}^{\infty} b_n \zeta(-n,a) t^n$ for the asymptotic expansion of g, and again the correct answer is the sum of these two:

$$\sum_{m=0}^{\infty} f((m+a)t) \;\sim\; \frac{I_f}{t} \,+\, \sum_{n=0}^{\infty} b_n\, \frac{B_{n+1}(a)}{n+1}\, t^n \tag{44}$$

(cf. equation (23)). The proof is similar to that of Proposition 3 and will be omitted. By taking rational values of a in (44) and forming suitable linear combinations, we can use this formula to give the asymptotic development of $\sum_{m\ge1} \chi(m) f(mt)$ as $t \to 0$ for any periodic function $\chi(m)$, such as a Dirichlet character.

2. Next, we can allow non-integral exponents of t in (38). If the expansion of $f(t)$ at the origin contains terms $b_\lambda t^\lambda$ with arbitrary real numbers $\lambda > -1$ (or complex numbers with real part greater than -1), then the formula for g need only be modified by adding the corresponding terms $b_\lambda \zeta(-\lambda) t^\lambda$. Terms with $\lambda < -1$ are not interesting since they can simply be subtracted from $f(t)$, since the sum $\sum_m b_m (mt)\lambda$ then converges absolutely. The limiting case $\lambda = -1$ is of interest because it occurs in various applications. Here the answer (proved most easily by taking one function, like $t^{-1}e^{-t}$, which has a $1/t$ singularity at the origin and for which $\sum f(mt)$ can be computed exactly) is

$$f(t) \sim \sum_{\lambda \ge -1} b_\lambda\, t^\lambda \quad\Rightarrow\quad \sum_{m=1}^{\infty} f(mt) \;\sim\; \frac{1}{t}\left(b_{-1}\,\log\frac{1}{t} \,+\, I_f^*\right) \,+\, \sum_{\lambda > -1} b_\lambda\, \zeta(-\lambda)\,(-t)^\lambda\,, \tag{45}$$

where $I_f^* \;=\; \int_0^\infty (f(t) - b_{-1}e^{-t}/t)dt$.

3. Finally, we can also allow terms of the form $t^\lambda (\log t)^n$ in the expansion of $f(t)$, the corresponding contribution to $g(t)$ being simply the nth derivative with respect to λ of $\zeta(-\lambda)t^\lambda$, e.g. a term $t^\lambda \log\frac{1}{t}$ in the expansion of $f(t)$ at 0 leads to a term $t^\lambda\big(\zeta(-\lambda)\log\frac{1}{t} + \zeta'(-\lambda)\big)$ in the expansion at 0 of $g(t)$. In particular, using the known value $\zeta'(0) = -\frac{1}{2}\log(2\pi)$ we find

$$f(t) \sim b\,\log\frac{1}{t} \,+\, \sum_{n=0}^{\infty} b_n t^n \quad\Rightarrow\quad \sum_{m=1}^{\infty} f(mt) \;\sim\; \frac{I_f}{t} \,-\, \frac{b}{2}\,\log\frac{2\pi}{t} \,+\, \sum_{n=0}^{\infty} b_n\,\zeta(-n)\,t^n\,. \tag{46}$$

We end by giving four examples—two easy and two harder—to illustrate how these asymptotic formulas work.

Example 1. Take $f(t) = e^{-\lambda t}$ with $\lambda > 0$. This function is smooth, small at infinity, and has an expansion (38) at $t = 0$ with $b_n = (-\lambda)^n/n!$. The integral I_f equals $1/\lambda$. Hence (42) gives $g(t) \sim \dfrac{1}{\lambda t} + \sum_{n=0}^{\infty} \dfrac{B_{n+1}}{(n+1)!} (\lambda t)^n$ as the asymptotic expansion of $g(t) = \sum_{m=1}^{\infty} f(mt) = \dfrac{1}{e^{\lambda t}-1}$, in accordance with the definition (6) of the Bernoulli numbers.

Example 2. Now take $f(t) = e^{-\lambda t^2}$ with $\lambda > 0$. This function is again smooth and small at infinity, and has an expansion (38) at $t = 0$ with $b_{2n} = (-\lambda)^n/n!$ and $b_n = 0$ for n odd. Since all Bernoulli numbers with odd indices > 1 vanish, the asymptotic expansion in (42) breaks off after two terms, and we find $g(t) = I_f/t - 1/2 + \mathrm{O}(t^N)$ for all N, with $I_f = \sqrt{\pi/4\lambda}$. In this case, of course, we know much more, because $g(t)$ is simply $\frac{1}{2}\bigl(\vartheta(\lambda t^2/\pi) - 1\bigr)$ with $\vartheta(t)$ as in eq. (9), and therefore equation (12) gives the much more precise statement $g(t) = I_f/t - 1/2 + \mathrm{O}\bigl(t^{-1}e^{-\pi^2/\lambda t^2}\bigr)$. The same applies to any function $f(t)$ whose expansion at $t = 0$ has only even powers of t. For such a function, the expansion (42) collapses to $g(t) \sim I_f/t - b_0/2$, but this is always just a weakening of the Poisson summation formula (13), because we can extend $f(t)$ by $f(t) = f(|t|)$ to a smooth even function on the real line and use (14) to get the exact formula

$$2g(t) + f(0) \;=\; \sum_{n\in\mathbb{Z}} f(nt) \;=\; \frac{1}{t}\sum_{n\in\zeta} \widehat{f}(nt) \;=\; \frac{2I_f}{t} + \frac{2}{t}\sum_{n=1}^{\infty} \widehat{f}(nt)\,,$$

and the smoothness of f implies that the function $\widehat{f}(y)$ decays at infinity more rapidly than any negative power of y. The right way to think of Proposition 3 is therefore as a replacement for the Poisson summation formula when one is confronted with a sum over only positive integers rather than a sum over all of $\mathbb{Z}$. Such sums are very much harder to study than sums over all integers—just think of the special values of the Riemann zeta function, where the numbers $\zeta(2k)$ can be obtained in closed form because they can be written as $\frac{1}{2}\sum'_{n\in\mathbb{Z}} n^{-k}$, while the numbers $\zeta(2k+1)$, which cannot be reduced to sums over all of $\mathbb{Z}$ in this way, are not known exactly.

Example 3. As our next example, we consider the function

$$g_k(q) \;=\; \sum_{n=1}^{\infty} \sigma_{k-1}(n)\, q^n \qquad (0 < q < 1)\,,$$

where k is an integer greater than 1 and $\sigma_{k-1}(n)$ denotes the sum of the $(k-1)$st powers of the divisors of a natural number n. In the theory of modular forms, it is shown that if k is even and larger than 2, then g_k satisfies the functional equation

$$-\frac{B_k}{2k} + g_k\bigl(e^{-2\pi t}\bigr) \;=\; \frac{(-1)^{k/2}}{t^k}\left(-\frac{B_k}{2k} + g_k\bigl(e^{-2\pi/t}\bigr)\right) \qquad (k = 4,\,6,\,8,\ldots;\; t > 0).$$

In particular, $g_k(e^{-2\pi t})$ has the terminating asymptotic expansion

$$g_k(e^{-2\pi t}) \;=\; -(-1)^{k/2}\frac{B_k}{2k}\,t^{-k} + \frac{B_k}{2k} + \mathrm{O}(t^N) \qquad (\forall N > 0) \tag{47}$$

as $t \to 0$ in these cases. Let us see how Proposition 3 permits us to recover this asymptotic formula using without knowing the modularity, and at the same time tells us why (47) fails for $k = 2$ or k odd and what replaces it in those cases.

We first note that

$$g_k(q) \;=\; \sum_{n=1}^{\infty}\Bigl(\sum_{m|n} m^{k-1}\Bigr)q^n \;=\; \sum_{m=1}^{\infty} m^{k-1}\bigl(q^m+q^{2m}+\cdots\bigr) \;=\; \sum_{m=1}^{\infty} m^{k-1}\,\frac{q^m}{1-q^m}$$

and hence that g_k can be written after a change of variables in the form

$$g_k\bigl(e^{-t}\bigr) \;=\; \frac{1}{t^{k-1}}\sum_{m=1}^{\infty} f_k(mt)\,, \qquad f_k(t) \;=\; \frac{t^{k-1}}{e^t-1}\,.$$

The function $f=f_k$ satisfies the hypotheses of Proposition 3, with integral

$$I_f \;=\; \int_0^\infty \frac{t^{k-1}}{e^t-1}\,dt \;=\; \int_0^\infty t^{k-1}\bigl(e^{-t}+e^{-2t}+\cdots\bigr)\,dt \;=\; (k-1)!\,\zeta(k)$$

and with Taylor expansion $f(t)=\sum_{r=0}^{\infty}\frac{B_r}{r!}t^{r+k-2}$ at zero. Proposition 3 therefore gives

$$g_k\bigl(e^{-t}\bigr) \;\sim\; \frac{(k-1)!\,\zeta(k)}{t^k} \;+\; \sum_{r=0}^{\infty}(-1)^{r+k}\,\frac{B_r}{r!}\,\frac{B_{r+k-1}}{r+k-1}\,t^{r-1} \qquad (t\to 0). \tag{48}$$

If k is even and ≥ 4, then all products B_rB_{r+k-1} with $r\neq 1$ vanish, since r and $r+k-1$ have opposite parity and all odd-index Bernoulli numbers except B_1 are zero. Thereofre only two terms of (48) survive, and replacing t by $2\pi t$ and using the well-known formula for $\zeta(k)$ in terms of B_k, we recover (47). If $k=2$, then the argument is the same except that now the $r=0$ term also gives a non-zero contribution, so that we find

$$g_2\bigl(e^{-2\pi t}\bigr) \;=\; \frac{1}{24t^2} \;-\; \frac{1}{4\pi t} \;+\; \frac{1}{24} \;+\; \mathrm{O}(t^N) \qquad (\forall N>0),$$

instead of (47), in accordance with the known near-modularity property of g_2. Finally, if k is odd then we still get an explicit asymptotic formula with rational coefficients, but it now no longer terminates. Thus for $k=3$ we find the expansion

$$g_3\bigl(e^{-t}\bigr) \;\sim\; \frac{2\,\zeta(3)}{t^3} \;-\; \frac{1}{12t} \;+\; \frac{t}{1440} \;+\; \frac{t^3}{181440} \;+\; \frac{t^5}{7257600} \;+\; \frac{t^7}{159667200} \;+\;\cdots$$

as $t\to 0$, even though g_k has no modularity properties in this case. We leave it as an exercise to the reader to calculate the corresponding expansion when $k=1$, where one has to use equation (45) instead of equation (42).

Example 4. As our final example, consider the function

$$P(q) \;=\; \prod_{m=1}^{\infty}\frac{1}{1-q^m} \qquad (|q|<1)\,,$$

the generating power series of the partition function. To study the behavior of the partition function, we need to know how $P(q)$ blows up as q approaches 1 from below (or, more generally, any root of unity from within the unit circle). Here again the known modularity properties of the function $P(q)$ imply the non-trivial functional equation

$$e^{\pi t/12}\,P\bigl(e^{-2\pi t}\bigr) \;=\; \sqrt{t}\,e^{\pi/12t}\,P\bigl(e^{-2\pi/t}\bigr) \qquad (t>0),$$

from which one immediately obtains the asymptotic expansion

$$\log P\big(e^{-2\pi t}\big) \;=\; \frac{\pi}{12t} \,+\, \frac{1}{2}\log t \,-\, \frac{\pi}{12}\,t \,+\, \mathrm{O}(t^N) \qquad (\forall N>0)\,. \tag{49}$$

To obtain this formula without knowing anything about the modularity of P, we observe that $\log P\big(e^{-t}\big)$ has the form $\sum_{m=1}^{\infty} f(mt)$ with $f(t) = -\log(1-e^{-t})$. This function is small at infinity, has integral $I_f = \zeta(2)$ (as one sees by integrating by parts once and then calculating as in Example 2), and has an asymptotic development

$$f(t) \;\sim\; \log\frac{1}{t} \,-\, \sum_{n=1}^{\infty} \frac{B_n}{n\cdot n!}\,t^n\,,$$

as one sees easily by differentiating once. Hence equation (47) applies and gives

$$\log P\big(e^{-t}\big) \;\sim\; \frac{\zeta(2)}{t} \,+\, \frac{1}{2}\log\frac{t}{2\pi} \,-\, \sum_{n=1}^{\infty}(-1)^n\,\frac{B_n}{n\cdot n!}\,\frac{B_{n+1}}{n+1}\,t^n\,.$$

Again all terms except for the first in the sum on the right vanish because n and $n+1$ have opposite parity, and replacing t by $2\pi t$ and using $\zeta(2)=\pi^2/6$ we recover (49). The same method using (44) with rational values of a lets us compute the exact asymptotics of $P(q)$ as q approaches any root of unity, recovering precisely the same result as that given by the modularity. Moreover, just as in the case of Example 2 for k odd, the method applies even when modularity fails. For example, if we define

$$P_2(q) \;=\; \prod_{m=1}^{\infty}\frac{1}{(1-q^m)^m} \qquad (|q|<1)\,,$$

a generating function that occurs in connection with the theory of plane partitions, then an analysis like the one just given for $P(q)$, but now with $f(t) = -t\log(1-e^{-t})$, produces the complete asymptotic expansion

$$P_2(e^{-t}) \;=\; c\,t^{1/12}\,e^{\zeta(3)/t^2}\left(1 \,-\, \frac{t^2}{2880} \,-\, \frac{17t^4}{12902400} \,-\, \cdots\right) \qquad (t\to 0)$$

with $c = e^{\zeta'(-1)} = 0.847536694\cdots$, and using (46) one can get the expansion of $P_2(\alpha e^{-t})$ for any root of unity α. Furthermore, for reasons similar to those which applied to g_k with k even, one finds that the corresponding expansions for the logarithm of $P_3(q) = \prod(1-q^n)^{-n^2}$ when q tends to a root of unity are terminating, even though there are no modularity properties in this case.

Warning. Equations numbering have to be checked, e.g., two eqs. (5), one now changed to $(4\frac{1}{2})$.