On Harder's $\mathrm{Sl}(2,\mathbb{R})$ - $\mathrm{Sl}(3,\mathbb{R})$ - identity

Appendix by Don Zagier

The object of this appendix to prove the experimentally obtained formula stated in Section 2.2.4 of the paper.

1 Notations and statement of the identity

Our notations differ slightly from those of the paper. Fix an integer $D \geq 0$. (This is the $d+2$ of the paper, but we assume neither D even nor $D > 2$.) We define coefficients

$$C_{a,b} \;=\; \sum_{\nu=0}^{\min(a,b)} (-1)^{a+b-\nu} \binom{a}{\nu} \binom{2D-a}{b-\nu} \qquad\qquad (0 \leq a,\, b \leq 2D)\,. \tag{1}$$

(This is $(-1)^a\, 2^D\, C^{(d)}_{a,b}$ in the notation of the paper.) For $n \in \mathbb{Z}$ and $z \in \mathbb{C}$ we define

$$\gamma_n(z) \;=\; \Gamma\Bigl(\frac{z+|n|+1}{2}\Bigr)^{-1} \sum_{k=0}^{|n|/2} (-1)^k \binom{|n|}{2k}\, \Gamma\bigl(k+\frac{1}{2}\bigr)\, \Gamma\Bigl(\frac{z+|n|}{2}-k\Bigr)\,. \tag{2}$$

In the paper the number $c(z,n) = i^{-n}\gamma_n(z-1)$ is used instead, but $\gamma_n(z)$ is slightly easier to work with since it is an even function of n and is real for z real, and the shift of z by 1 also simplifies some of the formulas. Harder's identity says that

$$\begin{aligned}\sum_{\substack{0\leq b\leq 2D\\ b\equiv m_\alpha \ (\mathrm{mod}\ 2)}} \; \sum_{\substack{0\leq c\leq 2D\\ c\equiv m_\beta \ (\mathrm{mod}\ 2)}} & i^{D+b+c}\, C_{2D,b}\, C_{b,c}\, C_{c,e}\, \gamma_{D-b}(z-D+1)\, \gamma_{D-c}(z) \\ &= \frac{2^{3D}\pi}{z} \Bigl(\delta_{e,2D} \;+\; (-1)^{m_\beta} \delta_{e,0}\Bigr)\end{aligned} \tag{3}$$

for any m_α, $m_\beta \in \mathbb{Z}/2\mathbb{Z}$ satisfying $m_\alpha + m_\beta \equiv D$, $e \in 2\mathbb{Z}$ satisfying $0 \leq e \leq 2D$, and $z \in \mathbb{Z}$ ($z = d - n_\alpha = n_\beta + 1$ in Harder's notation) satisfying $0 < z < D-1$ and $z \equiv m_\alpha$ (mod 2). In fact this identity is true without the restriction on the parity of $m_\alpha + m_\beta$ (but with the right-hand side replaced by 0 if $m_\alpha + m_\beta \not\equiv D$ (mod 2)) or of e, and for all complex numbers z (in the sense of meromorphic functions). If we restrict to integral values of z, then the individual terms on the left sometimes have poles and the identity does not make sense, but if $z \equiv m_\alpha$ (mod 2) then all terms are finite and the identity is true whenever $z > 0$, without the assumption $z < D-1$.

2 Properties of the coefficients $C_{a,b}$

The key property of the numbers (1) is the identity

$$\sum_{b=0}^{2D} C_{a,b}\, x^b \;=\; (x+1)^a\,(x-1)^{2D-a} \qquad (0 \le a \le 2D)\,, \tag{4}$$

which follows straight from the binomial theorem. This immediately implies the symmetry properties

$$C_{a,b} \;=\; (-1)^b\, C_{2D-a,b} \;=\; (-1)^a\, C_{a,2D-b} \qquad (0 \le a,\, b \le 2D) \tag{5}$$

(the first of which is also obvious from (1)). A further symmetry property is

$$\binom{2D}{a}\, C_{a,b} \;=\; \binom{2D}{b}\, C_{b,a}\,, \tag{6}$$

which follows either from (1) and the identity

$$\binom{2D}{a}\binom{a}{\nu}\binom{2D-a}{b-\nu} \;=\; \binom{2D}{b}\binom{b}{\nu}\binom{2D-b}{a-\nu}$$

or else from (4) and the generating function calculation

$$\sum_{a=0}^{2D}\sum_{b=0}^{2D}\binom{2D}{a} C_{a,b}\, x^a y^b \;=\; \sum_{a=0}^{2D}\binom{2D}{a}\, x^a (y{+}1)^a (y{-}1)^{2D-a} \;=\; (xy{+}x{+}y{-}1)^{2D}$$

Finally, substituting $x = \pm 1$ into (4) and using (6) gives two identities which we will use below:

$$\sum_{c=0}^{2D}\binom{2D}{c} C_{c,e} = 2^{2D}\delta_{e,2D}, \quad \sum_{c=0}^{2D}(-1)^c\binom{2D}{c} C_{c,e} = 2^{2D}\delta_{e,0} \qquad (0 \le e \le 2D) \tag{7}$$

Observe that (4) says simply that the $(2D{+}1)\times(2D{+}1)$ matrix $\big(C_{a,b}\big)_{0\le a,b\le 2D}$ is the image of the matrix $\left(\begin{smallmatrix}1 & 1\\ 1 & -1\end{smallmatrix}\right)$ under the $(2D)$th symmetric power map $GL(2) \to GL(2D+1)$. In particular, the square of this matrix is 2^{2D} times the identity.

3 Properties of the functions $\gamma_n(z)$

These are summarized in the following proposition and its corollary.

Proposition 1. *(i) For $n \in \mathbb{Z}$ and $z \in \mathbb{C}$ with $\Re(z) > 0$ we have the integral representation*

$$\gamma_n(z) \;=\; \int_{-\pi/2}^{\pi/2} e^{in\theta}\,(\cos\theta)^{z-1}\, d\theta\,. \tag{8}$$

(ii) For all $n \in \mathbb{Z}$ the function $\gamma_n(z)$ is given by the closed formula

$$\gamma_n(z) \;=\; \frac{\pi}{2^{z-1}}\,\frac{\Gamma(z)}{\Gamma\big(\frac{z+1+n}{2}\big)\Gamma\big(\frac{z+1-n}{2}\big)} \tag{9}$$

Collorary 1. *For n and z in $\mathbb{Z}$ the values of $\gamma_n(z)$ are given by*

$$\gamma_n(z) \;=\; \begin{cases} 2^{1-z}\pi \binom{z-1}{\frac{|n|+z-1}{2}} & \text{if } z \geq 1,\ z \equiv n+1 \pmod 2, \\ (-1)^{\frac{|n|-z}{2}}\, 2^{1-z}\, z^{-1} \,/\, \binom{\frac{|n|+z-1}{2}}{z} & \text{if } z \geq 1,\ z \equiv n \pmod 2, \\ (-1)^{\frac{|n|+z-1}{2}}\, 2^{-z}\, \pi \binom{\frac{|n|-z-1}{2}}{-z} & \text{if } z \leq 0,\ z \equiv n+1 \pmod 2, \\ \infty & \text{if } z \leq 0,\ z \equiv n \pmod 2. \end{cases} \tag{10}$$

In particular, $\gamma_n(z) = 0$ if $z \equiv n+1 \pmod 2$ and either $z < -|n|$ or $0 < z < |n|$.

Proof: Since the left- and right-hand sides of both (8) and (9) are even functions of n, we may assume that $n \geq 0$. Setting $m = 2k$ in (2) and using the beta integral followed by the substitution $t = \sin^2\theta$, we find

$$\begin{aligned} \gamma_n(z) \;&=\; \sum_{m=0}^{n} \binom{n}{m} \Re(i^m) \int_0^1 t^{(m-1)/2}\,(1-t)^{(z+n-m-2)/2}\,dt \\ &=\; 2\int_0^{\pi/2} \cos(n\theta)\,\cos(\theta)^{z-1}\,d\theta \end{aligned}$$

for $\Re(z) > 0$, establishing (8). If z is a positive integer congruent to $n+1$ modulo 2, then (8) gives

$$\gamma_n(z) \;=\; \frac{1}{2}\int_{-\pi}^{\pi} e^{in\theta}\,\Big(\frac{e^{i\theta}+e^{-i\theta}}{2}\Big)^{z-1}\,d\theta \;=\; \frac{\pi}{2^{z-1}}\binom{z-1}{\frac{z-1-n}{2}},$$

proving (9) for these values of z. But this is sufficient, since the quotient of each term in (2) by the right-hand side of (9) is a rational function of z and two rational function which agree at infinitely many arguments are equal. We can also give a purely combinatorial proof of (9), e.g., if $n = 2m \geq 0$ (the case of odd n is similar), then using the duplication formula of the gamma function we find

$$\frac{\gamma_n(z)}{\text{RHS of (9)}} \;=\; \binom{m-\frac{z+1}{2}}{m}^{-1} \sum_{k=0}^{m} \binom{m-\frac{1}{2}}{k}\binom{-\frac{z}{2}}{m-k} \;=\; 1\,.$$

The corollary follows immediately from (9) by computing the values or limiting values of the right-hand side as z tends to an integer. The second and third lines of (10) are identities given in Section 2.2.2 of the paper. ☐

4 Proof of the identity (3)

For $0 \leq c \leq 2D$ we define a meromorphic function $\widehat{\gamma}_c(z)$ by

$$\widehat{\gamma}_c(z) \;=\; \sum_{b=0}^{2D} i^{D+b+c}\, C_{c,b}\,\gamma_{D-b}(z-D+1)\,. \tag{11}$$

Note that this is a real function (in the sense that $\overline{\widehat{\gamma}_c(\overline{z})} = \widehat{\gamma}_c(z)$) because of the symmetry properties $C_{c,2D-b} = (-1)^c C_{c,b}$ and $\gamma_n(z) = \gamma_{-n}(z)$. The key fact is the identity

$$\widehat{\gamma}_c(z) \;=\; \frac{2^{D+1}\,\pi}{z\,\gamma_{D-c}(z)}\,. \tag{12}$$

To prove this, we may assume that $\Re(z) > D - 1$ since both sides of (12) are meromorphic functions of z. Then (8) and (4) together with the substitution $t = \cos\big(\frac{\pi}{4} - \frac{\theta}{2}\big)^2$ and the beta integral give

$$\begin{aligned}
\widehat{\gamma}_c(z) &= i^{D+c} \int_{-\pi/2}^{\pi/2} (\cos\theta)^{z-D}\, e^{iD\theta}\, (ie^{-i\theta}+1)^c\, (ie^{-i\theta}-1)^{2D-c}\, d\theta \\
&= 2^{z+D} \int_{-\pi/2}^{\pi/2} \cos\Big(\frac{\pi}{4} - \frac{\theta}{2}\Big)^{z+c-D} \sin\Big(\frac{\pi}{4} - \frac{\theta}{2}\Big)^{z+D-c} d\theta \\
&= 2^{z+D} \int_0^1 t^{(z+c-D-1)/2}\, (1-t)^{(z+D-c-1)/2}\, dt \\
&= 2^{z+D}\, \Gamma\Big(\frac{z+c-D+1}{2}\Big)\, \Gamma\Big(\frac{z-c+D+1}{2}\Big) \,/\, \Gamma(z+1)\,,
\end{aligned}$$

and together with (9) this proves the claim.

Combining equations (12) and (7), we obtain

$$\begin{aligned}
&\sum_{b=0}^{2D}\sum_{c=0}^{2D} i^{D+b\pm c}\, C_{2D,b}\, C_{b,c}\, C_{c,e}\, \gamma_{D-b}(z-D+1)\, \gamma_{D-c}(z) \\
&\quad = \sum_{c=0}^{2D} (\pm 1)^c \binom{2D}{c} \widehat{\gamma}_c(z)\, \gamma_{D-c}(z)\, C_{c,e} \qquad \Big(\text{by (6), since } C_{2D,b} = \binom{2D}{b}\Big) \\
&\quad = \frac{2^{D+1}\pi}{z} \sum_{c=0}^{2D} (\pm 1)^c \binom{2D}{c} C_{c,e} \qquad (\text{by (12)}) \\
&\quad = \frac{2^{3D+1}\pi}{z}\, \delta_{e,D\pm D} \qquad (\text{by (7)})\,,
\end{aligned}$$

and (3) follows by taking the real or imaginary part of the sum or difference of these two identities.