PROOF OF THE GAMMA CONJECTURE FOR FANO 3-FOLDS OF PICARD RANK ONE

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Dedicated to the memory of Andrei Andreevich Bolibrukh

Abstract. We verify the (first) Gamma Conjecture, which relates the gamma class of a Fano variety to the asymptotics at infinity of the Frobenius solutions of its associated quantum differential equation, for all of the 17 deformation classes of rank one Fano 3-folds. Doing this involves computing the corresponding limits ("Frobenius limits") for the Picard–Fuchs differential equations of Apéry type associated by mirror symmetry to the Fano families, and is achieved by two methods, one combinatorial and one using the modular properties of the differential equations. The Gamma Conjecture for Fano 3-folds always contains a rational multiple of the number $\zeta(3)$. We present numerical evidence suggesting that higher Frobenius limits of Apéry-like differential equations may be related to multiple zeta values.

Introduction

The goal of this paper is twofold. On the one hand, we will calculate certain coefficients of the transition, or central connection, matrix for the Laplace transform of a number of Apéry-like differential equations, finding the expansion coefficients of the most rapidly growing solution at infinity in terms of the Frobenius basis at 0. On the other hand, these calculations, the result of which involves the number $\zeta(3)$ each time, provides a verification of a prediction of mirror symmetry called the (first) Gamma Conjecture for each of the 17 deformation classes from the Iskovskikh classification of smooth Fano threefolds of Picard rank one.

We will present two approaches to computing the limits in question. One of them, which we carry out for the differential equation satisfied by the generating function of the numbers used by Apéry's in his famous proof of the irrationality of $\zeta(3)$, is based on the explicit hypergeometric formula that he gave for these numbers. This is the less satisfactory method, since it depends on complicated formulas that were found experimentally and whose proofs are somewhat artificial, but has nevertheless been included because it could apply also in other situations that are not modular, like the differential equations associated to most Fano 4-folds. It also works almost automatically whenever the differential equation is of hypergeometric type, which is the case for 10 of the 17 cases on Iskovskikh's list. The second method is based on the modular parametrizations of the differential equations in question, like the one found by Beukers many years ago for the Apéry case. This method is much nicer and explains the crucial constant $\zeta(3)$ as a period of an Eisenstein series. It works in a uniform way for each of the differential equations admitting a modular parametrization, which holds for 15 of the 17 families, including all of the non-hypergeometric ones. This gives our main result:

Theorem 1. *The Gamma Conjecture holds for all Fano 3-folds of Picard rank one.*

Since mes of the paper cover a wide spectrum and may not all be known to the same readers, we will include in Chapter 1 a review of the main ingredients of the story (Fano varieties, Iskovskikh classification, quantum differential equation, gamma class, Gamma Conjecture) for completeness. However, the actual calculations of the limits, which are given in Chapter 2, involve only classical tools from number theory and can be read independently of this material. In the rest of this introduction, we give a little more indication of the background of the problem and state the explicit prediction made by the Gamma Conjecture in the Apéry case.

Mirror symmetry predicts, among other things, that to each of the 17 families of Fano varieties in question there should be associated a family $\mathcal{E}$ of K3 surfaces over $\mathbb{P}^1$ (up to isogeny, it is the so-called Landau–Ginzburg model) in such a way that the "quantum differential equation" on the Fano side is the Laplace transform of the Picard–Fuchs differential equation satisfied by the periods of $\mathcal{E}$. These quantum differential equations, whose definition will be recalled briefly in §4, arise from counting embedded holomorphic curves (Gromov–Witten invariants), so that this is the "A-side" in the terminology of string theory (where one is usually interested in families of Calabi–Yau 3-folds), while the second family $\mathcal{E}$ with its Picard–Fuchs equation would be the "B-side." One further expects that the Picard rank of the Fano and that of the generic fiber of $\mathcal{E}$ add up to 20. For the 17 cases we are considering, these mirror symmetry predictions were made precise in [15] and proved in all cases in [15] and [27]. In each case, the family $\mathcal{E}$ is is a Kuga–Sato type family, with the base space equal to the modular curve[1] $X_0^*(N) = X_0(N)/W_N$ classifying unordered pairs (E, E') of N-isogenous elliptic curves for some N, and fibre equal to the smooth resolution of the quotient of $E \times E'$ by (-1), with Picard number $19 = 20 - 1$ as it should be (16 algebraic cycles coming from the resolutions of the 16 singularities and 3 more from the classes of E, E', and the graph of the isogeny). The fact that these families are of Kuga-Sato type means precisely that the solutions of their associated Picard–Fuchs differential equations have modular parametrizations. More specifically, the differential equation has a unique holomorphic solution $\Phi(t) = \sum_{n=0}^{\infty} A_n t^n$ at $t = 0$ with $A_0 = 1$, where t is a suitably chosen coordinate for the base space $\mathbb{P}^1$, and the modular parametrization says that $\Phi(t(\tau))^2$ is a modular form of weight 4 on $\Gamma_0^*(N)$ for some Hauptmodul $t(\tau)$ on $X_0^*(N)$. The Gamma Conjecture relates the asymptotics at infinity of the four Frobenius solutions of the differential equation satisfied by $\Psi(z) = \sum_{n=0}^{\infty} A_n z^n/n!$ to the so-called gamma class (a characteristic class in cohomology with real coefficients, whose definition will be recalled in §5) of the corresponding Fano 3-fold.

We now describe the picture in more detail in the Apéry case, corresponding to $N = 6$. Here we have

$$A_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2, \qquad \Phi(t) = 1 + 5t + 73t^2 + 1445t^3 + \cdots. \tag{0.1}$$

The corresponding recursion and differential equation are

$$(n+1)^3 A_{n+1} - P(n)A_n + n^3 A_{n-1} = 0, \quad \big(D^3 - tP(D) + t^2(D+1)^3\big)\,\Phi(t) = 0, \tag{0.2}$$

where $D = t\,\dfrac{d}{dt}$ and $P(n)$ is the polynomial $34n^3 + 51n^2 + 27n + 5 = (2n+1)(17n^2 + 17n + 5)$, while the modular parametrization, as found by Beukers [3], is given by

$$t = \Big(\frac{\eta(\tau)\eta(6\tau)}{\eta(2\tau)\eta(3\tau)}\Big)^{12} = q - 12q^2 + 66q^3 + \cdots, \qquad \Phi(t) = \frac{\big(\eta(2\tau)\eta(3\tau)\big)^7}{\big(\eta(\tau)\eta(6\tau)\big)^5} = 1 + 5q + 13q^2 + \cdots.$$

where $\eta(\tau)$ is the Dedekind eta-function and $q = e^{2\pi i\tau}$ as usual. For the Gamma Conjecture, we consider, not $\Phi(t)$, but the related power series

$$\Psi(z) = \sum_{n=0}^{\infty} A_n \frac{z^n}{n!} = 1 + 5z + \frac{73}{2}z^2 + \frac{1445}{6}z^3 + \cdots.$$

It again satisfies a linear differential equation, this time of order 4, with no singularities on $\mathbb{C}^*$, a regular singularity at 0 and an irregular one at infinity. (This is what we called the "Laplace transform" above, although "Borel transform" might be a better name.) The space of solutions of

[1] or a d–fold cover of it, where d is the "index", but in the Introduction we assume $d = 1$.

this transformed equation near $z = 0$ has the standard Frobenius basis $\Psi_j(z)$ $(0 \le j \le 3)$, where $\Psi_j(z)$ has a singularity like $(\log z)^j/j!$ near the origin. We define the *Frobenius limits* κ_j by

$$\kappa_j \ := \ \lim_{z\to\infty} \frac{\Psi_j(z)}{\Psi(z)} \,. \tag{0.3}$$

The Gamma Conjecture gives their values in terms of $\zeta(3)$ and the Chern numbers of one of the Fano varieties (the one called V_{12}) from the Iskovskikh list. Specifically, it predicts that

$$\kappa_1 \,=\, -\gamma\,, \qquad \kappa_2 \,=\, \frac{\gamma^2}{2} - \frac{3}{2}\zeta(2)\,, \qquad \kappa_3 \,=\, -\frac{\gamma^3}{6} + \frac{3}{2}\gamma\,\zeta(2) + \frac{5}{2}\zeta(3) \tag{0.4}$$

where γ is Euler's constant. (The easy explicit computation of the gamma class, here and for the other 16 cases, is given in Proposition 2 in Section 5.) This will be proved as a special case of our main theorem, Theorem 2 in Section 3, which gives the Frobenius limit in all seventeen cases.

We end by remarking that the Frobenius basis is part of a larger collection of functions $\Psi_j(z)$ for all $j \ge 0$, still with a singularity of type $(\log z)^j$ at the origin, but with Ψ_j for $j \ge 4$ satisfying an inhomogeneous version of the differential equation of $\Psi(z)$. The limits κ_j defined in (0.3) still exist and can be computed numerically to high precision by a method that will be explained in §9. We find that for $j \le 10$ each of these ratios is again a polynomial in γ and Riemann zeta values, but that this is false for κ_{11}, which involves a *multiple* zeta value as well. This point, although purely experimental at the moment, seems worth mentioning and will be described in more detail in the last section of the paper.

Since the main ingredient of our story is the monodromy of Fuchsian differential equations, we hope that it is a suitable homage to Andrei Andreevich Bolibrukh, who contributed so deeply to this subject.

Chapter 1. Fano varieties, Apéry-like differential equations, and mirror symmetry

In this chapter we will give a brief description of the Gamma Conjecture (see [23] and [12] for more detail) and a complete statement of the prediction it makes for Fano 3-folds whose Picard groups have rank 1. We also describe the associated Picard–Fuchs equations and their modular parametrizations, following [15].

1. Fano varieties and the Iskovskikh classification. A *Fano variety* X means in this paper a smooth complex projective algebraic variety with ample anticanonical class. The projective line $\mathbb{P}^1$ is the only Fano in dimension 1. Fanos in dimension 2 are called del Pezzo surfaces; these are either $\mathbb{P}^1 \times \mathbb{P}^1$ or blowups of $\mathbb{P}^2$ in $0 \le d \le 8$ points. By results of Mori and Mukai [25], there exist 105 deformation families of Fano 3-folds.

We will be interested in Fano 3-folds whose Picard lattices have rank 1. (Since degree 2 cohomology classes here are algebraic, this simply means that $H^2(X,\mathbb{Z}) \cong \mathbb{Z}$.) According to V. A. Iskovskikh [21, 22] (see also [20]), there are exactly 17 of these up to deformation. The relevant numerical invariants for this classification are the *index* $d = [H^2(X,\mathbb{Z}) : \mathbb{Z}\,c_1]$, where c_1 is the anticanonical divisor, and the *level* $N = \dfrac{1}{2d^2}\langle c_1^3, [X]\rangle$, which is always a positive integer. The 17 possible pairs of invariants (N,d) are then

$$\begin{gathered}(1,1),\ (2,1),\ (3,1),\ (4,1),\ (5,1),\ (6,1),\ (7,1),\ (8,1),\ (9,1),\ (11,1),\\ (1,2),\ (2,2),\ (3,2),\ (4,2),\ (5,2),\ (3,3),\ (2,4)\,.\end{gathered} \tag{1.1}$$

For instance, $(N,d) = (2,4)$ corresponds to the Fano variety $X = \mathbb{P}^3$ (so here the deformation family has a 0-dimensional base), for which the associated Picard–Fuchs differential equation is hypergeometric, while $(N,d) = (6,1)$, which is the family called V_{12} (whose geometric definition plays no role for us and will be omitted), corresponds to the Apéry differential equation as described in the Introduction and will be referred to hereafter as the "Apéry case."

2. Apéry-like differential equations. We will describe the Picard–Fuchs differential equations associated to our 17 cases in this section, their modular properties in the following one, and the relation of their Laplace transforms to the quantum cohomology of the Fano varieties in §4.

The differential operators occurring are of type D3. Here "type Dn" (the full name is "determinantal differential equations of order n") is a specific class of linear differential equations, introduced in [16], that includes the Picard–Fuchs differential equations of certain families of Calabi–Yau varieties of dimension $n-1$ and, in its Laplace-transformed version, the quantum differential equations of certain n-dimensional Fano varieties. The operators of type D2, which have the shape

$$D^2 \,+\, t\bigl(a_1\,D(D+1)\,+\,b_1\bigr) \,+\, a_2t^2\,(D+1)^2 + a_3t^3\,(D+1)(D+2) \qquad (D = t\,d/dt)\,,$$

are precisely the ones appearing in the "Apéry-like differential equations" studied in [4] and [32], of which the prototype was the order 2 differential equation coming from the coefficients used by Apéry in his new proof of the irrationality of $\zeta(2)$. Since the D3 class also contains the order 3 differential equation corresponding to Apéry's proof for $\zeta(3)$, we will use the terminology "Apéry-like equations of order n" as an alternative name for the class Dn. The generic D3 operator, which is the case that we will be studying, has the shape

$$\begin{aligned}\mathcal{L} \;=\; & D^3 \,+\, t\bigl(D+\tfrac12\bigr)\bigl(a_1\,D(D+1)\,+\,b_1\bigr) \,+\, t^2(D+1)\bigl(a_2(D+1)^2\,+\,b_2\bigr)\\ & +\, a_3t^3(D+1)(D+\tfrac32)(D+2) \,+\, a_4t^4(D+1)(D+2)(D+3)\,.\end{aligned} \tag{2.1}$$

This can also be written as tL where L is the differential operator

$$L \;=\; t^2Q\,\frac{d^3}{dt^3} \,+\, \frac32\,(t^2Q)'\,\frac{d^2}{dt^2} \,+\, \Bigl(\frac{t^2}{2}\,Q'' + 3tQ' + R\Bigr)\,\frac{d}{dt} \,+\, \Bigl(\frac{t}{2}\,Q'' + \tfrac12\,R'\Bigr)\,, \tag{2.2}$$

in which the polynomials

$$Q \;=\; Q(t) \;=\; 1 + a_1t + a_2t^2 + a_3t^3 + a_4t^4\,, \qquad R \;=\; R(t) \;=\; 1 + b_1t + b_2t^2 \tag{2.3}$$

of degree ≤ 4 and ≤ 2 describe the position of the singularities of the equation and the so-called accessory parameters, respectively.

In general, a Dn equation is obtained from an $(n+1)\times(n+1)$ matrix $\mathcal{A} = \bigl(a_{i,j}\bigr)_{0\le i,j\le n}$ whose coefficients (which in the quantum cohomology situation arise as correlators in a way recalled briefly in §4) satisfy

$$a_{i,j} \;=\; 0 \text{ for } j<i-1, \qquad a_{i,j} \;=\; 1 \text{ for } j=i-1, \qquad a_{i,j} \;=\; a_{n-j,n-i} \qquad (0\le i,\,j\le n),$$

so that this family of equations has $n^2/4\,+\,\mathrm{O}(n)$ parameters. The differential operator $\mathcal{L}$ corresponding to $\mathcal{A}$ and its Laplace transform $\widetilde{\mathcal{L}}$ (related to $\mathcal{L}$ by $\widetilde{\mathcal{L}}\Psi = 0 \Leftrightarrow \mathcal{L}\Phi = 0$ if $\Phi(t) = \sum A_nt^n$ and $\Psi(z) = \sum A_nz^n/n!$) are then given by

$$\mathcal{L} \;=\; D^{-1}\,\mathrm{det}_R\Bigl(\Bigl(\delta_{i,j}\,D \,-\, a_{i,j}\,(D\,t)^{j-i+1}\Bigr)_{0\le i,j\le n}\Bigr) \tag{2.4}$$

and (setting $D_z = z\,d/dz$)

$$\widetilde{\mathcal{L}} \;=\; \mathrm{det}_R\Bigl(\Bigl(\delta_{i,j}D_z \,-\, a_{i,j}z^{j-i+1}\Bigr)_{0\le i,j\le n}\Bigr)\,, \tag{2.5}$$

respectively, where the "right determinant" det_R of a matrix with non-commuting entries is defined inductively as the alternating sum of the right-most entries multiplied on the right by the right determinants of the corresponding minors. (The first formula makes sense because every term in the right-most column of the corresponding matrix is left divisible by D.) We will consider matrices $\mathcal{A}$ differing by a scalar cI as equivalent, because this corresponds merely to a translation $t^{-1} \mapsto t^{-1} + c$ on the t-side, or to multiplication by $\exp(cz)$ on the z–side, with no effect on the Frobenius limits in (0.3). We remark that the operators Dn are self-dual in the sense that the coefficient of t^i is $(-1)^{n-i}$-symmetric under $D \mapsto -D-i$. The equations of type D3 are also symmetric squares, which is important for us because being the symmetric power of a second order differential operator is a necessary condition for modularity. The corresponding assertion for higher Dn's is completely false, and indeed the D4 equations occurring as the Picard–Fuchs equations of families of Calabi–Yau 3-folds are almost never modular.

The 4×4 matrices corresponding to the 17 Iskovskikh families were listed in [15], e.g.

$$\begin{pmatrix} 5 & 96 & 1692 & 12816 \\ 1 & 12 & 216 & 1692 \\ 0 & 1 & 12 & 96 \\ 0 & 0 & 1 & 5 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 12/5 & 24 & 198 & 880 \\ 1 & 22/5 & 44 & 198 \\ 0 & 1 & 22/5 & 24 \\ 0 & 0 & 1 & 12/5 \end{pmatrix}$$

for the Apéry case $(N,d) = (6,1)$ and for the most complicated case $(N,d) = (11,1)$, in which the differential equation corresponds to a 5-term recursion for the coefficients of its holomorphic solution, and the scalar shifts are chosen so as to make the solutions become Eisenstein series. Instead of giving the matrices for the remaining cases, we give in Table 1 the polynomials Q and R as defined in (2.3) above, since these polynomials contain the same information and are much more compact to write. We give the data only for $d=1$, since there is a simple algebraic procedure to deduce the differential equation satisfied by a power series $\Phi(t^d)$ from the one satisfied by $\Phi(t)$. (In our cases, going from $(N,1)$ to (N,d) simply replaces $Q(t)$ by $Q(t^d)$, while the new R-polynomial equals $1+(4b_1-a_1)t^2$ for $d=2$ and just 1 for $d=3$ or 4, where a_1 and b_1 are the linear coefficients of the original Q and R.) The final three columns of the table contain certain invariants $(f_M)_{M|N}$, $(h_M)_{M|N}$, and $\mu_N = \frac{1}{2}\sum_M Mh_M$ that will be defined and explained in the next section (eq. (3.7)) and used later for the proof of the Gamma Conjecture.

Table 1

N	$Q(t)$	$R(t)$	$\{f_M\}$	$\{h_M\}$	μ_N
1	$1-1728t$	$1-240t$	—	—	62
2	$1-256t$	$1-48t$	$(24,-24)$	$(-80,80)$	40
3	$1-108t$	$1-24t$	$(12,-12)$	$(-30,30)$	30
4	$1-64t$	$1-16t$	$(8,0,-8)$	$(-16,0,16)$	24
5	$1-44t-16t^2$	$1-12t+4t^2$	$(6,-6)$	$(-10,10)$	20
6	$1-34t+t^2$	$1-10t$	$(5,-1,1,-5)$	$(-7,1,-1,7)$	17
7	$1-26t-27t^2$	$1-8t+3t^2$	$(4,-4)$	$(-5,5)$	15
8	$1-24t+16t^2$	$1-8t$	$(4,-2,2,-4)$	$(-4,1,-1,4)$	13
9	$1-18t-27t^2$	$1-6t$	$(3,0,-3)$	$(-3,0,3)$	12
11	$1-\frac{68}{5}t-\frac{616}{25}t^2-\frac{252}{125}t^3-\frac{1504}{625}t^4$	$1-\frac{24}{5}t-\frac{56}{25}t^2$	$\left(\frac{12}{5},-\frac{12}{5}\right)$	$(-2,2)$	10

3. Modular properties. That the operators we are studying are of type D3 is true by construction, but in fact they also have very specific modularity properties, discovered in [15], which we now describe.

For any integer $N\ge 1$, we have the modular curve $X_0(N)$, defined over $\mathbb{C}$ as $\mathbb{H}/\Gamma_0(N)\cup\{\text{cusps}\}$, the completed quotient of the upper half-plane by the level N congruence subgroup, and $X_0^*(N)$, the quotient of $X_0(N)$ by the Fricke involution W_N sending $\tau\in\mathbb{H}$ to $-1/N\tau$. We denote, as usual, the group generated by $\Gamma_0(N)$ and W_N by $\Gamma_0^*(N)$. As moduli spaces, $X_0(N)$ and $X_0^*(N)$ parametrize the ordered and unordered pairs, respectively, of elliptic curves related by a cyclic isogeny of degree N. The involution W_N acts of the space $M_k(\Gamma_0(N))$ of holomorphic modular forms of weight k on $\Gamma_0(N)$ by $(f|_kW_N)(\tau) = N^{k/2}\tau^k f(-1/N\tau)$ and splits this space into two eigenspaces $M_k^{\pm}(\Gamma_0(N))$, with $M_k^{+}(\Gamma_0(N)) = M_k(\Gamma_0^*(N))$ and $M_k^{-}(\Gamma_0(N)) = M_k(\Gamma_0^*(N),\chi)$, where $\chi:\Gamma_0^*(N)\to\{\pm1\}$ is the homomorphism sending $\Gamma_0(N)$ to $+1$ and W_N to -1. If $N=1$, then $W_N = S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ belongs to $\Gamma_0(N) = \mathrm{SL}(2,\mathbb{Z})$, so that W_N has only the eigenvalue $+1$. In this case, by abuse of notation we will write $F \in M_k^-(\Gamma_0(1))$ if $F\sqrt{E_4}$ is a modular form of weight $k+2$ on $\Gamma_0(1)$, where $E_4(\tau) = 1+240q+\cdots$ is the normalized Eisenstein series of weight 4. In that case F is not a one-valued function in $\mathbb{H}$, since E_4 has simple zeros at $\tau = \frac{\pm 1+i\sqrt{3}}{2}$, but it is well-defined and holomorphic in the union of the closed standard fundamental domain of $\mathrm{SL}(2,\mathbb{Z})$ and its image by S, which is closed under S and contains the fixed point i, and it satisfies the functional equation $(F|_k S) = -F$ in that domain.

We now formulate the main result describing the modular properties of the differential equations associated to the 17 families of the Iskovskikh list. The first statement in this theorem was proved in [15], which also contained the formulation of the second statement and a sketch of its proof in some cases. The remaining harder cases were checked by Przyjalkowski [27]. The ingredients used were the Quantum Lefschetz Theorem of Givental [14] together with the computations of the quantum multiplication by the first Chern class in grassmannians by Przyjalkowski [26] and by Fulton and Woodward [9], as well as Kuznetsov's calculation of GW invariants for varieties V_{22} (private communication) and Beauville's result on V_5 [2].

Theorem ([15], [27]). *Let N and d be natural integers, and suppose that $\Gamma_0^*(N)$ has genus 0 and that there is some modular form in $M_2(\Gamma_0(N))^-$ whose differential equation with respect to the d-th root of a Hauptmodul of $\Gamma_0^*(N)$ is of type D3. Then (N,d) belongs to the list (1.1) and the Laplace transform of this differential equation is, up to equivalence, the quantum differential equation (see Section 4 below) of the corresponding Fano variety from the Iskovskikh classification.*

The modular form F and Hauptmodul t occurring in this theorem can be given by a (nearly) uniform formula. If $N > 1$, we define a modular form $F_N \in M_2(\Gamma_0(N))^-$ as the unique Eisenstein series of weight 2 on $\Gamma_0(N)$ that takes on the values $+1$ at $\tau = \infty$, -1 at $\tau = 0$, and 0 at all other cusps. If $N = 1$, this definition makes no sense, but we can set $F_1 = \sqrt{E_4}$ and this again belongs to $M_2^-(\Gamma_0(1))$ in the sense just introduced. Then in every case the modular form $F = F_{N,d}$ and Hauptmodul $t = t_{N,d}$ whose existence is asserted by orem are given (after normalization by $F(\tau) = 1 + \mathrm{O}(q)$ and $t(\tau) = q + \mathrm{O}(q^2)$, and up to equivalences $t \mapsto t/(1+ct)$, $F \mapsto (1+ct)F$) by the uniform formulas

$$F_{N,d}(\tau) \;=\; F_N(d\tau)\,, \qquad t_{N,d}(\tau) \;=\; t_N(d\tau)^{1/d} \tag{3.1}$$

for all N and d, where $t_N(\tau)$ is the power series in q defined by

$$F_N(\tau)\, t_N(\tau)^{\frac{N+1}{12}} \;=\; \eta(\tau)^2\, \eta(N\tau)^2\,. \tag{3.2}$$

Conversely, it is an elementary exercise to show that the only pairs $(N,d) \in \mathbb{N}^2$ for which the function $t = t_{N,d}$ defined by (3.1) and (3.2) is a modular function on $\Gamma_0(dN)$ are those listed in (1.1). We sketch the argument. If N is sufficiently large then the well-known "valency formula" (the formula giving the number of zeros of a holomorphic modular form in a fundamental domain for the group) implies that the function F_N has at least one zero in $\mathbb{H}$ of order not divisible by $\frac{N+1}{(N+1,12)}$, and then the function $t_N(\tau)$ defined for $\mathrm{Im}(\tau)$ large by (3.2) does not even extend to a single-valued meromorphic function in the upper half-plane, let alone a modular function. Checking the remaining cases by computer, we find that the only N for which the needed root can be extracted are the ten values $N = 1,\ldots,9,11$ occurring in (1.1) and four further values $N = 12$, 16, 18 and 36, but in each of the latter cases the function $t_N(\tau)$ defined by (3.2) is not modular on $\Gamma_0(N)$. (In each of these four cases the function F_N is an eta-product and t^{N+1} is also an eta-product and is a modular function on $\Gamma_0^*(N)$ but t itself is not. For instance, for $N = 12$ then we find

$$F_{12}(\tau) \;=\; \frac{\eta(2\tau)^4\eta(3\tau)\eta(4\tau)\eta(6\tau)^4}{\eta(\tau)^3\eta(12\tau)^3}\,, \qquad t_{12}(\tau) \;=\; \left(\frac{\eta(\tau)^{60}\eta(12\tau)^{60}}{\eta(2\tau)^{48}\eta(3\tau)^{12}\eta(4\tau)^{12}\eta(6\tau)^{48}}\right)^{1/13}\,, \tag{3.3}$$

so t^{13} is a modular function on $\Gamma_0(12)$, but t itself is only invariant under a non-congruence subgroup of $\Gamma_0(12)$ of index 13.) This fixes the possible values of N, and then for each N a similar argument shows that for all values of d except those occurring in (1.1) the function $t_{N,d}$ defined by (3.1) is only a root of a modular function on $\Gamma_0(dN)$.

As a side remark we mention that there are exactly eight values of N for which the modular form $F_N \in M_2^-(\Gamma_0(N))$ can be written as a quotient of products of eta-functions, namely

N	4	6	8	9	12	16	18	36
F_N	$\frac{2^{20}}{1^8 4^8}$	$\frac{2^7 3^7}{1^5 6^5}$	$\frac{2^6 4^6}{1^4 8^4}$	$\frac{3^{10}}{1^3 3^3}$	$\frac{2^4 3^1 4^1 6^4}{1^3 12^3}$	$\frac{2^1 4^6 8^1}{1^2 16^2}$	$\frac{2^1 3^3 6^3 9^1}{2^2 18^2}$	$\frac{4^1 6^4 9^1}{1^1 36^1}$

where the notation is clear if one compares the $N = 12$ entry with the formula for F_{12} given in (3.3) or the N=6 entry with the formula for $\Phi(t)$ in the Apéry case that was given in the Introduction.

Now in each case we write the modular form $F = F_{N,d}$ as a power series $\Phi(t)$ in the Hauptmodul $t = t_{N,d}$, where $\Phi = \Phi_{N,d}$ is a power series with integral coefficients and with leading coefficient 1. (This integrality, which is clear from the modular description, is not at all obvious from the recursion for these coefficients coming from the differential equation of Φ, which was one part of the "Apéry miracle" that was demystified by Beukers's modular interpretation). The operator $\mathcal{L}$ annihilating Φ then has the form tL with L as in (2.2), as explained in the previous section. The following proposition expresses it in purely modular terms.

Proposition 1. *The differential operator $\mathcal{L}$ is given in terms of the modular variable τ by*

$$\mathcal{L} \;=\; \frac{1}{H(\tau)}\left(\frac{1}{2\pi i}\,\frac{d}{d\tau}\right)^3 \frac{1}{F(\tau)}\,, \tag{3.4}$$

where $F = F_{N,d} \in M_2^-(\Gamma_0(N)$ is the form defined above and $H = H_{N,d}$ is a modular form belonging in $M_4(\Gamma_0(N))^-$. This form is given explicitly by $H_{N,d}(\tau) = H_N(d\tau)$, where H_N for $N > 1$ is the unique Eisenstein series of weight 4 on $\Gamma_0(N)$ that takes on the values $+1$ at $\tau = \infty$, -1 at $\tau = 0$, and 0 at all other cusps, and $H_1 = E_6/\sqrt{E_4}$.

Proof: We first note that, since the space of solutions of the differential equation is spanned by $F(\tau)$, $\tau\,F(\tau)$ and $\tau^2 F(\tau)$, it consists of precisely those functions whose quotient by $F(\tau)$ is annihilated by $(d/d\tau)^3$, so $\mathcal{L}$ must have the form (3.4) for some function $H(\tau)$. By comparing the symbols (coefficient of $(d/dt)^3$) on the two sides of (3.4), using (2.2), we find that

$$t(\tau)^2\,Q(t(\tau)) \;=\; \frac{1}{H(\tau)}\,\frac{t'(\tau)^3}{t(\tau)F(\tau)}\,, \tag{3.5}$$

where $t' = \frac{1}{2\pi i}\,\frac{dt}{d\tau}$, and this is at least a meromorphic modular form of weight 4 on $\Gamma_0(N)$ with W_N-eigenvalue -1. On the other hand, a purely algebraic calculation shows that, because the differential operator is of type D3, one also has the formula

$$H(\tau) \;=\; F(\tau)\,\frac{t'(\tau)}{t(\tau)}\,, \tag{3.6}$$

and by comparing these two formulas one finds that the modular form H (for $N > 1$) is holomorphic and has the given values at the cusps, which suffices since $S_4(\Gamma_0(N))^-$ here is always 0. Alternatively, one can simply check case by case that the right-hand side of either of the above equations agrees with $E_{4,N}^-$ if $N > 1$ and with $E_6(\tau)/\sqrt{E_4(\tau)}$ if $N = 1$. The $d > 1$ cases follow easily from the $d = 1$ cases, with $H(\tau) = H_{N,d}(\tau)$ defined as $H_N(d\tau)$. $\square$

Note that the equality of the equations (3.5) and (3.6) for H implies that the curve $u^2 = Q(t)$, which is either rational or elliptic depending on the degree of Q, has a modular parametrization by $t = t(\tau)$, $u = t'/tF$, and in particular gives the Taniyama–Weil/Taylor–Wiles parametrization when deg $Q > 2$ (which here happens only in four cases, $(N, d) \in \{(11, 1), (5, 2), (3, 3), (2, 4)\}$). In this case the function $f = tF$ is the cusp form of weight 2 for which $f(\tau)\,d\tau = t'(\tau)/u(\tau)$ is the Weierstrass differential dt/u on the elliptic curve, and hence is a Hecke eigenform with multiplicative coefficients. In this connection we mention also the joint paper [17] of one of us with Masha Vlasenko, which contains among other things a classification of all $D3$ equations with 5 distinct singularities for which the function $t\Phi(t)$ has an expansion with respect to $\exp(\Phi_1/\Phi_0)$ with multiplicative coefficients (which in the modular case again means that it is a Hecke eigenform).

If $N > 1$, then, since all Eisenstein series on $\Gamma_0(N)$ for the cases in question come from level 1, the functions F_N and H_N are given by formulas of the form

$$F_N(\tau) \;=\; \sum_{M|N} M\, f_M\, G_2(M\tau)\,, \qquad H_N(\tau) \;=\; \sum_{M|N} M^2\, h_M\, G_4(M\tau)\,, \tag{3.7}$$

where

$$G_2(\tau) \;=\; -\frac{1}{24}\, E_2(\tau) \;=\; -\frac{1}{24} + \sum_{n=1}^{\infty} \frac{n\, q^n}{1-q^n}\,, \qquad G_4(\tau) \;=\; \frac{1}{240}\, E_4(\tau) \;=\; \frac{1}{240} + \sum_{n=1}^{\infty} \frac{n^3\, q^n}{1-q^n}$$

are the Hecke-normalized Eisenstein series of level 1 and weights 2 and 4 and where f_M, h_M $(M|N)$ are rational coefficients satisfying the antisymmetry property

$$f_{N/M} \;=\; -f_M\,, \qquad h_{N/M} \;=\; -h_M \qquad (\,M\,|\,N\,)\,, \tag{3.8}$$

since F and H are in the (-1)-eigenspace of W_N. (Note that the function $G_2(\tau)$ itself is only quasimodular, but since (3.8) implies that $\sum f_M$ vanishes, the right-hand side of the formula for F_N in (3.7) is modular.) It is these coefficients f_M and h_M that were tabulated in §2 together with the coefficients of the corresponding D3 operators. These coefficients, or rather the numbers

$$\mu_N \;=\; \begin{cases} \quad 62 & \text{if } N = 1 \\ \frac{1}{2} \sum\limits_{M|N} M h_M & \text{if } N > 1 \end{cases} \tag{3.9}$$

(which were also tabulated there), appear in the following theorem giving the values of the Frobenius limits occurring in the Gamma Conjecture.

Theorem 2. *Let $\Psi_0 = \Psi$, Ψ_1, Ψ_2, Ψ_3 be the Frobenius solutions of the fourth order linear differential equation satisfied by $\Psi(z) = \sum A_n z^n/n!$, where $F_{N,d}(\tau) = \sum A_n t_{N,d}(\tau)^n$ for one of the 17 pairs (N, d) from* (1.1)*. Then the Frobenius limits κ_j $(j = 1, 2, 3)$ defined by* (0.3) *are given by*

$$\kappa_1 = -\gamma, \quad \kappa_2 = \frac{\gamma^2}{2} - \Big(\frac{12}{d^2N} - \frac{1}{2}\Big)\zeta(2), \quad \kappa_3 = -\frac{\gamma^3}{6} + \Big(\frac{12}{d^2N} - \frac{1}{2}\Big)\gamma\zeta(2) + \Big(\frac{\mu_N}{d^3N} - \frac{1}{3}\Big)\zeta(3)\,,$$

where μ_N is defined by equation (3.9).

orem will be proved in Chapter 2, Sections 7 and 8. We remark that the formula in its statement can be written in the simpler-looking form

$$\Gamma(1+\varepsilon)^{-1} \sum_{j=0}^{\infty} \kappa_j \varepsilon^j \;=\; 1 + \frac{2}{d^2N}\, \pi^2\varepsilon^2 + \frac{\mu_N}{d^3N}\, \zeta(3)\, \varepsilon^3 + \mathrm{O}(\varepsilon^4)\,. \tag{3.10}$$

In fact, the expression which is naturally computed from the modular side is the expression on the left-hand side of this equation, which describes certain limits associated to the Frobenius solutions of the differential equation of Φ itself, rather than of its inverse Borel transform Ψ. On the topological side, this is related to the "modified gamma class" defined in (5.2) below.

4. Quantum differential equations of Fano manifolds. In this section we explain briefly how the differential equation whose asymptotic properties play a role in the Gamma Conjecture is defined. We first describe the meaning of the coefficients of the 4×4 matrix $\mathcal{A} = (a_{i,j})$ specifying the quantum differential equation via equation (2.5) and then for the benefit of the interested reader also explain very briefly how this determinantal equation arises.

Very roughly, the coefficient $a_{i,j}$ for $j \geq i$ (the other coefficients of $\mathcal{A}$ are 0 or 1 by definition) is meant to be the (correctly interpreted) "number" of rational curves of anticanonical degree $j-i+1$

that intersect algebraic cycles of codimensions specified by i and j. We state this a little more precisely in the case of interest to us when X is a Fano 3-fold with $H^2(X;\mathbb{Q}) = \mathbb{Q}\,c_1$. One defines the *three-point correlator* $\langle\alpha_1,\,\alpha_2,\,\alpha_3\rangle$ of three (effective and homogeneous) classes a_1, α_2, α_3 in $H^*(X)$ as the polynomial $\sum c_n z^n$, where c_n is the expected number of holomorphic maps $\mathbb{P}^1 \longrightarrow X$ of anticanonical degree n sending three fixed generic points P_i of $\mathbb{P}^1$ to cycles representing the Poincaré duals of the α_i. Using this correlator, one can define the *quantum cohomology* ring of X as $H^*(X)\otimes\mathbb{C}[z]$ together with a quantum multiplication $\star$, where $\alpha_1\star\alpha_2$ is defined by viewing $\langle\alpha_1,\alpha_2,\,\cdot\,\rangle$ as a linear functional on $H^*(X)\otimes\mathbb{C}[z]$ with values in $\mathbb{C}[z]$ and dualizing it with respect to the Poincaré pairing: $\int_{[X]}(\alpha_1\star\alpha_2)\cup\alpha_3 = \langle\alpha_1,\alpha_2,\alpha_3\rangle$. Then the matrix $G(z) = (g_{i,j}(z))$ of the operator $c_1\star$ with respect to the basis $\mathbf{c} = (1, c_1, c_1^2, c_1^3)$ has the form $g_{i,j} = a_{i,j}z^{j-i+1}$ for some matrix $\mathcal{A}$, and the fourth order differential operator $\widetilde{\mathcal{L}}$ that we want is the one associated to the system of first-order differential equations $D_z\,\vec{\zeta}(z) = \vec{\zeta}(z)\,G(z)$ in the usual way (with 1 as the cyclic vector). The Frobenius basis $\{\Psi_i\}$ of solutions of the differential equation $\widetilde{\mathcal{L}}\,\Psi = 0$ then corresponds to the basis of $H_*(X,\mathbb{Q})$ that is Poincaré-dual to $\mathbf{c}$. The expressions giving the coefficients of $\widetilde{\mathcal{L}}$, or the expansion coefficients of the Frobenius solutions, in terms of the Gromov–Witten invariants of X can be found in [28].

We can now consider the linear functional $\Psi \mapsto \lim\limits_{z\to\infty}\dfrac{\Psi(z)}{\Psi_0(z)}$ on the solution space of the differential equation $\widetilde{\mathcal{L}}\,\Psi = 0$ as a cohomology class A_X (called the "principal asymptotic class" of X) via the identification just described. In our case, it is given explicitly by

$$A_X \;=\; \sum_{j=0}^{3}\kappa_j\,c_1^j \in H^*(X;\mathbb{C})\,,$$

where the numbers κ_j are the Frobenius limits as defined in (0.3). The Gamma Conjecture, which we now describe, says that it should coincide with a certain characteristic class of X called its gamma class.

5. The gamma class and the Gamma Conjecture. The gamma class of a holomorphic vector bundle E over a topological space X is the multiplicative characteristic class, in the sense of Hirzebruch, associated to the power series expansion $\Gamma(1+x) = 1-\gamma x+\frac{\gamma^2+\zeta(2)}{2}\,x^2+\cdots$ of the gamma function at 1; in other words, it is the function that associates to a holomorphic bundle E/X the cohomology class $\widehat{\Gamma}(E) = \prod_i\Gamma(1+\tau_i)\in H^*(X,\mathbb{R})$, where the total Chern class of E has the formal factorization $c(E) = \prod(1+\tau_i)$ with τ_i of degree 2. If E is the tangent bundle of X, we write simply $\widehat{\Gamma}_X$ for $\widehat{\Gamma}(E)$. Its terms of degree ≤ 3, which are the only ones that will be needed for our purposes, are given by

$$\begin{aligned}\widehat{\Gamma}(E) \;=\; 1 \,-\, \gamma\,c_1 \,+\, \Bigl(-\zeta(2)\,c_2 \,+\, \frac{\zeta(2)+\gamma^2}{2}\,c_1^2\Bigr)&\\ +\,\Bigl(-\zeta(3)\,c_3 \,+\, \bigl(\zeta(3)+\gamma\,\zeta(2)\bigr)\,c_1c_2 \,-\, \frac{2\zeta(3)+3\gamma\,\zeta(2)+\gamma^3}{6}\,c_1^3\Bigr) \,+\,\cdots\,,&\end{aligned} \tag{5.1}$$

where $c_i = c_i(TX)\in H^{2i}(X)$ are the Chern classes of X. This formula becomes much simpler if we introduce the *modified gamma class* $\widehat{\Gamma}_X^0$, defined by

$$\widehat{\Gamma}_X \;=\; \Gamma(1+c_1)\,\widehat{\Gamma}_X^0\,, \tag{5.2}$$

in which case it reduces to

$$\widehat{\Gamma}_X^0 \;=\; 1\,-\,\zeta(2)\,c_2\,+\,\zeta(3)\,(c_1c_2-c_3)\,+\,\cdots\,. \tag{5.3}$$

(The passage from $\widehat{\Gamma}_X$ to $\widehat{\Gamma}_X^0$ in the cases that we will study reflects the relationship between the topology of the Fano variety to that of its K3-surface hyperplane sections, while for the other side of the Gamma Conjecture—as we already mentioned in connection with Theorem 2—it corresponds to the relation between the Frobenius limits for the quantum differential equation associated to the Fano 3-fold and the Frobenius limits of the Picard–Fuchs differential equation of its mirror dual.) The gamma class of a variety can be regarded as a "half" of the Todd class occurring in the Hirzebruch–Riemann–Roch theorem, or more precisely of the A-hat class $\widehat{A}_X = e^{-c_1}\mathrm{Td}_X$ occurring in the Atiyah–Hirzebruch theorem, since the Γ-function identity $\Gamma(1+z)\,\Gamma(1-z) = \frac{\pi z}{\sin \pi z}$ implies that we can factorize the A-hat class as $\mu_+(\widehat{\Gamma}_X)\mu_-(\widehat{\Gamma}_X)$, where $\mu_\pm$ denote the rescaling operators of multiplication by a factor $(\pm 2\pi i)^{-m}$ in $H^{2m}(X;\mathbb{C})$.

We now have all the ingredients necessary to state the Gamma Conjecture. Let $A_X \in H^*(X)$ be the principal asymptotic class of the quantum differential equation associated to a Fano variety X as explained in Section 4, and $\widehat{\Gamma}_X \in H^*(X)$ its gamma class.

Definition. If the equality

$$A_X \;=\; \widehat{\Gamma}_X \tag{5.4}$$

holds, then we will say that the *Gamma Conjecture*[2] holds for X.

Theorem 1 says that the Gamma Conjecture holds for the 17 Iskovskikh cases. To prove it, we will compute both sides of equation (5.4) independently and check that they agree in each case. The result on the Picard–Fuchs side was given in Theorem 2 above and will be proved in Chapter 2. The computation on the cohomology side is straightforward and will be given here.

Proposition 2. *The modified gamma class of a rank one Fano 3-fold X is given by*

$$\widehat{\Gamma}^0_X \;=\; 1 \;-\; \frac{12}{d^2N}\,\zeta(2)\,c_1^2 \;+\; \frac{h^{1,2}+10}{d^2N}\,\zeta(3)\,c_1^3\,,$$

where c_1 denotes the first Chern class of X and $h^{1,2}$ the dimension of $H^{1,2}(X)$.

Proof. The three Chern numbers of X are given by $\langle c_1^3,[X]\rangle = 2d^2N$ (by the definition of N), $\langle c_1c_2,[X]\rangle = 24$ (essentially because the hyperplane section of X is a K3-surface and because $\langle c_2,[S]\rangle = e(S) = 24$ for all K3-surfaces S) and $\langle c_3,[X]\rangle = e(X)$, the Euler characteristic of X. Here $e(X) = 4-2h^{1,2}$ because all Hodge numbers of X except $h^{i,i}=1$ and $h^{1,2}=h^{2,1}$ vanish. Since $c_i \in H^{i,i}_{\mathrm{alg}}(X;\mathbb{Q}) = \mathbb{Q}\,c_1^i$, this gives $c_2 = \dfrac{12}{d^2N}\,c_1^2$ and $c_1c_2 - c_3 = \dfrac{h^{1,2}+10}{d^2N}\,c_1^3$. Substituting these values into (5.3) gives the desired assertion. □

Proof of Theorem 1. The numbers $h^{1,2}$ for the 17 Iskovskikh families are given in the table below, which we took from the comprehensive source on Fano varieties [20]. We check that in all cases we

N	1	2	3	4	5	6	7	8	9	11	1	2	3	4	5	3	2
d	1	1	1	1	1	1	1	1	1	1	2	2	2	2	2	3	4
h^{12}	52	30	20	14	10	7	5	3	2	0	21	10	5	2	0	0	0

have $h^{1,2}+10 = \mu_N/d$, where μ_N is the integer tabulated in §2 and defined in (3.9). Combining Theorem 2 from §3 with Proposition 2, we obtain Theorem 1. □

Chapter 2. Computation of the Frobenius limits

In this chapter we will give two different approaches to calculating the Frobenius limits, one using the combinatorial description of the coefficients of the holomorphic solution as sums of binomial coefficients (like Apéry's formula in the V_{12} case or Landau–Ginzburg models in general) and one using the modular description of this solution. We will carry out the first one in detail for

[2]This is called "Gamma Conjecture I" in [12], but we will not discuss "Gamma Conjecture II" and will therefore simply speak of the "Gamma Conjecture"

the V_{12} case in §7, where the prediction of the Gamma Conjecture in the (easier) hypergeometric cases is also checked. The modular approach, which works uniformly in all of the cases with $N > 1$, will be treated in §8. In the final section, we describe briefly numerical calculations suggesting that the higher Frobenius limits beyond the dimension of the Fano variety are also interesting, with the next few still being polynomials in Riemann zeta values but the further values apparently involving more complicated kinds of periods like multiple zeta values.

6. The two Frobenius bases and their relationship. In this section we define the higher Frobenius functions for both the Picard–Fuchs type differential equations and their Laplace transforms. We will illustrate everything using the Apéry numbers, but the definitions given here work the same way for all cases.

The definition and recursion relation of the Apéry numbers $A_0 = 1$, $A_1 = 5$, $A_2 = 73$, ... were already given in the Introduction and will not be repeated here. We consider the Frobenius deformation of Apéry's recursion, i.e., the sequence of power series

$$A_n(\varepsilon) \;=\; \sum_{j=0}^{\infty} A_n^{(j)} \varepsilon^j \;\in\; \mathbb{Q}[[\varepsilon]] \qquad (n \,=\, 0,\, 1,\, \ldots)$$

defined by the initial condition $A_{-1}(\varepsilon) = 0$, $A_0(\varepsilon) = 1$ and the recursion

$$(n+\varepsilon+1)^3\, A_{n+1}(\varepsilon) \,-\, P(n+\varepsilon) A_n(\varepsilon) \,+\, (n+\varepsilon)^3\, A_{n-1}(\varepsilon) \;=\; 0\,, \tag{6.1}$$

with $P(x)$ as in (0.2). We assemble the rational numbers $A_n^{(j)}$ into further power series

$$\Phi_j^{\rm an}(t) \;=\; \sum_{n=0}^{\infty} A_n^{(j)} t^n\,, \qquad \Phi^{\rm an}(t,\varepsilon) \;=\; \sum_{j=0}^{\infty} \Phi_j^{\rm an}(t)\varepsilon^j \;=\; \sum_{n=0}^{\infty} A_n^{(\varepsilon)} t^n\,, \tag{6.2}$$

with the beginnings of the first few power series $\Phi_j(t)$ being given by

$$\begin{aligned}
\Phi_0^{\rm an}(t) &= \Phi(t) \;= 1 + 5\,t + 73\,t^2 + 1445\,t^3 + 33001\,t^4 + 819005\,t^5 \,+\, \cdots\,,\\
\Phi_1^{\rm an}(t) &= 12\,t \,+\, 210\,t^2 \,+\, 4438\,t^3 \,+\, 104825\,t^4 \,+\, \tfrac{13276637}{5}\,t^5 \,+\, \ldots\,,\\
\Phi_2^{\rm an}(t) &= 72\,t^2 \,+\, 2160\,t^3 \,+\, 59250\,t^4 \,+\, 1631910\,t^5 \,+\, \cdots\,,\\
\Phi_3^{\rm an}(t) &= -7\,t - \tfrac{1011}{8}\,t^2 - \tfrac{522389}{216}\,t^3 - \tfrac{90124865}{1728}\,t^4 - \tfrac{264872026721}{216000}\,t^5 \,+\, \cdots\,,\\
\Phi_4^{\rm an}(t) &= 9\,t + \tfrac{1437}{16}\,t^2 + \tfrac{182489}{144}\,t^3 + \tfrac{5753277}{256}\,t^4 + \tfrac{663266820361}{1440000}\,t^5 \,+\, \cdots\,.
\end{aligned} \tag{6.3}$$

Putting $t^\varepsilon = \exp(\varepsilon \log t)$, we define the *Frobenius functions* $\Phi_j(t)$ for all $j \geq 0$ by the expansions

$$\Phi(t,\varepsilon) \;=\; t^\varepsilon\, \Phi^{\rm an}(t,\varepsilon) \;=\; \sum_{j=0}^{\infty} \Phi_j(t)\, \varepsilon^j\,, \qquad \Phi_j(t) \;=\; \sum_{i=0}^{j} \Phi_i^{\rm an}(t)\, \frac{(\log t)^{j-i}}{(j-i)!}\,. \tag{6.4}$$

Then the recursion satisfied by the $A_n(\varepsilon)$ translates into the statement that the power series $\Phi_j(t)$ and $\Phi(t,\varepsilon)$ satisfy the differential equations

$$\mathcal{L}\big(\Phi(t,\varepsilon)\big) \;=\; \varepsilon^3 t^\varepsilon\,, \qquad \mathcal{L}\big(\Phi_j(t)\big) \;=\; \frac{(\log t)^{j-3}}{(j-3)!}\,, \tag{6.5}$$

respectively, where

$$\mathcal{L} \;=\; D^3 \,-\, t\,P(D) \,+\, t(D+1)^3 \qquad \Big(D = t\,\frac{d}{dt}\Big)$$

is the differential operator that annihilates $\Phi(t)$ and where the right-hand side of the second equation in (6.5) is to be interpreted as 0 if $j < 3$. In particular, Φ_0, Φ_1 and Φ_2 are solutions of the original differential equation $\mathcal{L}\Phi = 0$ and constitute, of course, the well-known Frobenius basis for the space of its solutions, but, as already mentioned in the Introduction, the higher Φ_j are also of interest. In any case, however, even for the Gamma Conjecture we will need Φ_3, which satisfies the inhomogeneous differential equation $\mathcal{L}\Phi = 1$, because under the Laplace transform there is a relationship between $\{\Phi_0,\dots,\Phi_J\}$ and $\{\Psi_0,\dots,\Psi_J\}$ for all J, as we will now discuss, and the operator $\widetilde{\mathcal{L}}$ has order four and thus *four* Frobenius solutions $\Psi_0,\dots,\Psi_3$.

We now do the same things on the Laplace transform side. The modified numbers $a_n = A_n/n!$ satisfy the modified recursion relation

$$(n+1)^4 a_{n+1} \;-\; P(n)a_n \;+\; n^2 a_{n-1} \;=\; 0\,, \tag{6.6}$$

with the same polynomial $P(n)$ as before, and their generating function $\Psi(z)=\sum a_n z^n$ therefore satisfies the modified differential equation (Laplace transform of $\mathcal{L}$)

$$\widetilde{\mathcal{L}}(\Psi) \;=\; 0\,, \qquad \widetilde{\mathcal{L}} \;=\; D_z^4 \;-\; z\,P(D_z) \;+\; z^2(D_z+1)^2\,,$$

where $D = z\,\frac{d}{dz}$ as before. The Frobenius deformation in this case is given by

$$(n+1+\varepsilon)^4\, a_{n+1}(\varepsilon) \;+\; P(n+\varepsilon)a_n(\varepsilon) \;+\; (n+\varepsilon)^2\, a_{n-1}(\varepsilon) \;=\; 0 \tag{6.7}$$

for $n\ge 0$, with initial condition $a_{-1}(\varepsilon)=0$ and $a_0(\varepsilon)=1$. Again we set $a_n(\varepsilon)=\sum_j a_n^{(j)}\varepsilon^j$ and define power series $\Psi_j^{\mathrm{an}}(z)=\sum_n a_n^{(j)} z^n$ and $\Psi^{\mathrm{an}}(z,\varepsilon)=\sum_j \Psi_j^{\mathrm{an}}(z)\,\varepsilon^j \;=\; \sum_n a_n(\varepsilon) z^n$, the first values being given this time by

$$\begin{aligned}
\Psi_0^{\mathrm{an}}(z) \;&=\; \Psi(z) \;=\; 1+5\,z+\tfrac{73}{2}\,z^2+\tfrac{1445}{6}\,z^3+\tfrac{33001}{24}\,z^4+\tfrac{163801}{24}\,z^5\;+\;\cdots\,,\\
\Psi_1^{\mathrm{an}}(z) \;&=\; \quad 7\,z+\tfrac{201}{4}\,z^2+\tfrac{10733}{36}\,z^3+\tfrac{432875}{288}\,z^4+\tfrac{47115959}{7200}\,z^5\;+\;\cdots\,,\\
\Psi_2^{\mathrm{an}}(z) \;&=\; -7\,z-\tfrac{461}{8}\,z^2-\tfrac{92323}{216}\,z^3-\tfrac{9220085}{3456}\,z^4-\tfrac{6108294133}{432000}\,z^5\;+\;\cdots\,,\\
\Psi_3^{\mathrm{an}}(z) \;&=\; \qquad\quad -\tfrac{15}{8}\,z^2+\tfrac{169}{4}\,z^3+\tfrac{4285465}{6912}\,z^4+\tfrac{3811075}{768}\,z^5\;+\;\cdots\,,\\
\Psi_4^{\mathrm{an}}(z) \;&=\; \quad 9\,z+\tfrac{2449}{32}\,z^2+\tfrac{441925}{864}\,z^3+\tfrac{52564099}{18432}\,z^4+\tfrac{259795048429}{19200000}\,z^5\;+\;\cdots\,.
\end{aligned} \tag{6.8}$$

Then just as before, putting $z^\varepsilon=\exp(\varepsilon\log z)$, we find that the Frobenius functions defined by

$$\Psi(t,\varepsilon) \;=\; z^\varepsilon\,\Psi^{\mathrm{an}}(z,\varepsilon) \;=\; \sum_{j=0}^{\infty}\Psi_j(z)\,\varepsilon^j\,, \qquad \Psi_j(z) \;=\; \sum_{i=0}^{j}\Psi_i^{\mathrm{an}}(z)\,\frac{(\log z)^{j-i}}{(j-i)!} \tag{6.9}$$

satisfy the inhomogeneous differential equations

$$\widetilde{\mathcal{L}}\bigl(\Psi(z,\varepsilon)\bigr) \;=\; \varepsilon^4 z^\varepsilon\,, \qquad \widetilde{\mathcal{L}}\bigl(\Psi_i(t)\bigr) \;=\; \frac{(\log t)^{j-4}}{(j-4)!}\,, \tag{6.10}$$

with the same convention as before. In particular, $\Psi_0,\dots,\Psi_3$ give a basis (again called the Frobenius basis) of solutions of the transformed differential equation.

The Gamma Conjecture concerns the limits κ_j defined by (0.3), but to calculate these it is more convenient to work with the numbers $A_n^{(j)}$ and functions $\Phi_j(t)$, which have better properties (regular singularities, modular parametrization). We therefore have to look how the two sequences of numbers and of functions are related. From the recursions we obtain

$$a_n(\varepsilon) \;=\; \frac{A_n(\varepsilon)}{(1+\varepsilon)_n} \;=\; \frac{A_n(\varepsilon)}{n!}\prod_{k=1}^{n}\Bigl(1+\frac{\varepsilon}{k}\Bigr)^{-1}\,.$$

Here $(1+\varepsilon)_n$ denotes the ascending Pochhamer symbol $(1+\varepsilon)(2+\varepsilon)\cdots(n+\varepsilon)$. We insert into this expression the expansion

$$\prod_{k=1}^{n}\Bigl(1+\frac{\varepsilon}{k}\Bigr)^{-1} = \exp\Bigl(-H_n\,\varepsilon \;+\; H_n^{(2)}\,\frac{\varepsilon^2}{2} \;-\; H_n^{(3)}\,\frac{\varepsilon^3}{3} \;+\;\dots\Bigr)\,,$$

where $H_m = 1 + \frac{1}{2} + \ldots + \frac{1}{m}$, $H_m^{(2)} = 1 + \frac{1}{4} + \ldots + \frac{1}{m^2}$, etc., and find

$$\begin{aligned} n!\, a_n^{(0)} &= A_n^{(0)} \\ n!\, a_n^{(1)} &= A_n^{(1)} - H_n A_n^{(0)}\,, \\ n!\, a_n^{(2)} &= A_n^{(2)} - H_n A_n^{(1)} + \frac{H_n^2 + H_n^{(2)}}{2} A_n^{(0)}\,, \\ &\ \vdots \end{aligned} \tag{6.11}$$

Now if we use that $H_n = \log n + \gamma + \mathrm{O}(1/n)$ and $H_n^{(m)} = \zeta(m) + \mathrm{O}(1/n)$ for $m > 1$ and that the maximum of $a_n^{(j)} z^n$ for large z and j fixed occurs for $n \approx Cz$ with $C = (1+\sqrt{2})^4 = 17 + 12\sqrt{2}$ (because $A_n \sim \mathrm{const} \cdot C^n / n^{3/2}$, as discussed in more detail in §9), then we see that we simply have to replace H_n by $\log(Cz) + \gamma$ and $H_n^{(m)}$ by $\zeta(m)$ for $m > 1$ to get the asymptotics of the Frobenius solutions. This means that if we define a sequence of Frobenius limits κ_j^0 and the corresponding generating function $\kappa^0(\varepsilon)$ for the regular case by

$$\kappa_j^0 \;=\; \sum_{i=0}^{j} \frac{(-\log C)^{j-i}}{(j-i)!} \lim_{n\to\infty} \left(\frac{A_n^{(i)}}{A_n} \right), \qquad \kappa^0(\varepsilon) \;=\; \sum_{j=0}^{\infty} \kappa_j^0\, \varepsilon^j \;=\; C^{-\varepsilon} \lim_{n\to\infty} \left(\frac{A_n(\varepsilon)}{A_n} \right) \tag{6.12}$$

then the relationship between the two generating functions $\kappa^0(\varepsilon)$ and

$$\kappa(\varepsilon) \;=\; \sum_{j=0}^{\infty} \kappa_j\, \varepsilon^j \;=\; \lim_{z\to\infty} \frac{\Psi(z,\varepsilon)}{\Psi(z)}\,, \tag{6.13}$$

is given simply by

$$\kappa^0(\varepsilon) \;=\; \frac{1}{\Gamma(1+\varepsilon)}\, \kappa(\varepsilon)\,, \tag{6.14}$$

exactly the same relationship (if we replace ε by $c_1(X)$) as that between the gamma class and modified gamma class of a Fano variety X as explained in §5. We will explain in §9 how the limits in both (6.12) and (6.13) (which in any case determine one another by (6.14)) can be computed quickly and to very high accuracy, and will discuss some of the results of the numerical computations.

7. Frobenius limits from the hypergeometric point of view. In this section we give a proof of the formula (0.4) for the Frobenius limits in the Apéry case based on Apéry's original formula (0.1) for his numbers as finite sums of products of binomial coefficients, or terminating hypergeometric series. This method is quite computational and uses identities that were found experimentally and whose proofs are not particularly enlightening, but has the advantages of being completely elementary and of applying in principle to any linear differential equations of this type, even if they are not modular. It also gives very easy proofs of the Gamma Conjecture for the 10 cases from the Iskovkikh list corresponding to hypergeometric differential equations.

The idea is to mimic Apéry's original proof of the irrationality of $\zeta(3)$, in which he studied the second solution A_n^* of the recursion (0.2) with initial values $A_0^* = 0$, $A_1^* = 1$ (whose generating function again satisfies an inhomogeneous version of the original differential equation, though this time with right-hand side t rather than 1) and proved that the limiting ratio $\lim_n A_n^*/A_n$ is equal to $\frac{1}{6}\,\zeta(3)$ by finding an explicit formula for A_n^* of the form $\sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 Q(n,k)$ for a suitably chosen elementary function $Q(n,k)$ involving partial sums of $\zeta(3)$. (See [1] and [31].) Looking for similar formulas for the numbers $A_n^{(i)}$ for small values of i, we found experimentally a number of identities of this type that gave the correct values for $1 \le i \le 3$ and for all n up to some large limit, and any of which could be used to evaluate the Frobenius limit in question. One choice, which had an especially simple form, is reproduced in Proposition 3 below together with its proof by the standard method of telescoping sums. To find both the identities and their proofs, we made an Ansatz for the functions denoted $Q_i(n,k)$ and $R_{n,k}(\varepsilon)$ in Proposition 3 that had the form given there, but with unknown coefficients (the three numerical coefficients of $H^{(i)}_{n+k}$, $H^{(i)}_k$ and $H^{(i)}_n$ in the case of $Q_i(n,k)$ and the four coefficients in $\mathbb{Q}(n,k)$ of ε^i in the case of $R_{n,k}(\varepsilon)$) and then determined the necessary values of these coefficients by a computer calculation. This proof is therefore not very aesthetic, but—as alredy mentioned—has the advantage that the method can in principle be applied to the Frobenius deformations of other differential equations, not necessarily having a modular parametrization.

Proposition 3. *For $n,k\ge 0$ set*

$$a_{n,k}(\varepsilon) \;=\; \binom{n}{k}^2\binom{n+k}{k}^2 \exp\Bigl(\sum_{i=1}^{3} Q_i(n,k)\frac{\varepsilon^i}{i!} \;+\; O(\varepsilon^4)\Bigr),$$

where $Q_1(n,k)=4H_{n+k}-4H_k$, $Q_2(n,k)=4H^{(2)}_k-8H^{(2)}_{n+k}$, *and* $Q_3(n,k)=32H^{(3)}_{n+k}-H^{(3)}_k-14H^{(3)}_n$. *Then*

$$A_n(\varepsilon) \;=\; \sum_{k=0}^{n} a_{n,k}(\varepsilon) \;+\; O(\varepsilon^4)\,.$$

Proof. Define $R_{n,k}(\varepsilon)\in\mathbb{Q}(n,k)[\varepsilon]/\varepsilon^4$ by

$$\begin{aligned}R_{n,k}(\varepsilon) \;=\;\; & 4(2n+1)(2k^2+k-(2n+1)^2)+\bigl(16k^2+8(4n+3)k-4(2n+1)(12n+5)\bigr)\varepsilon\\ & +16(2k-5n-2)\varepsilon^2+\bigl(-16+\tfrac{14(2n+1)k}{3n^2(n+1)^2}(2n^2+2n-k)\bigr)\varepsilon^3+O(\varepsilon^4).\end{aligned}$$

Using the easily checked identity

$$(n+1+\varepsilon)^3\,\frac{a_{n+1,k}(\varepsilon)}{a_{n,k}(\varepsilon)} \;-\; P(n+\varepsilon) \;+\; (n+\varepsilon)^3\,\frac{a_{n-1,k}(\varepsilon)}{a_{n,k}(\varepsilon)} \;=\; R_{n,k}(\varepsilon)-R_{n,k-1}(\varepsilon)\frac{a_{n,k-1}(\varepsilon)}{a_{n,k}(\varepsilon)}$$

in $\mathbb{Q}(n,k)[\varepsilon]/\varepsilon^4$ and induction on K, we find that

$$\sum_{k=0}^{K}\bigl((n+1+\varepsilon)^3a_{n+1,k}(\varepsilon)-P(n+\varepsilon)a_{n,k}(\varepsilon)+(n+\varepsilon)^3a_{n-1,k}(\varepsilon)\bigr) \;=\; R_{n,K}(\varepsilon)a_{n,K}(\varepsilon)$$

for all $K\ge 0$. Taking $K>n$ shows that $\sum_{k=0}^{n}a_{n,k}(\varepsilon)$ satisfies the defining recursion of $A_n(\varepsilon)$, and since it also has the same initial values (0 for $n=-1$, 1 for $n=0$), they are equal. $\square$

Corollary. One has

$$\lim_{n\to\infty}\frac{A_n(\varepsilon)}{C^\varepsilon A_n} \;=\; \exp\Bigl(-2\zeta(2)\varepsilon^2 \;+\; \frac{17}{6}\zeta(3)\varepsilon^3 \;+\; O(\varepsilon^4)\Bigr)\,.$$

Proof. The maximum of $a_{n,k}=\binom{n}{k}^2\binom{n+k}{k}^2$ over $0\le k\le n$ is sharply peaked at $k=n(1/\sqrt{2}+O(1))$ for n large, since the ratio $a_{n,k}/a_{n,k-1}$ equals $(n+k)^2(n-k+1)^2/k^4\approx(n^2/k^2-1)^2$. Hence

$$\sum_{k=0}^{n}a_{n,k}(\varepsilon)\sim\Bigl(\sum_{k=0}^{n}a_{n,k}\Bigr)\,\exp\Bigl(4\log(1+\sqrt{2})\,\varepsilon\;-\;2\zeta(2)\,\varepsilon^2\;+\;\frac{17}{6}\zeta(3)\,\varepsilon^3\;+\;O(\varepsilon^4)\Bigr)\,. \qquad\square$$

By virtue of (6.12) and (6.14), the statement of the Corollary is equivalent to the formula (0.4) predicted by the Gamma Conjecture, completing the proof of this conjecture for the Apéry case.

We mention that for each of our 17 families, and also for other DN equations arising as Picard–Fuchs equations, there exist formulas like (0.1) expressing the coefficients A_n as simple or multiple finite sums of products of binomial coefficients (terminating hypergeometric series), for example the ones coming from the "Landau–Ginzburg models," which express A_n as the constant term of $P(x,y,z)^n$ for some Laurent polynomial $P(x,y,z)$. One could therefore in principle study each of the other cases using the same idea of inserting appropriate factors into these formulas. We did not try to do this since the modular approach as discussed in the next section is much simpler and works in a uniform way for all cases. We can, however, use the combinatorial approach to give an easy direct proof of Theorem 2 in all ten cases from the Iskovskikh list for which the corresponding differential operators are hypergeometric, and this is useful since it includes the two $N = 1$ cases for which the modular approach fails (or at least needs modification) because the function $F_1 = \sqrt{E_4}$ is not a holomorphic modular form. Note, however, that the case $(N, d) = (2, 4)$ corresponds to the Fano variety $\mathbb{P}^3$ and therefore is a special case of the result of Dubrovin [7] (see also [12]) proving the Gamma Conjecture for all $\mathbb{P}^n$, and in fact all of the hypergeometric cases are essentially known by the work of Iritani (see §10). We nevertheless include a proof here since it is elementary and fits with the other cases considered.

We begin with a remark that applies to all 17 cases, not only the hypergeometric ones, namely that it suffices to prove Theorem 2 for the 10 cases of the Iskovskikh list with $d = 1$. Indeed, comparing the statement of Theorem 2 in the version (3.10) with the formula (6.14) relating the Frobenius limits for the differential operators $\mathcal{L}$ and $\widetilde{\mathcal{L}}$, we see that this theorem now takes on the simple form

$$\kappa^0_{N,d}(\varepsilon) \;=\; 1 \,+\, 0\,\varepsilon \,+\, \frac{2}{d^2N}\,\pi^2\varepsilon^2 \,+\, \frac{\mu_N}{d^3N}\,\zeta(3)\,\varepsilon^3 \,+\, \mathrm{O}(\varepsilon^4)\,, \tag{7.1}$$

where $\kappa^0_{N,d}(\varepsilon)$ denotes the function $\kappa^0(\varepsilon)$ as defined by (6.12) for the (N, d) case from the list (1.1) (so that the κ^0 for the Apéry case that was used as an illustration in §6 would be $\kappa^0_{6,1}$), but with the constant $C = 17+12\sqrt{2}$ appearing in (6.12) replaced in the other 16 cases by $\lim\limits_{n\to\infty} A_n^{1/n}$, which is the reciprocal of the smallest positive root of $Q(t)$. But the power series $\Phi_{N,d}(t)$ corresponding to this case is simply $\Phi_{N,1}(t^d)$, so passing from $(N, 1)$ to (N, d) replaces A_n and $A_n(\varepsilon)$ by $A_{n/d}$ and $A_{n/d}(\varepsilon)$, interpreted as 0 if $d \nmid n$, and C by $C^{1/d}$, and it therefore follows that $\kappa^0_{N,d}(\varepsilon) = \kappa^0_{N,1}(\varepsilon/d)$ to all orders, not just up to $\mathrm{O}(\varepsilon^4)$.

This remark reduces the number of hypergeometric cases to be proved from 10 to 4, namely the cases $1 \le N \le 4$ and $d = 1$. For these cases the coefficients of the power series $\Phi(t) = \sum_{m=0}^{\infty} A_n t^n$ are quotients of products of factorials, as given by the following table:

N	1	2	3	4
A_n	$\frac{(6n)!}{(3n)!n!^3}$	$\frac{(4n)!}{n!^4}$	$\frac{(3n)!(2n)!}{n!^5}$	$\frac{(2n)!^3}{n!^6}$

We write the expression for A_n in each case as $\prod_r (rn)!^{\nu_r}$ and notice that it makes sense also for non-integral values of n if we interpret $x!$ as $\Gamma(1+x)$. The Frobenius deformed numbers $A_n(\varepsilon)$, defined by the same recursion (here of length 2 rather than 3 as before) with initial value $A_0(\varepsilon) = 1$, can then be written simply as $A_n(\varepsilon) = A_{n+\varepsilon}/A_\varepsilon$. On the other hand, since $\sum_r r\nu_r = 0$ in all cases (otherwise $\Phi(t)$ could not have positive radius of convergence), Stirling's formula gives the asymptotics $A_x \sim \alpha x^{\nu/2} C^x$ for $x \to \infty$, with $\alpha = \prod_r (2\pi r)^{\nu_r/2}$, $\nu = \sum_r \nu_r$, $C = \prod_r r^{r\nu_r}$. Hence

$$\begin{aligned}\kappa^0(\varepsilon) \;&=\; C^{-\varepsilon} \lim_{n\to\infty} \frac{A_n(\varepsilon)}{A_n} \;=\; C^{-\varepsilon} \lim_{n\to\infty} \frac{A_{n+\varepsilon}}{A_n A_\varepsilon} \;=\; \frac{1}{A_\varepsilon} \;=\; \prod_r \Gamma(1+r\varepsilon)^{-\nu_r} \\ &=\; \exp\Bigl(\sum_r \nu_r \Bigl(-\gamma r\varepsilon \,+\, \frac{\zeta(2)}{2}\,r^2\varepsilon^2 \,-\, \frac{\zeta(3)}{3}\,r^3\varepsilon^3 \,+\, \cdots\Bigr)\Bigr)\,,\end{aligned}$$

and since $\sum r\nu_r = 0$, $\sum r^2\nu_r = 24/N$ and $\sum r^3\nu_r = 3\mu_N/N$ in all four cases $1 \le N \le 4$, this completes the proof of Theorem 2 for the ten hypergeometric cases.

8. Frobenius limits from the modular point of view. The method described in the preceding section for the Apéry case depended on a complicated and artificial-looking identity. We now give a more natural proof using the modular parametrizations of our diffential equations as discussed in §3. This approach works in a uniform way for all of the cases with $N > 1$ in the Iskovksikh list, i.e., for 15 of the 17 cases, and since the two $N = 1$ cases are hypergeometric and have already been proved, this completes the proofs of Theorem 2 and Theorem 1.

By the remark made at the end of the previous section, it suffices to prove the 9 cases with $d=1$ and N belonging to the list $\{2,\ldots,9,11\}$, and after all of our preparation this is now fairly easy. In each case the differential operator $\mathcal{L}$ has the form given in Proposition 1 (equation (3.4)). where $H(\tau)=H_N(\tau)$ is the weight 4 Eisenstein series given by (3.7). By (6.5), the first four Frobenius functions satisfy the differential equations $\mathcal{L}\Phi_0=\mathcal{L}\Phi_1=\mathcal{L}\Phi_2=0$ and $\mathcal{L}\Phi_3=1$, which then translate on the modular side into $(\Phi_j(t(\tau))/F(t))'''=\delta_{j,3}H(\tau)$ for $0\le j\le 3$. (Here $'$ means $\frac{1}{2\pi i}\frac{d}{d\tau}$, as in §3.) In view of the asymptotic form of the Frobenius functions $\Phi_j(t)$ near $t=0$, which corresponds to $q=0$ with $2\pi i\tau=\log q=\log t+\mathrm{O}(t)$, this means that

$$\Phi_j\big(t(\tau)\big)\;=\;\frac{(2\pi i\tau)^j}{j!}\,F(\tau)\quad(j=0,\,1,\,2),\qquad\Phi_3\big(t(\tau)\big)\;=\;\widetilde{H}(\tau)\,F(\tau)\,,\tag{8.1}$$

where $\widetilde{H}(\tau)$ is the *Eichler integral* of $H(\tau)$, defined by $\widetilde{H}'''=H$ and normalized by $\widetilde{H}(\tau)=\frac{(2\pi i\tau)^3}{3!}+\mathrm{O}(q)$ as $q\to0$. But from (3.7), we have

$$\widetilde{H}(\tau)\;=\;\sum_{M|N}\frac{h_M}{M}\,\widetilde{G}_4(M\tau)\,,\tag{8.2}$$

where $\widetilde{G}_4(\tau)=\frac{(2\pi i\tau)^3}{1440}+\sum_{n\ge1}\frac{n^{-3}q^n}{1-q^n}$ is the correspondingly normalized Eichler integral of $G_4(\tau)$. But it is well known, and elementary to prove, that $\widetilde{G}_4$ satisfies the functional equation

$$\widetilde{G}_4(\tau)\;-\;\tau^2\,\widetilde{G}_4\Big(-\frac{1}{\tau}\Big)\;=\;\frac{\zeta(3)}{2}\big(\tau^2\;-\;1\big)\;-\;\frac{\pi^3 i}{6}\,\tau\,.$$

for all $\tau\in\mathbb{H}$. (This follows from the transformation property $G_4|_4S=G_4$ by threefold integration using"Bol's identity," which gives that $(F|_{-2}\gamma)'''=F'''|_4\gamma$ for any holomorphic function F and any Möbius transformation γ, implying that the expression on the left is at most a quadratic polynomial, which one then calculates using that the L-function of G_4 equals $\zeta(s)\zeta(s-3)$.) Inserting this into (8.2) and using the antisymmetry property (3.8), we get

$$N\tau^2\,\widetilde{H}\Big(-\frac{1}{N\tau}\Big)\;+\;\widetilde{H}(\tau)\;=\;\sum_{M|N}\frac{h_M}{2M}\,\zeta(3)\,\big(M^2\tau^2\;-\;1\big)\;=\;\frac{\mu_N}{N}\,\zeta(3)\,(N\tau^2+1)\tag{8.3}$$

with μ_N as in (3.9). (A similar calculation is given in [3], 1.2.) The calculation of the Frobenius limit now follows. Indeed, from (8.1) and (8.3) and the anti-invariance of F under $\big|_2W_N$ it follows that, if we set $(k_0,k_1,k_2,k_3)=(1,\,0,\,\frac{2\pi^2}{N},\,\frac{\mu_N}{N}\zeta(3))$, then $\Phi_j(t(\tau))-k_jF(\tau)$ is a W_N-invariant function in the upper half-plane for each $j\in\{0,1,2,3\}$. But that means that, as functions of $t=t(\tau)$, which is W_N-invariant and hence has a double zero at the fixed point $\tau_N=i/\sqrt{N}$ of W_N, they have no singularities at the value $t=t(\tau_N)=1/C$, so that they have a larger radius of convergence than the radius of convergence $1/C$ of $\Phi(t)$. It follows that the Frobenius limit $\kappa_j^0=\lim_{t\to1/C}\Phi_j(t)/\Phi(t)$ equals k_j, completing the proof of equation (3.10) in all cases with $N>1$.

Finally, we make a few remarks about the missing case $N=1$. Here the function $H_1(\tau)=E_6(\tau)/\sqrt{E_4(\tau)}\;=\;1-624q+64368q^2-\cdots$ is no longer a modular form, but by virtue of the transformation property $H_1|_4S=-H_1$ and Bol's identity we still have

$$\widetilde{H}_1(\tau)\;+\;\tau^2\,\widetilde{H}_1\Big(-\frac{1}{\tau}\Big)\;=\;\mu\,\zeta(3)\,(\tau^2+1)\,.$$

for some complex number μ, so that the only thing missing is the evaluation $\mu=\mu_1=62$. This of course follows from our alternative proof of the Frobenius limit in question via the hypergeometric expansion of $\sqrt{E_4}$, and we have also checked it numerically to high precision, but have not given a purely modular proof. We note, however, that such a proof could probably be given by imitating the calculations in [8], where the Eichler integral of the very similar almost-modular form $\Delta(\tau)/\sqrt{E_6(\tau)}$ of weight 3 is related to the zeros of the Weierstrass $\wp$-function.

9. Higher Frobenius limits: beyond the Gamma Conjecture. In this final section we discuss very briefly the values of the Frobenius limits κ_j (or rather of the equivalent limits κ_j^0) for $j > 3$.

Since the results here are numerical only, we should first say briefly how to calculate κ_j^0 and κ_j to very high accuracy very quickly. (Of course one does not really need both, since they are related by (6.14), but being able to do the calculations in two ways provides a nice verification of the numerical correctness of the procedure.) For κ_j one simply uses (0.3) directly with a moderately large value of z like $z = 100$ (actually, a much smaller value also suffices), and since the convergence is exponential this works well. For κ_j^0 one cannot use (6.12) directly because the ratio of $C^{-\varepsilon}A_n(\varepsilon)$ to A_n converges to its limiting value $\kappa^0(\varepsilon)$ only like $1/n$. Instead we use that A_n has an asymptotic expansion (in the Apéry case; the others are of course similar) of the form

$$A_n \;\sim\; \mathbf{A}(n) \;:=\; 2^{-9/4}\pi^{-3/2}\,\frac{C^{n+1/2}}{(n+1/2)^{3/2}}\,P\Big(\frac{1}{64(n+\frac{1}{2}\sqrt{2})}\Big)$$

for a certain power series $P(X) = 1+30X+274X^2-17132X^3+\cdots$ with easily computable rational coefficients (determined by the property that $\mathbf{A}(n)$ has to satisfy the same recursion as A_n). Then $\kappa^0(\varepsilon) = C^{-\varepsilon}\lim_{n\to\infty} A_n(\varepsilon)/\mathbf{A}(n) = \lim_{n\to\infty} A_n(\varepsilon)/\mathbf{A}(n+\varepsilon)$, and in the latter expression the convergence is faster than any power of n, so that now by taking a moderately large number of coefficients of P and a moderately large value of n we get very precise values for the power series $\kappa^0(\varepsilon)$. (For instance, using 100 terms of P and taking $n = 100$ gives the first 15 coefficients κ_j^0 to 300 decimal digits in under 10 seconds on a normal PC.)

We did these calculations (to 300 digits) for both κ_j and κ_j^0 for V_{12} and several other cases, each time finding agreement of the two series in (6.14) to the precision of the calculation. We then tried to recognize the coefficients κ_j^0 beyond the values $j \le 3$ that were predicted by the Gamma Conjecture and proved by the calculations in the last two sections. It turned out that up to $j = 10$ (in the V_{12} case) these values were always polynomials in Riemann zeta values (or Riemann zeta values and the Euler constant γ if we work with the κ_j instead). The results are cleaner if we use the coefficients λ_j defined by the generating function $\sum_{j=1}^{\infty}\lambda_j\varepsilon^j = \log(\kappa^0(\varepsilon))$, in which case the first ten values are given (within the precision of the calculation) by

$$\lambda_1 \;=\; 0\,, \qquad \lambda_2 \;=\; -2\,\zeta(2)\,, \qquad \lambda_3 \;=\; \tfrac{17}{6}\,\zeta(3)\,, \qquad \lambda_4 \;=\; -3\,\zeta(4)\,,$$
$$\lambda_5 \;=\; \tfrac{7}{3}\,\zeta(5)\,, \qquad \lambda_6 \;=\; -\tfrac{2}{3}\,\zeta(6) \,-\, \tfrac{1}{72}\zeta(3)^2\,, \qquad \lambda_7 \;=\; -\tfrac{5}{3}\,\zeta(7) \,+\, \tfrac{1}{6}\zeta(3)\,\zeta(4)\,,$$
$$\lambda_8 \;=\; \tfrac{29}{12}\,\zeta(8) \,-\, \tfrac{11}{18}\,\zeta(3)\,\zeta(5)\,, \qquad \lambda_9 \;=\; \tfrac{8}{9}\,\zeta(9) \,+\, \tfrac{5}{3}\,\zeta(3)\,\zeta(6) \,+\, \tfrac{11}{3}\,\zeta(4)\,\zeta(5) \,+\, \tfrac{17}{648}\,\zeta(3)^3\,,$$
$$\lambda_{10} \;=\; -\,\tfrac{147}{5}\,\zeta(10) \,-\, \tfrac{59}{18}\,\zeta(3)\,\zeta(7) \,-\, \tfrac{121}{18}\,\zeta(5)^2 \,-\, \tfrac{17}{36}\,\zeta(4)\,\zeta(3)^2\,,$$

involving only Riemann zeta values, as already stated. But for the 11th coefficient we find

$$\lambda_{11} \;=\; 66\,\zeta(11) \,+\, \tfrac{59}{3}\,\zeta(4)\zeta(7) \,+\, \tfrac{110}{3}\,\zeta(5)\,\zeta(6) \,+\, \tfrac{215}{36}\,\zeta(8)\,\zeta(3) \,+\, \tfrac{187}{108}\,\zeta(3)^2\zeta(5) \,+\, \tfrac{2}{3}\,\zeta(3,5,3)\,,$$

where the final term involves not an ordinary zeta value but rather the multiple zeta value

$$\zeta(3,5,3) \;=\; \sum_{0<\ell<m<n}\frac{1}{\ell^3 m^5 n^3} \;=\; 0.002630072587647\cdots\;.$$

This suggests that the higher Frobenius limits might be interesting periods in general, and that at least in some cases they are connected with multiple zeta values. We make a few final remarks in this direction. First of all, the first weight in which the ring of multiple zeta values is not generated over $\mathbb{Q}$ by Riemann zeta values only is 8, the multiple zeta values space in both this weight and in weight 10 being 1 bigger (more properly ≤ 1 bigger, since the required linear independence statements are not actually rigorously known), but here neither of these values appear and the first non-trivial multiple zeta value that we see in this example is the number $\zeta(3,5,3)$ in weight 11. This suggests that there may be a connection with Brown's "single-valued multiple zeta values" [6], which also diverge from the ring of ordinary zeta values for the first time in weight 11. However, the connection is not quite clear since the new single-valued multiple zeta value in weight 11, modulo polynomials in Riemann zeta values is not a rational multiple of $\zeta(3,5,3)$, but rather a rational linear combination of $\zeta(3,5,3)$ and the product of $\zeta(3)$ with the non-trivial double zeta value $\zeta(3,5)$. However, a private communication from Brown suggests that there may be an explanation connected with the duality property of the D3 equations (already mentioned in §2) and with his older calculation [5]. Finally, we mention that in the other non-hypergeometric cases we looked at we again found polynomials in Riemann zeta values for the Frobenius limits up to a certain weight but not beyond, and that we could not always recognize the higher values (like κ_7 for the case $(N,d)=(9,1)$). This phenomenon is presumably related to the fact that Fano varieties like V_{12} can be obtained as successive hyperplane sections of Fano varieties of higher dimension, but only up to a certain point, so that it is only up to that limit that the Gamma Conjecture for these higher Fano would predict values that can be expressed by polynomials in Riemann zeta values. In the case of V_{12}, which can be obtained as a 7-fold iterated hyperplane section of a certain 10-dimensional Fano with Picard rank one (namely, the orthogonal Grassmannian of isotropic 5-planes in $\mathbb{C}^{10}$), we have checked that the prediction of the Gamma Conjecture indeed agrees with the numerically found values of λ_j as given above for all $j \leq 10$.

10. Related work/further references. The Gamma Conjecture in our hypergeometric cases (which correspond to complete intersections in toric varieties) follows essentially from the celebrated Quantum Lefschetz Theorem of Givental [13] and Iritani's work [19]; we have given a proof for the sake of completeness. Dubrovin [7] had computed the expansion of all other asymptotics in the case of projective spaces; one can find a different treatment of this subject in [23]. Przyjalkowski [29] defined weak Landau–Ginzburg models for Fano varieties, and proposed candidates for weak Landau–Ginzburg models in our 17 cases. He discovered that the number of the irreducible components in the resolution of the central fiber (which corresponds to the point $t=\infty$ in our notation) is one more than the $h^{1,2}$ of the respective Fano for all of his models. The relation of the Hodge numbers of Fano varieties and the reducible fibers of their Landau–Ginzburg models are explained in [24]. Galkin established modularity of "G-Fano varieties" [11]. He computed the "Apéry constants" for many homogeneous spaces and introduced what he called the "Apéry class" in [10]. The work of van Enckevort and van Straten [30] pertains to the case of Calabi–Yau, rather than Fano, 3-folds; there is an implicit relation to the topology of Fano 4-folds, again by the quantum Lefschetz principle. Finally, modularity for Fano threefolds of all Picard ranks has been recently announced by C. Doran, A. Harder, L. Katzarkov, J. Lewis and V. Przyjalkowski.

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