MULTIPLE ZETA VALUES

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Multiple zeta values are the numbers defined by the convergent series

$$\zeta(k_1, \ldots, k_n) = \sum_{0<m_1<\ldots<m_n} \frac{1}{m_1^{k_1} \ldots m_n^{k_n}}, \tag{1}$$

where $k_1, \ldots, k_n$ are positive integers with $k_n > 1$. Despite this simple-looking definition, these numbers have deep properties and have appeared in recent years in connection with a surprising diversity of topics, including knot invariants, Galois representations, periods of mixed Tate motives, and calculations of integrals associated to Feynman diagrams in perturbative quantum field theory.

We will call the number (1) a *multiple zeta value* of *depth* n and *weight* k, where $k = k_1 + \cdots + k_n$. Clearly $0 < n < k$. There are $\binom{k-2}{n-1}$ multiple zeta values of given weight k and depth n and 2^{k-2} altogether of given weight k, but the vector space they generate over $\mathbb{Q}$ turns out to have a much smaller dimension. For example, there are 1024 multiple zeta values of weight 12, but, according to our high-precision numerical calculations, only 12 linearly independent ones. The main goal of the theory (still very far from being achieved!) would be to understand all the linear relations over $\mathbb{Q}$ among the multiple zeta values of a given weight. More precisely, products of multiple zeta values can be expressed as a finite integral linear combination of other multiple zeta values, so that the $\mathbb{Q}$-vector space R spanned by all multiple zeta values forms a $\mathbb{Q}$-algebra, graded by the weight and filtered by the depth, and one would like to find its structure as an algebra.

The double zeta values ($n = 2$) were already studied by Euler, who—as well as his famous evaluation of the simple zeta values $\zeta(k)$ as rational multiples of π^k when k is even—discovered empirically, and proved in many cases, that these numbers could be written as rational linear combinations of products of two simple zeta values whenever $k_1 + k_2$ is odd. Many special identities can be proved. Two of these, which were conjectured by Hoffman and proved by the author and several other people, are a "duality formula" saying that $\zeta(k_1, \ldots, k_n) = \zeta(k'_1, \ldots, k'_{n'})$ for a certain involution $(k_1, \ldots, k_n) \to (k'_1, \ldots, k'_{n'})$ on index sets, and a "sum formula" saying that the sum of the multiple zeta values of weight k and depth n is equal to $\zeta(k)$ for any integers $0 < n < k$. There are also formulas describing the value of $\zeta(\underbrace{m, \ldots, m}_{n})$ and $\zeta(\underbrace{1, \ldots, 1}_{n}, m)$ for all positive integers n and m. Yet another identity, which I had conjectured a few years ago and which was proved only a few weeks ago by D. Broadhurst, says that

$$\zeta(\underbrace{1, 3, \ldots, 1, 3}_{n}) = \frac{2\pi^{4n}}{(4n+2)!}.$$

A simple proof of this using hypergeometric functions is sketched in the lecture.

The general theory, as already stated, is far from complete. One general result is that any multiple zeta value $\zeta(k_1, \ldots, k_n)$ with $k_1 + \ldots + k_n \not\equiv n \pmod 2$ is a rational linear combination of products (of the same total weight) of multiple zeta values of depth less than n (this includes the two results of Euler for $n = 1$ and $n = 2$ mentioned above). In general, one can show that the number of generators of the ring of multiple zeta values of weight k and depth n is bounded by $N_n(k - n)$, where $N_n(d)$ ($d > 0$) is defined as the dimension of a certain explicitly defined vector space $V_n(d)$ of homogeneous polynomials of degree d in n variables. For example, $V_3(d)$ is the space of homogeneous polynomials $f(x, y, z)$ of degree d satisfying the two relations

$$f(x, y, z) + f(x, z, y) + f(z, x, y) = 0, \quad \tilde{f}(x, y, z) + \tilde{f}(x, z, y) + \tilde{f}(z, x, y) = 0\,,$$

where $\tilde{f}(x, y, z) := f(x, x+y, x+y+z)$. The proof of this upper bound comes from the interaction between two sets of multiplicative relations (the "double shuffle relations"). For $n = 2$ and $n = 3$ the dimensions $N_n(d)$ are nearly completely known: for $n = 2$ (and d even; all $N_n(d)$ for d odd vanish) is equal to $\lfloor d/6 \rfloor$ and for $n = 3$ it is $\geq \lfloor (d^2 - 1)/48 \rfloor$, with conjectural equality. The first of these two assertions, though it has an elementary proof, is intimately related to modular forms on $PSL(2, \mathbb{Z})$ and to the cohomology of this group, and one expects that the theory for general n will be related in a similar way to the cohomology of $PSL(n, \mathbb{Z})$. The connection between double zeta functions and modular forms can be interpreted in several different ways and is one of the most intriguing aspects of the whole theory; it is closely related to the contents of Goncharov's lecture in this conference.