INTERPOLATED APÉRY NUMBERS, QUASIPERIODS OF MODULAR FORMS AND MOTIVIC GAMMA FUNCTIONS

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Dedicated to the memory of Boris Dubrovin, inspirational mathematician

Abstract. We extend the definitions of the sequences used by Apéry in his proof of the irrationality of $\zeta(3)$ to non-integral values of the index and relate the value with index $-1/2$ to the central value of the L-series of the unique normalized cusp form of weight 4 on $\Gamma_0(8)$. We also discuss the notion of quasiperiods of modular forms and relate the Apéry numbers of other half-integral indices to these. We further explain the conjectural relationship of the Taylor expansion around 0 of a different interpolation of the Apéry numbers to a generalized version of the Gamma Conjecture, and discuss interpretations of the various results with families of Calabi-Yau manifolds, mirror symmetry, and motivic gamma functions.

Introduction: the Apéry numbers and their interpolations

The Apéry numbers $A_0 = 1$, $A_1 = 5$, $A_2 = 73, \ldots$ are defined recursively by

$$(n+1)^3 A_{n+1} \,-\, (34n^3 + 51n^2 + 27n + 5)\, A_n \,+\, n^3 A_{n-1} \;=\; 0 \tag{1}$$

together with the initial value $A_0 = 1$ and an arbitrary value of A_{-1}. The fact that they are all integers, which is not at all obvious from this definition, played a key role in Apéry's famous proof of the irrationality of $\zeta(3)$ in 1978. It can be understood from at least three different points of view, one hypergeometric, one modular, and one algebraic-geometric. The first is due to Apéry himself, who gave the explicit closed formula

$$A_n \;=\; \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 \qquad (n \geq 0) \tag{2}$$

for his numbers. This makes their integrality obvious but does not really explain why it holds. A more conceptual explanation in terms of modular forms was found by Beukers [1]: the generating series

$$A(t) \;=\; \sum_{n=0}^{\infty} A_n\, t^n \;=\; 1 + 5t + 73t^2 + 1445t^3 + \cdots \tag{3}$$

of the Apéry numbers, which satisfies a third-order differential equation corresponding to the recursion (1), can be parametrized by

$$A(t_6(\tau)) \;=\; f_6(\tau), \tag{4}$$

where

$$t_6(\tau) \;=\; \frac{\eta(\tau)^{12}\, \eta(6\tau)^{12}}{\eta(2\tau)^{12}\, \eta(3\tau)^{12}} \;=\; q - 12q^2 + 66q^3 - 220q^4 + \cdots \tag{5}$$

($q = e^{2\pi i\tau}$, $\eta(\tau)$ = Dedekind eta-function) is a Hauptmodul for the group $\Gamma_0(6)$ and

$$f_6(\tau) \;=\; \frac{\eta(2\tau)^7\eta(3\tau)^7}{\eta(\tau)^5\,\eta(6\tau)^5} \;=\; 1 + 5q + 13q^2 + 23q^3 + \cdots \tag{6}$$

a certain modular form (Eisenstein series) of weight 2 on $\Gamma_0(6)$, and the integrality of the coefficients of A follows from that of t_6 and f_6. Finally, the algebraic-geometric explanation, which was given (slightly earlier) by Beukers and Peters [2], identifies the differential equation satisfied by $A(t)$ with the Picard-Fuchs differential equation of the family of K3 surfaces given (up to birational equivalence) by the equation

$$\frac{(y-1)(z-1)(x+z-1)(yz-x-z-1)}{xyz} \;=\; \frac{1}{t}\,. \tag{7}$$

More explicitly, one checks that A_n is the constant term of the nth power of the Laurent polynomial occurring on the left-hand side of (7) (so-called Landau-Ginzburg model), from which its integrality is again obvious. Each of these aspects is discussed in detail in [21] a long survey of the arithmetic and geometric properties of differential equations in which the Apéry numbers were used as the running example to illustrate the theory and in which many of the results of the present paper were announced.

In this paper we will extend these ideas in several different directions, essentially by adding a variable and thinking of the original results as those obtained by “fibering out” a larger motive over a torus and seeing how this affects the local or monodromy properties of the arithmetic of the individual fibers. Specifically, we will show:

• that there is a natural holomorphic extension of the Apéry numbers to complex values of the argument n, satisfying a modified form of the same recursion (1) and symmetric with respect to $n \mapsto -n-1$;

• that the interpolated value of A_n at the point of symmetry $n = -1/2$ is proportional to the central value of the Hecke L-series of a certain cusp form f_8 of weight 4 and level 8;

• that the interpolated values of A_n at other half-integral arguments of n are related to the periods and quasiperiods (whose definition we will review) of the same form f_8;

• that similar statements are true for the second Apéry-like sequence $\{B_n\} = \{1, 8, 88, \dots\}$ defined by the recurrence relation

$$(n+1)^3\,B_{n+1} \;-\; 8(4n^3+6n^2+4n+1)\,B_n \;+\; 256n^3\,B_{n-1} \;=\; 0\,, \tag{8}$$

with the interpolated values $B_{n-1/2}$ again related to the periods and quasiperiods of f_8;

• that these facts are related to the monodromy of the Picard-Fuchs differential equation of a certain family of 3-dimensional Calabi-Yau manifolds, one of 14 such families for which this differential equation is hypergeometric and for all of which a similar relationship was verified numerically in [13];

• that there is a second, and in some ways more natural, extension of the Apéry numbers (or the B_n) to complex arguments based on the asymptotics of the solutions of the recursion (1) (or (8)) at infinity, but now satisfying the same recursion as the original numbers without any modification; and finally,

• that the first Taylor coefficients at zero of this new interpolation have an interpretation in terms of the gamma class on a certain Fano variety. The recursion (1) appears as the Mellin transform of the regularized quantum differential equation of its plane in this context, see [12]. This phenomenon was found experimentally in the earlier paper [11] and is shown here to confirm the prediction of the so-called Gamma Conjecture for these varieties.

In the rest of the paper we will discuss each of these aspects in turn.

1. Hypergeometric interpolation, periods, and quasiperiods

As stated above, our purpose in this paper is to extend the definition of the sequence $\{A_n\}$ to non-integral values of n in two different ways and show how each of them leads to interesting extensions of some of the properties of the original numbers. The first, and most straightforward, interpolation arises by observing that we can rewrite (2) for $n \in \mathbb{Z}_{\geq 0}$ as

$$A_n \;=\; \sum_{k=0}^{\infty} \binom{n}{k}^2 \binom{n+k}{k}^2 , \tag{9}$$

since $\binom{n}{k}$ vanishes for $k > n$. The formula now makes sense for any complex value of n, and the function that it defines does indeed interpolate the original Apéry numbers A_n, but somewhat surprisingly it satisfies only a modification of the original recursion involving an additive correction term that vanishes at all integer arguments:

Theorem 1. *The series* (9) *converges absolutely and locally uniformly for all $n \in \mathbb{C}$ and defines a holomorphic function in the entire complex plane. This function is symmetric under $n \mapsto -n-1$ and satisfies the functional equation*

$$(n+1)^3 A_{n+1} \;-\; (34n^3+51n^2+27n+5)\, A_n \;+\; n^3 A_{n-1} \;=\; \frac{8}{\pi^2}\,(2n+1)\,\sin^2 \pi n\,. \tag{10}$$

A more interesting fact, which is the point of departure for the current paper, is that the value of the interpolated function A_n at its symmetry point $n = -1/2$ is a simple multiple of the central value of the Hecke L-series of a certain cusp form:

Theorem 2. *Define A_n for $n \in \mathbb{C}$ by* (9)*. Then*

$$A_{-1/2} \;=\; \frac{16}{\pi^2}\, L(f_8, 2)\,, \tag{11}$$

where

$$f_8(\tau) \;=\; \eta(2\tau)^4 \eta(4\tau)^4 \;=\; q - 4q^3 - 2q^5 + 24q^7 - \cdots \tag{12}$$

is the unique normalized Hecke eigenform in $S_4(\Gamma_0(8))$.

The statement of Theorem 2 is attractive, but at the same time somewhat mysterious, since it relates $A_{-1/2}$ to a modular form on a completely different modular group from the one occurring in the modular parametrization (4). The explanation was sketched in [21] and has to do with the fact that the double cover of the variety (7) obtained by replacing t^{-1} by w^2 has a Hasse-Weil zeta function containing $L(f_8, s)$ as a factor. But then there is a second mystery. The cusp form f_8, like any normalized Hecke eigenform, has two basic periods, which we can take to be ω_+ and $i\omega_-$ with $\omega_\pm \in \mathbb{R}$ defined by

$$\omega_+ \;=\; \int_0^\infty f_8(it)\, t\, dt \;=\; \frac{L(f_8,2)}{4\pi^2}\,, \qquad \omega_- \;=\; \int_0^\infty f_8(it)\, dt \;=\; \frac{L(f_8,1)}{2\pi} \;=\; \frac{2\,L(f_8,3)}{\pi^3}\,. \tag{13}$$

On the other hand, from (10) we see that the values of A_n for $n \in \mathbb{Z} + \frac{1}{2}$ lie in the $\mathbb{Q}$-vector space spanned by the three numbers $A_{-1/2}$, $A_{1/2}$, and π^{-2}. Since we know from Theorem 2 that $A_{-1/2} = 64\omega_+$, it is reasonable to guess that $A_{1/2}$, and therefore also $A_{n+1/2}$ for any $n \in \mathbb{Z}$, is in the $\mathbb{Q}$-linear span of the three numbers ω_+, ω_-, and π^{-2}. But this turns out not to be the case. We can compute all of the numbers involved to high precision (the series defining $A_{\pm 1/2}$ and $L(f_8, 3)$ converge very slowly and the one defining $L(f_8, 2)$ diverges, but they can be all computed rapidly and accurately by using standard convergence acceleration techniques and the integral representation of the L-function). When we do so, we indeed find

$$A_{-1/2} \;=\; 64\,\omega_+ \;=\; 1.1186363871641870683496192575256409167948575515293611914\,8\cdots\,,$$

in accordance with Theorem 2, but the two further numerical values

$$\begin{aligned} A_{1/2} &= 1.6719543024114185394024524459306932856672407502037904790003 8\cdots\,, \\ 64\,\omega_- &= 3.6109143132953392967615735950910519401218540144056782090236 7\cdots \end{aligned}$$

do not yield any $\mathbb{Q}$-linear dependencies among $A_{-1/2}$, $A_{1/2}$, π^{-2}, and ω_- with small coefficients.

The resolution of this mystery lies in the notion of *quasiperiods* of modular forms. These numbers, which can be viewed as "algebraic de Rham theory made concrete," are a simple generalization of differentials of the second kind, extending the well-known corresponding notion for elliptic curves discovered by Legendre in the 18th century. They were discovered in the context of modular forms and their entire theory worked out by Martin Eichler [8] in 1957, but were then forgotten and rediscovered some 60 years later by Francis Brown [5], [6] and also by us at about the same time (Spring 2015) in the context of the present paper, and are studied and calculated numerically in [13] in connection with the transition matrices of solutions of the Picard-Fuchs differential equation for hypergeometric families of Calabi-Yau 3-folds. We will summarize their definition and main properties briefly in Section 4, referring to any of the above-cited papers for more details. Roughly speaking, to any Hecke eigenform f one can associate two quasiperiods, well-defined up to an algebraic (and in this case rational) multiple of its two periods, defined by integrating a certain meromorphic modular form over appropriate cycles. Computing these two quasiperiods for f_8 numerically to high precision then suggests the following theorem, whose more precise statement and proof will be described briefly in Section 4 and which will follow essentially the same lines as the proof of Theorem 2 given in the next section:

Theorem 3. *The number $A_{1/2}$ defined by* (9) *is a rational linear combination of $A_{-1/2}$, $1/\pi^2$, and the real quasiperiod associated to the Hecke eigenform f_8.*

2. Proof of Theorems 1 and 2

The proofs of both Theorems 1 and 2 were given in [21] (and in fact, as mentioned there, an identity equivalent to (11) was proved independently at about the same time by Rogers, Wan, and Zucker [17]), but we will repeat them briefly here for the reader's convenience and because the same arguments with only minor modifications work also for Theorem 3 as well as for the corresponding statements for the B-sequence discussed in the next section.

It is convenient to shift the index by $\frac{1}{2}$, since this puts the symmetry point of the function at 0 and also makes the functional equation simpler. We therefore define a function $\mathcal{A}(x)$ with $\mathcal{A}(n+\frac{1}{2}) = A_n$ by

$$\mathcal{A}(x) \;=\; \sum_{k=0}^{\infty} \alpha_k(x), \qquad \alpha_k(x) \;=\; \binom{x-1/2}{k}^2 \binom{x+k-1/2}{k}^2. \tag{14}$$

The statements about absolute and locally uniform convergence follow easily from Stirling's formula and standard properties of the gamma function, which show that $\alpha_k(x) = \mathrm{O}(k^{-2})$ as $k \to \infty$, and the symmetry under $x \mapsto -x$ (corresponding to $n \mapsto -n-1$ for the numbers A_n) is obvious because $\alpha_k(x)$ can be written as $\binom{x-1/2}{k}^2\binom{-x-1/2}{k}^2$. Finally, induction on K gives

$$\sum_{k=0}^{K}\bigl((x+\tfrac{1}{2})^3\alpha_k(x+1) - (34\,x^3+\tfrac{3}{2}x)\alpha_k(x) + (x-\tfrac{1}{2})^3\alpha_k(x-1)\bigr) \;=\; 8x\,(2K^2+K-4x^2)\,\alpha_K(x)$$

for all integers $K \geq 0$, and by computing the limiting value of the right-hand side as $K \to \infty$ by Stirling's formula we find that

$$\big(x+\tfrac{1}{2}\big)^3\mathcal{A}(x+1) \;-\; \big(34\,x^3+\tfrac{3}{2}x\big)\mathcal{A}(x) \;+\; \big(x-\tfrac{1}{2}\big)^3\mathcal{A}(x-1) \;=\; \frac{16}{\pi^2}\,x\,\cos^2\pi x\,, \tag{15}$$

which is equivalent to equation (10). This completes the proof of Theorem 1.

For Theorem 2, we rewrite the hypergeometric series defining $A_{-1/2}$ as an integral involving modular functions and modular forms, but on a different group than the one occurring in the modular parametrization (4). Define a function $\beta(\lambda)$ in the closed unit disc by

$$\beta(\lambda) \;=\; F\big(\tfrac{1}{2},\tfrac{1}{2};1;\lambda\big) \;=\; \sum_{k=0}^{\infty}\binom{-1/2}{k}^2\lambda^k \qquad (|\lambda|\leq 1)\,, \tag{16}$$

where $F(a,b;c;x)$ denotes the Euler-Gauss hypergeometric function. Then we have

$$\mathcal{A}(0) \;=\; \sum_{k=0}^{\infty}\binom{-1/2}{k}^4 \;=\; \frac{1}{2\pi i}\oint_{|\lambda|=1}\beta(\lambda)\,\beta(1/\lambda)\,\frac{d\lambda}{\lambda}\,. \tag{17}$$

On the other hand, we have the well-known modular parametrization

$$\beta(\lambda(\tau)) \;=\; \vartheta_3(\tau)^2\,, \tag{18}$$

where $\lambda(\tau)$ (Legendre function) is the standard Hauptmodul for $\Gamma(2)$, defined by

$$\lambda(\tau) \;=\; 16\frac{\eta(\tau/2)^8\eta(2\tau)^{16}}{\eta(\tau)^{24}} \;=\; 1-\frac{\eta(\tau/2)^{16}\eta(2\tau)^8}{\eta(\tau)^{24}} \;=\; \left(\frac{\vartheta_2(\tau)}{\vartheta_3(\tau)}\right)^4,$$

in terms of the two Jacobi functions of weight 1/2

$$\vartheta_2(\tau) \;=\; \sum_{n\in\mathbb{Z}} q^{(n+\frac{1}{2})^2/2} \;=\; 2\,\frac{\eta(2\tau)^2}{\eta(\tau)}\,, \qquad \vartheta_3(\tau) \;=\; \sum_{n\in\mathbb{Z}} q^{n^2/2} \;=\; \frac{\eta(\tau)^5}{\eta(\tau/2)^2\,\eta(2\tau)^2}\,.$$

Using the modular transformation properties

$$\frac{1}{\lambda(\tau)} \;=\; \lambda\Big(\frac{\tau}{1-\tau}\Big)\,, \qquad \vartheta_3\Big(\frac{\tau}{1-\tau}\Big)^2 \;=\; (1-\tau)\,\vartheta_2(\tau)^2$$

and the modular form identity

$$\frac{1}{2\pi i}\,\vartheta_3(\tau)^2\,\vartheta_2(\tau)^2\,\frac{\lambda'(\tau)}{\lambda(\tau)} \;=\; 2\,f_8(\tau/4)\,,$$

with f_8 defined as in (12), we obtain the integral representation

$$\mathcal{A}(0) \;=\; 2\int_0^2 (1-\tau)\,f_8(\tau/4)\,d\tau\,, \tag{19}$$

where the integral is taken along the hyperbolic geodesic from 0 to 2 (= Euclidean semicircle with center 1 and radius 1), which is mapped by λ isomorphically to the unit circle. Since f_8 is a cusp form, we can replace this path of integration by the difference of the two vertical lines from 0 to $i\infty$ and from 2 to $i\infty$, and since $f_8(\tau+\frac{1}{2}) = -f_8(\tau)$ (because f_8 has a q-expansion containing only odd powers of q), this gives finally

$$\mathcal{A}(0) \;=\; 2\Big(\int_0^{\infty}-\int_2^{\infty}\Big)\,(1-\tau)\,f_8(\tau/4)\,d\tau \;=\; -4\int_0^{\infty}\tau\,f_8(\tau/4)\,d\tau \;=\; 64\,\omega_+$$

as desired.

3. A second Apéry-like sequence and the proof of Theorem 3

To understand the situation better we will have to consider not only the Apéry numbers A_n, but also a second sequence defined by the terminating hypergeometric sum

$$B_n \;=\; \sum_{k=0}^{n} \binom{2k}{k}^2 \binom{2n-2k}{n-k}^2 \qquad (n \ge 0) \tag{20}$$

and satisfying the recurrence relation (8). We will give the modular interpretation of this sequences in a moment, but first we do the analogue of what was done in §1 by describing an interpolation of B_n to non-integral values of n. This has both similarities and points of diference with the A_n case.

The most obvious way to interpolate the B_n, imitating what we did in the Apéry case, would be to rewrite the sum in (20) in the form

$$B_n \;=\; \binom{2n}{n}^2 \sum_{k=0}^{\infty} \binom{n}{k}^4 \binom{2n}{2k}^{-2}$$

and then take the right-hand side of this expression as a definition for all values of n, where the pre-factor $\binom{2n}{n}^2$ is interpreted as $\frac{\Gamma(2n+1)^2}{\Gamma(n+1)^4}$, since the sum again can be checked to converge like $1/k^2$. However, the function that we would obtain this way would be only meromorphic, with double poles at all half-integers. Instead, we replace the sum in (20) by

$$B_n \;=\; 16^n \sum_{k=0}^{\infty} \binom{n}{k}^2 \binom{k-\frac12}{n}^2 \;:=\; 16^n \sum_{k=0}^{\infty} \binom{-\frac12}{k}^2 \biggl(\frac{\Gamma(\frac12)}{\Gamma(n-k+1)\Gamma(k+\frac12-n)}\biggr)^2 ,$$

which again agrees with the original definition if n is a non-negative integer but now makes sense for any complex number n, since the sum converges like $1/k^2$ and has no poles. This gives a natural holomorphic continuation of the function $n \mapsto B_n$ to the complex plane. This extrapolation is actually simpler than the one for the Apéry numbers, because the value of B_n for integral $n < 0$ vanishes (as opposed to A_n for $n < 0$, which coincides with $A_{|n|-1}$) and the three-term functional equation satisfied by B_n remains true unchanged for complex values of n (as opposed to A_n, where the right-hand side of the identity had to be multiplied by a multiple of $\sin^2 \pi n$). We state these results in the form of a theorem, again shifting the index by 1/2 to make the functional equation and other properties of the function simpler, and also removing the factor 16^n in the above sum, which was useful for integrality when n was integral but is only a nuisance when it is not.

Theorem 4. *The sum*

$$\mathcal{B}(x) \;:=\; \sum_{k=0}^{\infty} \binom{-\frac12}{k}^2 \biggl(\frac{\Gamma(\frac12)}{\Gamma(x-k+\frac12)\Gamma(k+1-x)}\biggr)^2 \tag{21}$$

converges for all complex values of x and defines an entire function with $\mathcal{B}(n+\frac12) = 16^{-n} B_n$ for $n \in \mathbb{Z}_{\ge 0}$, $\mathcal{B}(n+\frac12) = 0$ for $n \in \mathbb{Z}_{<0}$ and $\mathcal{B}(-n) = \mathcal{B}(n)$ for $n \in \mathbb{Z}$, and satisfying the functional equation

$$\Bigl(x+\frac12\Bigr)^3 \mathcal{B}(x+1) \;-\; \Bigl(2\,x^3+\frac{x}{2}\Bigr)\mathcal{B}(x) \;+\; \Bigl(x-\frac12\Bigr)^3 \mathcal{B}(x-1) \;=\; 0\,. \tag{22}$$

for all $x \in \mathbb{C}$. The values of $\mathcal{B}(x)$ and $\mathcal{A}(x)$ at $x=0$ and $x=1$ are related by the formulas

$$\mathcal{B}(0) \;=\; \mathcal{A}(0)\,, \qquad 6\,\mathcal{B}(1) \;=\; \mathcal{A}(0) \;+\; \mathcal{A}(1) \;+\; \frac{1}{\pi^2}\,. \tag{23}$$

The proof of this is very similar to that of Theorem 1, so we skip the details. The $1/k^2$ convergence of the series (21) is checked as before and the interpolation properties $\mathcal{B}(n+\frac{1}{2}) = B_n$ for $n \in \mathbb{Z}_{\geq 0}$ and $\mathcal{B}(n+\frac{1}{2}) = 0$ for $n \in \mathbb{Z}_{<0}$ are obvious, since the series defining $\mathcal{B}(x)$ agrees term-for-term with that defining $B_{x-1/2}$ if x is a positive half-integer and vanishes term-for-term (actually doubly, so $\mathcal{B}$ has double zeros at these points) if x is a negative half-integer. The symmetry property for $x \in \mathbb{Z}$, which was obvious for $\mathcal{A}(x)$ because each terms of the series defining it was individually even, now follows from the recursion (22), which for $x=0$ gives $B(1) = \mathcal{B}(-1)$ and then for other integral values of x gives the equality between $\mathcal{B}(x)$ and $\mathcal{B}(-x)$ by induction. The functional equation is proved just the same way as for the Apéry numbers, with the sum obtained by inserting the definition into all three terms on the left telescoping. Finally, the last statement of the theorem follows by direct comparison of the hypergeometric sums defining the numbers in question: at $x=0$ we have

$$\mathcal{B}(0) \;=\; \sum_{k=0}^{\infty} \binom{-1/2}{k}^4 \;=\; \mathcal{A}(0)$$

while at $x=1$ we have

$$\mathcal{A}(1) \;=\; \sum_{k=0}^{\infty} \binom{1/2}{k}^2 \binom{-3/2}{k}^2 ,$$

and

$$\mathcal{B}(1) \;=\; \sum_{k=1}^{\infty} \binom{-1/2}{k}^2 \binom{-1/2}{k-1}^2 \;=\; \sum_{k=0}^{\infty} 4k^2 \binom{-1/2}{k}^2 \binom{1/2}{k}^2 , \tag{24}$$

so that the desired formula $\mathcal{A}(0) + \mathcal{A}(1) - 6\mathcal{B}(1) = \pi^{-2}$ follows as in the proof of Theorem 1 by taking the limit as $K \to \infty$ of the inductively proved identity

$$\sum_{k=0}^{K} \left[\binom{-1/2}{k}^4 + \binom{1/2}{k}^2 \binom{-3/2}{k}^2 - 24k^2 \binom{-1/2}{k}^2 \binom{1/2}{k}^2 \right] \;=\; 2\,(2K+1)^2 \binom{-1/2}{K}^4 .$$

We end by giving the modular interpretation of the numbers B_n, or rather of their generating series

$$B(t) \;=\; \sum_{n=0}^{\infty} B_n\, t^n \;=\; 1 + 8q + 88q^2 + 1088q^3 + \cdots .$$

Since $\binom{2k}{k} = 4^k \binom{-1/2}{k}$, we see that this series is simply $\beta(16t)^2$, with $\beta(\lambda)$ as in (16), so the identity (18) used in the proof of Theorem 2 immediately gives the modular parametrization

$$B(t_4(\tau)) \;=\; f_4(\tau) \tag{25}$$

of $B(t)$, where

$$t_4(\tau) \;=\; 16\lambda(2\tau) \;=\; \frac{\eta(\tau)^8 \eta(4\tau)^{16}}{\eta(2\tau)^{24}} \;=\; q - 8q^2 + 44q^3 - 192q^4 + \cdots \tag{26}$$

is a Hauptmodul for $\Gamma_0(4)$ (because $\lambda(\tau)$ is a Hauptmodul for $\Gamma(2)$) and

$$f_4(\tau) \;=\; \theta_3(2\tau)^2 \;=\; \frac{\eta(2\tau)^{20}}{\eta(\tau)^8 \eta(4\tau)^8} \;=\; 1 + 8q + 24q^2 + 32q^3 + 24q^4 + \cdots \tag{27}$$

is a modular form (Eisenstein series) of weight 2 on $\Gamma_0(4)$. All of this is exactly like the situation for the A's, and leads to the same mystery, though now with different numbers: while there the interpolated values of a sequence having a modular parametrization coming from level 6 were related to an L-value of a cusp form f_8 with the different level 8, here a sequence whose modular parametrization comes from level 4 is related (by virtue of the last line of Theorem 4) to the same level 8 L-value. The underlying geometry for this will be discussed in Section 5, but first we should describe the quasiperiods associated to the form f_8 and their relationship to the numbers $A_{n+1/2}$ and $B_{n+1/2}$.

4. Periods and quasiperiods of modular forms

In this section we review the definitions and basic properties of periods and quasiperiods of modular forms and then briefly describe the calculation of the quasiperiods in the case of the Hecke eigenform f_8 and the proof of Theorem 3.

We begin first by recalling the definition of the periods of a Hecke eigenform. For f_8 we could define the two periods ω_+ and $i\omega_-$ by equation (13), but for other eigenforms f of weight 4 this would not always work since $L(f,2)$ might vanish, and for forms f of weights k smaller or larger than 4 it is not the right approach at all since there are $k-1$ critical values $L(f,n)$ $(0 < n < k)$ but only two independent periods. A better way is to use Eichler integrals. Let f be a Hecke eigenform of weight k on a congruence subgroup Γ of $SL_2(\mathbb{Z})$ (in our cases, always $\Gamma_0(N)$ for some $N \in \mathbb{N}$). We will assume for convenience that f has Fourier coefficients in $\mathbb{Q}$, and will also not worry about integrality conditions, defining only a 2-dimensional $\mathbb{Q}$-vector space of periods with basis ω_+, $i\omega_-$ for some numbers $\omega_\pm \in \mathbb{R}^\times/\mathbb{Q}^\times$. For any $f \in M_k(\Gamma)$ we denote by $\widetilde{f}$ an Eichler integral of f, i.e., any holomorphic function in the upper half-plane satisfying $D^{k-1}\widetilde{f} = f$, where $D = q\,\frac{d}{dq} = \frac{1}{2\pi i}\,\frac{d}{d\tau}$ is the normalized differentiation operator. Bol's identity tells us that $D^{k-1}\big(\widetilde{f}|_{2-k}g\big) = f|_k g$ for any $g \in SL_2(\mathbb{R})$, where $|_k g$ denotes the usual "slash" operator $f|_k\big(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\big)(\tau) = (c\tau+d)^{-k} f\big(\frac{a\tau+b}{c\tau+d}\big)$, and hence that $r_{f,\gamma} := \widetilde{f} - \widetilde{f}|_{2-k}\gamma$ is a polynomial in τ of degree $\le k-2$ for every $\gamma \in \Gamma$. The map $r_f : \gamma \mapsto r_{f,\gamma}$ is then a cocycle on Γ with values in the space $V = V_{k-2}$ of all such polynomials (with the Γ-action $P \mapsto P|_{2-k}\gamma$), and the induced map from $M_k(\Gamma)$ to $H^1(\Gamma, V)$ is injective. If $f = \sum_{n=1}^\infty a_n q^n$ is a cusp form, we can normalize the choice of $\widetilde{f}$, and hence of the cocycle r_f within its cohomology class $[r_f]$, by choosing $\widetilde{f}(\tau) = \sum_{n=1}^\infty n^{1-k} a_n\, q^n$, and if f is also a Hecke eigenform as above, then with this choice all of the polynomials $r_f(\gamma)$ belong to $V(\mathbb{Q})\omega_+ \oplus V(\mathbb{Q})i\omega_-$ for some real numbers $\omega_\pm$ that are well-defined up to rational multiples. If Γ is $\Gamma_0(N)$ and we enlarge it by adding the Fricke involution $W_N = \frac{1}{\sqrt{N}}\big(\begin{smallmatrix} 0 & -1 \\ N & 0\end{smallmatrix}\big)$, under which f is automatically invariant up to sign, then the specialization of this statement to $\gamma = W_N$ gives the usual proportionality of the critical values of $L(f,n)$ (up to powers of π) with the numbers $\omega_\pm$.

For the quasiperiods we have to refine this story. The cocycle r_f for $f \in S_k(\Gamma)$ belongs to the subgroup $H^1_{\mathrm{par}}(\Gamma, V)$ of parabolic cohomology (meaning that $r_f(\gamma) \in V|_{2-k}(1-\gamma)$ for all parabolic elements γ of Γ), and the map $f \mapsto [r_f]$ together with its complex conjugate induce an isomorphism between $S_k(\Gamma) \oplus \overline{S_k(\Gamma)}$ and $H^1_{\mathrm{par}}(\Gamma, V)$ that is Hecke equivariant with respect to the natural action of Hecke operators for Γ on $H^1(\Gamma)$. But this is not the right way to get good arithmetic properties, because complex conjugation is not an algebraic operation. To see this, think of the case $k = 2$; then $S_2(\Gamma)$ is isomorphic to the space of holomorphic 1-forms on the compact modular curve $X_0(N)$, and the isomorphism just mentioned corresponds to the usual Hodge decomposition $H^1 = H^{1,0} \oplus H^{0,1} = \Omega^1 \oplus \overline{\Omega^1}$ valid for any compact Riemann surface, but this decomposition is transcendental and in algebraic contexts must be replaced by the isomorphism between H^1 and the quotient of the space of "differentials of the second kind" (meaning meromorphic 1-forms having residue 0 at all of their poles, or better, meromorphic 1-forms that are locally the derivatives of meromorphic functions) by the subspace of exact differentials (global derivatives of meromorphic functions). For higher weights this means that we will have to work with the spaces $M^!_k(\Gamma)$ and $M^{\mathrm{mer}}_k(\Gamma)$ of weakly holomorphic or meromorphic modular forms of weight k on Γ, rather than with just the holomorphic ones. (Recall that "weakly holomorphic" means "holomorphic in the upper half-plane but meromorphic at the cusps".) Instead of thinking of usual cusp forms as holomorphic modular forms that are small at infinity, we should see them as those that have vanishing constant terms at all cusps and hence can be integrated any number of times. We thus define the space $S_k^!(\Gamma)$ of *weakly holomorphic cusp forms of weight* k as the space of forms in $M_k^!(\Gamma)$ satisfying the same vanishing condition (i.e., having at every cusp a Fourier expansion of the form $\sum_{n\gg-\infty} c_n q^n$ with $c_0 = 0$), and the larger space $S_k^{\mathrm{mer}}(\Gamma)$ of *meromorphic cusp forms* or *cusp forms of the second kind* as the space of forms in $M_k^{\mathrm{mer}}(\Gamma)$ that are locally $(k-1)$st derivatives (i.e., that satisfy the above condition at cusps and that have a Laurent series with vanishing coefficients of $(\tau-\tau_0)^{-i}$ for all $0 < i < k$ at all poles τ_0 in the upper half-plane). For f belonging to either of these two spaces, we can define an Eichler integral $\widetilde{f}$ of f just as before (but replacing "holomorphic" by "meromorphic" in its definition in the case of $S_k^{\mathrm{mer}}(\Gamma)$), and Bol's identity implies just as before that r_f is a cocycle (and in fact a parabolic cocycle) in $H^1(\Gamma, V)$. We then get an identification of the parabolic cohomology group $H^1_{\mathrm{par}}(\Gamma, V)$ with the space $\mathbb{S}_k(\Gamma) = S_k^{\mathrm{mer}}(\Gamma)/D^{k-1}(M_{2-k}^{\mathrm{mer}}(\Gamma)) = S_k^!(\Gamma)/D^{k-1}(M_{2-k}^!(\Gamma))$. This space has twice the dimension of $S_k(\Gamma)$; more precisely, it contains $S_k(\Gamma)$ and the quotient is isomorphic to $S_k(\Gamma)$ as a Hecke module. This means that to each normalized Hecke eigenform f in $S_k(\Gamma)$ we can associate a meromorphic modular form F (which can be chosen if we like to be weakly holomorphic, or even to be holomorphic at all cusps except ∞) that has the same eigenvalues as f with respect to all Hecke operators modulo the $(k-1)$st derivatives of meromorphic (or weakly holomorphic) modular forms of complementary weight $2-k$. However, since $\mathbb{S}_k$ has only a canonical filtration rather than a canonical splitting, this F is only canonically defined up to a rational multiplicative factor (if we choose F to have rational Fourier coefficients) and the additition of a rational multiple of f and of a $(k-1)$st derivative. Once we have chosen F, we fix the choice of its Eichler integral $\widetilde{F}$ by imposing $k-1$ linearly independent conditions with coefficients in $\mathbb{Q}$ on the polynomials $r_F(\gamma) = \widetilde{F}|(1-\gamma)$ (e.g., by requiring that r_F vanishes identically for $\gamma = T = \left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$ and that the constant term of the polynomial $r_F(\gamma)$ vanishes for some chosen $\gamma \notin \langle T\rangle$; cf. [13] for more details and examples). If we make these choices, then all values of the associated cocycle r_F belong to $V(\mathbb{Q})\eta_+ \oplus V(\mathbb{Q})i\eta_-$ for some real numbers $\eta_\pm$ that are well-defined up to multiplication by rational numbers and the addition of rational multiples of the original periods $\omega_\pm$.

We now make this explicit for the special case of the cusp form f_8. This is actually only one of 14 cases of cusp forms $f_N \in S_4(\Gamma_0(N))$, corresponding to the 14 families of Calabi-Yau 3-folds whose corresponding Picard-Fuchs differential equations are hypergeometric (see the next section for more details), with values of N ranging from 8 to 864, the most famous of these being the case $N = 25$ corresponding to the hypergeometric function $\sum_n \frac{(5n)!}{n!^5} t^n$ and to the mirror quintic family that launched the whole field of mirror symmetry. These families are discussed in detail in [13], where an explicit choice of meromorphic cusp form f_N^m with the same Hecke eigenvalues as f_N is made for each of the 14 cases and the periods and quasiperiods computed numerically to high precision and shown to agree with entries of the transition matrices between bases of the spaces of solutions of the Picard-Fuchs equations at the conifold point and at the "MUM point" $t = 0$. For $N = 8$ we can choose as our meromorphic modular form F with the same eigenvalues as f_8 any of the three forms

$$F_1(\tau) = \frac{\eta(2\tau)^{12}\eta(4\tau)^4}{\eta(8\tau)^8}, \quad F_2(\tau) = \frac{\eta(2\tau)^{12}\eta(8\tau)^{16}}{\eta(4\tau)^{20}}, \quad F_3(\tau) = \frac{\eta(4\tau)^{28}}{\eta(2\tau)^4\eta(8\tau)^{16}},$$

where F_3 differs from F_1 by $16f_8$ and from $256F_2$ by the third derivative of a form in $M^!_{-2}(\Gamma_0(8))$. The forms $F_2(\tau)$ and $F_3(\tau)$ are equal to $t_4(2\tau)f_8(\tau)$ and to $t_4(2\tau)^{-1}f_8(\tau)$, respectively, with t_4 as in (26). It is now easy to give the proof of Theorem 3. For the proof of equation (23) we used the second of equations (24), but here the first is more convenient, since it tells us that we can compute the value of $\mathcal{B}(1)$ by an integral just like the one used for the value $\mathcal{B}(0)=\mathcal{A}(0)$ in equation (17), but with the differential $d\lambda/\lambda$ replaced by simply $d\lambda$. That means that we obtain $\mathcal{B}(1)$ simply by multiplying the integrand $(1-\tau)f_8(\tau/4)$ in (19) by $\lambda(\tau)=\frac{1}{16}t_4(\tau/2)$. This expresses $8\mathcal{B}(1)$ as the period integral $\int_0^2(1-\tau)F_2(\tau/4)\,d\tau$ of the weakly holomorphic cusp form F_2 as desired, the integral being convergent because $F_2(\tau)$ vanishes at the cusps $\tau=0$ and $\tau=1/2$ of $\Gamma_0(8)$.

5. Geometric interpretation: fibered motives

We first describe in abstract terms the geometric/motivic ideas underlying the calculations of the last sections, and then say in a little more detail how they look in our special case.

One way of proving an identity of the form $\int_{x=-\infty}^{\infty}\varphi(x)dx=\int_{y=-\infty}^{\infty}\psi(y)dy$ is to show that there exists a space $\mathcal{S}$ such that x and y are two functions on it, and $\varphi(x)$ resp. $\psi(x)$ are the measures of their level subspaces; then by Fubini, both integrals are equal to the measure of $\mathcal{S}$. This is a metaphoric rendering of a persistent theme in the arithmetician's study of differential equations that come from geometry as follows. Suppose we are given a hypergeometric variation of Hodge structures $\mathcal{V}$ on the torus $\mathbf{G}_m=\operatorname{Spec}\mathbb{C}[\lambda,\lambda^{-1}]$. Fix $\lambda=\lambda_0$ and write the period(s) of $H=\mathcal{V}_{\lambda_0}$ generically as $\sum\mathbf{\Gamma}(k)\lambda_0^k$, where the fact that $\mathcal{V}$ is hypergeometric simply means that $\mathbf{\Gamma}(k)=\prod_i\Gamma(l_i(k))^{n_i}$ for some linear functions $l_i(k)\in\mathbb{Z}k+\mathbb{Q}$ and some exponents $n_i\in\mathbb{Z}$. We choose a *lift* of $\mathbf{\Gamma}(k)$ to $\widetilde{\mathbf{\Gamma}}(n,k)$ given by the formula $\widetilde{\mathbf{\Gamma}}(n,k)=\prod_i\Gamma(\widetilde{l_i}(n,k))^{n_i}$, where $\widetilde{l_i}(n,k)$ are now linear functions in *two* variables (i.e. belonging to $\mathbb{Z}n+\mathbb{Z}k+\mathbb{Q}$) such that $\widetilde{l_i}(0,k)=l_i(k)$, and set formally

$$C_n\;=\;\sum_k\widetilde{\mathbf{\Gamma}}(n,k)\lambda_0^k\quad(n\in\mathbb{C}).\tag{28}$$

The expressions C_n satisfy a finite-length recursion $R(C_n)=0$ for n belonging to any coset $n_0+\mathbb{Z}$, and there is a corresponding linear differential operator L (the formal Mellin transform of R) that formally annihilates the doubly infinite series $\sum_{n\in n_0+\mathbb{Z}}C_nt^n$ for any $n_0\in\mathbb{C}$. (In practice, $\mathbf{\Gamma}(k)$ usually has the factor $k!$ in the denominator, as in the case of Gauss's ${}_{m+1}F_m$, so that the summation in (28) can be chosen to be over $k\in\mathbb{Z}_{\geq 0}$. Further, the series in n normally diverges, but the function $\Phi_{n_0}(t)=\sum_{n\in n_0+\mathbb{Z}_{\geq 0}}C_nt^n$ often has finite radius of convergence and then satisfies an inhomogeneous equation $L\Phi_{n_0}=r(t)$, where $r(t)$ is a finite Laurent series times t^{n_0}.)

Somewhat tautologically, Cauchy's formula says

$$C_n=\int\Phi(t)t^{-n}\frac{dt}{t}\,.\tag{29}$$

In particular, the same quantity

$$C_0=\int\Phi(t)\frac{dt}{t},\tag{30}$$

can be interpreted, depending on one's optic, as (a) the value at 0 of the solution of the recursion R or, alternatively, (b) as the period of the Hodge structure H, or else (c) a period in the Hodge structure arising in the cohomology of the t-torus $\operatorname{Spec}\mathbb{C}\big[t,t^{-1}\big]$ with coefficients in the Hodge module $\mathcal{H}$ given by L. We will say that the Hodge structure H is *fibered out* by t into a Hodge module $\mathcal{H}$ on $\mathbf{G}_m(t)$. The mental picture behind this wording is that of an invertible function, or a "unit", on the motive M that underlies H which turns it into a pencil of motives $\mathcal{M}$ over the t-torus. One of the benefits of the passage to a fibered motive from a variety equipped with a function fibering it into a pencil is that in real-life applications one will typically compactify and resolve in order to arrange a smooth proper morphism, in which case rubbish cohomology classes are often acquired along the way that obscure the picture and cause a lot of struggle with unnecessary details.

Sometimes we will have *two* different lifts of the same one-variable gamma product, corresponding (as we will discuss in a moment) to two different "fibering-outs" of the same motive over two tori $\mathbf{G}_m(t')$ and $\mathbf{G}_m(t'')$. Going back to our metaphor, the role of the space $\mathcal{S}$ is now played by the motive M, while the units t', t'' are analogues of x and y. One should view the triple (H, t', t'') as being a (Hodge) correspondence between $H^1(\mathbf{G}_m(t'), \mathcal{H}')$ and $H^1(\mathbf{G}_m(t''), \mathcal{H}'')$, and expect, generically, the orders of R' and R'' to be equal to the rank of H. The values at integer arguments of the solutions to R' and R'' that are in the respective $\mathbb{Q}$-Betti spaces should span the same space of periods. This conceptual picture can be used as a tool to produce and prove statements about the equality of periods. Relatively straightforward in the cases of pure Hodge structures, such as period matrices of elliptic curves or periods of a conifold Calabi–Yau fiber of the Landau–Ginzburg model of a rank 1 Fano 4-fold (the case treated in this paper), this quickly becomes a source of much less trivial identities as one passes to *iterated* variations of mixed Hodge structures and their special fibers.

Summing up these observations, one anticipates a relation between the value of the *motivic gamma function*

$$\Gamma^{\mathrm{mot}}_{\mathcal{H}}(s) := \int \Phi(t) t^s \frac{dt}{t} \tag{31}$$

at $s = 0$ and the entries of the period matrix of H — provided, of course, that one can make the meaning of the integral precise. We refer the reader to the recent paper [4] for the fundamentals of motivic gammas.

A final remark here is that the comparison of higher *derivatives* rather than values of the motivic gamma functions at integer arguments arising from different fiberings $\mathcal{H}$ of the same Hodge structure H

$$\frac{d^r \Gamma^{\mathrm{mot}}_{\mathcal{H}}(s)}{ds^r} = \int \Phi(t) t^s \log(t)^r \frac{dt}{t} \tag{32}$$

is an equally important subject, pertaining now to the study of mixed, rather than pure, motives. Special cases of Boyd's conjectures on Mahler measures of Laurent polynomials, for example, can be interpreted as statements about the existence of a single mixed motive that can be fibered out in two specific ways as above, see [3] and [9]. In what looks at first like a totally different setup, the *Gamma Conjecture* relates the motivic gamma derivatives to the expansion coefficients of the "gamma class" of a Fano variety whose regularized quantum differential equation is $\mathcal{H}$. In the final section of the paper we will give numerical evidence for this in the Apéry case.

The results of the preceding sections can be viewed as working out a case of the correspondence between two fiberings of a very specific H, and studying the respective motivic gammas. We start with the gamma product $\mathbf{\Gamma}(k) = 256^{-k}\binom{2k}{k}^4 = \binom{-1/2}{k}^4$, and the two lifts corresponding to our sequences $\{A_{n-1/2}\}$ and $\{B_{n-1/2}\}$ are given by $\mathbf{\Gamma}(n,k) = \binom{n-1/2}{k}^2\binom{-n-1/2}{k}^2$ and $\mathbf{\Gamma}(n,k) = \binom{n-1/2}{k}^2\binom{-1/2}{k}^2$ with $\lambda_0 = 1$. The generating series of our gamma product

$$\xi(\lambda) \;=\; \sum_{k=0}^{\infty} \mathbf{\Gamma}(k)\,\lambda^k \;=\; \sum_{k=0}^{\infty} \binom{2k}{k}\left(\frac{\lambda}{256}\right)^k \tag{33}$$

satisfies the differential equation

$$\big(D_\lambda^4 - \lambda(D_\lambda + 1/2)^4\big)\,\xi(\lambda) \;=\; 0 \tag{34}$$

with $D_\lambda = \lambda\frac{d}{d\lambda}$, one of the 14 hypergeometric DEs of order 4 that arise as regularized quantum differential equations of complete intersections in weighted projective spaces. The argument $\lambda_0 = 1$ is the *conifold singularity* of this equation. By Picard, hypergeometric differential equations with rational indices are known to be of geometric origin, or *Picard–Fuchs*, which simply means that their solutions can be represented as integrals of certain algebraic differential forms over relative cycles in a pencil of varieties. In our case, the pullback of (34) under the map of tori given by $\lambda = 256\,\Lambda^2$, i.e. the differential equation

$$\big(D_\Lambda^4 - 256\,\Lambda^2(D_\Lambda + 1)^4\big)\,\Xi(\Lambda) \;=\; 0\,, \tag{35}$$

corresponds to the pencil given by the Laurent polynomial on $\mathbf{G}_m^4$

$$\Lambda^{-1} \;=\; \prod_{i=1}^{4}\Big(x_i + \frac{1}{x_i}\Big)\,. \tag{36}$$

This shows that the space of solutions to (34) (or the dual space of flat sections) is a variation of $\mathbb{Q}$–Hodge structures.

The dimension of the space of analytic local solutions to (34) drops by 1 at the conifold point. The corresponding period matrix is of rank 3, but because of the symplectic polarization a rank 2 $\mathbb{Q}$–Hodge structure H splits off. The standard expectation is that it comes from a modular *newform*, so a correspondence should exist between the rigid Calabi–Yau threefold C given by

$$\prod_{i=0}^{3}\Big(x_i + \frac{1}{x_i}\Big) \;=\; 16\,. \tag{37}$$

and a Kuga-Sato variety. We refer the reader to papers [7], [15], [16], [18], [19], [20] for various aspects of the link between rigid Calabi–Yau threefolds and modular forms. The rank 2 motive M that underlies H can be fibered out "in different directions" that correspond to the choice of the lifts. In our situation one can be looking for special choices that turn $\mathcal{H}$ into a *modular* variation of Hodge structures. This means that in the differential equation $\mathcal{L}\Phi_{1/2}(t) = 0$ that controls $\mathcal{H}$, the solution $\Phi_{1/2}(t)$ can be chosen to be a weight 2 modular form, and t a Hauptmodul for the same congruence subgroup. The modular parameter τ could then be interpreted as the ratio $\frac{\Phi^*_{1/2}(t)}{\Phi_{1/2}(t)}$ of the normalized log to the analytic solution around $t = 0$; the functions $\Phi_{1/2}(t(\tau))$, $\tau\Phi_{1/2}(t(\tau)) = \Phi^*_{1/2}(t(\tau))$ and $\tau^2\Phi_{1/2}(t(\tau))$ form a basis of the space of local solutions of this differential equation around a cusp. Among the entries of the period matrix of H should then be the value of the *Eichler integral* of the weight 4 form $\Phi(t(\tau))t'(\tau)$. This can be made even more precise if one notices that the essential ingredient of the L-function of C computed by the point count is the Mellin transform of the level 4 form $f_8(\tau)$. In our examples, the weight 2 modular forms appearing in (4) and (25) arise exactly in this way from the two lifts of $\mathbf{\Gamma}(k)$ indicated above, and are related to the weight 4 form f_8 by

$$f_6(\tau)\,t_6(\tau)'/t_6(\tau)^{1/2} \;=\; f_8(\tau) \,-\, 9\,f_8(3\tau), \qquad f_4(\tau)\,t_4(\tau)'/t_4(\tau)^{1/2} \;=\; f_8(\tau). \tag{38}$$

In contrast, the forms $f_N(\tau)t'_N(\tau)$ and $f_N(\tau)t'_N(\tau)/t_N(\tau)$ for $N = 4$ or 6 are weight 4 Eisenstein series whose periods involve the number $\zeta(3)$ occurring in the Gamma Conjecture [11]. In our language, these would correspond to "fibering–outs" of a Hodge–Tate structure that arises in a special fiber in a highly reducible hypergeometric variation of mixed Hodge–Tate structures. Even though these two rank 2 Hodge structures appear to be completely dissimilar at first sight, both are members in a continuous family of integrals varying with the parameter s of the motivic gamma function. The reason to pay special attention to the cases where s is an integer or half–integer is that we want our Hodge structures to be defined over $\mathbb{Q}$, which requires $\exp(2\pi i s) \in \mathbb{Q}$.

6. Asymptotics

In this section we discuss the asymptotic properties of various solutions of the recursions (1) and (22). The two cases are very different. In the Apéry case, if we consider the homogeneous version of the shifted recursion (15) (with $\mathcal{A}(x)$ replaced by an unknown solution $F(x)$ and the right-hand side replaced by 0) and assume as Ansatz that $F(x) \sim C^x x^\mu P(1/x)$ for x large, where C and μ are constants P is a power series, then we find two asymptotic solutions, an exponentially large one and an exponentially small one, given by

$$\mathcal{L}_A(x) \;\sim\; \frac{C_0^x}{x^{3/2}}\,P\Big(\frac{1}{64x\sqrt{2}}\Big)\,, \qquad \mathcal{S}_A(x) \;\sim\; \frac{C_0^{-x}}{x^{3/2}}\,P\Big(-\frac{1}{64x\sqrt{2}}\Big)\,, \tag{39}$$

where $C_0 = 17 + 12\sqrt{2} = 33.9705\cdots$ and $P(X) \in \mathbb{Q}[[X]]$ is a power series beginning

$$P(X) \;=\; 1 \,+\, 30X \,+\, 274X^2 \,-\, 17132X^3 \,-\, 444234X^4 \,+\, 41390724X^5 \,+\, \cdots\,,$$

In particular, since the solution space of the recursion in a fixed residue class $\mathbb{Z}+x_0$ ($x_0 \in \mathbb{C}$) is 2-dimensional, we deduce that any unbounded solution of the recursion in this residue class grows exponentially and is asymptotically equal to a multiple of $\mathcal{L}_A(x)$ to all orders in $1/x$, while any bounded solution decays exponentially and is asymptotically equal to a multiple of $\mathcal{S}_A(x)$ to all orders. For example, from Stirling's formula and the Euler-Maclaurin formula one finds easily that $\mathcal{A}(x) \sim 2^{-9/4}\pi^{-3/2}\mathcal{L}_A(x)$ to all orders as $x \to \infty$. Notice that the small solution is asymptotically about 1154^n times smaller than the large one. It is this huge dichotomy that permitted Apéry to prove the irrationality of $\zeta(3)$, since it implies that the ratio of the two solutions of (1) with initial values $(0, 1)$ and $(1, 5)$ tends to its limit $\frac{1}{6}\,\zeta(3)$ with great exponential rapidity.

For the B-case the situation is different because if we make the same Ansatz $C^x x^\mu P(1/x)$ for the recursion (22), then the resulting characteristic equation $C^2 - 2C + 1$ for C has a double root at $C = 1$ (or $C = 16$ if we consider (8) instead), as opposed to the two different roots $C_0^{\pm 1}$ of the corresponding equation $C^2 - 34C + 1 = 0$ in the A-case. This means that there are now two asymptotic solutions of comparable size: a "small" solution

$$\mathcal{S}_B(x) \;\sim\; \frac{1}{x} \,-\, \frac{1}{2^4\,x^3} \,+\, \frac{17}{2^{10}\,x^5} \,-\, \frac{169}{2^{14}\,x^7} \,+\, \frac{50777}{2^{22}\,x^9} \,-\, \cdots \tag{40}$$

in $\frac{1}{x}\mathbb{Q}[[\frac{1}{x^2}]]$ and a "large" (or "logarithmic") solution

$$\mathcal{L}_B(x) \;\sim\; \big(\log(16x) + \gamma\big)\,\mathcal{S}_B(x) \,+\, \frac{5}{24\,x^3} \,-\, \frac{1219}{30720\,x^5} \,+\, \frac{304469}{10321920\,x^7} \,-\, \cdots \tag{41}$$

in $(\log x + \kappa)\,\mathcal{S}_B(x) \,+\, \frac{1}{x^3}\mathbb{Q}[[\frac{1}{x^2}]]$, where the constant $\kappa = \log 16 + \gamma$ ($\gamma =$ Euler's constant) has been chosen for later convenience. This in turn means that, whereas in the A-case we get only a *filtration* of the 2-dimensional solution space, with a uniquely defined solution that is asymptotically equal to $\mathcal{S}_A(x)$ but whose second solution that is asymptotically equal to $\sim$ $\mathcal{L}_A(x)$ that is defined only up to multiples of the first, in the B-case we get a well-defined *basis* of two solutions $S(x)$ and $L(x)$ having the asymptotics $\mathcal{S}_B(x)$ and $\mathcal{L}_B(x)$, respectively. More precisely, this means that the recursion relation (22) has two meromorphic solutions $L_B(x)$

and $S_B(x)$ (from now on we omit the subscript "B" for typographical convenience) determined uniquely by the requirements that

$$S(x) \;=\; \frac{1\,+\,\mathrm{o}(1)}{x}\,, \qquad L(x) \;=\; \frac{\log(16x)\,+\,\gamma\,+\,\mathrm{o}(1)}{x}$$

as $x\to\infty$ with x real. These functions then satisfy the asymptotic formulas $S(x)\sim\mathcal{S}_B(x)$ and $L(x)\sim\mathcal{L}_B(x)$ to all orders in $1/x$ as $x\to\infty$ with x real or as $\Re(x)\to\infty$ with $\Im(x)$ fixed, where $\mathcal{S}_B(x)$ and $\mathcal{L}_B(x)$ are the asymptotic solutions given above, and have poles (of order at most 3) at $x=-\frac12,\,-\frac32,\dots$ as their only singularities.

It follows that *any* solution $F(x)$ of the recursion on a fixed class in $\mathbb{C}/\mathbb{Z}$ is a linear combination $\lambda L(x)+\mu S(x)$, where the coefficients λ and μ are determined by $xF(x)=\lambda(\log(16x)+\gamma)+\mu+\mathrm{o}(1)$ as $x\to\infty$. In particular, it turns out that both $16^{-n}B_n$ and $\mathcal{B}(n)$ for $n\in\mathbb{Z}_{\ge0}$ are multiples of $L(x)$ (where $x=n+\frac12$ or n, respectively), with the coefficient of $S(x)$ being 0 in both cases (this was the reason for the choice of the constant $\log(16)+\gamma$ in the definition of $L(x)$), but that $\mathcal{B}(x)$ for non-half-integral values of x is *not* a multiple of $L(x)$. The full result is given as follows.

Theorem 5. *For general complex values of x, the solutions $\mathcal{B}(x)$ and $\mathcal{B}(-x)$ are given in terms of $L(x)$ and $S(x)$ by*

$$\mathcal{B}(x) \;=\; \frac{3-\cos2\pi x}{2\pi^2}\,L(x)\;+\;\frac{\sin2\pi x}{2\pi}\,S(x)\,,\qquad \mathcal{B}(-x)\;=\;\frac{\cos^2\pi x}{\pi^2}\,L(x)\,. \tag{42}$$

In particular, the numbers $16^{-n}B_n$ and $\mathcal{B}(n)$ ($n\in\mathbb{Z}_{\ge0}$) are multiples of the "large" solution:

$$B_n \;=\; \frac{2}{\pi^2}L(n+\tfrac12)\,,\qquad \mathcal{B}(n)\;=\;\frac{1}{\pi^2}L(n)\qquad (n=0,\,1,\,2,\dots)\,. \tag{43}$$

Corollary 5.1. *The functions $L(-x)$ and $S(-x)$ are given in terms of $L(x)$ and $S(x)$ by*

$$\begin{pmatrix}L(-x)/2\pi^2\\ S(-x)/2\pi\end{pmatrix} \;=\; \frac{1}{2\cos^2\pi x}\begin{pmatrix}3-\cos2\pi x & \sin2\pi x\\ 8\,\tan\pi x & 3-\cos2\pi x\end{pmatrix}\begin{pmatrix}L(x)/2\pi^2\\ S(x)/2\pi\end{pmatrix}\,.$$

Proof. We only need to prove equation (42), since (43) is just its specialization to half-integral x. Since both $\mathcal{B}(x)$ and $\mathcal{B}(-x)$ are solutions of the equation, by the previous remarks it suffices to prove the asymptotic formulas

$$\begin{aligned} x\,\mathcal{B}(x) \;&=\; \frac{3+\cos2\pi x}{2\pi^2}\,\bigl(\log(16x)+\gamma\bigr)\,\log x\;+\;\frac{\sin2\pi x}{2\pi}\;+\;\mathrm{o}(1)\,,\\ x\,\mathcal{B}(-x)\;&=\;\frac{\cos^2\pi x}{\pi^2}\,\bigl(\log(16x)+\gamma\bigr)\,\log x\;+\;\mathrm{o}(1)\end{aligned} \tag{44}$$

as $x\to\infty$. We prove the second statement only, the proof of the first being similar. Suppose that $|x|\to\infty$ with a fixed argument between $-\pi$ and π. From the formula $\Gamma(x)\Gamma(1-x)=\frac{\pi}{\sin\pi x}$ we have

$$\frac{\pi}{\cos^2\pi x}\,\mathcal{B}(-x)\;=\;\sum_{k=0}^{\infty}\binom{-\frac12}{k}^{2}\left(\frac{\Gamma(x+k+\frac12)}{\Gamma(k+1-x)}\right)^{2}\;=\;\sum_{k=0}^{\infty}\binom{-\frac12}{k}^{2}\left(\frac{1}{x+k}\;+\;\mathcal{O}\Bigl(\frac{1}{(x+k)^2}\Bigr)\right).$$

The second term can be estimated easily (by using that the binomial coefficient as $\mathcal{O}(k^{-1/2})$ and breaking up the sum into $k\le|x|$ and $k>|x|$) as $\mathcal{O}(\frac{\log x}{x^2})$, and the first is equal to

$$\sum_{k=0}^{\infty}\left(\binom{-\frac12}{k}^{2}\;-\;\frac{1}{\pi(k+1)}\right)\frac{1}{x+k}\;+\;\frac{1}{\pi(x-1)}\sum_{k=0}^{\infty}\left(\frac{1}{k+1}\;-\;\frac{1}{k+x}\right)\;=\;\frac{c+\log x+\gamma+\mathrm{o}(1)}{\pi x}$$

for a certain constant c. To calculate it, we use the modular parametrization:

$$\begin{aligned} c \;&=\; \sum_{k=0}^{\infty}\Bigl(\pi\binom{-\frac12}{k}^2 \;-\; \frac{1}{k+1}\Bigr) \;=\; \lim_{\lambda\to1}\Bigl(\pi\,\beta(\lambda) \;-\; \frac{1}{\lambda}\,\log\Bigl(\frac{1}{1-\lambda}\Bigr)\Bigr) \\ &=\; \lim_{\tau\to0}\Bigl(\pi\,\theta_3(\tau)^2 \;-\; \frac{1}{\lambda(\tau)}\,\log\Bigl(\frac{1}{1-\lambda(\tau)}\Bigr)\Bigr) \\ &=\; \lim_{\tau\to\infty}\Bigl(-\pi i\tau\,\theta_3(\tau)^2 \;+\; \frac{1}{1\,-\,\lambda(\tau)}\,\log\lambda(\tau)\Bigr) \;=\; \log 16\,, \end{aligned}$$

where we have used the modularity properties $\vartheta_3\bigl(-\frac{1}{\tau}\bigr)^2 = -i\tau\vartheta_3(\tau)^2$ and $\lambda\bigl(-\frac{1}{\tau}\bigr) = 1-\lambda(\tau)$. Finally, the corollary follows directly from the theorem, since

$$\begin{pmatrix} 3-\cos 2\pi x & -\sin 2\pi x \\ 2\,\cos^2\pi x & 0 \end{pmatrix}^{-1} \begin{pmatrix} 2\,\cos^2\pi x & 0 \\ 3-\cos 2\pi x & \sin 2\pi x \end{pmatrix} \;=\; \frac{1}{2\,\cos^2\pi x}\begin{pmatrix} 3-\cos 2\pi x & \sin 2\pi x \\ 8\,\tan\pi x & 3-\cos 2\pi x \end{pmatrix}\,.$$

7. The kappa series and the Gamma Conjecture

In [11], we considered the Frobenius deformation of Apéry's recursion (1), i.e.,

$$(n+\varepsilon+1)^3\,A_{n+1}(\varepsilon) \;-\; P(n+\varepsilon)A_n(\varepsilon) \;+\; (n+\varepsilon)^3\,A_{n-1}(\varepsilon) \;=\; 0$$

with $P(x) = 34x^3+51x^2+27x+5$, and its solution, the sequence of functions $A_n(\varepsilon)$ defined by the initial condition $A_{-1}(\varepsilon)=0$, $A_0(\varepsilon)=1$. We then defined the kappa function by

$$\kappa^0(\varepsilon) \;=\; \lim_{n\to\infty}\frac{A_n(\varepsilon)}{\mathcal{L}_A(n+1/2+\varepsilon)}.$$

The kappa function is meromorphic in the complex plane and holomorphic at 0. Expand kappa in Taylor series around 0 as $\sum_{j=0}^{\infty}\kappa_j^{(0)}\varepsilon^j$. The question was raised in [11] and later investigated in [4] of whether the expansion coefficients $\kappa_j^{(0)}$ are in the algebra of multiple zeta (or merely Riemann zeta) values. We found numerically with high precision that

$$\begin{aligned} &\kappa_0^{(0)} \;=\; 1, \quad \kappa_1^{(0)} \;=\; 0, \quad \kappa_2^{(0)} \;=\; -\frac13\,\pi^2, \quad \kappa_3^{(0)} \;=\; \frac{17}{6}\,\zeta\,(3)\,, \quad \kappa_4^{(0)} \;=\; \frac{1}{45}\,\pi^4, \\ &\kappa_5^{(0)} \;=\; -\frac{17}{18}\,\pi^2\zeta\,(3) + \frac73\,\zeta\,(5)\,, \quad \kappa_6^{(0)} = \frac{4}{945}\,\pi^6 + 4\,\zeta\,(3)^2\,, \\ &\kappa_7^{(0)} \;=\; -\frac79\,\pi^2\zeta\,(5) + \frac{7}{108}\,\pi^4\zeta\,(3) - \frac53\,\zeta\,(7)\,, \\ &\kappa_8^{(0)} \;=\; -\frac{11}{37800}\,\pi^8 + 6\,\zeta\,(5)\,\zeta\,(3) - \frac43\,\pi^2\zeta\,(3)^2\,, \\ &\kappa_9^{(0)} \;=\; \frac89\,\zeta\,(9) + \frac{34}{9}\,\zeta\,(3)^3 + \frac59\,\pi^2\zeta\,(7) + \frac{149}{11340}\,\pi^6\zeta\,(3) + \frac{5}{54}\,\pi^4\zeta\,(5)\,, \\ &\kappa_{10}^{(0)} \;=\; -\frac{107}{249480}\,\pi^{10} - 4\,\zeta\,(5)^2 - 8\,\zeta\,(3)\,\zeta\,(7) + \frac{4}{45}\,\pi^4\zeta\,(3)^2 - 2\,\pi^2\zeta\,(3)\,\zeta\,(5)\,, \end{aligned} \tag{45}$$

involving only Riemann zeta values, while $\kappa_{11}^{(0)}$ is given numerically by

$$\begin{aligned} \kappa_{11}^{(0)} \;=\; &-\frac{503}{680400}\,\pi^8\zeta\,(3) + \frac{199}{5670}\,\pi^6\zeta\,(5) + \frac{49}{270}\,\pi^4\zeta\,(7) - \frac{34}{27}\,\pi^2\zeta\,(3)^3 \\ &- \frac{8}{27}\,\pi^2\zeta\,(9) + \frac{28}{3}\,\zeta\,(3)^2\,\zeta\,(5) + 66\,\zeta\,(11) + \frac23\,\zeta\,(3,5,3)\,. \end{aligned}$$

The kappa function satisfies the original homogeneous (i.e. without the correction term) three–term recursion (1); in particular, its derivatives at any integer argument n can be expressed linearly in terms of the two sets of derivatives, at 0 *and* at 1. Computing further derivatives at 0 and continuing numerically, we find that for the first five expansion coefficients

$$\kappa_{1,j}^{(0)} := \frac{1}{j!}\frac{d^j}{dx^j}\left(\frac{\kappa^{(0)}(x)}{(1+x)^3}\right)\Big|_{x=1}$$

one has

$$\kappa_{1,0}^{(0)} = \frac{1}{6}\,\zeta(3)\,,\quad \kappa_{1,1}^{(0)} = -\frac{1}{90}\,\pi^4,\quad \kappa_{1,2}^{(0)} = -\frac{1}{18}\,\pi^2\zeta(3) + \frac{11}{3}\,\zeta(5)\,, \tag{46}$$

$$\kappa_{1,3}^{(0)} = -\frac{13}{1890}\,\pi^6,\quad \kappa_{1,4}^{(0)} = \frac{59}{3}\,\zeta(7) + \frac{19}{540}\,\pi^4\zeta(3) - \frac{11}{9}\,\pi^2\zeta(5)\,, \tag{47}$$

but

$$\kappa_{1,5}^{(0)} = -\frac{29}{56700}\,\pi^8 - 10\,\zeta(3)\,\zeta(5) - 4\zeta(3,5)\,.$$

(We have introduced the factor $(1+x)^{-3}$ in order to get rid of the "uninteresting" lower–weight components of the derivatives.) Reasoning in the spirit of the Gamma Conjecture, we predicted in [11] that the top–weight components of the leading expansion coefficients of kappa at 0 and at 1, namely the numbers $\kappa_j^{(0)}$, $j = 0, \dots, 10$ and $\kappa_{1,j}^{(0)}$, $j = 0, \dots, 4$ should be equal to the expansion coefficients of the normalized gamma class of the orthogonal Grassmannian $OG(5,10)$ of isotropic 5-planes in a 10-dimensional space endowed with a non-degenerate orthogonal form.

Recall that the normalized gamma class of a smooth variety V of index d is given by the formula

$$\widehat{\Gamma}^{\mathrm{norm}}(V) = \frac{\prod_\alpha \Gamma(1+r_\alpha)}{\Gamma(1+(-K_V)/d)^d},$$

where the product in the numerator is taken with respect to the formal Chern roots r_α of the tangent bundle $\mathcal{T}_V$ (so that $c(T_V) = \prod_\alpha(1+r_\alpha)$). We will explain in the particular case of $OG(5,10)$ how to compute the expansion coefficients of the normalized gamma class with respect to the elements of the Lefschetz basis.

Similarly to the case of the ordinary Grassmannian, where the classes of the Schubert cells are numbered by partitions and expressed as Schur polynomials in the Chern roots of the universal quotient bundle Q, the cell classes of OG are numbered by *strict* partitions—in our case, those that lie within the 4-by-4 box. As in the ordinary case, there is a map from the ring of symmetric functions to the cohomology ring, but the formulas are more complicated. In particular, the elementary symmetric functions e_i, $i = 1, \dots, 4$ evaluated on the Chern roots of Q give twice the cell classes that correspond to the special (one-strip) partitions: $\tau_i = \frac{c_i(Q)}{2}$, while the class $c_5(Q)$ vanishes. Thus, the τ_i's generate the cohomology as algebra, the ideal of relations being generated by the *Pragacz–Ratajski polynomials*:

$$\widetilde{P}_{i,i}(X) = \tau_i^2 + 2\sum_{k=1}^{i-1}(-1)^k\tau_{i+k}\tau_{i-k} + (-1)^i\tau_{2i} \qquad (i = 1, \dots, 4).$$

Looking (e.g. with the help of Gröbner bases) for a linear combination of τ_1^3 and τ_3 annihilated by multiplication by the class τ_1^5, we find that the class $\mathrm{P} = 7\tau_1^3 - 12\tau_3$ is primitive in weight 3, in the sense that $\mathrm{P}\tau_1^5 = 0$, $\mathrm{P}\tau_1^4 \neq 0$. Therefore, the two Lefschetz sl_2–submodules in $H^*(OG(5,10),\mathbb{Q})$ are spanned by $\{\tau_1^i \mid i = 0, \dots, 10\}$ and $\{\mathrm{P}\tau_1^i \mid i = 0, \dots, 4\}$ respectively. Passing from the full Chern class $\mathrm{Ch}(Q,t) = 1 + \sum_{i=1}^5 2\tau_i t^i$ to the Chern character in the usual way, using the isomorphism $\mathcal{T}_{OG} = \bigwedge^2 Q$, which implies the relation between the Chern characters

$$\mathrm{Ch}(\mathcal{T}_{OG},t) = \frac{1}{2}\Big(\mathrm{Ch}(Q,t)^2 - \mathrm{Ch}(Q,2t)\Big),$$

and expanding

$$\mathrm{Ch}(\mathcal{T}_{OG}, t) \;=\; \sum_{i\geq 0} \frac{p_i}{i!} t^i,$$

we compute the normalized gamma class with the formula

$$\widehat{\Gamma}^{\mathrm{norm}}(OG(5,10)) \;=\; \frac{\exp\Big(-p_1\gamma t + \sum_{i=2}^{10} (-1)^i \, \frac{\zeta(i)}{i} \, p_i t^i\Big)}{\exp\Big(-\gamma\tau_1 t + \sum_{i=2}^{10} (-1)^i \, \frac{\zeta(i)}{i} \tau_1^i \frac{t^i}{i}\Big)^8} \pmod{t^{11}}\,.$$

By straightforward computation we prove:

Theorem 6. *The expansion coefficients of the normalized gamma class of $OG(5,10)$ with respect to the Lefschetz basis $\{\tau_1^i \mid i = 0, \ldots, 10\} \cup \{\mathrm{P}\tau_1^i \mid i = 0, \ldots, 4\}$ coincide with the top–weight components of the leading expansion coefficients of the kappa function at* 0 *and at* 1 *as found experimentally and given above in* (45), (46)*:*

$$\widehat{\Gamma}^{norm}(OG(5,10)) \;=\; 1 + \sum_{i=2}^{10} \kappa_i^{(0)} \tau_1^i t^i \;+\; \sum_{j=0}^{4} \kappa_{1,j}^{(0)} \mathrm{P}\tau_1^j t^{j+3}.$$

Afterword

We would like to thank Spencer Bloch, Masha Vlasenko, and Duco van Straten for many illuminating discussions in connection with the present paper. All three are members of the group that undertook to pursue the subject of motivic gamma functions. We would like to mention in particular two papers that followed our discussions at meetings in Bures and at Herstmonceux Castle in June 2016. In [4], Bloch and Vlasenko work out the relation between the Mellin transform associated to a solution of a DN-type equation with the corresponding Apéry series, and link the Apéry constants with periods of limiting mixed Hodge structures. In [10], the present authors and van Straten consider dimensional interpolation of the Riemann-Roch–Hirzebruch formalism for Grassmannians and indicate how Apéry-like numbers can be interpolated with respect to the dimensional parameter as well as the index. In this context we refer the reader to [14] for the foundations of D-modules on tori and the algebraic Mellin transform. We would also like to thank Andrew Kresch for communicating to us facts about cohomology of the orthogonal Grassmannian and for organizing the activity in Konolfingen in Switzerland where parts of this work were first presented.

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