# A MODULAR IDENTITY ARISING FROM MIRROR SYMMETRY

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In this note we will prove the identity stated as Conjecture 7.2 in [2] and at the same time give other and simpler formulas for the modular forms involved.

We begin by recalling the situation in [2]. Let $G(z)$ be the modular form of weight 4 defined by

$$G(z) = \frac{1}{9}\, q^{\frac{4}{3}}\, \Theta_{E_8}(3z,\, z\gamma) = q^{\frac{4}{3}}\ \left(1 + 4q + 14q^2 + 28q^3 + \cdots\right) .$$

Here $z \in \mathfrak{H}$ = upper half-plane, $q = e^{2\pi i z}$ as usual, $E_8$ is the standard even unimodular lattice of rank 8, and $\gamma \in E_8$ a certain element of norm 8 (cf. [2] Conjecture 7.2). Equation (107) of [2] gives a representation of $18G(z)$ as

$$\Bigg(\sum_{(m,3)=1} q^{\frac{m^2}{6}}\Bigg)^8 + \Bigg(\sum_{(m,3)=1} (-1)^m q^{\frac{m^2}{6}}\Bigg)^8 + \Bigg(\sum_{(m,6)=1} q^{\frac{m^2}{24}}\Bigg)^8 - \Bigg(\sum_{(m,6)=1} \left(\frac{12}{m}\right)\, q^{\frac{m^2}{24}}\Bigg)^8$$

(with $m > 0$ in the summations) and the first 50 coefficients of $G(z)$ (multiplied by 9) are listed in [2] Table 1. On the other hand, define the power series

$$\begin{aligned}
\psi(u) &= \exp\left(\frac{\mathcal{F}_1(u)}{\mathcal{F}(u)}\right) = 1 + 11u + 215u^2 + 4757u^3 + \dots ,\\
q(u) &= u\,\exp\left(3\,\frac{\mathcal{F}_1(u) - \mathcal{F}_2(u)}{\mathcal{F}(u)}\right) = u + 15u^2 + 279u^3 + 5729u^4 + \dots ,
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{F}(u) &= F(\tfrac{1}{3}, \tfrac{2}{3}; 1; 27u) = \sum_{n=0}^{\infty} \frac{(3n)!}{n!^3}\, u^n = 1 + 6u + 90u^2 + 1680u^3 + \dots\\
\mathcal{F}_1(u) &= \sum_{n=0}^{\infty} \frac{(3n)!}{n!^3} \left(1 + \frac{1}{2} + \cdots + \frac{1}{3n}\right) u^n = 11u + \frac{441}{2}u^2 + \frac{14258}{3}u^3 + \dots\\
\mathcal{F}_2(u) &= \sum_{n=0}^{\infty} \frac{(3n)!}{n!^3} \left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right) u^n = 6u + 135u^2 + 3080u^3 + \dots
\end{aligned}$$

These functions are related to [2] eqs. (91) and (94): $\mathcal{F}(u) = \phi_0(u,0) = \xi(u,0)$ resp. $\mathcal{F}_1(u) = \frac{\partial}{\partial\rho}\xi(u,\rho)|_{\rho=0}$ resp. $\frac{\partial}{\partial\rho}\phi_0(u,\rho)|_{\rho=0} = 3\mathcal{F}_1(u) - 3\mathcal{F}_2(u)$. Thus $q(u)$ is $\overline{U}_1$ in [2] Proposition 7.5., while $\psi(u)$ here and in op. cit. are the same.

Define coefficients $\beta_n \in \mathbb{Z}$ by the expansion

$$\frac{u}{\psi(u)}\frac{q'(u)}{q(u)} = \sum_{n=0}^{\infty}\beta_n\, q(u)^n \;=\; 1 + 4q(u) + 14q(u)^2 + 40q(u)^3 + \dots .$$

The second part of the following theorem is the statement of Conjecture 7.2 of [2].

**Theorem 1.** **(i).** *The modular form $G(z)$ has the representations*

$$G(z) = \frac{1}{81}\Bigg(\sum_{n\geq1, n\equiv 1 \bmod 3}\sigma_3(n)\, q^{\frac{n}{3}} \;-\; \eta(z)^8\Bigg) = \frac{\eta(3z)^{12}}{\eta(z)^4}. \tag{1}$$

**(ii).** *The coefficients $\beta_n$ are the coefficients in the q-expansion*

$$\sum_{n=0}^{\infty}\beta_n q^{n-\frac{1}{6}} = \frac{G(z)}{\eta(3z)^{12}} = \frac{1}{\eta(z)^4}. \tag{2}$$

**Proof (i)** We claim that $G(z)$ belongs to the space $M_4(\Gamma_0(3),\chi)$ of modular forms of weight 4 and Nebentypus $\chi$ on $\Gamma_0(3)$, where $\chi : \Gamma_0(3) \to \mu_3$ is the character defined by $\chi(\gamma) = e^{2\pi i n/3}$ for $\gamma \equiv \pm\begin{pmatrix} 1 & n \\ 0 & 1\end{pmatrix}$ mod 3. To see this, we use the theory of Jacobi forms as developed in [1]. Theorem 7.1 there tells us that the function $\Theta_{E_8}(\tau, z\gamma) = \Theta_{E_8,\gamma}(\tau,z)$ is a Jacobi form of weight 4 (half the rank of $E_8$) and index 4 (half the norm of $\gamma$) for the full modular group (because $E_8$ is even and unimodular) and in particular satisfies

$$\begin{aligned}\Theta_{E_8,\gamma}(\tau+3, z+1) &= \Theta_{E_8,\gamma}(\tau,z),\\ \Theta_{E_8,\gamma}\left(\frac{\tau}{\tau+1},\frac{z}{\tau+1}\right) &= (\tau+1)^4\exp\left(2\pi i\frac{4z^2}{\tau+1}\right)\Theta_{E_8,\gamma}(\tau,z)\end{aligned}$$

for all $\tau \in \mathfrak{H}$, $z \in \mathbb{C}$. Applying these with $\tau = 3z$ gives the transformation laws

$$G(z+1) = e^{2\pi i/3}\, G(z), \qquad G\left(\frac{z}{3z+1}\right) = (3z+1)^4\, G(z).$$

This proves our claim since $\Gamma_0(3)$ is generated by $\begin{pmatrix} 1 & 1 \\ 0 & 1\end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 3 & 1\end{pmatrix}$. The representations (1) follow immediately since the space $M_4(\Gamma_0(3),\chi)$ is two-dimensional, generated by the two functions occurring in the first formula of (1) (which generate the Eisenstein and the cuspidal part, respectively). Indeed, any form in $M_4(\Gamma_0(3),\chi)$ has a Fourier expansion $aq^{1/3} + bq^{4/3} + \mathrm{O}(q^{7/3})$, and if $a$ and $b$ both vanished then the form $G(z)^3 \in M_{12}(\Gamma_0(3))$ would have a zero of order at least 7 at infinity, contradicting the fact that any non-zero modular form of weight 12 on $\Gamma_0(3)$ has precisely $[PSL_2(\mathbb{Z}) : \Gamma_0(3)] = 4$ zeros in a fundamental domain.

**(ii)** For the second (and main) part of the theorem, define

$$u(z) = \frac{\eta(3z)^{12}}{\eta(z)^{12} + 27\,\eta(3z)^{12}} = q - 15q^2 + 171q^3 - 1679q^4 + \ldots$$

and

$$\begin{aligned} E(z) = \sum_{n,m\in\mathbb{Z}} q^{n^2+nm+m^2} &= 1 + 6\sum_{n=1}^{\infty}\left(\sum_{d|n}\left(\frac{d}{3}\right)\right) q^n \\ &= 1 + 6q + 6q^3 + 6q^4 + 12q^7 + \cdots . \end{aligned}$$

Then $u(z)$ is a Hauptmodul for $\Gamma_0(3)$ sending the two cusps $\infty$ and 0 and the elliptic fixed point $z_0 = \frac{1}{2} + \frac{i}{2\sqrt{3}}$ of $\overline{\mathfrak{H}/\Gamma_0(3)}$ to 0, $\frac{1}{27}$ and $\infty$, respectively, while $E(z)$ is a modular form of weight 1 and quadratic Nebentypus on $\Gamma_0(3)$. We claim that the two functions $E(z)$ and $zE(z)$, expressed as power series in $u$, span the kernel of the second order differential operator

$$L = (27u^2 - u)\frac{d^2}{du^2} + (54u - 1)\frac{d}{du} + 6 .$$

Indeed, the space spanned by the functions $E(z)$ and $zE(z)$ is characterized by $(f/E)'' = 0$ (here $'$ denotes differentiation with respect to $z$), which in terms of $u$ becomes

$$0 = \frac{E}{u'^2}\frac{d^2}{dz^2}\left(\frac{f}{E}\right) = \frac{d^2 f}{du^2} + \frac{u''E - 2u'E'}{Eu'^2}\frac{df}{du} + \frac{2E'^2 - EE''}{E^2u'^2} f .$$

The coefficients of $df/du$ and $f$ are modular functions on $\Gamma_0(3)$: this can be checked directly or by noting that they are constant multiples of $F_1(E, u')/Eu'^2$ and $F_2(E, E)/E^2u'^2$, respectively, where $F_n(\cdot,\cdot)$ denotes the $n$th Cohen bracket (cf. [4] pp. 249–250). Hence they are rational functions of $u$. Computing them explicitly gives the assertion.

On the other hand, it is easily checked that the series $\mathcal{F}$, $\mathcal{F}_1$ and $\mathcal{F}_2$ are the unique power series solutions of the differential equations

$$L(\mathcal{F}) = 0, \qquad L(\mathcal{F}_1) = -(18u + \frac{1}{3})\frac{d\mathcal{F}}{du} - 9\mathcal{F}, \qquad L(\mathcal{F}_2) = -\frac{d\mathcal{F}}{du}$$

with initial conditions $\mathcal{F}(0) = 1$, $\mathcal{F}_1(0) = \mathcal{F}_2(0) = 0$. This implies that $\mathrm{Ker}(L)$ is spanned by $\mathcal{F}$ and $3\mathcal{F}_1 - 3\mathcal{F}_2 + \mathcal{F}\log u$ and that $\mathcal{F}_2 = 3\mathcal{F}_1 + \mathcal{F}\log(1-27u)$.

Comparing this with the previous assertion we deduce that

$$\mathcal{F}(u(z)) = E(z)\,, \qquad q(u(z)) = q\,, \qquad \psi(u(z)) = \frac{q^{-\frac{1}{6}}\,u(z)^{\frac{1}{6}}}{(1-27u(z))^{\frac{1}{2}}}\,. \qquad (3)$$

Equation (2) is therefore equivalent to the formula

$$u^{\frac{5}{6}}(1-27u)^{\frac{1}{2}}/(u'/2\pi i) = G(z)/\eta(3z)^{12}\,,$$

which is easy to verify: the function on the left of this formula is finite and non-zero at all points of $\mathfrak{H}/\Gamma_0(3)$ except $z_0$ because $u$ is a Hauptmodul and at $z_0$ because $u$ has a pole of order exactly 3 there, so the quotient of the two sides of the formula is a modular function with no zeros or poles.

**Remark** The link between modular forms and hypergeometric functions is a classical subject, going back to the work of Fricke and Klein at the beginning of the century, but has appeared more recently in connection with several other questions, including the theory of Mahler measures and the Bloch-Beilinson conjectures about special values of $L$-series as well as mirror symmetry. A survey of some of these connections can be found in the very interesting recent article [3] by F. Rodriguez Villegas. In particular, the first equation in (3), which in any case is undoubtedly not new, appears explicitly in n° 14 of [3] in connection with the computation of the Mahler measure (= average value of $\log|P_u(e^{i\alpha}, e^{i\beta})|$) of the polynomial $P_u(X,Y) = (X+Y+1)^3 - XY/u$ and, for certain rational values of $u$, with the value at $s=2$ of the $L$-series of the elliptic curve over $\mathbb{Q}$ defined by the equation $P_u(X,Y)=0$.

## References

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