ISRAEL MATHEMATICAL CONFERENCE PROCEEDINGS, Vol. 3, 1990

Hecke Operators and Periods of Modular Forms

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To my friend Ilya Piatetski-Shapiro on the occasion of his 60th birthday

1. Introduction. The main goal of this paper is to provide a new proof, based on the theory of periods of modular forms, of the Eichler-Selberg formula for traces of Hecke operators on the full modular group $SL_2(\mathbf{Z})$.

Recall that the *period polynomial* of a cusp form $f(\tau) = \sum_{l=1}^{\infty} a_f(l)\, e^{2\pi i l\tau}$ of weight k on $SL_2(\mathbf{Z})$ is the polynomial

$$r_f(X) = \sum_{n=0}^{k-2} (-1)^n \binom{k-2}{n} r_n(f)\, X^{k-2-n},$$

where the *periods* $r_n(f)$ of f are defined by

$$r_n(f) = \int_0^{i\infty} f(\tau)\,\tau^n\, d\tau = \left(\frac{i}{2\pi}\right)^{n+1} L(f, n+1)$$

($L(f,s)$ = Hecke L-series of f). If f is a normalized Hecke eigenform, i.e. $f|T_l = a_f(l)\, f$ for all l, then it is known from the Eichler-Shimura-Manin theory that $r_m(f) r_n(f)$ is an algebraic multiple of the Petersson scalar product (f,f) whenever n and m are of opposite parity. More precisely, $r_m(f) r_n(f)/i(f,f)$ transforms by σ when f is replaced by a conjugate form $f^\sigma = \sum a_f(l)^\sigma e^{2\pi i l\tau}$, $\sigma \in \mathrm{Aut}(\mathbf{C})$. It follows that the polynomials

$$c_{kl}^0(X,Y) = \sum_{\substack{f\in S_k \\ \text{eigenform}}} \frac{1}{(2i)^{k-3}(f,f)}\, a_f(l) \left(r_f(X) r_f(Y)\right)^- \qquad (l \geqslant 1),$$

where $\left(r_f(X) r_f(Y)\right)^- = \frac{1}{2}\left(r_f(X) r_f(Y) - r_f(-X) r_f(-Y)\right)$ is the odd part of $r_f(X) r_f(Y)$ and the sum is taken over a basis of Hecke eigenforms of S_k, have rational coefficients. The following closed formula for them was proved in [7]: add to $c_{kl}^0(X,Y)$ an "Eisenstein part" $c_{kl}^E(X,Y) \in (XY)^{-1}\mathbf{Q}[X,Y]$ defined by

$$c_{kl}^E(X,Y) = \frac{2k(k-2)!}{B_k}\, \sigma_{k-1}(l) \sum_{n=0}^{k} \frac{B_n}{n!} \frac{B_{k-n}}{(k-n)!} \left[X^{n-1}(Y^{k-2}-1) + (X^{k-2}-1)Y^{n-1}\right],$$

where B_k is the kth Bernoulli number and $\sigma_{k-1}(l) = \sum d^{k-1}$ (sum over positive divisors d of n); then the sum $c_{kl} = c_{kl}^0 + c_{kl}^E$ is given by

$$\begin{aligned} c_{kl}(X,Y) = &\sum_{\substack{ad-bc=l \\ ad>0>bc}} \operatorname{sgn}(bd)\,(cXY + dY + aX + b)^{k-2} \\ &- \frac{2}{k-1} \sum_{\substack{ad=l \\ a,\, d>0}} \Big(B_{k-1}^0(aX + dY) + X^{k-2} B_{k-1}^0\big(\frac{a}{X} - dY\big) \\ &\qquad + Y^{k-2} B_{k-1}^0\big(-aX + \frac{d}{Y}\big) - (XY)^{k-2} B_{k-1}^0\big(\frac{a}{X} + \frac{d}{Y}\big)\Big), \end{aligned} \tag{1}$$

where $B_{k-1}^0(X)$ denotes the modified Bernoulli polynomial $\sum_{r \neq 1} \binom{k-1}{r} B_r X^{k-1-r}$. This can be interpreted as saying that the generating function of the $c_{kl}(X,Y)$ is a quotient of products of Jacobi theta functions.

In this paper we use formula (1) to give a formula for the trace $\operatorname{Tr}(T_l, M_k)$ of the lth Hecke operator acting on the space of modular forms of weight k on $SL_2(\mathbf{Z})$. This approach is quite different from the classical method of integrating a kernel function for T_l. It also produces a formula for the traces which looks different from the usual one: class numbers of imaginary quadratic fields do not appear explicitly (instead, one must count solutions of a certain system of Diophantine equations and inequalities), and, at least in its initial form (stated in §3), the formula contains Bernoulli numbers. By using a certain Bernoulli number identity, we can cast it into a more elementary form. The special case of this identity needed for the case $l = 1$ of the trace formula is amusing enough to be stated here as an exercise for the reader:

PROPOSITION 1. *Let n be a positive odd number. Then*

$$\sum_{r=0}^{n} \binom{n+r}{2r} \frac{B_r}{n+r} = \begin{cases} \pm\frac{3}{4} & \text{if } n \equiv \pm 1 \pmod{12}, \\ \mp\frac{1}{4} & \text{if } n \equiv \pm 3 \text{ or } \pm 5 \pmod{12}. \end{cases}$$

The trace formula which we then obtain gives the generating function of the traces of T_l on M_k (l fixed, k variable) as a simple rational function:

THEOREM 1 (TRACE FORMULA). *Let l be a positive integer. Then*

$$\begin{aligned} -12 \sum_{\substack{k=2 \\ k \text{ even}}}^{\infty} \big[\operatorname{Tr}(T_l,\, M_k) - \tfrac{1}{2}\sigma_{k-1}(l)\big]\, T^{k-2} = &\sum_{\substack{ad-bc=l \\ ad>0>bc}} \frac{\operatorname{sgn}(cd)}{1 - (a+d-c)T + lT^2} \\ &+ \sum_{\substack{ad=l \\ a,\, d>0}} \Big({\sum_{|t| \leqslant a}}^{*} \frac{2}{1 - tT + lT^2} + {\sum_{|t| \leqslant a+d}}^{*} \frac{1}{1 - tT + lT^2} - \sum_{|t| = a+d} \frac{|t|/2}{1 - tT + lT^2} \Big), \end{aligned}$$

where the star on the summation means that the edge terms are to be counted with multiplicity 1/2.

For example, for $l = 1$ the right-hand side of this formula is

$$\frac{-1/2}{1-2T+T^2}+\frac{2}{1-T+T^2}+\frac{3}{1+T^2}+\frac{2}{1+T+T^2}+\frac{-1/2}{1+2T+T^2}$$

and we recover the standard formula for $\dim M_k$ as $\frac{k}{12}+\epsilon_k$ with ϵ_k periodic of period 12.

The usual formula for the trace of T_l (cf. [4], Appendix to Part I) involves the Kronecker-Hurwitz class number $H(N)$, defined for $N > 0$ as the number of $SL_2(\mathbf{Z})$-equivalence classes of positive definite binary quadratic forms of discriminant $-N$, forms equivalent to a multiple of X^2+XY+Y^2 or X^2+Y^2 being counted with multiplicity $\frac{1}{3}$ or $\frac{1}{2}$, respectively. The verification that the trace formula just stated agrees with the classical one relies on the following surprising class number formula, which we prove in §5:

PROPOSITION 2. *For $l \geqslant 1$ and $t \in \mathbf{Z}$, we have*

$$\sum_{\substack{a,b,c,d\in\mathbf{Z}\\ ad-bc=l\\ a+d-c=t\\ c\neq 0}} \big(sgn(a)+sgn(d)\big)\big(sgn(c)-sgn(b)\big) = \begin{cases} 24\,H(4l-t^2) & \text{if } 4l>t^2,\\ 2|t|-2 & \text{if } 4l=t^2,\\ 4|t|-12|u| & \text{if } t^2-4l=u^2>0,\\ 0 & \text{otherwise.}\end{cases} \qquad (2)$$

(Note that each summand equals ± 2, ± 4, or 0 and that only finitely many are non-zero.)

As a second application of (1), we will prove in §6 the following formula expressing the periods of $f|T_l$ as linear combinations of those of f:

THEOREM 2 (ACTION OF HECKE OPERATORS ON PERIODS). *Let l be a positive integer, f a cusp form of weight k on $SL_2(\mathbf{Z})$. Then*

$$r_{f|T_l}(X) = \sum_{\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)} (cX+d)^{k-2}\, r_f\Big(\frac{aX+b}{cX+d}\Big),$$

where the sum is over matrices $\begin{pmatrix}a&b\\c&d\end{pmatrix}$ of determinant l satisfying the conditions

$$a>|c|,\quad d>|b|,\quad bc\leqslant 0,\quad b=0\Rightarrow -\frac{a}{2}<c\leqslant\frac{a}{2},\quad c=0\Rightarrow -\frac{d}{2}<b\leqslant\frac{d}{2}, \qquad (3)$$

This theorem generalizes a result of Manin [5], proved using continued fractions, for the zeroth period $r_0(f|T_l)$.

2. The Petersson scalar product in terms of periods. The application of (1) to the trace formula depends on the fact that the Petersson scalar product (f,g) of two cusp forms f and g can be expressed as a linear combination of products of periods of f and periods of g of opposite parity. Define a map $\rho_k : \mathbf{Q}[X,Y] \to \mathbf{Q}$ (k even, $k > 2$) by

$$\rho_k(X^mY^n) = \begin{cases} \dfrac{m!\, n!}{(m+n-k+2)!(k-2)!} & \text{if } m+n \geqslant k-2,\, m \not\equiv n \pmod 2, \\ 0 & \text{otherwise;} \end{cases} \tag{4}$$

then if f and g have real Fourier coefficients (in particular, if they are normalized eigenforms)

$$\rho_k\big(r_f(X)r_g(Y)\big) = 3\,(2i)^{k-1}\,(f,g).$$

The formula was found by Haberland [2, Section 7] and is also proved in [3, pp. 243–245]. (Haberland states only the case when $f = g$ is a Hecke eigenform, the case we will be using; his formula contains a slight misprint. The definition of $r_n(f)$ in [3] differs from the one here by a factor i^{n+1}.) It follows that

$$\rho_k\big(c^0_{kl}(X,Y)\big) \;=\; -12 \sum_{\substack{f\in S_k\\ \text{eigenform}}} a_f(l) \;=\; -12\,\mathrm{Tr}(T_l, S_k)$$

or, substituting for c^0_{kl} from equation (1),

$$-12\,\mathrm{Tr}(T_l,S_k) \;=\; T_{\mathrm{I}} + T_{\mathrm{II}} + T_{\mathrm{III}} + T_{\mathrm{IV}} + T_{\mathrm{V}} + T_{\mathrm{VI}}, \tag{5}$$

where

$$T_{\mathrm{I}} = \sum_{\substack{ad-bc=l\\ ad>0>bc}} \mathrm{sgn}\,(bd)\,\rho_k\big[(cXY+dY+aX+b)^{k-2}\big]$$

$$T_{\mathrm{II}} = -\frac{2}{k-1}\sum_{\substack{ad=l\\ a,d>0}} \rho_k\big[B^0_{k-1}(aX+dY)\big]$$

$$T_{\mathrm{III}} = -\frac{2}{k-1}\sum_{\substack{ad=l\\ a,d>0}} \rho_k\big[X^{k-2}\,B^0_{k-1}(aX^{-1}-dY)\big]$$

$$T_{\mathrm{IV}} = -\frac{2}{k-1}\sum_{\substack{ad=l\\ a,d>0}} \rho_k\big[Y^{k-2}\,B^0_{k-1}(-aX+dY^{-1})\big]$$

$$T_{\mathrm{V}} = \frac{2}{k-1}\sum_{\substack{ad=l\\ a,d>0}} \rho_k\big[X^{k-2}Y^{k-2}\,B^0_{k-1}(aX^{-1}+dY^{-1})\big]$$

$$T_{\mathrm{VI}} = -\frac{4k(k-2)!}{B_k}\,\sigma_{k-1}(l)\sum_{n=0}^{k}\frac{B_n}{n!}\,\frac{B_{k-n}}{(k-n)!}\,\rho_k\big[(X^{k-2}-1)\,Y^{n-1}\big].$$

(Here we have extended ρ_k to $X^{-1}Y^{-1}\mathbf{Q}[X,Y]$ by $\rho_k(X^{-1}Y^n) = \rho_k(X^nY^{-1}) = 0$.) Since ρ_k is a completely explicit operator, this is already a trace formula: for each weight k and natural number l it expresses $\mathrm{Tr}(T_l, S_k)$ as a finite sum of computable rational numbers. However, it is not yet very pretty. In the next section we compute each of the terms occurring in (5) in terms of a generating function, obtaining a more attractive trace formula. The comparison between this trace formula and the classical one is given in §§4–5.

3. Computation of the trace. In (5), the first term is the main one, the other five being boundary terms coming from matrices $\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)$ of determinant l with $abcd = 0$. We now compute each of these terms in terms of generating functions.

First term. By the quadrinomial theorem,

$$\rho_k\big[(cXY + dY + aX + b)^{k-2}\big] = (k-2)! \sum_{\substack{\alpha,\beta,\gamma,\delta\geqslant 0\\ \alpha+\beta+\gamma+\delta=k-2}} \frac{a^\alpha}{\alpha!}\,\frac{b^\beta}{\beta!}\,\frac{c^\gamma}{\gamma!}\,\frac{d^\delta}{\delta!}\,\rho_k\big[X^{\gamma+\alpha}Y^{\gamma+\delta}\big]$$

$$= \sum_{\substack{\alpha,\beta,\gamma,\delta\geqslant 0\\ \alpha+\beta+\gamma+\delta=k-2\\ \alpha+\delta \text{ odd}}} \binom{\gamma+\alpha}{\gamma}\binom{\gamma+\delta}{\gamma}\binom{\gamma}{\beta} a^\alpha b^\beta c^\gamma d^\delta$$

$$= \mathbf{C}_{T^{k-2}}\Big[\sum_{\alpha,\beta,\gamma,\delta\geqslant 0} \frac{1-(-1)^{\gamma-\beta}}{2}\binom{\gamma+\alpha}{\gamma}\binom{\gamma+\delta}{\gamma}\binom{\gamma}{\beta}(aT)^\alpha(bT)^\beta(cT)^\gamma(dT)^\delta\Big]$$

(here and from now on, $\mathbf{C}_{T^n}[F(T)]$ will denote the coefficient of T^n in a power series $F(T)$)

$$= \frac{1}{2}\,\mathbf{C}_{T^{k-2}}\Big[\sum_{\gamma\geqslant 0}(cT)^\gamma\frac{(1+bT)^\gamma - (-1+bT)^\gamma}{(1-aT)^{\gamma+1}(1-dT)^{\gamma+1}}\Big]$$

(binomial theorem, applied to the sums over α, β and δ)

$$= \frac{1}{2}\,\mathbf{C}_{T^{k-2}}\Big[\frac{1}{1-(a+d-c)T+(ad-bc)T^2} - \frac{1}{1-(a+d+c)T+(ad-bc)T^2}\Big].$$

(geometric series). Hence

$$T_{\mathrm{I}} = \mathbf{C}_{T^{k-2}}\Big[\sum_{\substack{ad-bc=l\\ ad>0>bc}} \frac{\mathrm{sgn}\,(bd)}{1-(a+d-c)T+lT^2}\Big].$$

(We can drop the symmetrization $\frac{1}{2}(\ldots(c) - \ldots(-c))$ because it is already implied by the factor $\mathrm{sgn}\,(b) = -\mathrm{sgn}\,(c)$.)

Second term. We have

$$\frac{1}{k-1}B^0_{k-1}(aX+dY) = (k-2)! \sum_{\substack{p+r=k-1 \\ r \text{ even}}} \frac{B_r}{r!}\frac{(aX+dY)^p}{p!},$$

Both here and for the terms T_{III}, T_{IV}, and T_{V}, we will compute the contribution from each $(aX+dY)^p$ separately. In all four instances we will have to distinguish the cases $p \leqslant k-2$ $(r>0)$ and $p = k-1$ $(r=0)$. Here the contribution from $p \leqslant k-2$ vanishes because $(aX+dY)^p$ contains no monomials X^mY^n with $m \not\equiv n \pmod 2$ and $m+n \geqslant k-2$. For $p=k-1$ we find by the binomial theorem

$$\begin{aligned}\rho_k\big((aX+dY)^{k-1}\big) &= (k-1)\sum_{\substack{m,n\geqslant 0\\ m+n=k-1}} \frac{a^m}{m!}\frac{d^n}{n!}\,\rho_k(X^mY^n)\\ &= (k-1)\sum_{\substack{m,n\geqslant 0\\ m+n=k-1}} a^m d^n = (k-1)\mathrm{C}_{T^{k-1}}\left[\frac{1}{(1-aT)(1-dT)}\right]\end{aligned}$$

Therefore

$$T_{\mathrm{II}} \;=\; -2\,\mathrm{C}_{T^{k-1}}\left[\sum_{\substack{ad=l\\ a,d>0}} \frac{1}{(1-aT)(1-dT)}\right]$$

Third term. If $p \leqslant k-2$, p odd, then (setting $\tilde{p} = k-2-p$ for convenience)

$$\begin{aligned}\rho_k\big(X^{k-2}(-\tfrac{a}{X}+dY)^p\big) &= p!\sum_{\substack{m,n\geqslant 0\\ m+n=p}} \frac{a^m}{m!}\frac{d^n}{n!}\,\rho_k\big((-1)^m X^{k-2-m}Y^n\big)\\ &= \frac{p!\,\tilde{p}!}{(k-2)!}\sum_{\substack{n\geqslant m\geqslant 0\\ m+n=p}} (-1)^m\binom{n}{m}\binom{n+\tilde{p}}{n}a^m d^n\\ &= \frac{p!\,\tilde{p}!}{(k-2)!}\,\mathrm{C}_{T^p}\left[\sum_{n\geqslant m\geqslant 0} (-1)^m\binom{n}{m}\binom{n+\tilde{p}}{n}(aT)^m(dT)^n\right]\\ &= \frac{p!\,\tilde{p}!}{(k-2)!}\,\mathrm{C}_{T^p}\left[\sum_{n\geqslant 0}\binom{n+\tilde{p}}{n}(1-aT)^n(dT)^n\right]\\ &= \frac{p!\,\tilde{p}!}{(k-2)!}\,\mathrm{C}_{T^p}\left[\frac{1}{(1-dT+adT^2)^{\tilde{p}+1}}\right].\end{aligned}$$

For $p = k-1$, we have instead

$$\begin{aligned}
\rho_k\big(X^{k-2}(-\frac{a}{X}+dY)^{k-1}\big) &= (k-1)! \sum_{\substack{m,n\geqslant 0\\ m+n=k-1}} \frac{a^m}{m!}\frac{d^n}{n!}\,\rho_k\big((-1)^m X^{k-2-m}Y^n\big)\\
&= (k-1) \sum_{\substack{n>m\geqslant 0\\ m+n=k-1}} \frac{(-1)^m}{n}\binom{n}{m} a^m d^n\\
&= (k-1)\,\mathbf{C}_{T^{k-1}}\Big[\sum_{n>m\geqslant 0} \frac{(-1)^m}{n}\binom{n}{m}(aT)^m(dT)^n\Big]\\
&= (k-1)\,\mathbf{C}_{T^{k-1}}\Big[\sum_{n>0}\frac{1}{n}(1-aT)^n(dT)^n\Big]\\
&= (k-1)\,\mathbf{C}_{T^{k-1}}\Big[\log\Big(\frac{1}{1-dT+adT^2}\Big)\Big].
\end{aligned}$$

Inserting these expressions into the expansion of the polynomial B^0_{k-1}, we find the formula

$$T_{\rm III} \;=\; \mathbf{C}_{T^{k-1}}\Big[\sum_{\substack{ad=l\\ a,\,d>0}} \Psi_{l,d}(T)\Big],$$

where

$$\Psi_{l,s}(T) = \log\Big(\frac{1+sT+lT^2}{1-sT+lT^2}\Big) + \sum_{\substack{r>0\\ r \text{ even}}} \frac{B_r}{r}\Big(\frac{T^r}{(1-sT+lT^2)^r} - \frac{T^r}{(1+sT+lT^2)^r}\Big). \tag{6}$$

Fourth term. By symmetry (interchange the roles of X and Y and of a and d),

$$T_{\rm IV} \;=\; \mathbf{C}_{T^{k-1}}\Big[\sum_{\substack{ad=l\\ a,\,d>0}} \Psi_{l,a}(T)\Big]$$

(which of course is equal to $T_{\rm III}$).

Fifth term. If $p \leqslant k-2$, p odd, $\tilde p = k-2-p$ as before, then

$$\begin{aligned}
\rho_k\big(X^{k-2}Y^{k-2}(\frac{a}{X}+\frac{d}{Y})^p\big) &= p! \sum_{\substack{m,n\geqslant 0\\ m+n=p}} \frac{a^m}{m!}\frac{d^n}{n!}\,\rho_k\big(X^{k-2-m}Y^{k-2-n}\big)\\
&= \frac{p!\,\tilde p!}{(k-2)!} \sum_{\substack{m,n\geqslant 0\\ m+n=p}} \binom{m+\tilde p}{m}\binom{n+\tilde p}{n} a^m d^n\\
&= \frac{p!\,\tilde p!}{(k-2)!}\,\mathbf{C}_{T^p}\Big[\sum_{m\geqslant 0}\binom{m+\tilde p}{m}(aT)^m \cdot \sum_{n\geqslant 0}\binom{n+\tilde p}{n}(dT)^n\Big]\\
&= \frac{p!\,\tilde p!}{(k-2)!}\,\mathbf{C}_{T^p}\Big[\frac{1}{(1-aT)^{\tilde p+1}(1-dT)^{\tilde p+1}}\Big].
\end{aligned}$$

The term $p = k-1$ contributes nothing since it leads to a sum of terms $\rho_k(X^mY^n)$ with $m+n = 2(k-2)-p < k-2$. Hence

$$\rho_k\big[X^{k-2}Y^{k-2}\frac{1}{k-1}B^0_{k-1}(\frac{a}{X}+\frac{d}{Y})\big] = \mathbf{C}_{T^{k-1}}\Big[-\sum_{\substack{0<r<k\\ r \text{ even}}}\frac{B_r}{r}\frac{T^r}{(1-aT)^r(1-dT)^r}\Big].$$

Since $\log\frac{1}{(1-aT)(1-dT)} = \sum_{n=1}^{\infty}\frac{a^n+d^n}{n}T^n$, the right-hand side of this expression equals $(a^{k-1}+d^{k-1})/(k-1) - \frac{1}{2}\mathbf{C}_{T^{k-1}}\big[\Psi_{ad,a+d}(T)\big]$ with $\Psi_{l,s}$ as in (6). Thus the fifth term in (5) can be expressed as

$$T_{\rm V} \;=\; \frac{-4}{k-1}\sigma_{k-1}(l) + \mathbf{C}_{T^{k-1}}\Big[\sum_{\substack{ad=l\\ a,\,d>0}}\Psi_{l,a+d}(T)\Big].$$

Sixth term. Finally, we have

$$\sum_{\substack{0<n\leqslant k\\ n \text{ even}}}\frac{B_n}{n!}\frac{B_{k-n}}{(k-n)!}\rho_k\big(X^{k-2}Y^{n-1}-Y^{n-1}\big) = \sum_{\substack{0<n\leqslant k\\ n \text{ even}}}\frac{B_n}{n!}\frac{B_{k-n}}{(k-n)!}\big[1-(k-1)\delta_{n,k}\big]$$
$$= -\frac{2k-1}{k!}B_k\,.$$

(Here we have used the identity $\sum\frac{B_n}{n!}\frac{B_{k-n}}{(k-n)!} = -(k-1)\frac{B_k}{k!}$, which follows from $\big(\coth\frac{T}{2}\big)^2 = \frac{1}{4}-\frac{d}{dT}\coth\frac{T}{2}$ since $\sum_{n \text{ even}}\frac{B_n}{n!}T^{n-1} = \frac{1}{2}\coth\frac{T}{2}$.) Therefore

$$T_{\rm VI} \;=\; \frac{4(2k-1)}{k-1}\,\sigma_{k-1}(l).$$

Combining our formulas for $T_{\rm I}-T_{\rm VI}$, we obtain

TRACE FORMULA (FIRST FORM). *The trace of T_l on S_k $(k>2)$ is given by*

$$-12\,\mathrm{Tr}(T_l,S_k) \;= 8\,\sigma_{k-1}(l) + \mathbf{C}_{T^{k-2}}\Big[\sum_{\substack{ad-bc=l\\ ad>0>bc}}\frac{sgn(cd)}{1-(a+d-c)T+lT^2}\Big]$$
$$+\,\mathbf{C}_{T^{k-1}}\Big[\sum_{\substack{ad=l\\ a,\,d>0}}\Big(\Psi_{l,a}(T)+\Psi_{l,d}(T)+\Psi_{l,a+d}(T)-\frac{2}{(1-aT)(1-dT)}\Big)\Big]$$

where $\mathbf{C}_{T^n}[F(T)]$ denotes the coefficient of T^n in the Taylor expansion of $F(T)$ and $\Psi_{l,s}(T)$ is the function defined in (6).

We remark that if we replace $\mathrm{Tr}(T_l,S_k)$ by $\mathrm{Tr}(T_l,M_k)-\sigma_{k-1}(l)$ then the formula is valid for all $k\geqslant 2$, since (1) is true, suitably interpreted, also for modular forms of weight 2, by the discussion in [7]. However, we do not elaborate on this.

4. First simplification: a Bernoulli number identity. We now proceed to transform the formula for $\mathrm{Tr}(T_l)$ further. We start with the function $\Psi_{l,s}$ of equation (6).

PROPOSITION 3. *Let s be a natural number, $l \in \mathbf{C}$. Then*

$$\Psi_{l,s}(T) = \sum_{|t|\leqslant s}^{*} \frac{T}{1 - tT + lT^2}, \tag{7}$$

*where the * on the summation means that the boundary terms $t = \pm s$ are to be counted with multiplicity $\frac{1}{2}$.*

Remark. This is the generalization of Proposition 1 mentioned in the introduction. Indeed, for $l = 1$ and $s = 2$, Proposition 3 says that

$$\frac{T}{1-T+T^2} + \frac{T}{1+T^2} + \frac{T}{1+T+T^2}$$
$$= -\frac{1}{2}\left(\frac{T}{(1-T)^2} - \frac{T}{(1+T)^2}\right) + 2\log\frac{1+T}{1-T} + \sum_{r=2}^{\infty}\frac{B_r}{r}\left(\frac{T^r}{(1-T)^{2r}} - \frac{T^r}{(1-T)^{2r}}\right).$$

The coefficient of T^n on the right for n odd (for n even it is clearly 0) equals

$$-n + \frac{4}{n} + 2\sum_{2\leqslant r\leqslant n}\frac{B_r}{r}\binom{n+r-1}{n-r} = 4\sum_{r=0}^{n}\frac{B_r}{n+r}\binom{n+r}{2r},$$

while the coefficient on the left is 4 times the number given in Proposition 1.

PROOF: We start by proving the simpler power series identity

$$\sum_{|t|\leqslant s}^{*}\frac{T}{1-tT} = \log\frac{1+sT}{1-sT} + \sum_{r=2}^{\infty}\frac{B_r}{r}\left(\frac{T^r}{(1-sT)^r} - \frac{T^r}{(1+sT)^r}\right). \tag{8}$$

Both sides of (8) are odd power series in T. For n even, the coefficient of T^{n+1} on the left-hand side equals $\sum_{|t|\leqslant s}^{*} t^n$ and that on the right equals

$$2\left(\frac{s^{n+1}}{n+1} + \sum_{r=2}^{n}\frac{B_r}{r}\binom{n}{n-r+1}s^{n-r+1}\right) = \frac{2}{n+1}B_{n+1}^{0}(s) = \frac{1}{n+1}\Big(B_{n+1}^{0}(s) - B_{n+1}^{0}(-s)\Big);$$

these are equal because $B_{n+1}^{0}(x+1) - B_{n+1}^{0}(x) = \frac{n+1}{2}\left((x+1)^n + x^n\right)$ by the defining property of Bernoulli polynomials (which were introduced by Bernoulli precisely for the problem of summing powers of integers). Now to get from (8) to (7) simply replace T by $\frac{T}{1+lT^2}$. □

Substituting Proposition 3 and the identity

$$\begin{aligned}\mathbf{C}_{T^{k-1}}\left[\frac{1}{(1-aT)(1-dT)}\right] &= \mathbf{C}_{T^{k-1}}\left[\frac{1}{(1-aT)(1-dT)}-1\right]\\ &= \frac{1}{2}\mathbf{C}_{T^{k-2}}\left[\frac{a}{1-aT}+\frac{d}{1-dT}+\frac{a+d}{(1-aT)(1-dT)}\right]\\ &= \frac{1}{2}\left(a^{k-1}+d^{k-1}\right)+(a+d)\mathbf{C}_{T^{k-2}}\left[\frac{1}{(1-aT)(1-dT)}\right]\end{aligned}$$

into the trace formula given at the end of §3, we obtain the version formulated as a theorem in the introduction. Note that the quantities $\mathrm{Tr}\,(T_l, M_k) - \frac{1}{2}\sigma_{k-1}(l)$ which appear there are the averages of the traces of T_l in M_k and in S_k.

5. Second simplification: a class number identity. For $l \in \mathbf{N}$, $t \in \mathbf{Z}$ denote by $N(l,t)$ the left-hand side of equation (2). In the summation, terms with $ad < 0$ or $bc > 0$ give zero, so we have either $ad > 0 > bc$ or else $abcd = 0$. In the latter case, $\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)$ is one of the matrices $\left(\begin{smallmatrix} r & 0 \\ r+s-t & s\end{smallmatrix}\right)$, $\left(\begin{smallmatrix} -r & 0 \\ -r-s-t & -s\end{smallmatrix}\right)$, $\left(\begin{smallmatrix} s+t & -r \\ s & 0\end{smallmatrix}\right)$, $\left(\begin{smallmatrix} t-s & r \\ -s & 0\end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 0 & -r \\ s & s+t\end{smallmatrix}\right)$, or $\left(\begin{smallmatrix} 0 & r \\ -s & t-s\end{smallmatrix}\right)$, where r and s run over the solutions of $rs = l$, r, $s > 0$. Hence

$$\begin{aligned}N(l,t) =& 4\sum_{\substack{ad-bc=l\\ ad>0>bc\\ a+d-c=t}} \mathrm{sgn}\,(bd)\\ &+\sum_{\substack{rs=l\\ r,s>0}}\left(2\left[\mathrm{sgn}\,(r+s-t)+\mathrm{sgn}\,(r+s+t)\right]-4\left[\mathrm{sgn}\,(s+t)+\mathrm{sgn}\,(s-t)\right]\right)\end{aligned} \tag{9}$$

But $\mathrm{sgn}\,(x+t)+\mathrm{sgn}\,(x-t)$ for $x > 0$ equals 2, 1 or 0 according as $|t| < x$, $|t| = x$ or $|t| > x$, so

$$\begin{aligned}-\frac{1}{4}\sum_{t\in\mathbf{Z}}\frac{N(l,t)}{1-tT+lT^2} =& \sum_{\substack{ad-bc=l\\ ad>0>bc}}\frac{\mathrm{sgn}\,(cd)}{1-(a+d-c)T+lT^2}\\ &+\sum_{\substack{rs=l\\ r,s>0}}\left(2{\sum_{|t|\leqslant s}}^{*}\frac{1}{1-tT+lT^2}-{\sum_{|t|\leqslant r+s}}^{*}\frac{1}{1-tT+lT^2}\right).\end{aligned}$$

Therefore Proposition 2 transforms the trace formula we have just proved into the following (nearly) standard form:

Trace Formula (classical form). *Let l be a positive integer. Then*

$$\sum_{\substack{k=2\\ k \text{ even}}}^{\infty}\left(Tr(T_l,\, M_k)-\frac{1}{2}\sigma_{k-1}(l)\right)T^{k-2} = -\frac{1}{2}\sum_{t\in\mathbf{Z}}\frac{H(4l-t^2)}{1-tT+lT^2}, \tag{10}$$

where $H(N)$ for $N > 0$ is the Kronecker-Hurwitz class number of Proposition 2, $H(0) = -\frac{1}{12}$, $H(-r^2) = -\frac{1}{2}r$ for r positive, and $H(N) = 0$ if $-N$ is positive and not a square.

(Actually, this formula is not quite standard since the contribution from the terms with $t^2 - 4l = r^2 > 0$ is usually written in terms of decompositionsof l as a product ad. It is left to the reader as an exercise to check that the contribution to the coefficient of T^{k-2} of all terms with $4l - t^2 < 0$ on the right-hand side of (10) is $\frac{1}{4}\sum(\max\{a,d\}^{k-1} - \min\{a,d\}^{k-1})$, where the sum is over all decompositions $l = ad$ with a and d positive.)

It remains only to prove Proposition 2. The main feature of this proposition is that the expression $N(l,t) - 2|t|\,\nu(t^2 - 4l)$, depends only on the "discriminant" $t^2 - 4l$; here $\nu(\Delta)$ denotes the number of square roots of an integer Δ (i.e., 2 if $\Delta = r^2 > 0$, 1 if $\Delta = 0$, and 0 otherwise). This is explained by the following

LEMMA. *Suppose $l > 0$, $t \geqslant 0$. Then $N(l+t+1, t+2) - N(l,t) = 4\,\nu(t^2 - 4l)$.*

Assuming this, we now prove Proposition 2. Since both sides of (2) are invariant under $t \mapsto -t$, we may assume $t \geqslant 0$. Set $\Delta = t^2 - 4l$.

If $\Delta < 0$, then we can successively reduce t by 2 until it assumes the value 0 or 1, changing l at each stage to keep the discriminant invariant. Under each reduction we are comparing two values (l,t) and $(l+t+1, t+2)$ with $t \geqslant 0$ and $l > 0$ (since $4l > t^2$), so, by the lemma, the value of $N(l,t)$ does not change. Therefore the case $\Delta < 0$ reduces to the two assertions

$$N(l,0) = 24\,H(4l), \qquad N(l,1) = 24\,H(4l-1).$$

Both follow from reduction theory of definite binary quadratic forms. For example, the matrices $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ of determinant l with $ad \geqslant 0 \geqslant bc$ and $a + d = c$ have the form $\pm\left(\begin{smallmatrix} x & -z \\ x+y & y \end{smallmatrix}\right)$ where (x,y,z) is a solution in nonnegative integers of the equation $xy + xz + yz = l$, so we see (paying attention to multiplicities) that $N(l,0)$ equals 8 times the number of such solutions (solutions with $xyz = 0$ being counted with multiplicity 1/2). This number in turn equals $3H(4l)$, because putting $[A,B,C] = [x+y, 2x, x+z]$ sets up a correspondence between one-sixth of the solutions in question (namely, those with $z \geqslant y \geqslant x$) and one-half of all reduced forms of discriminant $-4l$ (namely, those with $C \geqslant A \geqslant B \geqslant 0$), the terms where some inequality is an equality being counted with suitable fractional multiplicities. (This expression for $H(4l)$ is also given in [6], Theorem 2, p. 292.) Similarly, the matrices counted by $N(l,1)$ $(l>0)$ can be put into correspondence with the reduced quadratic forms of discriminant $4l-1$ by mapping, for instance, $\left(\begin{smallmatrix} x & -z \\ x+y-1 & y \end{smallmatrix}\right)$ $(x,\, y,\, z \geqslant 0,\; xy+xz+yz-z = l)$ to $[y+z, 2z+1, x+z]$ if $x \geqslant y > z$, to $[x+y-1, 2x-1, x+z]$ if $z \geqslant y \geqslant x$, etc. Details are left to the reader.

If $\Delta \geqslant 0$, then we can keep reducing t by 2 at a time until $0 < l \leqslant t-1$, when a further reduction would make $l \leqslant 0$. The lemma implies that the correctness of Proposition 2 is not affected by this reduction. If $l < t-1$, then $t^2 > \Delta > (t-2)^2$ and therefore Δ, which has the same parity as t, is not a square, so in this case we must show that $N(l,t) = 0$. This is clear from (9), since if (a,b,c,d) occurs in the first sum then

$$t = a + d - c \leqslant |a| + |d| + |c| \leqslant |ad| + 1 + |bc| = ad - bc + 1 = l + 1,$$

while if r and s are positive integers with $rs = l$ then both r and $r + s$ are $\leqslant l + 1$. For $l = t - 1$ we must show $N(t-1,t) = 24 - 8t$ $(t > 0)$, which is proved by looking at the cases of equality in the argument just given.

We still have to prove the lemma. Let $\mathcal{A}$ denote the set of quadruples $(a,b,c,d) \in \mathbf{Z}^4$ with $ad - bc = l$ and $a + d - c = t$, and $\varepsilon(a,b,c,d)$ the quantity $\big(\operatorname{sgn}(a) + \operatorname{sgn}(d)\big)\big(\operatorname{sgn}(c) - \operatorname{sgn}(b)\big)$, so that $N(l,t) = \sum_{\mathcal{A}} \varepsilon(a,b,c,d)$. As (a,b,c,d) runs over the set $\mathcal{A}$, $(a+1, b+1, c, d+1)$ runs over the corresponding set for $l+t+1$ and $t+2$. Hence $N(l+t+1,t+2) - N(l,t)$ is the sum over $\mathcal{A}$ of the quantity $f(a,b,c,d) = \varepsilon(a+1,b+1,c,d+1) - \varepsilon(a,b,c,d)$. Clearly $f(a,b,c,d)$ is non-zero only when (at least) one of the integers a, b and d belongs to $\{0,-1\}$. Hence $N(l+t+1,t+2) - N(l,t) = S_1 + S_2 + S_3$ where S_1, S_2, S_3 denote the sum of $f(a,b,c,d)$ over the sets

$$\begin{aligned}
\mathcal{A}_1 &= \big\{(a,b,c,d) \in \mathcal{A} \mid \{a,d\} \cap \{0,-1\} \neq \emptyset,\ b \notin \{0,-1\}\big\},\\
\mathcal{A}_2 &= \big\{(a,b,c,d) \in \mathcal{A} \mid \{a,d\} \cap \{0,-1\} = \emptyset,\ b \in \{0,-1\}\big\},\\
\mathcal{A}_3 &= \big\{(a,b,c,d) \in \mathcal{A} \mid \{a,d\} \cap \{0,-1\} \neq \emptyset,\ b \in \{0,-1\}\big\}.
\end{aligned}$$

We have

$$\begin{aligned}
S_1 &= \sum_{\substack{rs=l\\ r\neq 1}} \begin{cases} f(0,-r,s,s+t) + f(s+t,-r,s,0) & \text{if } s+t \neq 0,\ -1\\ f(0,-r,s,0) & \text{if } s+t=0\\ f(0,-r,s,-1) & \text{if } s+t=-1 \end{cases}\\
&\quad + \sum_{\substack{rs=l+t+1\\ r\neq -1}} \begin{cases} f(-1,-r-1,s,s+t+1) + f(s+t+1,-r-1,s,-1) & \text{if } s+t \neq -1,-2\\ f(-1,-r-1,s,0) & \text{if } s+t=-1\\ f(-1,-r-1,s,-1) & \text{if } s+t=-2 \end{cases}\\
&= \sum_{\substack{rs=l\\ r\neq 1}} \big(4\operatorname{sgn}(r)\big) + \sum_{\substack{rs=l+t+1\\ r\neq -1}} \big(4\operatorname{sgn}(r)\big)\\
&= -4 + 4 = 0 \qquad \text{(combine } (r,s) \text{ and } (-r,-s) \text{ whenever } |r| \neq 1);\\
S_2 &= \sum_{\substack{rs=l\\ r,s\neq -1\\ r+s\neq t}} f(r,0,r+s-t,s) + \sum_{\substack{rs=l+t+1\\ r,s\neq 1\\ r+s\neq t+2}} f(r-1,-1,r+s-t-2,s-1)\\
&= \sum_{\substack{rs=l\\ r,s\neq -1\\ r+s\neq t}} \big(-2\operatorname{sgn}(r)\big) + \sum_{\substack{rs=l+t+1\\ r,s\neq 1\\ r+s\neq t+2}} \big(-2\operatorname{sgn}(r)\big)
\end{aligned}$$

$$= -4 + 2\delta_{l,1} + \sum_{\substack{rs=l \\ r+s=t}} 2 + 4 + \sum_{\substack{rs=l+t+1 \\ r+s=t+2}} 2 \qquad \text{(again by combining } (r,s) \text{ and } (-r,-s))$$

$$= 2\delta_{l,1} + 4\,\nu(t^2 - 4l);$$

$$S_3 = f(0,-1,l,l+t) + f(l+t,-1,l,0)$$

$$+ \begin{cases} f(-1,0,-l-t-1,-l) + f(-l,0,-l-t-1,-1) & \text{if } l > 1 \\ f(-1,0,-2-t,-1) & \text{if } l = 1 \end{cases}$$

$$= 0 + 0 + \begin{cases} 0+0 & \text{if } l > 1 \\ -2 & \text{if } l = 1 \end{cases} \quad = \quad -2\,\delta_{l,1}.$$

This completes the proof of the lemma and of Proposition 2.

6. The action of Hecke operators on periods. For each integer $l \geqslant 1$ define

$$\mathbf{Man}_l = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \, a,\ b,\ c,\ d \in \mathbf{Z},\ ad - bc = l,\ a,\ b,\ c,\ d \text{ satisfy (3)} \right\}.$$

The assertion of Theorem 2 is that the period polynomial of $f|T_l$ is given by

$$r_{f|T_l} = \sum_{M \in \mathbf{Man}_l} r_f|_{2-k} M,$$

where $(r|_h M)(X)$ for $h \in \mathbf{Z}$ and $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is defined as $(cX+d)^{-h} r\left(\frac{aX+b}{cX+d}\right)$. Because the set $\mathbf{Man}_l$ is invariant under $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mapsto \left(\begin{smallmatrix} a & -b \\ -c & d \end{smallmatrix}\right)$, this automatically implies the sharper statement that $r^{\pm}_{f|T_l} = \sum_{M \in \mathbf{Man}_l} r^{\pm}_f|_{2-k} M$, where $r^{\pm}_f(X) = \frac{1}{2}\big(r_f(X) \pm r_f(-X)\big)$ denote the "plus" and "minus" parts of the period polynomial r_f.

Examples. Clearly $\mathbf{Man}_1 = \left\{ \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \right\}$. For $l = 2$, 3, 5 we have

$$\mathbf{Man}_2 = \left\{ \left(\begin{smallmatrix} 2 & 0 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 \\ 0 & 2 \end{smallmatrix}\right), \left(\begin{smallmatrix} 2 & 0 \\ 1 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 1 \\ 0 & 2 \end{smallmatrix}\right) \right\},$$

$$\mathbf{Man}_3 = \left\{ \left(\begin{smallmatrix} 3 & 0 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 \\ 0 & 3 \end{smallmatrix}\right), \left(\begin{smallmatrix} 3 & 0 \\ 1 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 3 & 0 \\ -1 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 1 \\ 0 & 3 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & -1 \\ 0 & 3 \end{smallmatrix}\right) \right\},$$

$$\mathbf{Man}_5 = \left\{ \left(\begin{smallmatrix} 5 & 0 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 \\ 0 & 5 \end{smallmatrix}\right), \left(\begin{smallmatrix} 5 & 0 \\ 1 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 5 & 0 \\ -1 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 5 & 0 \\ 2 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 5 & 0 \\ -2 & 1 \end{smallmatrix}\right), \right.$$

$$\left. \left(\begin{smallmatrix} 1 & 1 \\ 0 & 5 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & -1 \\ 0 & 5 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 2 \\ 0 & 5 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & -2 \\ 0 & 5 \end{smallmatrix}\right), \left(\begin{smallmatrix} 2 & 1 \\ -1 & 2 \end{smallmatrix}\right), \left(\begin{smallmatrix} 2 & -1 \\ 1 & 2 \end{smallmatrix}\right) \right\}.$$

One can verify by direct computation that these act (via $|_{2-k}$) on the functions

$$X^{k-2} - 1 \qquad \text{and} \qquad \sum_{n=0}^{k-1} \frac{B_n}{n!} \frac{B_{k-n}}{(k-n)!} X^{n-1} \qquad (k > 2)$$

(the "plus" and "minus" parts of the period "polynomial" of the Eisenstein series of weight k) as multiplication by $\sigma_{k-1}(l)$, and (via $|_{-10}$) on the functions

$$X^2(X^2-1)^3 \qquad \text{and} \qquad X(X^2-1)^2(4X^2-1)(X^2-4)$$

(the "plus" and "minus" parts of the period polynomial of $\Delta \in S_{12}$) as multiplication by the Ramanujan tau-function $\tau(l)$, the lth Fourier coefficient of Δ (here -24, 252, or 4830).

Let $\mathcal{M}$ be the ring of formal finite linear combinations $\sum c_i M_i$ ($c_i \in \mathbf{Z}$, $M_i \in M_2^+(\mathbf{Z})$ = 2×2 matrices over $\mathbf{Z}$ with positive determinant). The action $r \mapsto r|_k M$ of $M_2^+(\mathbf{Z})$ on functions extends by linearity to an action of $\mathcal{M}$. For each $l \in \mathbf{N}$ we set $\tilde{T}_l = \sum_{M\in \mathrm{Man}_l} M \in \mathcal{M}$, so that the result we want to prove is $r_{f|T_l} = r_f|_{2-k}\tilde{T}_l$. The key identity for this purpose is the formula

$$(T-1)\,\tilde{T}_l = T_l^\infty\,(T-1) + (1-S)\,Y_l, \tag{11}$$

where $S = \left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$, $T = \left(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right) \in SL_2(\mathbf{Z})$, $T_l^0 = \sum_{\substack{ad=l\\0\leqslant b<d}} \left(\begin{smallmatrix}a&b\\0&d\end{smallmatrix}\right)$, and Y_l is the element

$$Y_l = \sum_{\substack{ps+qr=l\\ p>r>0,\,q>0,\,s>0\\ \text{or } p=r>0,\,q>s>0\\ \text{or } p>2r>0,\,s>q=0\\ \text{or } p>r=0,\,s>-2q>0}} \begin{pmatrix}p&q\\-r&s\end{pmatrix}(T-1) + \sum_{\substack{x,y>0\\2xy=l}}\left[\begin{pmatrix}0&-2y\\x&y\end{pmatrix} + \begin{pmatrix}x&-y\\x&y\end{pmatrix} + \begin{pmatrix}-x&y\\2x&0\end{pmatrix}\right]$$

of $\mathcal{M}$. This identity, which is tedious but not hard to verify directly, is proved in [1], except that Y_l is not worked out explicitly there.

We will show that

$$c_{kl}(X,Y) = c_{k1}(X,Y)|_{2-k}\tilde{T}_l\,, \tag{12}$$

where $\tilde{T}_l$ is taken to act on the variable X. In view of the formula

$$c_{kl}(X,Y) = \sum_{\substack{f\in M_k\\ \text{eigenform}}} \frac{1}{(2i)^{k-3}(f,f)}\,a_f(l)\left(r_f^+(X)r_f^-(Y) + r_f^-(X)r_f^+(Y)\right)$$

(where the summation is over the basis of normalized Hecke eigenforms in $M_k(SL_2(\mathbf{Z}))$) and of the linear independence of the functions $r_f^\epsilon(X)$ as $f =$ ranges over the eigenforms and ϵ over the set $\{\pm1\}$, this immediately implies the formula $r_f^\pm|\tilde{T}_l = a_f(l)\,r_f^\pm$ for all normalized Hecke eigenforms and hence the formula $r_f^\pm|\tilde{T}_l = r_{f|T_l}^\pm$ for all $f \in M_k$. (In the preceding sentence, $r_f^+(X)$ and $r_f^-(X)$ when f is the Eisenstein series of weight k must be interpreted as multiples of $X^{k-2}-1$ and $\sum\binom{k}{n}B_nB_{k-n}X^{n-1}$, respectively; cf. [7].)

We remark that the identity (12) is not actually necessary to deduce Theorem 2 once one has (11). Instead, we can use the alternate definition of r_f for $f \in S_k$ by $r_f(\tau) = \hat{f}(\tau) - \tau^{k-2}\hat{f}(-1/\tau)$ or $r_f = \hat{f}|_{2-k}(1-T)$, where

$$\hat{f}(\tau) = \frac{(k-2)!}{(-2\pi i)^{k-1}} \sum_{l=1}^{\infty} \frac{a_f(l)}{l^{k-1}} e^{2\pi i l\tau} = \int_\tau^\infty f(\tau')(\tau'-\tau)^{k-2}\, d\tau'$$

is the Eichler integral associated to f (cf. [4], Chapter 5). It is easily seen that replacing f by $f|T_l = l^{k-1}f|_kT_l^\infty$ replaces $\hat{f}$ by $\hat{f}|_{2-k}T_l^\infty$, so $r_{f|T_l} = \big(\hat{f}|_{2-k}T_l^\infty\big)|_{2-k}(1-T)$, and this equals $r_f|_{2-k}\tilde{T}_l$ by (11) because $\hat{f}|_{2-k}(1-S)$ vanishes. (For this argument we do not need the explicit formula for the operator Y_l in (11), which is why it was not given explicitly in [1].) We nevertheless give the proof of identity (12), since the calculation is amusing and since it was via this identity that the form of $\tilde{T}_l$ was discovered.

When $l=1$, formula (1) simplifies drastically, becoming simply

$$c_{k1}(X,Y) = -\frac{2}{k-1} B^0_{k-1}(X+Y)|_X(1-T)|_Y(1-T),$$

where $|_X$ and $|_Y$ denote the action of $M_2^+(\mathbf{Z})$ (the notation for the weight, which from now on will always be $2-k$, being omitted) with respect to the variables X and Y, respectively. Since the operators $|_XA$ and $|_YB$ obviously commute for any matrices A and B, we have by (11)

$$c_{k1}|_X\tilde{T}_l = \frac{2}{k-1} B^0_{k-1}(X+Y)|_X\big(T_l^\infty(T-1) + (1-S)Y_l\big)|_Y(1-T).$$

But

$$\begin{aligned} B^0_{k-1}(X+Y)|_XT_l^\infty &= \sum_{\substack{ad=l\\ a,d>0}} d^{k-2}\sum_{b=0}^{d-1} B^0_{k-1}\Big(\frac{aX+b}{d}+Y\Big) \\ &= \sum_{\substack{ad=l\\ a,d>0}} \Big[B^0_{k-1}(aX+dY) + \frac{k-1}{2}\sum_{0<b<d}(aX+dY+b)^{k-2}\Big] \end{aligned}$$

by the well-known distribution property $B_{k-1}(dt) = d^{k-2}\sum_{b=0}^{d-1} B_{k-1}(t+\frac{b}{d})$ of the standard Bernoulli polynomial $B_{k-1}(t) = B^0_{k-1}(t) - \frac{k-1}{2}t^{k-2}$. Also $B^0_{k-1}(X+Y)|_X(1-S) = -\frac{k-1}{2}(X+Y)^{k-2}|_X(1+S)$ (we already used this in the proof of Proposition 3). Hence

$$c_{k1}|_X\tilde{T}_l = -\frac{2}{k-1}\sum_{\substack{ad=l\\ a,d>0}} B^0_{k-1}(aX+dY)|_X(1-T)|_Y(1-T)$$

$$+ (X+Y)^{k-2}|_X\Big(\sum_{\substack{ad=l\\ 0<b<d}} \big(\begin{smallmatrix} a & b\\ 0 & d\end{smallmatrix}\big)(T-1) - (1+S)Y_l\Big)|_Y(1-T).$$

But this equals the right-hand side of (1) because one has the identities $(X+Y)^{k-2}|_X M|_Y T = (X+Y)^{k-2}|_X TM$ for any $M \in M_2^+(\mathbb{Z})$ and, by direct verification,

$$(1-T)\Big(\sum_{\substack{ad=l\\ 0<b<d}} \big(\begin{smallmatrix} a & b\\ 0 & d\end{smallmatrix}\big)(T-1) - (1+S)Y_l\Big) = \sum_{\substack{ad-bc=l\\ ad>0>bc}} \operatorname{sgn}(bd)\big(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\big) \qquad \text{in } \mathcal{M}.$$

Finally, we should compare Theorem 2 with the formula for the 0th period of $f|T_l$ given by Manin [5]. Setting $X = 0$ in the equation $r^+_{f|T_l}(X) = r^+_f(X)|\tilde{T}_l$, we find

$$a_f(l)\, r_0(f) = \sum_{\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)\in \mathrm{Man}_l} \sum_{\substack{0\leqslant n<\frac12 k-1\\ n \text{ even}}} \binom{k-2}{n} r_n(f)\,(b^{k-2-n}d^n - b^n d^{k-2-n})$$

if $f \in M_k$ is an eigenform. Applying this when f is the Eisenstein series of weight k, we see that the coefficient of $r_0(f)$ on the right equals $\sigma_{k-1}(l)$ (since this is the value of $a_f(l)$ in this case and all $r_n(f)$ for even $0 < n < \frac12 k - 1$ vanish). From this it is easy to check that the formula given here agrees with that given by Manin. It should also be remarked that the methods of Manin's paper suffice in principle to calculate $r_n(f|T_l)$ for $n > 0$ as well as $n = 0$, and could thus be used to give an alternate proof of Theorem 2 via continued fractions; however, this proof would be less direct than the one arising from equation (11).

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