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Quantum Modular Forms

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To Alain Connes on his 60th birthday, in friendship and admiration

A classical modular form is a holomorphic function f in the complex upper half-plane $\mathfrak{H}$ satisfying the transformation equation

$$(f|_k\gamma)(z) \;:=\; (cz+d)^{-k}\, f\Big(\frac{az+b}{cz+d}\Big) \;=\; f(z) \tag{1}$$

for all $z \in \mathfrak{H}$ and all matrices $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2,\mathbb{Z})$, where k, the weight of the modular form, is a fixed integer. Of course, there are many variants: one can replace the group $\mathrm{SL}(2,\mathbb{Z})$ by a group commensurable with it or by a more general Fuchsian subgroup of $\mathrm{SL}(2,\mathbb{R})$; the automorphy factor $(cz+d)^{-k}$ may be multiplied by a character or replaced by a more general multiplier system; the weight k may be half-integral or even rational; the function f can be vector-valued rather than scalar-valued; there may be a further additive correction on the right-hand side of (1); one can allow non-holomorphic functions of specified type (e.g., Maass wave forms); etc. But in all of these generalizations, as well as the higher-dimensional generalizations of modular forms to Hilbert or Siegel modular forms or to automorphic forms of more general type, the functions considered are defined on a symmetric space $X = G/K$ associated to a Lie group G and transform suitably with respect to the action of a discrete subgroup $\Gamma \subset G$ on X.

In this note we want to discuss, in the simplest cases, another type of modular object which, because it has the "feel" of the objects occurring in perturbative quantum field theory and because several of the examples come from quantum invariants of knots and 3-manifolds, we call *quantum modular forms*. These are objects which live at the boundary of the space X, are defined only asymptotically, rather than exactly, and have a transformation behavior of a quite different type with respect to some modular group. We will consider only the case when G is $\mathrm{SL}(2,\mathbb{R})$, X is $\mathfrak{H}$, and Γ is $\mathrm{SL}(2,\mathbb{Z})$ or a group commensurable with it. Then, as is well-known, the natural boundary of X is $\mathbb{P}^1(\mathbb{Q}) = \mathbb{Q} \cup \{\infty\}$, the set of "cusps" of Γ.

A quantum modular form should therefore be a complex-valued function f on $\mathbb{Q}$, or possibly on $\mathbb{P}^1(\mathbb{Q}) \smallsetminus S$ for some finite subset $S \subset \mathbb{P}^1(\mathbb{Q})$, having a certain behavior under the action of Γ on $\mathbb{P}^1(\mathbb{Q})$. Here neither of the properties which are

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required of classical modular forms—analyticity and Γ-covariance—are reasonable things to require: the former because $\mathbb{P}^1(\mathbb{Q})$, viewed as the set of cusps of the action on Γ on $\mathfrak{H}$, is naturally equipped only with the discrete topology, not with its induced topology as a subset of $\mathbb{P}^1(\mathbb{R})$, so that any requirement of continuity or analyticity is vacuous; and the latter because Γ acts on $\mathbb{P}^1(\mathbb{Q})$ transitively or with only finitely many orbits, so that any requirement of Γ-covariance of a function on this set would lead to a trivial definition. So we do not demand either continuity/analyticity or modularity, but require instead that the failure of one precisely offsets the failure of the other. In other words, our quantum modular form should be a function $f:\mathbb{Q}\to\mathbb{C}$ for which the function $h_\gamma:\mathbb{Q}\smallsetminus\{\gamma^{-1}(\infty)\}\to\mathbb{C}$ defined by

$$h_\gamma(x) \;=\; f(x) \,-\, (f|_k\gamma)(x) \tag{2}$$

has some property of continuity or analyticity (now with respect to the real topology) for every element $\gamma\in\Gamma$. This is purposely a little vague, since examples coming from different sources have somewhat different properties, and we want to consider all of them as being quantum modular forms. For the sake of definiteness we will take as our canonical definition of a quantum modular form a function $f:\mathbb{Q}\to\mathbb{C}$ for which the function h_γ defined by (2) extends to a real-analytic function on $\mathbb{P}^1(\mathbb{R})\smallsetminus S_\gamma$, where $S_\gamma\subset\mathbb{P}^1(R)$ is a finite set (typically just $\{\infty,\gamma^{-1}(\infty)\}$), for each $\gamma\in\Gamma$. Notice that this property need only be checked for a set of generators of Γ, and hence for only finitely many elements, because its validity for γ_1 and γ_2 automatically implies its validity for $\gamma_1\gamma_2$. In fact, the function $\gamma\mapsto h_\gamma$ is a cocycle on Γ (i.e., it satisfies $h_{\gamma_1\gamma_2}=h_{\gamma_1}|_k\gamma_2+h_{\gamma_2}$), so that any quantum modular form defines a cohomology class in the first cohomology group of Γ with coefficients in the space of piecewise analytic functions on $\mathbb{P}^1(\mathbb{R})$ with the action $h\mapsto h|_k\gamma$ of Γ.

The definition just given describes what one can call a *weak quantum modular form*. A *strong quantum modular form*—and most of our examples will belong to this category—is an object with a stronger (and more interesting) structure: it associates to each element of $\mathbb{Q}$ a formal power series over $\mathbb{C}$, rather than just a complex number, with a correspondingly stronger requirement on its behavior under the action of Γ. To describe this, we write the power series in $\mathbb{C}[[\varepsilon]]$ associated to $x\in\mathbb{Q}$ as $f(x+i\varepsilon)$ rather than, say, $f_x(\varepsilon)$, so that f is now defined in the union of (disjoint!) formal infinitesimal neighborhoods of all points $x\in\mathbb{Q}\subset\mathbb{C}$. Since the function h_γ in (2) was required to be real-analytic on the complement of a finite subset S_γ of $\mathbb{P}^1(\mathbb{R})$, it extends holomorphically to a neighborhood of $\mathbb{P}^1(\mathbb{R})\smallsetminus S_\gamma$ in $\mathbb{P}^1(\mathbb{C})$, and in particular has a power series expansion (convergent in some disk of positive radius) around each point $x\in\mathbb{Q}$. Our stronger requirement is now that the equation

$$f(z) \,-\, (f|_k\gamma)(z) \;=\; h_\gamma(z) \qquad (\gamma\in\Gamma,\quad z\to x\in\mathbb{Q}) \tag{3}$$

holds as an identity between countable collections of formal power series.

Finally, there is a further property which holds for all the examples of strong quantum modular forms that we know, namely, that the formal function $f(z)$ just described extends to an actual function $f:(\mathbb{C}\smallsetminus\mathbb{R})\cup\mathbb{Q}\to\mathbb{C}$ that is analytic on $\mathbb{C}\smallsetminus\mathbb{R}$ and whose asymptotic expansion as one approaches any rational point $x\in\mathbb{Q}$ vertically from above or below coincides to all orders with the formal power series f at x. (Here "analytic" can mean "holomorphic" or merely "real-analytic," depending on the example.) Of course such an extension, even if it exists, isn't canonical since it can be modified by adding an analytic function in $\mathfrak{H}^\pm$ which vanishes to infinite order as one approaches any rational point, but in our examples there will often be a natural choice. One then gets a peculiar kind of object: an analytic function in the upper half-plane which "leaks" into the lower half-plane through the infinitely many "holes" $\mathbb{Q} \subset \mathbb{R}$ in the real axis to another analytic function in $\mathfrak{H}^-$ in such a way that the combined function on $\mathfrak{H} \cup \mathbb{Q} \cup \mathfrak{H}^-$ is C^∞ on any vertical line passing through a rational point, or more generally on any smooth curve in $\mathbb{C}$ which intersects $\mathbb{R}$ only orthogonally and in rational points. The sheaf defined by functions of this type gives $(\mathbb{C} \smallsetminus \mathbb{R}) \cup \mathbb{Q}$ a bizarre "hybrid topology" in which it is a 1-dimensional complex manifold at all points outside of $\mathbb{Q}$ and a kind of 1-dimensional real C^∞-manifold at all points of $\mathbb{Q}$.

All of this sounds somewhat abstract. Let us turn for the rest of the paper to the examples, which are taken from a variety of fields: number theory, combinatorics (q-series) and, as already mentioned, quantum invariants of 3-manifolds and knots.

Example 0. We begin with a function which is is more of a prototype than a true example because it does not fit precisely into the scheme described above, but which is in the same spirit and is very familiar to number theorists. This is the classical Dedekind sum, defined on pairs of coprime integers (c, d) with $c > 0$ by the formula

$$s(d, c) \;=\; \sum_{0<k<c} \Big(\!\Big(\frac{k}{c}\Big)\!\Big) \Big(\!\Big(\frac{kd}{c}\Big)\!\Big)\,,$$

where $((x))$ denotes $x - [x] - \frac{1}{2}$ for $x \notin \mathbb{Z}$. It satisfies the well-known identities

$$s(d+c, c) = s(d, c), \quad s(-d, c) = -s(d, c), \quad s(d, c) + s(c, d) = \frac{c^2 + d^2 + 1 - 3cd}{12cd}\,,$$

which determine it completely. Hence the function $S : \mathbb{Q} \to \mathbb{Q}$ defined by $S(d/c) = 12s(d, c)$ satisfies the functional equations

$$S(x) - S(x{+}1) \;=\; 0, \qquad S(x) - S(-1/x) \;=\; x + \frac{1}{x} \pm 3 + \frac{1}{\mathrm{Num}(x)\mathrm{Den}(x)} \quad (x \lessgtr 0)\,.$$

If we ignore the last term, which is the reason why we said that this example does not quite fit in with our general scheme, then we see that we have precisely an example of the type of transformation property described above. (The reason for the anomaly is that this example is related to the Eisenstein series of weight 2 on $\mathrm{SL}(2, \mathbb{Z})$, which is a quasimodular rather than a modular form.)

We mention that a function with quantum modular properties very similar to those of the Dedekind sum occurs in a recent preprint of Brian Conrey [**5**].

Example 1. We consider the following two q-hypergeometric functions, the first of which was given in Ramanujan's "Lost" Notebook and the second, its partner, discovered later:

$$\begin{aligned} \sigma(q) &= \sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}}{(1+q)(1+q^2)\cdots(1+q^n)} \\ &= 1 + q - q^2 + 2\,q^3 - 2\,q^4 + q^5 + q^7 - 2\,q^8 + \cdots\,, \\ \sigma^*(q) &= 2\sum_{n=1}^{\infty} \frac{(-1)^n q^{n^2}}{(1-q)(1-q^3)\cdots(1-q^{2n-1})} \\ &= -2\,q - 2\,q^2 - 2\,q^3 + 2\,q^7 + 2\,q^8 + 2\,q^{10} + \cdots\,. \end{aligned}$$

In a beautiful paper by George Andrews, Freeman Dyson and Dean Hickerson [**2**]—the story is told in more detail in the last section of Dyson's famous survey article [**8**]—several identities expressing these two q-series as theta series associated to indefinite quadratic forms were proved, thereby explaining in particular the otherwise amazing experimental fact that the coefficients of both are very small, even though the individual terms have huge coefficients. (For instance, no coefficient of q^n in $\sigma(q)$ for $n \le 1600$ is greater than 4 in absolute value, even though some coefficients of the individual terms in the sum in the same range exceed 10^{13}.) A typical identity they proved is

$$q\,\sigma\big(q^{24}\big) \;=\; \sum_{\substack{a,\,b\,\in\,\mathbb{Z}\\ a>6|b|}} \left(\frac{12}{a}\right)\big(-1\big)^b\,q^{a^2-24b^2}\,, \tag{4}$$

the right-hand side of which is very similar to that of the modular identity

$$\frac{\eta(24z)^3}{\eta(48z)} \;=\; \sum_{\substack{a,\,b\,\in\mathbb{Z}\\ a>6|b|}} \left(\frac{-12}{a}\right)\big(-1\big)^b\,q^{a^2-24b^2}\,, \tag{5}$$

where $\eta(z) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n)$ denotes the classical Dedekind eta function.

In an equally beautiful paper [**4**] which appeared side-by-side with the Andrews-Dyson-Hickerson paper, Henri Cohen interpreted these identities in terms, first of algebraic number theory, and then of the theory of Maass wave forms. Define coefficients $\big\{T(n)\big\}_{n\in 24\mathbb{Z}+1}$ by

$$q\,\sigma\big(q^{24}\big) \;=\; \sum_{n\ge 0} T(n)\,q^n\,, \qquad q^{-1}\,\sigma^*\big(q^{24}\big) \;=\; \sum_{n<0} T(n)\,q^{|n|}\,. \tag{6}$$

Then the identities of [**2**] are equivalent to the fact that $T(n)$ is the coefficient of $|n|^{-s}$ in the Dirichlet series

$$L(s) \;=\; \prod_{\substack{p\equiv\pm 3\\ (\mathrm{mod}\ 8)}} \frac{1}{1-p^{-2s}} \prod_{\substack{p\equiv\pm 7\\ (\mathrm{mod}\ 24)}} \frac{1}{1+p^{-2s}} \prod_{\substack{p\equiv\pm 1\\ (\mathrm{mod}\ 24)}} \frac{1}{(1-\varepsilon(p)\,p^{-s})^2}\,,$$

where $\varepsilon(p)$ is defined for $p = |P|$ with $P \in 24\mathbb{Z}+1$ by $\varepsilon(p) = (-1)^b = \left(\frac{12}{c}\right) = \left(\frac{24}{f}\right)$ if P has the representations $P = a^2 - 72b^2 = c^2 - 96d^2 = e^2 - 192f^2$ as a norm in the three quadratic orders $\mathbb{Z}[6\sqrt{2}]$, $\mathbb{Z}[4\sqrt{6}]$ and $\mathbb{Z}[8\sqrt{3}]$, respectively. Cohen observed that this is an Artin L-function that can be expressed via the identities $L(s) = \zeta_{\mathbb{Q}\left(\sqrt{3+\sqrt{3}}\right)}(s)/\zeta_{\mathbb{Q}\left(\sqrt{3}\right)}(s) = \zeta_{\mathbb{Q}\left(\sqrt{3+\sqrt{6}}\right)}(s)/\zeta_{\mathbb{Q}\left(\sqrt{3}\right)}(s)$ as a quotient of Dedekind zeta functions. This implies the functional equation $\widehat{L}(s) = -\widehat{L}(1-s)$, where $\widehat{L}(s) = (24\sqrt{2}/\pi)^s\Gamma(s/2)^2L(s)$, and from this in turn one deduces that the function

$$u(z) \;=\; \sqrt{y}\sum_{n\in 24\mathbb{Z}+1} T(n)\,K_0\big(2\pi|n|y/24\big)\,e^{2\pi i n x/24} \qquad (z = x+iy \in \mathfrak{H}) \tag{7}$$

satisfies $u(-1/2z) = \overline{u(z)}$ as well as the more obvious functional equation $u(z+1) = e^{2\pi i/24}u(z)$, whence also $u(z/(2z+1)) = e^{2\pi i/24}u(z)$. Since $u(z)$ is also an eigenfunction of the hyperbolic Laplace operator $-y^2\big(\partial^2/\partial x^2 + \partial^2/\partial y^2\big)$ with eigenvalue $1/4$, this shows that $u(z)$ is a Maass wave form on the congruence subgroup $\Gamma_0(2)$ and thus that the identity (4) is just as modular in nature as the identity (5), but now using non-holomorphic rather than holomorphic modular forms.

All of this seems to have nothing to do with quantum modular forms. However, Cohen also observed a further phenomenon, and it is this which concerns us here. One has the two q-series identities (the first due to Andrews, the second derived in a similar way by Cohen)

$$\begin{aligned} \sigma(q) &= 1 + \sum_{n=0}^{\infty} (-1)^n \, q^{n+1} \, (1-q)(1-q^2)\cdots(1-q^n) \,, \\ \sigma^*(q) &= -2 \sum_{n=0}^{\infty} q^{n+1} \, (1-q^2)(1-q^4)\cdots(1-q^{2n}) \,. \end{aligned} \tag{8}$$

Cohen observed that the right-hand side of each of these expressions, as well as being a convergent series in the disk $|q| < 1$, also makes sense whenever q is a root of unity, because the series is then terminating in both cases. He then discovered the following surprising fact about these functions.

Lemma. *Define σ and σ^* at roots of unity by (8). Then $\sigma(q) = -\sigma^*(q^{-1})$ for every root of unity q.*

The first cases of this can be checked by hand: $\sigma(1) = -\sigma(1) = 2$, $\sigma(-1) = -\sigma^*(-1) = -2$, $\sigma(\omega) = -\sigma^*(\omega^2) = 2\omega + 6$ for $\omega^2 + \omega + 1 = 0$, and $\sigma(\pm i) = -\sigma^*(\mp i) = \mp 2i - 4$.

Proof. The Laurent series

$$S_k \;=\; \sum_{n=1}^{k} q^{-n(n-1)/2}(1+q)(1+q^2)\cdots(1+q^{k-n}) \quad \in \mathbb{Z}[q,q^{-1}]$$

satisfies the recursion $S_{k+1} - S_k = q^{k+1}\bigl(S_{k+1} - (1+q)\cdots(1+q^k)\bigr) - q^{-k(k+1)/2}$, so by induction

$$\sum_{n=0}^{k-1} \bigl(q^{-1}-1\bigr)\cdots\bigl(q^{-n}-1\bigr) - \sum_{n=0}^{k-1} q^{n+1}(1-q^2)\cdots(1-q^{2n}) \;=\; (1-q)\cdots(1-q^k)\, S_k \tag{9}$$

for every $k \geq 0$. If q is a root of unity and k is bigger than or equal to the order of q, then the right-hand side of (9) vanishes and the left-hand side is easily seen to be $\frac{1}{2}\sigma(q^{-1}) + \frac{1}{2}\sigma^*(q)$. ■

We can now define our quantum modular form. Define a function $f : \mathbb{Q} \to \mathbb{C}$ by

$$f(x) \;=\; q^{1/24}\,\sigma(q) \;=\; -q^{1/24}\,\sigma^*(q^{-1}) \qquad (x \in \mathbb{Q},\ q = e^{2\pi i x})\,, \tag{10}$$

where the equality of the two formulas is precisely the content of the lemma. This function, whose values for x with denominator ≤ 4 were given (up to the factor $q^{1/24}$) before the proof of the lemma, jumps around erratically as x runs through the rational numbers, but the cocycle defined by (3) with $\Gamma = \Gamma_0(2)$ and $k = 1$ is almost everywhere analytic:

Proposition. *The function $f : \mathbb{Q} \to \mathbb{C}$ defined by (10) satisfies*

$$f(x+1) \;=\; e^{2\pi i/24} f(x)\,, \qquad \frac{1}{2x+1}\, f\Bigl(\frac{x}{2x+1}\Bigr) \;=\; e^{2\pi i/24} f(x) \,+\, h(x) \tag{11}$$

where $h : \mathbb{R} \to \mathbb{C}$ is C^∞ on $\mathbb{R}$ and real-analytic except at $x = -1/2$.

We illustrate this behavior by plotting in Figure 1 the real part of $f(x)$ for all rational numbers $x \in [-1.7, 1.1]$ with denominator ≤ 100 (the imaginary part looks very similar), and in Figure 2 the values of the real and imaginary parts of $h(x)$ for the same values of x.

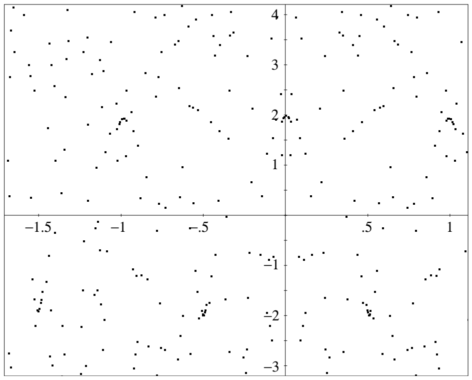

Figure 1. Graph of $\Re(f(x))$

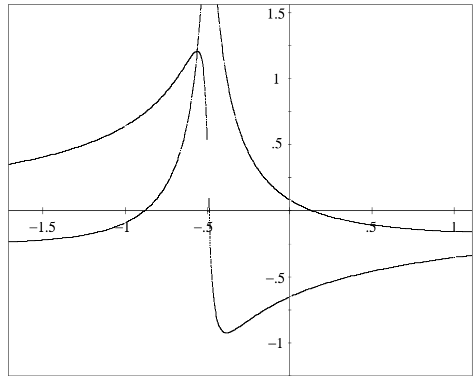

Figure 2. Graph of $\Re(h(x))$ and $\Im(h(x))$

This proposition, which we will prove in a moment, shows that f is a quantum modular form in the sense explained in the introduction, and the figures depict graphically what this means. In fact, f is a strong quantum modular form. Indeed, the two expressions in (8) are not only well-defined complex numbers when q is a root of unity, but well-defined power series in t, with coefficients in $\mathbb{Q}[\xi]$, when we take $q = \xi e^{-t}$ with ξ a root of unity. Furthermore, the identity $\sigma(q) = -\sigma^*(q^{-1})$ of the lemma remains true as an identity in $\mathbb{Q}[\xi][[t]]$, with the same proof, because the right-hand side of (7) is $\mathrm{O}(t^m)$ for k larger than m times the order of ξ. For instance, if we take $\xi = 1$ we find

$$\sigma\bigl(e^{-t}\bigr) \;=\; -\sigma^*\bigl(e^{t}\bigr) \;=\; 2 - 2\,t + 5\,t^2 - \frac{55}{3}\,t^3 + \frac{1073}{12}\,t^4 - \frac{32671}{60}\,t^5 + \frac{286333}{72}\,t^6 - \cdots\,.$$

If we extend the definition of f to infinitesimal neighborhoods of all rational points by interpreting (10) in the obvious way when x is replaced by $x + iy$ with $x \in \mathbb{Q}$ and y infinitesimal (so $q = \xi e^{-t}$ with $\xi = e^{2\pi i x}$ and $t = 2\pi y$), then (11) then still holds, where $h(x)$ is extended to a neighborhood of $\mathbb{R} \smallsetminus \{-1/2\}$ by analytic continuation. Here we can also clearly see the phenomenon of “leaking through the rational numbers” mentioned in the introduction, because we can extend the formally defined function f to a globally defined function $f : \mathfrak{H} \cup \mathfrak{H}^- \cup \mathbb{Q} \to \mathbb{C}$ by setting

$$f(z) \;=\; \begin{cases} q^{1/24}\,\sigma(q) & \text{if } z \in \mathfrak{H} \cup \mathbb{Q}\,, \\ -q^{1/24}\,\sigma^*(q^{-1}) & \text{if } z \in \mathfrak{H}^- \cup \mathbb{Q}, \end{cases} \tag{12}$$

where $q = e^{2\pi i z}$. Then the argument just given shows that f, which is obviously analytic in both $\mathfrak{H}$ and $\mathfrak{H}^-$, is C^∞ on any curve passing vertically through a rational point. In fact, the function $f(z)$ is the key to the proof of the proposition. Inserting the Fourier expansions (6) into (12) we can rewrite the definition of f in $\mathbb{C} \smallsetminus \mathbb{R}$ as

$$f(z) \;=\; \begin{cases} \sum_{n>0} T(n)\,q^n & \text{if } z \in \mathfrak{H}\,, \\ -\sum_{n<0} T(n)\,q^n & \text{if } z \in \mathfrak{H}^-, \end{cases}$$

which stands exactly is the same relation to the Maass wave form (7) as the functions denoted in the same way in the earlier work of J. Lewis and the author on Maass cusp forms on $\mathrm{SL}(2,\mathbb{Z})$ and their associated "period functions" [**12, 13**]. Making the needed minor changes in the results given there, we find that the holomorphic function f in $\mathbb{C} \smallsetminus \mathbb{R}$ can be expressed in terms of the Maass form u by the integral formulas

$$f(z) \;=\; \begin{cases} \int_z^\infty [u(\tau), r_z(\tau)] & \text{if } z \in \mathfrak{H}\,, \\ -\int_{\bar z}^\infty [r_z(\tau), u(\tau)] & \text{if } z \in \mathfrak{H}^-, \end{cases} \tag{13}$$

where the function $r_z : \mathfrak{H} \to \mathbb{C}$ is defined by $r_z(\tau) = (\Im(\tau)/(\tau - z)(\bar\tau - z))^{1/2}$ and, like u, is an eigenfunction of the hyperbolic Laplace operator (with respect to τ) with eigenvalue $1/4$, and where $[\,\cdot\,,\,\cdot\,]$ denotes the *Green's form*

$$\big[u(\tau), v(\tau)\big] \;=\; \frac{\partial u(\tau)}{\partial \tau}\, v(\tau)\, d\tau \;+\; u(\tau)\, \frac{\partial v(\tau)}{\partial \bar\tau}\, d\bar\tau\,,$$

which is a closed 1-form whenever u and v are eigenfunctions of the hyperbolic Laplace operator with the same eigenvalue. From this and the modularity property $u(\gamma\tau) = \chi(\gamma)u(\tau)$ for $\gamma \in \Gamma$ of $u(\tau)$, where $\chi : \Gamma_0(2) \to \mathbb{C}^*$ is the character sending both generators $\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}1&0\\2&1\end{smallmatrix}\right)$ to $e^{2\pi i/24}$, together with the easy equivariance property $r_{\gamma z}(\gamma\tau) = \pm(cz+d)r_z(\tau)$ for $\gamma = \left(\begin{smallmatrix}\cdot&\cdot\\c&d\end{smallmatrix}\right) \in \mathrm{SL}(2,\mathbb{R})$, we deduce, apart from the obvious periodicity property $f(z+1) = e^{2\pi i/24} f(z)$, the formula

$$(2z+1)\, f\Big(\frac{z}{2z+1}\Big) \;-\; e^{2\pi i/24} f(z) \;=\; -\int_{-1/2}^\infty [u(\tau), r_z(\tau)] \tag{14}$$

for z in either the upper or the lower half-plane, where the integral is taken along any path from $-1/2$ to ∞ passing to the left of z or $\bar z$. But the right-hand side now makes sense for any z lying to the right of both the chosen path and its reflection in the x-axis, so (if we push the path of integration far to the left) defines a holomorphic function on all of $\mathbb{C} \smallsetminus (-\infty, 0]$. The function $h(x)$ occurring in (11) for $x > 0$ is the restriction of this function to $\mathbb{R}_+$ and hence is real-analytic, and a similar argument works for $z \in \mathbb{C} \smallsetminus [0,\infty)$ and $x < 0$ if we change the minus sign on the left-hand side of (14) to a plus sign and take a path of integration passing to the right of z and $\bar z$. This establishes the real-analyticity of h on $\mathbb{R}^*$. The fact that it is C^∞ also at $x = -1/2$ follows by looking more closely at the integral and using that u is a cusp form, as was done in [**13**] for the period functions of Maass forms on the full modular group. ■

A similar discussion applies to other Maass wave forms on groups commensurable with $\mathrm{SL}(2,\mathbb{Z})$. We refer to the article [**3**] by R. Bruggeman for a treatment of this more general case.

Example 2. Our second example comes from [**14**], where the following elementary but rather surprising facts were proved.

1. Let $\mathcal{Q}_5$ denote the set of all quadratic functions $Q(x) = ax^2 + bx + c$ with $a, b, c \in \mathbb{Z}$, $a < 0$, and discriminant $b^2 - 4ac$ equal to 5. Then for every rational number x we have

$$\sum_{Q \in \mathcal{Q}_5} \max(Q(x), 0) \;=\; 2\,,$$

the sum always being finite. (For example, the only $Q \in \mathcal{Q}_5$ with $Q(\frac{1}{3}) > 0$ are $-x^2+x+1$, $-x^2-x+1$, $-5x^2+5x-1$ and $-11x^2+7x-1$ and the corresponding values $Q(\frac{1}{3}) = \frac{11}{9}, \frac{5}{9}, \frac{1}{9}, \frac{1}{9}$ add up to 2.) More generally, if for every positive non-square integer D we define $\mathcal{Q}_D$ like $\mathcal{Q}_5$ but with the discriminant of Q now being the given number D, then we have

$$\sum_{Q\in\mathcal{Q}_D} \max(Q(x),0) \;=\; \alpha_D \tag{15}$$

for all $x \in \mathbb{Q}$, where α_D is a rational number that depends only on D and is equal to a simple multiple of the value of the Dedekind zeta function of $\mathbb{Q}(\sqrt{D})$ at $s=2$.
2. If one replaces the expression $\max(Q(x),0)$ by its cube, then the same thing happens: one has

$$\sum_{Q\in\mathcal{Q}_D} \max(Q(x),0)^3 \;=\; \beta_D \tag{16}$$

for all $x \in \mathbb{Q}$, where $\beta_D \in \mathbb{Q}$ is related to $\zeta_{\mathbb{Q}(\sqrt{D})}(4)$. But for the fifth power one has instead

$$\sum_{Q\in\mathcal{Q}_D} \max(Q(x),0)^5 \;=\; \gamma_D \,+\, \delta_D\,\Phi(x) \tag{17}$$

where γ_D (again related to $\zeta_{\mathbb{Q}(\sqrt{D})}(6)$) and δ_D are rational numbers depending only on D and $\Phi : \mathbb{Q} \to \mathbb{Q}$ is an even periodic function satisfying $q^{10}\,\Phi(\frac{p}{q}) \in \mathbb{Z}$ for all $\frac{p}{q} \in \mathbb{Q}$, the first values being

$p/q \pmod 1$	0	1/2	$\pm1/3$	$\pm1/4$	$\pm1/5$	$\pm2/5$	$\pm1/6$
$q^{10}\,\Phi(p/q)$	1	-1049	-29399	12076	3132025	-8012423	30839551

The function Φ satisfies—and, if one fixes one value, is uniquely characterized by—the two functional equations

$$\Phi(x+1) \;=\; \Phi(x)\,, \qquad x^{10}\,\Phi(-1/x) \;=\; \Phi(x) \,+\, x^{10} \,-\, \frac{691}{36}x^2(x^2-1)^3 \,-\, 1\,. \tag{18}$$

Therefore $\Phi(x)$ (and hence also $\sum_{Q\in\mathcal{Q}_D}\max(Q(x),0)^5$ for any D) is a quantum modular form. This example is unusual in that the cocycle $r_\gamma = \Phi - \Phi|_{-10}\gamma$ is analytic on all of $\mathbb{R}$ (it is a polynomial) and that Φ itself extends continuously (and even differentiably, though not C^∞) from $\mathbb{Q}$ to $\mathbb{R}$.

Here, again, the explanation is modular, but much simpler than in our first example because now only holomorphic modular forms on the full modular group are involved. The reason for the different behavior of the functions in (15) and (16) and in (17) is that there are no holomorphic modular forms except for Eisenstein series of weight 4 or 8 on $\mathrm{SL}(2,\mathbb{Z})$, while in weight 12 one has the cusp form

$$\Delta(z) \;=\; q\prod_{n=1}^{\infty}\bigl(1-q^n\bigr)^{24} \;=\; \sum_{n=1}^{\infty}\tau(n)\,q^n \qquad \bigl(z\in\mathfrak{H},\ q=e^{2\pi iz}\bigr),$$

as well as the Eisenstein series. The existence of the quantum modular form Φ follows directly from the existence of the cusp form Δ, as a consequence the classical Eichler-Shimura-Manin theory of periods of holomorphic modular forms. Specifically, we associate to $\Delta(z)$ its *Eichler integral*

$$\widetilde{\Delta}(z) \;=\; \frac{(2\pi/i)^{11}}{10!}\int_z^{\infty}\Delta(z')\,(z'-z)^{10}\,dz' \;=\; \sum_{n=1}^{\infty}\frac{\tau(n)}{n^{11}}\,q^n \qquad (z\in\mathfrak{H})\,. \tag{19}$$

(For an arbitrary cusp form f of weight k, $\widetilde{f}$ would be defined the same way with 10 and 11 replaced by $k-2$ and $k-1$.) Then $\dfrac{d^{10}\widetilde{\Delta}(z)}{dz^{10}} = \Delta(z)$ and from this and the modularity of Δ one deduces easily that

$$\widetilde{\Delta}(z) \;-\; (cz+d)^{10}\,\widetilde{\Delta}\Big(\frac{az+b}{cz+d}\Big) \;=\; P_{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)}(z) \tag{20}$$

(or, more succinctly, $\widetilde{\Delta}\big|_{-10}(\gamma-1) = P_\gamma$) for all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_1 = \mathrm{PSL}(2,\mathbb{Z})$, where $P_\gamma(z)$ is a polynomial of degree ≤ 10, given explicitly by

$$P_\gamma(z) \;=\; \frac{(2\pi/i)^{11}}{10!} \int_{\gamma^{-1}(\infty)}^{\infty} \Delta(z')\,(z-z')^{10}\,dz' \;. \tag{21}$$

These polynomials satisfy the cocycle relation $P_{\gamma\gamma'} = P_\gamma\big|_{-10}\gamma' + P_{\gamma'}$ and hence are determined by their values for the generators $T = \left(\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ and $S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ of Γ_1, which are $P_T = 0$ (obviously) and

$$P_S(z) \;=\; -\Omega_1\left(z^{10} - \frac{691}{36}z^2(z^2-1)^3 - 1\right) + \Omega_2\left(z(z^2-1)^2(z^2-4)(4z^2-1)\right)$$

with $\Omega_1 = 0.98943291\cdots \in \mathbb{R}$, $\Omega_2 = 1.53908051\ldots i \in i\mathbb{R}$. From this and (18) we deduce that

$$\Phi(x) \;=\; \Re\big(\widetilde{\Delta}(x)/\Omega_1\big) \;=\; \frac{1}{\Omega_1}\sum_{n=1}^{\infty}\frac{\tau(n)}{n^{11}}\,\cos(2\pi n x) \tag{22}$$

for $x \in \mathbb{R}$, where $\widetilde{\Delta}(x)$ is defined by either of the formulas in (19), both of which remain convergent also when z lies on the real axis. The above-mentioned "continuous but not infinitely differentiable" properties of the function Φ follow from this: it is known that $\tau(n)$ is $\mathrm{O}\big(n^{11/2}\big)$ but not $\mathrm{o}\big(n^5\big)$ for n large, so the function $\Phi(x)$ on $\mathbb{R}$ is 4 times but not 6 times continuously differentiable.

In this example, too, we find a function that "leaks" from $\mathfrak{H}$ into $\mathfrak{H}^-$ through the rational holes in the real axis. To do this, we extend the definition (19) to the lower half-plane by

$$\begin{aligned}\widetilde{\Delta}(z) \;&=\; \frac{(2\pi/i)^{11}}{10!}\int_{\bar z}^{\infty}\Delta(z')\,(z'-z)^{10}\,dz' \\ &=\; \frac{1}{10!}\sum_{n=1}^{\infty}\frac{\tau(n)}{n^{11}}\,\gamma_{11}(4\pi n|y|)\,q^n \qquad (z\in\mathfrak{H}^-),\end{aligned} \tag{23}$$

where $z = x+iy \in \mathfrak{H}$ and $\gamma_{11}(t) = \int_t^\infty e^{-u}\,u^{10}\,du$, the incomplete gamma function (which is equal to e^{-t} times a polynomial in t). For $z = x \in \mathbb{R}$ the integrals in both (19) and (23) are convergent, because $\Delta(x+iy) = \mathrm{O}\big(y^{-6}\big)$ as $|y| \to 0$, so $\widetilde{\Delta}$ extends in this case to a continuous function in all of $\mathbb{C}$. This extended function still satisfies the functional equation (20), with the same polynomials P_γ as before, and because Δ is a cusp form and hence vanishes to infinite order as τ approaches any rational point, one sees easily that its restriction to any vertical line passing through a rational point is infinitely often differentiable. However, unlike the situation in our first example, here the function that "leaks" is only real-analytic, not holomorphic, in the lower half-plane.

Example 3. Our next example of a quantum modular form comes from the unusual series

$$F(q) \;=\; \sum_{m=0}^{\infty} (1-q)(1-q^2)\cdots(1-q^m)\,, \tag{24}$$

invented by Maxim Kontsevich, which has the peculiarity of not converging on any open subset of $\mathbb{C}$ but nevertheless makes sense as a function on the set of roots of unity because the series terminates after N terms if $q^N = 1$. We will be fairly brief in our treatment here, since this function was studied in detail in [**15**], and will only discuss the quantum modular aspect. Define $\varphi : \mathbb{Q} \to \mathbb{C}$ by $\varphi(x) = q^{1/24}F(q)$, where $q = e^{2\pi i x}$ as usual. Then $\varphi(1/n)$ has an asymptotic expansion of the form

$$\varphi(1/n) \;\sim\; n^{3/2}\, e^{2\pi i(3-n)/24} \;+\; \sum_{j=0}^{\infty} c_j\, (-2\pi i/n)^j \tag{25}$$

as $n \to \infty$, where $c_0 = 1$, $c_1 = \frac{23}{24}$, $c_2 = \frac{1681}{1152}, \ldots$ are certain rational coefficients. From the trivial functional equation $\varphi(x+1) = e^{2\pi i/24}\varphi(x)$ one sees that $e^{2\pi i(3-n)/24}$ equals $\sqrt{i}\,\varphi(-n)$, so (25) says that the function $g(x)$ defined by the second of the two equations

$$\varphi(x{+}1) = e^{2\pi i/24}\varphi(x)\,, \qquad \varphi(x) \mp i^{1/2}|x|^{3/2}\,\varphi(-1/x) \;=\; g(x) \qquad (x \in \mathbb{Q},\ \pm x > 0) \tag{26}$$

is smooth (i.e., has a well-defined Taylor expansion) at $x = 0$, and in fact it is real-analytic on the rest of the real axis, so that (26) presents $\varphi(x)$ as a quantum modular form.

The explanation is quite similar to that in the last example, except that the cusp form $\Delta(z)$ is replaced by its 24th root $\eta(z)$, which is a modular form of half-integral weight. Again we have a function $\widetilde{\eta}(z)$ in $\mathfrak{H} \cup \mathfrak{H}^-$, related to $\eta(z)$ in the same way as $\widetilde{\Delta}(z)$ in the previous example was related to $\Delta(z)$. (The direct analogues of the integrals in (19) and (23) diverge, because η has weight 1/2, so that the exponent "10" in the integrand would have to be replaced by "$-3/2$," but they can be made sense of by integrating by parts once, or alternatively, we can use the definitions via sums rather than integrals.) In particular, since $\eta(z) = \sum_{n=0}^{\infty} n\left(\frac{12}{n}\right) q^{n^2/24}$ (Euler), this gives that $\widetilde{\eta}(z)$, appropriately normalized, is given by

$$\widetilde{\eta}(z) \;=\; \sum_{n=0}^{\infty} n\left(\frac{12}{n}\right) q^{n^2/24} \;=\; q^{1/24}\left(1 \,-\, 5q \,-\, 7q^2 \,+\, 11q^5 \,+\, \cdots\right), \tag{27}$$

and now the relation to Kontsevich's function follows from the formula

$$\sum_{n=0}^{\infty}\left(q^{1/24}(1-q)(1-q^2)\cdots(1-q^n) \,-\, \eta(z)\right) \;=\; -\frac{1}{2}\,\widetilde{\eta}(z) \,+\, \eta(z)\left(\frac{1}{2} - \sum_{n=1}^{\infty}\frac{q^n}{1-q^n}\right)$$

([**15**], Theorem 2), which implies that $-2\varphi(x)$ for $x \in \mathbb{Q}$ is the limiting value of $\widetilde{\eta}(z)$ as z approaches x from either the upper or the lower half-plane. We also deduce (26), with an explicit formula for the cocycle function $g(x)$ as an integral of the Dedekind eta-function along a path from 0 to ∞ in the upper half-plane.

We observe in passing that the function of this example, like those of Examples 4 and 5, belongs to the Habiro ring of "analytic functions of roots of unity" [**10**]. These functions, which are also related to the (now so very fashionable) $\mathbb{F}_1$-story, occur in many contexts connected with quantum topological invariants and quantum groups, and would be a natural setting to look for more examples of quantum modular forms.

Example 4. The next example, taken from [**11**], is similar in many ways to the last one, but is interesting because it comes from topology and more particularly from the theory of quantum invariants of 3-manifolds. Again we shall be brief and refer to the original paper for details. To any 3-manifold one can associate the so-called Witten-Reshetikhin-Turaev invariant, defined by the first of these authors by a path integral that can be made sense of only perturbatively or in the sense of topological quantum field theory, and by the second two in a rigorous, but less illuminating, algebraic way. The invariant makes sense at roots of unity of the form $\zeta_K = e^{2\pi i/K}$ with $K > 0$ integral. For manifolds of very special types (such as torus knots or Seifert fibrations) there are explicit formulas for it, and in particular for the Poincaré homology sphere $\Sigma(2,3,5)$ it is given by

$$W(q) \;=\; \frac{1}{2G} \sum_{\substack{\beta \pmod{60K} \\ \beta\not\equiv 0 \pmod{K}}} \frac{(1-\alpha^{24\beta})(1-\alpha^{40\beta})}{1+\alpha^{60\beta}}\,\alpha^{-(\beta+1)^2}\,, \tag{28}$$

if $q = \zeta_K$, where $\alpha = \zeta_{120K}$ and $G = \sum_{\beta \bmod 60K} \alpha^{-\beta^2} \;=\; (1-i)\sqrt{30K}$ (Gauss sum). We extend this to other roots of unity by Galois invariance $W(q)^\sigma = W(q^\sigma)$, or equivalently by formula (28) for q equal to any primitive Kth root of unity, with α being any primitive $(120K)$-th root of unity with $\alpha^{120} = q$. Let $\chi_+(n)$ be the odd periodic function of period 60 defined by the formula

$$\chi_+(n) \;=\; \begin{cases} (-1)^{[n/30]} & \text{if } (n,6)=1 \text{ and } n\equiv \pm 1 \pmod 5, \\ \quad 0 & \text{otherwise,} \end{cases}$$

and let $\Theta_+(z)$ be the theta series

$$\Theta_+(z) \;=\; \sum_{n=1}^{\infty} n\,\chi_+(n)\,q^{\frac{n^2}{120}} \;=\; q^{\frac{1}{120}}\bigl(1+11q+19q^3+29q^7-31q^8-\cdots\bigr) \quad (z\in\mathfrak{H}),$$

which is a modular form of weight $3/2$ on a certain congruence subgroup of $\mathrm{SL}(2,\mathbb{Z})$ (and in fact is the first component of a 2-component vector-valued modular form of weight $3/2$ on the full group $\mathrm{SL}(2,\mathbb{Z})$). Then for every $x\in\mathbb{Q}$ the number

$$f(x) \;=\; 2\,e^{\pi i x/60}\,\bigl(1 \,-\, W\bigl(e^{2\pi i x}\bigr)\bigr) \tag{29}$$

is equal to the limit as $z\to x$ of the Eichler integral

$$\widetilde{\Theta}_+(z) \;=\; \sum_{n=1}^{\infty} \chi_+(n)\,q^{\frac{n^2}{120}} \;=\; q^{\frac{1}{120}}\bigl(1\,+\,q\,+\,q^3\,+\,q^7\,-\,q^8\,-\,\cdots\bigr) \tag{30}$$

([**11**], Theorem 1), and from this it follows that the function $f:\mathbb{Q}\to\mathbb{C}$ is a quantum modular form (in fact, a strong quantum modular form). The whole story is quite similar to that in Example 3 except that this time the modular form whose Eichler integral is involved has weight $3/2$ rather than $1/2$. There is also an expression

$$\sum_{n=0}^{\infty}\bigl[q^n(1-q)(1-q^6)\cdots(1-q^{5n-4}) \,+\, q^{4n+3}(1-q^4)(1-q^9)\cdots(1-q^{5n-1})\bigr]$$

for $q^{-1/120}\widetilde{\Theta}_+(q)$ (due to Zwegers) of the same type as (24), which terminates and hence gives a closed formula for $W(q)$ whenever q is a root of unity of order not divisible by 5, as well as relations (also pointed out by Zwegers) to the mock theta functions of Ramanujan. See [**11**] for more details.

Example 5. The last example, which again comes from topology, is the most mysterious and in many ways the most interesting. The function from $\mathbb{Q}/\mathbb{Z}$ to $\mathbb{R}$ that we obtain in this case is not a quantum modular form in the strict sense of the definition we gave in the introduction, let alone a strong quantum modular form, because the associated cocycle is no longer analytic or even continuous, but it nevertheless will turn out to have a clearly defined modularity property.

To any knot and any integer $n \geq 2$ one can associate a Laurent polynomial $J_n(q) \in \mathbb{Z}[q, q^{-1}]$, called the n-colored Jones polynomial. The definition, which involves the theory of quantum groups, will not be reviewed here since we will only look at one example and here the Jones polynomials can simply be given by an explicit formula. We will consider the figure-eight knot, the simplest hyperbolic knot. The Jones polynomial of this knot is given by

$$J_n(q) \;=\; \sum_{m=0}^{n-1} q^{-mn} \prod_{j=1}^{m} (1-q^{n-j})(1-q^{n+j})\;.$$

(Here the sum could also be taken from $m = 0$ to ∞ since the mth summand is 0 for $m \geq n$.) If we fix a root of unity q, then the function $n \mapsto J_n(q)$ is periodic, of period N if $q^N = 1$, so we can extrapolate it backwards to define $J_n(q)$ also for $n \leq 0$. Of particular interest to us is the $\overline{\mathbb{Q}}$-valued function on roots of unity defined by

$$J_0(q) \;:=\; J_N(q) \;=\; \sum_{m=0}^{\infty} \bigl|(1-q)(1-q^2)\cdots(1-q^m)\bigr|^2 \qquad (q \in \mathbb{C}^*,\;\; q^N = 1\,) \tag{31}$$

(compare the sum on the right-hand side to (24)), the first few values of which are as follows:

q	1	-1	$\zeta_3^{\pm 1}$	$\pm i$	$\zeta_5^{\pm 1}$	$\zeta_5^{\pm 2}$	$\zeta_6^{\pm 1}$
$J_0(q)$	1	5	13	27	$46+2\sqrt{5}$	$46-2\sqrt{5}$	89

The function J_0, which is related to perturbative $\mathrm{SL}(2,\mathbb{C})$ Chern-Simons theory (cf. [**7**]), is of a very different nature than the Jones polynomials themselves. For instance, the values of the Jones polynomials $J_n(q)$ when q is a root of unity are of only polynomial growth if $q^n \neq 1$, but the values of $J_0(\zeta_N) = J_N(\zeta_N)$ are exponentially big, as one can see in the following table:

N	100	200	300
$\max_{0<n<N} \lvert J_n(\zeta_N)\rvert$	12.07	18.62	24.99
$J_0(\zeta_N) = J_N(\zeta_N)$	8.20×10^{16}	2.48×10^{31}	4.89×10^{45}

Explicitly, $J_0(\zeta_N)$ is given by the the asymptotic formula [**1**]

$$J_0\bigl(e^{2\pi i/N}\bigr) \;\sim\; \frac{1}{\sqrt[4]{3}}\, N^{3/2}\, e^{CN} \qquad\qquad (n \to \infty)\,,$$

where $C = 0.3230659\ldots$ is $1/2\pi$ times the hyperbolic volume of the complement of the knot, and in fact one has a complete asymptotic expansion [**6, 9, 16**]

(32)
$$J_0\bigl(e^{2\pi i/N}\bigr) \;=\; \frac{1}{3^{1/4}}\,N^{3/2}\,e^{CN}\left(1+\frac{11}{36\sqrt{3}}\,\frac{\pi}{N}+\frac{697}{7776}\,\frac{\pi^2}{N^2}+\frac{724351}{4199040\sqrt{3}}\,\frac{\pi^3}{N^3}+\cdots\right)$$

as $N\to\infty$, where the factor in parentheses is a power series in $\pi/N\sqrt{3}$ with rational coefficients. Conjecturally [**7**], the corresponding expansion for an arbitrary hyperbolic knot would be a power series in $\pi i/N$ with coefficients in the trace field of the knot, this trace field being $\mathbb{Q}(\sqrt{-3})$ for the figure 8 knot.

But since $J_0(q)$ is defined for all roots of unity, we can look at its expansion near some other point than 1, e.g., we can consider the values $q=-\zeta_N$ rather than $q=\zeta_N$. It is here that the phenomenon of most interest to us appears: these values are given (experimentally) by the asymptotic series

$$(33)\quad J_0\bigl(-e^{2\pi i/N}\bigr) \;=\; \kappa(N)\cdot\frac{3^{1/4}}{2^{3/2}}\,N^{3/2}\,e^{CN/4}\left(1+\frac{41}{36\sqrt{3}}\,\frac{\pi}{N}+\frac{12625}{7776}\,\frac{\pi^2}{N^2}+\cdots\right),$$

of the same general form as (32), but this time involving an extra factor

$$\kappa(N) \;=\; \begin{cases} 27 & \text{if } N\equiv 1 \pmod 2,\\ 1 & \text{if } N\equiv 2 \pmod 4,\\ 5 & \text{if } N\equiv 0 \pmod 4\end{cases}$$

that depends on the value of $N \pmod 4$. Comparing with the table of values of $J_0(q)$ given above, we find that this factor is given in all cases by $\kappa(N) = J_0\bigl(i^{N+2}\bigr)$. What's more, if we now try *rational* rather than integral values for N, but with bounded denominators, then we find that (33) still holds, with $\kappa(N) = J_0\bigl(e^{\pi i(N+2)/2}\bigr)$. Going back to (32), we find exactly the same behavior there: if N goes to infinity, not through integers, but through rational numbers, say with denominator 2, 3 or 4, then (32) remains true if we multiply the right-hand side multiplied by 5, 13, and 27, respectively (and in general by $J_0(e^{2\pi iN})$). More generally, the experimentally found asymptotic behavior of the function

$$(34)\qquad \mathbf{J}\,:\,\mathbb{Q}/\mathbb{Z}\;\to\;\overline{\mathbb{Q}}\cap\mathbb{R}\,,\qquad\qquad \mathbf{J}(x)\;:=\;J_0(e^{2\pi ix})$$

as x tends to a fixed rational number $\alpha=a/c$ ($a,\,c\in\mathbb{Z}$, $(a,c)=1$) from the right or left is given by the formula

$$(35)\quad \mathbf{J}(\alpha\pm\varepsilon)\;=\;\mathbf{J}(\alpha^*\pm\beta)\cdot\frac{\exp(C/c^2\varepsilon)}{\varepsilon^{3/2}}\,\bigl(A_\pm(\alpha)\,+\,B_\pm(\alpha)\varepsilon\,+\,C_\pm(\alpha)\varepsilon^2\,+\,\cdots\bigr)$$

as ε tends to 0 through positive rational values with $1/c^2\varepsilon\equiv\beta \pmod 1$ for some fixed rational number β, where $\alpha^*=d/c$ with $d\equiv a^{-1} \pmod c$ and $A_\pm(\alpha) = A(\pm\alpha)$, $B_\pm(\alpha)=B(\pm\alpha)$, $\ldots$ are algebraic numbers depending only on α modulo 1. (In equations (32) and (33) we have $\alpha=\alpha^*=\beta=0$ and $\alpha=\alpha^*=1/2$, $\beta\equiv N/4 \pmod 1$, respectively, explaining the extra factor $\kappa(N)=\mathbf{J}((N+2)/4)$ in the latter case.)

The factor $\mathbf{J}(\alpha^*\pm\beta)$ in equation (35) looks odd at first sight, but in fact has a simple modular explanation: if we set $\gamma=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)$, where $b\in\mathbb{Z}$ is chosen so that $\gamma\in\mathrm{SL}(2,\mathbb{Z})$, then we have $\mathbf{J}(\alpha^*\pm\beta)=\mathbf{J}(-\alpha^*\pm 1/c^2\varepsilon)=\mathbf{J}(\gamma^{-1}(\alpha\pm\varepsilon))$, so that (35) can be seen as simply relating the values of $\mathbf{J}(X)$ and $\mathbf{J}(\gamma(X))$ as X $(=-\alpha^*\pm 1/c^2\varepsilon\,)$ tends to infinity through rational numbers with small denominator.

The asymptotic formula (35) is therefore equivalent to the first part of the following conjecture generalizing formulas (32) and (33):

CONJECTURE. *Let $\alpha \in \mathbb{Q}$ and choose $\gamma \in SL(2,\mathbb{Z})$ with $\gamma(\infty) = \alpha$. Then for suitable real numbers $S_0(\alpha)$, $S_1(\alpha)$, $\ldots$ depending only on $\alpha \pmod 1$ we have an asymptotic expansion*

$$(36) \qquad \frac{\mathbf{J}(\gamma(X))}{\mathbf{J}(X)} \;\sim\; \bigl(\pi/\hbar\bigr)^{3/2} \exp\biggl(\sum_{n=0}^{\infty} S_n(\alpha)\,\hbar^{n-1}\biggr), \qquad \hbar \;=\; \frac{\pi/\sqrt{3}}{X-\gamma^{-1}(\infty)}$$

as $X \to \infty$ through rational numbers with bounded denominators. The value of $S_0(\alpha)$ is independent of α and is equal to $\pi C/\sqrt{3}$, while $S_1(\alpha) \in \mathbb{Q}\log\bigl(K_\alpha^\times\bigr)$ and $S_n(\alpha) \in K_\alpha$ for $n \geq 2$, where K_α is the maximal real subfield of the cyclotomic field $\mathbb{Q}\bigl(\sqrt{-3},\, e^{2\pi i\alpha}\bigr)$.

We expect that a similar conjecture should hold for any hyperbolic knot complement, with $\hbar$ being defined as $\pi i/(X-\gamma^{-1}(\infty))$ (we divided this by $\sqrt{-3}$ in our special case to make everything real) and K_α being replaced by $K\bigl(e^{2\pi i\alpha}\bigr)$, where K is the trace field of the knot. The case when $\alpha = 0$ and $X \in \mathbb{Z}$ is precisely the arithmeticity conjecture from [**7**] which was cited earlier.

Observe that the correctness of (36) is unchanged by replacing (γ, X) by $(\gamma T, X-1)$ or $(T\gamma, X)$, where $T = \bigl(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\bigr)$, since these changes do not affect the left-hand side or the quantity $\hbar$, so the quantities $S_n(\alpha)$ really do depend only on α rather than on γ, and are periodic in α.

Here is a table of the numerically obtained values of $S_n(\alpha)$ for some small n and simple α, where in the last line ε_k ($k = 1, 2, 3$) denotes the real cyclotomic unit $\zeta^k + \zeta^{-k}$ with $\zeta = \zeta_3^{-1}\, e^{4\pi i a/5}$ and $\pi_{29} = 2 - \varepsilon_1 + \varepsilon_2 + 2\varepsilon_3$, a prime of $\mathbb{Q}(\zeta)$ of norm -29.

α	$\exp(S_1(\alpha))$	$S_2(\alpha)$	$S_3(\alpha)$	$S_4(\alpha)$
0	$\frac{1}{3}$	$\frac{11}{2^2 3^2}$	$\frac{2}{3^2}$	$\frac{1081}{2^1 3^5 5}$
$\frac{1}{2}$	$(2^3/3)^{1/2}$	$\frac{41}{2^4 3^2}$	$\frac{19}{2^3 3^2}$	$\frac{71089}{2^7 3^5 5}$
$\frac{1}{3}$	$2 \cdot 3^{2/3}$	$\frac{37}{2^2 3^3}$	$\frac{401}{2^1 3^6}$	$\frac{30767}{2^1 3^8 5}$
$\frac{2}{3}$	$3^{4/3}$	$\frac{25}{2^2 3^3}$	$\frac{182}{3^6}$	$\frac{29027}{2^1 3^8 5}$
$\frac{1}{6}$	$2^{7/2} \cdot 3^{5/6}$	$\frac{193}{2^4 3^3}$	$\frac{24691}{2^7 3^6}$	$\frac{8027957}{2^9 3^8 5}$
$\frac{5}{6}$	$2^{3/2} \cdot 3^{13/6}$	$\frac{67}{2^4 3^3}$	$\frac{1289}{2^3 3^6}$	$\frac{1759883}{2^7 3^8 5}$
$\pm\frac{1}{4}$	$\frac{2^3(2\sqrt{3}\pm 1)}{3(2\pm\sqrt{3})^{1/4}}$	$\frac{1855\pm 360\sqrt{3}}{2^6 3^2 11}$	$\frac{71132\pm 3123\sqrt{3}}{2^8 3^2 11^2}$	$\frac{1499191589\pm 43727850\sqrt{3}}{2^{11} 3^5 5^1 11^3}$
$\frac{a}{5},\ 5 \nmid a$	$\frac{(5^3/3)^{1/2}\lvert\pi_{29}\rvert}{(\varepsilon_1^3\varepsilon_3/\varepsilon_2)^{1/10}}$	$\frac{1}{2^2 3^2 5^3 \pi_{29}}(2678 - 943\varepsilon_1$	$\cdots$	$\cdots$
		$+1831\varepsilon_2 + 2990\varepsilon_3)$		

Formulas (35) or (36) say that the values of $\mathbf{J}(x)$ as x approaches any given rational number go exponentially rapidly to infinity and lie on certain smooth curves (countably many, all proportional to one another) depending on the rational number in question. This behavior can be seen clearly in the graph of the function $\mathbf{J}$, which looks as follows, where because of the very rapid growth we have plotted $f(x) = \log(\mathbf{J}(x))$ rather than $\mathbf{J}$ itself, so that now the different curves containing the points of the graph with argument near any fixed rational point differ by vertical translations:

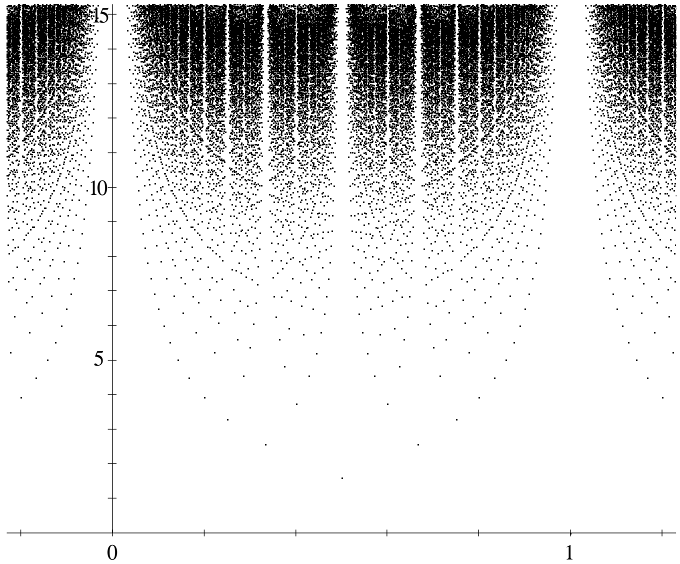

Figure 3. Graph of $f(x) = \log(\mathbf{J}(x))$

To make more sense of this graph, we do as in Examples 1–4 and compare the values of $f(x)$ at x and $1/x$. The graph of the difference indeed looks much better than the graph of f itself:

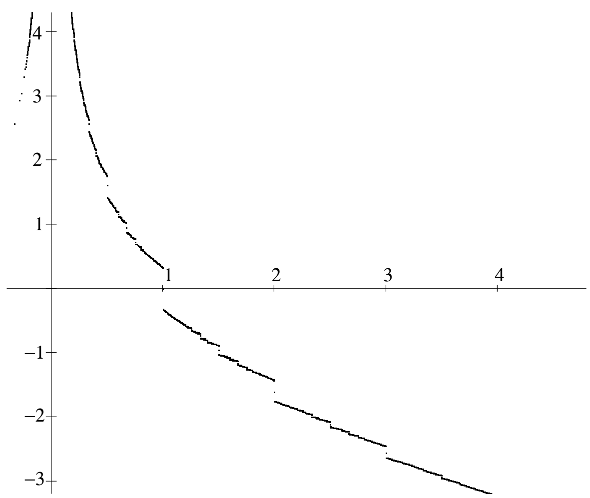

Figure 4. Graph of $h(x) = \log(\mathbf{J}(x)/\mathbf{J}(1/x))$

The behavior that we see here is a consequence of the conjecture above, which can easily be seen to imply that the function $h(x)$ has a jump at every rational point $\alpha = a/c$ but is C^∞ as we approach α from the left or from the right, with limiting values of the form $h^{\pm}(\alpha) = \pm C/ac + \log(\beta_{\pm}(\alpha))$ as x approaches α from the left or from the right, where $\beta_{\pm}(\alpha)$ are real algebraic numbers. This smoothness from the two sides can be seen more clearly by looking more closely at the graph of $h(x)$ in the neighborhood of a rational point α with small denominator, say $\alpha = 3/8$:

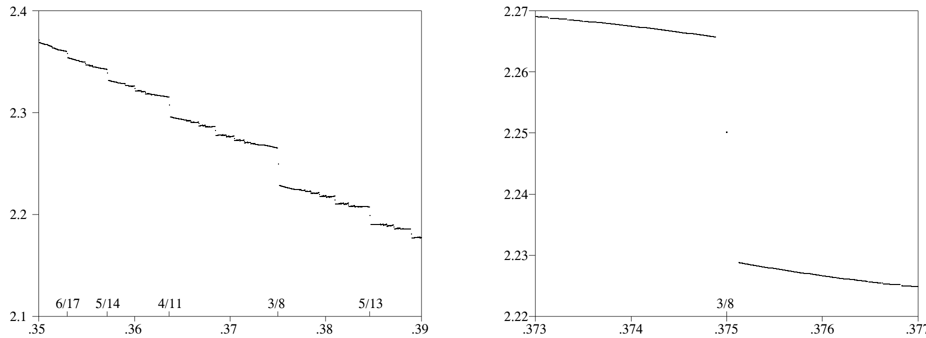

Figure 5. Graphs of $h(x)$ near $x = 3/8$

By contrast, in a small interval around the point $1/\phi$ ($\phi = (1+\sqrt{5})/2$ = golden ratio), where there are no points with particularly small denominator, we get the following picture

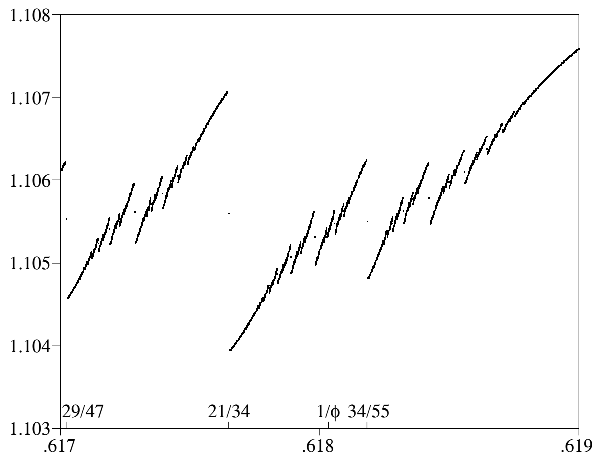

Figure 6. Graph of $h(x)$ near $x = 1/\phi$

showing that, unlike what the picture in Figure 4 might have suggested, $h(x)$ is not monotone decreasing on $\{x > 0\}$ and seeming to indicate that the function $h(x)$ is continuous but in general not differentiable at irrational values of x.

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