QUASIMODULARITY AND LARGE GENUS LIMITS OF SIEGEL-VEECH CONSTANTS

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Contents

Abstract. Quasimodular forms were first studied systematically in the context of counting torus coverings. Here we show that a weighted version of these coverings with Siegel-Veech weights also provides quasimodular forms. We apply this to prove conjectures of Eskin and Zorich on the large genus limits of Masur-Veech volumes and of Siegel-Veech constants.

In Part I we connect the geometric definition of Siegel-Veech constants both with a combinatorial counting problem and with intersection numbers on Hurwitz spaces. We also introduce certain modified Siegel-Veech weights whose generating functions will later be shown to be quasimodular.

Parts II and III are devoted to the study of the (quasi) modular properties of the generating functions arising from weighted counting of torus coverings. These two parts contain little geometry and can be read independently of the rest of the paper. The starting point is the theorem of Bloch and Okounkov saying that certain weighted averages, called q-brackets, of shifted symmetric functions on partitions are quasimodular forms. In Part II we give an expression for the growth polynomials (a certain polynomial invariant of quasimodular forms) of these q-brackets in terms of Gaussian integrals and use this to obtain a closed formula for the generating series of cumulants that is the basis for studying large genus asymptotics. In Part III we show that the even hook-length moments of partitions are shifted symmetric polynomials and prove a surprising formula for the q-bracket of the product of such a hook-length moment with an arbitrary shifted symmetric polynomial as a linear combination of derivatives of Eisenstein series. This formula gives a quasimodularity statement also for the (-2)-nd hook-length moments by an appropriate extrapolation, and this in turn implies the quasimodularity of the Siegel-Veech weighted counting functions.

Finally, in Part IV these results are used to give explicit generating functions for the volumes and Siegel-Veech constants in the case of the principal stratum of abelian differentials. The generating functions have an amusing form in terms of the inversion of a power series (with multiples of Bernoulli numbers as coefficients) that gives the asymptotic expansion of a Hurwitz zeta function. To apply these exact formulas to the Eskin-Zorich conjectures on large genus asymptotics (both for the volume and the Siegel-Veech constant) we provide in a separate appendix a general framework for computing the asymptotics of rapidly divergent power series.

Introduction

This paper grew out of an attempt to understand the algebraic and combinatorial nature of Siegel-Veech constants on flat surfaces (Part I) and culminates in a proof of the Eskin-Zorich conjecture ([20]) on large genus asymptotics of Masur-Veech volumes and Siegel-Veech constants for the case of the principal stratum of abelian differentials (Part IV). Along the way we discovered properties of Bloch-Okounkov correlators and growth polynomials of quasimodular forms, of interest independently of the geometric background. Consequently, we start with the motivation through Siegel-Veech constants but a reader with focus on Bloch-Okounkov correlators and quasimodular forms may skip to the presentation of Part II and Part III below, where no background on flat surfaces is required.

Part I: Siegel-Veech constants on Hurwitz spaces. The number of closed geodesics of bounded length on a flat surface, i.e. a Riemann surface with a flat metric (induced from an abelian differential), has quadratic growth. The moduli space of flat surfaces is stratified by the number and multiplicities of zeros of the differential and the leading term of the quadratic asymptotic is the same for all generic flat surfaces in a given stratum. This leading term is called the Siegel-Veech constant ([43], [14]). In fact, there are several variants of Siegel-Veech constants (e.g. [45] and [6]) obtained by counting the trajectories with different weights. Among them is the *area Siegel-Veech constant* (see Section 1 for the definition), whose importance is due to the connection with intersection numbers on the moduli space of curves and with Lyapunov exponents. We will focus on the area Siegel-Veech constant throughout the paper.

The strata of the moduli space of flat surfaces have an integral affine structure and thus a natural volume form, due to Masur and Veech. The area Siegel-Veech constants for strata have been computed recursively using Masur-Veech volumes of strata in low genera by Eskin-Masur-Zorich ([15]). This procedure is combinatorially quite involved and sheds little light on the algebro-geometric significance of Siegel-Veech constants.

Now consider the *Hurwitz space* $H_d(\Pi)$ of degree d torus coverings with ramification profile Π (see Section 2 for the background and notation). These spaces are dense in every stratum and the same definition of area Siegel-Veech constants through quadratic asymptotics applies here as well. The advantage of Hurwitz spaces is that there we can provide a transparent combinatorial and intersection-theoretic explanation of Siegel-Veech constants. We define the p-th *part-length moment* of a partition α to be

$$S_p(\alpha) \;=\; \sum_{i=1}^{r} \alpha_i^p, \qquad (\alpha = (\alpha_1, \alpha_2, \dots, \alpha_r),\ \ \alpha_i \geq 1)\,, \tag{1}$$

where p is any complex number. A torus covering in $H_d(\Pi)$ can be described by the associated Hurwitz tuple $(\alpha, \beta, \gamma_i)$ where α, β and γ_i are elements in S_d arising from monodromy of the covering (see (20) for the definition). Let the *p-weighted Siegel-Veech constant* $c_p^0(d, \Pi)$ of a Hurwitz space be the sum of $S_p(\alpha)$ over all Hurwitz tuples $(\alpha, \beta, \gamma_i)$ for $H_d(\Pi)$ and $N_d^0(\Pi)$ the number of these tuples. Using the case $p = -1$, in Theorem 3.1 we give the following combinatorial formula for Siegel-Veech constants on Hurwitz spaces:

- The area Siegel-Veech constant c_{area} for a Hurwitz space $H_d(\Pi)$ is equal to

$$c_{\mathrm{area}}(d, \Pi) \;=\; \frac{3}{\pi^2}\,\frac{c_{-1}^0(d, \Pi)}{N_d^0(\Pi)}\,.$$

The proof is a standard application of the Siegel-Veech transform. Defining and using the p-weighted Siegel-Veech constants for all $p \in \mathbb{Z}$ will be crucial in Part III, although we are not aware of a flat geometric interpretation of the counting functions for $p \neq -1$.

The sum of Lyapunov exponents for the Teichmüller geodesic flow is another quantity of dynamic origin, defined for strata and Hurwitz spaces (in fact for any $\mathrm{SL}(2, \mathbb{R})$-invariant submanifold of strata), whose algebraic nature still awaits to be understood completely. Here we do not rely on the dynamic definition of Lyapunov exponents via growth rates of cohomology classes (see [51] for more detail). The point of departure is rather the reinterpretation of Kontsevich-Zorich ([30]) for the sum of Lyapunov exponents as a ratio of two intersection numbers with a foliation class β (recalled in Section 4.2). Intersection with β is well-defined as a transverse measure class, but since an interpretation of β as a rational cohomology class is still missing, there is currently no direct algebraic proof of the rationality of the sum of

Lyapunov exponents for strata. However, Eskin-Kontsevich-Zorich ([13]) managed to prove this indirectly with a beautiful generalization of Noether's formula (recalled in (37) below), showing that the sum of Lyapunov exponents differs from the area Siegel-Veech constant by an easily computable rational number, an evaluation of the κ-class.

In the case of Hurwitz spaces we show that all of the above quantities have transparent algebro-geometric interpretations. In Section 5 we show that β is indeed proportional to a cohomology class and relate β to the tautological classes ψ_i (see Theorem 4.3):

- On the moduli space $\overline{\mathcal{M}}_{1,n}$ the classes β and $\psi_2 \cdots \psi_n$ are proportional.

Moreover, the Siegel-Veech constant $c^0_{-1}(d,\Pi)$ with weight $p=-1$ appears in the pushforward of the nodal locus in the universal curve over the Hurwitz space to $\overline{\mathcal{M}}_{1,n}$, namely as coefficient of the boundary divisor $\delta_{\rm irr}$ (Theorem 4.1). Combining these observations, we give a proof of the main result of [13] for Hurwitz spaces using only intersection theory calculations (Theorem 4.2).

In order to understand the combinatorial nature of the p-weighted Siegel-Veech constants we form the generating series

$$c_p(\Pi) = \sum_{d\geq 1} c^0_p(d,\Pi)\, q^d.$$

As we will explain in the motivation for Part II in more detail, the generating function for counting covers without weights is a *quasimodular form* for $\mathrm{SL}(2,\mathbb{Z})$, i.e. a polynomial in the Eisenstein series E_2, E_4, and E_6. Our first main structure result (Theorem 6.4) states that quasimodularity still holds with odd Siegel-Veech weight $p \geq -1$, despite the unusual counting involving inverses (if $p=-1$) of part-lengths:

- The generating series of Siegel-Veech constants $c_{-1}(\Pi)$ with weight $p=-1$ for Hurwitz spaces in the stratum $\Omega\mathcal{M}_g(m_1,\ldots,m_n)$ as well as its p-weighted variants for odd $p>0$ are quasimodular forms of mixed weight $\leq p+1+\sum_{i=1}^n(m_i+2)$.

The proof of this result requires all the material of Part III and will be completed only in Section 16. In covering theory it is a standard argument that coverings without unramified components can be counted by counting all coverings and then dividing by the partition function. The generating series for counting connected coverings is then obtained by their linear combinations. In Proposition 6.2 we show that a similar procedure works in the presence of a Siegel-Veech weight, though with a different formula since Siegel-Veech weights are additive (rather than multiplicative) on disjoint unions of partitions. Consequently, we need to understand q-brackets to prove Theorem 6.4.

Part II: Bloch-Okounkov correlators and their growth polynomials. The point of departure for Part II is a beautiful theorem of Bloch and Okounkov ([7]) saying that the q-bracket

$$\langle f\rangle_q \;=\; \frac{\sum_{\lambda\in\mathbf{P}} f(\lambda)\, q^{|\lambda|}}{\sum_{\lambda\in\mathbf{P}} q^{|\lambda|}} \;\in\; \mathbb{Q}[[q]]\,,$$

of any "shifted symmetric polynomial f" on the set of all partitions is a quasimodular form. This theorem continued the ideas of Dijkgraaf ([12], with a rigorous proof given in [27]), that the generating series for the number of connected covers of a torus with simple branching is a quasimodular form. Eskin and Okounkov ([18]) used the Bloch-Okounkov theorem to show that quasimodularity holds for any type of branching profile. We recall in Sections 7 and 8 the background on the ring $\mathcal{R}$ of shifted symmetric polynomials, on quasimodular forms, and the Bloch-Okounkov theorem. We refer to the generating functions $F(z_1,\dots,z_n)$ of q-brackets for a fixed number n of monomials as *Bloch-Okounkov correlators*.

Understanding the quasimodular forms arising this way is difficult, even though e.g. the top term as a polynomial in E_2 had been computed in [27]. However, there is a ring homomorphism Ev associating to each quasimodular form a "growth polynomial" (given on generators by $E_4 \mapsto X^2$, $E_6 \mapsto X^3$, while $E_2 \mapsto X+12$) that governs the growth of its Fourier coefficients and describes the asymptotic behavior of the quasimodular form near the cusp (Proposition 9.3):

- Let F be a quasimodular form of weight k with $\mathrm{Ev}[F] = AX^h + \cdots$ and the leading coefficient $A \neq 0$. Then the sum of the first N Fourier coefficients of F has the asymptotic behaviour

$$\sum_{n=1}^{N} a_n(F) \;=\; (-4\pi^2)^h A \frac{N^{h+k}}{(h+k)!} \;+\; \mathrm{O}(N^{h+k-1}\log(N))\,.$$

The growth polynomial is essentially equivalent to an expansion used by Eskin and Okounkov in [18], but we give a different presentation and several further properties.

The main new ideas of this part start in Section 10. Previously in [7] and [18], the focus had been on the generating functions $F(z_1,\dots,z_n)$ and their Ev-images. Instead, we introduce the partition function

$$\Phi(\mathbf{u})_q \;=\; \Big\langle \exp\Big(\sum_{\ell\geq 1} p_\ell\, u_\ell\Big) \Big\rangle_q \;=\; \sum_{\mathbf{n}\geq 0} \langle \underbrace{p_1,\dots,p_1}_{n_1}, \underbrace{p_2,\dots,p_2}_{n_2}, \dots\rangle_q \,\frac{\mathbf{u}^{\mathbf{n}}}{\mathbf{n}!}$$

where p_ℓ are power sum generators of the algebra of shifted symmetric polynomials. After passing to the growth polynomial (and only then!) the structure of this partition function becomes transparent (Theorem 10.2):

- The Ev-image $\Phi(\mathbf{u})_X = \mathrm{Ev}[\Phi(\mathbf{u})_q]$ of the partition function can be expressed as the formal Gaussian integral

$$\Phi(\mathbf{u})_X \;=\; \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-y^2/2 \,+\, \mathcal{B}(\mathbf{u},iy,X)}\, dy\,. \tag{2}$$

  where we use the coefficients of $\sum_{k\geq 0}\beta_k z^k = \frac{z/2}{\sinh(z/2)}$ to define

$$\mathcal{B}(\mathbf{u},y,X) \;=\; \sum_{\substack{\mathbf{a}>0\\ r\geq 0}} (a_1+2a_2+3a_3+\cdots)!\;\beta_{2-r+w(\mathbf{a})}\sqrt{X}^{\,2-r+w(\mathbf{a})}\,\frac{\mathbf{u}^{\mathbf{a}}}{\mathbf{a}!}\,\frac{y^r}{r!}\,,$$

  with $w(\mathbf{a}) = a_2 + 2a_3 + 3a_4 + \cdots$.

Note that the right hand side of (2) is purely algebraic and does not really involve integration. Our proof of this theorem uses the formula for $\mathrm{Ev}[F(z_1,\dots,z_n)]$ of Eskin-Okounkov, for which we also give an independent proof in Theorem 10.1.

In Section 11 we apply these results to the computation of *connected brackets*

$$\langle f_1|\dots|f_n\rangle_q \;=\; \sum_{\alpha\in\mathcal{P}(n)} (-1)^{|\alpha|-1}(|\alpha|-1)!\;\prod_{A\in\alpha}\Big\langle \prod_{a\in A} f_a\Big\rangle_q$$

of shifted symmetric polynomials f_i. (Here $\mathcal{P}(n)$ is the set of partitions of the set $\{1,\dots,n\}$.) Geometrically these arise when counting *connected* coverings (compare the definition to (50)). The connected brackets involving only the algebra generators p_ℓ all appear in the generating function

$$\Psi(\mathbf{u})_q \;:=\; \sum_{n=0}^{\infty}\frac{1}{n!}\sum_{\ell_1,\dots,\ell_n\ge 1}\langle p_{\ell_1}|\cdots|p_{\ell_n}\rangle_q\, u_{\ell_1}\cdots u_{\ell_n} \;=\; \log\Phi(\mathbf{u})_q\,,$$

the logarithm of the partition function introduced above. For all asymptotic questions the important quantities are the leading terms of the growth polynomials of these connected brackets, called *cumulants*. We denote the passage from brackets to cumulants by decorating the function with a subscript L. An efficient evaluation of these cumulants is not obvious, since the degree of the growth polynomial drops by one for every insertion of a slash into a bracket. This was observed in [18, Theorem 6.2], and we provide an independent proof in Proposition 11.1. Computationally, the key to the evaluation of cumulants is the following consequence of the representation of $\Phi(\mathbf{u})_X$ as Gaussian integral (Theorem 11.2):

- The generating series of cumulants is given by
$$\Psi(\mathbf{u})_L = \mathcal{B}(\mathbf{u},y_0) \;+\; \frac{y_0^2}{2}\,, \tag{3}$$
where $y_0 = y_0(\mathbf{u})$ is the unique power series with $\frac{\partial}{\partial y}\mathcal{B}(\mathbf{u},y_0)+y_0=0\,.$

This is shown using the principle of least action. We obtain as a corollary also the generating function of cumulants for a fixed number of variables (formula (120), equivalent to [18, Theorem 6.7]), but it is the formula for $\Psi(\mathbf{u})_L$ that turns out to be useful for all applications to asymptotic questions.

When we specialize to cumulants with only 2's with application to the counting problems for simple branching in mind, the computation of $\psi(u)=\Psi(\mathbf{u})_L|_{\mathbf{u}=(0,u,0,\cdots)}$ allows further simplification. While (3) is a two-variable expression even after this specialization, we show in Theorem 12.1 that the cumulants

$$v_n \;=\; \frac{1}{n!}\,\langle\underbrace{p_2|\cdots|p_2}_{n}\rangle_L \qquad (n>0),\quad v_{-2}=v_0=-\frac{1}{24},\; v_{-1}=0 \tag{4}$$

can be obtained by manipulating only one-variable series, as follows:

- Define a Laurent series
$$\mathfrak{B}_{1/2}(X) \;=\; X^{1/2} \;+\; \frac{X^{-3/2}}{96} \;-\; \frac{7X^{-7/2}}{6144} \;+\; \frac{31X^{-11/2}}{65536} \;-\; \cdots$$
in $X^{-1/2}$ as the unique solution in $X^{-1/2}\mathbb{Q}[[1/X]]$ of the functional equation
$$\mathfrak{B}_{1/2}(X+\tfrac12) \;-\; \mathfrak{B}_{1/2}(X-\tfrac12) \;=\; \frac{X^{-1/2}}{2}\,.$$
Then the rational cumulants v_n are given by the inversion formula
$$Y=\mathfrak{B}_{1/2}(X) \qquad\Longleftrightarrow\qquad X \;=\; \sum_{n=-2}^{\infty}\frac{2n+1}{2^{2n+1}}\,v_n\,Y^{-2n-2}\,. \tag{5}$$

We mention that the power series $\mathfrak{B}_{1/2}(X)$ gives the asymptotic expansion of the Hurwitz zeta function $-\frac12\,\zeta(\frac12,X+\frac12)$ as $X\to\infty$, and that its Taylor coefficients are simple multiples of the numbers β_n used above.

Similar one-variable inversion formulas are given in Theorems 12.2 and 12.3 for other linear combinations of cumulants involving 2's that are relevant for the computation of Siegel-Veech asymptotics. The precise form of the linear combinations is motivated by the operators T_p that appear in Part III.

Part III: The hook-length moments T_p. In the Bloch-Okounkov theorem, one obtains quasimodularity for the q-brackets of shifted symmetric polynomials. For our applications we need a quite different looking class of functions on partitions, the *hook-length moments*

$$T_p(\lambda) \;=\; \sum_{\sigma\in Y_\lambda} h(\sigma)^{p-1}\,,$$

where $h(\sigma)$ denotes the hook-length of the cell σ of the Young diagram of λ. Surprisingly, half of these functions *do* lie in the ring of shifted symmetric polynomials, as we show in Theorem 13.4:

- For $p \geq 1$ odd, the function $\widetilde{T}_p(\lambda) = T_p(\lambda) + \frac{1}{2}\zeta(-p)$ is a homogeneous shifted symmetric polynomial of weight $p+1$, given by

$$\widetilde{T}_p(\lambda) \;=\; \frac{(p-1)!}{2}\sum_{k=0}^{p+1}(-1)^k Q_k(\lambda)\, Q_{p+1-k}(\lambda)$$

  where $Q_0 = 1$ and $Q_{\ell+1} = \ell!\, p_\ell$.

The hook-length moments appear first in our study of Siegel-Veech constants in the amusing formula (Corollary 13.2)

$$\frac{1}{d!}\sum_{\mu\in\mathbf{P}(d)} S_p(\mu)\, z_\mu\, \chi^\lambda(\mu)^2 \;=\; T_p(\lambda)\,, \tag{6}$$

obtained by inserting the part-length moment (1) into the left hand side that would simplify to just 1 by Schur orthogonality without the weight.

A leitmotiv for Part III is the observation that many more functions on partitions than just shifted symmetric polynomial have quasimodular or nearly quasimodular q-brackets. In [49] identities like (6) were used to create many non-trivial examples. Here we are more specifically interested in T_{-1}, which is certainly not a shifted symmetric polynomial. Yet, we will show that for f any shifted symmetric polynomial, $\left\langle T_{-1}\, f\right\rangle_q$ is nearly quasimodular: it is in the 2-dimensional module over quasimodular forms generated additively by 1 and $\log(q^{-1/24}\eta)$.

The way we show this property of T_{-1} is very indirect, but reveals many beautiful properties of the operators T_p and $\widetilde{T}_p$. We show the quasimodularity of expressions of the form

$$\left\langle T_p\, f\right\rangle_q - \left\langle T_p\right\rangle_q \left\langle f\right\rangle_q \quad \text{for } f \text{ a shifted symmetric polynomial,}\ \ p \geq -1 \text{ odd} \tag{7}$$

by extrapolating a formula for $p > 0$ to $p = -1$. The key property of the operators $\widetilde{T}_p$, discovered experimentally and discussed in detail in Section 14, is that

$$\left\langle \widetilde{T}_p\, f\right\rangle_q \;=\; \sum_{i,\,j\geq 0}\left\langle \rho_{i,j}(f)\right\rangle_q G^{(j)}_{p+i+1} \qquad \text{for all odd } p \geq 1\,, \tag{8}$$

where $G_k^{(j)}$ is the j-th derivative of the Eisenstein series G_k and where $\rho_{i,j} : \mathcal{R} \to \mathcal{R}$ is a differential operator of the form

$$\rho_{i,j} \;=\; \sum_{k=0}^{\infty} Q_k\, \rho_{i,j}^{(k)}\Big(\frac{\partial}{\partial p_1}, \frac{\partial}{\partial p_2}, \dots\Big)$$

for some polynomials $\rho_{i,j}^{(k)}$ that are given explicitly in Theorem 14.1.

We will give several different descriptions of the operators $\rho_{i,j}$ in Section 14. For the proofs, it turns out to be convenient to reinterpret the key property (8) in terms of the Bloch-Okounkov correlators $F(z_1, \dots, z_n)$. We show in particular that (8) is equivalent to the following statement:

- A correlator with two arguments u and $-u$ that add up to zero can be expressed in terms of certain nearly-elliptic functions of one variable $Z_\ell(u)$ given explicitly in (167) and correlators not involving u by the formula

$$F(u, -u, \mathfrak{z}_N) \;=\; \sum_{\substack{I \subseteq J \subseteq N \\ \varepsilon \in \{\pm 1\}}} (-1)^{|J \smallsetminus I|}\, Z_{|J|}(z_I + \varepsilon u)\, F(\mathfrak{z}_{N \smallsetminus J}, z_J)\,, \tag{9}$$

  where $N = \{1, \dots, n\}$, $\mathfrak{z}_J = (z_j, j \in J)$, and $z_J = \sum_{j \in J} z_j$ for $J \subseteq N$.

The basic strategy to prove such identities is to show that both sides have the same elliptic transformation laws, the same poles, and that they agree at one point. This idea has already been used in [7] and the formulas for the elliptic transformation laws of F are given there. They involve summing the contributions of correlators for all subsets of arguments. But (9) as it stands is completely inadapted to recursive arguments. To overcome this, we give in Theorem 15.1 a formula to express a Bloch-Okounkov correlator involving two distinguished variables u and v as a linear combination of products of a correlator involving only $u + v$ and a nearly elliptic function Z_ℓ involving only one of the variables u and v. This formula specializes to (9) for $v = -u$ and allows for a straightforward (though somewhat tedious) proof following the basic strategy outlined above.

The formula (8) enables us to extrapolate in Section 16 the effect of T_p to $p = -1$, to prove the quasimodularity of (7) also for $p = -1$, and thus to complete the proof of Theorem 6.4 on the quasimodularity of Siegel-Veech weighted counting functions announced at the end of Part I.

Part IV: Volumes and Siegel-Veech constants for large genus. In this part we come back to the geometric applications. So far, in Part I, we have been studying one Hurwitz space at a time, but we have packaged the resulting functions into generating series. It was the motivation of the work of Eskin-Okounkov ([18]) that the Masur-Veech volume of a stratum can be expressed as the limit of volumes of the Hurwitz spaces contained in that stratum, and hence in terms of cumulants (see formula (189)). A similar statement also holds for Siegel-Veech constants. It appeared for arithmetic Teichmüller curves in the appendix of [9], and we give a self-contained statement and proof in Proposition 17.1. We also mention in Section 17 an interpretation of the non-varying phenomenom for the sum of Lyapunov exponents ([10]) in the light of the quasimodularity theorem for Siegel-Veech generating series.

The main goal of Part IV is to study the large genus limits of both Masur-Veech volumes and Siegel-Veech constants. Large genus geometry of the moduli space has already attracted a lot of attention in the parallel world of Weil-Peterson volumes

([28], [38]) and also in algebraic geometry in the form of the slope conjecture ([24], [21], see [9] for some connections), and it is natural to ask similar questions in Teichmüller geometry.

Based on numerical data Eskin and Zorich conjectured (around the time [15] was written, see [20] for more detail) the following asymptotic behaviour. The volumes of the stata (in the normalization of [15]) are conjecturally

$$\mathrm{vol}_{\mathrm{EMZ}}\left(\Omega\mathcal{M}_g(m_1,\dots,m_n)\right) \;\sim\; \frac{4}{(m_1+1)(m_2+1)\cdots(m_n+1)} \;+\; \mathrm{o}(1)$$

as $\sum m_i = 2g-2$ tends to infinity. Moreover, except for hyperelliptic components of strata, they conjecture

$$\lim c_{\mathrm{area}}(\Omega\mathcal{M}_g(m_1,\dots,m_n)) = \frac{1}{2}$$

uniformly as $\sum m_i = 2g-2$ tends to infinity. To avoid making this paper even longer than it already is, we have focused on the principal stratum to prove the two conjectures, with full asymptotic expansions in both cases.

For volumes, we are led by the Eskin-Okounkov formula to compute the asymptotics as $n \to \infty$ of the cumulants v_n introduced in (4). The formula (5) starts with a power series of known asymtotics (involving just factorials and Bernoulli numbers), but we are then required to perform operations such as taking powers and compositional inverses to arrive at v_n. Such a formula seems at first glance rather unsuitable for asymptotic calculations. However, the exact contrary is the case, by the following mechanism of asymptotics of rapidly divergent power series.

In the appendix we consider power series $f = \sum a_n x^n$ that have an asymptotic expansion of the form

$$a_n \;\sim\; n!^{\alpha}\beta^n n^{\gamma}\Big(A_0 + \frac{A_1}{n} + \frac{A_2}{n^2} + \cdots\Big) \tag{10}$$

for $\alpha > 0$ and $\beta > 0$. In all the applications to volumes and Siegel-Veech constants we will have $\alpha = 2$. Series of this type are sometimes called of *Gevrey order* α in the literature. The fact that products of such power series, and hence positive powers, are again of Gevrey order α is certainly well-known, but even for these cases the fact that the full asymptotic expansion can be calculated is hard to find in the literature. In fact, due to the rapid growth of $n!^{\alpha}$ only the first and the last terms in the formula for the product matter. Our new observation is that if $\alpha > 1$ then a similar principle holds also for the composition of two power series of Gevrey order α and for the functional inverse of such a series. The proofs in both cases require a more delicate uniform estimate of the asyptotic growth of the Taylor coefficients of large powers of power series of Gevrey order > 1, together with an application of Lagrange inversion for the case of the functional inverse. A typical result is:

- If $f = \sum a_n x^n$ ($a_0 = 0,\ a_1 = 1$) has coefficients with an asymptotic expansion of the form (10) with $\alpha = 2$, then the coefficients of the functional inverse $f^{-1} = \sum b_n x^n$ have an asymptotic expansion of the same form, beginning

$$b_n \;\sim\; n!^2\beta^n n^{\gamma}\Big(-A_0 \,+\, \frac{\beta^{-1}a_2A_0 - A_1}{n} \,+\, \cdots\Big)\,.$$

The full statement about which Gevrey classes are closed under composition is given in Theorem A.1.

In Section 18 we apply the results on rapidly divergent series to the asymptotics of cumulants. For example, we compute (Theorem 18.2 combined with Theorem 12.2) that for k fixed and $h \to \infty$

$$\langle p_{k-1}|\underbrace{p_2|\cdots|p_2}_{2h-k}\rangle_L \;\sim\; \frac{(-1)^h}{k\cdot 2^k}\,\frac{(2h)!^2}{h^{3/2}}\Bigl(\frac{2}{\pi}\Bigr)^{2h+\frac12}\Bigl(1\,-\,\frac{2\pi^2-6k^2-6k-3}{48h}\,+\,\cdots\Bigr).$$

For the volumes of the principal strata, it now suffices to put the pieces together. For Siegel-Veech constants, the remaining step is to write the leading coefficients $c^0_{-1}(\mathrm{Tr}^n)$ of the generating function of Siegel-Veech constants with weight $p=-1$ and ramification profile $\Pi=\mathrm{Tr}^n$ consisting of transpositions as well in terms of cumulants, see Theorem 19.5. From this, we deduce the final result:

- The Masur-Veech volume of the principal stratum is asymptotically

$$\mathrm{vol}(\Omega\mathcal{M}_g(\underbrace{1,\ldots,1}_{2g-2}))\;\sim\;\frac{4}{2^{2g-2}}\Bigl(1\,-\,\frac{\pi^2}{24g}\,-\,\frac{\pi^4-60\pi^2}{1152g^2}\,+\,\cdots\Bigr)$$

  and the area Siegel-Veech constants of these strata have the asymptotics

$$c_{\mathrm{area}}(\Omega\mathcal{M}_g(1^{2g-2}))\;\sim\;\frac{1}{2}\,-\,\frac{1}{8g}\,-\,\frac{5}{32g^2}\,-\,\frac{4\pi^2+75}{384g^3}\,+\,\cdots,$$

  as $g\to\infty$.

We remark that the extrapolation in Part III from $p>0$ to $p=-1$ works at the level of q-brackets only, not at the level of shifted symmetric functions (since $T_{-1}\notin\mathcal{R}$) and not at the level of cumulants either. To illustrate this, we compute the asymptotics of the p-weighted variant $c^0_p(\mathrm{Tr}^n)$ in Corollary 19.7.

To settle the Eskin-Zorich conjecture for all strata, one has to combine properties of the partition function with the base change from the f_k-generators of $\mathcal{R}$ to the p_ℓ-generators of $\mathcal{R}$ that appear in the partition function, see (183) for examples with small k. We plan to come back to this in a sequel to this paper.

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Part I: Siegel-Veech constants on Hurwitz spaces

We start in Sections 1 and 2 with an overview of Siegel-Veech constants and Hurwitz spaces of torus coverings to set the scene. The interpretation of Siegel-Veech constants for such Hurwitz spaces as combinatorial objects is provided in Section 3. The connections to algebraic geometry of these invariants are given in Section 4, where area Siegel-Veech constants are expressed as boundary contributions on Hurwitz spaces, and in Section 5, where the class β of the $\mathrm{SL}(2,\mathbb{R})$-foliation on Hurwitz spaces is expressed in terms of ψ-classes.

Starting with Section 6 we package the Siegel-Veech constants of the individual Hurwitz spaces into a generating series with respect to the degree of the coverings. The combinatorics of Siegel-Veech constants is then cast in the language of representation theory. This will be used for the proof of the quasimodularity Theorem 6.4 at the end of Part III.

1. Siegel-Veech constants and configurations

1.1. **Counting problems on flat surfaces.** Let (X,ω) be a *flat surface*, consisting of a Riemann surface X and an abelian differential ω on X. We visualize flat surfaces as planar polygons glued along their sides by parallel translation as in Figure 1. The zeros of ω are called saddles or singularities of the flat surface. With the billiard origin of studying flat surfaces in mind, natural counting problems arise from that of closed geodesics under the flat metric, as well as counting saddle connections which are geodesics joining two given (or any two) saddles on the flat surface.

For saddle connections we can most easily define the meaning of the counting problem. We are interested in properties of functions like

$$N_{\mathrm{sc}}(T) \;=\; |\{\gamma\subset X \text{ a saddle connection}, \ell(\gamma)\le T\}|$$

in the limit as $T\to\infty$, where $\ell(\gamma)$ is the flat length of γ.

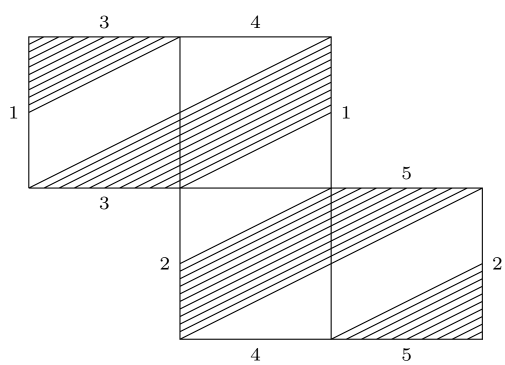

Figure 1. Some short cylinders on a flat surface

Quadratic upper and lower bounds for such counting functions were established by Masur ([36]). Fundamental works of Veech ([44]) and Eskin-Masur ([14]) showed that for almost every surface (X,ω) in the sense of the Masur-Veech measure ([35], [43]) there is a quadratic asymptotic, i.e. that

$$N_{\mathrm{sc}}(T) \;\sim\; c_{\mathrm{sc}}(X,\omega)T^2\,.$$

The constant $c_{\rm sc}(X,\omega)$ is the first example of a Siegel-Veech constant, the one for (any type of) saddle connection. This notion was formalized by Eskin-Masur ([14]) and also by Vorobets ([45]) with the result that many natural counting functions satisfy some Siegel-Veech axioms and consequently have precise quadratic asymptotics.

Counting of saddle connections will however not be considered in the sequel, and we refer to [2] for the latest results. Back to closed geodesics, note that they come in classes, homotopic to one another. In other words, one can slide a closed geodesic transversely on a flat surface (in both orientations) until its translate passes through a singularity. In this way, the translates sweep out cylinders as in Figure 1. Counting these cylinders, possibly with weight, is the right way to interpret counting of closed geodesics. We let

$$N_{\rm cyl}(T) \;=\; |\{Z \subset X \text{ a cylinder, } w(Z) \le T\}|\,,$$

where $w(Z)$ is the width of the cylinder, i.e. the flat length of its core curve.

Once we discuss (see Theorem 4.1) the connection of the corresponding Siegel-Veech constants and intersection numbers on moduli spaces, it will become clear that it is more natural to count the cylinders Z with weight $\operatorname{area}(Z)/\operatorname{area}(X)$, i.e.

$$N_{\rm area}(T) \;=\; \sum_{Z\subset X\,\text{cylinder},w(Z)\le T} \frac{\operatorname{area}(Z)}{\operatorname{area}(X)}\,. \tag{11}$$

The *Siegel-Veech constants* associated to the counting functions are

$$c_{\rm cyl}(X,\omega) \;=\; \lim_{T\to\infty}\frac{N_{\rm cyl}(T)}{\pi T^2}\,, \quad c_{\rm area}(X,\omega) \;=\; \lim_{T\to\infty}\frac{N_{\rm area}(T)}{\pi T^2}\,. \tag{12}$$

In view of Section 3 we remark that there are many interesting variants of the counting functions above with quadratic asymptotics and which moreover satisfy the axioms of Siegel-Veech constants in [14]. For example one could take

$$N_{{\rm area},p}(T) \;=\; \sum_{Z\subset X\,\text{cylinder},w(Z)\le T} \frac{\operatorname{area}(Z)^p}{\operatorname{area}(X)^p},$$

However, this does not correspond to the p-weighted Siegel-Veech constants defined in Section 3, which rather correspond to the counting problem

$$N_p(T) \;=\; \sum_{Z\subset X\,\text{cylinder},w(Z)\le T} \frac{w(Z)h(Z)^{p+2}}{\operatorname{area}(X)^{(p+3)/2}},$$

where $h(Z)$ is the height of the cylinder. Note that this counting function $N_p(T)$ is not $\mathrm{SL}(2,\mathbb{R})$-equivariant. In particular, it does not satisfy the Siegel-Veech axioms. The reason for studying $N_p(T)$ will become apparent in Section 16.

1.2. **The moduli space of flat surfaces and** $\mathrm{SL}(2,\mathbb{R})$ **action.** We denote by $\Omega\mathcal{M}_g$ the moduli space of flat surfaces of genus $g \ge 1$. It is the total space of the vector bundle $\pi_*(\omega_{\mathcal{X}/\mathcal{M}_g})$ over $\mathcal{M}_g$, called the *Hodge bundle*. Here $\omega_{\mathcal{X}/\mathcal{M}_g}$ is the relative dualizing sheaf associated to the universal curve $\pi : \mathcal{X} \to \mathcal{M}_g$. The group $\mathrm{SL}(2,\mathbb{R})$ acts on planar polygons and this action is well-defined also on the resulting flat surfaces. We may provide flat surfaces with a finite number of *marked points* $P_1,\dots,P_n$ that may coincide with zeros of ω and vary under the action of $\mathrm{SL}(2,\mathbb{R})$. The space $\Omega\mathcal{M}_g$ is stratified according to the number and multiplicities of zeros that we denote by $\Omega\mathcal{M}_g(\mathbf{m})$, where $\mathbf{m} = (m_1\,\dots,m_n)$ is a partition of

$2g-2$. Connected components of these strata have been classified in [31]. There are up to three connected components. We will often restrict our attention to the *principal stratum* $\Omega\mathcal{M}_g(1\ldots,1)$, the stratum where all zeros are simple, which is connected.

The action of $\mathrm{SL}(2,\mathbb{R})$ obviously preserves the area of a flat surface. For this reason, whenever talking about orbit closures, volumes etc, we may and will tacitly assume that the invariant manifold is contained in the subset $\Omega_1\mathcal{M}_g$ *of flat surfaces of area one*. We denote by $\mathcal{F}$ the foliation of $\Omega\mathcal{M}_g$ by orbits of $\mathrm{SL}(2,\mathbb{R})$.

The classification of $\mathrm{SL}(2,\mathbb{R})$-orbit closures is one of the major problems in the field, presently open to large extent. Significant progress has been made recently by Eskin-Mirzakhani-Mohammadi ([16], [17]) by showing that orbit closures have a nice geometric structure, i.e. they are linear submanifolds of $\Omega\mathcal{M}_g$. It has been further shown by Filip ([22]) that all linear submanifolds are algebraic varieties defined over $\overline{\mathbb{Q}}$.

The $\mathrm{SL}(2,\mathbb{R})$-orbit closures come with a natural $\mathrm{SL}(2,\mathbb{R})$-invariant measure that we will describe in more detail below in the cases that are relevant here. It follows from the Siegel-Veech axioms (see [14]) that Siegel-Veech constants for almost all flat surfaces (X,ω) in an $\mathrm{SL}(2,\mathbb{R})$-orbit closure M agree. We call these surfaces *generic* (for M). Consequently, we let

$$c_{\mathrm{area}}(M) = c_{\mathrm{area}}(X,\omega) \tag{13}$$

for any (X,ω) which is generic for M.[1]

We will be mainly interested in the Siegel-Veech constants for strata (since this is the most generic case) and for Hurwitz spaces, as introduced below, since they are combinatorially interesting, basically the only source of infinitely many proper closed $\mathrm{SL}(2,\mathbb{R})$-invariant subsets of strata for all genera and, most importantly, their Siegel-Veech constants approach the Siegel-Veech constants for strata, as we will show in Section 17.

1.3. **Cylinder configurations and Siegel-Veech constants for strata: the recursive procedure.** Eskin-Masur-Zorich ([15]) give a recipe to calculate Siegel-Veech constants for strata recursively. Their result is an effective algorithm which is nevertheless combinatorially quite involved. We now explain their basic idea. Moreover we formalize the notion of cylinder configurations, which appears for strata in [15], for general $\mathrm{SL}(2,\mathbb{R})$-invariant manifolds in order to apply it later for Hurwitz spaces.

We start by recalling the notion of Siegel-Veech transform. Let $V = V(X,\omega) \subset \mathbb{R}^2$ a function that associates with a flat surface a subset in $\mathbb{R}^2$ with (real) multiplicities, satisfying the Siegel-Veech axioms (see Section 2 in [14]). These axioms are roughly the $\mathrm{SL}(2,\mathbb{R})$-equivariance, the quadratic growth rate of V, and an integrability condition. The holonomies of all saddle connections and all closed geodesics (with multiplicity one or with multiplicity equal to the area of the ambient cylinder) are examples of such functions. Further examples come from the restriction to only those saddle connection vectors that belong to configurations as defined below. For any function $\chi:\mathbb{R}^2\to\mathbb{R}$ we denote by $\widehat{\chi}$ the Siegel-Veech transform with respect

[1]It is an interesting open problem, if $c_{\mathrm{area}}(M) = c_{\mathrm{area}}(X,\omega)$ for any flat surface (X,ω) such that the closure of $\mathrm{SL}(2,\mathbb{R})\cdot(X,\omega)$ is equal to M.

to V, i.e.

$$\widehat{\chi}(X,\omega) \;=\; \sum_{v\in V(X,\omega)} \chi(v)\,. \tag{14}$$

Let ν be a finite $\mathrm{SL}(2,\mathbb{R})$-invariant measure on a subset of $\Omega_1\mathcal{M}_g$ whose support we denote by H. The fundamental results of Veech and Eskin-Masur ([44], [14]) jointly imply that for appropriate V and ν there is a constant $c(\nu,V)$, such that for all functions χ we have

$$\frac{1}{\nu(H)}\int_H \widehat{\chi}d\nu \;=\; c(\nu,V)\int_{\mathbb{R}^2}\chi dxdy. \tag{15}$$

In this section we will use ν for the Masur-Veech measure ([35], [43]) on strata. In later sections the support of ν will be on Hurwitz spaces. Moreover, if the $\mathrm{SL}(2,\mathbb{R})$-orbit closure of (X,ω) is H and if V is the set of holonomy vectors of all closed geodesics with multiplicity one or multiplicity equal to the area of the ambient cylinder respectively, then $c(\nu,V) = c_{\mathrm{cyl}}(X,\omega)$ resp. $c(\nu,V) = c_{\mathrm{area}}(X,\omega)$ ([14, Theorem 2.1]).

In order to make use of (15), one takes as test function χ_ε, the characteristic function of a little disc of radius ε. The right hand side of the equation is then $\pi\varepsilon^2$ times the constant we are interested in. So we need to compute the left hand side, in fact up to terms of order $o(\varepsilon^2)$.

Roughly speaking, a cylinder configuration is the combinatorial datum encoding the cylinders in a direction θ on a flat surface (X,ω). More precisely, a *cylinder configuration* (on a genus g surface) is a closed subsurface $S\subset\Sigma_g$ together with a graph $\Gamma\subset S$ such that Γ contains the boundary of S and such that the complementary regions, the connected components of $S\smallsetminus\Gamma$, are open parallel cylinders. In particular, boundaries of the cylinders in a cylinder configuration stay parallel and the proportions of their lengths stay fixed under the action of $\mathrm{SL}(2,\mathbb{R})$.

We say that a direction θ on a flat surface (X,ω) *belongs to* the cylinder configuration $\mathcal{C}=(S,\Gamma)$, if there is a subset of the cylinders swept out by closed geodesics in the direction θ such that the closure of these cylinders is S and such that the saddle connections in S form the graph Γ.

Siegel-Veech constants can be refined by counting according to the configuration. That is, we define

$$N_{\mathrm{area}}(T,\mathcal{C}) \;=\; \sum_{\substack{Z\subset X\,\mathrm{cylinder},\,w(Z)\le T\\ Z\ \text{belongs to}\ \mathcal{C}}} \frac{\mathrm{area}(Z)}{\mathrm{area}(X)}. \tag{16}$$

and, as above,

$$c_{\mathrm{area}}(X,\omega,\mathcal{C}) \;=\; \lim_{T\to\infty}\frac{N_{\mathrm{area}}(T,\mathcal{C})}{\pi T^2}\,,\quad c_{\mathrm{area}}(M,\mathcal{C}) \;=\; c_{\mathrm{area}}(X,\omega,\mathcal{C}) \tag{17}$$

if (X,ω) is generic in M.

Counting according to the configuration will appear in this paper as a technical tool. We now formalize that we want to consider only relevant configurations and that we do not want to miss any configuration. A *full set of cylinder configurations* for an $\mathrm{SL}(2,\mathbb{R})$-invariant manifold H is a finite set of cylinder configurations $\mathcal{C}_i$, $i\in I$ with the following properties:

i) For each (X,ω) and each direction θ, the cylinders in the direction θ belong to at most one of the configurations $\mathcal{C}_i$.

ii) For each $i \in I$ there exists $\varepsilon_0 > 0$ such that for all ε in an interval $(0, \varepsilon_0)$ there exist flat surfaces (X, ω) in H that possess a cylinder of width $\leq \varepsilon$ in a direction θ belonging to the cylinder configuration $\mathcal{C}_i$. The set of such surfaces is denoted by $H^\varepsilon(\mathcal{C}_i)$.
iii) For each $i \in I$ the limit of $\frac{1}{\varepsilon^2}\nu(H^\varepsilon(\mathcal{C}_i))$ as $\varepsilon \to 0$ is positive.
iv) The contributions of the configurations $\mathcal{C}_i$ sum up to the area Siegel-Veech constants, i.e.
$$\sum_{i \in I} c_{\mathrm{area}}(H, \mathcal{C}_i) \;=\; c_{\mathrm{area}}(H)\,.$$

We refer to [3] for more background and related discussion regarding the above conditions. We also remark that in the case of strata, a full set of configurations defined above corresponds to a complete list of configurations of homologous cylinders (or saddle connections) in the literature.

We now discuss Siegel-Veech constants for a stratum $\Omega\mathcal{M}_g(\mathbf{m})$. Let $\nu_{\mathrm{str}} = \nu_{\Omega\mathcal{M}_g(\mathbf{m})}$ be the Masur-Veech measure on the stratum. Then the following formula is a direct consequence of the definition of configuration and the Siegel-Veech formula applied to a small disc:

$$c_{\mathrm{area}}(\Omega\mathcal{M}_g(\mathbf{m})) \;=\; \lim_{\varepsilon \to 0} \frac{1}{\pi\varepsilon^2} \sum_{\mathcal{C}} \frac{\nu_{\mathrm{str}}(\Omega\mathcal{M}_g^\varepsilon(\mathbf{m}, \mathcal{C}))}{\nu_{\mathrm{str}}(\Omega\mathcal{M}_g(\mathbf{m}))}\,, \tag{18}$$

where the summation ranges over a full set of cylinder configurations for the Masur-Veech measure supported on the stratum.

To compute c_{area} for more general cases one has to apply the Siegel-Veech formula to several test functions, as we will explain in Section 3 when computing these constants for Hurwitz spaces.

In any case, to make this formula useful, one has to overcome two problems. First, one has to be able to compute the volume in the numerator. It turns out for strata that this is a sum of volumes of strata obtained by cutting the surfaces along the core curves γ_i. This turns the computation of [15] into a recursive formula.

Second, one has to determine a full set of configurations for a stratum. For this purpose, recall that ([15, Proposition 3.1]) in any stratum two non-homologous saddle connections sharing the same holonomy vector exist only on a set of measure zero. This can be used to show that a full set of cylinder configurations consists of all possibilities of embedding disjoint closed cylinders into Σ_g such that no two core curves are homotopic, no cylinder is separating, but any pair of cylinders is separating. For each such collection we let S be the union of the closures of the cylinders and Γ be their boundary curves. Combinatorially one can describe such a cylinder configuration by the tuple of genera $(g_1, \ldots, g_s)$ of $\Sigma_g \smallsetminus S$ up to cyclic permutation. (See [15, Proposition 3.1, Sections 11 and 12] for more details and the values of many Siegel-Veech constants.)

2. Hurwitz spaces of torus covers and their configurations

We give a short introduction to Hurwitz spaces of torus coverings and recall some basic notions needed in the sequel. The main result in this section is a combinatorial description of a full set of cylinder configurations for these Hurwitz spaces.

2.1. **Admissible covers and torus coverings.** Harris and Mumford ([26]) came up with the notion of admissible covers to deal with degenerations of coverings of smooth curves to coverings of nodal curves. In general, denote by $p : X \to C$ a finite morphism of nodal curves such that

i) The smooth locus of X maps to the smooth locus of C and the nodes of X map to the nodes of C.
ii) Suppose that $p(s) = t$ for a node $s \in X$ and a node $t \in C$. Then there exist suitable local coordinates x, y for the branches at s and u, v for the branches at t, such that

$$u = p(x) = x^k, \quad v = p(y) = y^k \quad \text{for some} \quad k \in \mathbb{Z}^+ .$$

We say that p is an *admissible cover*. One useful thing to keep in mind is that adding admissible covers provides a natural compactification of Hurwitz spaces of ordinary branched covers, which is analogous to the Deligne-Mumford compactification of the moduli space of curves by adding stable nodal curves. We refer to [25, Chapter 3.G] for a detailed introduction to admissible covers.

Now we specialize to torus coverings. Let $\Pi = (\mu^{(1)}, \cdots, \mu^{(n)})$ consist of partitions $\mu^{(i)} = (\mu_1^{(i)}, \mu_2^{(i)}, \cdots)$ such that each entry $\mu_j^{(i)}$ is a non-negative integer and $\sum_{i,j}(\mu_j^{(i)} - 1) = 2g - 2$. We call such a tuple Π a *ramification profile*.

An admissible cover $p : X \to E$ has ramification profile Π, if it has n branch points and over the i-th branch point the sheets coming together form the partition $\mu^{(i)}$ (completed by singletons, if $|\mu^{(i)}| < \deg(p)$). Let $\overline{H}_d(\Pi)$ (or just $\overline{H}$ if the parameters are fixed) denote the n-dimensional *Hurwitz space* of degree d, genus g, connected admissible coverings $p : X \to E$ of a curve of genus one with n branch points and ramification profile Π. We use $H_d(\Pi)$ for the open subset of $\overline{H}_d(\Pi)$, where X is smooth.

Here we fix the notation for covers parameterized by this Hurwitz space and for counting problems. Let $\rho : \pi_1(E \smallsetminus \{P_1, \ldots, P_n\}) \to S_d$ be the monodromy representation in the symmetric group of d elements associated with a covering in $H_d(\Pi)$. We use the convention that loops (and elements of the symmetric group) are composed from right to left. The elements $(\alpha, \beta, \gamma_1, \cdots, \gamma_n)$ as in the left picture of Figure 2 generate the fundamental group $\pi_1(E \smallsetminus \{P_1, \ldots, P_n\})$ with the relation

$$\beta^{-1}\alpha^{-1}\beta\alpha \;=\; \gamma_n \cdots \gamma_1 . \tag{19}$$

Given such a homomorphism ρ, we let $\boldsymbol{\alpha} = \rho(\alpha)$, $\boldsymbol{\beta} = \rho(\beta)$, $\boldsymbol{\gamma}_i = \rho(\gamma_i)$, and call the tuple

$$h \;=\; (\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}_1, \cdots, \boldsymbol{\gamma}_n) \in (S_d)^{n+2} \tag{20}$$

the *Hurwitz tuple* corresponding to ρ and the choice of generators. Conversely, a Hurwitz tuple as in (20) satisfying (19) defines a homomorphism ρ and thus a covering p. If we are only interested in connected coverings, we require a Hurwitz tuple moreover to generate a transitive subgroup of S_d.

We say that a Hurwitz tuple *has profile* Π if the conjugacy class $[\boldsymbol{\gamma}_i] = \mu^{(i)}$ for $i = 1, \ldots, n$. Here we use the general convention to call two partitions of different sizes $d_1 \le d_2$ equal, if they differ by $d_2 - d_1$ parts of length one. The set of Hurwitz tuples of degree d and profile Π acting transitively on $\{1, \ldots, d\}$ is denoted by $\mathrm{Hur}_d^0(\Pi)$.

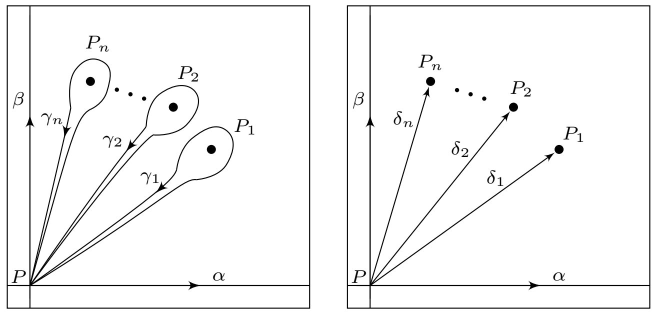

FIGURE 2. Standard presentation of $\pi_1(E \smallsetminus \{P_1, \ldots, P_n\})$ and standard choice of relative periods

The covering map p does not depend on the choice of the base point. Changing the base point results in simultaneous conjugation in S_d of the Hurwitz tuple. We call the conjugacy classes of Hurwitz tuples *Hurwitz classes* and refer to the cardinality of the set of Hurwitz classes of profile Π as $N^{\mathrm{na}} = N_d^{\mathrm{na}}(\Pi)$.

The upper index "na" indicates that no automorphisms of the coverings are taken into account. For counting problems, in particular when studying generating series, it is more natural to weight any Hurwitz classes by the factor $|\mathrm{Aut}(p)|^{-1}$. Such automorphisms correspond bijectively to elements of the centralizer of $\rho(p)$. We denote the number of weighted Hurwitz classes of profile Π by $N_d^0(\Pi)$ and we have the fundamental relation

$$N_d^0(\Pi) \;=\; \frac{|\mathrm{Hur}_d^0(\Pi)|}{d!}. \tag{21}$$

For asymptotics on connected covers, the weighting factor $|\mathrm{Aut}(p)|^{-1}$ is negligible (see [18, Section 3.1]).

We remark that for some branching profiles Π the space $\overline{H}_d(\Pi)$ can be disconnected, e.g. if the profile consists of cycles of odd length only, the parity of the spin structure of [31] distinguishes two components. Whether $\overline{H}_d(\Pi)$ decomposes into more components than the obvious ones is a hard problem that will not play any role in the sequel.

2.2. **Period coordinates, invariant measure, foliations.** Denote by $\mathcal{M}_{1,n}$ the moduli space of genus one curves with n ordered marked points. Let $\Omega\mathcal{M}_{1,n}$ be the Hodge bundle of holomorphic one-forms over $\mathcal{M}_{1,n}$. We introduce a coordinate system on $\Omega\mathcal{M}_{1,n}$ to define the $\mathrm{SL}(2,\mathbb{R})$-invariant measure ν, which we have already been referring to in the Siegel-Veech formula, and to define foliations we will argue with in the sequel.

We present a point $(E, \omega, P_1, \ldots, P_n)$ in $\Omega\mathcal{M}_{1,n}$ as a flat surface as in Figure 2 using the unique non-zero holomorphic one-form ω on E (up to scaling). Whereas the left picture gives a basis of $\pi_1(E \smallsetminus \{P_1, \ldots, P_n\})$ we indicate in the right picture a basis of relative homology $H_1(E, \{P_1, \ldots, P_n\}, \mathbb{Z})$.

Period coordinates are given by assigning to $(E', \omega', P'_1, \ldots, P'_n)$ in a neighborhood of $(E, \omega, P_1, \ldots, P_n)$ the tuple

$$(z_\alpha, z_\beta, z_2, \ldots, z_n) \;=\; \Big(\int_\alpha \omega', \int_\beta \omega', \int_{\delta_1} \omega', \ldots, \int_{\delta_{n-1}} \omega'\Big) \in \mathbb{C}^{n+1}. \qquad (22)$$

It is well-known that this defines a local coordinate system on $\Omega\mathcal{M}_{1,n}$.

Inside $\Omega\mathcal{M}_{1,n}$ there is a (real) hypersurface $\Omega_1\mathcal{M}_{1,n}$ of (pointed) flat tori with ω-area equal to one. Note that $\Omega_1\mathcal{M}_{1,n}$ is isomorphic to an open subset of the symmetric space $\mathrm{SL}(2,\mathbb{R}) \times (\mathbb{R}^2)^{n-1}/\mathrm{SL}(2,\mathbb{Z}) \times (\mathbb{Z}^2)^{n-1}$. Hence by Ratner's theorem there is a unique finite $\mathrm{SL}(2,\mathbb{R})$-*invariant ergodic measure* $\overline{\nu}_1$ on $\Omega_1\mathcal{M}_{1,n}$ (up to scaling). We will denote by $\overline{\nu}$ the push-forward of $\overline{\nu}_1$ under the quotient map by $\mathrm{SO}_2(\mathbb{R})$, i.e. on $\mathcal{M}_{1,n}$. There are two ways to construct $\overline{\nu}_1$ explicitly.

The first construction of $\overline{\nu}_1$ is completely analogous to the construction that works for (the connected components of) the strata. For each open subset $U \subset \Omega_1\mathcal{M}_{1,n}$ let $C(U)$ be the cone of flat surfaces over U, i.e. flat surfaces $(X, \omega) \in \mathbb{C}^* \cdot U$ with area ≤ 1. We take $\overline{\nu}_1(U)$ to be the Lebesgue measure of $C(U)$ with the normalization such that the unit cube of $\mathbb{Z}[i]^{n+1} \subset \mathbb{C}^{n+1}$ has volume one. A change of basis corresponds to an action of $\mathrm{SL}(2,\mathbb{Z}) \times (\mathbb{Z}^2)^{n-1}$ on period coordinates, thus preserving the integral lattice. Consequently, the unit cube normalization is well-defined.

The second construction provides a transverse measure on the following foliation. Denote by REL the foliation of $\Omega\mathcal{M}_{1,n}$ whose leaves are the preimages of the forgetful map $\Omega\mathcal{M}_{1,n} \to \Omega\mathcal{M}_{1,1}$. By definition the leaves are $\mathrm{SO}_2(\mathbb{R})$-invariant. Hence the foliation descends to a foliation on $\mathcal{M}_{1,n}$ which we also denote by REL. This foliation is transversal to the foliation $\mathcal{F}$ by $\mathrm{SL}(2,\mathbb{R})$-orbits. The leaf of REL over (E, P_1) is the $(n-1)$-fold product of E minus the diagonals. We provide $\Omega_1\mathcal{M}_{1,1} = \mathrm{SL}(2,\mathbb{R})/\mathrm{SL}(2,\mathbb{Z})$ with the Haar probability measure and define a transverse measure to $\mathcal{F}$ using the Euclidean volume on E^{n-1}, normalized so that $\mathrm{vol}(E^{n-1}) = 1$. The measure $\overline{\nu}$ is obtained by the direct integral of this transverse measure along the Haar measure on $\Omega_1\mathcal{M}_{1,1}$.

We let ΩH be the moduli space of pairs consisting of a covering $(p : X \to E) \in H$ and a non-zero holomorphic one-form ω on X that is a pullback from E via p. This is a $\mathbb{C}^*$-bundle over H and again we let $\Omega_1 H$ be the hypersurface of flat surfaces (X, ω) of area one. The space ΩH is a finite unramified cover of $\Omega\mathcal{M}_{1,n}$ and the same holds for the restriction to $\Omega_1 H$ as well as to the variants with constraints on the connectivity of p. Consequently, the above period coordinates are local coordinates on ΩH, too. Moreover, the measures $\overline{\nu}_1$ and $\overline{\nu}$ pull back to finite measures ν_1 and ν on $\Omega_1 H$ and H respectively. Finally, the foliation REL also defines a holomorphic foliation on ΩH, with leaves of codimension one.

2.3. **Configurations for Hurwitz spaces.** In this section we describe a full set of cylinder configurations for a Hurwitz space $H = H_d(\Pi)$ of torus coverings. For a given Hurwitz tuple h we define the (horizontal) *Dehn twist* around the curve α to be the map that sends

$$h = (\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}_1, \cdots, \boldsymbol{\gamma}_n) \quad \text{to} \quad (\boldsymbol{\alpha}, \boldsymbol{\beta}\boldsymbol{\alpha}, \boldsymbol{\alpha}^{-1}\boldsymbol{\gamma}_1\boldsymbol{\alpha}, \cdots, \boldsymbol{\alpha}^{-1}\boldsymbol{\gamma}_n\boldsymbol{\alpha}). \qquad (23)$$

Proposition 2.1. *There is a natural bijection between the set of equivalence classes of Hurwitz tuples up to simultaneous conjugation and Dehn twist action and a full set of cylinder configurations for the Hurwitz space H.*

Proof. First we associate to any Hurwitz tuple h a cylinder configuration $\mathcal{C}(h)$ as follows. We order and place the branch points on E with strictly decreasing vertical coordinates, as in Figure 2. The Hurwitz tuple defines a covering $p : X \to E$ with $g(X) = g$. The subsurface of the cylinder configuration is $S = \Sigma_g$ and Γ is the p-preimage of the union of closed horizontal loops through the points P_i. Obviously, the resulting cylinder configuration is unchanged under conjugation of the Hurwitz tuple and independent of the representative in the Dehn twist orbit.

Conversely, suppose that $(p : X \to E,\, \omega = p^*\omega_E)$ is a covering parameterized by $\Omega_1 H$ and that θ is a direction such that no two branch points in E lie on the same closed ω_E-geodesic. (Other directions need not be taken into account, since aligned branch points form a measure zero subset. They do not contribute to the Siegel-Veech constant and they do not satisfy the condition iii) of a full set of cylinder configurations.) We may assume moreover that there is a cylinder in the direction θ, hence the p-image of its core curve is a closed loop on E in the direction θ. We call this loop α and fix a base point on α. Next we choose a complementary direction θ_2 admitting a closed geodesic β. We label the branch points in decreasing height (with respect to the direction θ_2) and choose loops as in Figure 2. The monodromy of the cover defines a Hurwitz tuple. Its equivalence class up to conjugacy and Dehn twist action is independent of the choices we made. Finally we note that the two constructions are inverse to each other.

It remains to check that these cylinder configurations form a full set of such configurations. Condition i) is obvious and condition ii) holds by taking the base curve E of the covering sufficiently tall and thin. In fact, $\varepsilon_0 = 1/nd$ works. Condition iii) now follows immediately from the preceding description of the measure ν_1, since the location of the branch points is unconstrained except for a set of measure zero. To check condition iv) it suffices to notice that we only neglected cylinder configurations that appear on a set of ν_1-measure zero. □

Suppose the fundamental group of the punctured surface $E \smallsetminus \{P_1, \ldots, P_n\}$ is given in our standard presentation of Figure 2. We remark that the core curves of the horizontal cylinders are represented by the loops

$$\sigma_0 = \alpha,\ \sigma_1 = \alpha\,\gamma_1^{-1},\ \sigma_2 = \alpha\,(\gamma_2\gamma_1)^{-1}, \ldots,\ \sigma_{n-1} = \alpha\,(\gamma_{n-1}\cdots\gamma_1)^{-1}. \tag{24}$$

3. Weighted counting of Hurwitz classes

We will now count Hurwitz classes with a weight, that we call Siegel-Veech weight. The aim of this section is to show that this gives a combinatorial way to compute the area Siegel-Veech constants of Hurwitz spaces.

Let $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k)$ with $\lambda_i \geq 0$ be a partition. The *p-th part-length moment* of λ is defined as

$$S_p(\lambda) \;=\; \sum_{j=1}^{k} \lambda_j^p \tag{25}$$

for any $p \in \mathbb{C}$, but only the moments $p \in \mathbb{Z}$ will be used in this paper. If $h = (\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}_1, \cdots, \boldsymbol{\gamma}_n) \in (S_d)^{n+2}$ is a Hurwitz tuple, we consider the n permutations

$$\boldsymbol{\sigma}_0 = \boldsymbol{\alpha},\, \boldsymbol{\sigma}_1 = \boldsymbol{\alpha}\,\boldsymbol{\gamma}_1^{-1},\ \boldsymbol{\sigma}_2 = \boldsymbol{\alpha}\,(\boldsymbol{\gamma}_2\boldsymbol{\gamma}_1)^{-1}, \ldots,\ \boldsymbol{\sigma}_{n-1} = \boldsymbol{\alpha}\,(\boldsymbol{\gamma}_{n-1}\cdots\boldsymbol{\gamma}_1)^{-1} \tag{26}$$

in S_d that arise from monodromy of the core curves of the horizontal cylinders as represented in (24). Define the *p-th Siegel-Veech weight of a Hurwitz tuple h_j* to be

$$S_p(h_j) \;=\; \sum_{i=0}^{n-1} S_p(\boldsymbol{\sigma}_i(h_j))\,. \tag{27}$$

Geometrically speaking, the p-th Siegel-Veech weight $S_p(h_j)$ encodes the sum of moduli of the horizontal cylinders on the covering surface, each with weight given by raising to the p-th power. This weight is obviously independent of representative of the Hurwitz tuples in a given Hurwitz class. We define the *(combinatorial) p-weighted Siegel-Veech constant $c_p^0(d,\Pi)$* to be the sum of the weights over all Hurwitz classes for $H_d(\Pi)$, i.e.

$$c_p^0(d,\Pi) \;=\; \frac{1}{n\,d!}\sum_{j=1}^{|\mathrm{Hur}_d^0(\Pi)|} S_p(h_j) \;=\; \frac{1}{n}\sum_{j=1}^{N_d^0(\Pi)} S_p(h_j) \tag{28}$$

where the two equivalent definitions are linked by (21).

The upper zero in $c_p^0(d,\Pi)$ refers to the fact that here all covers are connected and all Hurwitz tuples generate a transitive subgroup of S_d by definition. (In Section 6 we will discuss the passage between the connected and possibly disconnected cases.)

Theorem 3.1. *Fix a degree d and a ramification profile Π. Then the combinatorial Siegel-Veech constant $c_{-1}^0(d,\Pi)$ defined in* (28) *and the area Siegel-Veech constant of the Hurwitz space $H_d(\Pi)$ satisfy the following relation*

$$c_{\mathrm{area}}(H_d(\Pi)) \;=\; \frac{3}{\pi^2}\,\frac{c_{-1}^0(d,\Pi)}{N_d^0(\Pi)}\,. \tag{29}$$

We will present two proofs of this formula. The first proof given below just uses the Siegel-Veech transform. It generalizes a combinatorial formula for Siegel-Veech constants of Teichmüller curves given in [13, Appendix]. A second proof is given in Section 4, which is more algebraic and uses the main result of [13] relating the area Siegel-Veech constant to the sum of Lyapunov exponents. We remark that combining the two proofs gives a new proof of the main result of [13] in the case of Hurwitz spaces by intersection theory only, without any reference to analytic techniques such as the determinant of the Laplacian etc.

Proof of Theorem 3.1. Let $\{\mathcal{C}_i\}$ for $i \in I$ be a full set of cylinder configurations for the Hurwitz space $H_d(\Pi)$. The left hand side of (29) is a sum of $c_{\mathrm{area}}(H_d(\Pi),\mathcal{C}_i)$. By Proposition 2.1 each cylinder configuration $\mathcal{C}_i$ corresponds to an orbit $\mathcal{O}_i$ of Hurwitz tuples under the Dehn twist action and conjugation. It thus suffices to show that $c_{\mathrm{area}}(H_d(\Pi),\mathcal{C}_i)$ equals the contribution of $\mathcal{O}_i$ to the numerator of the right hand side for each $i \in I$. We fix $\mathcal{C}_i$ and $\mathcal{O}_i$ from now on and let $N_i = |\mathcal{O}_i|$.

For a flat surface (X,ω) we let $V_i \subset \mathbb{R}^2$ be the subset of holonomy vectors of the core curves of cylinders belonging to the cylinder configuration $\mathcal{C}_i$. (We count them with multiplicity one, and area multiplicities will be introduced through (30) below.) Let $C^{(k)}$ for $k \in K$ be the cylinders of the configuration $\mathcal{C}_i$. Since we identified cylinder configurations with equivalence classes of Hurwitz tuples, the ratios of widths among the $C^{(k)}$ is determined by the configuration. In fact, the core curves of these cylinders are the p-preimages of the loops $\sigma_0,\ldots,\sigma_{n-1}$ of (24), so the cylinders correspond to the parts of the partitions $\boldsymbol{\sigma}_0,\ldots,\boldsymbol{\sigma}_{n-1}$ and the widths are proportional to the cardinality of the parts. Consequently, we may order the cylinders increasingly by their widths $w_k = w(C^{(k)})$, i.e. $w_1 \le w_2 \le \cdots \le w_{|K|}$.

We apply the Siegel-Veech formula (14) to two functions. The first function is the characteristic function χ_ε for a small disc of radius ε at the origin. We evaluate

$$\frac{1}{\nu(\Omega_1 H_d(\Pi))}\int_{\Omega_1 H_d(\Pi)} \widehat{\chi}_\varepsilon d\nu_1 = B_i \int_{\mathbb{R}^2} \chi_\varepsilon dxdy\,,$$

where B_i is the Siegel-Veech constant for V_i. (In fact, it is the cylinder Siegel-Veech constant for the configuration $\mathcal{C}_i$.) The integrand on the left hand side is constant along the REL-foliation, and hence its value equals N_i times the volume of an ε-neighborhood of the cusp in $\Omega\mathcal{M}_{1,1}$, which is $\pi\varepsilon^2$. The volume of $\Omega_1 H$ is N^0 times the volume of the modular surface, which is $\pi^2/3$. Since the integral on the right hand side is $\pi\varepsilon^2$, we conclude that

$$B_i = \frac{3}{\pi^2}\frac{N_i}{N^0}.$$

The second function we plug in the Siegel-Veech formula is the sum of characteristic functions for counting cylinders with fixed widths w_k and (as parameter) the tuple of heights $\mathrm{ht} = (\mathrm{ht}_1, \ldots, \mathrm{ht}_{|K|})$. That is, for $v \in \mathbb{R}^2$ we let

$$\chi_{r,\mathrm{ht}}(v,\mathcal{C}_i) = \begin{cases} 0 & \text{if} \quad w_1||v|| \ge r \\ \frac{\mathrm{ht}_1 w_1}{d} & \text{if} \quad w_2||v|| \ge r > w_1||v|| \\ \cdots & \qquad \cdots \\ \frac{\mathrm{ht}_1 w_1 + \cdots + \mathrm{ht}_j w_j}{d} & \text{if} \quad w_{j+1}||v|| \ge r > w_j||v|| \\ \cdots & \qquad \cdots \\ \frac{\mathrm{ht}_1 w_1 + \cdots + \mathrm{ht}_{|K|} w_{|K|}}{d} & \text{if} \quad r > w_{|K|}||v|| \end{cases} \tag{30}$$

and let χ_r be the function with "average" height, i.e. $\chi_r = \chi_{r,(1/n,\ldots,1/n)}$. Since

$$\int_{\mathbb{R}^2} \chi_r((x,y),\mathcal{C}_i)dxdy = \pi r^2 \frac{1}{nd}\sum_{k=1}^{|K|} w_k^{-1},$$

we obtain using the Siegel-Veech formula again and the value of B_i that

$$\begin{aligned} \frac{1}{\nu(\Omega_1 H_d(\Pi))}\int_{\Omega_1 H_d(\Pi)} \widehat{\chi}_r((X,\omega),\mathcal{C}_i)d\nu_1 &= \frac{3r^2}{\pi}\frac{N_i}{N^0}\frac{1}{nd}\sum_{k=1}^{|K|} w_k^{-1} \\ &= \frac{3r^2}{\pi}\frac{1}{N^0}\frac{1}{d}c^0_{-1}(H,\mathcal{C}_i), \end{aligned} \tag{31}$$

where an analog of (28) was used in the last step for the configuration $\mathcal{C}_i$.

It remains to show that

$$c_{\mathrm{area}}(H,\mathcal{C}_i) = d\lim_{r\to\infty}\frac{1}{\pi r^2}\frac{1}{\nu(\Omega_1 H_d(\Pi))}\int_{\Omega_1 H_d(\Pi)} \widehat{\chi}_r((X,\omega),\mathcal{C}_i)d\nu_1. \tag{32}$$

For this purpose, note that the integrand does not depend on the location of (X,ω) within the REL-foliation, equivalently within the fibers of the projection $\Omega_1 H_d(\Pi) \to \Omega_1\mathcal{M}_{1,1}$. We disintegrate ν_1 over this fibration as $d\mu_X d\overline{\nu}_1(X,\omega)$.

Then for any fixed r the sum over $V_i(X,\omega)$ is finite and we obtain that

$$\begin{aligned}\int_{\Omega_1 H_d(\Pi)} \widehat{\chi}_r((X,\omega),\mathcal{C}_i)d\nu_1 &= \int_{\Omega_1 H_d(\Pi)} \sum_{v\in V_i(X,\omega)} \chi_r(v,\mathcal{C}_i)d\nu_1 \\ &= N^0 \int_{\Omega_1\mathcal{M}_{1,1}} \sum_{v\in V_i(X,\omega)} \int_{X^{n-1}} \chi_r(v,\mathcal{C}_i)d\mu_X d\overline{\nu}_1(X,\omega).\end{aligned}$$

For every covering $p: X\to E$ and every v we slice the torus E parallel to v and some direction $v^\perp$ given by a primitive vector in the lattice of E which is not parallel to v. Instead of integrating over X^{n-1} we will integrate over E^{n-1} and take into account the degree d of the covering. Let $B=\{(\mathrm{ht}_1,\ldots,\mathrm{ht}_{|K|})\in[0,1]^{|K|}: \sum_{k=1}^{|K|}\mathrm{ht}_i=1\}$. Using v and $v^\perp$ as a basis, we place the first point P_1 at the corner $(0,0)$. Integrating over the points $P_2,\ldots,P_n$ in E can be done by placing these points at $a_iv/||v||+b_iv^\perp$ with $a_i,b_i\in[0,1]$ for $i=2,\ldots,n$. The cylinders in the direction v will have height b_i-b_{i+1} if the points are ordered by decreasing second coordinates and thus give a tuple in B. Using that χ_r is the average of the $\chi_{r,\mathrm{ht}}$ over B, we obtain that

$$\begin{aligned}\int_{\Omega_1 H_d(\Pi)} \widehat{\chi}_r((X,\omega),\mathcal{C}_i)d\nu_1 \;&=\; N^0\int_{\Omega_1\mathcal{M}_{1,1}} \sum_{v\in V_i(X,\omega)} \int_{[0,1]^{n-1}}\int_B \chi_r(v,\mathcal{C}_i)d\mu_X d\overline{\nu}_1(X,\omega) \\ &=\; N^0\int_{\Omega_1\mathcal{M}_{1,1}} \sum_{v\in V_i(X,\omega)} \int_{[0,1]^{n-1}}\int_{\mathrm{ht}\in B} \chi_{r,\mathrm{ht}}(v,\mathcal{C}_i)d\mu_X d\overline{\nu}_1(X,\omega) \\ &=\; \frac{1}{d}N^0\int_{\Omega_1\mathcal{M}_{1,1}} N_{\mathrm{area}}((X,\omega),r,\mathcal{C}_i)d\overline{\nu}_1(X,\omega).\end{aligned}$$

For r large, the integrand on the right hand side of the last step converges by (12) to $\frac{1}{d}\pi r^2 c_{\mathrm{area}}(H,\mathcal{C}_i)$, independent of the flat surface (X,ω), where the scaling factor $\frac{1}{d}$ is due to that of χ_r in its definition. Recall also that the volume of $\Omega_1 H$ is N^0 times the volume of the modular surface. Altogether this implies that (32) holds.

Finally, adding up the contributions from (32) using (31) gives the claim. □

4. The sum of Lyapunov exponents as a ratio of intersection numbers

In this section we justify geometrically why we give preference to area Siegel-Veech constants over other Siegel-Veech constants. The first answer, given in § 4.1 is that they appear as a coefficient of the push-forward of a boundary class. The second answer, given in § 4.2, relates area Siegel-Veech constants to the sum of Lyapunov exponents, which is further expressed as a ratio of intersection numbers on moduli spaces. We work on Hurwitz spaces throughout in this section and emphasize that the discussion is entirely algebraic. In particular, analytic tools such as determinants of the Laplacian as in [13] are not needed.

4.1. **Push-forward of the nodal locus.** We fix the degree d and the ramification profile Π. The moduli maps for the Hurwitz space and the universal family $\mathcal{X}$ over it give rise to the following commutative diagram

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{\;h\;} & \overline{\mathcal{M}}_{1,n+1} \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi_{n+1}} \\ \overline{H}_d(\Pi) & \xrightarrow{\;f\;} & \overline{\mathcal{M}}_{1,n} \end{array}$$

where f and h are finite morphisms of degree N and dN, respectively, and where π_{n+1} is the map forgetting the last marked point. Let $\delta_{\mathcal{X}} \subset \mathcal{X}$ be the (codimension two) locus of nodal singularities of the fibers. Recall that the Deligne-Mumford boundary of $\overline{\mathcal{M}}_{1,n}$ consists of the divisor $\delta_{\rm irr}$ that parametrizes generically irreducible nodal rational curves and the divisors $\delta_{0,S}$ for S a subset of $\{1,\ldots,n\}$ with $|S| \geq 2$ that parametrize generically curves with one separating node such that the marked points in S lie in the component of genus zero. We denote an undetermined linear combination of the divisors $\delta_{0,S}$ by $\delta_{\rm other}$.

Theorem 4.1. *The push-forward of the nodal locus in $\mathcal{X}$ to $\overline{\mathcal{M}}_{1,n}$ can be evaluated using the weighted sum of divisor classes introduced above as*

$$\pi_{n+1*}h_*\delta_{\mathcal{X}} \;=\; c^0_{-1}(d,\Pi)\,\delta_{\rm irr} + \delta_{\rm other}\,. \tag{33}$$

Proof of Theorem 4.1. The set theoretic image of $\delta_{\mathcal{X}}$ is of course contained in the union of boundary divisors, so the only content of the theorem is the multiplicity of $\delta_{\rm irr}$. The preimage of a tubular neighborhood of $\delta_{\rm irr}$ in H consists of the set of Hurwitz classes grouped to the orbits of the Dehn twist as in (23). (The tubular neighborhood is determined by α being short.) As above, we denote these orbits by $\mathcal{O}_i$ for $i = 1,\ldots,m$ and let $N_i = |\mathcal{O}_i|$. Suppose that $\mathcal{O}_i$ consists of the Hurwitz classes $\{h_j\}$. It suffices to compare both sides of (33) in the neighborhood specified by each of these orbits $\mathcal{O}_i$ separately and then add their contributions together.

We want to show that the intersection number with a test curve agree on both sides of (33) in the boundary neighborhood determined by $\mathcal{O}_i$. For this purpose we use the Teichmüller curve C generated by a square-tiled surface (X,ω) of dn rectangles constructed as follows. Pile n rectangles of width 1 and height $1/n$ from top to bottom to produce a torus $E = \mathbb{C}/(\mathbb{Z}+i\mathbb{Z})$, and place the point P_l in the middle of the upper boundary of the l-th rectangle. Take a degree d cover $p : X \to E$ with monodromy given by a Hurwitz class $h_j \in \mathcal{O}_i$ (using the presentation of the fundamental group as in Figure 2, with the base point in the left part of the bottom rectangle) and let $\omega = p^*\omega_E$. The $\mathrm{SL}(2,\mathbb{R})$-orbit of (X,ω) defines a Teichmüller curve $\varphi : C \to \overline{H}_d(\Pi)$.

The horizontal cylinders of the flat surface (X,ω) are in bijection with the union of the cycles $c_{s,j}$ of the permutations $\boldsymbol{\sigma}_s$, $s = 0,\ldots,n-1$, introduced in (26), associated with a Hurwitz class h_j in the above. We denote these cylinders by $C^{(k)}$ for $k \in K(j)$. These cylinders (possibly not maximal cylinders) have height $1/n$ and width $\ell(c_{s,j})$. (Recall that the modulus $m(C^{(k)})$ of a cylinder is defined as the ratio "height over width".) To sum up, we have for all j the relation

$$\sum_{s=0}^{n-1} S_{-1}(\boldsymbol{\sigma}_s) \;=\; \sum_{k\in K(j)} \ell(C^{(k)})^{-1} \;=\; n \sum_{k\in K(j)} m(C^{(k)})\,. \tag{34}$$

Let ℓ be the least common multiple of all the $\ell(C^{(k)})$ for $k \in K(j)$. The parabolic element $N(\ell) = \left(\begin{smallmatrix} 1 & n\ell \\ 0 & 1\end{smallmatrix}\right)$ is in the affine group of (X,ω) and fixes the horizontal direction. Consequently, the corresponding diffeomorphism acts on the surface X as the product of Dehn twists

$$N(\ell) \;=\; \prod_{k\in K(j)} D_{C^{(k)}}^{\ell/\ell(C^{(k)})}\,, \tag{35}$$

where $D_{C^{(k)}}$ is the Dehn twist around the core curve of $C^{(k)}$.

We start by determining the intersection number of the Teichmüller curve φ with the right hand side of (33) in a neighborhood U of the cusp determined by the horizontal direction on (X, ω). On E the action of $N(\ell)$ is an ℓ-fold Dehn twist of each of the n horizontal cylinders of $E \smallsetminus \{P_1, \ldots, P_n\}$ or, equivalently, it is an (ℓn)-fold Dehn twist of the unique horizontal cylinder of E. In both viewpoints, the local contribution of U to the intersection $\delta_{\mathrm{irr}} \cdot (f \circ \varphi)(C)$ is equal to ℓn.

On the other hand, the local contribution of U to the intersection $\pi_* \delta_{\mathcal{X}} \cdot \varphi(C)$ is equal to $\sum_{k \in K(j)} \ell / \ell(C^{(k)})$ by (35).

Note that the Siegel-Veech weights of two Hurwitz classes related by (23) agree. Moreover, the (local) degree of f restricted to $\varphi(U)$ is N_i. Comparing the two calculations above and using (34), we obtain on $V = (f \circ \varphi)(U)$ that

$$f_* \pi_* \delta_{\mathcal{X}}|_V \;=\; \frac{N_i}{n} \sum_{k \in K(j)} \ell(C^{(k)})^{-1} \delta_{\mathrm{irr}}|_V = \frac{1}{n} \sum_{h_j \in \mathcal{O}_i} S_{-1}(h_j) \delta_{\mathrm{irr}}|_V .$$

Summing over all the m Dehn twist orbits of Hurwitz classes $\mathcal{O}_i$ thus completes the proof. □

4.2. **From Siegel-Veech to Lyapunov: an algebraic proof.** Lyapunov exponents measure the growth rate of cohomology classes on flat surfaces under parallel transport along the Teichmüller geodesic flow. They agree for any two flat surfaces with the same $\mathrm{SL}(2, \mathbb{R})$-orbit closure. Hence they are important invariants of orbit closures, in particular of Hurwitz spaces and strata. We refer e.g. to [51] and [39] for the motivation and definition of Lyapunov exponents. In general not much is known about number theoretic properties of individual Lyapunov exponents. Their sum, however, is always a rational number. This was shown in full generality in [13], if one uses [3] to remove a technical hypothesis on regularity of $\mathrm{SL}(2, \mathbb{R})$-orbit closures. The proof of [13] uses a large detour via Siegel-Veech constants and many analytic tools.

On the other hand, shortly after Zorich's discovery of the rationality behavior, Kontsevich interpreted the sum of Lyapunov exponents as the ratio of a transverse measure β integrated against two natural first Chern classes ([30]). This interpretation rather than the definition will be our starting point to compute the sum of Lyapunov exponents. If the class β could be interpreted as a rational cohomology class on a suitable compactification of an orbit closure, this would give a more conceptual proof of the rationality of the sum of Lyapunov exponents. Finding such an interpretation of β in the case of strata is currently a central open problem.

We will identify β for Hurwitz spaces as a rational cohomology class. This will be stated in Theorem 4.3 below and proven in Section 5. The main result in this section is a proof of the following result, using intersection theory only. Suppose that the smallest stratum that contains $H_d(\Pi)$ is $\Omega\mathcal{M}_g(m_1, \ldots, m_n)$.

Theorem 4.2. *For the sum of Lyapunov exponents of the Hurwitz space $H_d(\Pi)$ and the combinatorial Siegel-Veech constant $c^0_{-1}(d, \Pi)$ we have the relation*

$$\lambda_1 + \cdots + \lambda_g \;=\; \frac{c^0_{-1}(d, \Pi)}{N^0_d(\Pi)} + \kappa, \quad \textit{where} \quad \kappa \;=\; \frac{1}{12} \sum_{i=1}^{n} \frac{m_i(m_i + 2)}{m_i + 1} . \tag{36}$$

The proof uses Theorem 4.1 as its only ingredient besides intersection theory. Theorem 4.2 should be compared to the main result of [13] which states that for an $\mathrm{SL}(2, \mathbb{R})$-invariant submanifold H that is minimally contained in the stratum

$\Omega\mathcal{M}_g(m_1,\ldots,m_n)$ the Lyapunov exponents and the area Siegel-Veech constant are related by

$$\lambda_1+\cdots+\lambda_g \;=\; \frac{\pi^2}{3}\,c_{\text{area}}(H)+\kappa\,. \tag{37}$$

Consequently, combining Theorems 3.1 and 4.2 provides an algebraic proof of the formula of Eskin-Kontsevich-Zorich in the case of Hurwitz spaces.

We first introduce the formula for the sum of Lyapunov exponents as a ratio of two integrals. The projectivized Hodge bundle $\mathbb{P}\Omega\overline{\mathcal{M}}_g$ comes with a tautological line bundle $\mathcal{O}(-1)$. Its fiber over a point (X,ω) is the $\mathbb{C}$-span of ω. The first Chern class of this line bundle is denoted by γ_1 in [30]. We use the same notation for a vector bundle on the whole moduli space and its restriction to any algebraic $\mathrm{SL}(2,\mathbb{R})$-invariant submanifold H. A second tautological class is the first Chern class of the Hodge bundle, denoted by λ. The fiber of the Hodge bundle over a point (X,ω) is the vector space $H^0(X,\Omega^1_X)$. (Note that λ is denoted by γ_2 in [30].)

The third key player is not quite a class in cohomology, but a transverse measure. Recall that an $\mathrm{SL}(2,\mathbb{R})$-invariant submanifold H has a natural projection $\pi : H \to \mathbb{P}H$, quotienting by $\mathrm{SO}_2(\mathbb{R})$ (or quotienting the $\mathrm{GL}(2,\mathbb{R})$-orbit closure by $\mathbb{C}^*$, explaining the notation). Let $\mathcal{F}$ be the π-image of the (non-holomorphic) foliation of H by $\mathrm{SL}(2,\mathbb{R})$-orbits. Then β is the transverse measure to the foliation $\mathcal{F}$ which is obtained by disintegrating the Masur-Veech measure ν_1. With these notations the main formula sketched in [30] becomes

$$\lambda_1+\cdots+\lambda_g \;=\; \frac{\int_{\mathbb{P}H}\beta\wedge\lambda}{\int_{\mathbb{P}H}\beta\wedge\gamma_1}\,, \tag{38}$$

where H is the $\mathrm{SL}(2,\mathbb{R})$-orbit closure of the flat surface whose Lyapunov spectrum we are interested in. A full proof of the above formula, stated as the "Background Theorem", appears in [13, Section 3], along with references to various other cases where this formula has been established rigorously before. In the case of the moduli space of pointed elliptic curves, obviously $\mathbb{P}\Omega\overline{\mathcal{M}}_{1,n} = \overline{\mathcal{M}}_{1,n}$. We will show in the next section that on $\overline{\mathcal{M}}_{1,n}$ integration against β is represented by a rational cohomology class. More precisely, we can identify the class as follows. We define the tautological classes ψ_i on $\overline{\mathcal{M}}_{g,n}$ by having the value $-\pi_*(\sigma_i^2)$ on any family of stable genus g curves $\pi : \mathcal{X} \to C$ with sections σ_i corresponding to the marked points.

Theorem 4.3. *As elements of $H^{2n-2}(\overline{\mathcal{M}}_{1,n},\mathbb{C})$, the classes β and $\psi_2\cdots\psi_n$ are proportional.*

Since the foliation by $\mathrm{SL}(2,\mathbb{R})$-orbits and the measure ν_1 on the Hurwitz space are defined as pullbacks from $\Omega_1\mathcal{M}_{1,n}$, the integration of first Chern classes of line bundles against β on $\overline{H}_d(\Pi)$ is proportional to the intersection product with the class $f^*(\psi_2)\cdots f^*(\psi_n)$ where $f : \overline{H}_d(\Pi) \to \overline{\mathcal{M}}_{1,n}$ is the forgetful map.

4.3. **Tautological class calculations on $\overline{\mathcal{M}}_{1,n}$.** To identify β we next summarize known results on the cohomology ring of $\overline{\mathcal{M}}_{1,n}$. Recall that λ is the first Chern class of the Hodge bundle on $\overline{\mathcal{M}}_{g,n}$. Recall also the definition of $\delta_{0,S}$ and π_{n+1} from Section 4.1 and the following result from [1].

Proposition 4.4. *$H^2(\overline{\mathcal{M}}_{1,n},\mathbb{C})$ is freely generated by λ and the boundary classes $\delta_{0,S}$ for $2 \le |S| \le n$.*

We use $\langle\mu\rangle_{1,n}$ to denote the degree of a given class μ in $H^{2n}(\overline{\mathcal{M}}_{1,n},\mathbb{C})$.

As a special case of the preceding proposition, $H^2(\overline{\mathcal{M}}_{1,1},\mathbb{C})$ is of rank one, and in fact (see [46, (2.46)])

$$\psi_1 \;=\; \lambda \;=\; \frac{1}{12}\delta_{\mathrm{irr}} \quad \text{and} \quad \langle\delta_{\mathrm{irr}}\rangle_{1,1} \;=\; \frac{1}{2}. \tag{39}$$

The aim of this subsection is to deduce the following relations in the cohomology ring of $\overline{\mathcal{M}}_{1,n}$ from well-known properties.

Lemma 4.5. *For any subset $S \subset \{1,\cdots,n\}$ such that $2 \le |S| \le n$, we have*

$$\langle\delta_{0,S}\psi_2\cdots\psi_n\rangle_{1,n} \;=\; 0\,.$$

Since the statement and proof is symmetric in the marked points, we may replace here and in the subsequent lemmas $\psi_2\cdots\psi_n$ by any product of $(n-1)$ distinct ψ-classes.

Lemma 4.6. *On $\overline{\mathcal{M}}_{1,n}$ we have*

$$\langle\delta_{\mathrm{irr}}\psi_2\cdots\psi_n\rangle_{1,n} = \frac{(n-1)!}{2} \quad \textit{and} \quad \langle\psi_i\psi_2\cdots\psi_n\rangle_{1,n} = \frac{(n-1)!}{24}$$

for $1 \le i \le n$.

Before starting with the proofs, recall that

$$\psi_i\delta_{0,\{i,j\}} = 0\,, \tag{40}$$

$$\pi_{n+1*}\psi_{n+1} \;=\; (2g-2+n)[\overline{\mathcal{M}}_{g,n}]\,. \tag{41}$$

Equation (40) follows from the fact that a $\mathbb{P}^1$-tail with two marked points has no non-trivial moduli, and (41) holds because ψ_{n+1} restricted to a fiber of π_{n+1} has degree $2g-2+n$.

Since $\psi_i = \pi_{n+1}^*\psi_i + \delta_{0,\{i,n+1\}}$ for $i \ne n+1$, by (40) and the projection formula we obtain that

$$\pi_{n+1*}(\psi_1^{a_1}\cdots\psi_n^{a_n}\psi_{n+1}^{a_{n+1}}) \;=\; \pi_{n+1*}(\psi_{n+1}^{a_{n+1}})(\psi_1^{a_1}\cdots\psi_n^{a_n})\,. \tag{42}$$

As a special case when $a_{n+1}=1$, by (41) we obtain the *dilaton equation* (see [46, (2.45)])

$$\langle\prod_{i=1}^{n}\psi_i^{a_i}\psi_{n+1}\rangle_{g,n+1} \;=\; (2g-2+n)\,\langle\prod_{i=1}^{n}\psi_i^{a_i}\rangle_{g,n}\,. \tag{43}$$

Proof of Lemma 4.5. In the case when $|S|=2$ or $n=2$, the result follows from (40). Suppose it holds for all S on $\overline{\mathcal{M}}_{1,k}$ with $k<n$ and for S on $\overline{\mathcal{M}}_{1,n}$ with $|S|<j$. Without loss of generality, assume that $n \in S$ and let $S' = S \smallsetminus \{n\}$. Since $\pi_n^*\delta_{0,S'} = \delta_{0,S'} + \delta_{0,S}$, we obtain that

$$\begin{aligned}\langle\delta_{0,S}\psi_2\cdots\psi_n\rangle_{1,n} \;&=\; \langle\delta_{0,S'}\pi_{n*}(\psi_2\cdots\psi_n)\rangle_{1,n}\\ &=\; (n-1)\langle\delta_{0,S'}\psi_2\cdots\psi_{n-1}\rangle_{1,n-1} \quad = \; 0\,,\end{aligned}$$

using (41), (42) and the induction hypothesis. □

Proof of Lemma 4.6. To prove the first formula we use that $\pi_n^*\delta_{\mathrm{irr}} = \delta_{\mathrm{irr}}$ and hence

$$\langle\delta_{\mathrm{irr}}\psi_2\cdots\psi_n\rangle_{1,n} \;=\; (n-1)\langle\delta_{\mathrm{irr}}\psi_2\cdots\psi_{n-1}\rangle_{1,n-1}$$

by the projection formula and (41). The result follows by induction and from (39).

For the second formula we can assume without loss of generality that $i = 1$ or $i = 2$. The dilation equation (43) implies that

$$\langle \psi_i \psi_2 \cdots \psi_n \rangle_{1,n} \;=\; (n-1)\langle \psi_i \psi_2 \cdots \psi_{n-1} \rangle_{1,n-1} \,.$$

For $i = 1$, the result follows by induction and from (39). For $i = 2$, note that

$$\langle \psi_2^2 \rangle_{1,2} = \langle \psi_1^2 \rangle_{1,2} = \langle \psi_1 \rangle_{1,1} = \frac{1}{24} \,,$$

which can be seen by using the relation $\psi_1 = \pi_2^* \psi_1 + \delta_{0,\{1,2\}}$ and the projection formula for the map $\pi_2 : \overline{\mathcal{M}}_{1,2} \to \overline{\mathcal{M}}_{1,1}$. Then the result follows similarly by induction. □

For the proof of Theorem 4.2 we also need the following statement.

Lemma 4.7. *Let $\omega_{\pi_{n+1}}$ be the first Chern class of the relative dualizing sheaf associated to π_{n+1}. Then $\langle \pi_{n+1*}(\omega^2_{\pi_{n+1}}) \psi_2 \cdots \psi_n \rangle_{1,n} = 0$.*

Proof. From the relation (see e.g. [34])

$$\psi_{n+1} \;=\; \omega_{\pi_{n+1}} + \sum_{i=1}^{n} \delta_{0,\{i,n+1\}}$$

in the tautological ring, we deduce

$$\begin{aligned}
\pi_{n+1*}(\psi^2_{n+1}) \;&=\; \pi_{n+1*}\left(\psi_{n+1}\Big(\omega_{\pi_{n+1}} + \sum_{i=1}^{n} \delta_{0,\{i,n+1\}}\Big)\right) \\
&=\; \pi_{n+1*}\left(\psi_{n+1}\omega_{\pi_{n+1}}\right) \\
&=\; \pi_{n+1*}(\omega^2_{\pi_{n+1}}) + \sum_{i=1}^{n} \psi_i
\end{aligned}$$

where we used $\omega_{\pi_{n+1}} \delta_{0,\{i,n+1\}} = -\delta^2_{0,\{i,n+1\}}$, $\pi_{n+1*}(\delta^2_{0,\{i,n+1\}}) = -\psi_i$ (see e.g. [34, Table 1]) and (40) in the above. It follows that

$$\begin{aligned}
\langle \pi_{n+1*}(\omega^2_{\pi_{n+1}}) \psi_2 \cdots \psi_n \rangle_{1,n} \;&=\; \langle \pi_{n+1*}(\psi^2_{n+1}) \psi_2 \cdots \psi_n \rangle_{1,n} - \sum_{i=1}^{n} \langle \psi_i \psi_2 \cdots \psi_n \rangle_{1,n} \\
&=\; \langle \psi_{n+1} \psi_2 \cdots \psi_{n+1} \rangle_{1,n+1} - \sum_{i=1}^{n} \frac{(n-1)!}{24} \;=\; 0,
\end{aligned}$$

where we applied (42) and Lemma 4.6 in the last two steps. □

Assuming Theorem 4.3 for the moment, we can prove formula (37) and thus the rationality of the sum of Lyapunov exponents for Hurwitz spaces using intersection theory only.

Proof of Theorem 4.2. By Theorem 4.3 and Kontsevich's formula (38) we need to evaluate the quotient

$$L \;=\; \frac{\langle \lambda (f^*\psi_2) \cdots (f^*\psi_n) \rangle_{\overline{H}}}{\langle (f^*\lambda)(f^*\psi_2) \cdots (f^*\psi_n) \rangle_{\overline{H}}} \,,$$

where the class γ_1 in (38) is $f^*\lambda$ in this case, since the generating differentials on the covering curves are pulled back from the target elliptic curves and on $\overline{\mathcal{M}}_{1,1}$ the

Hodge bundle is a line bundle with first Chern class λ. By the projection formula, the denominator is equal to

$$\begin{aligned} N^0\langle\lambda\psi_2\cdots\psi_n\rangle_{1,n} &= N^0\langle(\pi_n^*\lambda)\psi_2\cdots\psi_n\rangle_{1,n} \\ &= N^0\langle\lambda\pi_{n*}(\psi_2\cdots\psi_n)\rangle_{1,n-1} \\ &= N^0(n-1)\langle\lambda\psi_2\cdots\psi_{n-1}\rangle_{1,n-1} \quad = \cdots \\ &= \frac{N^0(n-1)!}{24} \end{aligned}$$

by recursion. Next, we evaluate the numerator. Noether's formula states that $12\lambda = \pi_*(\delta_{\mathcal{X}} + \omega_\pi^2)$ where $\delta_{\mathcal{X}}$ is the class of the nodal locus in the universal curve $\mathcal{X}$ over the Hurwitz space. Hence the numerator is equal to

$$\langle(f_*\lambda)\psi_2\cdots\psi_n\rangle_{1,n} = \frac{\langle(\pi_{n+1*}h_*\delta_{\mathcal{X}} + \pi_{n+1*}h_*(\omega_\pi^2))\psi_2\cdots\psi_n\rangle_{1,n}}{12}.$$

Using Lemmas 4.5, 4.6, and Theorem 4.1, we obtain that

$$\langle(\pi_{n+1*}h_*\delta_{\mathcal{X}})\psi_2\cdots\psi_n\rangle_{1,n} = \frac{(n-1)!}{2}\, c^0_{-1}(d,\Pi).$$

For the other term involving ω_π^2, we apply the Riemann-Hurwitz formula

$$\omega_\pi = h^*\omega_{\pi_{n+1}} + \sum_{i,j} m_{ij}\Gamma_{ij},$$

where Σ_i is the section of the i-th branch point and $\Gamma_{ij} \subset \mathcal{X}$ is the section of ramification order m_{ij} in the inverse image of Σ_i. Consequently,

$$h_*(\omega_\pi^2) = h_*(h^*\omega_{\pi_{n+1}})^2 + 2\sum_{i,j} m_{ij}(h_*\Gamma_{ij})\omega_{\pi_{n+1}} + \sum_{i,j} m_{ij}^2 h_*(\Gamma_{ij}^2).$$

Using the relations

$$h^*\Sigma_i = \sum_j (m_{ij}+1)\Gamma_{ij}, \quad h_*\Gamma_{ij} = N^0\Sigma_i, \quad \text{and} \quad \Gamma_{ij}\Gamma_{kl} = 0$$

for $(i,j) \neq (k,l)$, we obtain that

$$h_*(\Gamma_{ij}^2) = \frac{1}{m_{ij}+1}(h^*\Sigma_i)\Gamma_{ij} = \frac{N^0}{m_{ij}+1}\Sigma_i^2.$$

Moreover, we have

$$\omega_{\pi_{n+1}}\Sigma_i = -\Sigma_i^2, \quad h_*(h^*\omega_{\pi_{n+1}})^2 = dN^0\omega_{\pi_{n+1}}^2.$$

Using these equalities, we obtain that

$$h_*(\omega_\pi^2) = dN^0\omega_{\pi_{n+1}}^2 - N^0\Big(\sum_{i,j}\frac{m_{ij}(m_{ij}+2)}{m_{ij}+1}\Sigma_i^2\Big),$$

$$\pi_{n+1*}h_*(\omega_\pi^2) = dN^0\pi_{n+1*}(\omega_{\pi_{n+1}}^2) + N^0\Big(\sum_{i,j}\frac{m_{ij}(m_{ij}+2)}{m_{ij}+1}\psi_i\Big).$$

Applying Lemmas 4.6 and 4.7, we conclude that

$$\begin{aligned} \langle\pi_{n+1*}h_*(\omega_\pi^2)\psi_2\cdots\psi_n\rangle_{1,n} &= \frac{N^0(n-1)!}{24}\Big(\sum_{i,j}\frac{m_{ij}(m_{ij}+2)}{m_{ij}+1}\Big) \\ &= \frac{N^0(n-1)!}{2}\,\kappa. \end{aligned}$$

Assembling all the ingredients we computed, we thus obtain the desired equality. □

5. Identifying the β-class

The first aim of this section is to justify, as we claimed in the previous section, that the integration against the transverse measure β used to define the sum of Lyapunov exponents is proportional to the cup product with a rational cohomology class. We treat the case of the $\mathrm{SL}(2,\mathbb{R})$-invariant manifold $\Omega_1\mathcal{M}_{1,n}$. The proof is to some extent parallel to that in [5]. However in our situation, periods cannot be used at every point to provide coordinates of the locus. The use of cross-ratio coordinates is a new ingredient here. Both the proofs here and in [5] rely on the fact that the REL-foliation is of complex codimension one, transverse to the foliation of $\mathrm{SL}(2,\mathbb{R})$-orbits. Such an $\mathrm{SL}(2,\mathbb{R})$-invariant manifold is called of *rank one* and presumably, the identification of β as a multiple of a rational cohomology class can be achieved for all rank-one $\mathrm{SL}(2,\mathbb{R})$-invariant manifolds.

Recall that $\mathrm{SL}(2,\mathbb{R})$-invariant manifolds H have a natural projection $\pi : H \to \mathbb{P}H$ by modulo $\mathbb{C}^*$. For such a manifold $\mathbb{P}H$ the disintegration along the image of the $\mathrm{SL}(2,\mathbb{R})$-foliation of the π-pushforward of the Masur-Veech measure ν_1 (and here, for $\mathbb{P}\Omega_1\mathcal{M}_{1,n} = \mathcal{M}_{1,n}$, even more concretely, the symmetric space measure $\overline{\nu}_1$, see § 2.2) can be made explicit. In general, let M be a manifold with a measure ν and a foliation $\mathcal{F}$ whose leaves are Riemannian manifolds. For a p-form ω we define a function $||\omega||_\mathcal{F}$ by

$$||\omega||_\mathcal{F} \;=\; \sup_{v_1,\ldots,v_p\in T\mathcal{F}} \frac{\omega(v_1,\ldots,v_p)}{||v_1||\cdots||v_p||}$$

and we let

$$\int_\mathcal{F}\omega \;=\; \int_M ||\omega||_\mathcal{F} d\nu\,. \tag{44}$$

We first apply this definition to $M = \mathbb{P}\Omega_1\mathcal{M}_{1,n}$, the push-forward of ν_1, and the image foliation $\mathcal{F}$ of $\mathrm{SL}(2,\mathbb{R})$-orbits. Its leaves are quotients of $\mathbb{H}$, provided with the Poincaré metric. It follows from a local calculation and the definition of ν_1 that on 2-forms the functionals $\omega \mapsto \int_\mathcal{F}\omega$ and $\omega \mapsto \int_{\mathcal{M}_{1,n}}\beta\wedge\omega$ are proportional.

Proposition 5.1. *The integration along $\mathcal{F}$, i.e. the map $\omega \mapsto \int_\mathcal{F}\omega$, defines a closed current of dimension 2 on $\overline{\mathcal{M}}_{1,n}$.*

By slight abuse of notation and suppressing the proportionality constant we denote the current defined by integration along $\mathcal{F}$ by β.

The second aim of this section is the proof of Theorem 4.3. In view of Proposition 4.4 and Lemma 4.5, it is equivalent to show the following proposition.

Proposition 5.2. *Let $S \subset \{1,\ldots,n\}$ with $|S| \geq 2$. Then $\langle\delta_{0,S}\beta\rangle_{1,n} = 0$.*

We prepare for the proof of Proposition 5.1 and recall Mumford's notion of forms of Poincaré growth. For this purpose we provide open sets isomorphic to $(\Delta^*)^k\times\Delta^n$ with a metric ρ by putting the Euclidean metric on the Δ-factors and the Poincaré metric on the Δ^*-factors. We say that a p-form ω on a manifold X has *Poincaré growth* with respect to a divisor D, if X can be covered by polydiscs $V_\alpha \cong \Delta^n$ such that $U_\alpha = V_\alpha \cap (X \smallsetminus D) \cong (\Delta^*)^k \times \Delta^{n-k}$ and $||\omega||_\rho$ is bounded on each of the U_α.

Since the volume form on $\overline{\mathcal{M}}_{1,n}$ has Poincaré growth with respect to the divisor $\delta_{\rm irr}$, the following is the main step towards proving Proposition 5.1.

Lemma 5.3. *For any 2-form ω on $\overline{\mathcal{M}}_{1,n}$ of Poincaré growth with respect to the divisor $\delta_{\rm irr}$, the norm $||\omega||_{\mathcal{F}}$ is bounded.*

Proof of Lemma 5.3. For each boundary point of $\overline{\mathcal{M}}_{1,n}$ let $U \cong \Delta^n$ be a sufficiently small open neighborhood such that $U \cap \mathcal{M}_{1,n} \cong (\Delta^*)^r \times \Delta^{n-r}$. Recall that on U we consider the metric ρ as the product of the Poincaré metrics on the Δ^*-factors and the Euclidean metric on the Δ-factors. It suffices to check that $||v||_\rho/||v||_{\mathcal{F}}$ is bounded for any vector field v on U. Since $\mathcal{F}$ has complex dimension one it suffices to check for any vector field tangent to $\mathcal{F}$ that each of the factors contributing to $||v||_\rho$ is bounded.

We first consider a neighborhood of a generic point in $\delta_{\rm irr}$. As coordinates in $\Omega\mathcal{M}_{1,n}$ we use the period coordinates $(z_\alpha, z_\beta, z_2, \ldots, z_n)$ as defined in (22). We choose the representative of our point in $\mathcal{M}_{1,n} = \mathbb{P}\Omega\mathcal{M}_{1,n}$ to have $z_\alpha = 1$. We take v to be the tangent vector field to the action of the diagonal subgroup of $\mathrm{SL}(2,\mathbb{R})$ given by the matrices $a_t = \mathrm{diag}(e^{-t/2}, e^{t/2})$. Then, in terms of the coordinates $\tau_1 = z_\beta/z_\alpha,\ v_2 = z_2/z_\alpha, \ldots, v_n = z_n/z_\alpha$, the action is given by

$$a_t(\tau_1, v_2, \ldots, v_n) \;=\; (\Re(\tau_1) + ie^t\Im(\tau_1),\ \Re(v_2) + ie^t\Im(v_2), \ldots, \Re(v_n) + ie^t\Im(v_n)).$$

Consequently, a unit tangent vector field is given by

$$v \;=\; i\Im(\tau_1)\frac{\partial}{\partial\tau_1} + \sum_{j=2}^{n} i\Im(v_j)\frac{\partial}{\partial v_j}. \tag{45}$$

In the polydisc coordinates $q_1 = e^{2\pi i\tau_1}$, $v_2, \ldots, v_n$ around $\delta_{\rm irr} = \{q_1 = 0\}$ the tangent vector is

$$v \;=\; q_1\log|q_1|\frac{\partial}{\partial q_1} + \sum_{j=2}^{n} i\Im(v_j)\frac{\partial}{\partial v_j}. \tag{46}$$

The first summand is bounded by definition of the Poincaré metric and the boundedness is obvious for all the remaining summands.

Next we consider a neighborhood of a generic point Q in $\delta_{0,S}$. We introduce the following convenient coordinate system. Denote by s the cardinality of S and relabel the marked points so that $S = \{n-s+1, \ldots, n\}$. The point Q parameterizes a flat surface (E,ω) of genus one with the marked points $P_1, \ldots, P_{n-s}$ and a rational tail with the marked points $P_{n-s+1}, \ldots, P_n$ attached to Q at a point K. Let $(\tau_1^{(0)}, v_2^{(0)}, \ldots, v_{n-s}^{(0)}, v_K^{(0)})$ be the period coordinates of $(E, \omega, P_1, \ldots, P_{n-s}, K)$, normalized as above such that $z_\alpha^{(0)} = 1$. Smooth surfaces in a neighborhood of Q are represented by flat surfaces $(E, \omega, P_1, \ldots, P_n)$ such that the normalized coordinates $(\tau_1, v_2, \ldots, v_n)$ have the following properties. The coordinates τ_1 and v_i for $i = 2, \ldots, n-s$ are close to their initial values (denoted by an upper index (0)), the coordinate v_{n-s+1} is close to $v_K^{(0)}$, and v_{n-s+j} is close to v_{n-s+1} for $j = 2, \ldots, s$. Let $t_{n-s+j} = v_{n-s+j} - v_{n-s+1}$ for $j = 2, \ldots, s$ and let $u_{n-s+j} = t_{n-s+j}/t_{n-s+2}$ for $j = 3, \ldots, s$. Here u_{n-s+j} measures the approaching rate of z_{n-s+j} to z_{n-s+1} with respect to that of z_{n-s+2} to z_{n-s+1}.

With this normalization, the cross-ratio coordinate system on a polydisc neighborhood around Q we use is $(\tau_1, v_2, \ldots, v_{n-s}, v_{n-s+1}, t_{n-s+2}, u_{n-s+3}, \ldots, u_n)$. In this coordinate system, t_{n-s+2} measures the distance from the boundary $\delta_{0,S}$ and the corresponding disc is provided with the Poincaré metric, while all the other discs are provided with the Euclidean metric. Relabeling these points in S and using that Q is generic in $\delta_{0,S}$ we may assume moreover that u_{n-s+j} is bounded near Q.

We use the action of the diagonal flow a_t as above. In our chosen coordinates a unit tangent vector is

$$\begin{aligned} v &= i\Im(\tau_1)\frac{\partial}{\partial \tau_1} + \sum_{j=2}^{n} i\Im(v_j)\frac{\partial}{\partial v_j} \\ &= i\Im(\tau_1)\frac{\partial}{\partial \tau_1} + \sum_{j=2}^{n-s+1} i\Im(v_j)\frac{\partial}{\partial v_j} + i\Im(t_{n-s+2})\frac{\partial}{\partial t_{n-s+2}} + \sum_{k=n-s+3}^{n} f_k \frac{\partial}{\partial u_k}\,, \end{aligned}$$

where

$$f_k = \frac{i\Im(t_{n-s+j})t_{n-s+2} - i\Im(t_{n-s+2})t_{n-s+j}}{t_{n-s+2}^2}$$

for $k = n-s+j$ and $j \geq 3$. From this it is clear that $||v||_\rho$ is bounded near Q.

The case that the boundary point lies in the intersection of several boundary divisors directly follows from the combination of these calculations, since ρ is defined as the product metric. □

We will be brief in the remaining steps, following [5]. The preceding lemma and the finite total volume show that for any two-form ω of Poincaré growth along δ_{irr} we have $\int_{\mathcal{F}} |\omega| < \infty$ and hence integration over $\mathcal{F}$ defines a current β on $\overline{\mathcal{M}}_{1,n}$. (Details are given in loc. cit., Corollary 8.4.)

The final step in the *proof of Proposition 5.1* consists of showing that the current is closed. To achieve this we need to show that $\int_{\mathcal{F}} d\eta = 0$ for any smooth one-form η. This follows as in [5, Theorem 8.1], by an application of Stokes' theorem from the following existence statement of suitable cusp neighborhoods. Let $N \subset \mathrm{SL}(2,\mathbb{R})$ be the subgroup of upper triangular matrices and H the horocycle subgroup.

Lemma 5.4. *For any $\epsilon > 0$ there is a closed H-invariant neighborhood W of δ_{irr} such that* $\mathrm{vol}(W) < \epsilon$ *and such that ∂W is transversal to $\mathcal{F}$.*

Here two submanifolds are called transversal if the sum of their tangent spaces generates the whole tangent space at every point of their intersection. Orbits of N are of course both contained in W and $\mathcal{F}$.

Proof. As in [5] we take a decomposition of $(E, P_1, \ldots, P_n)$ into horizontal cylinders C_i and let $f((E, P_1, \ldots, P_n)) = \sum \rho(\mathrm{height}(C_i))$ where $\rho : \mathbb{R} \to \mathbb{R}$ is a bump function to make the function f smooth near zero.

Let $W_\ell = f^{-1}(\ell, \infty)$. Since the height is N-invariant, W_ℓ is also N-invariant. Since the total ν_1 volume of $\Omega_1\mathcal{M}_{1,n}$ is finite, the volume of W_ℓ as $\ell \to 0$ is eventually smaller than ϵ. □

We now come to the *proof of Proposition 5.2*. Morally, this is due to the fact that the foliation can be extended to $\overline{\mathcal{M}}_{1,n} \smallsetminus \delta_{\mathrm{irr}}$ and that $\delta_{0,S}$ is a leaf of the foliation, which gets no mass from a transverse measure. A precise argument, inspired by [5], will be given in the remainder of this section.

By definition of the intersection product of the cohomology class of a two-current and a divisor we have to show that $\int_{\mathcal{F}} \mathrm{PD}(\delta_{0,S}) = 0$, where $\mathrm{PD}(\delta_{0,S})$ is the two-form Poincaré dual to the divisor $\delta_{0,S}$. The idea of the proof is that this Poincaré dual two-form can be represented by a smooth form with compact support on a tubular neighborhood N of $\delta_{0,S}$.

We use the coordinates around $\delta_{0,S}$ as in the proof of Lemma 5.3, in particular $t = t_{n-s+1}$ measures the distance to the boundary. By [8, Proposition 6.24 b) and p. 70], the Poincaré dual of $\delta_{0,S}$ is represented by a smooth compactly supported two-form $\Psi = d(\rho(t)\psi)$ where ψ is a one-form (constructed by patching angular forms) and where $\rho : [0,\infty) \to \mathbb{R}$ is a bump function, identically one near zero and with support in $[0,1]$.

We cut off the integration over N on two sets. The first set is $C_n = \pi_1^{-1}(C)$, where C is a neighborhood of the cusp in $\overline{\mathcal{M}}_{1,1}$ bounded by the horocycle H and π_1 is the morphism forgetting all markings but the first one. The second set is a small neighborhood N_s of $\delta_{0,S}$ given by $t \le s$. The horocycle flow defines a foliation $\mathcal{F}_H$ whose leaves are contained in the leaves of $\mathcal{F}$. The boundary H_n of C_n and the boundary B_s of N_s are both foliated by horocycles.

Recall the definition of foliated integrals from (44). By Stokes' theorem

$$\int_{\mathcal{F}} \Psi \;=\; \int_N ||\Psi||_{\mathcal{F}} \;=\; \int_{C_n \cup N_s} ||\Psi||_{\mathcal{F}} \;=\; \int_{H_n} ||\rho\,\psi||_{\mathcal{F}_H} + \int_{B_s} ||\rho\,\psi||_{\mathcal{F}_H}.$$

Since Ψ is smooth, in particular of Poincaré growth, its $\mathcal{F}$-norm is bounded by Lemma 5.3. We can thus estimate the last two integrals by a constant depending only on Ψ times the length $\ell(H)$ and times the volume $\nu(B_s)$, respectively. These contributions can both be made arbitrarily small by shrinking $\ell(H)$ and s. The following two lemmas consequently conclude the proof of Proposition 5.2.

Lemma 5.5. *There exists a constant A_1, depending only on n, such that*

$$\int_{H_n} ||\rho\,\psi||_{\mathcal{F}_H} < A_1\,\ell(H)\,.$$

Lemma 5.6. *There exists a constant A_2, depending only on n, such that*

$$\int_{B_s} ||\rho\,\psi||_{\mathcal{F}_H} < A_2\,s^2\,.$$

Proof of Lemma 5.5. As in [5, Lemma 2.4], one shows using the local coordinates $(q_1 = e^{2\pi i\tau_1}, v_2, \ldots, v_{n-s+1}, t_{n-s+2}, u_{n-s+3}, \ldots, u_n)$ of Lemma 5.3 that for any smooth one-form η compactly supported on N the norm $||\eta||_{\mathcal{F}_H}$ is bounded. Next, one shows as in [5, Lemma 2.5] that $||\psi||_{\mathcal{F}_H}$ is bounded on compact subsets of N by compensating the singularities with another one-form of bounded $\mathcal{F}_H$-norm. Both calculations happen essentially in the two variables (q_1, t_{n-s+2}) as in loc. cit, and the other variables are irrelevant.

It now suffices to show that there is a constant $C(\rho)$ such that

$$\int_{H_n} \mathrm{supp}(\rho)d\mu \le C(\rho)\ell(H)$$

where μ is the product of the arc length measure on the horocycle and the transverse measure. This is an exercise in hyperbolic geometry that is solved in [5, Lemma 2.6]. $\square$

Proof of Lemma 5.6. The claim follows from the boundedness of $||\psi||_{\mathcal{F}_H}$ shown in the previous lemma and $\nu(B_s) = \pi s^2 \operatorname{vol}(\mathcal{M}_{1,1})$. □

6. Generating series for counting problems

The standard procedure to count connected Hurwitz numbers is to first count all covers (a problem for which functions involved are nice, e.g. shifted symmetric), then to pass to covers without unramified components (which involves taking q-brackets), and finally to apply inclusion-exclusion to reduce to the connected case. We show in this section that this procedure applies in principle also to the counting problems with Siegel-Veech weight, if one takes into account that the Siegel-Veech weight is *additive* on a disjoint product of permutations, in contrast to the constant weight 1 which is *multiplicative*.

We provide first examples of all these generating series and state at the end of the section in Theorem 6.4 one of our main results, the quasimodularity of generating functions of Siegel-Veech constants.

For the application to Siegel-Veech asymptotics for strata, we will often restrict to the ramification profile where each $\mu^{(i)}$ is a *cycle* μ_i, i.e. there is only one ramification point in each fiber over P_i.

6.1. **Counting connected and possibly disconnected coverings.** So far, we have imposed the connectivity constraint on the coverings. We remove the upper index zero, if we take all coverings (of profile Π) into consideration. As technical intermediate notion we will also consider coverings without unramified components and reflect this in the notation by a prime. Consequently, we define $\operatorname{Hur}_d(\Pi)$ to be the set of all Hurwitz tuples $h \in S_d^{n+2}$ (without the transitivity hypothesis) and we let $\operatorname{Hur}'_d(\Pi)$ be the subset of Hurwitz tuples $h = (\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}_1, \cdots, \boldsymbol{\gamma}_n)$ in $\operatorname{Hur}_d(\Pi)$ where the action of the subgroup $\langle \boldsymbol{\gamma}_1, \cdots, \boldsymbol{\gamma}_n \rangle$ is non-trivial on every $\langle h \rangle$-orbit. We denote by $N_d(\Pi)$ and $N'_d(\Pi)$ the number of the corresponding Hurwitz classes including the usual weight of $1/\operatorname{Aut}(p)$, i.e.

$$N^*_d(\Pi) \;=\; \frac{|\operatorname{Hur}^*_d(\Pi)|}{d!} \quad \text{for} \quad * \in \{', 0, \emptyset\}. \tag{47}$$

To express the passage between these counting problems we work with the generating series

$$N(\Pi) \;=\; \sum_{d=0}^{\infty} N_d(\Pi) q^d, \quad N'(\Pi) \;=\; \sum_{d=0}^{\infty} N'_d(\Pi) q^d, \quad N^0(\Pi) \;=\; \sum_{d=0}^{\infty} N^0_d(\Pi) q^d$$

for all (resp. without unramified components, resp. connected) coverings. For the empty branching profile, we drop the argument Π, in particular,

$$N() \;=\; (q)_\infty^{-1} = \sum_{\lambda} q^{|\lambda|} \;=\; 1 + q + 2q^2 + 3q^3 + 5q^4 + 7q^5 + \cdots$$

is the partition function, where $(q)_\infty = \prod_{n\geq 1}(1-q^n)$. From

$$|\operatorname{Hur}_d(\Pi)| = \sum_{j=0}^{d} \binom{d}{j} |\operatorname{Hur}'_j(\Pi)|\, |\operatorname{Hur}_{d-j}()|$$

we derive the passage between the generating functions

$$N'(\Pi) = N(\Pi)/N()\,, \tag{48}$$

see e.g. [18].

Next, we recall the passage from $N'(\Pi)$ to $N^0(\Pi)$. We denote by $\mathcal{P}(n)$ or $\mathcal{P}(N)$ the set of partitions of the set $N = \{1, \ldots, n\}$. Recall also the notation $\mathbf{P}(n)$ which is the set of partitions of n (not of the set N). We now use our assumption that each $\mu^{(i)}$ is a cycle, i.e. there is only one ramification point in each fiber over the branch point P_i, which is sufficient for later applications in the paper. Under this assumption any covering p without unramified components induces a partition $\alpha \in \mathcal{P}(n)$ corresponding to the ramification points of the connected components of the covering. This implies

$$N'(\Pi) \;=\; \sum_{\alpha\in\mathcal{P}(n)} \prod_{j=1}^{\ell(\alpha)} N^0(\Pi_{\alpha_j})\,, \tag{49}$$

where Π_{α_k} is the subset of the ramification profile corresponding to the indices appearing in the k-th subset α_k of α. We are rather interested in expressing $N^0(\Pi)$ in terms of $N'(\Pi_\alpha)$. It follows from (49) and Möbius inversion that

$$N^0(\Pi) \;=\; \sum_{\alpha\in\mathcal{P}(n)} (-1)^{\ell(\alpha)-1}(\ell(\alpha)-1)! \prod_{j=1}^{\ell(\alpha)} N'(\Pi_{\alpha_j})\,. \tag{50}$$

Finally, we recall the classical Burnside Lemma (see e.g. [32, Theorem A.1.10]) that the number of coverings with ramification profile Π and any permutation $\mu^{(i)}$ is given by

$$N_d(\Pi) \;=\; \sum_{\lambda\in\mathbf{P}(d)} \prod_{i=1}^{n} f_{\mu^{(i)}}(\lambda)\,, \tag{51}$$

where a conjugacy class σ is completed with singletons to form a partition of $|\lambda|$ and where

$$f_\sigma(\lambda) \;=\; z_\sigma \chi^\lambda(\sigma)/\dim\chi^\lambda\,. \tag{52}$$

Here z_σ denotes the size of the conjugacy class of σ and $\dim\chi^\lambda$ is the dimension of representation λ. We also write f_k for the special case that σ is a k-cycle.

We specialize now even further for the case of simply branched coverings, i.e. μ_i being the class Tr of a transposition for all i. In this case the number of branch points $n = 2k = 2g-2$ is even. For small values of k the generating series are

$$\begin{aligned} N(\mathrm{Tr}^2) &= 2q^2 + 18q^3 + 80q^4 + 258q^5 + \cdots \\ N'(\mathrm{Tr}^2) &= 2q^2 + 16q^3 + 60q^4 + 160q^5 + \cdots \\ N^0(\mathrm{Tr}^2) &= N'(\mathrm{Tr}^2) \end{aligned}$$

$$\begin{aligned} N(\mathrm{Tr}^4) &= 2q^2 + 162q^3 + 2624q^4 + 21282q^5 + \cdots \\ N'(\mathrm{Tr}^4) &= 2q^2 + 160q^3 + 2460q^4 + 18496q^5 + \cdots \\ N^0(\mathrm{Tr}^4) &= 2q^2 + 160q^3 + 2448q^4 + 18304q^5 + \cdots \end{aligned}$$

6.2. **Generating series for Siegel-Veech counting.** Recall from (28) the combinatorial definition of the p-weighted Siegel-Veech constant $c_p^0(d,\Pi)$ for connected covers. In the same way as (28) we can define the p-weighted Siegel-Veech constants $c_p(d,\Pi)$ for all covers and $c'_p(d,\Pi)$ for covers without unramified components, by taking the Hurwitz tuples ranging over all covers and over covers without unramified components, respectively.

As in the classical counting case, we introduce for counting with Siegel-Veech weight the generating series

$$c_p(\Pi) = \sum_{d\geq 0} c_p(d,\Pi)q^d, \quad c'_p(\Pi) = \sum_{d\geq 0} c'_p(d,\Pi)q^d, \quad c_p^0(\Pi) = \sum_{d\geq 0} c_p^0(d,\Pi)q^d \tag{53}$$

for counting all (resp. without unramified components, resp. connected) covers with p-weighted Siegel-Veech constants and study the passage between them.

We first simplify the sum (27) by reducing from n terms per Hurwitz tuple to just one summand.

Lemma 6.1. *For $* \in \{', 0, \emptyset\}$ and any ramification profile Π, we have*

$$c_p^*(d,\Pi) \;=\; \sum_{j=1}^{N_d^*(\Pi)} S_p(\alpha^{(j)})\,, \tag{54}$$

where $\alpha^{(j)}$ is the first element of the Hurwitz tuple h_j.

Proof. If $(\alpha,\beta,\gamma_1,\cdots,\gamma_n)$ is a Hurwitz tuple of profile Π, i.e. satisfying the relation

$$[\beta^{-1},\alpha^{-1}] \;=\; \beta^{-1}\alpha^{-1}\beta\alpha \;=\; \gamma_n\cdots\gamma_1$$

then

$$[\beta^{-1},\gamma_1\alpha^{-1}] \;=\; (\beta^{-1}\gamma_1\beta)\cdot\gamma_n\cdots\gamma_2$$

gives rise to a Hurwitz tuple $(\alpha\gamma_1^{-1},\beta,\gamma_2,\cdots,\gamma_n,(\beta^{-1}\gamma_1\beta))$ of the profile $\Pi' = (\mu^{(2)},\ldots,\mu^{(n)},\mu^{(1)})$. This map is a bijection between Hurwitz tuples, which is equivariant with respect to simultaneous conjugation. On the other hand,

$$[\beta^{-1},\alpha^{-1}] \;=\; (\gamma_n\gamma_{n-1}\gamma_n^{-1})\gamma_n\gamma_{n-2}\cdots\gamma_1$$

is a Hurwitz tuple with the same (α,β) and with the profile where the last two points are swapped. Iterating the use of such transforms in the profile gives a bijection between Hurwitz tuples of profile Π and Π' that preserves (α,β). The combination of the two observations shows that the sums over all Hurwitz tuples of the contribution of $\sigma_0 = \alpha$ to (28) and the contribution of $\sigma_1 = \alpha\gamma_1^{-1}$ coincide. Iterating this comparison n times for all σ_i proves the claim. □

The passage from $c_p(\Pi)$ to $c'_p(\Pi)$ in the following proposition uses essentially that Siegel-Veech weights are additive on disjoint cycles in the sense that $S_p(\lambda) = \sum_{i\geq 0} S_p(\lambda_i)$ for a partition $\lambda = (\lambda_1,\lambda_2,\ldots)$.

Proposition 6.2. *Let $\Pi = (\mu_1,\ldots,\mu_n)$ be a ramification profile with n branch points and each μ_i being a cycle. Then for any p the generating series for Siegel-Veech counting without unramified components and for Siegel-Veech counting with connected coverings are related by*

$$c'_p(\Pi) \;=\; \sum_{\sigma\in\mathcal{P}(n)} \sum_{k=1}^{\ell(\sigma)} c_p^0(\Pi_{\sigma_k}) \prod_{j=1,j\neq k}^{\ell(\sigma)} N^0(\Pi_{\sigma_j}) \tag{55}$$

where $\sigma = (\sigma_1, \ldots, \sigma_{\ell(\sigma)})$ and $\Pi_{\sigma_k} = (\{\mu_i\}_{i\in\sigma_k})$. The generating series for Siegel-Veech counting without unramified components and for Siegel-Veech counting of all coverings are related by

$$c_p(\Pi) \;=\; c'_p(\Pi)N() + N'(\Pi)c_p(). \tag{56}$$

Proof. For the first relation, suppose a covering without unramified components corresponds to the partition $\sigma \in \mathcal{P}(n)$ of the n branch points. Such a covering is given by the data

$$(\alpha^{(k)}, \beta^{(k)}, \{\gamma_i\}_{i\in\sigma_k})_{k=1,\ldots,\ell(\sigma)}\,.$$

By Lemma 6.1 its contribution to the left hand side is $S_p(\alpha^{(1)}\cdots\alpha^{(\ell(\sigma))})$, whereas each summand of the interior sum on the right hand side gives a contribution of $S_p(\alpha^{(k)})$. Additivity of the function S_p on disjoint cycles implies that these contributions are equal.

The second relation follows from the same argument, by decomposing a covering into its unramified components and into the remaining components. □

The proposition below uses representation theory to reduce the computation of Siegel-Veech counting from a sum over all Hurwitz tuples to just a sum over pairs of partitions. This expression will be simplified further in Part II. We emphasize for future use that the following proposition does not require the additional hypothesis that each μ_i is a cycle.

Proposition 6.3. *If $\Pi = (\mu_1, \ldots, \mu_n)$ with $\mu_i \in \mathbf{P}(d)$ any partitions, then*

$$c_p(d,\Pi) \;=\; \sum_{\lambda\in\mathbf{P}(d)}\prod_{i=1}^{n} f_{\mu_i}(\lambda)\frac{1}{d!}\sum_{\tau\in\mathbf{P}(d)} z_\tau S_p(\tau)\chi^\lambda(\tau)^2. \tag{57}$$

Proof. We start by recalling the proof of the Burnside Lemma to count coverings. If we want to count all factorizations $\prod_{i=1}^{n+2}\gamma_i = 1$ with $\gamma_i \in S_d$ belonging to a fixed conjugacy class C_i, then the number of such factorizations is

$$|\mathrm{Hur}_d(C_1,\ldots,C_{n+2})| \;=\; \sum_{\lambda\in\mathbf{P}(d)}\frac{(\dim\chi^\lambda)^2}{d!}\prod_{i=1}^{n+2} f_{C_i}(\lambda),$$

which can be checked by comparing the trace of the action of $\sum_{g\in C_i} g \in \mathbb{C}[S_d]$ on the decomposition of $\mathbb{C}[S_d]$ into irreducible representations (e.g. [32, Theorem A.1.9]). We will apply this to $C_i = \mu_i$ for $i = 1,\ldots,n$, for $\gamma_{n+1} = \alpha$ belonging to any conjugacy class, and for $\gamma_{n+2} = \beta\alpha^{-1}\beta^{-1}$ being a conjugate of α^{-1}. Since there are $d!/z_\alpha$ elements that conjugate a given α^{-1} into a given element $\alpha'^{-1} = \beta\alpha^{-1}\beta^{-1}$, we deduce that the number of factorizations $[\alpha,\beta] = \prod_{i=1}^n \gamma_i$ with γ_i in the conjugacy class μ_i is

$$|\mathrm{Hur}_d(\Pi)| \;=\; \sum_{\lambda\in\mathbf{P}(d)}\prod_{i=1}^{n} f_{\mu_i}(\lambda)\Big(\sum_{\alpha\in\mathbf{P}(d)} z_\alpha\chi^\lambda(\alpha)^2\Big).$$

If we count with Siegel-Veech weight, using Lemma 6.1 we see that the innermost bracket is $\sum_{\alpha\in\mathbf{P}(d)} z_\alpha S_p(\alpha)\chi^\lambda(\alpha)^2$ instead. Finally recall the relation $N_d^*(\Pi) = |\mathrm{Hur}_d^*(\Pi)|/d!$ and similarly for the Siegel-Veech count, thus proving the desired formula. □

Our initial motivation for the analysis of q-brackets in Part II is to prove the following theorem, as one of our main results. The special case that $\mu_i = \mathrm{Tr}$ will be analyzed in detail in this paper, since it corresponds to the counting problems for the principal stratum in genus $g = k+1 = \frac{n}{2}+1$.

Theorem 6.4. *For each μ_i being a cycle, the two counting functions $c^0_p(\mu_1,\ldots,\mu_n)$ and $c'_p(\mu_1,\ldots,\mu_n)$ are quasimodular forms of mixed weight $\leq \sum_{i=1}^n(|\mu_i|+1)+p+1$ for $\mathrm{SL}(2,\mathbb{Z})$ and for any $p \geq -1$.*

If $\mu_i = \mathrm{Tr}$ for all i, then the counting functions $c^0_p(\mathrm{Tr}^n)$ and $c'_p(\mathrm{Tr}^n)$ are quasimodular forms of pure weight $3n+p+1$ for $\mathrm{SL}(2,\mathbb{Z})$ and for any $p \geq -1$.

The proof of this theorem will be completed in Section 16.

6.3. **Examples of the Siegel-Veech counting functions.** We specialize to the case $\mu_i = \mathrm{Tr}$, the class of a transposition, and give examples of the series introduced above.

For $p = 1$, the Siegel-Veech counting function $c^*_1(\Pi) = D(N^*(\Pi))$ is just the $D = q\frac{\partial}{\partial q}$-derivative for $* \in \{\emptyset, ', 0\}$. If $\Pi = \mathrm{Tr}^n$, then by the Riemann-Hurwitz formula $n = 2k = 2g-2$ has to be even. For small k and for $p = -1$ the first several series are

$$\begin{aligned}
c_{-1}(\mathrm{Tr}^2) &= \tfrac{5}{2}q^2 + \tfrac{49}{2}q^3 + 121q^4 + \tfrac{2593}{6}q^5 + \cdots \\
c'_{-1}(\mathrm{Tr}^2) &= \tfrac{5}{2}q^2 + 20q^3 + 75q^4 + 200q^5 + \cdots \\
c^0_{-1}(\mathrm{Tr}^2) &= c'_{-1}(\mathrm{Tr}^2) \\
\\
c_{-1}(\mathrm{Tr}^4) &= \tfrac{5}{2}q^2 + \tfrac{441}{2}q^3 + 3764q^4 + \tfrac{194107}{6}q^5 + \cdots \\
c'_{-1}(\mathrm{Tr}^4) &= \tfrac{5}{2}q^2 + 216q^3 + 3378q^4 + 25664q^5 + \cdots \\
c^0_{-1}(\mathrm{Tr}^4) &= \tfrac{5}{2}q^2 + 216q^3 + 3348q^4 + 25184q^5 + \cdots
\end{aligned} \tag{58}$$

It was shown by Eskin-Okounkov in [18] based on work of [7] that $N'(\Pi)$ and $N^0(\Pi)$ are quasimodular forms for any Π. We will recall these notions and results in Part II.

Part II: Bloch-Okounkov correlators and their growth polynomials

The point of departure for Part II is a beautiful theorem of Bloch and Okounkov saying that the q-bracket (a certain weighted average) of any "shifted symmetric polynomial" on the set $\mathbf{P}$ of all partitions is a quasimodular form. In the first section of this part we review some of the many ways to describe elements of $\mathbf{P}$ and the definition of shifted symmetric polynomials. Section 8 contains the statement of the Bloch-Okounkov theorem and various complementary results, as well as a review of the definitions and main properties of quasimodular forms. The following section shows how to associate to each quasimodular form a "growth polynomial" that contains information both about the growth of the function near cusps and about the growth of its Fourier coefficients. This notion is essentially equivalent to one used by Eskin and Okounkov in [18], but since these polynomials can also be useful in other contexts in the theory of modular forms we give a different and considerably more detailed presentation, including alternative descriptions and other basic properties of growth polynomials.

The new results of this chapter are contained in the last three sections. The main fact is that the growth polynomials of the quasimodular forms defined by the Bloch-Okounkov theorem, unlike these forms themselves, can be given in terms of explicit generating functions. This was discovered by Eskin and Okounkov in [18] in terms of the so-called "n-point correlators". In Section 10 we give their formula with a different and simpler proof, as well as a second formula in terms of an all-variable generating function that we show can be represented by a formal Gaussian integral vaguely reminiscent of the path integrals of quantum field theory. This is then applied in Section 11 to give a new formula for certain special combinations of q-brackets called "cumulants", which are the expressions that we will need for the applications to the calculation of invariants of moduli spaces and Siegel-Veech constants. A result of this type was also given in [18], but here we find a direct proof and thus as a corollary a much simpler proof of their result. Finally, in Section 12 we show how to express the main quantities of interest to us for the geometric applications in terms of some special power series in one variable, related to the Hurwitz zeta functions, whose Taylor coefficients are simple multiplies of Bernoulli numbers.

7. Partitions and shifted symmetric polynomials

Let $\mathbf{P}$ denote the set of all partitions. We use λ to denote a generic element of $\mathbf{P}$ and $\lambda^\vee$ to denote the dual partition. The *size* of λ (i.e. the number of which it is a partition) will be denoted by $|\lambda|$, and $\mathbf{P}(d)$ denotes the set of all partitions of d.

There are (at least) six elementary ways to view a partition, all of which will be used in the sequel.

(a) *Parts.* We write $\lambda = (\lambda_1, \lambda_2, \ldots)$, with $\lambda_1 \geq \lambda_2 \geq \cdots$ and $\sum_{i=1}^\infty \lambda_i = |\lambda|$. If k is the largest index such that $\lambda_k > 0$, we call $k = \ell(\lambda)$ the *length* of λ.

(b) *Multiplicities.* Let $r_1, r_2, r_3, \cdots$ be non-negative integers, almost all equal to 0, and write $\lambda = 1^{r_1}2^{r_2}3^{r_3}\ldots$, so that $r_m = |\{j \geq 1 : \lambda_j = m\}|$. In these coordinates the size of λ is given by $\sum m r_m$ and its length by $\sum r_m$.

(c) *Young diagram.* To any $\lambda \in \mathbf{P}$ we associate the Young diagram

$$Y_\lambda = \{(i,j) : 1 \leq j \leq k, 1 \leq i \leq \lambda_j\} \subset \mathbb{N}^2 .$$

This clearly gives a bijection between $\mathbf{P}$ and the set of finite subsets of $\mathbb{N}^2$ that are closed under making either coordinate smaller. The set Y_λ is usually denoted pictorially by replacing the elements of $\mathbb{N}^2$ by boxes of unit size, oriented so that increasing i moves one to the right and increasing j moves one downwards. The Young diagram of $\lambda^\vee$ is the transpose of Y_λ.

(d) *Frobenius coordinates.* We encode a partition $\lambda \in \mathbf{P}(d)$ by a collection of numbers

$$(s;\, a_1 \geq \cdots \geq a_s \geq 0,\ b_1 \geq \cdots \geq b_s \geq 0) \tag{59}$$

with $a_i, b_i \in \mathbb{Z}$ and $\sum_{i=1}^{s}(a_i + b_i + 1) = d$. These are given in terms of the Young diagram by setting s equal to the length of the main diagonal of Y_λ (i.e. the largest i with $(i,i) \in Y_\lambda$) and by defining a_i and b_i to be the number of boxes of Y_λ to the right of or below the diagonal box (i,i), i.e. $a_i = \lambda_i - i$ and $b_i = \lambda_i^\vee - i = |\{j : \lambda_j \geq i\}| - i$.

(e) *Semibounded subsets.* We let $X_\lambda = \{\lambda_i - i + \frac{1}{2} \mid i \geq 1\} \subset \mathbb{Z} + \frac{1}{2}$. Subsets that arise in this way are bounded above (by $\lambda_1 - \frac{1}{2}$) and have a complement in $\mathbb{Z} + \frac{1}{2}$ that is bounded below (by $\frac{1}{2} - k$, where k is the length of λ). They have the further property that the number of positive elements of X_λ is equal to the number of negative elements of $X_\lambda^c = (\mathbb{Z} + \frac{1}{2}) \smallsetminus X_\lambda$. This leads to the last description:

(f) *Balanced subsets.* There is a bijection between $\mathbf{P}$ and the set of all finite subsets $C \subset \mathbb{Z} + \frac{1}{2}$ with $\sum_{c \in C} \operatorname{sgn}(c) = 0$. The set C_λ associated to λ under this bijection is given in terms of the Frobenius coordinates (59) of λ by $C_\lambda = \{a_i + \frac{1}{2}\} \cup \{-b_j - \frac{1}{2}\}$, and conversely we recover the Frobenius coordinates by defining the a_i to be the non-negative elements of $C - \frac{1}{2}$ and the $-b_i$ to be the non-positive elements in $C + \frac{1}{2}$. In terms of the set X_λ of (e), we have $C_\lambda = (\mathbb{Z} + \frac{1}{2})_{>0} \cap X_\lambda \ \cup\ (\mathbb{Z} + \frac{1}{2})_{<0} \smallsetminus X_\lambda$.

For each integer $\ell \geq 0$ we define the *power sum* function $P_\ell : \mathbf{P} \to \mathbb{Q}$ by

$$P_\ell(\lambda) \;=\; \sum_{c \in C_\lambda} \operatorname{sgn}(c)\, c^\ell \;=\; \sum_{i=1}^{s} \left[\, (a_i + \tfrac{1}{2})^\ell \,-\, (-b_i - \tfrac{1}{2})^\ell \right], \tag{60}$$

where we have used the descriptions (f) and (d). The first three values are

$$P_0(\lambda) = 0, \quad P_1(\lambda) = |\lambda|, \quad P_2(\lambda) = 2\, f_{\mathrm{Tr}}(\lambda) = 2\, z_{\mathrm{Tr}}\, \frac{\chi^\lambda(\mathrm{Tr})}{\chi^\lambda(1)}$$

with the notations as in Section 6. Using the correspondence between (e) and (f) we can express $P_\ell(\lambda)$ in terms of the elements of X_λ in the form

$$P_\ell(\lambda) \;=\; \sum_{i=1}^{\infty} \left((\lambda_i - i + \tfrac{1}{2})^\ell - (-i + \tfrac{1}{2})^\ell \right) . \tag{61}$$

This shows that P_ℓ is an element of the algebra Λ^* of *shifted symmetric functions* ([29], [40]), which is defined as $\Lambda^* = \varprojlim \Lambda^*(n)$ where $\Lambda^*(n)$ is the algebra of symmetric polynomials in the n variables $\lambda_1 - 1, \ldots, \lambda_n - n$ and the projective limit is taken with respect to the homomorphisms setting the last variable to zero. In fact, a theorem of Okounkov and Olshanski ([40]) states that the algebra Λ^* is freely generated by the P_ℓ with $\ell \geq 1$.

We will also work with a differently normalized set of functions $Q_k : \mathbf{P} \to \mathbb{Q}$ that are related to the power sum functions by

$$Q_0(\lambda) \;=\; 1\,, \qquad Q_k(\lambda) \;=\; \frac{P_{k-1}(\lambda)}{(k-1)!} + \beta_k \quad \text{if } k \ge 1\,, \tag{62}$$

where the constants $\beta_k \in \mathbb{Q}$, with $\beta_0 = 1$, $\beta_1 = 0$, $\beta_2 = -\frac{1}{24}, \ldots$ are defined by the power series expansion

$$B(z) \;:=\; \frac{z/2}{\sinh(z/2)} \;=\; \sum_{k=0}^{\infty} \beta_k\, z^k \;=\; 1 \,-\, \frac{1}{24}\, z^2 \,+\, \frac{7}{5760}\, z^4 \,+\, \cdots \tag{63}$$

The somewhat unnatural-looking definition (62) can be explained by noting that $\ell!\,\beta_{\ell+1}$ equals $(1-2^{-\ell})\,\zeta(-\ell)$, which is the natural regularization of the divergent sum $\sum_{i=1}^{\infty}(-i+\frac{1}{2})^{\ell}$ in (61), so that $\ell!\,Q_{\ell+1}$ can be thought of as the regularization of the divergent sum $\sum_{x\in X_\lambda} x^\ell$. Another way to understand the relationship between the P's and the Q's is in terms of generating functions: if we set

$$w^0_\lambda(t) \;=\; \sum_{c\in C_\lambda} \operatorname{sgn}(c)\, t^c\,, \qquad w_\lambda(t) \;=\; \sum_{x\in X_\lambda} t^x \qquad (|t|>1)\,, \tag{64}$$

then the functions $W^0_\lambda(z) := w^0_\lambda(e^z)$ and $W_\lambda(z) := w_\lambda(e^z)$ have Laurent series expansions given by

$$W^0_\lambda(z) \;=\; \sum_{\ell=0}^{\infty} P_\ell(\lambda)\, \frac{z^\ell}{\ell!}\,, \qquad W_\lambda(z) \;=\; \sum_{k=0}^{\infty} Q_k(\lambda) z^{k-1} \tag{65}$$

and the relationship between X_λ and C_λ described above implies that $w_\lambda(t) = w^0_\lambda(t) + \frac{\sqrt{t}}{t-1}$ or $W_\lambda(z) = W^0_\lambda(z) + \frac{1/2}{\sinh(z/2)}$.

For later purposes we also introduce yet a third normalization, namely

$$p_\ell(\lambda) \;=\; P_\ell(\lambda) \,+\, (1-2^{-\ell})\,\zeta(-\ell) \;=\; \ell!\, Q_{\ell+1}(\lambda)\,. \tag{66}$$

This then agrees with the notation in [18] (whereas in [7] the symbol p_ℓ is used for what we call P_ℓ) and will be used in Sections 10 and 11.

Now let $\mathcal{R}$ be the ring $\mathbb{Q}[Q_1, Q_2, \ldots]$, with the grading $\mathcal{R} = \bigoplus \mathcal{R}_k$ given by assigning to Q_k the weight k. (It is in order to define this grading that we work with the Q_k rather than the P_ℓ.) To any element $f \in \mathcal{R}$ we associate a function on $\mathbf{P}$, denoted by the same letter, by setting $f(\lambda) = f(Q_1(\lambda), Q_2(\lambda), \ldots)$. By the result quoted above, this function lies in Λ^* and all elements of Λ^* arise this way. (The ring Λ^* is isomorphic to the quotient $\mathcal{R}_*/Q_1\mathcal{R}_*$.)

8. Quasimodular forms and the Bloch-Okounkov theorem

Let $f : \mathbf{P} \to \mathbb{Q}$ be an arbitrary function on the set $\mathbf{P}$ of all partitions. Motivated by the averaging operators encountered in classical statistical physics, Bloch and Okounkov associate to f the formal power series

$$\langle f \rangle_q \;=\; \frac{\sum_{\lambda\in\mathbf{P}} f(\lambda)\, q^{|\lambda|}}{\sum_{\lambda\in\mathbf{P}} q^{|\lambda|}} \;\in\; \mathbb{Q}[[q]]\,, \tag{67}$$

which we will call the *q-bracket*, and prove that this q-bracket is a quasimodular form whenever f belongs to Λ^*. More precisely, their theorem says:

Theorem 8.1 (Bloch-Okounkov)**.** *If f is a shifted symmetric function of weight k, then $\langle f \rangle_q$ is a quasimodular form of weight k.*

In view of the description of the grading given in the previous section, this says that if f is a weighted homogeneous polynomial of degree K in the functions Q_k, where Q_k has weight k, then $\langle f\rangle_q \in \widetilde{M}_K$. To calculate these q-brackets, it clearly suffices to calculate them for monomials $Q_{k_1}\cdots Q_{k_n}$. We therefore introduce the generating Laurent series

$$W(z) \;=\; \sum_{k=0}^{\infty} Q_k\, z^{k-1} \qquad \in\; \mathscr{R}[z^{-1},z]] \tag{68}$$

corresponding to the function $W_\lambda(z)$ in (65), and define the *n-point correlator*

$$\begin{aligned} F_n(z_1,\dots,z_n) \;&=\; \langle W(z_1)\cdots W(z_n)\rangle_q \\ &=\; \sum_{k_1,\dots,k_n\ge 0} \langle Q_{k_1}\cdots Q_{k_n}\rangle_q\, z_1^{k_1-1}\cdots z_n^{k_n-1} \end{aligned} \tag{69}$$

for each n. (Here the dependence on τ and $q=e^{2\pi i\tau}$ has been omitted from the notation on the left, and the subscript n could also be omitted, since it is simply equal to the number of variables.) Bloch and Okounkov give a beautiful identity to compute the functions F_n in terms of the Jacobi theta series

$$\theta(z) \;=\; \theta_\tau(z) \;=\; \sum_{\nu\in\mathbb{Z}+\frac12} (-1)^{[\nu]}\, e^{\nu z}\, q^{\nu^2/2} \qquad\in\; q^{1/8}\,\mathbb{Q}[[q]][[z]]\,, \tag{70}$$

the first three cases of this formula being given (with G_2 as in (72)) by

$$\begin{aligned} F_1(z_1) \;&=\; \frac{\theta'(0)}{\theta(z_1)}\,, \qquad F_2(z_1,z_2) \;=\; \frac{\theta'(0)}{\theta(z_1+z_2)}\,\mathrm{Sym}_2\Bigl(\frac{\theta'}{\theta}(z_1)\Bigr)\,, \\ F_3(z_1,z_2,z_3) \;&=\; \frac{\theta'(0)}{\theta(z_1+z_2+z_3)}\,\mathrm{Sym}_3\Bigl(\frac{\theta'}{\theta}(z_1)\,\frac{\theta'}{\theta}(z_1+z_2) - \frac{\theta''}{2\theta}(z_1) - G_2\Bigr)\,, \end{aligned} \tag{71}$$

where "Sym_n" denotes complete symmetrization of a function of n variables.

An elementary and very short proof of Theorem 8.1 is given in [49], together with several complementary results concerning the correlators F_n. Since several of these will be useful for us later, we list some of them here briefly. First, however, we begin by reviewing the definition and main properties of quasimodular forms.

We recall first that a modular form of weight k on the full modular group $\Gamma = \mathrm{SL}(2,\mathbb{Z})$ is a holomorphic function φ from the complex upper half-plane $\mathbb{H}$ to $\mathbb{C}$ satisfying $\varphi\bigl(\frac{a\tau+b}{c\tau+d}\bigr) = (c\tau+d)^k\phi(\tau)$ for all $\gamma = \bigl(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\bigr)\in\Gamma$, an example being the Eisenstein series

$$G_k(\tau) \;=\; -\,\frac{B_k}{2k} \;+\; \sum_{n=1}^{\infty}\sigma_{k-1}(n)\,q^n \qquad (k>0 \text{ even}), \tag{72}$$

if $k\ge 4$. (Here B_k denotes the k-th Bernoulli number and $\sigma_{k-1}(n)=\sum_{d|n} d^{k-1}$.) We denote by M_k the space of all modular forms of weight k on Γ and by $M_* = \bigoplus_k M_k$ the corresponding graded ring. For $k\ge 4$ we have $M_k = \mathbb{C}G_k\oplus S_k$, where S_k is the subspace of cusp forms (modular forms with no q^0 term). For k odd we have $M_k=0$ and set $G_k=0$.

A quasimodular form is defined by imposing only a weaker transformation law under the action of Γ, typical examples being G_2 and the derivatives of modular forms, but we can omit the intrinsic definition since it is known that the algebra $\widetilde{M}_*$ of all quasimodular forms on Γ is freely generated over M_* by the quasimodular form G_2 of weight 2. More explicitly, using Ramanujan's convenient notations P,

Q and, R (also often denoted by E_2, E_4, and E_6) for $-24G_2 = 1 - 24q - \cdots \in \widetilde{M}_2$, $240G_4 = 1 + 240q + \cdots \in M_4$, and $-504G_6 = 1 - 504q - \cdots \in M_6$, we have

$$M_* \;=\; \mathbb{Q}[Q,R]\,, \qquad \widetilde{M}_* \;=\; M_*[P] \;=\; \mathbb{Q}[P,\,Q,R]\,. \tag{73}$$

Examples of the first equality are $G_8 = Q^2/480$ and $G_{12} = (441Q^3 + 250R^2)/65520$. A basic fact is that the ring $\widetilde{M}_*$ is closed under the differentiation operator

$$D \;=\; \frac{1}{2\pi i}\,\frac{d}{d\tau} \;=\; q\,\frac{d}{dq}\,,$$

as can be seen either directly from the definition that we have omitted or else from the structure theorem (73) together with Ramanujan's formulas

$$D(P) \;=\; \tfrac{1}{12}\,(P^2 - Q), \quad D(Q) \;=\; \tfrac{1}{3}\,(PQ - R), \quad D(R) \;=\; \tfrac{1}{2}\,(PR - Q^2)\,. \tag{74}$$

The operator D acts on $\widetilde{M}_*$ as a derivation of degree $+2$ (i.e. it raises the weight of a quasimodular form by 2). Another important operator is the derivation $\mathfrak{d}$ of degree -2 defined in terms of the isomorphisms (73) as $12\,\partial/\partial P$. (There is also an intrinsic definition.) Together with D and the weight operator $\boldsymbol{W}$ sending f to kf for $f \in \widetilde{M}_k$ they span a Lie algebra of derivations of $\widetilde{M}_*$ isomorphic to $\mathfrak{sl}_2$, namely,

$$[\boldsymbol{W},\, D] \;=\; 2D\,, \qquad [\boldsymbol{W},\, \mathfrak{d}] \;=\; -2\mathfrak{d}\,, \qquad [\mathfrak{d},\, D] \;=\; \boldsymbol{W}\,, \tag{75}$$

where the first two equations simply say that D and $\mathfrak{d}$ have degree 2 and -2.

A collection of examples of quasimodular forms that will be important for us is given by the Taylor expansion

$$\theta(z) \;=\; \theta'(0)\sum_{n=0}^{\infty} H_n(\tau)\, z^{n+1} \tag{76}$$

of the Jacobi theta series (70), in which $\theta'(0) = \eta(\tau)^3$, where η is the Dedekind eta function defined by $\eta(\tau) = q^{1/24}\prod(1-q^n)$ or by $1728\eta(\tau)^{24} = Q^3 - R^2$. We have $H_n \in \widetilde{M}_n$ for all n, the first few values being $H_0 = 1$, $H_2 = P/24$, $H_4 = (5P^2 - 2Q)/5760$ (and of course $H_n = 0$ for n odd), and the later values being computable recursively by the formula

$$4n(n+1)H_n \;=\; 8D(H_{n-2}) \,+\, PH_{n-2}\,. \tag{77}$$

The expansion (76) is at the base of the proof of Theorem 8.1 given in [49]. More precisely, it is shown there by a very simple combinatorial argument that

$$\langle \theta(\partial) g\rangle_q \;=\; 0 \qquad \text{for all} \quad g \in \mathbb{Q}[Q_2, Q_3, \ldots] \subset \mathcal{R}\,, \tag{78}$$

where $\partial : \mathcal{R} \to \mathcal{R}$ is the derivation sending Q_k to Q_{k-1}; then the quasimodularity of the Taylor coefficients of θ together with an easy induction on the weight suffices to prove the quasimodularity of $\langle f\rangle_q$ for all $f \in \mathcal{R}$. The identity (78) is also used to prove the inductive formula (equivalent to the explicit formulas in Bloch-Okounkov)

$$\sum_{J\subseteq N} (-1)^{n-|J|}\,\theta^{(n-|J|)}\big(z_J\big)\, F_{|J|}\big(\mathfrak{z}_J\big) \;=\; 0 \qquad (n \geq 1) \tag{79}$$

for the correlators, where $N = \{1, \ldots, n\}$ and for any subset $J \subseteq N$ (including the empty set) we denote by $\mathfrak{z}_J$ and z_J the set of z_j with $j \in J$ and the sum of these elements, respectively. Yet a third equivalent version given in [49] is the following axiomatic characterization of the correlators, in which the symbol $[G]^+$ in the final axiom denotes the strictly-positive-exponent part of a Laurent series G in several variables:

Theorem 8.2. *The Bloch-Okounkov correlators $F_n(z_1,\dots,z_n)$ $(n\ge 0)$ are the unique Laurent series satisfying:*

(i) $F_0(\,)\;=\;1\,.$

(ii) $F_n(z_1,\dots,z_n)$ *is symmetric in all* n *arguments.*

(iii) $F_n(z_1,\dots,z_n)\;=\;\dfrac{1}{z_n}\,F_{n-1}(z_1,\dots,z_{n-1})\;+\;\mathrm{O}(z_n)$ *as* $z_n\to 0\,.$

(iv) $\bigl[\theta(z_1+\cdots+z_n)\,F_n(z_1,\dots,z_n)\bigr]^+\;=\;0$ *for all* $n\ge 0\,.$

An important aspect of the Bloch-Okounkov map $\langle\ \rangle_q:\mathcal{R}_*\to\widetilde{M}_*$, which will be used several times in the sequel, is its relation to the $\mathfrak{sl}_2$-action on $\widetilde{M}_*$ as defined above. For two of the generators of $\mathfrak{sl}_2$ this is easy: since $\langle\ \rangle_q:\mathcal{R}_*\to\widetilde{M}_*$ preserves the grading we have $\boldsymbol{W}\langle f\rangle_q=\langle Ef\rangle_q$, where $E=\sum Q_k\partial/\partial Q_k$ is the Euler operator, and we also have the formula

$$D\,\langle f\rangle_q\;=\;\langle Q_2\,f\rangle_q\;+\;\frac{P}{24}\,\langle f\rangle_q\qquad(f\in\mathcal{R})\tag{80}$$

as an immediate consequence of the definition (67) and the formulas $Q_2(\lambda)=|\lambda|-\frac{1}{24}$ and $D(\eta)=P\eta/24$. The action of the third generator $\mathfrak{d}$ of $\mathfrak{sl}_2$, which is much harder to compute, was found in [49] and is given as follows.

Proposition 8.3. *The action of the derivation* $\mathfrak{d}:\widetilde{M}_*\to\widetilde{M}_{*-2}$ *as defined above on* q*-brackets is given by*

$$\mathfrak{d}\,\langle f\rangle_q\;=\;\bigl\langle\tfrac12\bigl(\Delta\,-\,\partial^2\bigr)\,f\bigr\rangle_q\qquad(f\in\mathcal{R})\,,\tag{81}$$

where $\Delta:\mathcal{R}\to\mathcal{R}$ *is the second order differential operator of degree* -2 *defined by*

$$\Delta\;=\;\sum_{k,\,\ell\,\ge\,0}\binom{k+\ell}{k}\,Q_{k+l}\,\frac{\partial^2}{\partial Q_{k+1}\,\partial Q_{\ell+1}}\tag{82}$$

and ∂ *the derivation defined in* (78)*. Moreover, the actions of* Δ *and* ∂ *commute.*

Proof. The first statement is Theorem 3 of [49] and the second is an easy consequence of the definitions of ∂ and Δ using $\binom{k+\ell}{k}=\binom{k+\ell-1}{k}+\binom{k+\ell-1}{\ell}$. ☐

A corollary of this proposition ([49, Theorem 2]) is that, if we define the "top coefficient" $\mathbf{T}(F)$ of a quasimodular form F of weight $2n$ as the coefficient of P^n in the expression of F as a polynomial in P, Q, and R, then

$$\mathbf{T}(\langle f\rangle_q)\;=\;-\frac{(2n-3)!!}{(-12)^n}\,\mu(f)\qquad\text{for all } f\in\mathcal{R}_{2n},\tag{83}$$

where $(2n-3)!!:=1\times3\times\cdots\times(2n-3)$ (resp. $(-1)!!=1$, $(-3)!!=-1$) and $\mu:\mathcal{R}\to\mathbb{Q}$ is the ring homomorphism sending Q_n to $(1-n)/n!$ for every $n\ge0$.

To illustrate the statement of the Bloch-Okounkov theorem we end this section by giving a short list of the q-brackets of all monomials in the Q's of even weight ≤ 6:

$$\langle Q_2\rangle_q\;=\;\frac{-P}{24}\,,\qquad\langle Q_2^2\rangle_q\;=\;\frac{-P^2+2Q}{576}\,,\qquad\langle Q_4\rangle_q\;=\;\frac{5P^2+2Q}{5760}\,,$$
$$\langle Q_2^3\rangle_q\;=\;\frac{-3P^3+18QP-16R}{13824}\,,\qquad\langle Q_2Q_4\rangle_q\;=\;\frac{15P^3-6QP-16R}{138240}\,,$$
$$\langle Q_3^2\rangle_q\;=\;\frac{5P^3-3QP-2R}{25920}\,,\qquad\langle Q_6\rangle_q\;=\;\frac{-35P^3-42QP-16R}{2903040}\,.$$

A slightly longer list, up to weight 8, can be found in [49].

9. The growth polynomials of quasimodular forms

In this section we introduce a polynomial (actually two polynomials, related to each other by a simple transformation) that describes the growth of a quasimodular form $F(\tau)$ near $\tau = 0$ and at the same time the average growth of its Fourier coefficients. In the following section this polynomial will be computed for the image of the Bloch-Okounkov map. The latter calculation is equivalent to a result of Eskin and Okounkov ([18]), of which we will then be able to give simpler alternative proofs, and the idea of considering the asymptotic growth of quasimodular forms near the origin is already contained in their work, but not explicitly worked out in this generality. Since the construction is very natural and will undoubtedly be useful also in other situations involving quasimodular forms, we present it here in fair detail, including some further properties. The map assigning to a quasimodular form its growth polynomial is a ring homomorphism that can be thought of as a kind of polynomial evaluation map, and we will denote the two versions of this map by the symbols Ev and ev.

For a quasimodular form $F \in \widetilde{M}$ we write $F(\infty)$ ($:= \lim_{\tau\to i\infty} F(\tau) = a_0(F)$, where $F(\tau) = \sum_{n=0}^{\infty} a_n(F)\, q^n$ is the Fourier expansion of F) for the constant term. We write $E_k \in \widetilde{M}_k$ for the normalized Eisenstein series $G_k/G_k(\infty)$ for $k \in 2\mathbb{N}$ (so $E_2 = P$, $E_4 = Q$, and $E_6 = R$ are the generators of the algebra $\widetilde{M}_*$), and set $E_0 = 1$ and $E_k = 0$ for k odd. As before we write Df or f' for the derivative $\frac{1}{2\pi i}\frac{df}{d\tau}$ of $f \in \widetilde{M}_*$ and use the notations $f^{(r)}$ and $D^r(f)$ interchangeably. The space $\widetilde{M}_*$ of quasimodular forms with coefficients in $\mathbb{Q}$ is the direct sum of the subspace **DE** spanned by all derivatives of all Eisenstein series E_k and the subspace **DS** spanned by all derivatives of all cusp forms. We can therefore define a linear map $\mathrm{Ev} : \widetilde{M}_* \to \mathbb{Q}[X]$ by setting

$$\mathrm{Ev}[F] \,=\, 0 \text{ for } F \in \mathbf{DS}, \qquad \mathrm{Ev}\big[E_{2\ell}^{(r)}\big](X) \,=\, \begin{cases} \delta_{r,0} & \text{if } \ell = 0, \\ (r+1)!\,X \,+\, 12\,r! & \text{if } \ell = 1, \\ \frac{(r+2\ell-1)!}{(2\ell-1)!}\,X^\ell & \text{if } \ell \geq 2. \end{cases}$$

Presented like this, the definition looks somewhat unnatural, but in fact the map Ev has very nice properties, as given in the next five propositions.

Proposition 9.1. *The map* Ev *is the algebra homomorphism from* $\widetilde{M}_*$ *to* $\mathbb{Q}[X]$ *sending* E_2 *to* $X + 12$, E_4 *to* X^2, *and* E_6 *to* X^3.

Proof. It is clear by induction on r that Ev is characterized axiomatically by the three properties

(i) $\mathrm{Ev}[f](X) \,=\, a_0(f)X^k$ for $f \in M_{2k}$;
(ii) $\mathrm{Ev}[E_2](X) \,=\, X \,+\, 12$;
(iii) $\mathrm{Ev}[DF] \,=\, \big(X\frac{d}{dX} + k\big)\mathrm{Ev}[F]$ for $F \in \widetilde{M}_{2k}$.

It therefore suffices to show that the algebra homomorphism $\Phi : \widetilde{M}_* \to \mathbb{Q}[X]$ defined by $E_2 \mapsto X + 12$, $E_4 \mapsto X^2$, $E_6 \mapsto X^3$ has the same three properties. The first one is obvious since it holds for the generators E_4 and E_6 of the ring M_* and since $f \mapsto a_0(f)X^{\mathrm{wt}(f)/2}$ is a ring homomorphism, and the second is true by definition. For the third, we have to check the commutativity of the diagram

$$\begin{array}{ccc} \widetilde{M}_* & \xrightarrow{\mathrm{Ev}} & \mathbb{Q}[X] \\ {\scriptstyle D-H}\big\downarrow & & \big\downarrow{\scriptstyle X\frac{d}{dX}} \\ \widetilde{M}_* & \xrightarrow{\mathrm{Ev}} & \mathbb{Q}[X] \end{array}$$

where $H:\widetilde{M}_*\to\widetilde{M}_*$ is the operator sending $F\in\widetilde{M}_{2k}$ to kF. This commutativity follows for the generators P, Q, R from Ramanujan's formulas (74), since

$$\begin{aligned} P &\overset{D-H}{\mapsto} \frac{P^2-Q}{12}-P \overset{\Phi}{\mapsto} \frac{(X+12)^2-X^2}{12} - (X+12) \;=\; X \;=X\frac{d}{dX}\Phi(P)\,,\\ Q &\overset{D-H}{\mapsto} \frac{PQ-R}{3}-2Q \overset{\Phi}{\mapsto} \frac{(X+12)X^2-X^3}{3} - 2X^2 \;=\; 2X^2 \;=X\frac{d}{dX}\Phi(Q)\,,\\ R &\overset{D-H}{\mapsto} \frac{PR-Q^2}{2}-3R \overset{\Phi}{\mapsto} \frac{(X+12)X^3-X^4}{2} - 3X^3 \;=\; 3X^3 \;=X\frac{d}{dX}\Phi(R)\,, \end{aligned}$$

and then holds in general because the horizontal maps in the diagram are ring homomorphisms and the vertical maps are derivations. □

The next proposition expresses the map $\mathrm{Ev}:\widetilde{M}_*\to\mathbb{Q}[X]$ explicitly in terms of the action of the Lie algebra $\mathfrak{sl}_2=\langle \boldsymbol{W},D,\mathfrak{d}\rangle$ on $\widetilde{M}_*$ described in (75) and the constant term map $a_0: F\mapsto F(\infty)$ from $\widetilde{M}_*$ to $\mathbb{Q}$.

Proposition 9.2. *For any $F\in\widetilde{M}_*$ we have $\mathrm{Ev}[F](X)=a_0\big(X^{\boldsymbol{W}/2}e^{\mathfrak{d}}F\big)$.*

Proof. This is a corollary of Proposition 9.1 since the maps $e^{\mathfrak{d}}$, $X^{\boldsymbol{W}/2}$, and a_0 are all algebra homomorphisms (because $\mathfrak{d}$ and $\boldsymbol{W}$ are derivations) and since the identity in question holds by inspection for the generators P, Q, and R of $\widetilde{M}_*$. Note that $X^{\boldsymbol{W}/2}e^{\mathfrak{d}}$ can also be written as $e^{\mathfrak{d}/X}X^{\boldsymbol{W}/2}$. □

The third proposition relates $\mathrm{Ev}[F]$ directly to the behavior of $F(\tau)$ as $\tau\to 0$.

Proposition 9.3. *For $F\in\widetilde{M}_{2k}$ the polynomial $\mathrm{Ev}[F](X)$ describes the asymptotic behavior of $F(\tau)$ near the cusp $\tau=0$. More precisely, we have*

$$F\big(i\varepsilon\big) \;=\; \frac{1}{(2\pi\varepsilon)^k}\,\mathrm{Ev}[F]\Big(-\frac{2\pi}{\varepsilon}\Big) \;+\; (\text{small}) \qquad \big(\varepsilon\searrow 0\big), \tag{84}$$

where “small” *means terms that tend exponentially quickly to* 0 *as* ε *tends to* 0.

Proof. Suppose first that $F=f^{(r)}$ with $f\in M_{2\ell}$. The modular transformation property $f(-1/\tau)=\tau^{2\ell}f(\tau)$ gives

$$f(i\varepsilon) \;=\; \frac{(-1)^\ell}{\varepsilon^{2\ell}}\sum_{n=0}^{\infty}a_n(f)\,e^{-2\pi n/\varepsilon}$$

and hence

$$\begin{aligned} F(i\varepsilon) \;&=\; \Big(-\frac{1}{2\pi}\frac{d}{d\varepsilon}\Big)^r f(i\varepsilon) \;=\; \frac{(-1)^\ell}{(2\pi)^r}\,\frac{(r+2\ell-1)!}{(2\ell-1)!}\,\frac{a_0(f)}{\varepsilon^{2\ell+r}} \;+\; \mathrm{O}\big(\varepsilon^{-2\ell-r}e^{-2\pi/\varepsilon}\big)\\ &=\; \frac{(r+2\ell-1)!}{(2\ell-1)!}\,a_0(f)\,\frac{(-2\pi/\varepsilon)^\ell}{(2\pi\varepsilon)^{r+\ell}} \;+\; (\text{small})\,, \end{aligned}$$

confirming the statement in this case. If $F = E_2^{(r)}$, then the modular transformation property $E_2(-1/\tau) = \tau^2 E_2(\tau) + 6\tau/\pi i$ gives

$$E_2(i\varepsilon) \;=\; -\,\frac{1}{\varepsilon^2} \,+\, \frac{6}{\pi\varepsilon} \,+\, \frac{24}{\varepsilon^2}\sum_{n=1}^{\infty}\sigma_1(n)\,e^{-2\pi n/\varepsilon}$$

and hence

$$F(i\varepsilon) \;=\; \Bigl(-\frac{1}{2\pi}\,\frac{d}{d\varepsilon}\Bigr)^r E_2(i\varepsilon) \;=\; (r+1)!\,\frac{-2\pi/\varepsilon}{(2\pi\varepsilon)^{r+1}} \,+\, \frac{12\,r!}{(2\pi\varepsilon)^{r+1}} \,+\, (\text{small})\,,$$

again in accordance with the statement of the proposition. Since $\widetilde{M}_*$ is spanned by the derivatives of modular forms and of E_2, this completes the proof. □

Note that Proposition 9.3 also gives an alternative proof of Proposition 9.1, since sending a function to its asymptotic development near 0 is obviously a ring homomorphism. We nevertheless preferred to give an independent and purely algebraic proof to emphasize the axiomatic description of the map Ev and in particular its relation to differentiation.

As already mentioned, for the applications to the quasimodular forms coming from the Bloch-Okounkov theorem, and for the results of Eskin and Okounkov that we will reprove and generalize in the next section, it is convenient to work with a different normalization of the growth polynomials that we now introduce. Replace the variable ε in the proposition by $\hbar = h/2\pi$ (where the letters h and $\hbar$ are meant to suggest Planck's constant and quantum mechanics). Then if we define

$$\mathrm{ev}[F](h) \;=\; \frac{1}{h^k}\,\mathrm{Ev}[F]\Bigl(-\frac{4\pi^2}{h}\Bigr) \;\in\; \mathbb{Q}[\pi^2][1/h] \tag{85}$$

for $F \in \widetilde{M}_{2k}$, the statement of Proposition 9.3 is that $F(\tau)$ equals $\mathrm{ev}[F](h)$ plus exponentially small terms as $q = e^{2\pi i\tau} = e^{-h}$ tends to 1. Although the two polynomials $\mathrm{Ev}[F]$ and $\mathrm{ev}[F]$ are equivalent, it is useful to retain both versions because each has convenient properties: the former because it has rational coefficients and no extraneous powers of the variable, and the latter because it describes the growth of $F(\tau)$ near $\tau = 0$ directly. We will refer to both $\mathrm{Ev}[F]$ and $\mathrm{ev}[F]$ as the *growth polynomials* of the quasimodular form F. This terminology is justified not only by Proposition 9.3, relating these polynomials to the growth of $F(\tau)$ near $\tau = 0$, but also by the following result, which says that their leading terms determine the average asymptotic growth of the Fourier coefficients of F.

Proposition 9.4. *Let F be a homogeneous element of $\widetilde{M}_*$ satisfying $\mathrm{ev}[F] = Ah^{-p} + \mathrm{O}(h^{1-p})$ as $h \to 0$ for some integer $p \geq 0$ and constant $A \neq 0$. Then the sum of the first N Fourier coefficients of F has the asymptotic behavior*

$$\sum_{n=1}^{N} a_n(F) \;=\; A\,\frac{N^p}{p!} \,+\, \mathrm{O}\bigl(N^{p-1}\log N\bigr) \qquad (N\to\infty)\,. \tag{86}$$

Proof. Let the weight of F be $2k$. If $k = 0$ then there is nothing to prove, since $a_n(F) = 0$ for all $n \geq 1$. If $k \geq 1$, then we can write F as a linear combination of derivatives $D^{k-\ell}G_{2\ell}$ and $D^{k-\ell}f_\ell$ with $1 \leq \ell \leq k$, where $G_{2\ell}$ is the Eisenstein series and f_ℓ a cusp form of weight 2ℓ. Since we are assuming that $\mathrm{ev}[F]$ is not identically zero, there is at least one Eisenstein contribution, and since the degree of $\mathrm{ev}(D^{k-\ell}G_{2\ell})$ in h^{-1} is $k+\ell$, it follows that $k+1 \leq p \leq 2k$. The terms $D^{k-\ell}f_\ell$ do not affect the estimate (86), since by a result of Hafner and Ivić ([23]) we have $\sum_{n\le N} a_n(f_\ell) = \mathrm{O}(N^{\ell-\frac{1}{6}})$ (the weaker estimate $\mathrm{O}(N^\ell \log N)$ would be enough for our purposes) and by partial summation we deduce that $\sum_{n\le N} a_n(D^{k-\ell} f_\ell) = \sum_{n\le N} n^{k-\ell} a_n(f_\ell) = \mathrm{O}(N^{k-\frac{1}{6}})$, which gets absorbed into the error term in (86) since $k \le p-1$. It therefore suffices to consider the case $F = G_{2\ell}^{(r)}$ with $\ell \ge 1$, $r \ge 0$, $k = \ell + r$. For this form we have from the original definition of the growth polynomial the formula

$$\begin{aligned}\mathrm{Ev}\big[G_{2\ell}^{(r)}\big] \;&=\; -\,\frac{B_{2\ell}}{4\ell}\,\frac{(r+2\ell-1)!}{(2\ell-1)!}\,X^\ell \;-\; \frac{r!}{2}\,\delta_{\ell,1}\\ &=\; (r+2\ell-1)!\,\frac{\zeta(2\ell)}{(2\pi i)^{2\ell}}\,X^\ell \;-\; \frac{r!}{2}\,\delta_{\ell,1}\,,\end{aligned}$$

which we can rewrite in terms of ev as

$$\mathrm{ev}\big[G_{2\ell}^{(r)}\big] \;=\; (r+2\ell-1)!\,\frac{\zeta(2\ell)}{h^{r+2\ell}} \;-\; r!\,\frac{\delta_{\ell,1}}{2h^{r+1}}\,, \tag{87}$$

and on the other hand

$$\begin{aligned}\sum_{n=1}^{N} a_n\big(G_{2\ell}^{(r)}\big) \;&=\; \sum_{n=1}^{N} n^r\,\sigma_{2\ell-1}(n) \;=\; \sum_{\substack{a,\,b\ge1\\ ab\le N}} a^{r+2\ell-1} b^r\\ &=\; \sum_{b=1}^{N} b^r\left(\frac{(N/b)^{r+2\ell}}{r+2\ell} \;+\; \mathrm{O}\big((N/b)^{r+2\ell-1}\big)\right)\\ &=\; \zeta(2\ell)\,\frac{N^{r+2\ell}}{r+2\ell} \;+\; \mathrm{O}\big(N^{r+2\ell-1}\log N\big)\,.\end{aligned}$$

(Here the "$\log N$" factor is needed only for $\ell = 1$.) This confirms (86) in this case and hence also in general. □

Our final statement about the growth polynomials associated to a quasimodular form F is that the number of monomials they contain equals the number of poles of the meromorphic continuation of the L-series $L(s,F) = \sum_{n=1}^{\infty} a_n(F) n^{-s}$, with the corresponding exponents and coefficients corresponding to the positions and residues of these poles.

Proposition 9.5. *Let F be a quasimodular form of weight $2k$. Then the L-series of F has a meromorphic continuation to the whole complex plane, with at most simple poles at $s = k, \dots, 2k$ as its only singularities, and the growth polynomial $\mathrm{ev}[F]$ of F is given in terms of the residues of $L(s,F)$ by the formula*

$$\mathrm{ev}[F](h) \;=\; \sum_{m=k}^{2k} (m-1)!\,\mathrm{Res}_{s=m}\big[L(s,F)\big]\,h^{-m}\,. \tag{88}$$

Proof. Again we verify this by looking at the cases of derivatives of cusp forms and of Eisenstein series separately. For the first case the assertion is trivial, since if $F = f^{(r)}$ for some cusp form f then $L(s,F) = L(s-r,f)$ extends to an entire function of s and the polynomial $\mathrm{Ev}(F)$ vanishes identically. If $F = G_{2\ell}^{(r)}$, then we have $L(F,s) = L(s-r,G_{2\ell}) = \zeta(s-r)\,\zeta(s-r-2\ell+1)$, which extends to a meromorphic function having only simple poles, one at $s = r+2\ell$ with residue $\zeta(2\ell)$ and a second one at $s = r+1$ with residue $-\frac{1}{2}$ if $\ell = 1$, so that equation (88) agrees with equation (87). □

It is perhaps worth noting that an alternative proof of Proposition 9.5 could be given using Proposition 9.3, since if $F(it) = \sum_{m=1}^{M} c_m t^{-m} + \mathrm{O}(t^N)$ for t small, where M is fixed and N can be chosen arbitarily large, then we have (initially for s with sufficiently large real part)

$$\begin{aligned}(2\pi)^{-s}\Gamma(s)\, L(s,F) \;&=\; \int_0^\infty \bigl(F(it) - a_0(F)\bigr)\, t^{s-1} dt \\ &=\; \int_0^{t_0} \Biggl(\sum_{m=1}^{M} c_m t^{-m} - a_0(F) + \mathrm{O}(t^N)\Biggr)\, t^{s-1}\, dt \;+\; \int_{t_0}^\infty \mathrm{O}(e^{-2\pi t})\, t^{s-1} dt \\ &=\; \sum_{m=1}^{M} \frac{c_m}{s-m} \;-\; \frac{a_0(F)}{s} \;+\; \bigl(\text{holomorphic for } \Re(s) > -N\bigr)\,,\end{aligned}$$

giving a meromorphic continuation of $L(F,s)$ to the whole complex plane with simple poles of residue $(2\pi)^m c_m/(m-1)!$ at integers $m \geq 1$ and no other poles. Proposition 9.5 also explains where Proposition 9.4 comes from, using the standard expression for $\sum_{n=1}^{N} a_n(F)$ as $\frac{1}{2\pi i}\int_{C-i\infty}^{C+i\infty} L(F,s)\, N^s\, \frac{ds}{s}$ for C sufficiently large and then shifting the path of integration to the left to pick up a residue from the rightmost pole of $L(F,s)$ and using the functional equation of the L-series and the Phragmén-Lindelöf theorem to estimate the integrand on the shifted contour. We omit the details.

We end this section with a simple illustrative example.

Proposition 9.6. *The growth polynomial of the quasimodular form $H_{2k} \in \widetilde{M}_{2k}$ defined by* (76) *is given by*

$$\mathrm{Ev}[H_{2k}](X) \;=\; \sum_{\substack{m,\,n\geq 0\\ m+n=k}} \frac{1}{2^m m!}\, \frac{(X/4)^n}{(2n+1)!}\,. \tag{89}$$

Proof. We give two proofs of equation (89), to illustrate the use of the different properties of growth polynomials. Write h_{2k} for $\mathrm{Ev}[H_{2k}]$. Then the recursion (77) and the differentiation property (iii) in the proof of Proposition 9.1 give

$$k\,(2k+1)\, h_{2k}(X) \;=\; \Bigl(X\,\frac{d}{dX} \;+\; k-1 \;+\; \frac{X+12}{8}\Bigr)\, h_{2k-2}(X)\,,$$

and (89) follows easily by induction on k starting with the value $h_0(X) = 1$. Alternatively, from equation (76) and Proposition 9.3 we have

$$\frac{\theta_{i\varepsilon}(z)}{z\,\theta'_{i\varepsilon}(0)} \;=\; \sum_{k=0}^{\infty} H_{2k}(i\varepsilon)\, z^{2k} \;=\; \sum_{k=0}^{\infty} h_{2k}\Bigl(-\frac{2\pi}{\varepsilon}\Bigr)\, \frac{z^{2k}}{(2\pi\varepsilon)^k} \;+\; (\text{small})\,,$$

where "small" denotes terms decreasing faster than any power of ε, and since from the modular transformation property of θ we have

$$\frac{\theta_{i\varepsilon}(z)}{z\,\theta'_{i\varepsilon}(0)} \;=\; e^{z^2/4\pi\varepsilon}\, \frac{\theta_{i/\varepsilon}(iz/\varepsilon)}{iz/\varepsilon \cdot \theta'_{i/\varepsilon}(0)} \;=\; e^{z^2/4\pi\varepsilon}\, \frac{\sin(z/2\varepsilon)}{z/2\varepsilon} \;+\; (\text{small})\,,$$

we obtain a second proof of (89) by comparing the coefficients of z^{2k}. □

The second proof above gives the generating series for the h_{2k} explicitly:

$$\sum_{k=0}^{\infty} h_{2k}(X)\, z^{2k+1} \;=\; \mathrm{Ev}\Big[\frac{\theta(z)}{\theta'(0)}\Big](X) \;=\; e^{z^2/2}\,\frac{\sinh(z\sqrt{X}/2)}{\sqrt{X}/2} \tag{90}$$
$$=\; z \,+\, \Big(\frac{X}{4}+3\Big)\frac{z^3}{3!} \,+\, \Big(\frac{X^2}{16}+\frac{5X}{2}+15\Big)\frac{z^5}{5!} \,+\, \Big(\frac{X^3}{64}+\frac{21X^2}{16}+\frac{105X}{4}\Big)\frac{z^7}{7!} \,+\, \cdots\,.$$

10. The growth polynomials of q-brackets

In this section we will consider the growth polynomials of q-brackets, for which we use the notations $\langle f\rangle_X := \mathrm{Ev}[\langle f\rangle_q](X)$ and $\langle f\rangle_h := \mathrm{ev}\big[\langle f\rangle_q\big](h)$ ($f\in\mathcal{R}$) and the terminology *X-brackets* and *h-brackets*, respectively. It turns out that, whereas there is no really practical "closed formula" for the q-brackets of arbitrary elements of $\mathcal{R}$, there *is* such a formula for their growth polynomials. In fact, there are two, each in terms of a suitable generating function. One of them, which is due to Eskin and Okounkov ([18]) but of which we will give a simpler proof and also a slight refinement, gives the growth polynomial $F(z_1,\dots,z_n)_X := \mathrm{Ev}[F(z_1,\dots,z_n)](X)$ of the correlator function (69) for each integer $n\ge 1$. The other, which we will state as Theorem 10.2 and which is the principal result of this section, gives all of the X-brackets simultaneously as a single generating function in infinitely many variables ("partition function") that we express as a one-dimensional formal Gaussian integral.

To motivate these formulas, we first look at small values of n. For $n=1$ we find from the first of equations (71) together with equation (90) the result

$$F_1(z)_X \;=\; \frac{x\,e^{-z^2/2}}{\sinh xz} \qquad (x := \sqrt{X}/2)\,,$$

and similarly for $n=2$ the second of equations (71) together with (90) and the addition law for the hyperbolic sine function give

$$\begin{aligned} F_2(z_1,z_2)_X \;&=\; \frac{x\,e^{-(z_1+z_2)^2/2}}{\sinh x(z_1+z_2)}\Big(z_1+\frac{x}{\tanh xz_1}+z_2+\frac{x}{\tanh xz_2}\Big)\\ &=\; e^{-z_{12}^2/2}\Big(\frac{xz_{12}}{\sinh xz_{12}} \,+\, \frac{x}{\sinh xz_1}\,\frac{x}{\sinh xz_2}\Big) \qquad (z_{12} := z_1+z_2)\,,\end{aligned}$$

while for $n=3$ a similar calculation using the third of equations (71) gives

$$\begin{aligned} F_3(z_1,z_2,z_3)_X \;&=\; \frac{x\,e^{-z_{123}^2/2}}{\sinh xz_{123}}\,\mathrm{Sym}_3\Big[\Big(z_1+\frac{x}{\tanh xz_1}\Big)\Big(z_{12}+\frac{x}{\tanh xz_{12}}\Big)\\ &\qquad\qquad -\,\Big(\frac{1+x^2+z_1^2}{2}\,+\,\frac{xz_1}{\tanh xz_1}\Big)\,+\,\frac{4x^2+12}{24}\Big]\\ &=\; e^{-z_{123}^2/2}\Big(\frac{xz_{123}^2}{\sinh xz_{123}} \,+\, \frac{x}{\sinh xz_1}\,\frac{xz_{23}}{\sinh xz_{23}} \,+\, \frac{x}{\sinh xz_2}\,\frac{xz_{13}}{\sinh xz_{13}}\\ &\qquad\qquad +\, \frac{x}{\sinh xz_3}\,\frac{xz_{12}}{\sinh xz_{12}} \,+\, \frac{x}{\sinh xz_1}\,\frac{x}{\sinh xz_2}\,\frac{x}{\sinh xz_3}\Big)\end{aligned}$$

with $z_{123} := z_1+z_2+z_3$ etc. These special cases suggest the following result in which, as in Section 6, $\mathcal{P}(n)$ denotes the set of unordered partitions of the set $\{1,\dots,n\}$.

Theorem 10.1 ([18], Theorem 4.7)**.** *The X-evaluation of the n-point Bloch-Okounkov correlator is given by*

$$F_n\left(z_1,\ldots,z_n\right)_X \;=\; e^{-z_N^2/2}\sum_{\alpha\in\mathcal{P}(n)}\prod_{A\in\alpha}\frac{z_A^{|A|-1}\sqrt{X}/2}{\sinh(z_A\sqrt{X}/2)}\,, \tag{91}$$

where $N=\{1,\ldots,n\}$ and $z_A=\sum_{a\in A}z_a$ for $A\subseteq N$.

Theorem 10.1, which we will prove below, gives a formula for the growth polynomial of the Bloch-Okounkov correlator functions $F_n(z_1,\ldots,z_n)$ defined in (69), and thus for the q-bracket of a product of n generators Q_k of $\mathcal{R}$ for a fixed value of n. It turns out that these formulas can be expressed more simply, and in a way that is better suited for our applications, if we organize them into a different kind of generating function, namely the *partition function*

$$\begin{aligned}\Phi(\mathbf{u})_q \;&=\; \Big\langle\exp\Big(\sum_{\ell\geq1}p_\ell u_\ell\Big)\Big\rangle_q \;=\; \sum_{\mathbf{n}\geq0}\langle\underbrace{p_1,\ldots,p_1}_{n_1},\underbrace{p_2,\ldots,p_2}_{n_2},\ldots\rangle_q\,\frac{\mathbf{u}^{\mathbf{n}}}{\mathbf{n}!}\\ &=\;\sum_{n=0}^{\infty}\frac{1}{n!}\sum_{\ell_1,\ldots,\ell_n\geq1}\langle p_{\ell_1}\cdots p_{\ell_n}\rangle_q\,u_{\ell_1}\cdots u_{\ell_n}\,.\end{aligned} \tag{92}$$

Here $p_\ell=\ell!\,Q_{\ell+1}$ as in (66) and we have used standard multi-variable notation: $\mathbf{u}=(u_1,u_2,\cdots)$ denotes a tuple of countably many independent variables u_i and $\mathbf{n}\geq0$ denotes a multi-index $\mathbf{n}=(n_1,n_2,\cdots)$ of non-negative integers n_i, with $\mathbf{u}^{\mathbf{n}}=\prod_{m\geq0}u_m^{n_m}$ and $\mathbf{n}!=\prod_{m\geq0}n_m!\,$. The definition (92) can be compared to Witten's generating function for intersection numbers of ψ-classes on the moduli spaces $\overline{\mathcal{M}}_{g,n}$:

$$\Phi_{\text{Witten}}(u_0,u_1,\ldots)\;=\;\sum_{n=0}^{\infty}\frac{1}{n!}\sum_{m_1,\ldots,m_n\geq0}\langle\tau_{m_1}\cdots\tau_{m_n}\rangle\,u_{m_1}\cdots u_{m_n}$$

with

$$\langle\tau_{m_1}\cdots\tau_{m_n}\rangle\;=\;\int_{\overline{\mathcal{M}}_{g,n}}\psi_1^{m_1}\cdots\psi_n^{m_n}\,,$$

in which the formal variables u_i are also attached to the number of occurences of ψ_i in the product rather than to the index of the marked point.

Our main result below gives an explicit formula for the growth polynomial generating function $\Phi(\mathbf{u})_X:=\mathrm{Ev}[\Phi(\mathbf{u})_q](X)$. To state it, we introduce an auxiliary generating function defined by

$$\mathcal{B}(\mathbf{u},y,X)\;=\;\sum_{\substack{\mathbf{a}>0\\ r\geq0}}(a_1+2a_2+3a_3+\cdots)!\;\beta_{2-r+w(\mathbf{a})}\sqrt{X}^{2-r+w(\mathbf{a})}\,\frac{\mathbf{u}^{\mathbf{a}}}{\mathbf{a}!}\,\frac{y^r}{r!}\,, \tag{93}$$

with β_m as in (63) and $w(\mathbf{a})=a_2+2a_3+3a_4+\cdots$. (Alternative and simpler expressions for $\mathcal{B}$ are given in equations (103) and (104) below.) Note that the exponents of X are all non-negative and integral, since $\beta_k=0$ for $k<0$ or k odd, and also that the coefficient of each monomial in X and y contains only finitely many monomials in the u_i, so that $\mathcal{B}(\mathbf{u},y,X)$ belongs to $\mathbb{Q}[\mathbf{u}][[y,X]]$.

Theorem 10.2. *The growth coefficient polynomial of the generating function $\Phi(\mathbf{u})_q$ can be expressed as the formal Gaussian integral*

$$\Phi(\mathbf{u})_X \;=\; \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-y^2/2\,+\,\mathcal{B}(\mathbf{u},iy,X)}\,dy\,. \tag{94}$$

Note that the expression on the right hand side of (94) is purely algebraic and does not really involve integration, since we can state the theorem equivalently as

$$\Phi(\mathbf{u})_X \;=\; \mathfrak{I}\big[e^{\mathcal{B}(\mathbf{u},y,X)}\big] \quad \in\; \mathbb{Q}[X][[\mathbf{u}]]\,, \tag{95}$$

where $\mathfrak{I}$ is the functional on power series in y defined on monomials by

$$\mathfrak{I}\big[y^n\big] \;=\; \begin{cases} (-1)^{n/2}(n-1)!! & \text{for } n \text{ even,} \\ 0 & \text{for } n \text{ odd.} \end{cases} \tag{96}$$

Equation (95) makes sense because the terms of $\mathcal{B}$ all have strictly positive degree in the u_i and the coefficient of any monomial $u_1^{\ell_1}u_2^{\ell_2}\cdots$ in $\mathcal{B}$, and hence also in $e^{\mathcal{B}}$, is a polynomial in X and y to which the functional $\mathfrak{I}$ can be applied to get a polynomial in X.

We now prove Theorems 10.1 and 10.2. Our proof of the former will use the axiomatic characterization of $F_n(z_1,\ldots,z_n)$ given in Theorem 8.2. Theorem 10.2 will then be deduced from Theorem 10.1, the argument being a purely formal one in the sense that if we replaced the power series $B(z)=\frac{z/2}{\sinh z/2}$ in equation (91) by any other even power series with constant coefficient 1, and replaced the numbers β_m in the definition (93) by the coefficients of this power series, then the new equation (91) would imply the new equation (94) in exactly the same way.

Proof of Theorem 10.1. From the axiomatic description of $F_n(z_1,\ldots,z_n)$ given in Theorem 8.2 and the fact that $\mathrm{Ev}:\widetilde{M}_*\to\mathbb{Q}[X]$ is a ring homomorphism it follows immediately that there is a similar axiomatic description of $F_n(z_1,\ldots,z_n)_X$ in which the function $\theta(z)$ in (iv) is replaced by its X-evaluation as given in (90) and everything else is unchanged. We thus need to verify that the right hand side of (91), which we denote by $F_n^*(z_1,\ldots,z_n)_X$ in the proof below, satisfies these modified axioms. First, note that F^* is indeed a Laurent series in the variables $z_1,\ldots,z_n$, since the exponent of $z_A=\sum_{a\in A}z_a$ in (91) is negative only if $|A|=1$. The property (i) and the symmetry in the arguments stated in (ii) are immediate for F^* from its definition. For (iii) (with n replaced by $n+1$ and z_{n+1} by z), we observe that, since any element of $\mathcal{P}(n+1)$ is obtained from a unique element α of $\mathcal{P}(n)$ either by adding the one-element set $\{n+1\}$ to α or by replacing some element of α by its union with $\{n+1\}$, we have

$$\begin{aligned} F_{n+1}^*(z_1,\ldots,z_n,z)_X \;=\; & e^{-(z_N+z)^2/2}\sum_{\alpha\in\mathcal{P}(n)}\Bigg[\frac{\sqrt{X}/2}{\sinh z\sqrt{X}/2}\prod_{A\in\alpha}\frac{z_A^{|A|-1}\sqrt{X}/2}{\sinh z_A\sqrt{X}/2} \\ & \qquad + \sum_{B\in\alpha}\frac{(z_B+z)^{|B|}\sqrt{X}/2}{\sinh((z_B+z)\sqrt{X}/2)}\prod_{A\in\alpha\smallsetminus\{B\}}\frac{z_A^{|A|-1}\sqrt{X}/2}{\sinh(z_A\sqrt{X}/2)}\Bigg] \\ =\; & e^{-z_N^2/2}\Big(1-zz_N+\mathrm{O}(z^2)\Big)\sum_{\alpha\in\mathcal{P}(n)}\Big(\frac{1}{z}+\sum_{B\in\alpha}z_B+\mathrm{O}(z)\Big)\prod_{A\in\alpha}\frac{z_A^{|A|-1}\sqrt{X}/2}{\sinh z_A\sqrt{X}/2} \\ =\; & \frac{1}{z}\,F_n^*(z_1,\ldots,z_n)_X\;+\;\mathrm{O}(z) \qquad \text{as } z\to 0\,, \end{aligned}$$

as desired. Finally, to show (iv) we multiply the right hand side of (91) with (90) (with z replaced by z_N). The positive part of this expression is zero if we can show that the positive part of $\sinh(s_N)\prod_{A\in\alpha}\frac{s_A^{|A|-1}}{\sinh s_A}$ is 0 for each $\alpha\in\mathcal{P}(n)$ individually, where we set $s_A = z_A\sqrt{X}/2$. But since $s_N=\sum_{A\in\alpha}s_A$, we have

$$\sinh(s_N) \;=\; \sum_{\substack{\beta\subseteq\alpha\\ \ell(\beta)\text{ odd}}}\;\prod_{A\in\alpha\smallsetminus\beta}\cosh s_A\cdot\prod_{A\in\beta}\sinh s_A$$

and hence

$$\sinh(s_N)\prod_{A\in\alpha}\frac{s_A^{|A|-1}}{\sinh s_A} \;=\; \sum_{\substack{\beta\subseteq\alpha\\ \ell(\beta)\text{ odd}}}\;\prod_{A\in\beta}s_A^{|A|-1}\cdot\prod_{A\notin\beta}\frac{s_A^{|A|-1}}{\tanh s_A}\,.$$

We want to show that the coefficient of $z_1^{r_1}\cdots z_n^{r_n}$ in each summand of this expression vanishes if all of the r_i are strictly positive. Since each i belongs to only one of the sets in α, and since every set β occurring has odd cardinality and hence is non-empty, it suffices to prove this statement for a single term $s_A^{|A|-1}$ $(A\in\beta)$, and this is obvious since a homogeneous polynomial of degree $|A|-1$ cannot contain every variable z_a $(a\in A)$ to a strictly positive power. □

Proof of Theorem 10.2. Both for this proof and for use later in the paper, it is convenient to define a linear map Ω_n from the space of Laurent polynomials or Laurent series in n variables $z_1,\dots,z_n$ to the space of polynomials or power series in infinitely many variables $u_1,u_2,\dots$ by the formula

$$\Omega_n\bigl[z_1^{\ell_1}\,\cdots\,z_n^{\ell_n}\bigr] \;=\; \begin{cases}\dfrac{\ell_1!\,\cdots\,\ell_n!}{n!}\,u_{\ell_1}\cdots u_{\ell_n} & \text{if } \ell_1,\dots,\ell_n\ge 1,\\[2mm] \qquad 0 & \text{otherwise.}\end{cases} \tag{97}$$

With this notation, we can compute our two generating functions (92) and (69), or their X-bracket versions, by

$$\Phi(\mathbf{u})_q \;=\; \sum_{n=0}^{\infty}\Omega_n\bigl[F_n(z_1,\dots,z_n)\bigr]\,,\quad \Phi(\mathbf{u})_X \;=\; \sum_{n=0}^{\infty}\Omega_n\bigl[F_n(z_1,\dots,z_n)_X\bigr]\,. \tag{98}$$

(Recall that $p_\ell=\ell!\,Q_{\ell+1}$ for $\ell\ge 1$.) On the other hand, if $\alpha\in\mathcal{P}(n)$ is a partition of $N=\{1,\dots,n\}$ and if to each $A\in\alpha$ we have associated a power series $G_A(z)$ in one variable, then from the multinomial theorem we find that $\Omega_n\bigl[\prod_{A\in\alpha}G_A(z_A)\bigr]$ equals the product over all $A\in\alpha$ of $G_A(d/dt)(U(t)^{|A|})|_{t=0}$, where $z_A=\sum_{a\in A}z_a$ as before and where

$$U(t) \;=\; u_1t\,+\,u_2t^2\,+\,\cdots \tag{99}$$

is the generating power series of the u_ℓ. In particular, if $G_A(z)=G_{|A|}(z)$ depends only on the cardinality of A, then

$$\Omega_n\Biggl[\sum_{\substack{\alpha\in\mathcal{P}(n)\\ \ell(\alpha)=m}}\prod_{A\in\alpha}G_{|A|}(z_A)\Biggr] \;=\; \frac{1}{m!}\sum_{\substack{s_1,\dots,s_m\ge 1\\ s_1+\cdots+s_m=n}}\frac{\gamma_{s_1}(\mathbf{u})}{s_1!}\cdots\frac{\gamma_{s_m}(\mathbf{u})}{s_m!} \tag{100}$$

with $\gamma_s(\mathbf{u}):=G_s(d/dt)(U(t)^s)|_{t=0}$, because if α is a partition of $N=\{1,\dots,n\}$ with m parts, then we can order them in precisely $m!$ ways (since they are non-empty and distinct), and there are $\frac{n!}{s_1!\cdots s_m!}$ ordered partitions $N=A_1\sqcup\cdots\sqcup A_m$

of given sizes $s_1, \ldots, s_m \geq 1$. Summing (100) over m and then over n gives

$$\sum_{n=0}^{\infty} \Omega_n \left[\sum_{\alpha \in \mathcal{P}(n)} \prod_{A \in \alpha} G_{|A|}(z_A) \right] \;=\; \exp\left(\sum_{s \geq 1} \frac{\gamma_s(\mathbf{u})}{s!} \right). \tag{101}$$

On the other hand, observing that for any z independent of y we have from (96)

$$e^{-z^2/2} \;=\; \sum_{\ell=0}^{\infty} \frac{(-1/2)^\ell}{\ell!} z^{2\ell} \;=\; \mathfrak{I}\big[e^{zy}\big]\,, \tag{102}$$

we can rewrite (91) as

$$F_n(z_1, \ldots, z_n)_X \;=\; \mathfrak{I}\left[\sum_{\alpha \in \mathcal{P}(n)} \prod_{A \in \alpha} \Big(z_A^{|A|-2} B\big(z_A \sqrt{X}\big) e^{z_A y} \Big) \right]$$

with $B(z)$ defined as in (63). We insert this into the second equation of (98) and note that the maps Ω_n and $\mathfrak{I}$ commute. We apply (101) with

$$G_s(z) \;=\; z^{s-2}\big(B(z\sqrt{X})e^{zy} - \delta_{s,1}\big)\,.$$

(Here we are allowed to delete the pole term $1/z$ for $s = 1$ because negative powers in (97) are discarded.) This gives equation (95) with $\mathcal{B}$ defined by

$$\mathcal{B}(\mathbf{u}, y, X) \;=\; \sum_{\substack{k,r \geq 0 \\ k+r \geq 2}} \beta_k \, X^{k/2} \, \frac{y^r}{r!} \sum_{s \geq 1} \frac{d^{k+r+s-2}}{dt^{k+r+s-2}} \Big(\frac{U(t)^s}{s!} \Big) \Big|_{t=0}\,, \tag{103}$$

which is easily seen to be equivalent to the definition (93). □

The inner sum in the formula (103) used above can be calculated in a more explicit form using the Lagrange inversion theorem. This leads to the following two propositions, special cases of which will be used in Section 12 to write down various one-variable power series that will be needed for the asymptotic calculations in Part IV.

Proposition 10.3. *Let*

$$T(y) \;=\; T(\mathbf{u}, y) \;=\; \frac{u_1}{1-u_1}\, y \;+\; \frac{u_2}{(1-u_1)^3}\, y^2 \;+\; \frac{2u_2^2 + (1-u_1)u_3}{(1-u_1)^5} y^3 \;+\; \cdots$$

be the solution of $T = U(y + T)$, *with* $U(t)$ *as in* (99). *Then*

$$\mathcal{B}(\mathbf{u}, y, X) \;=\; \int_0^y T(y')dy' \;+\; \sum_{k \geq 2} \beta_k \, T^{(k-1)}(y) \, X^{k/2}\,. \tag{104}$$

Proof. This follows (independently of the definition of the coefficients β_k) directly from equation (103) together with the formula

$$T \;=\; U(y+T) \quad\Longleftrightarrow\quad T \;=\; \sum_{s=1}^{\infty} \frac{1}{s!} \, \frac{d^{s-1}}{dy^{s-1}} \, U(y)^s\,,$$

which is one of the forms of the Lagrange inversion theorem. □

Remark. From either (93) or (104) we see that the function $\mathcal{B}(\mathbf{u},y,X)$ has a very special form: if we denote by $c_n(\mathbf{u})$ the coefficient of y^n in $T(\mathbf{u},y)$, then

$$\mathcal{B}(\mathbf{u},y,X) \;=\; c_1(\mathbf{u})\Big(\frac{y^2}{2}-\frac{X}{24}\Big)+c_2(\mathbf{u})\Big(\frac{y^3}{3}-\frac{Xy}{12}\Big)+c_3(\mathbf{u})\Big(\frac{y^4}{4}-\frac{Xy^2}{8}+\frac{7X^2}{960}\Big)+\cdots$$

in which the ratio of the coefficients of X^iy^j and y^{2i+j} for any $i,j\geq 0$ is independent of $\mathbf{u}$.

Proposition 10.4. *Let $t(y)=t(\mathbf{u},y)$ be the inverse power series of $y=t-U(t)$. Then for all $\ell\geq 1$ we have*

$$\frac{\partial\mathcal{B}(\mathbf{u},y,X)}{\partial u_\ell} \;=\; \sum_{k=0}^{\infty}\beta_k\,\frac{\partial^k}{\partial y^k}\Big(\frac{t(\mathbf{u},y)^{\ell+1}}{\ell+1}\Big)\,X^{k/2}\,. \tag{105}$$

Proof. We first observe that the power series $t(y)$ of this proposition is related to the $T(y)$ of the previous proposition by $t(y)=y+T(y)$. Then

$$0 \;=\; \frac{\partial}{\partial u_\ell}\Big(T(\mathbf{u},y)\;-\;U\big(y+T(\mathbf{u},y)\big)\Big) \;=\; (1-U'(t))\,\frac{\partial T}{\partial u_\ell}\;-\;t^\ell$$

or

$$\frac{\partial T}{\partial u_\ell}\;=\;\frac{t^\ell}{1-U'(t)}\;=\;\frac{t^\ell}{\partial y/\partial t}\;=\;\frac{\partial}{\partial y}\Big(\frac{t^{\ell+1}}{\ell+1}\Big)\,.$$

Combining this with (104), we find

$$\frac{\partial^2\mathcal{B}(\mathbf{u},y,X)}{\partial y\,\partial u_\ell}\;=\;\frac{\partial}{\partial u_\ell}\Bigg(\sum_{k=0}^{\infty}\beta_k\,\frac{\partial^kT}{\partial y^k}\,X^{k/2}\Bigg)\;=\;\sum_{k=0}^{\infty}\beta_k\,\frac{\partial^{k+1}}{\partial y^{k+1}}\Big(\frac{t^{\ell+1}}{\ell+1}\Big)\,X^{k/2}\,,$$

and (105) follows by integrating with respect to y, the constant term being 0. □

We next present a result that gives a small refinement of Theorem 10.1. At the end of Section 8 we introduced two differential operators ∂ and Δ on the ring of shifted symmetric polynomials and explained their relationship to q-brackets (Propostion 8.3). The following proposition describes their surprisingly simple action on the n-point generating function $W(z_1)\cdots W(z_n)$ (see (65) for the notation $W(z)$).

Proposition 10.5. *We have*

$$g(\partial)\big(W(z_1)\cdots W(z_n)\big)\;=\;g(z_N)\,W(z_1)\cdots W(z_n)$$

for any power series $g(t)\in\mathbb{C}((t))$, and

$$e^{\varepsilon\Delta/2}\big(W(z_1)\cdots W(z_n)\big)=\sum_{\alpha\in\mathcal{P}(n)}\prod_{A\in\alpha}(\varepsilon z_A)^{|A|-1}\,W(z_A)\,.$$

Proof. The definition of ∂ implies that $W(z_1)\cdots W(z_n)$ is an eigenvector of ∂ with eigenvalue z_N. This gives the first formula. For Δ we find, using $\dfrac{\partial W(z_i)}{\partial Q_{k+1}}=z_i^k$,

$$\begin{aligned}\Delta\Bigg(\prod_{i=1}^{n}W(z_i)\Bigg)\;&=\;\sum_{1\leq i\neq j\leq n}\Bigg(\sum_{k,\ell\geq 0}\binom{k+\ell}{k}Q_{k+\ell}\,z_i^kz_j^\ell\Bigg)\prod_{\substack{1\leq h\leq n\\ h\neq i,\,j}}W(z_h)\\ &=\;2\sum_{1\leq i<j\leq n}(z_i+z_j)W(z_i+z_j)\prod_{\substack{1\leq h\leq n\\ h\neq i,\,j}}W(z_h)\,.\end{aligned}\tag{106}$$

By induction we obtain a formula for the action of Δ^r on $W(z_1)\cdots W(z_n)$, and then multiplying by $(\varepsilon/2)^r/r!$ and summing over r we obtain the claim. □

Now take $g = e^{-\varepsilon t^2/2}$ in the first formula of the proposition and then replace ε and z_i by $1/X$ and $z_i\sqrt{X}$, respectively, in both formulas. Then from the two assertions of Proposition 8.3 we obtain (with $\boldsymbol{W}$ and $\mathfrak{d}$ as in (75))

$$\begin{aligned} X^{\boldsymbol{W}/2}e^{\mathfrak{d}}\big\langle W(z_1)\cdots W(z_n)\big\rangle_q &= e^{\mathfrak{d}/X}\,\big\langle \sqrt{X}\,W\big(z_1\sqrt{X}\big)\ \cdots\ \sqrt{X}\,W\big(z_n\sqrt{X}\big)\big\rangle_q \\ &= X^{n/2}\,\big\langle e^{\Delta/2X}\,e^{-\partial^2/2X}\,\big(W\big(z_1\sqrt{X}\big)\cdots W\big(z_n\sqrt{X}\big)\big)\big\rangle_q \\ &= e^{-z_N^2/2}\sum_{\alpha\in\mathcal{P}(n)}\big\langle\prod_{A\in\alpha} z_A^{|A|-1}\sqrt{X}W(z_A\sqrt{X})\big\rangle_q\,. \end{aligned} \tag{107}$$

This is the above-mentioned strengthening of Theorem 10.1, since (91) follows immediately from (107) and Proposition 9.2 by applying the ring homomorphism $\mathcal{R}_* \xrightarrow{\langle\cdot\rangle_q} \widetilde{M}_* \xrightarrow{a_0} \mathbb{Q}$ which sends $Q_k \mapsto \beta_k$ and $W(z)$ to $1/2\sinh(z/2)$.

We end this section by giving a statement about the "degree drop" of the growth polynomials of certain q-brackets. It says, for instance, that the X-bracket $\langle Q_3^{2n}\rangle_X$, which *a priori* could have degree up to $3n$ in X since the weight of Q_3^{2n} is $6n$, in fact has degree at most $2n$. A related and even stronger statement for the "connected brackets" studied in the next section will lead to the definitions of cumulants that will be crucial for the asymptotic calculations given in Part IV.

Proposition 10.6. *The degree of the growth polynomial of an element of $\mathcal{R}$ of weight $2k$ that is a product of $2n$ elements of odd weight is at most $k-n$. In particular, the degree of the X-bracket of a monomial $p_1^{r_1}p_2^{r_2}\cdots$ in the p_i is bounded by $r_1+r_2+2(r_3+r_4)+3(r_5+r_6)+\cdots$.*

Proof. We will in fact prove the second statement of the proposition, which is clearly equivalent to the first. To any monomial $\mathbf{u}^{\mathbf{a}} = u_1^{a_1}u_2^{a_2}\cdots$ we associate the invariants $w(\mathbf{a}) = a_2+2a_3+3a_4+\cdots$ (as in (93)), $s(\mathbf{a}) = a_1+a_2+a_3+\cdots$ (= the s of (103)), $K(\mathbf{a}) = 2a_1+3a_2+4a_3+\cdots$ (the modular weight), $O(\mathbf{a}) = a_2+a_4+a_6+\cdots$ (corresponding to the number of occurrences of p_ℓ of odd weight), and $\varepsilon(\mathbf{a}) = 0$ or 1 depending on whether $O(\mathbf{a})$ is even or odd. They are related by $K = 2s+w$, $s \ge O \ge 0$, and $w \equiv \varepsilon \pmod 2$. If a monomial $u_1^{a_1}u_2^{a_2}\cdots X^d$ occurs in (93), then we have $s \ge 1$ and $2d = 2+w-r \le 2+w-\varepsilon$, because $r \ge 0$ and r must be strictly positive if $w(\mathbf{a})$ is odd since β_k vanishes for k odd. It follows that $K-2d \ge 2s-2+\varepsilon$, which is always $\ge O$. (If $O=0$ then $2s-2+\varepsilon \ge 2s-2 \ge 0$; if $O=1$ then $2s-2+\varepsilon \ge 2s-1 \ge 1$, and if $O \ge 2$ then $2s-2+\varepsilon \ge 2s-2 \ge 2O-2 \ge O$.) Thus the X-degree of the monomial in question is always $\le \frac{1}{2}(K-O)$, and since both the X-degree and the invariants K and O are additive, it follows that the same estimate is true for any monomial occurring in any power of $\mathcal{B}$, and hence also for every monomial occurring in our formula $\mathfrak{I}\big[e^{\mathcal{B}}\big]$ for $\Phi(\mathbf{u})_X$. □

11. The generating series of cumulants

In this section we study the connected q-brackets and cumulants of [18], which encode the counting functions for counting of connected covers and their leading terms. Our main result is Theorem 11.2, which gives an expression for the generating series of cumulants as the value of the function $\mathcal{B}(\mathbf{u},y)+y^2$ of the previous section at a stationary point. The proof relies on the principle of least action applied to the formal Gaussian integral formula for $\Phi(\mathbf{u})_X$.

We begin by describing a general algebraic formalism that is relevant in many geometric counting problems when we pass from disconnected to connected objects. Let R and R' be two commutative $\mathbb{Q}$-algebras with unit and $\langle\ \rangle : R \to R'$ a linear map sending 1 to 1. (Of course the cases of interest to us will be when R is the Bloch-Okounkov ring $\mathcal{R}$ and $\langle\ \rangle$ is the q-, X-, or h-bracket to $R' = \widetilde{M}_*$, $\mathbb{Q}[X]$, or $\mathbb{Q}[\pi^2][h]$, respectively.) Then we extend $\langle\ \rangle$ to a multi-linear map $R^{\otimes n} \to R'$ for every $n \geq 1$, the image of $f_1 \otimes \cdots \otimes f_n$ being denoted by either $\langle f_1 | \cdots | f_n \rangle$ or $\langle | f_1 \otimes \cdots \otimes f_n | \rangle$, that are defined by the formula

$$\langle f_1 | \ldots | f_n \rangle \;=\; \sum_{\alpha \in \mathcal{P}(n)} (-1)^{\ell(\alpha)-1} (\ell(\alpha)-1)! \prod_{A \in \alpha} \Big\langle \prod_{a \in A} f_a \Big\rangle \tag{108}$$

(cf. (50)), where $\ell(\alpha)$ denotes the length (cardinality) of the partition α. For instance, for $n=2$ and $n=3$ we have

$$\begin{aligned} \langle f|g \rangle \;&=\; \langle fg \rangle \,-\, \langle f \rangle \langle g \rangle\,, \\ \langle f|g|h \rangle \;&=\; \langle fgh \rangle \,-\, \langle f \rangle \langle gh \rangle \,-\, \langle g \rangle \langle fh \rangle \,-\, \langle h \rangle \langle fg \rangle \,+\, 2\, \langle f \rangle \langle g \rangle \langle h \rangle\,. \end{aligned}$$

Following [18], we call $\langle f_1 | \cdots | f_n \rangle$ the *connected bracket* of the functions $f_1, \ldots, f_n$ corresponding to the original bracket $\langle\ \rangle$. Note that the connected bracket is symmetric, so defines a map from $S^n(R)$ to R', and that it vanishes if any f_i equals 1, so in fact descends to a map $S^n(R/\mathbb{Q}) \to R'$, and also that the definition can be inverted to express all brackets in terms of connected ones, e.g.

$$\begin{aligned} \langle fg \rangle \;&=\; \langle f|g \rangle \,+\, \langle f \rangle \langle g \rangle\,, \\ \langle fgh \rangle \;&=\; \langle f|g|h \rangle \,+\, \langle f \rangle \langle g|h \rangle \,+\, \langle g \rangle \langle f|h \rangle \,+\, \langle h \rangle \langle f|g \rangle \,+\, \langle f \rangle \langle g \rangle \langle h \rangle \end{aligned}$$

and in general

$$\langle f_1 \cdots f_n \rangle \;=\; \sum_{\alpha \in \mathcal{P}(n)} \prod_{A \in \alpha} \big\langle \big| \otimes_{a \in A} f_a \big| \big\rangle\,. \tag{109}$$

This formula is a special case of Proposition 11.5 below.

Perhaps the most important property of connected brackets is their appearance in the logarithm of the original bracket applied to an exponential:

$$\begin{aligned} \log\!\big(\big\langle e^{f_1+f_2+f_3+\cdots} \big\rangle\big) \;&=\; \log\Big(1 \,+\, \sum_i \langle f_i \rangle \,+\, \frac{1}{2!} \sum_{i,j} \langle f_i f_j \rangle \,+\, \frac{1}{3!} \sum_{i,j,k} \langle f_i f_j f_k \rangle \,+\, \cdots \Big) \\ &=\; \sum_i \langle f_i \rangle \,+\, \frac{1}{2!} \sum_{i,j} \langle f_i | f_j \rangle \,+\, \frac{1}{3!} \sum_{i,j,k} \langle f_i | f_j | f_k \rangle \,+\, \cdots\,, \end{aligned}$$

which explains by a well-known principle why the connected brackets correspond to the counting functions of connected objects. This gives us yet a third definition of the connected bracket $\langle f_1 | \cdots | f_n \rangle$, as the coefficient of the monomial $x_1 \cdots x_n$ in $\big\langle \exp(x_1 f_1 + \cdots + x_n f_n) \big\rangle$. Applying it to the rings $R = \mathcal{R}$ and $R' = \widetilde{M}_*$ and the q-bracket $\langle\ \rangle_q$, we find that the generating series of the connected q-brackets is equal to the logarithm of the partition function $\Phi(\mathbf{u})_q$ defined in (92):

$$\begin{aligned} \Psi(\mathbf{u})_q :=\; &\sum_{n>0}^{\infty} \frac{1}{n!} \sum_{\ell_1, \ldots, \ell_n \geq 1} \langle p_{\ell_1} | \cdots | p_{\ell_n} \rangle_q \, u_{\ell_1} \cdots u_{\ell_n} \\ =\; &\sum_{\mathbf{n}>0} \langle \underbrace{p_1 | \cdots | p_1}_{n_1} | \underbrace{p_2 | \cdots | p_2}_{n_2} | \cdots \rangle_q \, \frac{\mathbf{u}^{\mathbf{n}}}{\mathbf{n}!} \;=\; \log \Phi(\mathbf{u})_q\,, \end{aligned} \tag{110}$$

and similarly

$$\Psi(\mathbf{u})_X := \mathrm{Ev}[\Psi(\mathbf{u})_q] = \log \Phi(\mathbf{u})_X \tag{111}$$

for the generating function of the connected X-brackets $\langle p_{m_1}|\cdots|p_{m_n}\rangle_X$.

Our main concern is the X-evaluation of connected brackets. The first result, which is due to Eskin and Okounkov ([18, Theorem 6.3]) but will also follow from our proof of Theorem 11.2 below, is that the degree of the connected X-brackets as a polynomial in X, which for the original X-bracket was at most half of the weight, drops by one for every $|$-insertion.

Proposition 11.1. *Let $f_i \in \mathcal{R}_{k_i}$ $(i = 1, \ldots, n)$ be homogeneous elements of the ring $\mathcal{R}$ and $k = k_1+\cdots+k_n$ the total weight. Then* $\deg(\langle f_1|\cdots|f_n\rangle_X) \le 1-n+k/2$.

Motivated by this, we define the *leading coefficient* of the growth polynomial of $\langle f_1|\cdots|f_n\rangle_q$ for f_i and k as in the proposition by

$$\langle f_1|\cdots|f_n\rangle_L \;=\; [X^{1-n+k/2}]\,\langle f_1|\cdots|f_n\rangle_X \;=\; \lim_{X\to\infty} \frac{\mathrm{Ev}[\langle f_1|\cdots|f_n\rangle_q](X)}{X^{1-n+k/2}}\,.$$

We will be especially interested in the case when each of the f_i is one of the standard generators $p_\ell = \ell!\,Q_{\ell+1}$ of $\mathcal{R}$. We define the rational numbers

$$\langle\langle\, \ell_1, \cdots, \ell_n \,\rangle\rangle_{\mathbb{Q}} \;=\; \langle p_{\ell_1}|\cdots|p_{\ell_n}\rangle_L \qquad (\ell_1,\ldots,\ell_n \ge 1)\,, \tag{112}$$

which we call *rational cumulants*,[2] with the corresponding generating series

$$\begin{aligned} \Psi(\mathbf{u})_L \;&=\; \sum_{\mathbf{n}\ge 0} \langle\langle\, \underbrace{1,\ldots,1}_{n_1}, \underbrace{2,\ldots,2}_{n_2}, \ldots \,\rangle\rangle_{\mathbb{Q}}\, \frac{\mathbf{u}^{\mathbf{n}}}{\mathbf{n}!} \\ &=\; \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\ell_1,\ldots,\ell_n\ge 1} \langle\langle\, \ell_1,\ldots,\ell_n \,\rangle\rangle_{\mathbb{Q}}\; u_{\ell_1}\cdots u_{\ell_n}\,. \end{aligned} \tag{113}$$

Our main result in this section is a formula for this generating function that will be used in Section 12 and in Part IV. Its statement uses the function $\mathcal{B}(\mathbf{u}, y, X)$ defined in (93). We write $\mathcal{B}(\mathbf{u}, y)$ for the polynomial $\mathcal{B}(\mathbf{u}, y, 1)$ and denote by $\mathcal{B}'(\mathbf{u}, y, X)$ and $\mathcal{B}'(\mathbf{u}, y)$ the derivatives of $\mathcal{B}(\mathbf{u}, y, X)$ and $\mathcal{B}(\mathbf{u}, y)$ with respect to y.

Theorem 11.2. *The generating series of cumulants is given by*

$$\Psi(\mathbf{u})_L = \mathcal{B}(\mathbf{u}, y_0) \;+\; \frac{y_0^2}{2}\,, \tag{114}$$

where $y_0 = y_0(\mathbf{u}) \in \mathbb{Q}[[\mathbf{u}]]$ is the unique power series satisfying $\mathcal{B}'(\mathbf{u}, y_0) + y_0 = 0$.

Proof of Proposition 11.1 and Theorem 11.2. We first note that there is a unique power series $y = y_0(\mathbf{u}, X)$ as solution of the equation $\mathcal{B}'(\mathbf{u}, y, X)+y = 0$, as one can see either by Newton's method or by iterating $y \mapsto -\mathcal{B}(\mathbf{u}, y, X)$ (starting in either case with $y = 0$), or alternatively by noting that the latter map is a contraction and hence has a unique fixed point. The special case $y_0(\mathbf{u}, 1)$ is the function $y_0(\mathbf{u})$ occurring in the theorem, and in fact the two functions are equivalent because from its definition $\mathcal{B}(\mathbf{u}, y, X)$ has the homogeneity property

$$\mathcal{B}(\mathbf{u}, ty, t^2X) \;=\; t^2\,\mathcal{B}(t\circ\mathbf{u}, y, X), \tag{115}$$

[2]We avoid powers of π here. The real cumulants $\langle\langle\, \ell_1, \ldots, \ell_s \,\rangle\rangle \in \mathbb{Q}[\pi]$ will be defined in Part IV.

where $t\circ\mathbf{u} := (u_1, tu_2, t^2u_3, \ldots)$, and therefore $y_0(\mathbf{u}, X) = X^{1/2}y_0(X^{1/2}\circ\mathbf{u})$. From the beginning of the Taylor expansion of $\mathcal{B}$, as given either by its definition or by Proposition 10.3, we find that the expansion of $y_0(\mathbf{u}, X)$ begins with

$$y_0(\mathbf{u}, X) \;=\; \frac{u_2}{12(1-u_1)^2}X - \Bigl(\frac{u_2^3}{9(1-u_1)^6} + \frac{u_2u_3}{8(1-u_1)^5} + \frac{7u_4}{240(1-u_1)^4}\Bigr)X^2 + \cdots,$$

in which the homogeneity property just mentioned is reflected in the fact that the coefficient of X^n is homogeneous of weight $2n-1$ for each n, where u_i has weight $i-1$ (or equivalently, if y_0 is thought of as an element of $\mathbb{Q}[X][[\mathbf{u}]]$, that the coefficient of any monomial $\mathbf{u}^{\mathbf{a}}$ is a multiple of $X^{1+w(\mathbf{a})/2}$).

We now use formula (111) together with Theorem 10.2, which expresses $\Phi(\mathbf{u})_X$ as a formal Gaussian integral. To evaluate the logarithm of this integral, guided by the *principle of least action*, we shift the integration variable y by $y_0(\mathbf{u}, X)$ so that the exponent of the integrand has no linear term. The procedure is justified because the translational invariance of the Gaussian integral (or a simple combinatorial calculation using the formal definition (96)) gives the transformation property

$$\mathfrak{I}[F(y+z)] \;=\; e^{z^2/2}\,\mathfrak{I}[e^{-yz}F(y)] \tag{116}$$

for polynomials $F(y)$ (equation (102) is the special case $F = 1$ of this), and we can also apply this when $F(y)$ is a power series in y with coefficients in $\mathbf{u}$ so long as the coefficient of the monomial $\mathbf{u}^{\mathbf{a}}$ vanishes for $\mathbf{a} = 0$ and is a polynomial in y for $\mathbf{a} > 0$. We apply it with $z = y_0(\mathbf{u}, X)$, using the Taylor expansion

$$\mathcal{B}(\mathbf{u}, y_0+y, X) + \frac{(y_0+y)^2}{2} \;=\; \mathcal{B}(y_0) + \frac{y_0^2}{2} + \bigl(\mathcal{B}''(y_0)+1\bigr)\frac{y^2}{2} + \mathcal{B}'''(y_0)\,\frac{y^3}{6} + \cdots,$$

where $\mathcal{B}^{(n)}(y_0)$ is shorthand for $\frac{\partial^n\mathcal{B}}{\partial y^n}(\mathbf{u}, y_0(\mathbf{u}, X), X)$. Here the coefficient of y is 0 by the definition of y_0 and the other terms have expansions beginning with

$$\begin{aligned}
\mathcal{B}(\mathbf{u}, y_0, X) + \frac{y_0^2}{2} \;&=\; \frac{u_1}{24(1-u_1)}X + \Bigl(\frac{u_2^2}{90(1-u_1)^5} + \frac{7u_3}{960(1-u_1)^4}\Bigr)X^2 + \cdots,\\
\mathcal{B}''(\mathbf{u}, y_0, X) + 1 \;&=\; \frac{1}{1-u_1} - \Bigl(\frac{u_2^2}{3(1-u_1)^5} + \frac{u_3}{4(1-u_1)^4}\Bigr)X + \cdots,\\
\mathcal{B}'''(\mathbf{u}, y_0, X) \;&=\; \frac{2u_2}{(1-u_1)^3} - \Bigl(\frac{4u_2^3}{3(1-u_1)^7} + \frac{9u_2u_3}{2(1-u_1)^6} + \frac{u_4}{(1-u_1)^5}\Bigr)X + \cdots,
\end{aligned}$$

in which the coefficient of X^k in $\mathcal{B}^{(n)}(y_0)$ is homogeneous of weight $2k+n-2$ in $\mathbf{u}$. Making the substitution $\mathbf{u} \mapsto X^{-1/2}\circ\mathbf{u}$ and using this homogeneity property, we therefore find

$$\Phi(X^{-\frac{1}{2}}\circ\mathbf{u})_X \;=\; e^{\left(\mathcal{B}(y_0)+\frac{1}{2}y_0^2\right)X}\,\mathfrak{I}\Bigl[\exp\Bigl(\frac{\mathcal{B}''(y_0)}{2}y^2 + \frac{\mathcal{B}'''(y_0)}{6\sqrt{X}}y^3 + \frac{\mathcal{B}^{\mathrm{iv}}(y_0)}{24X}y^4 + \cdots\Bigr)\Bigr]$$

(now with $\mathcal{B}^{(n)}(y_0) = \frac{\partial^n\mathcal{B}}{\partial y^n}(\mathbf{u}, y_0(\mathbf{u})))$, and expanding the first few terms of this by (96) and taking logarithms gives

$$\begin{aligned}
\Psi(X^{-1/2}\circ\mathbf{u})_X \;=\; &\Bigl(\mathcal{B}(y_0) + \frac{1}{2}y_0^2\Bigr)X - \frac{1}{2}\log\bigl(1+\mathcal{B}''(y_0)\bigr)\\
&+ \Bigl(\frac{B^{\mathrm{iv}}(y_0)}{8(1+\mathcal{B}''(y_0))^2} - \frac{5B'''(y_0)^2}{24(1+\mathcal{B}''(y_0))^3}\Bigr)\frac{1}{X} + \cdots.
\end{aligned}$$

The fact that this Laurent series has no powers $X^{>1}$ implies Proposition 11.1, the fact that the coefficient of X is $\mathcal{B}(y_0)+\frac{1}{2}y_0^2$ gives Theorem 11.2, and the further terms of the expansion give as many subleading terms of $\Psi(\mathbf{u})_X$ as desired. □

Equation (114) gives an effective way to evaluate cumulants, since y_0 is given as a fixed point and can be computed rapidly by iteration. The next proposition, which is also suitable for practical calculations, gives an alternative formula for the generating series $\Psi(\mathbf{u})_L$, reminiscent of the formula for $\mathcal{B}(\mathbf{u},y,X)$ in Proposition 10.3.

Proposition 11.3. *The generating series of rational cumulants is given by*

$$\Psi(\mathbf{u})_L \;=\; \mathcal{B}(\mathbf{u},0)+\sum_{m=2}^{\infty}\frac{(-1)^{m-1}}{m(m-1)}\left[y^{m-2}\right](\mathcal{B}'(\mathbf{u},y)^m)\,. \tag{117}$$

Proof. We need to prove the identity

$$\mathcal{B}(0)+\sum_{m=2}^{\infty}\frac{(-1)^{m-1}}{m(m-1)}\,[y^{m-2}](\mathcal{B}'(y)^m) \;=\; \mathcal{B}(y_0)+\frac{y_0^2}{2}\,, \tag{118}$$

where $\mathcal{B}(y)=\mathcal{B}(\mathbf{u},y)$ and y_0 is the solution of $\mathcal{B}'(y_0)=-y_0$. Write $z=y/\mathcal{B}'(y)$ and expand y^2 in powers of z, i.e. we define a_m by $y^2=\sum_{m\ge 2}a_m z^m$. Then

$$\begin{aligned}\left[y^{m-2}\right](\mathcal{B}'(y)^m) \;&=\; \mathrm{Res}_{y=0}\Big(\frac{\mathcal{B}'(y)^m}{y^{m-1}}dy\Big) \;=\; \mathrm{Res}_{y=0}\Big(z^{-m}\,d\big(\frac{y^2}{2}\big)\Big)\\ &=\; -\frac{1}{2}\,\mathrm{Res}_{z=0}\Big(y^2 d(z^{-m})\Big) \;=\; \frac{m}{2}\,\mathrm{Res}_{z=0}\Big(\frac{y^2}{z^{m+1}}dz\Big) \;=\; \frac{m}{2}a_m.\end{aligned}$$

Let S be the left hand side of the expression in (118). Then

$$\begin{aligned} S-\mathcal{B}(0) \;&=\; \frac{1}{2}\sum_{m=2}^{\infty}\frac{(-1)^{m-1}}{m-1}a_m \qquad = \frac{1}{2}\int_0^{-1}\frac{y(z)^2}{z^2}dz\\ &=\; \frac{1}{2}\int_{z=0}^{z=-1}y^2\frac{d}{dy}\Big(\frac{-1}{z(y)}\Big)dy \;=\; \frac{1}{2}\int_{z=0}^{z=-1}\big(2\mathcal{B}(y)-y\mathcal{B}'(y)\big)'dy\\ &=\; \big(\mathcal{B}(y)-\frac{y}{2}\mathcal{B}'(y)\big)\Big|_{z(y)=0}^{z(y)=-1} \;=\; \mathcal{B}(y_0)\,+\,\frac{y_0^2}{2}\,-\,\mathcal{B}(0)\,,\end{aligned}$$

since $z=0$ corresponds to $y=0$ and $z=-1$ to $y=y_0$. □

Finally, just as Theorems 10.1 and 10.2 in the previous section, one also has a version of the formula for cumulants with a fixed number of variables, i.e. for the generating function

$$\begin{aligned} C_n(z_1,\dots,z_n) \;&=\; \langle|W(z_1)\otimes\cdots\otimes W(z_n)|\rangle_L\\ &=\; \sum_{k_1,\dots,k_n\ge 0}\langle Q_{k_1}|\dots|Q_{k_n}\rangle_L\; z_1^{k_1-1}\cdots z_n^{k_n-1}\\ &=\; \frac{\delta_{n,1}}{z_1}\,+\sum_{\ell_1,\dots,\ell_n\ge 1}\langle\langle\,\ell_1,\cdots,\ell_n\,\rangle\rangle\;\frac{z_1^{\ell_1}\cdots z_n^{\ell_n}}{\ell_1!\,\cdots\,\ell_n!}\,.\end{aligned} \tag{119}$$

(Here the last equality holds because $Q_0=1$, $Q_1=0$, and $Q_{\ell+1}=p_\ell/\ell!$ for $\ell\ge 1$ and because all connected brackets having some argument equal to 1 vanish except for $\langle\langle\, 1\,\rangle\rangle = 1$.) This formula, which can be deduced from Theorem 10.1, is equivalent to [18, Theorem 6.7], where it is stated in a somewhat different form, but here we will deduce it instead from Proposition 11.3.

Proposition 11.4. *The generating function* (119) *is given by*

$$C_n(z_1,\dots,z_n) \;=\; \sum_{\alpha\in\mathcal{P}(n)} (-1)^{\ell(\alpha)-1}\, z_N^{\ell(\alpha)-2} \prod_{A\in\alpha} \frac{z_A^{|A|}/2}{\sinh(z_A/2)}\,, \tag{120}$$

where N and z_A for $A\subseteq N$ have the same meaning as in Theorem 10.1.

Proof. We use the same formalism and notations as in the proof of Theorem 10.2. In view of equations (113), (119), and (97) we have $\Omega_n[C_n(z_1,\dots,z_n)] = \psi(\mathbf{u})_L^{(n)}$, the degree n part of $\psi(\mathbf{u})_L$, so if we denote by $\psi(\mathbf{u})_L^{(n,m)}$ $(1\le m\le n)$ the degree n part of the m-th term in (117) and by $C_{n,m}(z_1,\dots,z_n)$ the subsum of the right hand side of (120) corresponding to partitions $\alpha\in\mathcal{P}(n)$ with $\ell(\alpha)=m$, then it suffices to prove that $\Omega_n[C_{n,m}] = \psi(\mathbf{u})_L^{(n,m)}$ for each m. Instead of (102) we now use that $z^{m-2} = (m-2)!\,[y^{m-2}]e^{zy}$ to get

$$C_{n,m}(z_1,\dots,z_n) \;=\; (-1)^{m-1}(m-2)!\,\big[y^{m-2}\big] \sum_{\substack{\alpha\in\mathcal{P}(n)\\ \ell(\alpha)=m}} \prod_{A\in\alpha} \Big(z_A^{|A|-1}\,B(z_A)\,e^{z_A y}\Big)$$

for $m\ge 2$, with $B(x)$ as in (63). Then using (100) and the fact that the operations Ω_n and $[y^{m-2}]$ commute, we find

$$\Omega_n\big[C_{n,m}\big] \;=\; \frac{(-1)^{m-1}}{m(m-1)} \left(\big[y^{m-2}\big]\left(\sum_{s\ge1}\frac{\gamma_s(\mathbf{u})}{s!}\right)^{m}\right)^{(n)}$$

where $\gamma_{s,y}(\mathbf{u}) := G_{s,y}(d/dt)(U(t)^s)|_{t=0}$ with $G_{s,y}(z) = z^{s-1}B(z)e^{yz}$. But

$$\sum_{s\ge1}\frac{\gamma_s(\mathbf{u})}{s!} \;=\; \sum_{s\ge1}\left(\sum_{\substack{k,r\ge0\\ k+r\ge1}} \beta_k\,\frac{y^r}{r!}\,\frac{d^{k+r+s-1}}{dt^{k+r+s-1}}\Big(\frac{U(t)^s}{s!}\Big)\Big|_{t=0}\right) \;=\; \mathcal{B}'(\mathbf{u},y)$$

by (103) with $X=1$. This completes the proof of the cases $m\ge2$. The case $m=1$ is similar but easier, using

$$\sum_{n\ge1}\Omega_n\big[C_{n,1}\big] \;=\; \sum_{n\ge1}\left(\sum_{k\ge2}\beta_k\,\frac{d^{k+n-2}}{dt^{k+n-2}}\Big(\frac{U(t)^n}{n!}\Big)\Big|_{t=0}\right) \;=\; \mathcal{B}(\mathbf{u},0)\,.$$

□

Theorem 11.2 or either of the last two propositions can let us compute the leading terms of connected brackets whose arguments are single generators p_ℓ. For the leading terms of more general connected brackets, we need a formula that expresses mixed brackets, involving both products and slashes, as products of connected brackets of single variables. A special case of this formula is (109) above, and a simple mixed example is

$$\langle f|gh\rangle \;=\; \langle f|g|h\rangle \,+\, \langle g\rangle\langle f|h\rangle \,+\, \langle h\rangle\langle f|g\rangle\,.$$

The general result is stated, for arbitrary rings and brackets, in the following proposition. Certain versions of the result were known before (e.g. it is equivalent to [42, Proposition 4.3]; cf. [37, Chapter 6], p. 279), but the proof is not easy to find in the literature, and hence we give a short one here. For the formulation we need some terminology. If α and β are partitions of a finite set N, we denote by $\alpha\vee\beta$ the finest partition coarser than both (i.e. if we think of partitions as equivalence relations, the equivalence relation generated by α and β). We denote by $\mathbf{1}_N$ the one-element partition $\{N\}$. If $\alpha\vee\beta=\mathbf{1}_N$, then it is easy to see that $|\alpha|+|\beta|\le|N|+1$. If equality holds, then the partitions are called *complementary.* The pairs with $\alpha\vee\beta=\mathbf{1}_N$ will play a role in the following proposition, while complementary partitions appear in the corollary concerning leading terms.

Proposition 11.5. *Let $f_1,\dots,f_n$ be elements of $\mathcal{R}$. Then for any partition β of $N=\{1,\dots,n\}$ we have*

$$\big\langle\big|\bigotimes_{B\in\beta} f_B\big|\big\rangle \;=\; \sum_{\substack{\alpha\in\mathcal{P}(n)\\ \alpha\vee\beta=\mathbf{1}_N}} \prod_{A\in\alpha}\big\langle\big|\bigotimes_{a\in A} f_a\big|\big\rangle\,, \tag{121}$$

where $f_B=\prod_{b\in B}f_b$ for $B\subseteq N$.

Proof. We first recall the generalized *Möbius inversion formula* for partially ordered sets in the special case of the lattice of partitions of N, ordered by $\alpha\le\beta$ if α is finer than β (cf. [41], especially Example 1 of Section 7). If g is any function on $\mathcal{P}(n)$ and G is the associated *cumulative function*

$$G(\alpha)=\sum_{\beta\le\alpha}g(\beta),\quad\text{ then }\quad g(\beta)=\sum_{\alpha\le\beta}\mu(\alpha,\beta)G(\alpha)$$

with the *Möbius function* $\mu(\alpha,\beta)$ given by $\prod_{B\in\beta}(-1)^{|\alpha_B|-1}(|\alpha_B|-1)!$, where α_B for $B\subseteq N$ is the partition on B induced by α. In this notation the definition (108) of the connected bracket can be written as

$$\langle|\otimes_{i\in N}f_i\,|\rangle \;=\; \sum_{\alpha\in\mathcal{P}(N)}\mu(\alpha,\mathbf{1}_N)\prod_{A\in\alpha}\langle f_A\rangle\,,$$

whose Möbius inversion is (109). We apply this with N replaced by β, noting that $\mathcal{P}(\beta)$ can be identified with $\{\gamma\in\mathcal{P}(N)\mid\gamma\ge\beta\}$, to obtain

$$\langle|\otimes_{B\in\beta}f_B\,|\rangle \;=\; \sum_{\gamma\ge\beta}\mu(\gamma,\mathbf{1}_N)\prod_{C\in\gamma}\langle f_C\rangle\,.$$

We now apply (109) to each factor on the right hand side and identify $\prod_{C\in\gamma}\mathcal{P}(C)$ with $\{\alpha\in\mathcal{P}(N)\mid\alpha\le\gamma\}$ to obtain

$$\langle|\otimes_{B\in\beta}f_B\,|\rangle \;=\; \sum_{\alpha}\Big(\sum_{\gamma\ge\alpha\vee\beta}\mu(\gamma,\mathbf{1}_N)\Big)\prod_{A\in\alpha}\langle|\otimes_{a\in A}f_a\,|\rangle\,.$$

The proposition follows since $\sum_{\gamma\ge\alpha\vee\beta}\mu(\gamma,\mathbf{1}_N)=\delta_{\mathbf{1}_N,\alpha\vee\beta}$. □

Proposition 11.5 can be used in particular with $f_i=p_{\ell_i}\in\mathcal{R}$ for integers $\ell_i\in\mathbb{N}$ to compute arbitrary connected q- or X-brackets in terms of those whose arguments are single p_ℓ's. In the case of the X-brackets, we see from Proposition 11.1 that the total degree drop on the left hand side of (121) (i.e. the minimal difference between the degree of this polynomial with respect to X and $K/2$, where $K=\sum(\ell_i+1)$ is the total weight) is $|\beta|-1$, while the degree drop for the α-th term on the right is $\sum_{A\in\alpha}(|A|-1)=n-|\alpha|$, which is strictly smaller than $|\beta|-1$ unless α and β are complementary. We therefore obtain the following expression for the leading terms of arbitrary connected X-brackets in terms of rational cumulants.

Corollary 11.6. *Let $\ell_1,\dots,\ell_n$ be natural numbers and for $B\subseteq N=\{1,\dots,n\}$ set $f_B=\prod_{b\in B}p_{\ell_b}\in\mathcal{R}$. Then for any partition $\beta=\{B_1,\dots,B_s\}$ of N we have*

$$\left\langle f_{B_1}|\cdots|f_{B_s}\right\rangle_L \;=\; \sum_{\alpha}\prod_{A\in\alpha}\langle\langle p_{\ell_a},\ a\in A\rangle\rangle_{\mathbb{Q}}\,, \tag{122}$$

where the sum is over all partitions α of N that are complementary to β.

We single out one important special case of this corollary. If $|\beta|=n-1$, so that β has the form $\{\{1,2\},\{3\},\dots,\{n\}\}$, then the partitions α with $\alpha\vee\beta=\mathbf{1}_N$ are the one-set partition $\mathbf{1}_N$ and the two-set partitions $\{A_1,A_2\}$ with $1\in A_1$ and $2\in A_2$, with all but the first of these being complementary to β. Therefore Corollary 11.6 in this case tells us that for any $f,g,h_i\in\mathcal{R}$ we have

$$\left\langle fg|h_1|\cdots|h_m\right\rangle_L \;=\; \sum_{I\sqcup J=\{1,\dots,m\}}\left\langle|\,f\otimes\prod_{i\in I}h_i\,|\right\rangle_L\left\langle|\,g\otimes\prod_{j\in J}h_j\,|\right\rangle_L.$$

In particular, for any $n_1,n_2,\ldots\geq 0$ we have

$$\begin{aligned}&\left\langle fg|\underbrace{p_1|\cdots|p_1}_{n_1}|\underbrace{p_2|\cdots|p_2}_{n_2}|\cdots\right\rangle_L\\ =&\sum_{\mathbf{n}=\mathbf{n}'+\mathbf{n}''}\frac{\mathbf{n}!}{\mathbf{n}'!\,\mathbf{n}''!}\left\langle f|\underbrace{p_1|\cdots|p_1}_{n_1'}|\underbrace{p_2|\cdots|p_2}_{n_2'}|\cdots\right\rangle_L\left\langle g|\underbrace{p_1|\cdots|p_1}_{n_1''}|\underbrace{p_2|\cdots|p_2}_{n_2''}|\cdots\right\rangle_L.\end{aligned}$$

Making a generating series, we obtain the following proposition.

Proposition 11.7. *The map $\mathcal{R}\to\mathbb{Q}[\mathbf{u}]$ defined by*

$$\Psi(f;\mathbf{u}) \;=\; \sum_{\mathbf{n}\geq 0}\left\langle f|\underbrace{p_1|\cdots|p_1}_{n_1}|\underbrace{p_2|\cdots|p_2}_{n_2}|\cdots\right\rangle_L\frac{\mathbf{u}^n}{\mathbf{n}!} \tag{123}$$

is a homomorphism of $\mathbb{Q}$-algebras.

Note that the generating series (123) for $f=p_\ell$ takes the value

$$\Psi(p_\ell;\mathbf{u}) \;=\; \sum_{\mathbf{n}\geq 0}\left\langle\underbrace{p_1|\cdots|p_1}_{n_1}|\cdots|\underbrace{p_\ell|\cdots|p_\ell}_{n_\ell+1}|\cdots\right\rangle_L\frac{\mathbf{u}^n}{\mathbf{n}!} \;=\; \frac{\partial\Psi(\mathbf{u})_L}{\partial u_\ell}\,, \tag{124}$$

and since $\mathcal{R}$ is generated by the p_ℓ, this also gives the general values. A more explicit formula for $\Psi(p_\ell;\mathbf{u})$ will be given in equation (145) below.

12. One-variable generating series for cumulants

The main generating series identities of the last two sections were expressed in terms of a multi-variable $\mathbf{u}=(u_1,u_2,\dots)$. For our main applications to the calculations of volumes and Siegel-Veech constants, we will be particularly interested in the specialization to the case when this multi-variable has the special form $(0,u,0,0,\dots)$ for a single variable u. The basic invariants here are the special cumulants

$$v_n \;=\; \frac{1}{n!}\,\langle\langle\underbrace{2,\dots,2}_{n}\rangle\rangle_{\mathbb{Q}} \;=\; \frac{1}{n!}\,\langle\underbrace{p_2|\cdots|p_2}_{n}\rangle_L \qquad (n>0) \tag{125}$$

involving only 2's (corresponding to coverings of a torus having only simple branch points), which will be used in Part IV for the computation of the volume of the principal stratum of abelian differentials, and their generating series

$$\begin{aligned}\psi(u) &= \Psi(0,u,0,0,\dots)_L = \sum_{n=2}^{\infty} v_n\, u^n \\ &= \frac{1}{90}u^2 - \frac{7}{162}u^4 + \frac{377}{810}u^6 - \frac{23357}{2430}u^8 + \frac{16493303}{51030}u^{10} - \cdots .\end{aligned} \tag{126}$$

Note that v_n in (125) vanishes unless n is even, and then corresponds to genus g coverings of a torus, where $n = 2g-2$. To take into account genus 0 and 1, it turns out to be appropriate to extend (125) to all n by defining

$$v_{-2} = v_0 = -\frac{1}{24}, \qquad v_n = 0 \text{ for } n \text{ odd or } n < -2\,. \tag{127}$$

The next most important numbers for us are the mixed cumulants defined by

$$v_{n,k} = \frac{k}{n!}\,\langle\langle \underbrace{2,\dots,2}_{n}, k-1\,\rangle\rangle_{\mathbb{Q}} = \frac{k!}{n!}\,\langle \underbrace{p_2|\cdots|p_2}_{n}\,|Q_k\rangle_L \tag{128}$$

for $n \ge 0$, $k \ge 1$ and by $v_{n,0} = \delta_{n,0}$ if $k = 0$ (which agrees with (128) in that case since $Q_0 = 1$), with corresponding generating series

$$\psi_k(u) = \sum_{n=0}^{\infty} v_{n,k}\, u^n = k!\,\Psi(Q_k; 0, u, 0, 0, \dots)\,, \tag{129}$$

where $\Psi(F;\mathbf{u})$ is the power series associated to $F \in \mathcal{R}$ in Proposition 11.7. The values of $v_{n,k}$ for $k \in \{0,1,2,3\}$ and $n \ge 0$ are given in terms of v_n by

$$v_{n,0} = \delta_{n,0}, \quad v_{n,1} = 0, \quad v_{n,2} = (4n+2)v_n, \quad v_{n,3} = (3n+3)v_{n+1}\,, \tag{130}$$

and the first numerical values of $v_{n,k}$ for $4 \le k \le 6$ are given by

$$\begin{aligned}\psi_4(u) &= \frac{7}{240} - \frac{5u^2}{18} + \frac{259u^4}{54} - \frac{110773u^6}{810} + \frac{2220941u^8}{378} - \cdots, \\ \psi_5(u) &= -\frac{13u}{126} + \frac{179u^3}{81} - \frac{33415u^5}{486} + \frac{26367046u^7}{8505} - \frac{29692284359u^9}{153090} + \cdots, \\ \psi_6(u) &= -\frac{31}{1344} + \frac{587u^2}{720} - \frac{38525u^4}{1296} + \frac{84696203u^6}{58320} - \frac{12981245593u^8}{136080} + \cdots .\end{aligned}$$

Finally, we want to study the particular combinations of cumulants defined by

$$\kappa_n = \sum_{k=0}^{n} 2^k\, v_{n-k,k} = \delta_{n,0} + \sum_{k=1}^{n} \frac{2^k k}{(n-k)!}\,\langle\langle \underbrace{2,\dots,2}_{n-k}, k-1\,\rangle\rangle_{\mathbb{Q}}\,, \tag{131}$$

which will be related in Part IV to the area Siegel-Veech constants $c_{\mathrm{area}}(\Omega\mathcal{M}_g(1^{2g-2}))$, and the corresponding generating function

$$\begin{aligned}K(u) &= \sum_{n=0}^{\infty} \kappa_n\, u^n = \sum_{k=0}^{\infty} (2u)^k\, \psi_k(u) \\ &= 1 - \frac{1}{3}u^2 + \frac{13}{9}u^4 - \frac{445}{27}u^6 + \frac{142333}{405}u^8 - \frac{975203}{81}u^{10} + \cdots .\end{aligned} \tag{132}$$

In this section, which uses all of the results proved in Part II, we give explicit formulas allowing for the numerical calculation of the coefficients of each of the generating series ψ, ψ_k, and K (and also, as we will see in Part IV, for the asymptotic evaluation of these coefficients). It turns out that all of these generating functions can be expressed by a single sequence of Laurent series, which we now introduce.

We begin by defining a polynomial of degree and parity n for each integer $n \geq 0$ by $\mathfrak{B}_n(X) = B_n(X+\frac{1}{2})$, the n-th Bernoulli polynomial with its argument shifted by one-half. Its first values are 1, X, $X^2-\frac{1}{12}$, $X^3-\frac{X}{4}$, and $X^4-\frac{X^2}{2}+\frac{7}{240}$, and its expansion for general n is given by

$$\mathfrak{B}_n(X) \;=\; \sum_{k=0}^{n} (n)_k\,\beta_k\,X^{n-k} \qquad (n=0,1,2,\dots), \tag{133}$$

where β_k is as in (63) and $(n)_k = n(n-1)\cdots(n-k+1)$ is the descending Pochhammer symbol. We extend this definition to arbitrary complex values of n by setting

$$\mathfrak{B}_n(X) \;=\; \sum_{k=0}^{\infty} (n)_k\,\beta_k\,X^{n-k} \;\in\; X^n\mathbb{C}[X^{-1}] \qquad (n\in\mathbb{C})\,, \tag{134}$$

a shifted Laurent series whose expansion begins with

$$\mathfrak{B}_n(X) \;=\; X^n \,-\, \frac{n(n-1)}{24}X^{n-2} \,+\, \frac{7n(n-1)(n-1)(n-3)}{5760}X^{n-4} \,-\, \cdots\,.$$

The fact that this series is divergent for all $n \notin \mathbb{Z}_{\geq 0}$ is not important for us, since we will use it only as a formal series, but it is worth mentioning that $\mathfrak{B}_n(X)$ can be defined as an actual function of n and X by the formula

$$\mathfrak{B}_n(X) \;=\; -\,n\,\zeta(1-n, X+\tfrac{1}{2}) \qquad (n\in\mathbb{C},\; X\in\mathbb{C}\smallsetminus(-\infty,-\tfrac{1}{2}]), \tag{135}$$

where $\zeta(s,\alpha)$ denotes the Hurwitz zeta-function, defined by the convergent series $\sum_{m=0}^{\infty}(m+\alpha)^{-s}$ for $\alpha\in\mathbb{C}\smallsetminus(-\infty,0]$ and $\Re(s)>1$ and then for all s by meromorphic continuation. This new function $\mathfrak{B}_n(X)$ is entire in n (since $\zeta(s,\alpha)$ has a simple pole at $s=1$ as its only singularity), reduces to the previous definition if n is a non-negative integer, and has the asymptotic expansion (134) for all $n\in\mathbb{C}$, as one can see for instance for $\Re(n)<0$ from the integral representation $\frac{1}{\Gamma(-n)}\int_0^\infty \frac{t^{-n}e^{-tX}dt}{2\sinh t/2}$ valid in that case. From the formula (135), or from the definition (134) and a simple calculation with Bernoulli numbers, we see that $\mathfrak{B}_n(X)$ satisfies the functional equation

$$\mathfrak{B}_n(X+\tfrac{1}{2}) \,-\, \mathfrak{B}_n(X-\tfrac{1}{2}) \;=\; n\,X^{n-1} \tag{136}$$

for all n, and for $n\notin\mathbb{Z}_{\geq 0}$ this property characterizes $\mathfrak{B}_n(X)$ uniquely as an element of $X^n\mathbb{C}[X^{-1}]$, giving us an alternative and less computational definition.

For our purposes we need only the cases $n\in\mathbb{Z}_{\geq 0}$, where $\mathfrak{B}_n(X)$ is a polynomial, and $n\in\mathbb{Z}_{\geq 0}-\frac{1}{2}$, the first three cases here being

$$\begin{aligned}
\mathfrak{B}_{-1/2}(X) \;&=\; X^{-1/2} - \frac{1}{32}X^{-5/2} + \frac{49}{6144}X^{-9/2} - \frac{341}{65536}X^{-13/2} + \cdots\,,\\
\mathfrak{B}_{1/2}(X) \;&=\; X^{1/2} + \frac{1}{96}X^{-3/2} - \frac{7}{6144}X^{-7/2} + \frac{31}{65536}X^{-11/2} - \cdots\,,\\
\mathfrak{B}_{3/2}(X) \;&=\; X^{3/2} - \frac{1}{32}X^{-1/2} + \frac{7}{10240}X^{-5/2} - \frac{31}{196608}X^{-9/2} + \cdots\,.
\end{aligned}$$

We can now state our final formulas for the generating functions ψ, ψ_k, and K.

Theorem 12.1. *Let the Laurent series $X(u) = (4u)^{-1} + \cdots$ be defined by*

$$X \;=\; X(u) \quad \Longleftrightarrow \quad \frac{1}{2\sqrt{u}} \;=\; \mathfrak{B}_{1/2}(X)\,. \tag{137}$$

Then the numbers v_n defined by equations (125) *and* (127) *are given either by the generating series*

$$\sum_{n=-2}^{\infty} (4n+2) v_n \, u^{n+1} \;=\; X(u) \tag{138}$$

or by the generating series

$$\sum_{n=-2}^{\infty} (3n+3)\, v_n \, u^{n+1/2} \;=\; \mathfrak{B}_{3/2}(X(u))\,. \tag{139}$$

Theorem 12.2. *Define $X = X(u)$ as in Theorem 12.1. Then the generating series ψ_k defined by* (128) *and* (129) *is given for all $k \geq 0$ by*

$$\psi_k(u) \;=\; \frac{1}{(2u)^k} \sum_{m=0}^{k} (-1)^m \binom{k}{m} (4u)^{m/2}\, \mathfrak{B}_{m/2}(X(u))\,. \tag{140}$$

Theorem 12.3. *Let X and u be as above. Then the generating series* (132) *is given by*

$$2\, u^{1/2}\, K(u) \;=\; \mathfrak{B}_{-1/2}(X(u))\,. \tag{141}$$

We make a few remarks on these theorems before giving their proofs.

1. By taking a linear combination of equations (138) and (139) we can also obtain the explicit, though not very attractive, closed formula

$$\psi(u) \;=\; \frac{2}{3\sqrt{u}}\, \mathfrak{B}_{3/2}(X(u)) \;-\; \frac{X(u)}{2u} \;+\; \frac{1}{24} \;+\; \frac{1}{24u^2} \tag{142}$$

for the original generating series $\psi(u)$ defined in (126).

2. The right hand side of equation (140) reduces to 1 and to $\frac{1}{2u}(1-\sqrt{4u}\,\mathfrak{B}_{1/2}(X))$ for $k = 0$ and $k = 1$, respectively, so Theorem 12.2 gives the correct values $\psi_0(u) = 1$ and $\psi_1(u) = 0$ in these two cases. In fact, if we wished we could rewrite the whole theorem as the assertion that there is *some* Laurent series $X = X(u) = \frac{1}{4u} + \cdots$ such that (140) holds for all $k \geq 0$, since then the special case $k = 1$ combined with the fact that ψ_1 vanishes identically would force the relation $\sqrt{4u}\,\mathfrak{B}_{1/2}(X) = 1$.

3. Similarly, using that $\mathfrak{B}_1(X) = X$ we find that equation (140) for $k = 2$ and $k = 3$ reduces to $4u^2\psi_2(u) = -1+4uX$ and $8u^3\psi_3(u) = -2+12uX-8u^{3/2}\mathfrak{B}_{3/2}(X)$, respectively, in agreement with equations (130), (138), and (139).

4. The individual terms on the right hand side of (140) have poles of order k in u, but all the negative powers of u cancel in the sum because the coefficient of u^i in $(4u)^m\mathfrak{B}_{m/2}(X)$ is a polynomial of degree i in m for all $i \geq 0$ and the k-th difference of such a polynomial vanishes if $i < k$.

5. This same delicate cancellation means that one cannot deduce equation (141) from equation (140) simply by plugging the latter into (132), because in the double series obtained by this substitution one cannot interchange the order of summation.

For the proof of Theorems 12.1–12.3 we use the formalism of the previous two sections and in particular the power series $\mathcal{B}(\mathbf{u}, y)$ and its specialization

$$\mathcal{B}(u,y) \;=\; \mathcal{B}((0,u,0,0,\dots),y) \;=\; \sum_{k\geq 0}\beta_k \sum_{\substack{a\geq 1,\, r\geq 0\\ a-r=k-2}} \frac{(2a)!}{a!r!}\, u^a y^r\,. \tag{143}$$

to $\mathbf{u} = (0,u,0,0,\dots)$. It is obvious from the definition that the specialization of $\mathcal{B}(u,y)$ to $y=0$ can be expressed in terms of the function $\mathfrak{B}_{3/2}(X)$ by

$$\mathcal{B}(u,0) \;=\; \sum_{k=3}^{\infty}\frac{(2k-4)!}{(k-2)!}\,\beta_k u^{k-2} \;=\; \frac{2}{3\sqrt{u}}\mathfrak{B}_{3/2}\Big(\frac{1}{4u}\Big) - \frac{1}{12u^2} + \frac{1}{24}\,.$$

What is more surprising is that the *whole* two-variable function $\mathcal{B}(u,y)$ can be expressed in terms of the one-variable function $\mathfrak{B}_{3/2}(X)$, as stated in the following proposition. This is the reason why the whole story works.

Proposition 12.4. *The two-variable function $\mathcal{B}(u,y)$ defined by* (143) *can be expressed in terms of the one-variable function $\mathfrak{B}_{3/2}(X)$ by the formula*

$$\mathcal{B}(u,y) \;=\; \frac{2}{3\sqrt{u}}\,\mathfrak{B}_{3/2}\Big(\frac{1-4uy}{4u}\Big) \;-\; \frac{y^2}{2} \;+\; \frac{y}{2u} \;-\; \frac{1}{12u^2} \;+\; \frac{1}{24}\,. \tag{144}$$

Proof. We have

$$\begin{aligned}\mathcal{B}(u,y) \;&=\; \sum_{k\geq 0}\beta_k \sum_{\substack{a\geq 1,\, r\geq 0\\ a-r=k-2}} \frac{(2a-1)!!}{r!}\,(2u)^a y^r\\ &=\; \frac{4}{3}\sum_{k\geq 0}(3/2)_k\,\beta_k\,(4u)^{k-2} \sum_{\substack{r\geq 0\\ k+r\geq 3}} \binom{r+k-\frac{5}{2}}{r}(4uy)^r\,.\end{aligned}$$

The proposition then follows since the internal sum is equal to $(1-4uy)^{3/2-k}$ by the binomial theorem in all cases except $k=2$ and $k=0$, where we must subtract one or three monomials corresponding to $0\leq r\leq 2-k$. (The term $k=1$ does not enter since $\beta_k=0$ for k odd.) □

We observe that the above proposition and its proof are just the specialization of Proposition 10.3 to $\mathbf{u} = (0,u,0,0,\dots)$, since in that case the function $U(t)$ reduces to ut^2, the solution $T(y) = T(u,y)$ of $T = U(y+T)$ is given by

$$T(y) \;=\; \frac{1-2uy-\sqrt{1-4uy}}{2u}\,,$$

and the integral and derivatives of $T(y)$ are given by

$$\begin{aligned}\int_0^y T(y')dy' \;&=\; \frac{(1-4uy)^{3/2}-(1-6uy+6u^2y^2)}{12u^2}\,,\\ T^{(k-1)}(y) \;&=\; -\,\delta_{k,2} \;+\; \frac{2}{3\sqrt{u}}\,(3/2)_k\left(\frac{4u}{1-4uy}\right)^{k-3/2} \qquad (k\geq 2)\,.\end{aligned}$$

Using Proposition 12.4 we can now give the proofs of all three theorems above.

Proof of Theorem 12.1. We calculate $\psi(u)$ using Theorem 11.2. Differentiating (144) and using the obvious formula $\mathfrak{B}'_n(X) = n\mathfrak{B}_{n-1}(X)$ for any n, we find that

$$\mathcal{B}'(u,y) + y = \frac{1}{2u} - \frac{1}{\sqrt{u}}\,\mathfrak{B}_{1/2}\Big(\frac{1-4uy}{4u}\Big),$$

which vanishes if $X = \frac{1-4uy}{4u}$ is related to u by $\mathfrak{B}_{1/2}(X) = \frac{1}{2\sqrt{u}}$. Thus the function $y_0 = y_0(u)$ occurring in Theorem 11.2 is related to the function $X(u)$ defined in Theorem 12.1 by $X(u) = \frac{1-4uy_0(u)}{4u}$. Substituting this into Theorem 11.2 and using Proposition 12.4 again gives equation (142) after a short computation, and differentiating this equation and using the definition of $y_0(u)$ once again lets us then deduce the nicer formulas (138) and (139). □

Proof of Theorem 12.2. From equations (124) and (114) we have, for any $\ell \geq 1$,

$$\begin{aligned}\Psi(p_\ell, \mathbf{u}) &= \frac{\partial \Psi(\mathbf{u})_L}{\partial u_\ell} = \frac{\partial}{\partial u_\ell}\Big(\mathcal{B}\big(\mathbf{u}, y_0(\mathbf{u})\big) + \frac{1}{2}y_0(\mathbf{u})^2\Big)\\ &= \frac{\partial \mathcal{B}(\mathbf{u},y)}{\partial u_\ell}\Big|_{y=y_0(\mathbf{u})} + \Big(\mathcal{B}'\big(\mathbf{u}, y_0(\mathbf{u})\big) + y_0(\mathbf{u})\Big)\frac{\partial y_0(\mathbf{u})}{\partial u_\ell}\\ &= \frac{\partial \mathcal{B}(\mathbf{u},y)}{\partial u_\ell}\Big|_{y=y_0(\mathbf{u})}. \end{aligned} \tag{145}$$

On the other hand, specializing (105) to $\mathbf{u} = (0,u,0,0,\dots)$ and $X=1$ we get

$$n\,\frac{\partial \mathcal{B}(\mathbf{u},y)}{\partial u_{n-1}}\Big|_{\mathbf{u}=(0,u,0,0,\dots)} = \sum_{k=0}^{\infty} \beta_k\,\frac{\partial^k}{\partial y^k}\big(t(u,y^n)\big).$$

Substituting into this the formula

$$\begin{aligned}\frac{\partial^k}{\partial y^k}\big(t(u,y)^n\big) &= \frac{\partial^k}{\partial y^k}\bigg(\Big(\frac{1-\sqrt{1-4uy}}{2u}\Big)^n\bigg)\\ &= \frac{\partial^k}{\partial y^k}\bigg(\frac{1}{(2u)^n}\sum_{m=0}^{n}(-1)^m\binom{n}{m}(1-4uy)^{m/2}\bigg)\\ &= \frac{1}{(2u)^n}\sum_{m=0}^{n}(-1)^m\binom{n}{m}(m/2)_k\,(4u)^k(1-4uy)^{m/2-k}\end{aligned}$$

and interchanging the order of summation we get (after changing n to k)

$$k\,\frac{\partial \mathcal{B}(\mathbf{u},y)}{\partial u_{k-1}}\Big|_{\mathbf{u}=(0,u,0,0,\dots)} = \frac{1}{(2u)^k}\sum_{m=0}^{k}(-1)^m\binom{k}{m}(4u)^{m/2}\,\mathfrak{B}_{m/2}\Big(\frac{1-4uy}{4u}\Big),$$

and now combining this with (145) and remembering that $\frac{1-4uy_0(u)}{4u} = X(u)$ we obtain the formula (140) for the power series $\psi_k(u) = k\Psi(p_{k-1}; 0,u,0,0,\dots)$. □

Proof of Theorem 12.3. Just as in Remark **4.** above, equation (140) gives

$$2^k\, v_{n-k,k} = [u^n]\big((2u)^k\psi_k(u)\big) = \sum_{m=0}^{k}(-1)^m\binom{k}{m}P_n(m)$$

for any $n \geq k \geq 0$, where $P_n(m)$ is the coefficient of u^n in $(4u)^{m/2}\mathfrak{B}_{m/2}(X(u))$, which is a polynomial of degree $\leq n$ in m. Summing over $0 \leq k \leq n$, we find

$$\kappa_n \;=\; \sum_{0\leq m\leq k\leq n} (-1)^m \binom{k}{m} P_n(m) \;=\; \sum_{m=0}^{n} (-1)^m \binom{n+1}{m+1} P_n(m) \;=\; P_n(-1)\,,$$

where the last equality holds because the $(n+1)$st difference of a polynomial of degree $\leq n$ vanishes. It follows that the generating function $K(u) = \sum \kappa_n u^n$ equals $(2u)^{-1/2}\mathfrak{B}_{-1/2}(X(u))$, as asserted. □

Part III: The hook-length moment T_p

The heros of Part III are the hook-length moments

$$T_p(\lambda) \;=\; \sum_{\xi\in Y_\lambda} h(\xi)^{p-1} \qquad (p>0 \text{ odd}), \tag{146}$$

where Y_λ is the Young diagram of λ and $h(\xi)$ is the hook-length of the cell ξ. We will show that these functions belong to the ring of shifted symmetric polynomials and study their effect on q-brackets of functions on partitions.

In Section 13 we show that T_p appears naturally in a spectral decomposition of Schur's orthogonality relation and as a natural function whose q-brackets are Eisenstein series. The q-brackets are linear, but are far from ring homomorphisms. In Section 14 we give a remarkable formula that expresses the multiplicative effect of T_p inside a q-bracket only in terms of Eisenstein series and a collection of differential operators. The proof of these formula relies on a two-step recursive expression for the Bloch-Okounkov functions that we will give in Section 15. Finally, in Section 16 we apply our knowledge about T_p, which we extend to include T_{-1}, to prove the quasimodularity of Siegel-Veech generating series that appeared at the end of Part I.

13. From part-length moments to hook-length moments

The function T_p defined in (146) has three remarkable properties that we discuss in this section. (We continue to use the notations for partitions given at the beginning of Section 7.) The first property appears in the problem of decomposing Schur's orthogonality relation, which states that for $\lambda_1,\lambda_2\in\mathbf{P}(d)$

$$\frac{1}{d!}\sum_{\mu\in\mathbf{P}(d)} z_\mu \left(\sum_{m=1}^{\infty} m r_m(\mu)\right)\chi^{\lambda_1}(\mu)\chi^{\lambda_2}(\mu) \;=\; d\delta_{\lambda_1,\lambda_2},$$

where $z_\mu = d!\cdot(\prod_{m=1}^\infty m^{r_m(\mu)}\prod_{m=1}^\infty r_m(\mu)!)^{-1}$ is the size of the conjugacy class of the partition μ. What is the contribution, if we fix m in the inner sum?

To give the answer, we denote by $h(\xi)$ the hook-length of a cell $\xi\in Y_\lambda$ in the Young diagram of λ and define the hook-length count and the related counting polynomial to be

$$N_m(\lambda) \;=\; |\{\xi\in Y_\lambda \mid h(\xi)=m\}|\,, \quad H_\lambda(t) \;=\; \sum_{m=1}^{\infty} N_m(\lambda)\,t^m \;=\; \sum_{\xi\in Y_\lambda} t^{h(\xi)} \;\in\; t\,\mathbb{Z}[t]\,.$$

Theorem 13.1. *For each $d\in\mathbb{N}$, $\lambda\in\mathbf{P}(d)$, and $m\in\mathbb{N}$, we have the identity*

$$\frac{1}{d!}\sum_{\mu\in\mathbf{P}(d)} z_\mu\, m r_m(\mu)\chi^\lambda(\mu)^2 \;=\; N_m(\lambda) \tag{147}$$

We define, as in Part I, the p-th weight $S_p(\lambda)=\sum_{j=1}^k \lambda_j^p$ of a partition $\lambda=(\lambda_1,\dots,\lambda_k)$. Multiplying (147) by t^m, summing over all $m\ge 1$, and taking the $(p-1)$-st moment, that is, applying $p-1$ times the differential operator $D=z\frac{\partial}{\partial z}$ and substituting $z=1$, we thus obtain the following statement, which is the original motivation for this section and will be used crucially in Section 16.

Corollary 13.2. *For every $\lambda\in\mathbf{P}(d)$*

$$\frac{1}{d!}\sum_{\mu\in\mathbf{P}(d)} S_p(\mu)\,z_\mu\,\chi^\lambda(\mu)^2 \;=\; T_p(\lambda)\,.$$

We remark that one can introduce a transformation $f \mapsto \mathbf{M}f$ on the functions $\mathbf{P} \to \mathbb{Q}$ to be

$$\mathbf{M}f(\lambda) \;=\; \frac{1}{d!} \sum_{\mu \in \mathbf{P}(d)} z_\mu f(\mu) \chi^\lambda(\mu)^2$$

for $\lambda \in \mathbf{P}(d)$. This transformation has the feature $\langle f \rangle_q = \langle \mathbf{M}f \rangle_q$ and, by the preceding results, the image of S_p under the transformation $\mathbf{M}$ is T_p. This was used in [49] as one of several examples to point out that the set of functions with quasimodular q-brackets is much larger than the ring of shifted symmetric functions.

The proof of Theorem 13.1 will actually give a more general formula. For two partitions σ and λ with $\sigma_i \leq \lambda_i$ we define a *skew Young diagram* λ/σ by removing the cells of Y_σ from the cells of Y_λ. We call λ/σ a *border strip* or *rim hook*, if it is connected (through edges of boxes, not only through vertices) and if it does not contain a 2×2 block. We also write $\lambda \setminus \gamma$ for the smaller partition σ after removing the rim hook γ from λ. The *height* $\mathrm{ht}(\gamma)$ of a rim hook γ is the number of its rows minus one. There is an obvious bijection between hooks and rim hooks that fixes the end-points of the hook. For $m \leq d$ we define a $|\mathbf{P}(d-m)| \times |\mathbf{P}(d)|$ matrix by

$$(D^d_m)_{\sigma,\lambda} = \begin{cases} (-1)^{\mathrm{ht}(\gamma)} & \text{if } \lambda/\sigma = \gamma \text{ is a rim hook} \\ 0 & \text{otherwise,} \end{cases}$$

where $\lambda \in \mathbf{P}(d)$ and $\sigma \in \mathbf{P}(m)$. Theorem 13.1 then follows from the result below.

Proposition 13.3. *For each pair $\lambda_1, \lambda_2 \in \mathbf{P}(d)$, we have*

$$\frac{1}{d!} \sum_{\mu \in \mathbf{P}(d)} z_\mu \, m r_m(\mu) \chi^{\lambda_1}(\mu) \chi^{\lambda_2}(\mu) = \big((D^d_m)^T D^d_m\big)_{\lambda_1,\lambda_2}. \tag{148}$$

Note that the matrix $(D^d_m)^T D^d_m$ does not depend on the choice of ordering the elements in $\mathbf{P}(d-m)$ that we used to form the matrix D^d_m.

Proof of Proposition 13.3. The proof will be based on the Murnaghan-Nakayama rule. To recall this, we say that $\alpha = (\alpha_1, \alpha_2, \ldots)$ is a *composition* of d, if $\alpha_i \in \mathbb{N}$ and $\sum_{i=1}^\infty \alpha_i = d$. (A partition is thus a composition with weakly decreasing α_i.) Let $\alpha \setminus \alpha_1$ denote the composition $(\alpha_2, \alpha_3, \ldots)$ of $d - \alpha_1$. The Murnaghan-Nakayama rule states that if $\lambda \in \mathbf{P}(d)$ and α is a composition of d, then

$$\chi^\lambda(\alpha) = \sum_{|\gamma| = \alpha_1} (-1)^{\mathrm{ht}(\gamma)} \chi^{\lambda \setminus \gamma}(\alpha \setminus \alpha_1),$$

where the sum is over all rim hooks γ of λ with α_1 cells.

The left hand side of (148) is a sum over $\mu \in \mathbf{P}(d)$ and only those with a part of length m contribute. We may thus use a composition $\mu = (m, \alpha_2(\mu), \alpha_3(\mu), \ldots)$ to evaluate the left hand side. Let $\mu' = (\alpha_2(\mu), \alpha_3(\mu), \ldots)$ and use γ_i to denote rim hooks of λ_i below. Then we have

$$\frac{1}{d!}\sum_{\mu\in\mathbf{P}(d)} z_\mu m r_m(\mu)\chi^{\lambda_1}(\mu)\chi^{\lambda_2}(\mu)$$
$$= \sum_{\mu\in\mathbf{P}(d)} \frac{z_\mu}{d!} m r_m(\mu)\left(\sum_{|\gamma_1|=m}(-1)^{\mathrm{ht}(\gamma_1)}\chi^{\lambda_1\smallsetminus\gamma_1}(\mu')\right)\left(\sum_{|\gamma_2|=m}(-1)^{\mathrm{ht}(\gamma_2)}\chi^{\lambda_2\smallsetminus\gamma_2}(\mu')\right)$$
$$= \sum_{\mu'\in\mathbf{P}(d-m)} \frac{z_{\mu'}}{(d-m)!}\left(\sum_{|\gamma_1|=m}(-1)^{\mathrm{ht}(\gamma_1)}\chi^{\lambda_1\smallsetminus\gamma_1}(\mu')\right)\left(\sum_{|\gamma_2|=m}(-1)^{\mathrm{ht}(\gamma_2)}\chi^{\lambda_2\smallsetminus\gamma_2}(\mu')\right)$$
$$= \sum_{|\gamma_1|=m}\sum_{|\gamma_2|=m}(-1)^{\mathrm{ht}(\gamma_1)+\mathrm{ht}(\gamma_2)}\sum_{\mu'\in\mathbf{P}(d-m)}\frac{z_{\mu'}}{(d-m)!}\chi^{\lambda_1\smallsetminus\gamma_1}(\mu')\chi^{\lambda_2\smallsetminus\gamma_2}(\mu')$$
$$= \sum_{|\gamma_1|=m}\sum_{|\gamma_2|=m}(-1)^{\mathrm{ht}(\gamma_1)+\mathrm{ht}(\gamma_2)}\delta_{\lambda_1\smallsetminus\gamma_1,\lambda_2\smallsetminus\gamma_2}.$$

This agrees with the right hand side by the definition of D^d_m. □

The second remarkable property is that the q-brackets of T_p are Eisenstein series.

Proposition 13.4. *For all $p\in\mathbb{Z}$*

$$\langle T_p\rangle_q \;=\; \sum_{d\geq 1}\sigma_p(d)\,q^d.$$

Note in particular, that $\langle T_{-1}\rangle_q = -\log((q)_\infty)$ is not a quasimodular form, but almost, in the sense that its derivative is quasimodular. This statement can also be deduced from Corollary 13.2, and below we give an elementary proof.

Proof. For any given $d\in\mathbb{N}$, the multiset of hook-lengths of all partitions of d is equal to the multiset that is the union over all $|\lambda| = d$ of λ_i repeated λ_i times. This fact appears in many guises in the combinatorics literature, e.g. in [4]. In our notation

$$\sum_{d=1}^{\infty}\sum_{|\lambda|=d} H_\lambda(z)q^d \;=\; \sum_{\lambda}\sum_{j\geq 0}\lambda_j z^{\lambda_j} q^{|\lambda|}\,.$$

In the right hand side of this expression, the coefficient in front of z^m equals

$$\sum_\lambda m r_m(\lambda) q^{|\lambda|} \;=\; q^m\sum_{j=1}^{\infty}\Big(\sum_{\lambda_1\geq\cdots\geq\lambda_{j-1}\geq m} q^{\lambda_1+\cdots+\lambda_{j-1}}\Big)\Big(\sum_{m\geq\lambda_{j+1}\geq\lambda_{j+2}\geq\cdots} q^{\lambda_{j+1}+\lambda_{j+2}+\cdots}\Big)$$
$$= q^m\prod_{d\geq m}(1-q^d)^{-1}\prod_{d\leq m}(1-q^d)^{-1} \;=\; \frac{q^m}{(q)_\infty(1-q^m)}.$$

Consequently,

$$(q)_\infty\sum_{d=1}^{\infty}\sum_{|\lambda|=d} H_\lambda(z)q^d \;=\; \sum_{m\geq 1}\frac{q^m}{1-q^m}z^m \;=\; \sum_{d,m\geq 1} q^{md}z^m.$$

The claim follows by taking the $(p-1)$-st moment, that is, applying $p-1$ times the differential operator $D = z\frac{\partial}{\partial z}$ and plugging in $z=1$. □

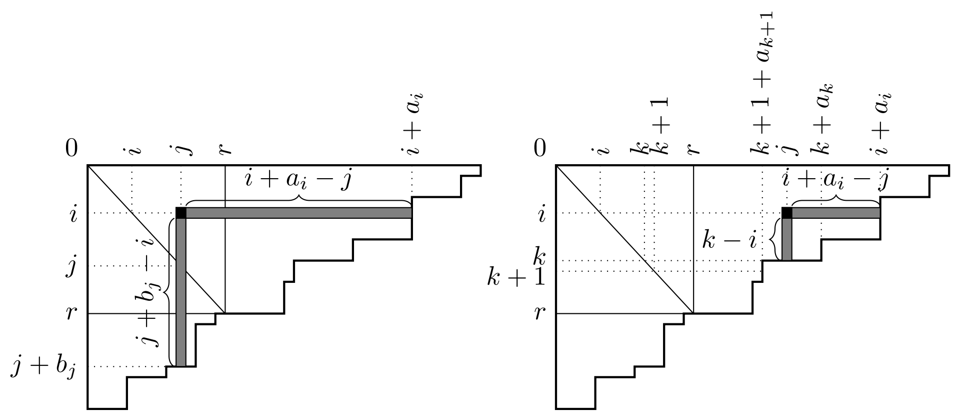

FIGURE 3. Hooks in Frobenius coordinates, for a cell inside (left) and outside (right) the central square

Finally we define $\widetilde{T}_p : \mathbf{P} \to \mathbb{Q}$ by

$$\widetilde{T}_p(\lambda) \;=\; \begin{cases} T_p(\lambda) \;+\; \frac{1}{2}\zeta(-p) & \text{for } p \geq 1 \text{ odd} \\ \qquad 0 & \text{for } p \text{ even,} \end{cases}$$

or equivalently by the generating function

$$\frac{1}{z^2} \;+\; 2\sum_{p=1}^{\infty} \widetilde{T}_p(\lambda)\,\frac{z^{p-1}}{(p-1)!} \;=\; H_\lambda(e^z) \;+\; H_\lambda(e^{-z}) \;+\; \frac{1}{4\,\sinh^2(z/2)} \qquad \in\; z^{-2}\mathbb{Q}[[z^2]]\;. \tag{149}$$

The following result describes these functions in terms of the basic invariants $Q_k(\lambda)$.

Theorem 13.5. *The function $T_p : \mathbf{P} \to \mathbb{Z}$ belongs to the ring Λ^* of shifted symmetric functions for every odd $p \geq 1$. Explicitly, the function $\widetilde{T}_p : \mathbf{P} \to \mathbb{Q}$ is the homogeneous element of weight $p+1$ given by*

$$\frac{\widetilde{T}_p(\lambda)}{(p-1)!} \;=\; \frac{1}{2}\sum_{k=0}^{p+1}(-1)^k Q_k(\lambda)\,Q_{p+1-k}(\lambda) \qquad (p \geq 1)\,. \tag{150}$$

In terms of generating functions, we can restate formula (150) as

$$\frac{1}{z^2} \;+\; 2\sum_{p=1}^{\infty} \widetilde{T}_p(\lambda)\,\frac{z^{p-1}}{(p-1)!} \;=\; -\,W_\lambda(z)\,W_\lambda(-z) \quad \big(=\, W_\lambda(z)W_{\lambda^\vee}(z)\big) \tag{151}$$

where $W_\lambda(z)$ is defined as in (65).

Proof. Denote by $H_\lambda^{(1)}(t)$, $H_\lambda^{(2)}(t)$, and $H_\lambda^{(3)}(t)$ the contributions to $H_\lambda(t)$ coming from the cells $s = (i,j) \in Y_\lambda$ with $1 \leq i,\, j \leq r$, $1 \leq i \leq r < j$, and $1 \leq j \leq r < i$, respectively, where $(r; a_1, \ldots, a_r; b_1, \ldots, b_r)$ are the Frobenius coordinates of λ. If $1 \leq i,\, j \leq r$, then (see Figure 3, left picture) the hook from s has end-points

$(i,i+a_i)$ and $(j+b_j,j)$, so $h(s)=(i+a_i-j)+(j+b_j-i)+1=a_i+b_j+1$. Hence

$$H_\lambda^{(1)}(t) \;=\; \sum_{i,\,j=1}^{r} t^{a_i+b_j+1} \;=\; \sum_{\substack{c,\,c'\in C_\lambda\\ c>0>c'}} t^{c-c'}\,, \quad H_\lambda^{(1)}(t)\,+\,H_\lambda^{(1)}(1/t) \;=\; \sum_{\substack{c,\,c'\in C_\lambda\\ cc'<0}} t^{c-c'}\,.$$

If $1\le i\le r<j$, then (see Figure 3, right picture) the end-points of the hook from s are at $(i,i+a_i)$ and (k,j), where $i\le k\le r$ is the unique index with $k+1+a_{k+1}<j\le k+a_k$ (resp. $r<j\le r+a_r$ if $k=r$), so here $h(s)=(i+a_i-j)+(k-i)+1=a_i+k-j+1$. Hence

$$\begin{aligned} H_\lambda^{(2)}(t) \;&=\; \sum_{i=1}^{r} t^{a_i+1}\biggl(\sum_{k=i}^{r-1}\frac{t^{-a_{k+1}-1}\,-\,t^{-a_k}}{t\,-\,1} \,+\, \frac{1\,-\,t^{-a_r}}{t\,-\,1}\biggr) \\ &=\; -\sum_{1\le i\le k\le r} t^{a_i-a_k} \,+\, \sum_{i=1}^{r}\frac{t^{a_i+1}\,-\,1}{t\,-\,1} \;=\; -\sum_{\substack{c,\,c'\in C_\lambda\\ c\ge c'>0}} t^{c-c'} \,+\, \sum_{\substack{c\in C_\lambda\\ c>0}} \frac{t^{c+\frac12}\,-\,1}{t\,-\,1}\,, \\ H_\lambda^{(2)}(t)\,+\,H_\lambda^{(2)}(1/t) \;&=\; -\sum_{\substack{c,\,c'\in C_\lambda\\ c,\,c'>0}} t^{c-c'} \,+\, \sum_{\substack{c\in C_\lambda\\ c>0}} \frac{t^{c}\,-\,t^{-c}}{t^{1/2}\,-\,t^{-1/2}}\,. \end{aligned}$$

Similarly, or by interchanging the roles of the a_i and b_j (i.e. replacing λ by $\lambda^\vee$),

$$H_\lambda^{(3)}(t)\,+\,H_\lambda^{(3)}(1/t) \;=\; -\sum_{\substack{c,\,c'\in C_\lambda\\ c,\,c'<0}} t^{c-c'} \,-\, \sum_{\substack{c\in C_\lambda\\ c<0}} \frac{t^{c}\,-\,t^{-c}}{t^{1/2}\,-\,t^{-1/2}}\,.$$

Adding all three formulas we get

$$\begin{aligned} H_\lambda(t)\,+\,H_\lambda(1/t) \;&=\; -\sum_{c,\,c'\in C_\lambda} \operatorname{sgn}(cc')\,t^{c-c'} \,+\, \sum_{c\in C_\lambda}\operatorname{sgn}(c)\,\frac{t^{c}\,-\,t^{-c}}{t^{1/2}\,-\,t^{-1/2}} \\ &=\; -\,w_\lambda^0(t)\,w_\lambda^0(1/t) \,+\, \frac{t^{1/2}}{t\,-\,1}\bigl(w_\lambda^0(t)\,+\,w_\lambda^0(1/t)\bigr) \\ &=\; -\,w_\lambda(t)\,w_\lambda(1/t) \,-\, \frac{t}{(t\,-\,1)^2}\,, \end{aligned}$$

where $w_\lambda^0(t)$ and $w_\lambda(t)$ are defined in (64). In view of (149) the above identity is equivalent to equation (151). □

14. A formula for q-brackets involving $\widetilde{T}_p$

With the applications to Siegel-Veech constants in mind, the most important among the functions T_p is the case $p=-1$. Here T_p is *not* a shifted symmetric function and the Bloch-Okounkov theorem does not apply. The motivation for this section is to isolate the p-dependence outside the q-brackets and to interpolate the quasimodularity proven for $p\ge1$ to $p=-1$. This is achieved by discovering a general formula for the q-bracket of the product of T_p ($p\ge1$ odd) with an arbitrary shifted symmetric function.

The basic observation, first made experimentally, is that the q-brackets $\langle\widetilde{T}_p\,f\rangle_q$ for a fixed element $f\in\mathscr{R}$ and varying odd numbers p is a linear combination of derivatives of Eisenstein series with coefficients that are independent of p, i.e.

$$\big\langle \widetilde{T}_p\, f\big\rangle_q \;=\; \sum_{i,\,j\geq 0} \rho_{i,j}(f)_q\, G^{(j)}_{p+i+1} \qquad \text{for all odd } p\geq 1\,, \tag{152}$$

where $G^{(j)}_k := D^j G_k$ and $\rho_{i,j}(f)_q \in \widetilde{M}_*$. Notice that the quasimodular forms $\rho_{i,j}(f)_q$ are uniquely determined by this for i even (since p takes on infinitely many values), while those for i odd are completely free (since $G_k\equiv 0$ for k odd). We then find that the quasimodular forms $\rho_{i,j}(f)_q$ have natural lifts from $\widetilde{M}_*$ to $\mathcal{R}$, i.e. there exist linear operators $\rho_{i,j}$ from the Bloch-Okounkov ring to itself such that

$$\big\langle \widetilde{T}_p\, f\big\rangle_q \;=\; \sum_{i,\,j\geq 0} \big\langle\rho_{i,j}(f)\big\rangle_q\, G^{(j)}_{p+i+1} \qquad \text{for all odd } p\geq 1\,. \tag{153}$$

In view of the formula for $\widetilde{T}_p$ as a quadratic polynomial in the Q_k's given in the previous section (equation (151)), we can rewrite (153) in terms of the Q_k-generating series $W(z)$ as

$$\begin{aligned} &F(u,-u,z_1,\dots,z_n) + \frac{1}{u^2}F(z_1,\dots,z_n) \\ &=\; -\,2 \sum_{\substack{i,\,j\geq 0\\ p\geq 1 \text{ odd}}} \big\langle \rho_{i,j}\big(W(z_1)\cdots W(z_n)\big)\big\rangle_q\, G^{(j)}_{p+i+1}\,\frac{u^{p-1}}{(p-1)!}\,, \end{aligned} \tag{154}$$

where W and F are defined in (68) and (69). It is in this form that we will prove in Section 15. For this purpose, however, we need to know explicit formulas for the maps $\rho_{i,j}$. We remark that finding these formulas required a combination of numerical computation, interpolation, and guesswork, because the q-bracket from $\mathcal{R}$ to $\widetilde{M}_*$ is far from injective and (153) gives only the q-brackets, not the maps $\rho_{i,j}$ themselves. It eventually turned out that there is a natural lift. The maps $\rho_{i,j}$ admit two quite different-looking descriptions, one as differential operators on the ring $\mathcal{R}$ and one via a closed formula for $\rho_{i,j}\big(W(z_1)\cdots W(z_n)\big)$ for each fixed value of n, analogous to the two types of generating functions (correlators and partition functions) used in Section 10.

We begin with some preliminary observations. For compatibility with the weight we require that $\rho_{i,j}$ has weight $-i-2j$. We also require the initial values

$$\rho_{i,0} = \delta_{i,0}\cdot \mathrm{Id}\,, \qquad \rho_{0,1} = \partial_2\,, \tag{155}$$

where ∂_2 is the derivation of degree -2 on Λ_* sending Q_k to Q_{k-2}. Next, for compatibility with (80) we require that $[\rho_{i,j}, Q_2] = \rho_{i,j-1}$, or equivalently, that

$$\rho_{i,j}(Q_2 f) \;=\; Q_2\,\rho_{i,j}(f) \;+\; \rho_{i,j-1}(f) \tag{156}$$

for all $f\in\Lambda^*$ and $i,\,j\geq 0$, where $\rho_{i,j-1}(f)=0$ if $j=0$. Finally, for the effect of $\rho_{i,j}$ on powers of the generator Q_3 (which are the only important ones for the case of the principal stratum, since $f_2 = \frac{1}{2}P_2 = Q_3$) we find the simple formula

$$\rho_{i,j}\Big(\frac{Q_3^n}{n!}\Big) \;=\; \begin{cases} \dfrac{Q_3^{n-j}Q_{j-i}}{2^i(i+1)!\,(n-j)!} & \text{if } 0\leq i\leq j\leq n \\ 0 & \text{otherwise.}\end{cases} \tag{157}$$

(with $Q_k=0$ for $k<0$) which together with (156) already describes the action of $\rho_{i,j}$ on $\mathbb{Q}[Q_1,Q_2,Q_3]\subset\mathcal{R}$.

We now observe that equation (157) can be rewritten as

$$\rho_{i,j}\Big|_{\mathbb{Q}[Q_3]} \;=\; \frac{Q_{j-i}}{2^i(i+1)!}\,\frac{\partial^j}{\partial Q_3^j}\,. \tag{158}$$

This suggests that $\rho_{i,j}$ may be expressed as a differential operator on $\mathscr{R}$, and further experiments suggest that it is linear in the generators Q_k, but polynomial in the derivations $\frac{\partial}{\partial Q_k}$. We therefore write

$$\rho_{i,j} \;=\; \sum_{k=0}^{\infty} Q_k\,\rho_{i,j}^{(k)}\Big(\frac{\partial}{\partial p_1},\frac{\partial}{\partial p_2},\dots\Big)\,, \tag{159}$$

where to simplify later formulas we have used Q_k for the linear part (including $Q_0 = 1$), but $p_\ell = \ell!\,Q_{\ell+1}$ for the derivations. Here the polynomial $\rho_{i,j}^{(k)}$ in the variables u_ℓ has weight $i+2j+k$ and degree j, where u_ℓ has degree 1 and weight $\ell+1$ (and therefore, since Q_k has weight k and degree 0 or 1 depending on whether $k=0$ or $k>0$, that the full operator $\rho_{i,j}$ has weight $-i-2j$ and mixed degree $-j$ and $1-j$). Because of this bi-homogeneity property, there is no loss of information if we consider only the power series $\rho^{(k)} = \sum_{i,j}\rho_{i,j}^{(k)}$. In this language, equation (155) says that the constant and linear terms of $\rho^{(k)}$ are $\delta_{k,0}$ and $u_{k+1}/(k+1)!$, respectively; the differentiation property (156) translates into the property $\rho^{(k)}(u_1,u_2,\dots) = e^{u_1}\rho^{(k)}(0,u_2,\dots)$; and equation (158) says that $\rho^{(k)}(0,u,0,0,\dots) = 2^k u^{k-1}e^u$ for $k>0$.

To find the full formula, the key observation is that $\rho^{(k+1)} = \mathbf{d}\rho^{(k)}(\mathbf{u})$ for all k, where $\mathbf{d}$ is the derivation $\sum_{i=0}^{\infty}(i+1)\,u_{i+1}\partial/\partial u_i$ on $\mathbb{Q}[[\mathbf{u}]]$. It follows that $\rho^{(k)} = \mathbf{d}^k\rho^{(0)}$ for all $k\ge 1$. We were not able to recognize the coefficients of the power series $\rho^{(0)}$ directly, but the next case $\rho^{(1)}$ turned out to be easy to recognize, since if we made the choice

$$\rho^{(1)}(\mathbf{u}) \;=\; 2\,\exp(u_1+u_2+u_3+\cdots) \tag{160}$$

and then defined the other $\rho^{(k)}$ as $\mathbf{d}^{k-1}\rho^{(1)}$ (meaning in the case of $k=0$ that we have to integrate once with respect to $\mathbf{d}$), then we obtained operators having the right properties. To get the $k=0$ term, we note that, since we are free to choose the operators $\rho_{i,j}$ for i odd in any way we want, we can replace (160) by its odd part

$$\rho^{(1)}(\mathbf{u}) \;=\; \exp(u_1+u_2+u_3+\cdots) \,-\, \exp(u_1-u_2+u_3-\cdots)\,. \tag{161}$$

This can now be integrated to give the formula $\rho^{(0)}(\mathbf{u}) = \int_0^1 e^{U(t)-U(t-1)}\,dt$, where $U(t) = \sum u_n t^n$ as in (99), because from $\mathbf{d}(U(t)) = U'(t)$ and $U(0)=0$ we obtain

$$\mathbf{d}\Big(\int_0^1 e^{U(t)-U(t-1)}\,dt\Big) \;=\; \int_0^1 d\Big(e^{U(t)-U(t-1)}\Big) \;=\; e^{U(1)} \,-\, e^{-U(-1)}\,.$$

Now applying powers of $\mathbf{d}$ to get formulas for the higher $\rho^{(k)}$, we are led to the following final formulation of the experimentally obtained expression for the operators $\rho_{i,j}$, which includes all of the special cases discussed above:

Theorem 14.1. *Define power series $\rho^{(k)}(\mathbf{u})$ for $k\ge 0$ by the generating series*

$$\sum_{k=0}^{\infty}\rho^{(k)}(\mathbf{u})\,\frac{v^k}{k!} \;=\; \int_v^{v+1} e^{U(t)-U(t-1)}\,dt\,, \tag{162}$$

with $U(t)=\sum u_n t^n$ as in (99), and let $\rho^{(k)}_{i,j}$ for $i,\,j\ge 0$ be the part of $\rho^{(k)}$ of degree j and weight $i+2j+k$. Then equation (153) holds with $\rho_{i,j}$ defined by (159).

This theorem can also be expressed as a formula for the action of the maps $\rho_{i,j}$ on the generating function $\Phi(\mathbf{u})=\exp(p_1u_1+p_2u_2+\cdots)$ whose q- and X-brackets $\Phi(\mathbf{u})_q$ and $\Phi(\mathbf{u})_X$ were studied in Section 10. For the reasons of weight and degree explained above, it is enough to specify the action of the total operator $\rho=\sum_{i,j}\rho_{i,j}$ on Φ. In view of (66), this action is given simply in terms of a *first-order* differential operator in the u's

$$\rho(\Phi(\mathbf{u})) \;=\; \biggl(\int_0^1 e^{U(t)-U(t-1)}\,dt \,+\, \sum_{\ell=1}^{\infty}\frac{1}{\ell!}\frac{d^\ell}{dt^\ell}\Bigl(e^{U(t)-U(t-1)}\Bigr)\Big|_{t=0}^{t=1}\frac{\partial}{\partial u_\ell}\biggr)\,\Phi(\mathbf{u})\,.$$

In the rest of this section we give a proof of the following for the action of $\rho_{i,j}$ on products $W(z_1)\cdots W(z_n)$, which is what we need for (154).

Theorem 14.2. *The effect of the operator $\rho_{i,j}$ defined in Theorem 14.1 on monomials $Q_{k_1}\cdots Q_{k_n}$ of fixed length n is given in terms of the generating function $W(z)=\sum Q_k z^{k-1}$ by*

$$\rho_{i,j}\bigl(W(z_1)\cdots W(z_n)\bigr) \;=\; \sum_{\substack{J\subset N\\ |J|=j}} W(z_J)\,R_i(\mathfrak{z}_J)\prod_{\nu\in N\smallsetminus J}W(z_\nu) \tag{163}$$

where $N=\{1,\ldots,n\}$, $z_J=\sum_{j\in J}z_j$, $\mathfrak{z}_J=\{z_j,\,j\in J\}$, and the polynomials $R_i(\mathfrak{z}_J)$ are given by the generating function

$$\sum_{i=0}^{\infty}R_i(\mathfrak{z}_J)\,t^i \;=\; \frac{e^{tz_J}-1}{t}\prod_{\nu\in J}\frac{1-e^{-tz_\nu}}{t} \;=\; \frac{\sinh(tz_J/2)}{t/2}\prod_{\nu\in J}\frac{\sinh(tz_\nu/2)}{t/2}\,.$$

Note that formula (163) makes sense, even though $W(z)$ is a Laurent series beginning with $1/z$, because the polynomial $R_i(\mathfrak{z}_J)$ is divisible by z_J. Notice also that the formula implies $\rho_{i,j}\bigl(W(z_1)\cdots W(z_n)\bigr)=0$ if $n<j$.

Proof. Write the polynomials $R_i(\mathfrak{z}_J)$ as $R_{i,j}(\mathfrak{z}_J)$ ($j=|J|$) for clarity, and for $k\ge 0$ set

$$R^{(k)}_{i,j}(\mathfrak{z}_J) \;=\; z_J^{k-1}\,R_{i,j}(\mathfrak{z}_J)\,,$$

which is a homogeneous polynomial of degree $i+j+k$ (even for $k=0$, as just pointed out). In view of the definition (159), the equation to be proved is equivalent to

$$\rho^{(k)}_{i,j}\bigl(W(z_1)\cdots W(z_n)\bigr) \;=\; \sum_{\substack{J\subset N\\ |J|=j}} R^{(k)}_{i,j}(\mathfrak{z}_J)\prod_{\nu\in N\smallsetminus J}W(z_\nu)\,. \tag{164}$$

To prove (164) we will use the linear map $\Omega_j:\mathbb{Q}[z_1,\ldots,z_j]\to\mathbb{Q}[\mathbf{u}]$ defined in (97). This map satisfies the general formula

$$\Omega_j(R)\Bigl(\frac{\partial}{\partial p_1},\frac{\partial}{\partial p_2},\ldots\Bigr)\bigl(W(z_1)\cdots W(z_n)\bigr) \;=\; \sum_{\substack{J\subset N\\ |J|=j}} R(\mathfrak{z}_J)\prod_{\nu\in N\smallsetminus J}W(z_\nu)$$

for any symmetric function R in j variables, because

$$\frac{\partial^j\bigl(W(z_1)\cdots W(z_n)\bigr)}{\partial p_{\ell_1}\cdots\partial p_{\ell_j}} \;=\; \sum_{\substack{1\le i_1,\ldots,i_j\le n\\ i_1,\,\ldots,\,i_j\text{ distinct}}} \frac{z_{i_1}^{\ell_1}\cdots z_{i_j}^{\ell_j}}{\ell_1!\,\cdots\,\ell_j!}\prod_{\substack{1\le\nu\le n\\ \nu\notin\{i_1,\ldots,i_j\}}}W(z_\nu)\,.$$

by induction on j (since $\partial W(z)/\partial p_\ell = z^\ell/\ell!$). Therefore (164) will follow if we show that

$$\rho^{(k)}_{i,j}(u_1,u_2\ldots) \;=\; \Omega_j\bigl(R^{(k)}_{i,j}(z_1,\ldots,z_j)\bigr)\,. \tag{165}$$

This is true for $k=1$ because the definition of $R_i(\mathfrak{z}_J)$ via generating functions can be expanded as

$$R_i(z_1,\ldots,z_j) \;=\; \bigl(1+(-1)^i\bigr)\sum_{\substack{n_1,\ldots,n_j\geq 1\\ n_1+\cdots+n_j=i+j+1}} \frac{z_1^{n_1}\cdots z_j^{n_j}}{n_1!\,\cdots\, n_j!} \tag{166}$$

or, in view of the definition of Ω_j, as

$$\Omega_j(R_i)(\mathbf{u}) \;=\; \frac{1+(-1)^i}{j!}\sum_{\substack{n_1,\ldots,n_j\geq 1\\ n_1+\cdots+n_j=i+j+1}} u_{n_1}\cdots u_{n_j}\,,$$

which agrees with $\rho^{(1)}_{i,j}(\mathbf{u})$ by virtue of either (160) or (161). The case $k\geq 1$ then follows because $\rho^{(k)}_{i,j}=\mathbf{d}\rho^{(k-1)}_{i,j}$ and because the map Ω_j satisfies

$$\Omega_j\bigl(z_J\,R(\mathfrak{z}_J)\bigr) \;=\; \mathbf{d}\bigl(\Omega_j R(\mathfrak{z}_J)\bigr)$$

for any polynomial R, as one verifies easily. The case $k=0$ follows from the same observation together with the fact that the representation of a function of $\mathbf{u}$ as $\Omega_j(R)$ is unique if R is assumed to be symmetric in its arguments and divisible by their product. □

It is perhaps amusing to note that the polynomials $R^{(k)}_{i,j}$ are virtually impossible to recognize numerically, whether they are written in the variables z_ν or in their elementary symmetric polynomials, unless $k=1$, which is the one case that one cannot find experimentally, because the coefficient Q_1 in (159) vanishes identically. In practice, we found expressions in terms of elementary symmetric polynomials, such as

$$R^{(k)}_{i,2}(z_1,z_2) \;=\; \frac{2}{(i+2)!}\sum_{a+2b=i}\frac{(-1)^b\,(a+b+1)!}{(a+1)!\,(b+1)!}\,(z_1+z_2)^{k+a}(z_1z_2)^{b+1}$$

for $j=2$ and a much more complicated expression for $j=3$, and then worked backwards from there.

15. Correlators with two distinguished variables

The information we need to calculate the effect of T_p on q-brackets will all follow from the theorem below and its corollary. To formulate this theorem, we let $Z_\ell(u)$ ($\ell\geq 0$) be the functions defined by

$$\frac{\theta(u+v)\,\theta'(0)}{\theta(u)\,\theta(v)} \;=\; \frac{1}{v}+\sum_{\ell=0}^{\infty} Z_\ell(u)\,\frac{v^\ell}{\ell!}\,.$$

By [48, equation (15)], these functions are given by

$$\begin{aligned} Z_0(u) \;&=\; \frac{\theta'(u)}{\theta(u)} \;=\; \zeta(u) \;=\; \frac{1}{u}\;-\;2\sum_{r\geq 0} G_{r+1}\,\frac{u^r}{r!}\,,\\ Z_\ell(u) \;&=\; -\,2\sum_{r\geq 0} G^{(\min(r,\ell))}_{|r-\ell|+1}\,\frac{u^r}{r!}\qquad (\ell\geq 1)\,. \end{aligned} \tag{167}$$

Theorem 15.1. *A Bloch-Okounkov correlator involving two distinguished variables u and v can be written as a linear combination of products of a correlator involving only $u+v$ and a function Z_ℓ involving only one of the variables u and v. More precisely, we have*

$$F(u,v,\mathfrak{z}_N) \;=\; \sum_{J\subseteq N} F(u+v+z_J,\,\mathfrak{z}_{J^c}) \sum_{I\subseteq J}(-1)^{|J\smallsetminus I|}\bigl(Z_{|J|}(u+z_I)\;+\;Z_{|J|}(v+z_I)\bigr)\,. \tag{168}$$

This will be proved at the end of the section.

Corollary 15.2. *A correlator with two variables u and $-u$ that add up to zero can be expressed in terms of the nearly-elliptic functions Z_j and correlators not involving u by the formula*

$$F(u,-u,\mathfrak{z}_N) \;=\; \sum_{J\subseteq N} F(z_J,\,\mathfrak{z}_{J^c})\,M(u,\mathfrak{z}_J)\,, \tag{169}$$

where $M(u,\mathfrak{z}_J)$ is defined as $\zeta'(u)$ if $J=\emptyset$ and by

$$M(u,\mathfrak{z}_J) = \sum_{I\subseteq J}(-1)^{|J\smallsetminus I|}\bigl(Z_{|J|}(z_I+u)\;+\;Z_{|J|}(z_I-u)\bigr) \tag{170}$$

if $|J|\geq 1$.

Proof. The terms with $J\neq\emptyset$ in (169) are obtained from (168) by specializing to $v=-u$. For the $J=\emptyset$ term, we use Theorem 8.2 (iii) to obtain

$$\lim_{v\to -u} F(u,v,\mathfrak{z}_N) \;=\; F(\mathfrak{z}_N)\,\lim_{\varepsilon\to 0}\frac{Z_0(u)-Z_0(u-\varepsilon)}{\varepsilon} \;=\; F(\mathfrak{z}_N)\,M(u)\,,$$

because $Z_0'(u)=\zeta'(u)=M(u)$. □

We remark that for the following proof of Theorem 14.1 we only need this corollary, but its statement seems not to allow an inductive proof (since after applying the recursion (79) we are left with correlators involving the variable u just once), so that we are forced to show the more general result (168).

Proof of Theorem 14.1. In view of Theorem 14.2 we have to prove (153) with $\rho_{i,j}$ defined by (163). Applying the q-bracket to the latter and using the definition of correlators we obtain

$$\bigl\langle \rho_{i,j}\bigl(W(z_1)\cdots W(z_n)\bigr)\bigr\rangle_q \;=\; [t^{i+j+1}]\sum_{\substack{J\subseteq N\\ |J|=j}}(1-e^{-tz_J})\;\cdot\;\prod_{\nu\in J}(e^{tz_\nu}-1)\;\cdot\;F(z_J,\,\mathfrak{z}_{J^c})\,.$$

Substituting this and (169) into (154), we see that the formula to be proved reduces to the two identities

$$M(u)+\frac{1}{u^2} \;=\; -2\sum_{p\geq 1\,\mathrm{odd}} G_{p+1}\,\frac{u^{p-1}}{(p-1)!}$$

and

$$M(u,\mathfrak{z}_J) \;=\; -2\sum_{\substack{i\geq 0\\ i\,\mathrm{even}}}\sum_{\substack{p\geq 1\\ p\,\mathrm{odd}}} G^{(j)}_{p+i+1}\;\cdot\;\frac{u^{p-1}}{(p-1)!}\;\cdot\;[t^{i+1}](1-e^{-tz_J})\left(\prod_{\nu\in J}\frac{e^{tz_\nu}-1}{t}\right) \tag{171}$$

for $|J| = j \geq 1$. The first of these follows from (167) and the second follows by noting that

$$[t^{i+1}]\left(\prod_{\nu\in J}\frac{e^{tz_\nu}-1}{t}\right) = [t^{i+j+1}]\sum_{I\subseteq J}(-1)^{|J|-|I|}e^{tz_I} = \sum_{I\subseteq J}(-1)^{|J|-|I|}\frac{z_I^{i+j+1}}{(i+j+1)!}\,,$$

$$[t^{i+1}](-e^{-tz_J})\left(\prod_{\nu\in J}\frac{e^{tz_\nu}-1}{t}\right) = \sum_{I\subseteq J}(-1)^{|J|-|I|+i}\frac{z_I^{i+j+1}}{(i+j+1)!},$$

and then calculating

$$\begin{aligned}\text{RHS of (171)} &= -4\sum_{\substack{k\geq 2\\ k\text{ even}}} G_k^{(j)}\sum_{I\subseteq J}(-1)^{j-|I|}\sum_{\substack{i+p=k-1\\ i,p\geq 0,\ i\text{ even}}}\frac{z_I^{i+j+1}}{(i+j+1)!}\frac{u^{p-1}}{(p-1)!}\\ &= -2\sum_{I\subseteq J}(-1)^{j-|I|}\sum_{\substack{k\geq 2\\ k\text{ even}}} G_k^{(j)}\frac{(z_I+u)^{k+j-1}+(z_I-u)^{k+j-1}}{(k+j-1)!}\,.\end{aligned}$$

Now the claim follows from (170) and (167) together with the fact that

$$\sum_{I\subseteq J}(-1)^{|I|}P(z_I) = 0$$

for any polynomial P of degree smaller than $|J| = j$. □

Proof of Theorem 15.1. We define $\widehat{F}(\mathfrak{z}_N) = \frac{\theta(z_N)}{\theta'(0)}F(\mathfrak{z}_N)$. We can change F to $\widehat{F}$ everywhere in the theorem without affecting the truth of the statement, since the sum of the arguments of F is the same in all terms. We denote by $G(u,v,\mathfrak{z}_N)$ the right hand side of (168), so that we have to show that $G(u,v,\mathfrak{z}_N) = F(u,v,\mathfrak{z}_N)$, or equivalently that $\widehat{G}(u,v,\mathfrak{z}_N) := \theta(u+v+z_N)G(u,v,\mathfrak{z}_N)/\theta'(0) = \widehat{F}(u,v,\mathfrak{z}_N)$. We will do this by comparing poles and elliptic transformation properties.

It is easy to see that the residue at $z_n = 0$ of the function $F(u,v,z_1,\ldots,z_n)$ equals $F(u,v,z_1,\ldots,z_{n-1})$ and that the residue at $z_n = 0$ of $G(u,v,z_1,\ldots,z_n)$ equals $G(u,v,z_1,\ldots,z_{n-1})$, so by induction on $|N|$ the difference $\widehat{F}-\widehat{G}$ has no poles at $z_n = 0$. For the poles at $u = 0$ the calculation is even easier: the residue of $F(u,v,z_1,\ldots,z_n)$ at $u = 0$ is $F(v,z_1,\ldots,z_n)$, and the residue of $G(u,v,z_1,\ldots,z_n)$ at $u = 0$ is easily seen to have the same value. In fact, only the term $J = \emptyset$ in the definition of G contributes, since Z_j is holomorphic for $j \neq 0$. Finally, if $u+v+z_J = 0$ for some $J \subseteq N$ (which we can assume is unique, since we can assume that all the variables are generic), then the left hand side of (168) has no pole and the right hand side is also non-singular because the terms for I and $I^c = J \smallsetminus I$ cancel since $(-1)^{|I^c|}Z_{|J|}(v+z_I) = (-1)^{|J\smallsetminus I|}Z_{|J|}(-u-z_{I^c}) = -(-1)^{|I|}Z_{|J|}(u+z_{I^c})$.

For the elliptic transformation properties, we recall that Bloch and Okounkov have shown in [7] the elliptic transformation law

$$\widehat{F}(z_1+\tau,z_2,\ldots,z_n) = \sum_{1\in J\subseteq N}(-1)^{|J|-1}\widehat{F}(z_J,\mathfrak{z}_{J^c})\,. \tag{172}$$

We introduce the difference operator Δ_x which associates to any function $f(x)$, possibly depending on other variables, the difference $(\Delta_x f)(x) = f(x+\tau)-f(x)$.

Then (172) says that $\Delta_{z_1}\widehat{F}$ equals the right hand side of (172) with the term $J=\{1\}$ omitted, while for Z_ℓ we have

$$\Delta_u Z_\ell(u) \;=\; \sum_{k=1}^{\ell}(-1)^k\binom{\ell}{k}\,Z_{\ell-k}(u) \;+\; \frac{(-1)^{\ell+1}}{(\ell+1)!}\,. \tag{173}$$

Since both sides in (168) are symmetric in the variables $z_1,\dots,z_n$ and also in u and v, is suffices to show that the differences of $\widehat{F}$ and $\widehat{G}$ with respect to (say) z_1 and u agree, in which case $\widehat{F}-\widehat{G}$ is periodic and holomorphic in all variables, hence is a constant.

We start with the variable u. We have

$$\Delta_u\widehat{F}(u,v,\mathfrak{z}_N) \;=\; \sum_{\emptyset\neq J\subseteq N}(-1)^{|J|}\widehat{F}(u+z_J,v,\mathfrak{z}_{J^c}) - \sum_{H\subseteq N}(-1)^{|H|}\widehat{F}(u+v+z_H,\mathfrak{z}_{H^c})\,.$$

We can compute the first summand using the identity we claim, which is true by induction on $|N|$, since $|J^c|<|N|$ for $J\neq\emptyset$. This gives

$$\sum_{\emptyset\neq J\subseteq N}(-1)^{|J|}\widehat{F}(u+z_J,v,\mathfrak{z}_{J^c}) \;=\; \sum_{\emptyset\neq J\subseteq K\subseteq H\subseteq N}(-1)^{|H|+|K\smallsetminus J|}\widehat{F}(u+v+z_H,\,\mathfrak{z}_{H^c})\cdot$$
$$\cdot\big(Z_{|H\smallsetminus J|}(u+z_K) \;+\; Z_{|H\smallsetminus J|}(v+z_K-z_J)\big)\,.$$

Combining the terms we obtain

$$\Delta_u\widehat{F}(u,v,\mathfrak{z}_N) = \sum_{H\subseteq N}(-1)^{|H|}\,\alpha(H)\,\widehat{F}(u+v+z_H,\,\mathfrak{z}_{H^c})\,,$$

with

$$\alpha(H) \;=\; (-1) \;+ \sum_{\substack{I\subseteq H\\ |H\smallsetminus I|\leq\lambda<|H|}}(-1)^{|H|-\lambda+|I|}\binom{|I|}{|H|-\lambda}Z_\lambda(u+z_I)$$
$$+\;\sum_{\substack{I\subseteq H\\ |I|\leq\lambda<|H|}}(-1)^{|I|}\binom{|H|-|I|}{\lambda-|I|}Z_\lambda(v+z_I)\,.$$

On the other hand, using (172) and (173), we obtain

$$\Delta_u\widehat{G}(u,v,\mathfrak{z}_N) = \sum_{H\subseteq N}(-1)^{|H|}\beta(H)\widehat{F}(u+v+z_H,\,\mathfrak{z}_{H^c})$$

with

$$\begin{aligned}
\beta(H) \;=\; & \sum_{I\subseteq J\subseteq H} (-1)^{|I|} \left(\sum_{k=0}^{|J|} (-1)^k \binom{|J|}{k} Z_{|J|-k}(u+z_I) \;+\; Z_{|J|}(v+z_I) \right) \\
& + \sum_{I\subseteq J\subseteq H} (-1)^{|I|} \frac{(-1)^{|J|+1}}{(|J|+1)!} \\
& - \sum_{I\subseteq H} (-1)^{|I|} \big(Z_{|H|}(u+z_I) \;+\; Z_{|H|}(v+z_I) \big) \\
= \; & \sum_{\substack{I\subseteq H \\ 0\le\lambda<|H|}} (-1)^{|I|} \left[\sum_{|I|\le n\le |H|} (-1)^{n-\lambda} \binom{n}{\lambda} \binom{|H|-|I|}{|H|-n} \right] Z_\lambda(u+z_I) \\
& + \sum_{\substack{I\subseteq H \\ |I|\le\lambda<|H|}} (-1)^{|I|} \binom{|H|-|I|}{\lambda-|I|} Z_\lambda(v+z_I) \\
& + \sum_{J\subseteq H} \frac{(-1)^{|J|+1}}{(|J|+1)!} \sum_{I\subseteq J} (-1)^{|I|} \,.
\end{aligned}$$

Notice that the expression in the square brackets equals

$$(-1)^{|H|-\lambda} \binom{|I|}{|H|-\lambda} \;=\; \big[x^{|H|-\lambda}\big] \Big(\frac{1}{(1+x)^{\lambda+1}} \, (1+x)^{|H|-|I|} \Big) \,, \tag{174}$$

which is zero when $\lambda < |H \smallsetminus I|$, and the summation of the constant terms is equal to -1 with the only non-zero contribution coming from $J = \emptyset$. It follows that the formulas for $\alpha(H)$ and $\beta(H)$ agree.

The difference with respect to the variable z_1 behaves similarly. We have

$$\begin{aligned}
& \Delta_{z_1} \widehat{F}(z_1, z_2, \ldots, z_n, u, v) \\
= \; & \sum_{\{1\}\subsetneq J\subseteq N} (-1)^{|J|-1} \widehat{F}(z_J, u, v, \mathfrak{z}_{J^c}) \;+\; \sum_{1\in J\subseteq N} (-1)^{|J|+1} \widehat{F}(z_J+u+v, \mathfrak{z}_{J^c}) \\
& + \sum_{1\in J\subseteq N} (-1)^{|J|} \big(\widehat{F}(z_J+u, v, \mathfrak{z}_{J^c}) \;+\; \widehat{F}(z_J+v, u, \mathfrak{z}_{J^c}) \big) \\
= \; & \sum_{\{1\}\subsetneq J\subseteq N} \; \sum_{I\subseteq J_2\subseteq J^c} (-1)^{|J|+|I|+|J_2|+1} \widehat{F}(z_{J_2}+u+v, z_J, \mathfrak{z}_{(J\cup J_2)^c}) \cdot \\
& \qquad\qquad \cdot \big(Z_{|J_2|}(z_I+u) + Z_{|J_2|}(z_I+v) \big) \\
& + \sum_{1\in H\subseteq N} (-1)^{|H|} \widehat{F}(z_H+u+v, \mathfrak{z}_{H^c}) \, \alpha(H)
\end{aligned}$$

with

$$\begin{aligned}
\alpha(H) \;=\; & \sum_{1\notin I\subseteq K\subsetneq H} (-1)^{|I|}\big(Z_{|K|}(z_I+u)+Z_{|K|}(z_I+v)\big) \\
& + (-1) \\
& + \sum_{\substack{I,J\subseteq H,\ I\cap J=\emptyset \\ 1\in J\cup I,\ J\neq\emptyset}} (-1)^{|I|}\big(Z_{|H\smallsetminus J|}(z_I+z_J+u)+Z_{|H\smallsetminus J|}(z_I+z_J+v)\big) \\
=\; & \sum_{1\notin I\subseteq H}\sum_{\ell=|I|}^{|H|-1} \alpha_0(\ell,I,H)\big(Z_\ell(z_I+u)+Z_\ell(z_I+v)\big) \\
& + \sum_{1\in I\subseteq H}\sum_{\ell=|H|-|I|}^{|H|-1} \alpha_1(\ell,I,H)\big(Z_\ell(z_I+u)+Z_\ell(z_I+v)\big) \\
& + (-1)
\end{aligned}$$

where, taking $I\cup J$ as the new I and $\ell=|H\smallsetminus J|$ for the transformation from the second summand to the fourth,

$$\alpha_0(\ell,I,H) = (-1)^{|I|}\binom{|H|-|I|}{\ell-|I|}$$

and

$$\alpha_1(\ell,I,H) = (-1)^{|I|+|H|-\ell}\binom{|I|}{|H|-\ell}.$$

Computing Δ_{z_1} of the right hand side we distinguish the case $1\in J^c$ which is the only one where terms of the form $\widehat{F}(z_J+u+v,z_{J_2},\mathfrak{z}_{(J\cup J_2)^c})$ appear, the case $1\in I$ and the remaining case $1\in J\smallsetminus I$. After simplifying, we obtain that

$$\begin{aligned}
& \Delta_{z_1}\widehat{G}(z_1,z_2,\ldots,z_n,u,v) \\
=\; & \sum_{\substack{I\subseteq J\subseteq N\\ 1\notin J}}\ \sum_{\{1\}\subsetneq J_2\subseteq J^c} (-1)^{|J|+|I|+|J_2|+1}\widehat{F}(z_J+u+v,z_{J_2},\mathfrak{z}_{(J\cup J_2)^c})\cdot \\
& \qquad\qquad \cdot\big(Z_{|J|}(z_I+u)+Z_{|J|}(z_I+v)\big) \\
& + \sum_{1\in H\subseteq N} (-1)^{|H|}\widehat{F}(z_H+u+v,\mathfrak{z}_{H^c})\,\beta(H)
\end{aligned}$$

with

$$\begin{aligned}
\beta(H) \;=\; & \sum_{1\notin I\subseteq J\subseteq H} (-1)^{|I|}\big(Z_{|J|}(z_I+u)+Z_{|J|}(z_I+v)\big) \\
& + \sum_{I\subseteq H}(-1)^{|I|+1}\big(Z_{|H|}(z_I+u)+Z_{|H|}(z_I+v)\big) \\
& + \sum_{1\in I\subseteq J\subseteq H}(-1)^{|I|}\left[\sum_{k=1}^{|J|}(-1)^k\binom{|J|}{k}Z_{|J|-k}(z_I+u)+Z_{|J|-k}(z_I+v)+2\cdot\frac{(-1)^{|J|+1}}{(|J|+1)!}\right] \\
& + \sum_{1\in I\subseteq J\subseteq H}(-1)^{|I|}\big(Z_{|J|}(z_I+u)+Z_{|J|}(z_I+v)\big)\,.
\end{aligned}$$

Notice that the summation of $2 \cdot \frac{(-1)^{|I|+|J|+1}}{(|J|+1)!}$ ranging over $1 \in I \subseteq J \subseteq H$ equals -1, where the only non-zero contribution comes from $J = \{1\}$ (for fixed J and for varying I). In addition, set $\ell = |J| - k$ in the summation of the terms with subscript $|J| - k$, and apply (174) to simplify. We conclude that

$$\begin{aligned}\beta(H) \;=\; & \sum_{1 \notin I \subseteq H} \sum_{\ell=|I|}^{|H|-1} \beta_0(\ell, I, H)\big(Z_\ell(z_I + u) + Z_\ell(z_I + v)\big) \\ & + \sum_{1 \in I \subseteq H} \sum_{\ell=|H|-|I|}^{|H|-1} \beta_1(\ell, I, H)\big(Z_\ell(z_I + u) + Z_\ell(z_I + v)\big) \\ & + (-1)\end{aligned}$$

where

$$\beta_0(\ell, I, H) = (-1)^{|I|} \binom{|H| - |I|}{\ell - |I|}$$

and

$$\beta_1(\ell, I, H) = (-1)^{|H|+|I|+\ell} \binom{|I|}{|H| - \ell}.$$

We see that $\alpha_0 = \beta_0$ and $\alpha_1 = \beta_1$, hence $\alpha(H) = \beta(H)$. □

16. Applications to T_{-1} and to Siegel-Veech constants

We saw in Proposition 13.4 that $\langle T_{-1} \rangle_q$ is not a quasimodular form, but the τ-derivative is. In this section we use the formula (153) on the effect of T_p to deduce that a certain linear combination of brackets involving T_{-1} is indeed quasimodular. We apply this to prove the quasimodularity of area Siegel-Veech constants.

Theorem 16.1. *For all $f \in \Lambda_k$ the modified q-bracket*

$$\langle f \rangle_q^\star \;=\; \langle T_{-1}\, f \rangle_q \;-\; \langle T_{-1} \rangle_q \,\langle f \rangle_q \;-\; \frac{1}{24}\, \langle \partial_2(f) \rangle_q \tag{175}$$

is a quasimodular form of weight k. More precisely, we have

$$\langle f \rangle_q^\star \;=\; \sum_{i \ge 2,\, j \ge 0} G_i^{(j)} \,\langle \rho_{i,j}^\star(f) \rangle_q\,,$$

where $\rho_{i,j}^\star \;=\; \rho_{i,j} + \delta_{i,2}\, \rho_{0,j+1}$.

Proof. From (153) we get, for $p > 0$ odd,

$$\langle T_p\, f \rangle_q \;=\; \sum_{i \ge 0,\, j \ge 1} \langle \rho_{i,j}(f) \rangle_q \left(G_{p+i+1}^{(j)} \;-\; \delta_{i+j,0} \frac{\zeta(-p)}{2} \right).$$

Using $\rho_{i,0} = 0$ for $i > 0$ and $T_p(\lambda) = \sum_m m^{p-1} N_m(\lambda)$, where $\lambda \mapsto N_m(\lambda)$ is the hook-length counting function of Section 13), we can rewrite this as

$$\sum_{m>0} m^{p-1} \langle N_m f \rangle_q \;=\; \sum_{\substack{i,j \ge 0 \\ i \text{ even}}} \langle \rho_{i,j}(f) \rangle_q \sum_{n=1}^{\infty} n^j \sigma_{p+i}(n)\, q^n\,.$$

Since a function of the form $p \mapsto \sum_m a_m m^p$ on $\{p \in \mathbb{N}, p \text{ odd}\}$ determines all the a_m uniquely, we deduce

$$\big\langle N_m f \big\rangle_q \;=\; \sum_{\substack{i,\,j\geq 0\\ i \text{ even}}} \Big(\sum_{r>0} m^{i+1}(mr)^j q^{mr} \Big) \big\langle \rho_{i,j}(f) \big\rangle_q \,.$$

Therefore,

$$\begin{aligned} \big\langle T_{-1}\, f \big\rangle_q \;&=\; \sum_{m=1}^{\infty} m^{-2} \big\langle N_m f \big\rangle_q \;=\; \sum_{\substack{i,\,j\geq 0\\ i \text{ even}}} \Big(\sum_{m,\,r>0} m^{i-1}(mr)^j q^{mr} \Big) \big\langle \rho_{i,j}(f) \big\rangle_q \\ &=\; \sum_{i\geq 2,\, j\geq 0} G_i^{(j)} \big\langle \rho_{i,j}(f) \big\rangle_q \,+\, \Big(\sum_{m,r>0} \frac{q^{mr}}{m} \Big) \big\langle\, f \big\rangle_q \\ &\quad +\, \sum_{j\geq 1} \big(G_2^{(j-1)} + \frac{1}{24}\delta_{j,1} \big) \big\langle \rho_{0,j}(f) \big\rangle_q \,. \end{aligned}$$

In view of the formula $\rho_{0,1} = \partial_2$ and Proposition 13.4 this gives the claim. □

We can now prove Theorem 6.4 of Part I. Recall that Eskin and Okounkov have shown ([18]) the quasimodularity of the generating function of Hurwitz numbers

$$N'(\Pi) \in \widetilde{M}_{\leq \mathrm{wt}(\Pi)}, \quad N^0(\Pi) \in \widetilde{M}_{\leq \mathrm{wt}(\Pi)} \tag{176}$$

where $\mathrm{wt}(\Pi) = \sum \mathrm{wt}(\mu_i)$ for $\Pi = (\mu_1, \ldots, \mu_n)$ and the weight of a b-cycle is defined to be $b+1$. This is a consequence of the Burnside formula (51) and the Bloch-Okounkov Theorem 8.1 using the formula (48) and the fact ([29]) that the character functions f_k in (52) are shifted symmetric functions. (We give examples in Section 17.) The formula (49) provides the passage to the connected case. Moreover, since $f_2 = Q_3$ is a shifted symmetric function of pure weight three, the modular forms $N'(\mathrm{Tr}^n)$ and $N^0(\mathrm{Tr}^n)$ are pure of weight $3n$.

Proof of Theorem 6.4. We start with the case $\mu_i = \mathrm{Tr}$ for all i, that is $\Pi = \mathrm{Tr}^n$. Combining the passage from counting all covers to counting covers without unramified components in (56), the Siegel-Veech analog of the Burnside formula (57), and Corollary 13.2, we deduce that

$$c'_p(\mathrm{Tr}^n) = \big\langle T_p f_2^n \big\rangle_q - \big\langle f_2^n \big\rangle_q \big\langle T_p \big\rangle_q \tag{177}$$

and the preceding remarks together with (151) imply that for p positive $c'_p(\mathrm{Tr}^n)$ is quasimodular of weight $3n+p+1 = 6g-6+p+1$. Moreover, $\partial_2(Q_3^n) = 0$ implies that

$$c'_{-1}(\Pi) \;=\; \big\langle f_2^n \big\rangle_q^{\star} \tag{178}$$

and this is a quasimodular form of weight $3n = 6g-6$ by Theorem 16.1.

For all odd $p \geq -1$ we can use (55) to recursively conclude that the generating functions $c_p^0(\mathrm{Tr}^n)$ for counting connected covers with Siegel-Veech weight are also quasimodular forms of weight $3n+p+1$.

In the general case, the same argument works, except that now f_k is not pure, but a linear combination of shifted symmetric functions of weight $\leq k+1$. Since ∂_2 also decreases weight, we conclude that $c_p^0(\mu_1, \ldots, \mu_n)$ is quasimodular of mixed weight $\leq p+1+\sum_{i=1}^n(|\mu_i|+1)$. □

Part IV: Volumes and Siegel-Veech constants for large genus

We return here to the geometric set-up around Siegel-Veech constants in Part I. Using all of the results of Parts I−III, we find closed formulas for both the Masur-Veech volumes and the Siegel-Veech constants of the principal stratum in terms of generating functions related to Hurwitz zeta functions.

While the focus in Part I was on Hurwitz spaces, we show in Section 17 that the large degree asymptotics also provide the Siegel-Veech constants for strata. In addition, this section contains a short digression on interpreting the non-varying phenomenon for the sum of Lyapunov exponents in terms of our quasimodularity results.

Finally, Sections 18 and 19 prove the Eskin-Zorich conjecture for the large genus asymptotics of Masur-Veech volumes and Siegel-Veech constants for the case of principal stratum.

17. From Hurwitz spaces to strata

We have worked out in Part I a combinatorial formula for Siegel-Veech constants and proved in Part III the quasimodularity of their generating functions. We now show that we can determine the area Siegel-Veech constants of strata (i.e. of any generic flat surface in a stratum $\Omega\mathcal{M}_g(m_1,\dots,m_n)$) as limits of Siegel-Veech constants of Hurwitz spaces. In this context, the non-varying phenomenon for sums of Lyapunov exponents (or, equivalently, for area Siegel-Veech constants) discovered in [11] turns out to be just a proportionality of two quasimodular forms. We will discuss this in the second part of this section.

To determine Siegel-Veech constants of strata we use Hurwitz spaces with ramification profile $\Pi = (\mu_1,\dots,\mu_n)$ where each μ_i is an m_i-cycle.

Proposition 17.1. *For any stratum $\Omega\mathcal{M}_g(m_1,\dots,m_n)$ the normalized combinatorial area Siegel-Veech constants converge to the area Siegel-Veech constant of a generic surface (X,ω) in that stratum, i.e.*

$$\frac{3}{\pi^2}\frac{\sum_{d=1}^{D} c^0_{-1}(d,\Pi)}{\sum_{d=1}^{D} N^0_d(\Pi)} \to c_{\mathrm{area}}(X,\omega) \quad \textit{for} \quad D\to\infty. \tag{179}$$

The proof is an adaptation of the argument of Eskin written for the case of arithmetic Teichmüller curves in [9, Appendix].

Proof. We abbreviate $\mathbf{m} = (m_1,\dots,m_n)$ and $\mathrm{vol} = \nu_{\mathrm{str}}(\Omega_1\mathcal{M}_g(\mathbf{m}))$. We let $V = V(X,\omega) \subset \mathbb{R}^2$ be the weighted subset of holonomy vectors of core curves of cylinders on (X,ω) with multiplicity equal to the area of each cylinder. We denote by $\widehat{f}$ the Siegel-Veech transform (cf. (14)) of a compactly supported function $f:\mathbb{R}^2\to\mathbb{R}$ with respect to V. Then (15) applied to the stratum and the Hurwitz spaces gives

$$\frac{1}{\mathrm{vol}}\int_{\Omega\mathcal{M}_g(\mathbf{m})} \widehat{f}(X)\,d\nu_{\mathrm{str}}(X) \;=\; c_{\mathrm{area}}(\Omega_1\mathcal{M}_g(\mathbf{m}))\int_{\mathbb{R}^2} f\,dxdy \tag{180}$$

and

$$\frac{1}{\nu_1(\Omega_1 H_d(\Pi))}\int_{\Omega_1 H_d(\Pi)} \widehat{f}(X)\,d\nu_1(X) \;=\; c_{\mathrm{area}}(H_d(\Pi))\int_{\mathbb{R}^2} f\,dxdy \tag{181}$$

for any fixed generic flat surface (X_d,ω_d) in the Hurwitz space $H_d(\Pi)$.

The key step is that the uniform density of rational lattice points in period coordinates implies by the arguments in [18, Section 3.2] that for every pointed elliptic curve E in $\mathcal{M}_{1,n}$

$$\lim_{D\to\infty} \frac{\sum_{d=1}^{D} \widehat{f}_d(\Pi)}{\sum_{d=1}^{D} N_d^0(\Pi)} \;=\; \frac{1}{\mathrm{vol}} \int_{\Omega_1\mathcal{M}_g(\mathbf{m})} \widehat{f}(X)\, d\nu_{\mathrm{str}}(X) \tag{182}$$

where

$$\widehat{f}_d(\Pi) \;=\; \sum_{\substack{\pi:X\to E \\ \in \mathrm{Hur}_d^0(\Pi)/\sim}} \widehat{f}(X)$$

and where $\pi : X \to E$ is the covering topologically specified by the equivalence class of a Hurwitz tuple in $\mathrm{Hur}_d^0(\Pi)$ (i.e. up to simultaneous conjugation on the Hurwitz tuples). We use the claim and (180) together with an extra averaging over $\Omega_1\mathcal{M}_{1,n}$ and interchange limit and integral by dominated convergence to obtain that

$$\begin{aligned}
\frac{1}{\mathrm{vol}} \int_{\Omega_1\mathcal{M}_g(\mathbf{m})} \widehat{f}(X)\, d\nu_{\mathrm{str}}(X) \;&=\; \lim_{d\to\infty} \frac{1}{\overline{\nu}_1(\Omega_1\mathcal{M}_{1,n})} \int_{\Omega_1\mathcal{M}_{1,n}} \frac{1}{N_d^0(\Pi)} \sum_{\substack{\pi:X\to E \\ \in \mathrm{Hur}_d^0(\Pi)/\sim}} \widehat{f}(X)\, d\overline{\nu}_1(X) \\
&=\; \lim_{d\to\infty} \frac{1}{\nu_1(\Omega_1 H_d(\Pi))} \int_{\Omega_1 H_d(\Pi)} \widehat{f}(X)\, d\nu_1(X) \\
&=\; \lim_{d\to\infty} c_{\mathrm{area}}(H_d(\Pi)) \int_{\mathbb{R}^2} f\, dxdy\,.
\end{aligned}$$

The proposition now follows from Theorem 3.1 by comparing the preceding equality to (181). □

In the remainder of this section we relate the non-varying phenomenon for strata in low genus and the quasimodularity theorem for Siegel-Veech constants. In [11] we called a connected component of a stratum $\Omega\mathcal{M}_g(\mathbf{m})$ *non-varying* if for every Teichmüller curve C generated by a Veech surface in that component the sum of Lyapunov exponents for C is the same as the sum of Lyapunov exponents for the whole component. Since the main theorem of [13], as recalled in (37), holds for all $\mathrm{SL}(2,\mathbb{R})$-invariant submanifolds and since κ depends on the stratum only, we may replace "sum of Lyapunov exponents" by "area Siegel-Veech constant" in the definition of non-varying.

The non-varying phenomenon holds for a number of connected components of strata in low genus and was discovered experimentally by Kontsevich and Zorich. It was first proved in [11] by exhibiting geometrically defined divisors in the moduli spaces of (pointed) stable curves that are disjoint from Teichmüller curves in a given stratum. Later on another proof was given by Yu and Zuo in [47] using filtrations of the Hodge bundle over Teichmüller curves.

For a connected stratum $\Omega\mathcal{M}_g(\mathbf{m})$ non-varying implies that the quasimodular forms $N^0(\mu_1,\ldots,\mu_n)$ and $c_{-1}^0(\mu_1,\ldots,\mu_n)$, where μ_i is a cycle of length m_i+1, are simply *proportional*. In fact, the Hurwitz spaces $H_d(\Pi)$ considered in this paper contain a dense set of Teichmüller curves and the argument of Proposition 17.1 in the form of [9] implies the claim. Conversely, we expect that the non-varying phenomenon restricted to the class of arithmetic Teichmüller curves can be shown by extending the quasimodularity theorem to Hurwitz spaces with more than one ramification point in the fiber over a branch point. Note that the case of non-arithmetic Teichmüller curves is not in the scope of the discussion here, because they do not arise from a covering construction.

We present examples for all strata in genus two and three. To compute volumes and Siegel-Veech constants using the formulas in the preceding sections, we first need to express the functions f_i defined in (52) as polynomials in our standard generators of the ring of shifted symmetric functions. This goes back to work of Kerov and Olshanski ([29]). Explicit formulas have been compiled e.g. by Lassalle ([33]). The first few of these functions are

$$\begin{aligned} f_1 &= p_1 + \frac{1}{24} & f_2 &= \frac{1}{2}p_2 \\ f_3 &= \frac{1}{3}p_3 - \frac{1}{2}p_1^2 + \frac{3}{8}p_1 + \frac{9}{640} & f_4 &= \frac{1}{4}p_4 - p_2p_1 + \frac{4}{3}p_2 \\ f_5 &= \frac{1}{5}p_5 - p_3p_1 - \frac{1}{2}p_2^2 + \frac{5}{6}p_1^3 - \frac{175}{48}p_1^2 + \frac{25}{8}p_3 + \frac{2375}{1152}p_1 + \frac{40625}{580608}\,. \end{aligned} \tag{183}$$

The counting functions with and without Siegel-Veech weight for the principal stratum in genus two and three have been given in (58). By Theorem 6.4 we can now confirm that

$$c_{-1}^0(\mathrm{Tr}^2) \;=\; \frac{5}{4}N^0(\mathrm{Tr}^2) \;=\; \frac{5}{4}\frac{1}{25920}(5P^2 - 3PQ - 2R)\,.$$

The modular forms

$$N^0(\mathrm{Tr}^4) \;=\; \frac{-6P^6 + 15QP^4 + 4RP^3 - 12Q^2P^2 - 12RQP + 7Q^3 + 4R^2}{1492992}$$

and

$$c_{-1}^0(\mathrm{Tr}^4) \;=\; \frac{-34P^6 + 87QP^4 + 20RP^3 - 72Q^2P^2 - 60RQP + 39Q^3 + 20R^2}{5971968}$$

are not proportional, but since the principal stratum in genus three does not have the non-varying property, we did not expect them to be proportional, either.

In the stratum $\Omega\mathcal{M}_2(2)$ we let Π be a single 3-cycle σ_3. The Siegel-Veech constant is given as the ratio of

$$\begin{aligned} N^0(\sigma_3) &= \langle f_3 \rangle_q \;=\; \frac{1}{384}P^2 - \frac{1}{960}Q - \frac{1}{64}P + \frac{9}{640} \\ &= 3x^3 + 9x^4 + 27x^5 + 45x^6 + 90x^7 + 135x^8 + 201x^9 + \cdots \end{aligned}$$

and

$$\begin{aligned} c_{-1}^0(\sigma_3) &= \langle T_1 f_3 \rangle_q - \langle T_{-1} \rangle_q \langle f_3 \rangle_q \;=\; \frac{10}{9}N^0(\Pi) \\ &= \frac{10}{3}x^3 + 10x^4 + 30x^5 + 50x^6 + 100x^7 + 150x^8 + \frac{670}{3}x^9 + \cdots \end{aligned}$$

confirming the proportionality expected by the non-varying property.

Similarly, in the stratum $\Omega\mathcal{M}_3(3,1)$ we let Π consist of a 4-cycle σ_4 and a 2-cycle Tr. As expected we find the proportionality of

$$N^0(\sigma_4, \mathrm{Tr}) \;=\; \frac{1}{272160}\left(-35P^4 + 140P^3 + 42QP^2 - 84Q + 8RP - 15Q^2 - 56R\right)$$

and

$$c_{-1}^0(\sigma_4, \mathrm{Tr}) \;=\; \langle T_{-1} f_4 f_2 \rangle_q - \langle T_{-1} \rangle_q \langle f_4 f_2 \rangle_q = \frac{21}{16}N^0(\Pi)\,.$$

In the stratum $\Omega\mathcal{M}_3(2,1,1)$, the non-varying phenomenon is again confirmed by

$$\begin{aligned} N^0(\sigma_3,\mathrm{Tr},\mathrm{Tr}) \;&=\; \big\langle f_3 f_2^2\big\rangle_q - \big\langle f_3\big\rangle_q \big\langle f_2^2\big\rangle_q \\ &=\; \frac{1}{55296}\left(-P^5+P^4+2QP^3-32QP^2-Q^2P+Q^2\right) \end{aligned}$$

and

$$\begin{aligned} c_{-1}^0(\sigma_3,\mathrm{Tr},\mathrm{Tr}) \;&=\; \big\langle T_{-1} f_3 f_2^2\big\rangle_q - \big\langle T_{-1}\big\rangle_q \big\langle f_3 f_2^2\big\rangle_q \\ &\quad - N^0(\mathrm{Tr}^2)c_{-1}^0(\sigma_3) - c_{-1}^0(\mathrm{Tr}^2)N^0(\sigma_3) \;=\; \frac{49}{36}N^0(\sigma_3,\mathrm{Tr},\mathrm{Tr})\,. \end{aligned}$$

The stratum $\Omega\mathcal{M}_3(4)$ has two connected components. Both are non-varying, with area Siegel-Veech constants 7/5 and 6/5, respectively. However, the quasimodular forms $N^0(\sigma_5)$ and $c_{-1}^0(\sigma_5)$ are not proportional, since

$$\begin{aligned} N^0(\sigma_5) \;&=\; \frac{-875P^3+13125P^2+714Q-49875P-3570Q-144R+40625}{580608} \\ c_{-1}^0(\sigma_5) \;&=\; \frac{-3875P^3+58125P^2+3102Q-219375P-15510Q-592R+178125}{2073600}\,. \end{aligned}$$

This is not a contradiction, since the volumes of the two components are not equal and our definition of Siegel-Veech constant only gives the total contribution.

The same happens in the stratum $\Omega\mathcal{M}_3(2,2)$. Again the stratum has two connected components, both non-varying, with different Siegel-Veech constants, and the quasimodular forms $N^0(\sigma_3,\sigma_3)$ and $c_{-1}^0(\sigma_3,\sigma_3)$ are not proportional.

In [19] the volumes of the connected components of strata have been calculated individually. The generating functions are quasimodular forms for the subgroup $\Gamma_0(2)$ of $\mathrm{SL}(2,\mathbb{Z})$. It seems likely that the counting functions with Siegel-Veech weight c_{-1}^0 for these components are also quasimodular forms for $\Gamma_0(2)$.

18. Asymptotics of series related to Hurwitz zeta functions

In this section we apply the general results about asymptotics proved in the appendix to the special one-variable generating series that were introduced in Section 12. Specifically, we will prove the following asymptotic formulas for the coefficients of the power series $uX(u)$ and $(4u)^{m/2}\mathfrak{B}_{m/2}(X(u))$ ($m\in\mathbb{Z}_{\geq -1}$) occurring in Theorems 12.1−12.3.

Theorem 18.1. *The coefficients v_n ($n\geq -2$ even) defined by* (138) *have the asymptotic expansion*

$$v_n \;\sim\; (-1)^{\frac{n}{2}-1}\,\frac{n!}{8\sqrt{2n}}\left(\frac{2}{\pi}\right)^{n+\frac{5}{2}}\left(1-\frac{2\pi^2+3}{24\,n}+\frac{4\pi^4-36\pi^2+9}{1152\,n^2}+\cdots\right), \quad (184)$$

where the last factor is a (divergent) power series in $1/n$ with coefficients in $\mathbb{Q}[\pi^2]$.

Theorem 18.2. *For $m\in\mathbb{Z}_{\geq -1}$ the coefficients $b_m(h)$ defined by*

$$(4u)^{m/2}\mathfrak{B}_{m/2}(X(u)) \;=\; \sum_{h=0}^{\infty} b_m(h)\,u^{2h}$$

have asymptotics given by

$$b_{-1}(h) \;\sim\; (-1)^h\,\frac{(2h)!}{h^{5/2}}\left(\frac{2}{\pi}\right)^{2h+\frac{1}{2}}\left(1-\frac{2\pi^2+15}{48\,h}+\frac{4\pi^4+12\pi^2-207}{4608\,h^2}+\cdots\right)$$

for $m = -1$ and by

$$b_m(h) \sim (-1)^h \frac{(2h)!}{h^{3/2}} \Big(\frac{2}{\pi}\Big)^{2h+\frac{1}{2}} \Big(A_0(m) + \frac{A_1(m)}{h} + \frac{A_2(m)}{h^2} + \cdots\Big) \qquad (185)$$

for $m \geq 0$, where each $A_i(m)$ belongs to $\mathbb{Q}[\pi^2]$. The coefficient $A_i(m)$ has the form $A_i(m) = (-1)^i m(P_i(m) - \varepsilon_i(m))$ with $P_i(m) \in \mathbb{Q}[\pi^2][m]$ and a correction term $\varepsilon_i(m)$ that is non-zero only for $m \in \{1, 3, \ldots, 2i+1\}$, as illustrated in Table 1.

i	$P_i(m)$	$\varepsilon_i(3)$	$\varepsilon_i(5)$	$\varepsilon_i(7)$
0	$\frac{1}{2^2}$	—	—	—
1	$\frac{P-3}{2^6}$	$\frac{1}{2^4}$	—	—
2	$m(m-5)\frac{P}{2^8} + \frac{P^2+2P+25}{2^{11}}$	$\frac{P-15}{2^8}$	$\frac{3}{2^6}$	—
3	$m(m-5)\frac{P^2-35P}{2^{12}} + \frac{\frac{1}{3}P^3+61P^2-735P-105}{2^{15}}$	$\frac{P^2-70P+385}{2^{13}}$	$\frac{3P-105}{2^{10}}$	$\frac{15}{2^8}$

TABLE 1. Coefficients in the expansion of $b_m(h)$. Here $P = 2\pi^2/3$.

We observe that the first of these two theorems is a special case of the second, since by Theorem 12.1 we can write v_n not only as the coefficient $(4n+2)u^{n+1}$ in $X(u)$ but also as the the coefficient $24(n+1)u^{n+2}$ in $(4u)^{3/2}\mathfrak{B}_{3/2}(X(u))$. We have stated it as a separate theorem, not only because it is the most important case for our applications (to volumes of strata), but also because it must be proved separately and then used for the proof of Theorem 18.2. The case $m = 2$ of Theorem 18.2 also includes Theorem 18.1, because $\mathfrak{B}_1(X) \equiv X$. Besides the cases $m = 2$ and $m = 3$, we also note the special cases $m = 0$ and $m = 1$ where $(4u)^{m/2}\mathfrak{B}_{m/2}(X(u))$ is identically 1 and all coefficients of the asymptotic expansion (185) vanish. Because of the latter observation, we have omitted the values of $\varepsilon_i(1) = P_i(1)$ from Table 1. We also wrote the asymptotic formula for $b_{-1}(h)$ separately in Theorem 18.2 because this case is of special interest to us as the one giving the coefficients of the power series $K(u)$ in Theorem 12.3 related to the area Siegel-Veech constants, and also because the asymptotic expansion in this case has a different leading power of h, compared to the case for $m \geq 0$.

Proof. The proof consists of successive applications of the rules for operation with power series of Gevrey class $\alpha = 2$, as given in the appendix, using in each case the explicit values for the small orders of which the first few were listed there. There is one important preliminary point. The series $\mathfrak{B}_n(X)$ for $n \in \frac{1}{2}\mathbb{Z}$ is a Laurent series in $X^{-1/2}$, but up to a factor X^n it is actually an even Laurent series in X^{-1}. We therefore make the substitution $x = X^{-2}$, writing $\mathfrak{B}_n(X)$ as $X^n\mathfrak{b}_n(x)$ where $\mathfrak{b}_n(x) = \sum (n)_{2k}\beta_{2k}x^k$, a power series in x. This is important because the effect of replacing X^{-1} by its square root is to change the series in question from even power series of Gevrey order 1 to power series of Gevrey order 2, to which the results about composition and functional inverse apply. We must therefore work with *three* variables x, X, and u, related by $X = X(u) = \frac{1}{4u} - \frac{u}{12} + \cdots$ and $x = X^{-2}$.

We first note that the number β_k equals $2/(2\pi i)^k$ to all orders for k even. (The two numbers differ by a factor $(1 - 2^{1-k})\zeta(k) = 1 + \mathrm{O}(2^{-k})$.) It follows from Stirling's formula that $(n)_k\beta_k$, the coefficient of $x^{k/2}$ in $\mathfrak{b}_n(x)$, has the asymptotic expansion

$$(n)_k\beta_k \sim \frac{k^{-n-1}}{\Gamma(-n)} \frac{2k!}{(2\pi i)^k} \Big(1 + \frac{n(n+1)}{2k} + \frac{n(n+1)(n+2)(3n+1)}{24k^2} + \cdots\Big)$$

to all orders in h as $k = 2h \to \infty$ with n fixed. Note that the right hand side vanishes identically if n is a non-negative integer, which is as it should be since $\mathfrak{b}_n(x)$ is a polynomial of degree n in this case. Note also that we can use Stirling's formula again to replace the asymptotic expansion on the right by one involving $h!^2$ rather than $(2h)!$, making explicit the fact that the power series $\mathfrak{b}_n(x)$ has Gevrey class 2, but the expression in terms of $(2h)!$ is simpler and more convenient for the applications. We will be concerned only with the case when $n = m/2 \geq -1/2$ is half-integral, since these are the cases occurring in Section 12, and the different behavior of the coefficients $A_i(m)$ for even and odd m is a direct consequence of this remark.

Specializing the above to the case $n = 1/2$ and applying the rules for reciprocals f^{-1} from the appendix, we obtain the asymptotics of the coefficients of $16u^2 = x/\mathfrak{b}_{1/2}(x)$ as an invertible power series in x. Applying to this the rule for the functional inverse we obtain the asymptotics of the expansion coefficients of x as an even power series in u, and then applying again the rule for powers f^λ, this time for $\lambda = -1/2$, we obtain the asymptotics of the coefficients of $X = x^{-1/2}$ as an odd power series in u. They are as given in Theorem 18.1.

Exactly the same type of calculation gives the proof for Theorem 18.2. Since we now have the asymptotics of the coefficients of both power series $\mathfrak{b}_n(x)$ and $x = x(u)$, we obtain the asymptotics of the coefficients of $\mathfrak{b}_n(x(u))$ by applying the rule for the composition of power series of Gevrey class 2, the asymptotics for the series $(x/16u^2)^{-m/4} = (4uX)^{m/2}$ by applying the rules for powers to either of the monic power series $x(u)/16u^2$ or $4uX(u)$, and the asymptotics for their product $(4u)^{m/2}\mathfrak{B}_{m/2}(X(u)) = (16u^2/x)^{m/4}\mathfrak{b}_{m/2}(x)$ by applying the rule for products. The results of these computations are the ones given in the theorem. The difference between the cases of odd and even m, as already noted, comes from the fact that the power series $\mathfrak{b}_{m/2}(x)$ terminates in the former case, so that when we apply the rule for composition to the two series $\mathfrak{b}_{m/2}$ and $x(u) = 4u + \cdots$, the "last" contributions in (A.3) (in the terminology explained there) all vanish and we get only the "first" ones. These lead to the polynomial part $P_i(m)$ of the expansion coefficients $A_i(m)$. For m odd (and also for non-integral values of m, which we are not considering), one also has to include the "last" contributions in (A.3) as well, and for a fixed odd value of m this gives a second infinite expansion in powers of $1/h$ contributing to (185). This second expansion starts a little later than the first one, which is why for each value of i there are only finitely many odd values of m for which the term $\varepsilon_i(m)$ is non-zero. □

19. Asymptotics of Masur-Veech volumes and Siegel-Veech constants

In this section we prove two conjectures of Eskin and Zorich on the large genus asymptotics of the Masur-Veech volumes and the area Siegel-Veech constants for the principal stratum. Our strategy, based on the results of the previous two sections, gives not only the top terms of the asymptotics conjectured by Eskin and Zorich, but all terms (or as many as one is willing to compute).

We start with a discussion on the normalizations of the measure. The Masur-Veech measure of a subset S of $\Omega_1\mathcal{M}_g(m_1,\dots,m_n)$ is the volume in the N-dimensional Lebesgue measure ($N = 2g-1+n$) in period coordinates of the cone under S in $\Omega\mathcal{M}_g$. The viewpoint adopted in [18] is to define the unit cube in the lattice $\mathbb{Z}[i]^N \subset \mathbb{C}^N$ to have volume one. We denote by $\mathrm{vol}(\Omega_1\mathcal{M}_g(m_1,\dots,m_n))$ the volumes with respect to this normalization.

An alternative normalization (used in the key reference [15] for Siegel-Veech constants) is to compute for $t \in \mathbb{R}$ the function $\mathrm{vol}(S,t)$ giving the volume of the cone over S intersected with the set $\{\mathrm{area}(X,\omega) \le t\} \subset \Omega_1\mathcal{M}_g(m_1,\dots,m_n)$ and then to declare $2\frac{\partial}{\partial t}\mathrm{vol}(S,t)$ to be the Masur-Veech volume of S. This definition mimics the relation between the area and volume of a sphere in $\mathbb{C}^N$. We denote by $\mathrm{vol}_{\mathrm{EMZ}}(\Omega_1\mathcal{M}_g(m_1,\dots,m_n))$ the volumes with respect to this normalization. This normalization is discussed in [50] and it is shown there that

$$\mathrm{vol}_{\mathrm{EMZ}}(\Omega_1\mathcal{M}_g(m_1,\dots,m_n)) \;=\; 2N\mathrm{vol}(\Omega_1\mathcal{M}_g(m_1,\dots,m_n))\,.$$

We follow the idea of Zorich and Eskin-Okounkov ([18]) to compute volumes by counting lattice points with finer and finer mesh size. It will be convenient to introduce cumulants that involve the appropriate powers of π. Hence we define $\langle\langle\, \ell_1,\dots,\ell_s \,\rangle\rangle$ as the leading term (in $1/h$) of an h-evaluation. More precisely, let

$$\mathrm{ev}[\langle p_{\ell_1}|\cdots|p_{\ell_s}|\rangle] \;=\; \frac{1}{h^{1+\sum_{i=1}^s(\ell_i+1)}}\;\langle\langle\, \ell_1,\dots,\ell_s \,\rangle\rangle\,(1+\mathrm{O}(h))\,,$$

so that by Proposition 11.1 and (85)

$$\langle\langle\, \ell_1,\dots,\ell_s \,\rangle\rangle \;=\; (-4\pi^2)^{1+\sum_{i=1}^s(\ell_i-1)/2}\,\langle\langle\, \ell_1,\dots,\ell_s \,\rangle\rangle_{\mathbb{Q}}\,. \tag{186}$$

The volumes and the cumulants for small genera are listed in Table 2, taken from work of Eskin and Okounkov.

$n = 2g-2$	2	4	6	8	10
vol	$\frac{1}{1350}\pi^4$	$\frac{1}{87480}\pi^6$	$\frac{29}{134719200}\pi^8$	$\frac{23357\,\pi^{10}}{5359129776000}$	$\frac{16493303\,\pi^{12}}{179616593572416000}$
$\mathrm{vol}_{\mathrm{EMZ}}$	$\frac{1}{135}\pi^4$	$\frac{1}{4860}\pi^6$	$\frac{377}{67359600}\pi^8$	$\frac{23357\,\pi^{10}}{157621464000}$	$\frac{16493303\,\pi^{12}}{4276585561248000}$
$\langle\langle\,\underbrace{2,\dots,2}_{n}\,\rangle\rangle$	$\frac{16}{45}\pi^4$	$\frac{1792}{27}\pi^6$	$\frac{772096}{9}\pi^8$	$\frac{10715070464\,\pi^{10}}{27}$	$\frac{43236204216320\,\pi^{12}}{9}$

Table 2. Masur-Veech volumes of the principal stratum

Proposition 19.1. *The volume of the principal stratum can be expressed in terms of cumulants as*

$$(4n+2)\,\mathrm{vol}\,(\Omega\mathcal{M}_g(1^n)) \;=\; \mathrm{vol}_{\mathrm{EMZ}}\,(\Omega\mathcal{M}_g(1^n)) \;=\; \frac{\langle\langle \overbrace{2,\ldots,2}^{n} \rangle\rangle}{2^{n-1}\,(2n)!}\,. \tag{187}$$

Proof. The definition of connected brackets in (108), and hence the definition of cumulants as their leading terms, are made to reproduce the passage from counting covers without unramified components to counting connected covers in (50). Consequently, the combination the definitions (51), (47), and (48) gives $N'(\mathrm{Tr}^n) = \langle f_2^n \rangle_q^\star$ and together with $f_2 = \frac{1}{2}p_2$ this implies

$$\mathrm{ev}(N^0(\mathrm{Tr}^n)) \;=\; \frac{\langle\langle \overbrace{2,\ldots,2}^{n} \rangle\rangle}{2^n}\, h^{-(2n+1)}\,(1+\mathrm{O}(h))\,. \tag{188}$$

The volume of the stratum can be computed as the limit as $D \to \infty$ of the number of points with period coordinates in $\mathbb{Z}[D^{-1}]$. The precise version of this idea is the following formula by Eskin and Okounkov ([18, Formula 3.2])

$$\mathrm{vol}\,(\Omega\mathcal{M}_g(1^n)) \;=\; \lim_{D\to\infty} D^{-(2n+1)} \sum_{d=1}^{D} N_d^0(\mathrm{Tr}^n)\,. \tag{189}$$

The proposition now follows from Proposition 9.4. □

On the basis of numerical values obtained from the algorithms in [18], Eskin and Zorich made the following conjecture.

Conjecture 19.2 ([20]). *Let*

$$V(\mathbf{m}) \;=\; \frac{(m_1+1)(m_2+1)\cdots(m_n+1)}{4}\;\mathrm{vol}_{\mathrm{EMZ}}\,(\Omega\mathcal{M}_g(m_1,\ldots,m_n))\,.$$

Then $V(\mathbf{m}) = 1 + \mathrm{o}(1)$ as $\sum m_i = 2g-2$ tends to infinity.

Theorem 19.3. *Conjecture 19.2 holds for the principal stratum.*

Proof. By (187), the conversion (186) from cumulants to rational cumulants and via (125) to v_n, and the asymptotics of v_n given in Theorem 18.1 we have

$$V(\underbrace{1,\ldots,1}_{2g-2}) \;\sim\; \Bigl(1 \,-\, \frac{\pi^2}{24g} \,-\, \frac{\pi^4 - 60\pi^2}{1152g^2} \,+\, \cdots\Bigr)$$

as $g \to \infty$. □

We now discuss the large genus asymptotics of the area Siegel-Veech constants $c_{\mathrm{area}}(\Omega\mathcal{M}_g(1^{2g-2}))$, again restricted to the case of the principal stratum. Values for small g are given in the table below.

$g = \frac{n}{2}+1$	2	3	4	5	6
$\frac{\pi^2}{3}c_{\mathrm{area}}(\Omega\mathcal{M}_g(1^{2g-2}))$	$\frac{5}{4}$	$\frac{39}{28}$	$\frac{2225}{1508}$	$\frac{142333}{93428}$	$\frac{102396315}{65973212}$

The leading order in the following theorem had also been conjectured by Eskin and Zorich ([20]).

Theorem 19.4. *For $g \to \infty$*

$$c_{\text{area}}(\Omega\mathcal{M}_g(1^{2g-2})) \sim \frac{1}{2} - \frac{1}{8g} - \frac{5}{32g^2} - \frac{4\pi^2+75}{384g^3} + \cdots,$$

where the coefficient of $1/g^\ell$ is a polynomial in π^2 of degree $\ell-2$ for all $\ell \geq 2$.

It is remarkable that although the individual area Siegel-Veech constants all have a factor of $1/\pi^2$, the dominating term of the asymptotics is rational.

Proof. We will show at the end of this section that

$$c_{\text{area}}(\Omega\mathcal{M}_g(1^{2g-2})) \;=\; -\frac{1}{8\pi^2}\,\frac{\kappa_n}{v_n} \qquad (n = 2g-2), \tag{190}$$

where κ_n and v_n are as in Section 12. The assertion then follows immediately from the asymptotic results in Section 18 since the asymptotics of v_n is given in Theorem 18.1, while the generating series K for the κ_n was expressed in Theorem 12.3 in terms of $\mathfrak{B}_{-1/2}$, and the asymptotics of its coefficients is given in Theorem 18.2. □

To prove (190), we will use the approximation of the Siegel-Veech constants that we gave in Proposition 17.1. By the asymptotic formula for the coefficients of a modular form in Proposition 9.4 it suffices to compute the leading terms of the X-evaluations of the modular forms whose coefficients are summed up in the numerator and denominator of (179) respectively. The denominator has been taken care of by (188) and we now treat the numerator. Recall that we defined $c^0_{-1}(\mathrm{Tr}^{2k})$ in Section 6 as the generating function of covers with (-1)-Siegel-Veech weight and that we showed in Theorem 6.4 that this generating series is a quasimodular form.

Theorem 19.5. *The X-evaluation of the quasimodular form $c^0_{-1}(\mathrm{Tr}^n)$ has degree $\frac{n}{2}+1$. Its leading term $c^0_{-1}(\mathrm{Tr}^n)_L = [X^{\frac{n}{2}+1}]\,\mathrm{Ev}[c^0_{-1}(\mathrm{Tr}^n)]$ is given by*

$$c^0_{-1}(\mathrm{Tr}^n)_L \;=\; n!\,(-B_2)\sum_{k=2}^{n}\frac{k\,\langle\langle\overbrace{2,\ldots,2}^{n-k},k-1\,\rangle\rangle_{\mathbb{Q}}}{2^{n-k+2}\,(n-k)!} \;=\; -\frac{1}{24}\,\frac{n!}{2^n}\,\kappa_n \tag{191}$$

where $B_2 = \frac{1}{6}$ is the second Bernoulli number.

From the formula for κ_n given in Section 12 we find the following values.

n	2	4	6	8	10
$c^0_{-1}(\mathrm{Tr}^n)_L$	$\frac{1}{144}$	$-\frac{13}{144}$	$\frac{2225}{288}$	$-\frac{996331}{432}$	$\frac{170660525}{96}$

We introduced p-Siegel-Veech weight and $c^0_p(\mathrm{Tr}^n)$ in Part III as a crucial tool for interpolation and to prove the quasimodularity of $c^0_{-1}(\mathrm{Tr}^n)$. For comparison we give the analogous statement to Theorem 19.5 for $p \geq 1$.

Proposition 19.6. *Let $p \geq 1$ be odd and $n \geq 2$ even. Then the X-evaluation of the quasimodular form $c^0_p(\mathrm{Tr}^n)$ has degree $\frac{n+p+1}{2}$ and the leading term $c^0_p(\mathrm{Tr}^n)_L = [X^{\frac{n+p+1}{2}}]\,\mathrm{Ev}[c^0_p(\mathrm{Tr}^n)]$ is given either in terms of the mixed cumulants (128) as*

$$\frac{1}{n!}c^0_p(\mathrm{Tr}^n)_L \;=\; \frac{n!}{2^n}\,\langle T_p|\underbrace{p_2|\cdots|p_2}_{n}\rangle_{\mathbb{Q}} \tag{192}$$

or explicitly in terms of Bernoulli numbers by formula (199) *below.*

We emphasize that, although the statements of Theorem 19.6 and Theorem 19.5 are quite parallel, we cannot deduce the latter from the former, because the leading terms correspond to different powers of X. Moreover, we cannot deduce the asymptotics of $c^0_{-1}(\mathrm{Tr}^n)_L$ by extrapolation the asymptotics of $c^0_p(\mathrm{Tr}^n)_L$ to $p=-1$, as the following corollary shows.

Corollary 19.7. *For $p \geq 1$ odd*

$$c^0_p(\mathrm{Tr}^{2h})_L \;\sim\; \frac{(-1)^h}{\sqrt{\pi}}\,\frac{(2h)!^2}{h^{3/2}\pi^{2h}}\;\cdot h^{p+1}\cdot\Big(\frac{2}{\pi}\Big)^{p+1}\,\frac{(-1)^{(p+1)/2}}{p(p+1)}$$

as $h\to\infty$, while

$$c^0_{-1}(\mathrm{Tr}^{2h})_L \;\sim\; \frac{(-1)^h}{\sqrt{\pi}}\,\frac{(2h)!^2}{h^{3/2}\pi^{2h}}\cdot\frac{-1}{24}\,.$$

Proof. The second line follows by (191) from the asymptotics of κ_n that we already discussed.

For the first line we use (192). That is, we first compute the asymptotics of the cumulants $\langle p_{k-1}|p_2|\cdots|p_2\rangle$ encoded in the generating series $\psi_k(u)$ (see (129)) by linearly combining with the help of (140) the asymptotics given in Theorem 18.2. (The result is stated in the introduction.) Since T_p is a quadratic polynomial in the Q_k's by Theorem 13.5, the generating series of the cumulants we are interested in is by Proposition 11.7 a linear combination of products of the ψ_k's. Consequently, we can apply the product rule from the appendix to conclude. □

To prove the main results of this section, we form the generating series of Siegel-Veech constants for the principal stratum (as power series with quasimodular form coefficients)

$$C'_p(u) \;:=\; \sum_{n=0}^{\infty} c'_p(\mathrm{Tr}^n)\frac{u^n}{2^n n!}\,,\qquad C^0_p(u)\;:=\;\sum_{n=0}^{\infty} c^0_p(\mathrm{Tr}^n)\frac{u^n}{2^n n!} \tag{193}$$

for coverings without unramified components and for connected covers, where c'_p and c^0_p are the generating series defined in (53). By definition these power series are even and have no constant term. Note that our notation emphasizes that so far, in Parts I and III, we have been working with Siegel-Veech constants for a fixed ramification pattern (i.e. in fixed genus) and we studied the generating series as *the number d of sheets is growing*, denoted by small letters c with appropriate decorations. Only now, the *number of branch points is growing* and the corresponding generating series are denoted by decorated capital letters C.

Recall that $f_2 = \frac{p_2}{2} = Q_3$ and that by (177) the series for coverings without unramified components is given for any $p \geq -1$ by

$$C'_p(u) \;=\; \sum_{n=1}^{\infty}\big(\langle T_p f_2^n\rangle_q - \langle T_p\rangle_q\langle f_2^n\rangle_q\big)\,\frac{u^n}{2^n n!} \;=\; \sum_{n=1}^{\infty}\langle T_p|p_2^n\rangle\frac{u^n}{n!} \;=\; \sum_{n=1}^{\infty}\langle \widetilde{T}_p|p_2^n\rangle\frac{u^n}{n!}\,.$$

(This explains why we included the factor 2^{-n} in (193)). For $p=-1$, using the definition (175) of the bracket $\langle\,\rangle^\star_q$ and noting $\partial_2 f_2 = 0$, we have instead the identity

$$C'_{-1}(u) \;=\; \sum_{n=1}^{\infty}\big\langle f_2^n\big\rangle^\star_q\,\frac{u^n}{n!}\,.$$

Since $N'(\mathrm{Tr}^n) = \langle f_2^n \rangle_q$ (as recalled in Proposition 19.1) the two generating series are related by

$$C_p^0(u) \;=\; \frac{C'_p(u)}{\sum_{n\geq 0} N'(\mathrm{Tr}^n)\frac{u^n}{2^n n!}} \;=\; \frac{\langle\, \widetilde{T}_p \mid \exp(up_2)\,\rangle_q}{\langle\, \exp(up_2)\,\rangle_q}\,, \tag{194}$$

since (55) specializes to this identity in the case that all elements in the ramification profile are equal.

Proposition 19.8. *The generating series of Siegel-Veech constants for the principal stratum is given for $p > 0$ by*

$$C_p^0(u) = \sum_{\substack{i,\,k\geq 0,\\ i+k>0}} G_{p+i+1}^{(i+k)} \frac{u^{i+k}}{2^i(i+1)!} \sum_{m=0}^{\infty} \langle Q_k | \underbrace{p_2|\ldots|p_2}_{m} \rangle_q \frac{u^m}{m!} \tag{195}$$

and for $p = -1$ by

$$\begin{aligned} C_{-1}^0(u) = &\sum_{i\geq 2,k\geq 0} G_i^{(i+k)} \frac{u^{i+k}}{2^i(i+1)!} \sum_{m=0}^{\infty} \langle Q_k | \underbrace{p_2|\ldots|p_2}_{m} \rangle_q \frac{u^m}{m!} \\ &+ \sum_{k\geq 2} G_2^{(k-1)} u^k \sum_{m=0}^{\infty} \langle Q_k | \underbrace{p_2|\ldots|p_2}_{m} \rangle_q \frac{u^m}{m!}\,. \end{aligned} \tag{196}$$

Proof. For $p \geq 1$ we use Theorem 14.1 with (155) and the effect of the $\rho_{i,j}$-operator on powers of Q_3 given in (157) to deduce from the preceding formulas that

$$\begin{aligned} C_p^0(u) &= \frac{\sum_{n=0}^{\infty} \sum_{i\geq 0,j\geq 1} \langle \rho_{i,j}(p_2^n)\rangle_q \, G_{p+i+1}^{(j)} \frac{u^n}{n!}}{\sum_{n=0}^{\infty} \langle p_2^n \rangle_q \frac{u^n}{n!}} \\ &= \sum_{i\geq 0,j\geq 1} G_{p+i+1}^{(j)} \frac{u^j}{2^i(i+1)!} \frac{\sum_{n=0}^{\infty} \langle Q_{j-i} p_2^n \rangle_q \frac{u^n}{n!}}{\sum_{n=0}^{\infty} \langle p_2^n \rangle_q \frac{u^n}{n!}}\,. \end{aligned}$$

The equality to the statement in the lemma follows from the definition of cumulants.

For $p = -1$ recall that by Theorem 16.1

$$\begin{aligned} &\langle T_{-1} f\rangle_q - \langle T_{-1}\rangle \langle f\rangle_q \\ &\quad= \sum_{j\geq 1} \big(D^{j-1}(G_2) + \frac{1}{24}\delta_{j,1}\big) \langle \rho_{0,j}(f)\rangle_q + \sum_{i\geq 2,j\geq 1} D^j(G_i) \langle \rho_{i,j}(f)\rangle_q \end{aligned}$$

and since $\partial_2 Q_3^n = 0$ the extra term given by $\delta_{j,1}$ disappears here. ∎

The leading coefficient of the expression in the preceding lemma differs upon $p \geq 1$ or not as we now discuss.

Proof of Theorem 19.5 and Proposition 19.6. By the definition of cumulants

$$\langle Q_k | \underbrace{p_2|\ldots|p_2}_{m} \rangle_X = \frac{\langle\langle \overbrace{2,\ldots,2}^{m}, k-1 \rangle\rangle_{\mathbb{Q}}}{(k-1)!}\; X^{\frac{k+m}{2}} \;+\; \mathrm{O}(X^{\frac{k+m}{2}-1})\,. \tag{197}$$

On the other hand, the leading term of the derivative of an Eisenstein series is determined by

$$\langle D^{i+\ell}(G_{p+i+1})\rangle_X \;=\; \frac{(2i+\ell+p)!}{(p+i)!} \frac{-B_{p+i+1}}{2(p+i+1)} X^{\frac{p+i+1}{2}} \;+ \mathrm{O}(X^{\frac{p+i+1}{2}-1})\,. \tag{198}$$

Consequently, the degree of the X-evaluation of all the summands in (195) is $k+\frac{p+1}{2}$ and all of them contribute to the leading term. Adding the contributions gives

$$\frac{1}{n!}c_p^0(\mathrm{Tr}^n)_L = \sum_{i=1}^{n-2}\sum_{k=2}^{n-i}\frac{(2i+k+p)!}{(p+i+1)!}\,\frac{-B_{p+i+1}}{2^{n-k-i+1}(i+1)!(k-1)!}\,\frac{\langle\langle\, k-1,\overbrace{2,\ldots,2}^{n-i-k}\,\rangle\rangle_{\mathbb{Q}}}{(n-i-k)!}$$
$$+\frac{(2n+p)!}{(p+n+1)!}\,\frac{-B_{p+n+1}}{2^{n+1}(n+1)!}\,. \tag{199}$$

This is the alternative formula mentioned in the proposition. The formula stated in (192) follows directly from (194) and the definition of cumulants.

Now we address the case $p=-1$. For all the terms with $i>0$ the preceding formulas are also valid in this case and contribute to the X^k-term. However, the summands in the last line of (196) contributes to the X^{k+1}-term of the X-evaluation. Applying (198) and (197) gives the formula in the theorem. □

Proof of (190). By Proposition 17.1 we need to take $3/\pi^2$ times the ratio of the asymptotics of the sum of the coefficients of $c_{-1}(\mathrm{Tr}^{2k})$ and the asymptotics of the sum of the coefficients of $N^0(\mathrm{Tr}^{2k})$. By Proposition 9.4 we can equivalently take the ratio of the leading coefficients of ev applied to the two modular forms. Since the numerator and denominator are of the same degree in h, we can work as well with the Ev-images. The claim now follows from (191) and (188), together with the definition of v_n in (125). □

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Appendix: Asymptotics of very rapidly divergent series

The aim of this appendix is to study the asymptotic behaviour of powers, inverses, functional inverses, products, and compositions of power series whose coefficients have very rapid growth. More specifically, we will verify that each of these operations preserves the class of functions having coefficients that grow like $n!^\alpha$, or that have an asymptotic expansion of the form

$$a_n \sim n!^\alpha \beta^n n^\gamma \Big(A_0 + \frac{A_1}{n} + \frac{A_2}{n^2} + \cdots \Big) \tag{A.1}$$

for some real constants $\alpha > 1$, $\beta > 0$, and $\gamma \in \mathbb{R}$, and where "asymptotic expansion" has the usual meaning that the series in (A.1) may be divergent but that $a_n/n!^\alpha\beta^n n^\gamma$ equals $A_0 + \cdots + A_{r-1}n^{-r+1} + \mathrm{O}(n^{-r})$ as $n \to \infty$ for any fixed $r > 0$. For multiplication and powers we need only $\alpha > 0$ ("rapidly divergent"), but for composition and functional inverse the assumption $\alpha > 1$ ("very rapidly divergent") is crucial. For each of these operations we will give explicit formulas for the asymptotics of the corresponding coefficients in the case $\alpha = 2$, which is the case that is of interest for our applications to the asymptotics of Siegel-Veech constants.

The results that we give in the case of products or fixed powers may be known in the literature, though even here we could not find any convenient reference, but for the cases of composition and functional inverse we could not find any reference at all, and it seemed best to give a self-contained account. Our proofs depend on a simple estimate for the coefficients of powers of series with coefficient growth of type $n!^\alpha$, given as Lemma A.2 below. This estimate is good enough for our applications, but out of curiosity we did numerical computations to study the actual asymptotic behavior, and since the results are of some interest we report on them briefly at the end of this appendix.

For real numbers $\alpha > 0$, $\beta > 0$, and $\gamma \in \mathbb{R}$ we denote by $\mathfrak{G}(\alpha, \beta, \gamma)$ the class of power series (say, with complex coefficients) $\sum a_n x^n$ whose Taylor coefficients a_n satisfy the bound $a_n = \mathrm{O}(n!^\alpha \beta^n n^\gamma)$ and by $\mathfrak{G}_{\mathrm{asy}}(\alpha, \beta, \gamma)$ the subclass for which a_n has a full asymptotic development as in (A.1). We also write $\mathfrak{G}(\alpha, \beta)$ for $\cup_\gamma \mathfrak{G}(\alpha, \beta, \gamma)$ and $\mathfrak{G}(\alpha)$ for $\cup_\beta \mathfrak{G}(\alpha, \beta)$. (The letter $\mathfrak{G}$ stands for Gevrey, who first studied series of these types.) We also define the class $\mathfrak{G}_{\mathrm{asy}}(\alpha, \beta)$, but here it is too restrictive to simply take the union of the $\mathfrak{G}_{\mathrm{asy}}(\alpha, \beta, \gamma)$ for all $\gamma \in \mathbb{R}$, since this class would not be closed under multiplication or even under addition. Instead, we define it to be the space of power series whose coefficients have an asymptotic expansion

$$a_n \sim n!^\alpha \beta^n \big(A_0\, n^{\gamma_0} + A_1\, n^{\gamma_1} + A_2\, n^{\gamma_2} + \cdots \big) \tag{A.2}$$

with real exponents $\gamma_0 > \gamma_1 > \gamma_2 > \cdots$, $\gamma_i \to \infty$. In our applications all of the exponents γ_i are rational, with bounded denominators. Note that any two classes $\mathfrak{G}(\alpha)$, $\mathfrak{G}(\alpha, \beta)$, or $\mathfrak{G}(\alpha, \beta, \gamma)$ have the property that one (namely, the one with the larger exponents (α, β, γ) in lexicographical order) contains the other. Note also that in both the expansions (A.1) and (A.2), we do not require that A_0, or for that matter any of the coefficients A_i, be non-zero, since otherwise these classes would not form vector spaces, let alone rings. This means that any space $\mathfrak{G}(\alpha', \beta')$ with $\alpha' < \alpha$ or with $\alpha' = \alpha$ and $\beta' < \beta$ can be considered as a subspace of $\mathfrak{G}_{\mathrm{asy}}(\alpha, \beta)$ (or of any $\mathfrak{G}_{\mathrm{asy}}(\alpha, \beta, \gamma)$) having an expansion (A.1) or (A.2) with all A_i equal to 0. This is convenient because it means that in statements about, say, the product of two functions of these types, we can assume without loss of generality that both belong to the same Gevrey class, thus avoiding fussy notational distinctions.

Theorem A.1. *Let $\alpha > 1$, $\beta > 0$, and γ be real numbers. Then each of the classes $\mathfrak{G}(\alpha)$, $\mathfrak{G}(\alpha,\beta)$, $\mathfrak{G}(\alpha,\beta,\gamma)$, $\mathfrak{G}_{\text{asy}}(\alpha,\beta,\gamma)$, and $\mathfrak{G}_{\text{asy}}(\alpha,\beta)$ is closed under the operations*

(i) *addition ($f(x)+g(x)$),*
(ii) *multiplication ($f(x)g(x)$),*
(iii) *composition ($g(f(x))$, where $f(x) = x + \mathrm{O}(x^2)$),*
(iv) *complex powers ($f(x)^r$, where $f(x) = 1 + \mathrm{O}(x)$), and*
(v) *functional inverse ($f^{-1}(x)$, where $f(x) = x + \mathrm{O}(x^2)$),*

where in the cases of $\mathfrak{G}_{\text{asy}}(\alpha,\beta,\gamma)$ and $\mathfrak{G}_{\text{asy}}(\alpha,\beta)$ the asymptotic expansion to any fixed order of the result of the operation depends only on the asymptotic expansions to the same order and on a bounded number of initial values of the Taylor coefficients of the input function or functions.

We have formulated the theorem in a purely qualitative way, without writing out the full asymptotic expansions of the result of each of the operations, in order to keep the statement reasonably short and to emphasize the main point, but in the course of the proof we will write out explicitly the first few terms of the asymptotics for each operation in the case $\alpha = 2$.

Sums. This case is trivial, since one just adds the asymptotic expansions.

Products. This is the next easiest case. Let f and g belong to the Gevrey class $\mathfrak{G}(\alpha,\beta)$ (without restriction of generality with the same α and β, for the reasons explained above). We want to show that fg also belongs to $\mathfrak{G}(\alpha,\beta)$ and that it has an asymptotic expansion of the form (A.1) if f and g do. Set

$$f(x) \;=\; \sum_{n=0}^{\infty} a_n x^n\,, \qquad g(x) \;=\; \sum_{n=0}^{\infty} b_n x^n\,, \qquad f(x)\,g(x) \;=\; \sum_{n=0}^{\infty} c_n x^n\,.$$

It is convenient, here and in the later proofs, to introduce the rescaled variables $\widetilde{a}_n = a_n/n!^\alpha \beta^n$, and similarly for $\widetilde{b}_n$ and $\widetilde{c}_n$. Then

$$\widetilde{c}_n \;=\; \sum_{m=0}^{n} \binom{n}{m}^{-\alpha} \widetilde{a}_m\, \widetilde{b}_{n-m}\,.$$

To study the asymptotics of this for large n, we fix an integer $L > 0$ and break up the sum into three subsums (which we call "first", "middle", and "last") according to $m < L$, $L \le m \le n-L$, and $m > n-L$, respectively. It is clear that the "first" and "last" sums are bounded by $\mathrm{O}(n^\gamma)$ if $\widetilde{a}_n$ and $\widetilde{b}_n$ satisfy this bound, and also that they have asymptotic expansions in $n^\gamma\mathbb{C}[[1/n]]$ if $\widetilde{a}_n$ and $\widetilde{b}_n$ do. For instance, if a_n has the expansion (A.1) with $\alpha = 2$ then the "last" sum has the asymptotic expansion

$$\begin{aligned}\sum_{m=n-L+1}^{n}\binom{n}{m}^{-2}\widetilde{a}_m\widetilde{b}_{n-m} \;&=\; \widetilde{b}_0\left(A_0n^\gamma+A_1n^{\gamma-1}+A_2n^{\gamma-2}+\cdots\right)\\ &\quad+\;\frac{\widetilde{b}_1}{n^2}\left(A_0(n-1)^\gamma+A_1(n-1)^{\gamma-1}+\cdots\right)\\ &\quad+\;\frac{4\,\widetilde{b}_2}{n^2(n-1)^2}\left(A_0(n-2)^\gamma+\cdots\right)\;+\;\cdots\\ &=\; n^\gamma\left(A_0\widetilde{b}_0\;+\;\frac{A_1\widetilde{b}_0}{n}\;+\;\frac{A_2\widetilde{b}_0+A_0\widetilde{b}_1}{n^2}\;+\;\frac{A_3\widetilde{b}_0+A_1\widetilde{b}_1-\gamma A_0\widetilde{b}_1}{n^3}\;+\;\cdots\right)\end{aligned}$$

as $n\to\infty$, and if b_n has an expansion like (A.1) with A_i replaced by B_i then the "first" sum is given by a similar expression with A_i and $\widetilde{b}_j$ replaced by B_i and $\widetilde{a}_j$. For the "middle" sum, we note that because each row of Pascal's triangle is unimodal (rising to a maximum and then falling), we have $\binom{n}{m}\le\binom{n}{L}=\mathrm{O}_L(n^L)$ for $L\le m\le n-L$ and hence $\sum_{m=L}^{n-L}\widetilde{a}_m\widetilde{b}_{n-m} \;=\; \mathrm{O}_L(n^{2\gamma+1-\alpha L})$, which is smaller than any fixed negative power of n if L is large enough. It follows that the coefficients c_n have an asymptotic expansion of the same form (A.1) with the same parameters α, β, and γ as for a_n and b_n, the beginning of this expansion being

$$n!^2\beta^n n^\gamma\Big(A_0b_0+a_0B_0+\frac{A_1b_0+a_0B_1}{n}+\frac{A_2b_0+a_0B_2+(A_0b_1+a_1B_0)/\beta}{n^2}+\cdots\Big)$$

in the case $\alpha=2$.

Compositions. Since we can only compose two series if the second one has zero constant term, we will write our composed power series as $g(xf(x))=\sum c_nx^n$ for some power series $f=\sum a_nx^n$ and $g=\sum b_nx^n$. We assume that $a_0\neq 0$ and can further assume (by replacing the power series $g(x)$ by $g(a_0x)$) that $a_0=1$. Then

$$c_n \;=\; [x^n]\big(g(xf(x))\big) \;=\; \sum_{k=1}^{n} b_k\,a_{n-k}^{(k)} \qquad (n\ge 1)\,, \tag{A.3}$$

where the coefficients $a_m^{(k)}$ $(m\ge 0)$ are defined by the generating series

$$\sum_{m=0}^{\infty} a_m^{(k)}\,x^m \;=\; f(x)^k \;=\; 1\;+\;ka_1\,x\;+\;\Big(ka_2+\frac{k(k-1)a_1}{2}\Big)\,x^2\;+\;\cdots\,. \tag{A.4}$$

Now we want to apply the same decomposition "first + middle + last" of the sum in (A.3) as we did for multiplication, with the first and last terms of the sum dominating the whole sum for n large. But unlike the case of multiplication, where it would have sufficed to assume $\alpha>0$, here the assumption $\alpha>1$ is crucial. For instance, if $\alpha=1$ then the last two terms b_n and $(n-1)a_1b_{n-1}$ of the sum have the same order of magnitude, and if $\alpha<1$ then each successive term starting at the end is actually larger than its predecessor, so that we do not get the desired asymptotic expansion. If, on the other hand, α is larger than 1, then it is clear from the expressions for the first few $a_m^{(k)}$ as given in (A.4) that each of the first and last terms of the sum (A.3), counting from the ends, is of a smaller order than its predecessor, so that the "first" and "last" subsums have well-defined asymptotic expansions by the same principle as we used for products. But this is not enough for our purposes. We are assuming that both f and g belong to the same growth class $\mathfrak{G}(\alpha,\beta,\gamma)$, and since we have already proved that this class is closed under multiplication, it follows that the coefficients $a_m^{(k)}$ have the same order of growth $\mathrm{O}(m!^\alpha\beta^m m^\gamma)$ as $m\to\infty$ for each *fixed* k, but since the summand k in (A.3) goes all the way up to n we need an estimate that is uniform in k. Such an estimate is provided by the following lemma, which, as already mentioned, is not sharp but is sufficient for proving the required growth properties of the coefficients c_n. We will formulate this lemma in detail for the specific growth estimate $|a_n|\le n!^\alpha$, in order to keep its statement and proof short and clean, and then indicate briefly afterwards the modifications needed for the general case.

Lemma A.2. *Suppose that* $f(x)=\sum_{n=0}^\infty a_n x^n$ *with* $|a_n|\le n!^\alpha$ *for all* $n\ge 0$ *for some* $\alpha\ge 1$. *Then the coefficients* $a_n^{(k)}$ *defined by* (A.4) *satisfy the estimates*

$$\big|a_n^{(k)}\big| \;\le\; C(\alpha)^{k-1}\,n!^\alpha\,, \qquad \big|a_n^{(k)}\big| \;\le\; \frac{(n+k-1)!\,n!^{\alpha-1}}{(k-1)!} \tag{A.5}$$

for all $n\ge 0$ *and* $k\ge 1$, *where* $C(\alpha)$ (*e.g.* $C(1)=\frac{8}{3}$, $C(2)=\frac{9}{4}$) *denotes the maximum over all integers* $n\ge 1$ *of the quantity* $\sum_{m=0}^n\binom{n}{m}^{-\alpha}$.

Proof. We rewrite the estimates (A.5) as $\big|\widetilde{a}_n^{(k)}\big|\le C(\alpha)^{k-1}$ and $\big|\widetilde{a}_n^{(k)}\big|\le\binom{n+k-1}{n}$, where $\widetilde{a}_n^{(k)}:=n!^{-\alpha}a_n^{(k)}$. Both of them follow by induction on k: the case $k=1$ (i.e. $\big|\widetilde{a}_n^{(1)}\big|\le 1$) is true by assumption, and if (A.5) holds for all $n\ge 0$ then from

$$\big|\widetilde{a}_n^{(k+1)}\big| \;=\; \Bigg|\sum_{m=0}^n\binom{n}{m}^{-\alpha}\widetilde{a}_m^{(k)}\,\widetilde{a}_{n-m}^{(1)}\Bigg| \;\le\; \sum_{m=0}^n\binom{n}{m}^{-\alpha}\big|\widetilde{a}_m^{(k)}\big|$$

we get the upper bounds

$$\big|\widetilde{a}_n^{(k+1)}\big| \;\le\; C(\alpha)^{k-1}\sum_{m=0}^n\binom{n}{m}^{-\alpha} \;\le\; C(\alpha)^k$$

and

$$\big|\widetilde{a}_n^{(k+1)}\big| \;\le\; \sum_{m=0}^n\binom{m+k-1}{m} \;=\; \binom{n+k}{n} \tag{A.6}$$

as required. ☐

For the general case, we first note that if a series $f=\sum a_n x^n$ with $a_0=1$ belongs to $\mathfrak{G}(\alpha,\beta)$ for some real numbers $\alpha\ge 1$, $\beta>0$, then its coefficients can be estimated by both $|a_n|\le n!^\alpha\beta^n(n+1)^c$ and $|a_n|\le n!^\alpha\beta^n\binom{n+c}{n}$ for some integer $c\ge 0$. We then replace the two estimates (A.5) by two different estimates involving these two different hypotheses, namely

$$|a_n|\le n!^\alpha\beta^n(n+1)^c \;\Rightarrow\; \big|a_n^{(k)}\big| \;\le\; C^{k-1}\,n!^\alpha\,\beta^n\,(n+1)^c\,, \tag{A.7}$$

$$|a_n|\le n!^\alpha\beta^n\binom{n+c}{c} \;\Rightarrow\; \big|a_n^{(k)}\big| \;\le\; k^C\,n!^\alpha\,\beta^n\binom{n+k+c-1}{n} \tag{A.8}$$

for some sufficiently large constant C depending only on α and c. The proof of (A.7) mimics the one in the lemma, with $\widetilde{a}_n^{(k)}$ defined as $a_n/n!^\alpha\beta^n(n+1)^c$ and C defined as the maximum of $\sum_{m=0}^n\binom{n}{m}^{-\alpha}\big(\frac{(m+1)(n-m+1)}{n+1}\big)^c$ over all $n\ge 0$. For (A.8) we first note that the case $n<c$ is trivial (even with k^C in (A.7) replaced by a constant 2^C for all $k\ge 2$) since $a_k^{(n)}$ for n fixed is a polynomial of degree n in k. For $n\ge c$

we define $\widetilde{a}_n^{(k)}$ as $a_n/n!^\alpha\beta^n$, so that $\widetilde{a}_\ell^{(1)} \le \binom{\ell+c}{\ell} \le \binom{n}{\ell}^\alpha$ for $n \ge \ell + c$, and then use the induction assumption and (A.6) with k replaced by $k+c$ to obtain the upper bound

$$\begin{aligned}\big|\widetilde{a}_n^{(k+1)}\big| \le \sum_{m=0}^{n} \binom{n}{m}^{-\alpha} \big|\widetilde{a}_{n-m}^{(1)}\, \widetilde{a}_m^{(k)}\big| \;\le\; k^C \sum_{m=c}^{n} \binom{m+k+c-1}{m} \,+\, \mathrm{O}\big(n^c k^{c-1}\big) \\ \le\; k^C \binom{n+k+c}{n} \Big(1 \,+\, \mathrm{O}\big(1/k\big)\Big) \;\le\; (k+1)^C \binom{n+k+c}{n}\end{aligned}$$

for sufficiently large C and all $k \ge 1$.

Using these estimates, we find easily that the sum of the “middle” terms in the sum in (A.3) is of smaller order of magnitude than the first and last terms. (More precisely, for any $H > 0$ there is a constant K depending on H such that each term with $K < k < n-K$ in (A.3) is $\mathrm{O}(n^{-H})$ times the dominant asymptotic $n!^\alpha\beta^n n^{\gamma_0}$ for n sufficiently large if a_n and b_n both satisfy estimates of the type (A.2), and this implies the assertion since the number of terms in the sum is also bounded by n.) As an explicit example, if a_n has an asymptotic expansion of the form (A.1) with $\alpha = 2$, $\beta = 1$, $\gamma = 0$ and b_n has an asymptotic expansion of the same form with A_j replaced by B_j, then the coefficient c_n of $g(xf(x))$ has the asymptotic expansion

$$c_n \;\sim\; n!^2 \left(B_0 \,+\, \frac{B_1 + a_1 B_0}{n} \,+\, \frac{B_2 + a_1(B_1 - B_0) + a_1^2 B_2/2 + b_1 A_0}{n^2} \,+\, \cdots\right)$$

as $n \to \infty$.

Arbitrary powers. This is a special case of the preceding case, but important enough to be stated separately. If $f(x) = 1 + \cdots$ belongs to the class $\mathfrak{G}(\alpha,\beta)$ and c and λ are arbitrary complex numbers, then one can obtain f^λ by writing $f(x) = 1 + cxf_1(x)$ with $f_1(x) = 1 + \mathrm{O}(x)$ and applying the previous result for $g(xf_1(x))$ to the power series $g(x) = (1+cx)^\lambda = \sum_k \binom{\lambda}{k} c^k x^k$. Here only the “first” coefficients (corresponding to small k) contribute, because the power series g is of Gevrey order zero. As an explicit example, if $f(x) = \sum a_n x^n$ with $a_0 = 1$ satisfies (A.1) with $\alpha = 2$, then the coefficients $a_n^{(\lambda)}$ of $f(x)^\lambda$ have the asymptotic expansion

$$a_n^{(\lambda)} \;\sim\; n!^2\beta^n n^\gamma \left(\lambda A_0 + \frac{\lambda A_1}{n} + \frac{\lambda A_2 + a_1\lambda(\lambda-1)A_0/\beta}{n^2} \,+\, \cdots\right)$$

as $n \to \infty$.

Inverse power series. Let $h(x)$ be a power series beginning with x belonging to the Gevrey class $\mathfrak{G}(\alpha,\beta)$ with some $\alpha > 1$. We want to show that the inverse power series $h^{-1}(x)$ also belongs to this class, and to give an explicit formula for the asymptotic expansion of its coefficients. Write $h(x) = x - F(x)$ where $F(x) = \mathrm{O}(x^2)$. Then the inverse power series is given by

$$h^{-1}(x) \;=\; x \,+\, \sum_{k=1}^{\infty} \frac{1}{k!}\, \frac{d^{k-1}}{dx^{k-1}}\, F(x)^k$$

by one of the forms of the Lagrange inversion formula (essentially the same one as we already used in the proof of Proposition 10.3), so if we write $F(x) = cx^{r+1}f(x)$ with $r \ge 1$, $c \ne 0$, and $f(x)$ a power series beginning with 1, and define coefficients

$a_m^{(k)}$ by (A.4), then

$$h^{-1}(x) \;=\; x \,+\, \sum_{n=2}^{\infty} c_n x^n\,, \qquad c_n \;=\; \sum_{0<k<n/r} \frac{c^k}{k} \binom{n+k-1}{k-1} a_{n-rk-1}^{(k)}\,.$$

We can now apply the same estimates as for the case of composition (Lemma A.2 and its extensions) to show that the asymptotic expansion of c_n is given to any given order by summing the first O(1) terms of this sum. Once again, only the "first" coefficients (k small) contribute, and we give as a concrete example the expansion of c_n for a_n satisfying (A.1) with $\alpha = 2$, namely,

$$c_n \;\sim\; n!^2 \beta^n n^{\gamma-4} \Bigl(cA_0 + \frac{cA_1 + 2c(c+2)A_0}{n} \,+\, \cdots \Bigr)\,.$$

This completes the proof of Theorem A.1.

True asymptotics. As mentioned in the introductory paragraphs, we end this appendix by describing the complete asymptotic behavior of the coefficients $a_n^{(k)}$ defined by (A.4) when n and k tend to infinity independently of one another, even though this is not used in the paper, because it is surprisingly subtle and because finding it even numerically is not easy. We will concentrate on the special but typical case $a_n = n!^2$. We will work with the renormalized values $\widetilde{a}_n^{(k)} = a_n^{(k)}/n!^2$ as before, since these are bounded as functions of n for fixed k, and will also describe the large k asymptotics of the numbers $M_k = \max_n \widetilde{a}_n^{(k)}$. We have not given complete analytic proofs of all results.

We first consider small n. The coefficient $a_n^{(k)}$ for n fixed is a polynomial in k of degree n with leading term $k^n/n!$, the first values being given by

$$\begin{aligned}\sum_{n=0}^{\infty} a_n^{(k)}\, x^n \;&=\; \bigl(1 \,+\, x \,+\, 4x^2 \,+\, 36x^3 \,+\, \cdots\bigr)^k \\ &=\; 1 \,+\, k\,x \,+\, \frac{k^2+7k}{2}\, x^2 \,+\, \frac{k^3+21k^2+194k}{6}\, x^3 \,+\, \cdots\,.\end{aligned}$$

This gives the asymptotics of $a_n^{(k)}$ for n fixed, e.g.

$$\widetilde{a}_3^{(k)} \;=\; \frac{k^3}{6^3}\left(1 \,+\, \frac{21}{k} \,+\, \frac{194}{k^2}\right) \;=\; \frac{k^3}{6^3}\,\exp\!\left(\frac{21}{k} \,-\, \frac{53}{2k^2} \,-\, \frac{987}{k^3} \,+\, \cdots\right)$$

and for general n

$$\begin{aligned}\widetilde{a}_n^{(k)} \;\sim\; A_1(n,k) \;:=\; \frac{k^n}{n!^3}\,\exp\Bigl[\frac{n(n-1)}{k}\Bigl(\frac{7}{2} \,+\, \frac{94n-335}{12k} \\ \,+\, \frac{1711n^2-11215n+16272}{12k^2} \,+\, \cdots\Bigr)\Bigr]\,,\end{aligned} \tag{A.9}$$

as one sees by writing $f(x)^k$ as $\exp(k\log f(x))$ and expanding the power series. This approximation is valid not only for n fixed and $k \to \infty$, but also for large n and experimentally gives the correct asymptotic behavior of $\widetilde{a}_n^{(k)}$ as long as $n \ll k^{1/3}$.

At the opposite extreme, when n tends to ∞ with k fixed, we have a quite different asymptotic expansion. The estimates for products given above show that when we write $a_n^{(k)}$ as a sum of products of k coefficients a_{n_i}, the dominant terms are those where all but one of the n_i are bounded, so

$$a_n^{(k)} \;\sim\; \sum_{r\geq 0}\Biggl(\sum_{j=1}^{k}\sum_{\substack{n_1,\dots,n_k\geq 0\\ n_1+\cdots+\widehat{n}_j+\cdots+n_k=r\\ n_j=n-r}} a_{n_1}\cdots a_{n_k}\Biggr) = k\sum_{r\geq 0} a_{n-r}a_r^{(k-1)}$$

in the sense that for any $C>0$ the sum of the terms on the right with $0\leq r\leq R$ approximates $a_n^{(k)}$ to within a relative error of $\mathrm{O}(n^{-C})$ if R is sufficiently large. Thus in the case $a_n = n!^2$ we find

$$\begin{aligned}\widetilde{a}_n^{(k)} \;&\sim\; k\,\Bigl(1+\frac{k-1}{n^2}+\frac{(k-1)(k+5)}{2n^2(n-1)^2}+\frac{(k-1)(k^2+19k+174)}{6n^2(n-1)^2(n-2)^2}+\cdots\Bigr)\\ &=\; k\,\exp\Bigl[\frac{k-1}{n^2}\Bigl(1+\frac{7}{2n^2}+\frac{k+6}{n^3}+\frac{9k+248}{6n^4}+\cdots\Bigr)\Bigr]\,.\end{aligned}$$

The series in $\mathbb{Q}[k][[1/n]]$ occurring in the exponent in the last expression on the right is an asymptotic series (in the sense that there are only finitely many terms of order greater than n^{-C} for any $C>0$) not only for k fixed but as long as $k\ll n^3$, and in that range it continues (experimentally) to give the correct asymptotic expansion of $\widetilde{a}_n^{(k)}$ to all orders in $1/n$. If k has the same order of magnitude as n^3, then the series contains infinitely many terms of any given order in $1/n$. If we collect them together we get the expansion

$$\widetilde{a}_n^{(k)} \;\sim\; A_2(n,k) \;:=\; k\,\exp\Biggl(\sum_{i=-1}^{\infty} G_i\Bigl(\frac{k}{n^3}\Bigr)\,n^{-i}\Biggr) \tag{A.10}$$

where the $G_i(t)$ are power series with radius of convergence $\frac{4}{27}$, the first few being

$$\begin{aligned}G_{-1}(t) \;&=\; t+t^2+\frac{7}{3}t^3+\frac{15}{2}t^4+\frac{143}{5}t^5+\frac{364}{3}t^6+\frac{3876}{7}t^7+\cdots\,,\\ G_0(t) \;&=\; \frac{3}{2}t^2+10t^3+\frac{243}{4}t^4+366t^5+2218t^6+13554t^7+\cdots\,,\\ G_1(t) \;&=\; \frac{7}{2}t+16t^2+94t^3+\frac{1271}{2}t^4+\frac{9141}{2}t^5+33608t^6+\cdots\,,\\ G_2(t) \;&=\; -1+5t+\frac{131}{2}t^2+621t^3+\frac{11209}{2}t^4+50042t^5+\cdots\,.\end{aligned}$$

We can easily recognize the coefficients of $G_{-1}(t)$ and then use Lagrange inversion to write it in closed form:

$$G_{-1}(t) \;=\; \sum_{n=1}^{\infty}\frac{2\,(3n-2)!}{n!\,(2n)!}\,t^n \;=\; 3a\,+\,2\log(1-a)\,,$$

where $a = t+2t^2+7t^3+30t^4+143t^5+\cdots$ is related to t by

$$t \;=\; a\,(1-a)^2 \qquad \text{with } 0\,<\,a\,<\,\frac{1}{3}\,. \tag{A.11}$$

Making the same substitution in the other G_i, we can recognize them too:

$$\begin{aligned}G_0(t) \;&=\; \log\frac{(1-a)^{3/2}}{(1-3a)^{1/2}}\,,\qquad G_1(t) \;=\; \frac{7a-59a^2+191a^3-204a^4+9a^5}{2(1-a)^2(1-3a)^3}\,,\\ G_2(t) \;&=\; \frac{-2+50a-433a^2+1884a^3-4065a^4+4122a^5-1458a^6}{2(1-a)^2(1-3a)^6}\,,\;\cdots\,.\end{aligned}$$

(Rigorous proofs of each of these expansions are not hard to give.)

We have now found two approximations $A_1(n,k)$ and $A_2(n,k)$ to $\widetilde{a}_n^{(k)}$, the first of which makes sense as an asymptotic series to all orders if $n \ll k^{1/2}$ and is (experimentally) correct to all orders if $n \ll k^{1/3}$ and the second of which makes sense as an asymptotic series to all orders if $n > ck^{1/3}$ for any $c > 2^{-2/3}3$ and is (experimentally) correct to all orders if $n \gg k^{1/3}$. In the transition region where $k = tn^3$ for fixed $t \in (0, \frac{4}{27})$, we have

$$A_1(n,k) \;\sim\; \frac{k^n}{n!^3} \;=\; \frac{(tn^3)^n}{n!^3} \;\sim\; (2\pi)^{-3/2} \cdot n^{-3/2} \cdot (e^3 t)^n \qquad (k = tn^3 \to \infty)$$

by Stirling's formula and

$$A_2(n,k) \;\sim\; C(t) \cdot n^3 \cdot B(t)^n \qquad (k = tn^3 \to \infty)$$

by the formulas given above, where $B(t)$ and $C(t)$ are given by

$$B(t) \;=\; e^{G_{-1}(t)} \;=\; (1-a)^2 e^{3a}, \qquad C(t) \;=\; te^{G_0(t)} \;=\; a\,\frac{(1-a)^{7/2}}{(1-3a)^{1/2}}$$

with a and t related by (A.11). Thus $A_1(n,tn^3)$ is exponentially larger than $A_2(n,tn^3)$ for $t > t_0$ fixed and $n \to \infty$, and $A_2(n,tn^3)$ is exponentially larger than $A_1(n,tn^3)$ for $t < t_0$ fixed and $n \to \infty$, where $t_0 = 0.0526457\cdots$ is the unique solution in $(0, \frac{4}{27})$ of the equation $B(t) = e^3 t$, given by $t_0 = a_0(1-a_0)^2$ where $a_0 = 0.0595202\cdots$ is the unique solution in $(0, \frac{1}{3})$ of the equation $e^{3a-3} = a$. Near $k = t_0 n^3$ both approximations have the same order of magnitude and the true value of $\widetilde{a}_n^{(k)}$ is given to high accuracy by their sum. Thus our final heuristic asymptotic formula is that

$$\widetilde{a}_n^{(k)} \;\sim\; \begin{cases} A_2(n,k) & \text{for } k/n^3 < t_0 - \varepsilon, \\ A_1(n,k) + A_2(n,k) & \text{for } t_0 - \varepsilon < k/n^3 < t_0 + \varepsilon, \\ A_1(n,k) & \text{for } k/n^3 > t_0 + \varepsilon \end{cases}$$

to all orders in n. That this works well in practice is illustrated by the following table, in which $k = 50000$ is fixed and we let n vary near $\sqrt{k/t_0} = 98.29\cdots$:

n	$\widetilde{a}_n^{(k)}$	$A_1(n,k)/\widetilde{a}_n^{(k)}$	$A_2(n,k)/\widetilde{a}_n^{(k)}$	sum
80	3.517×10^{19}	1.00000000	0.00000000	1.00000000
85	1.909×10^{14}	0.99999944	0.00000056	1.00000000
90	4.732×10^{8}	0.91404303	0.08595697	1.00000000
91	6.347×10^{7}	0.45791582	0.54208418	1.00000000
92	3.119×10^{7}	0.06059598	0.93940402	1.00000000
93	2.524×10^{7}	0.00471615	0.99528385	1.00000000
94	2.167×10^{7}	0.00033500	0.99966500	1.00000000
95	1.879×10^{7}	0.00002283	0.99997717	1.00000000
100	9.945×10^{6}	0.00000000	1.00000000	1.00000000

Here, of course, the last columns of the table are not rigorously defined, since both $A_1(n,k)$ and $A_2(n,k)$ are given only by divergent asymptotic series, but in both cases the approximations obtained by breaking off the series after a few terms is insensitive (to high order) to where we break it off: for the values in the table, the numbers $A_1(n,k)/\widetilde{a}_n^{(k)}$ and $A_2(n,k)/\widetilde{a}_n^{(k)}$ have the value given to the indicated number of digits if we take m terms of the defining series for any m between 7 and 57.

Finally, if the above asymptotics are correct, then we can give the precise asymptotics of the optimal constant $M_k = \max_n \widetilde{a}_n^{(k)}$ in the uniform estimate $a_n^{(k)} \le M_k n!^2$ for k fixed and all n: this value is attained for $n = k^{1/3} + \mathrm{O}(1)$ and is given by

$$M_k \;=\; \frac{\exp(3k^{1/3})}{(2\pi)^{3/2}\, k^{1/2}} \left(1 + \mathrm{O}\big(k^{-1/3}\big)\right),$$

whereas Lemma A.2 gave only the much cruder estimate $M_k < (9/4)^{k-1}$.