INTEGRABLE LAGRANGIANS AND PICARD MODULAR FORMS

FABIEN CLÉRY, EVGENY V. FERAPONTOV, ALEXANDER ODESSKII, AND DON ZAGIER

Abstract. We consider first-order Lagrangians whose Euler-Lagrange equations belong to the class of 3D dispersionless integrable systems. The integrability conditions are known to impose an involutive system of fourth-order differential constraints for the Lagrangian density, implying that the parameter space of integrable Lagrangians is 20-dimensional, supplied with a locally free action of a 20-dimensional equivalence group having one open orbit ("master-Lagrangian"). In this paper we give several explicit constructions of the corresponding master-density, one in terms of generalised hypergeometric functions and two in terms of Picard modular forms. In fact, we construct Picard modular forms for the Picard modular group $U(1,d)$ over the ring of integers of the field $\mathbb{Q}(\sqrt{-3})$ for every positive integer d, with the master density and its degenerations given by the cases $1 \leq d \leq 3$.

Contents

1. Introduction

Integrable systems in 1+1 dimensions (equations of KdV type) have been thoroughly investigated in the mathematical physics community. The key classification principle here is the so-called *symmetry approach* based on the requirement of the existence of infinitely many higher symmetries, a property that can be efficiently verified. The scarcity of classification results in 2+1 dimensions (equations of KP type) is due to non-locality of higher symmetries in multi-dimensions, which makes them difficult to compute. The only multi-dimensional exceptions where partial classification results are currently available are dispersionless equations, where several methods have recently been introduced (hydrodynamic reductions, dispersionless Lax pairs, integrable conformal geometry). A new feature here is a parametrisation in terms of multi-dimensional hypergeometric functions, sometimes with remarkable modular properties. The case treated in this paper (integrable first-order Lagrangians) is a particular manifestation of this phenomenon leading to a parametrisation of the corresponding integrable Lagrangian densities via Picard modular forms.

Date: November 28, 2022.
2020 *Mathematics Subject Classification.* 11F03, 11F55, 33C65, 35L71, 35Q51, 37K10, 37K20.

In more detail, we investigate first-order Lagrangian densities $f(v_1, v_1, v_3)$ for which the Euler-Lagrange equations

$$\sum_{i=1}^{3} \frac{\partial}{\partial t_i}\left(\frac{\partial f}{\partial v_i}\right) = 0 \tag{1.1}$$

associated to the Lagrangians $\int f(v_1, v_1, v_3)\,dt$ belong to the class of 3D dispersionless integrable systems. Here $v_i = \partial v/\partial t_i$, where $v = v(t_1, t_2, t_3)$ is a function of three independent variables, so that (1.1) for a given function f is a non-linear second order differential equation for v. Familiar examples include the dispersionless Kadomtsev-Petviashvili (KP) equation $v_{13} - v_1 v_{11} - v_{22} = 0$ and the Boyer-Finley equation $v_{11} + v_{22} - e^{v_3} v_{33} = 0$, corresponding respectively to the Lagrangian densities $f = \frac{1}{3}v_1^3 + v_2^2 - v_1 v_3$ and $f = v_1^2 + v_2^2 - 2e^{v_3}$. A Lagrangian is said to be *non-degenerate* if the determinant of the Hessian matrix of its Lagrangian density is non-zero. Non-degenerate Euler-Lagrange equations (1.1) arise in a wide range of applications in continuum mechanics, general relativity and differential geometry.

The integrability of the Euler-Lagrange equations (1.1) has been investigated by a variety of techniques:

- *The method of hydrodynamic reductions* [10, 11], based on the requirement that equation (1.1) has infinitely many multi-phase solutions of special type.
- *The method of dispersionless Lax pairs* [40, 11], based on the representation of equation (1.1) as the compatibility condition of two Hamilton-Jacobi equations.
- *Integrable conformal geometry* (or 'integrability on solutions') [12], based on the condition that the characteristic variety of equation (1.1) defines a conformal structure which is Einstein-Weyl on every solution.

Without going into technical details, let us note that all these seemingly different approaches are equivalent [11, 5, 12], leading to one and the same set of differential constraints for the Lagrangian density f, the so-called integrability conditions. If we denote by $h_{i_1\ldots i_n}$ the partial derivatives $\partial^n h/\partial v_{i_1}\cdots\partial v_{i_n}$ of a function $h(v_1, v_2, v_3)$ and by $d^n h$ its nth symmetric differential $\sum_{i_1,\ldots,i_n} f_{i_1\cdots i_n}\,dv_{i_1}\cdots dv_{i_n}$, then these conditions can be given in a remarkably compact form:

Theorem 1.1. [11] *For a non-degenerate Lagrangian, the Euler-Lagrange equation (1.1) is integrable if and only if the Lagrangian density f satisfies the relation*

$$d^4 f = d^3 f\,\frac{dH}{H} + \frac{3}{H}\det(dM), \tag{1.2}$$

where H for the determinant of the Hessian matrix $Hess(f)$ of f and M its bordered Hessian matrix:

$$M = \begin{pmatrix} 0 & f_1 & f_2 & f_3 \\ f_1 & f_{11} & f_{12} & f_{13} \\ f_2 & f_{12} & f_{22} & f_{23} \\ f_3 & f_{13} & f_{23} & f_{33} \end{pmatrix} = \begin{pmatrix} 0 & (\nabla f)^t \\ \nabla f & \mathrm{Hess}(f) \end{pmatrix}.$$

The non-degeneracy condition is equivalent to $H \neq 0$.

Both sides of relation (1.2) are homogeneous symmetric quartics in dv_1, dv_2, dv_3. Equating similar terms we obtain expressions for all fourth-order partial derivatives of the Lagrangian density f in terms of its second-order and third-order partial derivatives (fifteen relations altogether). For example, equating the coefficients of dv_1^4 in (1.2) leads to the following PDE

$$\begin{aligned}
&(f_{11}f_{22}f_{33} - f_{11}f_{23}^2 - f_{12}^2 f_{33} + 2f_{12}f_{13}f_{23} - f_{13}^2 f_{22})\,f_{1111} = \\
&(f_{22}f_{33} - f_{23}^2)f_{111}^2 + \big[(-2f_{12}f_{33} + 2f_{13}f_{23})f_{112} + (2f_{12}f_{23} - 2f_{13}f_{22})f_{113} + (f_{11}f_{33} - 4f_{13}^2)f_{122} \\
&\quad + (-2f_{11}f_{23} + 8f_{12}f_{13})f_{123} + (f_{11}f_{22} - 4f_{12}^2)f_{133}\big]\,f_{111} - 3f_{11}^2 f_{122}f_{133} + 3f_{11}^2 f_{123}^2 + 3f_{12}^2 f_{113}^2 \\
&\quad + 6f_{11}f_{12}f_{133}f_{112} - 6f_{11}f_{13}f_{123}f_{112} - 6f_{11}f_{12}f_{123}f_{113} + 6f_{11}f_{13}f_{122}f_{113} + 3f_{13}^2 f_{112}^2 - 6f_{12}f_{13}f_{112}f_{113}\,,
\end{aligned} \tag{1.3}$$

which shows the complexity of the system of PDEs we are facing. The resulting over-determined system for f is in involution (meaning that we get the same formulas for the fifth and higher order derivatives of f in whichever order we differentiate (1.2), so that the system is consistent), and its solution space is 20-dimensional, because the values of the partial derivatives of f up to order 3 at any generic point give a total of $1+3+6+10=20$ arbitrary constants and determine all the higher-order derivatives. Thus, we have a 20-dimensional parameter space of integrable Lagrangians.

It was observed in [13] that the integrability conditions (1.2) are invariant under the fractional transformations

$$v_j \;\mapsto\; \widetilde{v}_i = \frac{l_i(v_1,v_2,v_3)}{l_0(v_1,v_2,v_3)} \quad (i=1,2,3), \qquad f \;\mapsto\; \widetilde{f} = \frac{l_4(v_1,v_2,v_3)+f}{l_0(v_1,v_2,v_3)} \tag{1.4}$$

where $l_0,\dots,l_4$ are arbitrary affine functions. This means that we can multiply $(1:v_1:v_2:v_3:f)$ on the right by a 5×5 matrix with bottom row $(0\,0\,0\,0\,1)$, providing a 20-dimensional symmetry group of the problem. It was proved in [13] that the action of the symmetry group on the 20-dimensional parameter space of integrable Lagrangians has an open orbit. Any Lagrangian density from the open orbit (such densities will be called *generic*) gives rise to a *master-Lagrangian* whose modular properties were touched on in [13], and which will be the main subject of this paper. The equality of the dimensions of the solution space and the group of symmetries means that the master-density has no continuous symmetries, but it is remarkable that its stabiliser is a lattice in the real part of the complex symmetry group.

The system of equations (1.2) has a trivial solution $f(v_1,v_2,v_3)=v_1v_2v_3$ as noted in [11] and a hierarchy of four kinds of solutions that can be seen either as successive deformations of this trivial one or in the reverse order as successive degenerations of (any) master solution, namely

$$f(v_1,v_2,v_3) \;=\; g_d(v_1,\dots,v_d)\,v_{d+1}\cdots v_3 \qquad (0\le d\le 3)\,, \tag{1.5}$$

where $g_0()=1$ and each g_d $(1\le d\le 3)$ is supposed to be generic in the sense that the solution (1.5) is not equivalent under the symmetry group (1.4) to one of type $d-1$. In each case, the function g_d must satisfy a system of fourth-order non-linear differential equations, of which (1.2) (or (1.3) and its 14 companions) is the special case $d=3$, while the equation for g_1 in the case $d=1$ is simply

$$g\left(gg''-2g'^2\right)g'''' \;-\; 9g'^2g''^2 \;+\; 2gg'g''g''' \;+\; 8g'^3g''' \;-\; g^2g'''^2 \;=\; 0\,. \tag{1.6}$$

In Part II of the paper we will give descriptions of the generic solution g_d for each value $d=1$, 2 or 3. These descriptions are of two essentially different types.

- *Parametric representation.* We will prove that the nonlinear integrability conditions (1.2) can be linearised, leading to a parametrisation of the generic integrable Lagrangian density f via generalised hypergeometric functions. This is a direct corollary of symmetry properties of the underlying integrability conditions. This representation uses a basis of solutions of a generalised hypergeometric system comprised of Picard-Fuchs equations for periods of the associated family of Picard trigonal curves. These curves, along with the equations for their periods, were studied by É. Picard [31] already in 1883.
- *Explicit formulas.* In each case we will give three different versions, all of which are equivalent under the group of transformations. The simplest and most attractive are given by the symmetric power series expansions

$$g_1(x) \;=\; \sum_{i\geqslant 0} C_i^2\,\frac{x^{6i+1}}{(6i+1)!}\,, \qquad g_2(x,y) \;=\; \sum_{i,j\geqslant 0} C_i\,C_j\,C_{i+j}\,\frac{x^{6i+1}}{(6i+1)!}\,\frac{y^{6j+1}}{(6j+1)!}\,,$$

$$g_3(x,y,z) \;=\; \sum_{i,j,k\geqslant 0} C_i\,C_j\,C_k\,C_{i+j+k}\,\frac{x^{6i+1}}{(6i+1)!}\,\frac{y^{6j+1}}{(6j+1)!}\,\frac{z^{6k+1}}{(6k+1)!}\,,$$

where $\{C_n\}_{n\geqslant 0}$ is a certain remarkable sequence of integers beginning

n	0	1	2	3	4	5	6	7
C_n	1	−1	−6	552	18600	9831240	−5308476480	−1843562401920

that was introduced in [37] in connection with the problem of representing primes as sums of two cubes and that has a number of different interpretations in terms of hypergeometric functions, elliptic functions, polynomial recursions, and modular forms which will be reviewed in Section 2.

The above formulas for g_1, g_2 and g_3 generalize immediately to a definition of a function g_d of d complex variables for every integer $d \geqslant 1$. The second main result of the paper, developed in Part I, is that these functions (rescaled to make them holomorphic in the complex unit d-ball) have transformation properties under the Picard modular group $U(1,d;\mathcal{O}_F)$ over the ring of integers $\mathcal{O}_F$ of the field $F=\mathbb{Q}(\sqrt{-3})$ for every value of d, even though the connection with dispersionless integrable systems is limited only to the first three values. Possible extensions of these ideas are discussed briefly in the final section of the paper.

2. A remarkable sequence of integers

In this paper we will derive concrete expressions for both the general ('master') and degenerate solutions (depending on only one or two of the three variables) of the system (1.2) in a number of different ways. All of them turn out to be linked to a single sequence of integers $(c_k)_{k\geqslant 0}$, the first few of which are given by the table

k	0	1	3	6	7	9	12	13	15	18	19
c_k	1	1	1	2	6	−8	−152	−216	1240	6848	−119232

(and $c_k = 0$ for $k \not\equiv 0,1,3 \mod 6$). These numbers, which originally appeared in [37] in connection with a classical problem of Diophantine analysis (identifying the primes that are sums of two cubes), occur in a remarkable variety of different mathematical contexts: hypergeometric functions, polynomials satisfying linear differential recursions, elliptic functions, modular forms, special values of L-functions, and non-linear differential equations. It is this wealth of different manifestations that leads to the different concrete descriptions of the solutions of our differential system, and eventually to the Picard modular transformation properties of the generating functions g_d introduced above. In this section we describe, in a more or less random order, the different manifestations of the integers c_k listed above.

Actually, the only $c_k$needed for the applications in this paper are those with $k\equiv 1 \bmod 6$, which we can renormalise slightly without destroying their integrality by setting

$$C_n \;=\; \frac{c_{6n+1}}{(-6)^n} \qquad (n\geqslant 0)\,, \tag{2.1}$$

the first few values of the C_n being those tabulated in the introduction. The growth of these numbers, which is relevant for determining the radius of convergence of some of the power series occurring below, is given by

$$C_n \;=\; \mathrm{O}\big((3n)!\,\Omega^{3n}\sqrt{n}\,\big)\,, \qquad \Omega \;=\; 2^{11/3}3^{-5/6}\pi^3\Gamma(1/3)^{-6}\,. \tag{2.2}$$

We now proceed to the various descriptions and definitions of the integers c_k. References for all of these results are the paper [37] by the fourth author and Fernando Rodriguez Villegas, the earlier paper [36] on which it is based, and, at a more expository level, Sections 5 and 6 of [39].

2.1. **Hypergeometric functions.** The numbers c_k appear in several different ways as the coefficients of hypergeometric series, re-expressed as power series in the quotient of two hypergeometric functions. In particular, we have the following generating series for the three sequences (c_{6n}), (c_{6n+1}), and (c_{6n+3}):

$$(1-x)^{1/24}\,u^{1/2}=\sum_{n=0}^{\infty}\frac{c_{6n}}{(3n)!}\Big(\frac{v}{2u}\Big)^{3n},\quad (1-x)^{1/8}\,u^{3/2}=\sum_{n=0}^{\infty}\frac{c_{6n+1}}{(3n)!}\Big(\frac{v}{2u}\Big)^{3n}\;=\;\frac{2}{x^{1/3}}\sum_{n=0}^{\infty}\frac{c_{6n+3}}{(3n+1)!}\Big(\frac{v}{2u}\Big)^{3n+1},$$

where

$$u \;=\; F\big(\tfrac13,\tfrac13;\tfrac23;x\big) \;=\; 1+\frac{x}{6}+\frac{4x^2}{45}+\cdots\,, \qquad v \;=\; x^{1/3}\,F\big(\tfrac23,\tfrac23;\tfrac43;x\big) \;=\; x^{1/3}+\frac{x^{4/3}}{3}+\frac{25\,x^{7/3}}{126}+\cdots$$

are a basis of solutions of the hypergeometric differential equation

$$x(1-x)\,y'' + \frac{2-5x}{3}\,y' - \frac{1}{9}\,y \;=\; 0\,. \tag{2.3}$$

These equations express the powers $u^{1/2}$ and $u^{3/2}$ of u, multiplied by simple algebraic functions of x, as power series in the ratio v/u of two linearly independent solutions of the hypergeometric equation (2.3). Rather surprisingly, when we expand u itself with respect to the same variable, we find expansions that involve the *squares* of the same numbers c_k:

$$u \;=\; \sum_{n=0}^{\infty}\frac{c_{3n}^2}{(3n)!}\Big(\frac{v}{u}\Big)^{3n} \;=\; \frac{1}{x^{1/3}}\sum_{n=0}^{\infty}\frac{c_{6n+1}^2}{(6n+1)!}\Big(\frac{v}{u}\Big)^{6n+1}.$$

2.2. **Polynomials satisfying linear differential recursions.** The numbers c_k do not satisfy any linear recursion relation of bounded length with polynomial coefficients. Instead, they are the constant terms of a sequence of polynomials of one variable that do satisfy such a recursion, but now involving derivatives. Define a sequence of polynomials $P_n = P_n(t) \in \mathbb{Z}[t]$ recursively by

$$P_n(t) = (1-8t^3)P'_{n-1} + (16n-7)\,t^2P_{n-1} - 4(n-1)(2n-1)\,tP_{n-2} \qquad (n\geqslant 1)$$

with the initial condition $P_0(t)=1$. (The value of P_{-1} is not needed.) Then we have

$$(-6)^nC_n \;=\; c_{6n+1} \;=\; P_{3n}(0) \tag{2.4}$$

for all $n\geqslant 0$ (the constant terms $P_n(0)$ vanish if $3\nmid n$), and there are similar sequences of recursively defined polynomials whose constant terms give c_{6n} and c_{6n+3}. Moreover, just as in the hypergeometric description of the numbers c_k, we find that their squares have exactly analogous formulas, the ones for $k=3n$ for example being $c_{3n}^2 \;=\; Q_{3n}(0)$ with $Q_n = Q_n(t) \in \mathbb{Z}[t]$ defined recursively by $Q_0(t)=1$ and $Q_{n+1}(t) = (1-t^3)\,Q'_n(t) + (2n+1)\,t^2\,Q_n(t) - n^2\,t\,Q_{n-1}(t)$ for $n\geqslant 0$.

2.3. **Elliptic functions.** Define a function

$$\sigma(z) \;=\; \sum_{m\in\mathbb{Z}}\Big(\frac{-8}{m}\Big)\,\mathbf{e}\Big(\frac{m^2\sqrt{-3}}{16}+\frac{mz}{2}+\frac{z^2}{2\sqrt{-3}}\Big) \tag{2.5}$$

where $\mathbf{e}(x)$ denotes $e^{2\pi i x}$ as usual and $\big(\frac{-8}{m}\big)$ is the Dirichlet character taking the value ±1 for $m\equiv\pm1$ mod 8 or ±3 mod 8 and zero for m even. Up to rescaling (see Remark 3.6) this is the Weierstrass sigma-function (the function whose second logarithmic derivative is the Weierstrass $\wp$-function) associated with the lattice

$$\mathcal{O} \;=\; \mathbb{Z}+\mathbb{Z}\rho\,, \quad \text{where} \quad \rho = (-1+\sqrt{-3})/2 \tag{2.6}$$

($=$ring of integers of the imaginary quadratic field $\mathbb{Q}(\sqrt{-3})$). It has the double-periodicity property

$$\sigma(z+\alpha) \;=\; e^{\frac{2\pi}{\sqrt3}(\bar\alpha z-\rho N(\alpha))}\,\sigma(z) \quad \text{for any } \alpha\in\mathcal{O} \tag{2.7}$$

where $N(\alpha)=\alpha\bar\alpha\in\mathbb{Z}$ denotes the norm of a number α in $\mathcal{O}$, and also satisfies $\sigma(\mu z)=\mu\sigma(z)$ for any unit μ of $\mathcal{O}$, i.e., for any $\mu\in\mathcal{O}^\times=\big\{\pm1,\pm\rho,\pm\rho^2\big\}$.

The Taylor expansion of $\sigma(z)$ about the origin is given in terms of the numbers (2.1) by

$$\sigma(z) \;=\; \lambda_0\sum_{n=0}^{\infty}\frac{C_n}{(6n+1)!}\,(\lambda_1 z)^{6n+1} \tag{2.8}$$

where λ_0 and λ_1 are given in terms of the gamma function by

$$\lambda_0 = i\,\frac{3^{5/24}}{2^{7/6}\pi}\,\Gamma(1/3)^{3/2} \quad \text{and} \quad \lambda_1 = \frac{6^{1/6}}{2\pi}\,\Gamma(1/3)^3\,.$$

This expansion will play an important role in this paper. It also gives an easy recursive way to calculate the numbers C_n. Indeed, the function $\sigma(z)$ is related to the Weierstrass $\wp$-function $\wp(z)$ associated to the lattice $\mathcal{O}\subset\mathbb{C}$ by $(\sigma'/\sigma)' = -\wp$, so its Taylor expansion is given in terms of the Laurent expansion $\wp(z) = z^{-2} + \sum_{n\geqslant 1} p_n z^{6n-2}$ of $\wp(z)$ by $\sigma(z) = \lambda_0\lambda_1 z\,\exp\bigl(-\sum_{n\geqslant 1}\frac{p_n z^{6n}}{6n(6n-1)}\bigr)$, and the familiar differential equation $\wp'(z)^2 = 4\wp(z)^3 + (\text{const.})$ of $\wp(z)$ gives $\wp''(z) = 6\wp(z)^2$ under differentiation and hence the recursive formula $p_{n+1} = \frac{1}{n(6n+7)}\sum_{k=1}^{n} p_k p_{n+1-k}$ $(n\geqslant 1)$ for the coefficients p_n. Even more directly, the differential equation of $\wp$ implies the (also classical) differential equation $\sigma\sigma'''' - 4\sigma'\sigma''' + 3\sigma''^2 = 0$ for σ, and this together with (2.8) implies the recursive formula

$$C_n \;=\; -\,\frac{(6n-2)!}{n-1}\sum_{k+l=n,\;k,\,l>0}\frac{(k-l)\,(18k-6l+7)\;+\;1}{(6k+1)!\,(6l-1)!}\,C_k\,C_l \qquad (n\geqslant 2)$$

for the numbers C_n.

2.4. **Special values of Hecke L-functions.** For $k\geqslant 1$ denote by $L_k(s)$ the *Hecke L-function* defined by

$$L_k(s) = \sum_{\alpha\in 1+3\mathcal{O}}\frac{\alpha^{k-1}}{N(\alpha)^s} \quad \text{if} \quad \mathrm{Re}(s) > \frac{k+1}{2} \tag{2.9}$$

where $\mathcal{O}$ is as in (2.6). This L-function has a known analytic continuation to all $s\in\mathbb{C}$ and a functional equation relating its values at the arguments s and $k-s$, and its central value is given by the formula

$$L_{2n+2}(n+1) \;=\; \frac{c_n^2}{n!\,\epsilon_n}\left(\frac{3\sqrt{3}}{2\pi}\right)^n\left(\frac{\Gamma(1/3)^3}{2\pi\sqrt{3}}\right)^{2n+1} \tag{2.10}$$

where ϵ_n is 3 for $n\equiv 1 \bmod 6$ and 6 for $n\equiv 0 \bmod 3$. (For other values of n both $L_{2n+1}(n+1)$ and c_n vanish.) It is this occurrence of the c_n (or their squares) that, in conjunction with the conjecture of Birch and Swinnerton-Dyer (BSD), leads to the Diophantine application mentioned above. A question going back to Sylvester in the 19th century is to know which primes p can be expressed as a sum of two rational cubes (the analogous question for squares had been solved by Fermat). For $p>3$ and five of the six possible values of p mod 9, the answer is known (unconditionally in some cases and always if BSD is true): p is always so expressible if it is congruent to 4, 7 or 8 modulo 9 and never if it is congruent to 2 or 5 modulo 9. For the remaining case $p=9k+1$ the answer depends on p, but only through the number c_k: p is expressible as u^3+v^3 with u and $v\in\mathbb{Q}$ if and only if $p\,|\,c_{6k}$, with the direction 'only if' being known unconditionally and the direction 'if' as a consequence of the BSD conjecture. More precisely, $p\,|\,c_{6k}$ is the necessary and sufficient condition for the vanishing of the central value of the L-function of the elliptic curve $u^3+v^3=p$, and this according to the BSD conjecture should be equivalent to the existence of a rational solution of the equation defining this curve.

2.5. **Modular functions and modular forms.** We recall that a modular form of weight k on a discrete subgroup Γ of $\mathrm{SL}(2,\mathbb{R})$ is a holomorphic function $f(\tau)$ in the complex upper half-plane $\mathfrak{H}$ satisfying $f\bigl(\frac{a\tau+b}{c\tau+d}\bigr) = v(\gamma)(c\tau+d)^k f(\tau)$ for all $\gamma = \bigl(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\bigr)\in\Gamma$ and $\tau\in\mathfrak{H}$, where $v(\gamma)$ (the "multiplier") is a suitable root of unity. If Γ is commensurable with the group $\mathrm{SL}(2,\mathbb{Z})$, which will always be the case for us, then such a function has a Fourier expansion as a sum of (in general fractional) powers of $q := e^{2\pi i\tau}$, a simple example being the Dedekind eta function

$$\eta(\tau) \;=\; q^{1/24}\prod_{n=1}^{\infty}\bigl(1-q^n\bigr) \;=\; \sum_{n\in\mathbb{Z}}(-1)^n\,q^{(6n+1)^2/24} \tag{2.11}$$

(the equality of these two formulas being a famous identity of Euler), which satisfies $\eta(\tau+1) = e^{\pi i/12}\eta(\tau)$ and $\eta(-1/\tau) = \sqrt{\tau/i}\,\eta(\tau)$ and hence is a modular form of weight $1/2$ on $\mathrm{SL}(2,\mathbb{Z})$. Two important classes of modular forms are *unary theta series*, like the second formula for $\eta(\tau)$ just given or the expansion

$$\eta(\tau)^3 \;=\; \sum_{n\in\mathbb{Z}} \Bigl(\frac{-4}{n}\Bigr)\, n\, q^{n^2/8} \tag{2.12}$$

due to Jacobi (here $\bigl(\frac{-4}{n}\bigr)$ equals ±1 if $n\equiv\pm1 \pmod 4$ and 0 if n is even), and *binary theta series* like

$$\Theta(\tau) \;=\; \sum_{a,\,b\,\in\,\mathbb{Z}} q^{a^2+ab+b^2} \;=\; \sum_{\alpha\in\mathcal{O}} q^{N(\alpha)}, \tag{2.13}$$

which satisfies $\Theta(\tau+1)=\Theta(\tau)$ and $\Theta(-1/3\tau) = (\tau/\sqrt{-3})\,\Theta(\tau)$ and hence is a modular form of weight 1 on the group $\Gamma_1^*(3)$ generated by $\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$ and $\frac{1}{\sqrt3}\left(\begin{smallmatrix}0&-1\\3&0\end{smallmatrix}\right)$, or more generally

$$\Theta_k(\tau) \;=\; \sum_{\alpha\in\mathcal{O}} \alpha^{k-1}\, q^{N(\alpha)} \qquad (k\geqslant 1,\;\; k\equiv 1 \pmod 6\,), \tag{2.14}$$

which is a modular form cusp form of weight k on the same group.

A property that is of key importance for us is that modular forms not only have interesting Fourier expansions as power series in $q=e^{2\pi i\tau}$, but also interesting Taylor developments around "CM points" (points $\alpha\in\mathfrak{H}$ that satisfy a quadratic equation over $\mathbb{Q}$), after identifying the upper half-plane with the unit disk via $w=\frac{\tau-\alpha}{\tau-\bar\alpha}$, $\tau=\frac{\alpha-\bar\alpha w}{1-w}$ and then rescaling w to get an expansion with algebraic coefficients. Explicit examples for the modular forms introduced above are

$$(1-w)^{-1/2}\,\eta\Bigl(\frac{\rho-\bar\rho w}{1-w}\Bigr) \;=\; \frac{e^{-\pi i/24}\,\Gamma(\frac13)}{2^{2/3}\pi^{5/8}\Gamma(\frac56)} \sum_{n=0}^{\infty} \frac{c_{6n}}{(3n)!}\Bigl(-\frac{\sqrt3\,\Gamma(\frac13)^6}{2^4\pi^3}\,w\Bigr)^{3n}, \tag{2.15}$$

$$(1-w)^{-3/2}\,\eta\Bigl(\frac{\rho-\bar\rho w}{1-w}\Bigr)^3 \;=\; \Bigl(\frac{e^{-\pi i/24}\,\Gamma(\frac13)}{2^{2/3}\pi^{5/8}\Gamma(\frac56)}\Bigr)^3 \sum_{n=0}^{\infty} \frac{c_{6n+1}}{(3n)!}\Bigl(\frac{\sqrt3\,\Gamma(\frac13)^6}{2^4\pi^3}\,w\Bigr)^{3n}, \tag{2.16}$$

$$(1-w)^{-1}\Theta\Bigl(\frac{\tau_0-\overline{\tau}_0 w}{1-w}\Bigr) \;=\; \frac{\mu}{\sqrt2}\sum_{j\geqslant0} \frac{C_j^2}{(6j+1)!}(\Omega w)^{6j+1} \tag{2.17}$$

and more generally

$$(1-w)^{-k}\Theta_k\Bigl(\frac{\tau_0-\overline{\tau}_0 w}{1-w}\Bigr) \;=\; \frac{(-1)^n\mu^k}{\sqrt2}\sum_{j\geqslant0} \frac{C_jC_{j+n}}{(6j+1)!}(\Omega w)^{6j+1} \tag{2.18}$$

for $k=6n+1$ with $n\geqslant0$ and $|w|<1$, where

$$\tau_0 \;=\; \frac12+\frac{i}{2\sqrt3} \;=\; \frac{\rho}{\sqrt{-3}}\in\mathfrak{H}, \quad \mu \;=\; \frac{2^{1/6}\,3^{2/3}\,\Gamma(1/3)^3}{(2\pi)^2}, \quad \Omega \;=\; \frac{2^{1/3}\,3^{5/6}\,\Gamma(1/3)^6}{(2\pi)^3}\,.$$

It is above all this appearance of the numbers c_k in the Taylor expansions of modular forms that is responsible for all of the other identities in this section and all of the applications in this paper.

2.6. **Nonlinear differential equations.** Finally, from several of the above descriptions of the integers c_k, we can deduce nonlinear differential equations for various generating functions involving them, two examples being the equation

$$(F^2F''' - 15\,FF'F'' + 30\,F'^3)^2 + 32\,(FF'' - 3\,F'^2)^3 \;=\; 4\,F^{30}$$

for the generating function

$$F=\sum_{n=0}^{\infty} c_{6n}\frac{x^{3n}}{(3n)!} = 1+\frac{x^3}{3}-\frac{19x^6}{90}+\frac{107x^9}{5670}+\dots$$

and the equation (1.6) for the power series $g(x) = \sum_{n=0}^{\infty} C_n^2 \frac{x^{6n+1}}{(6n+1)!}$, which as explained in the introduction is the special case $d = 1$ of the differential systems studied in this paper.

3. Integrable Lagrangian densities via modular forms

Here we summarise various representations of integrable Lagrangian densities via special functions introduced in the previous section.

3.1. **Lagrangian densities of the form $f = v_{x_1}v_{x_2}g(v_{x_3})$.** The corresponding Euler-Lagrange equation takes the form

$$(v_{x_2}g(v_{x_3}))_{x_1} + (v_{x_1}g(v_{x_3}))_{x_2} + \left(v_{x_1}v_{x_2}g'(v_{x_3})\right)_{x_3} = 0, \tag{3.1}$$

and the integrability conditions (1.2) simplify to the single fourth-order ODE (1.6) for $g(z)$, recall that we set $z = v_{x_3}$. Equation (1.6) has a remarkable $\mathrm{GL}(2,\mathbb{C})$-invariance,

$$\tilde{z} = \frac{az+b}{cz+d}, \quad \tilde{g} = (cz+d)g, \quad \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}(2,\mathbb{C}) \tag{3.2}$$

which underlies its integration procedure. It was shown in [13] that the generic solution of equation (1.6) is a modular form of weight 1 and level 3 which can be parametrised in terms of two linearly independent solutions of the auxiliary hypergeometric equation

$$u(1-u)h_{uu} + (1-2u)h_u - \frac{2}{9}h = 0. \tag{3.3}$$

The geometry behind hypergeometric equation (3.3) is the 1-parameter family of genus 2 trigonal curves

$$r^3 = t(t-1)(t-u)^2, \tag{3.4}$$

supplied with the holomorphic differential $\omega = dt/r$ (these curves, also known as Borwein's curves, have appeared in the context of generalised arithmetic-geometric mean iteration [4]). The corresponding periods, $h = \int_a^b \omega$ where $a, b \in \{0, 1, \infty, u\}$, form a 2-dimensional vector space and satisfy the Picard-Fuchs equation (3.3). The following theorem provides three equivalent parametrisations of a generic solution.

Theorem 3.1. *A generic solution of (1.6) can be represented in any of the three equivalent forms:*

(1) *Theta representation:*

$$g(z) = \sum_{\alpha\in\mathcal{O}} q^{N(\alpha)} = \sum_{a,b\in\mathbb{Z}} q^{a^2-ab+b^2} = 1 + 6q + 6q^3 + 6q^4 + 12q^4 + \dots, \qquad q = e^{2\pi i z}, z \in \mathfrak{H}. \tag{3.5}$$

(2) *Power series:*

$$g(z) = \sum_{k\geqslant 0} C_k^2 \frac{z^{6k+1}}{(6k+1)!} \tag{3.6}$$

where the integers C_k are defined by (2.1) and $z \in \mathbb{C}^2$ such that $|z|^2 < |\chi_1|^2$ with $\chi_1 = \frac{\Gamma(1/3)^6\, 2^{1/3}\, 3^{5/6}}{(2\pi)^3}$.

(3) *Parametric form:*

$$z = \frac{h_1(u)}{h_2(u)}, \quad g = h_2(u) \tag{3.7}$$

where h_1 and h_2 are 2 linearly independent solutions of the hypergeometric equation (3.3). The $\mathrm{GL}(2,\mathbb{C})$-invariance (3.2) corresponds to the freedom in the choice of the basis $\{h_1, h_2\}$.

Remark 3.2. To be precise, generic solutions of (1.6) coming from representations (3.5), (3.6) and (3.7) are not identical and should not all be denoted by g. But as they are all related by transformations from the $\mathrm{GL}(2,\mathbb{C})$-invariance (3.2), we abused notations and denoted them all by g.

3.2. **Lagrangian densities of the form $f = v_{x_1} g(v_{x_2}, v_{x_3})$.** The corresponding Euler-Lagrange equation takes the form

$$(g)_{x_1} + \left(v_{x_1} g_{v_{x_2}}\right)_{x_2} + \left(v_{x_1} g_{v_{x_3}}\right)_{x_3} = 0. \tag{3.8}$$

The integrability conditions (1.2) lead to a system of five equations expressing all fourth-order partial derivatives of g in terms of its lower-order derivatives. In compact form, this system is given by

$$d^4 g = d^3 g \frac{dh}{h} + 6\frac{dg}{h}\det(dm) + 3\frac{(dg)^2}{h}\det(dn), \tag{3.9}$$

with $m = \begin{pmatrix} 0 & g_y & g_z \\ g_y & g_{yy} & g_{yz} \\ g_z & g_{yz} & g_{zz} \end{pmatrix}$, $n = \begin{pmatrix} g_{yy} & g_{yz} \\ g_{yz} & g_{zz} \end{pmatrix}$ and $h = -\det(m) = g_z^2 g_{yy} - 2g_y g_z g_{yz} + g_y^2 g_{zz}$, see [11]. The non-degeneracy condition is equivalent to $h \neq 0$. System (3.9) is in involution and its solution space is 10-dimensional. Furthermore, it is invariant under the 10-dimensional group

$$\tilde{y} = \frac{l_1(y,z)}{l_0(y,z)}, \quad \tilde{z} = \frac{l_2(y,z)}{l_0(y,z)}, \quad \tilde{g} = \alpha g + \beta, \tag{3.10}$$

where l_0, l_1 and l_2 are arbitrary affine transformations. This symmetry underlies the linearisation procedure of (3.9). We show that the function g can be parametrised in terms of three linearly independent solutions of the auxiliary hypergeometric system

$$\begin{aligned}
h_{u_1u_2} &= \frac{1}{3}\frac{h_{u_1} - h_{u_2}}{u_1 - u_2} \\
h_{u_1u_1} &= -\frac{h}{9u_1(u_1-1)} + \frac{h_{u_2}}{3(u_1-u_2)}\frac{u_2(u_2-1)}{u_1(u_1-1)} - \frac{h_{u_1}}{3}\left(\frac{1}{u_1-u_2} + \frac{2}{u_1} + \frac{2}{u_1-1}\right) \\
h_{u_2u_2} &= -\frac{h}{9u_2(u_2-1)} + \frac{h_{u_1}}{3(u_2-u_1)}\frac{u_1(u_1-1)}{u_2(u_2-1)} - \frac{h_{u_2}}{3}\left(\frac{1}{u_2-u_1} + \frac{2}{u_2} + \frac{2}{u_2-1}\right)
\end{aligned} \tag{3.11}$$

which is a particular case of hypergeometric system of Appell's type [2]. The geometry behind system (3.11) is the 2-parameter family of genus 3 Picard trigonal curves

$$r^3 = t(t-1)(t-u_1)(t-u_2)$$

supplied with the holomorphic differential $\omega = dt/r$. It was shown by Picard [31] that the corresponding periods, $h = \int_a^b \omega$ where $a, b \in \{0, 1, \infty, u_1, u_2\}$, form a 3-dimensional vector space and provide a basis of solutions of the Picard-Fuchs system (3.11). Note that periods of the differential ω can also be interpreted as periods of a holomorphic 2-form on the associated 2-parameter family of elliptic K3-surfaces [33], see also [34] for further generalisations of this construction. The following theorem provides three equivalent parametrisations of a generic solution of (3.9).

Theorem 3.3. *A generic solution of (3.9) can be represented in any of the three equivalent forms:*

(1) *Theta representation:*

$$g(y,z) = \frac{y}{b} + \sum_{\alpha \in \mathcal{O}^*} \frac{\sigma(\alpha y)}{\alpha} e^{2\pi i N(\alpha) z}, \tag{3.12}$$

where $b = -\frac{(2\pi)^2}{3^{3/8}\Gamma(1/3)^{9/2}} i$ *and* $(y,z) \in \mathbb{C}^2$ *such that* $|y|^2 < 2\sqrt{3}\,\mathrm{Im}(z)$.

(2) *Power series:*

$$g(y,z) = \sum_{i,j \geq 0} C_i C_j C_{i+j} \frac{y^{6i+1}}{(6i+1)!}\frac{z^{6i+1}}{(6j+1)!} \tag{3.13}$$

where $(y,z) \in \mathbb{C}^2$ *such that* $|y|^2 + |z|^2 < |\chi_1|^2$ *with* $\chi_1 = \frac{\Gamma(1/3)^6\, 2^{1/3}\, 3^{5/6}}{(2\pi)^3}$.

(3) *Parametric form:*

$$y=\frac{h_1(u_1,u_2)}{h_3(u_1,u_2)},\quad z=\frac{h_2(u_1,u_2)}{h_3(u_1,u_2)},\quad g=G(w),\quad w=\frac{u_1(u_2-1)}{u_2(u_1-1)} \tag{3.14}$$

where h_1, h_2, h_3 are three linearly independent solutions of the hypergeometric system (3.11) and $G'=[w(w-1)]^{-2/3}$. The $\mathrm{GL}(3,\mathbb{C})$-invariance part in (3.10) corresponds to the freedom in the choice of the basis $\{h_1,h_2,h_3\}$.

Remark 3.4. Here also generic solutions of (3.9) coming from representations (3.12), (3.13) and (3.14) are not identical and should not all be denoted by g. But as they are all related by transformations from the invariance (3.10), we abused notations and denoted them all by g. Lagrangian densities $f=v_{x_1}g(v_{x_2},v_{x_3})$ are related to Hamiltonian systems studied previously in [14], see Section 8 for more details. The constant b appearing in (3.12) will also appear in the case of generic Lagrangian densities and will be explained in Remark 3.6.

3.3. **Generic Lagrangian densities $f(v_{x_1},v_{x_2},v_{x_3})$.** The corresponding Euler-Lagrange equation takes the form (1.1). The integrability conditions (1.2) form a system of fifteen equations expressing all fourth-order partial derivatives of the Lagrangian density f in terms of its second-order and third-order partial derivatives. This system is in involution and its solution space is 20-dimensional. Furthermore, (1.2) is invariant under the 20-dimensional transformation group (1.4), which allows for its explicit integration. We will prove that the Lagrangian density f can be parametrised by four linearly independent solutions of the auxiliary hypergeometric system

$$\begin{aligned} h_{u_iu_j}&=\frac{1}{3}\frac{h_{u_i}-h_{u_j}}{u_i-u_j}\\ h_{u_iu_i}&=-\frac{1}{3u_i(u_i-1)}\Big(\frac{2\,h}{3}+\sum_{j\neq i}^{3}\frac{u_j(u_j-1)}{u_j-u_i}h_{u_j}\Big)-\frac{1}{3}\sum_{j\neq i}^{3}\Big(\frac{1}{u_i-u_j}+\frac{2}{u_i}+\frac{2}{u_i-1}\Big)h_{u_i}\end{aligned} \tag{3.15}$$

where $i,j\in\{1,2,3\}$ are pairwise distinct. The geometry behind the system of equations (3.15) is the 3-parameter family of genus 4 trigonal curves

$$r^3=t(t-1)(t-u_1)(t-u_2)(t-u_3),$$

supplied with the holomorphic differential $\omega=dt/r$. The corresponding periods, $h=\int_a^b\omega$ where $a,b\in\{0,1,\infty,u_1,u_2,u_3\}$, form a 4-dimensional vector space and provide a basis of solutions of the Picard-Fuchs system (3.15). To state one of our results we will also need the inhomogeneous extension of the hypergeometric system (3.15):

$$\begin{aligned} F_{u_iu_j}&=\frac{1}{3}\frac{F_{u_i}-F_{u_j}}{u_i-u_j}+\epsilon_{ijk}\frac{u_k(u_k-1)(u_i-u_j)}{U^{2/3}}\\ F_{u_iu_i}&=-\frac{1}{3u_i(u_i-1)}\Big(\frac{2\,F}{3}+\sum_{j\neq i}^{3}\frac{u_j(u_j-1)}{u_j-u_i}F_{u_j}\Big)-\frac{1}{3}\sum_{j\neq i}^{3}\Big(\frac{1}{u_i-u_j}+\frac{2}{u_i}+\frac{2}{u_i-1}\Big)F_{u_i}\end{aligned} \tag{3.16}$$

where $i,j,k\in\{1,2,3\}$ are pairwise distinct, ϵ_{ijk} is the totally antisymmetric tensor and

$$U=u_1u_2u_3(u_1-1)(u_2-1)(u_3-1)(u_1-u_2)(u_2-u_3)(u_3-u_1).$$

The following theorem provides three equivalent parametrisations of a generic solution of (1.2).

Theorem 3.5. *A generic solution of (1.2) can be represented in any of the three equivalent forms:*

(1) *Theta representation:*

$$f(x,y,z)=\frac{xy}{b^2}+\sum_{\alpha\in\mathcal{O}^*}\frac{\sigma(\alpha x)\sigma(\alpha y)}{\alpha^2}e^{2\pi iN(\alpha)z} \tag{3.17}$$

where $b = -\frac{(2\pi)^2}{3^{3/8}\Gamma(1/3)^{9/2}} i$ *and* $(x,y,z) \in \mathbb{C}^3$ *such that* $|x|^2 + |y|^2 < 2\sqrt{3}\,\mathrm{Im}(z)$.

(2) *Power series:*

$$f(x,y,z) = \sum_{i,j,k\geq 0} C_i C_j C_k C_{i+j+k} \frac{x^{6i+1}}{(6i+1)!} \frac{y^{6j+1}}{(6j+1)!} \frac{z^{6k+1}}{(6k+1)!} \tag{3.18}$$

where $(x,y,z) \in \mathbb{C}^3$ *such that* $|x|^2 + |y|^2 + |z|^2 < |\chi_1|^2$ *with* $\chi_1 = \frac{\Gamma(1/3)^6\, 2^{1/3}\, 3^{5/6}}{(2\pi)^3}$.

(3) *Parametric form:*

$$x = \frac{h_1(u_1,u_2,u_3)}{h_4(u_1,u_2,u_3)}, \quad y = \frac{h_2(u_1,u_2,u_3)}{h_4(u_1,u_2,u_3)}, \quad z = \frac{h_3(u_1,u_2,u_3)}{h_4(u_1,u_2,u_3)}, \quad f = \frac{F(u_1,u_2,u_3)}{h_4(u_1,u_2,u_3)} \tag{3.19}$$

where h_1, h_2, h_3, h_4 *are four linearly independent solutions of the hypergeometric system (3.15) and* F *is a solution of the inhomogeneous system (3.16). The* $\mathrm{GL}(4,\mathbb{C})$*-invariance (first equations of (1.4)) corresponds to the freedom in the choice of the basis* $\{h_1, h_2, h_3, h_4\}$*, while the further action of* $\mathbb{C}^4$ *(last equation of (1.4)) corresponds to the freedom in the choice of* F*, which is defined up to a linear combination of solutions of the homogeneous system (3.15).*

Remark 3.6. Based on computer experiments, the following expression introduced in Section 3.3 of [13],

$$g(x,y,z) = xy + \sum_{\substack{(k,l)\in\mathbb{Z}^2 \\ (k,l)\neq(0,0)}} \frac{\tilde{\theta}((k-\varepsilon l)x)\tilde{\theta}((k-\varepsilon l)y)}{(k-\varepsilon l)^2} e^{2\pi i(k^2-kl+l^2)z/3} \tag{3.20}$$

where $\varepsilon = \rho + 1$ and $\tilde{\theta}(z) = \frac{1}{\theta'(0)} e^{\pi(z^2/\sqrt{3}-iz)}\theta(z)$ with $\theta(z) = \sum_{k\in\mathbb{Z}}(-1)^k e^{2\pi i k(z-\rho^2(k-1)/2)}$ has been conjectured to solve the integrability conditions (1.2). By using the bijection $\lambda \mapsto \rho\lambda$ of $\mathcal{O}$, we easily see that

$$g(x,y,z) = xy + \sum_{\alpha\in\mathcal{O}^*} \frac{\tilde{\theta}(\alpha x)\tilde{\theta}(\alpha y)}{\alpha^2} e^{2\pi i N(\alpha)z/3}. \tag{3.21}$$

The authors of [13] did not realise that the function $\tilde{\theta}$ is the Weierstrass sigma-function associated with the lattice $\mathcal{O}$:

$$\tilde{\theta}(z) = \sigma_{\mathcal{O}}(z) \;=\; z \prod_{\lambda\in\mathcal{O}^*} (1 - \frac{z}{\lambda}) e^{\frac{z}{\lambda} + \frac{z^2}{2\lambda^2}}$$

It is also related to the function σ defined by (2.5):

$$\tilde{\theta} = b\,\sigma \quad \text{with} \quad b = -\frac{(2\pi)^2}{3^{3/8}\Gamma(1/3)^{9/2}} i \tag{3.22}$$

In this paper we prove that the function f given by (3.17) indeed satisfies the integrability conditions (1.2) and also explain how the expression (3.17) is equivalent to (3.21) eventually proving the main conjecture of [13].

4. Picard modular forms

In Section 5, we are going to prove the 'Theta representation' (3.17) given in Theorem 3.5 by using some properties of Picard modular forms i.e. modular forms on a unitary group over an imaginary quadratic field originally discovered by Picard hence their name (see [30] and [31]). The unitary groups that we consider are of signature $(2,1)$ or $(3,1)$ and the imaginary quadratic field will always be $\mathbb{Q}(\sqrt{-3})$. In the case of signature $(2,1)$, scalar-valued Picard modular forms have been explicitly studied by Feustel, Finis, Holzapfel and Shiga, see [15, 17, 22, 35] while the case of vector-valued ones has been studied in [7] and by Shintani in an unpublished work [38]. The case of signature $(3,1)$ has attracted less interest but useful works have been made by Freitag, Salvati Manni and Matsumoto, see [18, 24]. This section is organised as follows. We start with preliminaries about Picard modular forms then explain modular properties of integrabilty conditions viewed as differential operators.

4.1. **Picard modular forms.**

4.1.1. ***Backgrounds.*** Recall that we have denoted by $\mathcal{O} = \mathbb{Z}[\rho]$ the ring of integers of the imaginary quadratic field $K = \mathbb{Q}(\sqrt{-3})$. A convenient choice for a Hermitian form for our purposes is the following one:

$$h_n(v,w) \;=\; \bar{v}_1 w_2 + \bar{v}_2 w_1 + \sum_{i=1}^{n-1} \bar{v}_{i+2} w_{i+2} \quad (v, w \in \mathbb{C}^{n+1})$$

whose associated Hermitian matrix is given by $H_n = \left(\begin{smallmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1_{n-1} \end{smallmatrix}\right)$, where 1_n denotes the identity matrix of size n. Note that the Hermitian form h_n is $\mathcal{O}$-valued on the lattice $\mathcal{O}^{n+1}$. Let G_n be the algebraic group, defined over $\mathbb{Q}$, of unitary similitudes associated to h_n:

$$G_n \;=\; \{g \in \mathrm{GL}(n+1, K) \mid \exists\, \nu(g) \in \mathbb{Q}^* \,:\, h_n(gz, gz) = \nu(g)\, h_n(z,z)\}$$

and $G_n^+ = \{g \in G_n \mid \nu(g) > 0\}$ be the connected component of the identity in G_n. Recall that the unitary group and the special unitary group of signature $(n,1)$ over the field K are defined by

$$\begin{aligned} \mathrm{U}(n,1;K) &= \left\{g \in \mathrm{GL}(n+1,K) \;:\; h_n(gz,gz) = h_n(z,z) \text{ for any } z \in \mathbb{C}^{n+1}\right\}, \\ \mathrm{SU}(n,1;K) &= \{g \in \mathrm{U}(n,1;K) : \det g = 1\}\,. \end{aligned}$$

By choosing an embedding of K into $\mathbb{C}$, we identify $K \otimes_{\mathbb{Q}} \mathbb{R}$ with $\mathbb{C}$ and we consider the set V_n of complex lines on which h_n is negative definite:

$$V_n \;=\; \left\{\ell \subset \mathbb{C}^{n+1} : \dim \ell = 1, h_n|_\ell < 0\right\} \subset \mathrm{Gr}(1, \mathbb{C}^{n+1}) = \mathbb{P}^n\mathbb{C}$$

where $\mathrm{Gr}(1,\mathbb{C}^{n+1})$ denotes the Grassmannian of lines in $\mathbb{C}^{n+1}$. The group G_n^+ naturally acts on $\mathrm{Gr}(1,\mathbb{C}^{n+1})$ and so on V_n. Note that the set V_n can be identified with the complex n-ball. We are only interested in the cases $n=3$ and $n=2$ for which we describe explicitly the previous construction. For $n=3$, the set V_3 is given by

$$V_3 \;=\; \left\{z \in \mathbb{C}^4 \,:\, \bar{z}^t H_3 z < 0\right\} \;=\; \left\{z \in \mathbb{C}^4 : 2\,\mathrm{Re}(z_1\bar{z}_2) + |z_3|^2 + |z_4|^2 < 0\right\}$$

and by embedding $\mathbb{C}^3$ in $\mathbb{P}^3\mathbb{C}$ via the map $(u,v,w) \mapsto [v : 1 : u : w]$, note that second coordinate of a point in V_3 is never zero, we get the following Siegel domain:

$$B_3 \;=\; \left\{b = (u,v,w) \in \mathbb{C}^3 : 2\,\mathrm{Re}(v) + |u|^2 + |w|^2 < 0\right\}.$$

It is acted on by the group G_3^+ via:

$$\begin{array}{rcl} G_3^+ \times B_3 & \to & B_3 \\ (g,(u,v,w)) & \mapsto & g \cdot b = \left(\frac{g_{31}v + g_{32} + g_{33}u + g_{34}w}{g_{21}v + g_{22} + g_{23}u + g_{24}w}, \frac{g_{11}v + g_{12} + g_{13}u + g_{14}w}{g_{21}v + g_{22} + g_{23}u + g_{24}w}, \frac{g_{41}v + g_{42} + g_{43}u + g_{44}w}{g_{21}v + g_{22} + g_{23}u + g_{24}w}\right). \end{array} \tag{4.1}$$

This action defines two automorphy factors given by

$$j_{1,3}(g,b) = g_{21}v + g_{22} + g_{23}u + g_{24}w \quad \text{and} \quad j_{2,3}(g,b)^{-1} = j_{1,3}(g,b) J(g,b)^t$$

where $J(g,b)$ denotes the Jacobian of the previous action viewed as a biholomorphism of B_3. Note that we have $\det(J(g,b)) = \det(g)\, j_{1,3}(g,b)^{-4}$.
Similarly for $n=2$, we get

$$V_2 \;=\; \left\{z \in \mathbb{C}^3 \,:\, \bar{z}^t H_1 z < 0\right\} \;=\; \left\{z \in \mathbb{C}^3 : 2\,\mathrm{Re}(z_1\bar{z}_2) + |z_3|^2 < 0\right\}$$

and by embedding $\mathbb{C}^2$ in $\mathbb{P}^2\mathbb{C}$ via the map $(u,v) \mapsto [v : 1 : u]$, note that second coordinate of a point in V_2 cannot be zero, we get the following Siegel domain:

$$B_2 \;=\; \left\{b = (u,v) \in \mathbb{C}^2 : 2\,\mathrm{Re}(v) + |u|^2 < 0\right\}.$$

It is acted on by the group G_2^+ via:

$$\begin{array}{rcl} G_2^+ \times B_2 & \to & B_2 \\ (g,(u,v)) & \mapsto & g\cdot b = \left(\frac{g_{31}v+g_{32}+g_{33}u}{g_{21}v+g_{22}+g_{23}u}, \frac{g_{11}v+g_{12}+g_{13}u}{g_{21}v+g_{22}+g_{23}u}\right). \end{array} \tag{4.2}$$

Again this action defines two automorphy factors given by

$$j_{1,2}(g,b) = g_{21}v + g_{22} + g_{23}u \quad \text{and} \quad j_{2,2}(g,b)^{-1} = j_{1,2}(g,b)J(g,b)^t.$$

For $n = 2$ or 3, a pair of integers (j,k) and $g \in G_n^+ \cap \ker \nu$, we define a slash operator on functions $f : B_n \to \mathrm{Sym}^j(\mathbb{C}^n)$ by

$$(f|_{j,k}g)(b) \;=\; j_{1,n}(g,b)^{-k}\mathrm{Sym}^j(j_{2,n}(g,b)^{-1})f(g\cdot b).$$

For a discrete subgroup Γ of $G_n^+ \cap \ker\nu$ and χ a character of finite order of Γ, we define the space of modular forms of weight (j,k) on Γ with character χ as

$$M_{j,k}(\Gamma,\chi) \;=\; \left\{f : B_n \to \mathrm{Sym}^j(\mathbb{C}^n) \;:\; f \text{ holomorphic}, \; f|_{j,k}g = \chi(g)f \text{ for any } g \in \Gamma\right\}$$

and denote by $S_{j,k}(\Gamma,\chi)$ its subspace of cusp forms. For $j = 0$, case of scalar-valued modular forms, we shorten these notations:

$$M_{0,k}(\Gamma,\chi) = M_k(\Gamma,\chi) \quad \text{and} \quad S_{0,k}(\Gamma,\chi) = S_k(\Gamma,\chi).$$

The discrete subgroups of $G_n^+ \cap \ker\nu$ interesting us are:

$$\begin{aligned} \Gamma_n &= \mathrm{U}(H_n;\mathcal{O}) = \left\{g \in \mathrm{GL}(n+1,\mathcal{O}) \;:\; \bar{g}^t H_n g = H_n\right\}, \\ \mathrm{S}\Gamma_n &= \mathrm{SU}(H_n;\mathcal{O}) = \left\{g \in \mathrm{SL}(n+1,\mathcal{O}) \;:\; \bar{g}^t H_n g = H_n\right\}, \\ \Gamma_n[\sqrt{-3}] &= \left\{g \in \Gamma_n \;:\; g \equiv 1_{n+1} \bmod \sqrt{-3}\right\}, \\ \mathrm{S}\Gamma_n[\sqrt{-3}] &= \left\{g \in \mathrm{S}\Gamma_n \;:\; g \equiv 1_{n+1} \bmod \sqrt{-3}\right\}. \end{aligned}$$

Remark 4.1. It is known that the group $\Gamma_3[\sqrt{-3}]$ is the monodromy group of the hypergeometric differential system (3.15) (see Theorem 6.3.2 of [23]) while $\Gamma_2[\sqrt{-3}]$ is the monodromy group of the hypergeometric differential system (3.11) (see [22]).

Since the slash operator introduced above defines a group action, for understanding some modular behaviour of a function for a group, it is sufficient to understand it for a system of generators of this group. In order to describe systems of generators of the groups Γ_3 and $\Gamma_3[\sqrt{-3}]$, we introduce some special elements of Γ_3:

$$D_{\varepsilon_1,\varepsilon_2,\varepsilon_3} = \begin{pmatrix} \varepsilon_1 & 0 & 0 & 0 \\ 0 & \varepsilon_1 & 0 & 0 \\ 0 & 0 & \varepsilon_2 & 0 \\ 0 & 0 & 0 & \varepsilon_3 \end{pmatrix}, \quad T_{\alpha_1,\alpha_2,\alpha_3} = \begin{pmatrix} 1 & \alpha_1 & -\bar{\alpha}_2 & -\bar{\alpha}_3 \\ 0 & 1 & 0 & 0 \\ 0 & \alpha_2 & 1 & 0 \\ 0 & \alpha_3 & 0 & 1 \end{pmatrix}, \quad \widetilde{m} = \begin{pmatrix} a & b\sqrt{-3} & 0 \\ -c\sqrt{-3} & d & 0 \\ 0 & 0 & 1_2 \end{pmatrix}, \quad \iota = \begin{pmatrix} 1_2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \tag{4.3}$$

where $\varepsilon_1, \varepsilon_2, \varepsilon_3 \in \mathcal{O}^\times$, $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{O}$ such that $\mathrm{Tr}(\alpha_1) = -|\alpha_2|^2 - |\alpha_3|^2$ where $\mathrm{Tr}(\alpha) = \alpha + \bar{\alpha} \in \mathbb{Z}$ denotes the trace of a number α in $\mathcal{O}$ and $m = \left(\begin{smallmatrix} a & b \\ 3c & d \end{smallmatrix}\right) \in \Gamma_0(3)$. By using the results of [1] (Theorem 6.6), we get:

Theorem 4.2. *The group Γ_3 is generated by $r_0 = -1_4$ and the following five elements*

$$r_1 = D_{1,-\rho^2,1}, \quad r_2 = H_3 D_{1,-\rho^2,1}\, T_{\rho,-\rho^2,0}\, H_3, \quad r_3 = \iota r_2 \iota, \quad r_4 = D_{1,1,-\rho^2}, \quad r_5 = D_{\rho,1,1}\,\widetilde{\left(\begin{smallmatrix} 1 & 0 \\ 3 & 1 \end{smallmatrix}\right)} H_3.$$

These elements are explicitly given by

$$r_1 = \begin{pmatrix} 1_2 & 0 & 0 \\ 0 & 1+\rho & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad r_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \rho & 1 & \rho & 0 \\ \rho & 0 & 1+\rho & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad r_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \rho & 1 & 0 & \rho \\ 0 & 0 & 1 & 0 \\ \rho & 0 & 0 & 1+\rho \end{pmatrix}, \quad r_4 = \begin{pmatrix} 1_2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1+\rho \end{pmatrix}, \quad r_5 = \begin{pmatrix} 0 & \rho & 0 \\ \rho & 2+\rho & 0 \\ 0 & 0 & 1_2 \end{pmatrix}.$$

They are reflections of order 6 along the following vectors:

$$w_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad w_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \quad w_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \quad w_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad w_5 = \begin{pmatrix} 1 \\ -\rho \\ 0 \\ 0 \end{pmatrix},$$

and viewed as an automorphism of $\mathbb{C}^4$, we have $r_i(v) = v - (1+\rho^2)\frac{h_3(w_i,v)}{h_3(w_i,w_i)}w_i = v - (1+\rho^2)h_3(w_i,v)w_i$ since any w_i has norm 1 with respect to h_3. Note that we use vectors of norm 1; in [1], they use vectors of norm -1 since they deal with signature $(1,3)$. We will also need a system of generators of $\Gamma_3[\sqrt{-3}]$ that can again be deduced from the results of [1]:

Theorem 4.3. *The group $\Gamma_3[\sqrt{-3}]$ is generated by the following fifteen elements*

$$\begin{array}{lll}
g_1 = D_{1,1,\rho}, & g_6 = \iota g_5 \iota, & g_{11} = D_{\rho^2,1,1}\widetilde{\left(\begin{smallmatrix} -2 & -1 \\ 3 & 1 \end{smallmatrix}\right)}, \\
g_2 = D_{1,\rho,1}, & g_7 = H_3 g_3 H_3, & g_{12} = D_{-\rho^2,1,1}\, T_{-1,-\rho^2,-\rho^2}\widetilde{\left(\begin{smallmatrix} -1 & 1 \\ -3 & 2 \end{smallmatrix}\right)}\, T_{-1,-1,-1}, \\
g_3 = D_{1,\rho^2,\rho^2} H_3\, T_{\rho-1,\sqrt{-3},0}\, H_3 D_{1,\rho^2,\rho}, & g_8 = \iota g_7 \iota, & g_{13} = D_{1,-1,1} g_{12} D_{1,-1,1}, \\
g_4 = \iota g_3 \iota, & g_9 = H_3 g_5 H_3, & g_{14} = D_{1,1,-1} g_{12} D_{1,1,-1}, \\
g_5 = H_3\, D_{1,\rho^2,\rho^2} T_{\rho-1,0,-\sqrt{-3}}\, D_{1,\rho,\rho^2}\, H_3, & g_{10} = \iota g_9 \iota, & g_{15} = D_{1,-1,-1} g_{12} D_{1,-1,-1}.
\end{array}$$

The previous generators of $\Gamma_3[\sqrt{-3}]$ are reflections of order 3 along the following vectors:

$$\begin{array}{lllll}
v_1 = (0,0,0,1)^t, & v_2 = (0,0,1,0)^t, & v_3 = (0,1,-1,0)^t, & v_4 = (0,1,0,-1)^t, & v_5 = (0,1,0,1)^t, \\
v_6 = (0,1,1,0)^t, & v_7 = (1,0,-1,0)^t, & v_8 = (1,0,0,-1)^t, & v_9 = (1,0,0,1)^t, & v_{10} = (1,0,1,0)^t, \\
v_{11} = (\rho,-1,0,0)^t, & v_{12} = (\rho,1,1,1)^t, & v_{13} = (\rho,1,-1,1)^t, & v_{14} = (\rho,1,1,-1)^t, & v_{15} = (\rho,1,-1,-1)^t
\end{array}$$

and viewed as automorphisms of $\mathbb{C}^4$, we have $g_i(v) = v - (1-\rho)\frac{h_3(v_i,v)}{h_3(v_i,v_i)}v_i = v - (1-\rho)h_3(v_i,v)v_i$ since any v_i has norm 1 with respect to h_3. The fixed point locus of g_i acting on B_3 is biholomorphic to B_2, for example for g_1, we have

$$\{b \in B_3 \,|\, g_1 \cdot b = b\} = \{(u,v,w) \in B_3 \,|\, w = 0\}$$

which is clearly biholomorphic to B_2.

Remark 4.4. The previous theorem also appears in [24], see p. 425. In [24] they deal with the Hermitian matrix $1_{3,1} = diag(1,1,1,-1)$. The unitary group associated with $1_{3,1}$ over $\mathcal{O}$ is isomorphic to Γ_3 via $(\mathrm{U}(1_{3,1};\mathcal{O}) \to \Gamma_3; g \mapsto PgP^{-1})$ where $P = \left(\begin{smallmatrix} 1 & 0 & 0 & 1 \\ -\rho & 0 & 0 & \rho^2 \\ 0 & 0 & \rho & 0 \\ 0 & 1 & 0 & 0 \end{smallmatrix}\right) \in \mathrm{GL}(4,\mathcal{O})$.

A useful fact that we are going to use is the following isomorphism:

$$\Gamma_2/\Gamma_2[\sqrt{-3}] \simeq \mathfrak{S}_4 \times \mu_2 \tag{4.4}$$

where $\mathfrak{S}_4$ stands for the symmetric group on four letters and μ_2 the group of square roots of unity. The μ_2-part is generated by -1_3 while the $\mathfrak{S}_4$-part by

$$R_1 = -\left(\begin{smallmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right), \quad R_2 = \left(\begin{smallmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right), \quad R_3 = \left(\begin{smallmatrix} 1 & \rho^2 & 1 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{smallmatrix}\right) \tag{4.5}$$

(see pp.153–154 of [17] for more details about this isomorphism). This allows us to turn the space $M_{j,k}(\Gamma_2[\sqrt{-3}],\det^l)$ into a $\mathfrak{S}_4\times\mu_2$-representation space. The $(\mathfrak{S}_4\times\mu_2)$-invariant part of $M_{j,k}(\Gamma_2[\sqrt{-3}],\det^l)$ denoted by $M_{j,k}(\Gamma_2[\sqrt{-3}],\det^l)^{\mathfrak{S}_4\times\mu_2}$ equals $M_{j,k}(\Gamma_2,\det^l)$.

4.1.2. ***Fourier-Jacobi expansion of Picard modular forms.*** In this section, we briefly review the notion of Fourier-Jacobi expansion that we will also call q_v-expansion, of Picard modular forms. The case of signature $(2,1)$ has been described in Section 3 of [17] for scalar-valued modular forms while the case of vector-valued ones has been described in Section 5 of [7]. We briefly review this notion for a modular form f of weight (j,k) on $\mathrm{S}\Gamma_2[\sqrt{-3}]$ that we write $f = (f_1,\ldots,f_{j+1})^t$, note that $\mathrm{Sym}^j(\mathbb{C}^2) \simeq \mathbb{C}^{j+1}$. Let

$$T_n = \left\{ h : \mathbb{C} \to \mathbb{C} : h \text{ holomorphic}, h(u+c) = e^{\frac{2\pi n}{\sqrt{3}}(\bar{c}u - \rho N(c))}h(u), c \in \sqrt{-3}\mathcal{O} \right\},$$

then the last component of f has the following q_v-expansion:

$$f_{j+1}(u,v) = \sum_{n\geqslant 0} t_n(u) q_v^n \quad \text{with} \quad q_v = e^{\frac{2\pi}{\sqrt{3}}v} \quad \text{and} \quad t_n \in T_n.$$

The graded ring $T = \oplus_{n\geqslant 0} T_n$ is isomorphic to the projective coordinate ring of the elliptic curve $X^3 = \rho(Y^3 - Z^3)$ in $\mathbb{P}^2\mathbb{C}$, see Lemma 2 of [17]. Under this isomorphism, following Finis, we identify X with σ_2, Y with σ_6 and Z with σ_{-6} where

$$\sigma_2(u) = c^{-1} e^{\frac{\pi}{\sqrt{3}}u^2} \vartheta_{\frac{1}{2},\frac{1}{2}}(-\rho^2, u), \quad \sigma_6(u) = c^{-1} e^{\frac{\pi}{\sqrt{3}}u^2} \vartheta_{\frac{1}{6},\frac{1}{6}}(-\rho^2, u), \quad \sigma_{-6}(u) = \sigma_6(-u) \tag{4.6}$$

and $\vartheta_{a,b}(\tau,z) = \sum_{n\in\mathbb{Z}} e^{\pi i[(n+a)^2\tau + 2(n+a)(z+b)]}$ denotes the Jacobi theta function with characteristics and $c = \vartheta_{\frac{1}{6},\frac{1}{6}}(-\rho^2, 0) = \frac{3^{3/8}}{2\pi}\Gamma(1/3)^{3/2} e^{\frac{5\pi i}{72}}$ is introduced in the expressions (4.6) only for normalisation reasons: the Taylor expansion of Y about 0 starts with 1. We also mention that the function σ defined by (2.5) is connected to X via

$$\sigma = \beta_0 X \quad \text{where} \quad \beta_0 = -\frac{3^{3/8}}{2\pi}\Gamma(1/3)^{3/2} e^{4\pi i/9}. \tag{4.7}$$

Therefore the function X satisfies the following identities, compare with (2.7):

$$\begin{cases} X(u+\alpha) = e^{\frac{2\pi}{\sqrt{3}}(\bar{\alpha}u - \rho N(\alpha))} X(u) & \text{for any} \quad \alpha \in \mathcal{O}, \\ X(\mu u) = \mu X(u) & \text{for any} \quad \mu \in \mathcal{O}^\times = \left\{\pm 1, \pm\rho, \pm\rho^2\right\}. \end{cases} \tag{4.8}$$

Note that a basis of T_n is given by $\left\{X^a Y^b Z^c : 0 \leqslant a \leqslant 2, 0 \leqslant b \leqslant n-a, a+b+c = n\right\}$. Another property of the function X we will need is the following one:

$$XX'''' - 4X'X''' + 3(X'')^2 = 0. \tag{4.9}$$

This can be proved as follows: the function X is proportional to $\sigma_{\mathcal{O}}$, compare (3.22) and (4.7), and the Weierstrass σ_Λ-function associated with a lattice $\Lambda \subset \mathbb{C}$ satisfies the differential equation $\sigma_\Lambda \, \sigma_\Lambda'''' - 4\sigma_\Lambda' \, \sigma_\Lambda''' + 3(\sigma_\Lambda'')^2 - g_{2,\Lambda}\sigma_\Lambda^2 = 0$ but for $\Lambda = \mathcal{O}$, we have that $g_{2,\mathcal{O}} = 60 \sum_{\lambda\in\mathcal{O}^*} \lambda^{-4} = 0$ since $(\mathcal{O} \to \mathcal{O}; \lambda \mapsto \rho\lambda)$ is a bijection. Yet another proof, more systematic, of (4.9): let $f = XX'''' - 4X'X''' + 3(X'')^2$, then f is clearly holomorphic on $\mathbb{C}$ and a quick computation shows that f belongs to the space T_2. A basis of T_2 is given by $\left\{Y^2, YZ, Z^2, XY, XZ, X^2\right\}$ but $f(\mu u) = \mu^{-2} f(u)$ for any $\mu \in \mathcal{O}^\times$ and none of elements of the previous basis satisfies such an identity, so $f = 0$. In this way we obtain other examples of differential equations satisfied by X, Y and Z:

$$XX'' - (X')^2 = -YZ/(b\beta_0)^2 \quad \text{and} \quad XYZ' + XY'Z - 2X'YZ = -(Y^3 + Z^3)/b\beta_0. \tag{4.10}$$

(Note that $X'(0) = 1/b\beta_0 = \Gamma(1/3)^3 \, e^{-17\pi i/18}/2\pi$.) From these identities and since X is proportional to $\sigma_{\mathcal{O}}$, we have (recall $(\sigma_{\mathcal{O}}'/\sigma_{\mathcal{O}})' = -\wp_{\mathcal{O}}$)

$$\wp_{\mathcal{O}} = YZ/(b\beta_0 \, X)^2 \quad \text{and} \quad \wp_{\mathcal{O}}' = -(Y^3 + Z^3)/(b\beta_0 \, X)^3$$

and we deduce $(\wp_{\mathcal{O}}')^2 - 4\,\wp_{\mathcal{O}}^3 = \rho/(b\beta_0)^6 = \Gamma(1/3)^{18}/(2\pi)^6 = g_{3,\mathcal{O}}$ and is real as it should be.

To smooth the way for apprehending Picard modular forms, let us end this paragraph with few facts about them and give some concrete examples. We restrict ourselves to what we need. Let us start by an example of modular form on $\Gamma_2[\sqrt{-3}]$. It is known that the one–dimensional space $S_6(\Gamma_2[\sqrt{-3}], \det)$ is generated by a form ζ whose q_v-expansion (see Corollary 2 of [17]) is as follows:

$$\zeta(u,v) = \frac{1}{6}\sum_{\alpha\in\mathcal{O}} \alpha^5 X(\alpha u) q_v^{N(\alpha)} = X(u)q_v + 9\sqrt{-3}\,X(\sqrt{-3}u)q_v^3 + 32\,X(2u)q_v^4 + \ldots$$

and using the so-called Shintani operators (see Section 6 of [17]), we get (omitting the variable u as we will often do)

$$\zeta(u,v) = Xq_v - 27\,XYZq_v^3 + 32\,X(Y^3+Z^3)q_v^4 - (211\,XY^6 - 136\,XY^3Z + 211\,XZ^6)q_v^7 \ldots$$

Let us continue by describing the structures of the rings of modular forms on $\Gamma_2[\sqrt{-3}]$ and $\mathrm{S}\Gamma_2[\sqrt{-3}]$ as obtained by Feustel and Holzapfel (see [15, 22]). The ring of modular forms on $\Gamma_2[\sqrt{-3}]$, $M(\Gamma_2[\sqrt{-3}]) = \oplus_{k\geqslant 0} M_k(\Gamma_2[\sqrt{-3}])$, is given by

$$M\big(\Gamma_2[\sqrt{-3}]\big) = \mathbb{C}[\varphi_0, \varphi_1, \varphi_2]$$

where $\varphi_j \in M_3(\Gamma_2[\sqrt{-3}])$ given by $\varphi_j = \vartheta_j^3$ with $\vartheta_j(u,v) = \sum_{\alpha\in\mathcal{O}} \rho^{-j\mathrm{Tr}(\alpha)} Y(\alpha u)\, q_v^{N(\alpha)}$. The Fourier-Jacobi expansion of φ_0 starts with

$$\varphi_0(u,v) = 1 + (9Y + 9Z)q_v + (27\,Y^2 + 54\,YZ + 27\,Z^2)q_v^2 + (36\,Y^3 + 81\,Y^2Z + 81\,YZ^2 + 36\,Z^3)q_v^3 + \dots$$

The expansion of φ_j is obtained by substituting $(\rho^j Y, \rho^{2j} Z)$ for (Y,Z) in those of φ_0.

The ring $M(\mathrm{S}\Gamma_2[\sqrt{-3}]) = \oplus_{k\geqslant 0} M_k(\mathrm{S}\Gamma_2[\sqrt{-3}])$ is an extension of degree 3 of $M(\Gamma_2[\sqrt{-3}])$ by the cusp form ζ satisfying the relation

$$\zeta^3 = -\frac{\rho}{3^7\sqrt{-3}}\varphi_0\varphi_1\varphi_2(\varphi_1-\varphi_0)(\varphi_2-\varphi_0)(\varphi_2-\varphi_1).$$

The action of the group $\mathfrak{S}_4$ on the generators $\varphi_0, \varphi_1, \varphi_2$ of the graded ring $M(\Gamma_2[\sqrt{-3}])$ and on the form ζ is given in the next table:

i	$\varphi_0\vert_{0,3}R_i^{-1}$	$\varphi_1\vert_{0,3}R_i^{-1}$	$\varphi_2\vert_{0,3}R_i^{-1}$	$\zeta\vert_{0,6}R_i^{-1}$
1	φ_0	$\varphi_0-\varphi_1$	$\varphi_0-\varphi_2$	$-\zeta$
2	$-\varphi_0$	$-\varphi_2$	$-\varphi_1$	$-\zeta$
3	φ_2	φ_0	φ_1	ζ

and we deduce $M_3(\Gamma_2[\sqrt{-3}]) = s[2,1^2] = \mathrm{Span}_{\mathbb{C}}(\varphi_0,\varphi_1,\varphi_2)$ and $S_6(\Gamma[\sqrt{3}],\det) = s[1^4]$. Recall that the irreducible representations of $\mathfrak{S}_4$ are in bijection with the partitions of 4, we denote them by $s[4]$, $s[3,1]$, $s[2,2]$, $s[2,1^2]$ and $s[1^4]$. The ring of modular forms on Γ_2 is the ring of invariants $\mathbb{C}[\varphi_0,\varphi_1,\varphi_2]^{\mathfrak{S}_4\times\mu_2}$ and is given by $\mathbb{C}[E_6, E_{12}, E_9^2]$ where E_k is an Eisenstein series of weight k:

$$\begin{aligned}
E_6 &= \varphi_0^2+\varphi_1^2+\varphi_2^2 - \frac{2}{3}(\varphi_0\varphi_1+\varphi_0\varphi_2+\varphi_1\varphi_2),\\
E_9 &= (-\varphi_0+\varphi_1+\varphi_2)(\varphi_0-\varphi_1+\varphi_2)(\varphi_0+\varphi_1-\varphi_2),\\
E_{12} &= -\frac{1}{3}(\varphi_0+\varphi_1+\varphi_2)(-3\varphi_0+\varphi_1+\varphi_2)(\varphi_0-3\varphi_1+\varphi_2)(\varphi_0+\varphi_1-3\varphi_2).
\end{aligned}$$

Note that we have (see (18) in [17])

$$S_k(\Gamma_2[\sqrt{-3}],\det) = \zeta\, M_{k-6}(\Gamma_2[\sqrt{-3}]) \quad\text{and}\quad S_k(\Gamma_2[\sqrt{-3}],\det^2) = \zeta^2\, M_{k-12}(\Gamma_2[\sqrt{-3}]).$$

For examples of vector-valued modular forms, we refer to Section 9 of [7], let us briefly recall how such modular forms can be obtained (variant of the Rankin-Cohen brackets): set

$$\Phi_0 = -\frac{[\varphi_1,\varphi_2]}{6\pi i},\quad \Phi_1 = -\frac{[\varphi_2,\varphi_0]}{6\pi i},\quad \Phi_2 = -\frac{[\varphi_0,\varphi_1]}{6\pi i},\quad \text{where}\quad [\varphi_j,\varphi_l] = (\varphi_j\nabla\varphi_l - \varphi_l\nabla\varphi_j)/3.$$

(Here ∇f stands for the gradient of $f: B_2 \to \mathbb{C}$.) The form Φ_0, Φ_1 and Φ_2 generate the space $S_{1,7}(\Gamma_2[\sqrt{-3}])$ and the Fourier-Jacobi expansion of the last component of Φ_0 starts with

$$\Phi_0^{(2)}(u,v) = (Y-Z)q_v - 6(Y^2-Z^2)q_v^2 + \dots$$

while the corresponding expansions for $\Phi_j^{(2)}$ are obtained from this one by substituting $(\rho^j Y, \rho^{2j} Z)$ for (Y,Z). As a $\mathfrak{S}_4$-representation space, by using the previous table, we see that $S_{1,7}(\Gamma_2[\sqrt{-3}]) = s[2,1^2]$. We can now define the generator of the 1-dimensional space $M_{1,1}(\Gamma[\sqrt{-3}],\det)$:

$$E_{1,1} = (\varphi_0(\varphi_1-\varphi_0)\Phi_0 - \varphi_2(\varphi_2-\varphi_1)\Phi_2)/\zeta^2. \tag{4.11}$$

The form $E_{1,1}$ is an Eisenstein series whose Hecke eigenvalues are, surprisingly, not rational (see formulas (9a) and (9b) of Section 12 of [7]). This Eisenstein series is connected to the gradient of a generic solution of (3.9), see Remark 5.18. Note that $E_{1,1}$ is $\mathfrak{S}_4$-invariant. Another interesting Eisenstein series, $E_{1,4}$, is the one generating the 1-dimensional subspace of $\mathfrak{S}_4$-invariant form of $M_{1,4}(\Gamma[\sqrt{-3}],\det)$, it is defined as follows:

$$E_{1,4} = (\varphi_0(\varphi_0-\varphi_1)(\varphi_0+\varphi_1-3\varphi_2)\Phi_0 - \varphi_2(\varphi_1-\varphi_2)(\varphi_1+\varphi_2-3\varphi_0)\Phi_2)/\zeta^2$$

whose symmetric square belongs to $M_{2,8}(\Gamma[\sqrt{-3}],\det^2)$, see [7], Section 14.3.

4.1.3. ***Some properties of Picard modular forms.*** In this section we give some properties of Picard modular forms needed for the proof of the 'Theta representation' (3.17) given in Theorem 3.5. The first property of Picard modular forms that we will need is a corollary of the Koecher principle for unitary groups (see Theorem 1.1.30 of [21]).

Corollary 4.5. *For any $j \in \mathbb{Z}_{\geqslant 0}$, any character of finite order χ of Γ_n and any $k \in \mathbb{Z}_{<0}$, we have $M_{j,k}(\Gamma_n,\chi) = (0)$.*

A first lemma that we need is about restrictions to B_2 of modular forms on B_3. Let us consider the following modular embeddings:

$$\begin{array}{ccc} B_2 & \overset{\varpi}{\to} & B_3 \\ (u,v) & \mapsto & (u,v,0) \end{array} \quad \text{and} \quad \begin{array}{ccc} \Gamma_2 & \overset{\sigma}{\to} & \Gamma_3 \\ g & \mapsto & \left(\begin{smallmatrix} g & 0 \\ 0 & 1 \end{smallmatrix}\right) \end{array} \tag{4.12}$$

by modular embedding, we mean $\varpi(g\cdot b) = \sigma(g)\cdot\varpi(b)$ for any $b\in B_2$ and any $g\in\Gamma_2$.

Lemma 4.6. *Let $f \in M_{j,k}(\Gamma_3,\chi)$ where χ is a character of finite order of Γ_3. Assume that f vanishes at order l along B_2. Then*

$$\left.\frac{\partial^l f}{\partial w^l}\right|_{B_2} \in \bigoplus_{i=0}^{j} M_{i,k+l}(\Gamma_2,\chi\circ\sigma).$$

The same statement holds true for cusp forms.

Proof. Let $g\in\Gamma_2$ and $b=(u,v,w)\in B_3$, then direct computations give

$$\begin{aligned} \sigma(g)\cdot b &= \Big(\frac{g_{31}v+g_{32}+g_{33}u}{g_{21}v+g_{22}+g_{23}u}, \frac{g_{11}v+g_{12}+g_{13}u}{g_{21}v+g_{22}+g_{23}u}, \frac{w}{g_{21}v+g_{22}+g_{23}u}\Big), \\ j_{1,3}(\sigma(g),b) &= j_{1,2}(g,(u,v)), \\ j_{2,3}(\sigma(g),b)^{-1} &= \left(\begin{array}{cc|c} \multicolumn{2}{c|}{j_{2,2}(g,(u,v))^{-1}} & \begin{matrix} -j_{1,2}(g,(u,v))^{-1}g_{23}w \\ -j_{1,2}(g,(u,v))^{-1}g_{21}w \end{matrix} \\ \hline 0 & 0 & 1 \end{array}\right). \end{aligned}$$

By definition, we have the following functional equation satisfied by any $f\in M_{j,k}(\Gamma_3,\chi)$:

$$\chi(\sigma(g))f(b) = j_{1,3}(\sigma(g),b)^{-k}\mathrm{Sym}^j(j_{2,3}(\sigma(g),b)^{-1})f(\sigma(g)\cdot b)$$

so we get

$$\chi(\sigma(g))f(b) = j_{1,2}(g,b)^{-k}\mathrm{Sym}^j\left(\begin{array}{cc|c} \multicolumn{2}{c|}{j_{2,2}(g,(u,v))^{-1}} & \begin{matrix} -j_{1,2}(g,(u,v))^{-1}g_{23}w \\ -j_{1,2}(g,(u,v))^{-1}g_{21}w \end{matrix} \\ \hline 0 & 0 & 1 \end{array}\right) f(g\cdot b, \frac{w}{j_{1,2}(g,b)}).$$

This functional equation and the uniqueness of the Taylor expansion of f about B_2 i.e. about $w=0$, give the result. $\square$

Another technical lemma that we will need is the following one (M^{-t} denotes the transpose of the inverse of a matrix M). The proof is computational and left to the reader.

Lemma 4.7. *Let $f : B_3 \to \mathbb{C}$ be a holomorphic function, $k \in \mathbb{Z}$, $g = (g_{ij}) \in G_3^+$, $b \in B_3$ and set $g \cdot b = (\phi_1(b), \phi_2(b), \phi_3(b))$, then we have*

$$\begin{aligned}
Hess(f|_{0,k}\gamma)(b) &= j_{1,3}(g,b)^{-k-2}\, j_{2,3}(g,b)^{-1} Hess(f)(g\cdot b)\, j_{2,3}(g,b)^{-t} \\
&\quad + k(k+1) j_{1,3}(g,b)^{-k-2} \begin{pmatrix} g_{23}^2 & g_{23}g_{21} & g_{23}g_{24} \\ g_{23}g_{21} & g_{21}^2 & g_{21}g_{24} \\ g_{23}g_{24} & g_{21}g_{24} & g_{24}^2 \end{pmatrix} f(g\cdot b) \\
&\quad + (k+1)\Big(Hess(\phi_1)(b)\frac{\partial f}{\partial u}(g\cdot b) + Hess(\phi_2)(b)\frac{\partial f}{\partial v}(g\cdot b) + Hess(\phi_3)(b)\frac{\partial f}{\partial w}(g\cdot b)\Big).
\end{aligned}$$

We end this section by giving two examples of cusp forms defined on B_3. The first one (in fact a family) is explicitly given by its q_v-expansion while for the second one we only describe its main properties. These two forms will be discussed in a later publication. For any $(u, v, w) \in B_3$ and $n \in \mathbb{Z}_{\geqslant 0}$, we define

$$\chi_{n+1}(u, v, w) = \sum_{\alpha \in \mathcal{O}} \alpha^n X(\alpha u) X(\alpha w) q_v^{N(\alpha)}. \tag{4.13}$$

In the next section, as a byproduct of the proof of Theorem 5.10 (see Remark 5.12), we will see that $\chi_{n+1} \in M_{n+1}(\Gamma[\sqrt{-3}], \det)$. Note that if $n \not\equiv 4 \bmod 6$ then χ_{n+1} is identically zero (use the second identity in (4.8)) and there is noting to prove in that case. The second one is a cusp form that we denote by ξ_{15}. Its main properties can be summarised as follows. We have $\xi_{15} \in S_{15}(\Gamma_3[\sqrt{-3}])$: this cusp form can be constructed by using the theorems 5.6 and 6.2 of [18]: it is obtained as the product of the fifteen Borcherds products described in the Theorem 5.6 and Definition 8.1 of [18], they are modular forms of weight 1 on $\Gamma_3[\sqrt{-3}]$ with a certain muliplier system; its description ([18], Section 6) implies that the character of ξ_{15} is trivial. By using the results of Sections 7 and 8 of [18], we deduce that this cusp form vanishes on the $\Gamma_3[\sqrt{-3}]$-orbit of $B_2 \simeq \{b \in B_3 \,|\, r_4 \cdot b = b\} = \{(u, v, w) \in B_3 \,|\, w = 0\}$ with multiplicity 3. The $\Gamma_3[\sqrt{-3}]$-orbit of B_2 consists of fifteen components that are transitively permuted by the group Γ_3. We deduce

Lemma 4.8. *Let f be a cusp form of weight (j, k) with character χ on Γ_3 vanishing to order $3n$ along $B_2 \subset B_3$ then f/ξ_{15}^n belongs to $M_{j,k-15n}(\Gamma_3[\sqrt{-3}], \chi)$.*

4.2. **Integrability conditions as differential operators.** A first step towards the proof of the 'Theta representation' (3.17) given in Theorem 3.5 is to interpret the integrability conditions (1.2) as a differential operator on functions on B_3 and to understand its modular behaviour. For any function $f : B_3 \to \mathbb{C}$, we set

$$\mathcal{D}^{(3)} f = H\, d^4 f - d^3 f dH - 3\det(dM) = [\mathcal{D}_1^{(3)} f, \dots, \mathcal{D}_{15}^{(3)} f]^t$$

here we consider differentiations with respect to the variables u, v and w. Since we are assuming that the Hessian H is non-degenerate, a function f satisfies the integrability conditions (1.2) if and only if $\mathcal{D}^{(3)} f = 0$. Note that $\mathcal{D}^{(3)} f : B_3 \to \mathrm{Sym}^4(\mathbb{C}^3) \simeq \mathbb{C}^{15}$ for any $f : B_3 \to \mathbb{C}$ and if furthermore f is holomorphic on B_3 then so is $\mathcal{D}^{(3)} f$. A tedious computation shows that the differential operator $\mathcal{D}^{(3)}$ has the following modular behaviour.

Proposition 4.9. *For any $f : B_3 \to \mathbb{C}$, any $\gamma \in \mathrm{GL}(4, \mathbb{C})$ and any $(x, y, z) \in \mathbb{C}^3$ such that $\gamma \cdot (x, y, z)$ belongs to B_3, we have*

$$(\mathcal{D}^{(3)}(f|_{0,-1}\gamma)(x, y, z)) = \det(\gamma)^2((\mathcal{D}^{(3)} f)|_{4,8}\gamma)(x, y, z).$$

In this proposition, we have extended the action of the group G_3^+ on B_3 given by (4.1) to a so-called local action of the Lie group $\mathrm{GL}(4, \mathbb{C})$ on $\mathbb{C}^3$ by using the same formula. This proposition tells us that if $f : B_3 \to \mathbb{C}$ solves the integrability conditions (1.2) then, for any $\gamma \in \mathrm{GL}(4, \mathbb{C})$ such that $f|_{0,-1}\gamma$ is defined, the function $f|_{0,-1}\gamma$ yields another solution of (1.2).

Remark 4.10. Note that if $\gamma \in G_3^+ \cap \ker \nu$ and $b \in B_3$ then $\gamma \cdot b \in B_3$, so the equality $\mathcal{D}^{(3)}(f|_{0,-1}\gamma) = \det(\gamma)^2(\mathcal{D}^{(3)} f)|_{4,8}\gamma$ holds on B_3.

In a similar way, we view the integrability conditions (3.9) as a differential operator on functions on B_2: for any $f : B_2 \to \mathbb{C}$, we set:

$$\mathcal{D}^{(2)} f = h\, d^4 f - d^3 f\, dh - 6 df \det(dm) - 3(df)^2 \det(dn) = [\mathcal{D}_1^{(2)} f, \ldots, \mathcal{D}_5^{(2)} f]^t$$

and also the ordinary differential equation (1.6) as a differential operator on functions on $\mathfrak{H}$: for any $f : \mathfrak{H} \to \mathbb{C}$, we set:

$$\mathcal{D}^{(1)} f = f f''''(f f'' - 2\,(f')^2) - 9\,(f')^2 (f'')^2 + 2\, f f' f'' f''' + 8\,(f')^3 f''' - f^2 (f''')^2.$$

Similar computations as made for Proposition 4.9 lead to

Proposition 4.11. *For any holomorphic function f on B_2, any $\gamma \in \mathrm{GL}(3, \mathbb{C})$ and any $(x, y) \in \mathbb{C}^2$ such that $\gamma \cdot (x, y) \in B_2$, we have $(\mathcal{D}^{(2)}(f|_{0,0}\gamma)(x, y)) = \det(\gamma)^2((\mathcal{D}^{(2)} f)|_{4,10}\gamma)(x, y)$.*

For any holomorphic function f on $\mathfrak{H}$, any $\gamma \in \mathrm{GL}(2, \mathbb{C})$ and any $x \in \mathbb{C}$ such that $\gamma \cdot x \in \mathfrak{H}$, we have $(\mathcal{D}^{(1)}(f|_1\gamma)(x)) = \det(\gamma)^2((\mathcal{D}^{(1)} f)|_{16}\gamma)(x)$.

In the first statement of this proposition, we have again extended the action of the group G_2^+ on B_2 given by (4.2) to a local action of the Lie group $\mathrm{GL}(3, \mathbb{C})$ on $\mathbb{C}^2$ by using the same formula. While in the second statement, we have extended the classical action of the group $\mathrm{SL}(2, \mathbb{R})$ on $\mathfrak{H}$ to a local action of the Lie group $\mathrm{GL}(2, \mathbb{C})$ on $\mathbb{C}$ by using the same formula. From this proposition, we deduce the following two facts:

- if $f : B_2 \to \mathbb{C}$ solves the integrability conditions (3.9) then, for any $\gamma \in \mathrm{GL}(3, \mathbb{C})$ such that $f|_{0,0}\gamma$ is defined, the function $f|_{0,0}\gamma$ yields another solution of (3.9).
- if $f : \mathfrak{H} \to \mathbb{C}$ solves the differential equation (1.6) then, for any $\gamma \in \mathrm{GL}(2, \mathbb{C})$ such that $f|_1\gamma$ is defined, the function $f|_1\gamma$ yields another solution of (1.6).

The following proposition makes a connection between the differential operators $\mathcal{D}^{(3)}$ and $\mathcal{D}^{(2)}$. Its proof is just computational.

Proposition 4.12. *Let $f : B_3 \to \mathbb{C}$ be a holomorphic function and assume that its Taylor expansion about $w = 0$ starts with $f(u, v, w) = f_1(u, v)w + f_7(u, v)w^7 + O(w^{13})$ then*

$$\mathcal{D}^{(3)} f(u, v, w) = - \begin{pmatrix} \mathcal{D}_1^{(2)} f_1(u,v) \\ \mathcal{D}_2^{(2)} f_1(u,v) \\ 0 \\ \mathcal{D}_3^{(2)} f_1(u,v) \\ 0 \\ 0 \\ \mathcal{D}_4^{(2)} f_1(u,v) \\ 0 \\ 0 \\ 0 \\ \mathcal{D}_5^{(2)} f_1(u,v) \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} w^2 + O(w^5)^{15}.$$

In the same vein, the following proposition makes a connection between the differential operators $\mathcal{D}^{(2)}$ and $\mathcal{D}^{(1)}$. Its proof is again just computational.

Proposition 4.13. *Let $f : B_2 \to \mathbb{C}$ be a holomorphic function and assume that its Taylor expansion about $u = 0$ starts with $f(u, v) = f_1(v)u + f_7(v)u^7 + O(u^{13})$ then*

$$\mathcal{D}^{(2)} f(u, v) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \mathcal{D}^{(1)} f_1(v) \end{pmatrix} u^2 + O(u^5)^5.$$

These two propositions are going to be used in the next section to deduce the expression (3.12) from (3.17) and the expression (3.5) from (3.12).

Remark 4.14. Same kind of computations leads to the following fact: assume that $f : B_3 \to \mathbb{C}$ has a Taylor expansion about $u = w = 0$ of the form

$$f(u,v,w) = f_{1,1}(v)uw + f_{1,7}(v)uw^7 + f_{7,1}(v)u^7w + f_{7,7}(v)u^7w^7 + \dots$$

then the Taylor expansion of $\mathcal{D}^{(3)}f$ starts with

$$\mathcal{D}^{(3)}f(u,v,w) = -(0,\dots,0,\mathcal{D}^{(1)}f_{1,1}(v),0,0,0,0)^t u^2w^2 + \dots$$

5. Theta representation

In this section we prove that the expression (3.17) satisfies the integrability conditions (1.2). By applying the last two propositions we will deduce that (3.12) (resp. (3.5)) satisfies the integrability conditions (3.9) (resp. (1.6)).

5.1. **Reformulation of the problem.** In order to use the results about Picard modular forms obtained in Section 4, we reformulate the problem as follows. Let us define the following function on B_3:

$$F(u,v,w) = \frac{uw}{a^2} + \sum_{\alpha\in\mathcal{O}^*} \frac{X(\alpha u)X(\alpha w)}{\alpha^2} q_v^{N(\alpha)} \tag{5.1}$$

where $a = b\beta_0 = 1/X'(0)$, b given in (3.22) and β_0 in (4.7). From Proposition 4.9, we deduce the following corollary.

Corollary 5.1. *The expression (3.17) satisfies the integrability conditions (1.2) if and only if F also does.*

Proof. Recall that the expression (3.17) is given by

$$f(x,y,z) = \frac{xy}{b^2} + \sum_{\alpha\in\mathcal{O}^*} \frac{\sigma(\alpha x)\sigma(\alpha y)}{\alpha^2} e^{2\pi i N(\alpha)z}$$

Assume that f satisfies the integrability conditions (1.2), so we have $\mathcal{D}^{(3)}f = 0$. A short computation shows that (recall that $\sigma = \beta_0 X$, see (4.7), and $q_v = e^{\frac{2\pi}{\sqrt{3}}v}$) we have

$$F(u,v,w) = \frac{1}{\beta_0^2} f(u,w,\frac{v}{\sqrt{-3}}) = \frac{1}{\beta_0^2}(f|_{0,-1}\gamma)(u,v,w)$$

where $\gamma = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1/\sqrt{-3} & 0 & 0 & 0 \end{pmatrix} \in \mathrm{GL}(4,\mathbb{C})$. From Proposition 4.9, we get

$$\mathcal{D}^{(3)}F = \mathcal{D}^{(3)}(\frac{1}{\beta_0^2}f|_{0,-1}\gamma) = \frac{1}{\beta_0^8}\mathcal{D}^{(3)}(f|_{0,-1}\gamma) = \frac{\det(\gamma)^2}{\beta_0^8}(\mathcal{D}^{(3)}f)|_{4,8}\gamma = -\frac{1}{3\beta_0^8}(\mathcal{D}^{(3)}f)|_{4,8}\gamma = 0.$$

This shows one implication of the corollary. The other implication is obtained in the same way by using $f = \beta_0^2 F|_{0,-1}\gamma^{-1}$. □

Remark 5.2. The main conjecture of [13] states that the expression (3.21) satisfies the integrability conditions (1.2). Noticing that the expressions (3.21) and (3.17) are connected via $f = \frac{1}{b^2}(g|_{0,-1}\delta)$ with $\delta = \mathrm{diag}(1,1,1,3)$ with b as in (3.22), we get, same proof as for the previous corollary, that (3.17) satisfies the integrability conditions (1.2) if and only if (3.21) also does.

5.2. **Holomorphicity of F on B_3.** In this section, we prove that the function F defined by (5.1) is holomorphic on the domain B_3.

Proposition 5.3. *The function $F : B_3 \to \mathbb{C}$ is holomorphic.*

Proof. In order to prove that the function F is holomorphic on B_3, we just need to prove that the series $\sum_{\alpha\in\mathcal{O}^*} \frac{X(\alpha u)X(\alpha w)}{\alpha^2} e^{\frac{2\pi N(\alpha)v}{\sqrt{3}}}$ $(*)$ converges uniformally on any compact subset of B_3. Recall that the Jacobi theta function with characteristics $(\frac{1}{2}, \frac{1}{2})$ can be written as $\vartheta_{\frac{1}{2},\frac{1}{2}}(\tau, z) = \sum_{n\in\mathbb{Z}} \left(\frac{-4}{n}\right) e^{\frac{\pi i n^2 \tau}{4}} e^{\pi i n z}$ where $\left(\frac{-4}{n}\right) = \pm 1$ for $n \equiv \pm 1 \bmod 4$ and 0 otherwise and that the function X is defined on $\mathbb{C}$ by $X(u) = c^{-1} e^{\pi u^2/\sqrt{3}} \vartheta_{\frac{1}{2},\frac{1}{2}}(-\rho^2, u)$. By substituting these expressions in the series $(*)$, we get

$$\sum_{\alpha\in\mathcal{O}^*} \frac{X(\alpha u)X(\alpha w)}{\alpha^2} e^{\frac{2\pi \mathrm{N}(\alpha)v}{\sqrt{3}}} = c^{-2} \sum_{\substack{\alpha\in\mathcal{O}^* \\ m,n\in\mathbb{Z}}} \alpha^{-2} \left(\frac{-4}{mn}\right) e^{-\frac{\pi\sqrt{3}}{8}\mathcal{Q}(\alpha,m,n)}$$

where $\mathcal{Q}(\alpha, m, n) = (m^2+n^2)(1+\frac{1}{\sqrt{-3}}) + \frac{8\alpha}{\sqrt{-3}}(mu+nw) - \frac{8}{3}\alpha^2(u^2+w^2) - \frac{16}{3}N(\alpha)v$. A direct computation gives

$$\mathrm{Re}(Q(\alpha, m, n)) = (m + \frac{4}{\sqrt{3}}\mathrm{Im}(\alpha u))^2 + (n + \frac{4}{\sqrt{3}}\mathrm{Im}(\alpha w))^2 - \frac{8\, N(\alpha)}{3}(2\mathrm{Re}(v) + |u|^2 + |w|^2)$$

which is positive definite at any point $(u, v, w) \in B_3$. This proves the proposition. □

Remark 5.4. The last equality of the proof of Corollary 5.1 reads as $f(x, y, z) = \beta_0^2\, F(x, \sqrt{-3}z, y)$ where f is defined by the expression (3.17). This shows that f defines a holomorphic function on the domain $\left\{(x, y, z) \in \mathbb{C}^3 : |x|^2 + |y|^2 < 2\sqrt{3}\,\mathrm{Im}(z)\right\}$ which is clearly biholomorphic to B_3. From Remark 5.2, $g = f|_{0,-1}\delta^{-1}$ so g is holomorphic on $\left\{(x, y, z) \in \mathbb{C}^3 : |x|^2 + |y|^2 < \frac{2}{\sqrt{3}}\mathrm{Im}(z)\right\}$. This domain appeared in Section 3.3 of [13].

In the next paragraph we are going to describe the modular properties of F. In order to do so, we need another expression of F which comes from the Taylor expansion of σ given in (2.8):

$$X(u) = \frac{1}{\beta_0}\sigma(u) = \frac{\lambda_0}{\beta_0} \sum_{l\geqslant 0} \frac{C_l}{(6l+1)!}(\lambda_1 u)^{6l+1} = \kappa_0 \sum_{l\geqslant 0} a_l\, u^{6l+1} \tag{5.2}$$

with $\kappa_0 = \frac{\lambda_0}{\beta_0}$ and $a_l = \frac{C_l\, \lambda_1^{6l+1}}{(6l+1)!}$ for any $l \in \mathbb{Z}_{\geqslant 0}$. This gives the following expression for F:

$$F(u, v, w) = \kappa_0^2 \sum_{l,m\geqslant 0} a_l a_m \Theta_{6(l+m)+1}\left(\frac{v}{\sqrt{-3}}\right) u^{6l+1} w^{6m+1} \tag{5.3}$$

where Θ_k has been introduced in Section 2.5. Note that if $(u, v, w) \in B_3$ then $\frac{v}{\sqrt{-3}} \in \mathfrak{H}$ so evaluating the function Θ_k at $\frac{v}{\sqrt{-3}}$ for $(u, v, w) \in B_3$ makes sense. We will also need the first few terms of the q_v-expansion of F:

$$\begin{aligned} F(u, v, w) &= \frac{uw}{a^2} + \sum_{\alpha\in\mathcal{O}^*} \frac{X(\alpha u)X(\alpha w)}{\alpha^2} q_v^{N(\alpha)} \\ &= \frac{uw}{a^2} + 6X_1X_2\, q_v + 6\, X_1Y_1Z_1X_2Y_2Z_2\, q_v^3 + \frac{3}{2}X_1(Y_1^3 + Z_1^3)X_2(Y_2^3 + Z_2^3)\, q_v^4 + \dots \end{aligned} \tag{5.4}$$

where the index 1 (resp. 2) refers to the variable u (resp. w).

5.3. **Modular properties of the function F.** In this section, we study the modular behaviour of the function F defined by (5.1) under the elements (4.3) of Γ_3. The needed data for understanding this behaviour are summarised in the following table:

γ	$\gamma \cdot b$	$j_{1,3}(\gamma, b)$	$\det \gamma$
$D_{\varepsilon_1,\varepsilon_2,\varepsilon_3}$	$(\frac{\varepsilon_2}{\varepsilon_1}u, v, \frac{\varepsilon_3}{\varepsilon_1}w)$	ε_1	$\varepsilon_1^2\varepsilon_2\varepsilon_3$
$T_{\alpha_1,\alpha_2,\alpha_3}$	$(u+\alpha_2, -\bar\alpha_2 u + v - \bar\alpha_3 w + \alpha_1, w + \alpha_3)$	1	1
$\widetilde{m}$	$(\frac{u}{-c\sqrt{-3}v+d}, \frac{av+b\sqrt{-3}}{-c\sqrt{-3}v+d}, \frac{w}{-c\sqrt{-3}v+d})$	$-c\sqrt{-3}v+d$	1
H_3	$(\frac{u}{v}, \frac{1}{v}, \frac{w}{v})$	v	-1
ι	(w, v, u)	1	-1

Lemma 5.5. *For any units $\varepsilon_1, \varepsilon_2, \varepsilon_3$ of $\mathcal{O}$, we have $F|_{0,-1}D_{\varepsilon_1,\varepsilon_2,\varepsilon_3} = \frac{\varepsilon_2\varepsilon_3}{\varepsilon_1}F$.*

Proof. Set $D = D_{\varepsilon_1,\varepsilon_2,\varepsilon_3}$ then by definition, we have

$$(F|_{0,-1}D)(b) = j_{1,3}(D,b)F(D\cdot(u,v,w)) = \varepsilon_1\Big(\frac{\varepsilon_2\varepsilon_3}{\varepsilon_1^2}\frac{uw}{a^2} + \sum_{\alpha\in\mathcal{O}^*}\frac{X(\alpha\frac{\varepsilon_2}{\varepsilon_1}u)X(\alpha\frac{\varepsilon_3}{\varepsilon_1}w)}{\alpha^2}q_v^{N(\alpha)}\Big)$$
$$= \frac{\varepsilon_2\varepsilon_3}{\varepsilon_1}F(b)$$

where the last equality comes from the second equation in (4.8). □

Lemma 5.6. *Let $b = (u,v,w) \in B_3$ and $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{O}$ such that $\mathrm{Tr}(\alpha_1) = -|\alpha_2|^2 - |\alpha_3|^2$ then we have $(F|_{0,-1}T_{\alpha_1,\alpha_2,\alpha_3})(b) = F(b) + \frac{1}{a^2}(\alpha_2\alpha_3 + \alpha_3 u + \alpha_2 w)$.*

Proof. Set $T_{\alpha_1,\alpha_2,\alpha_3} = T$ then by definition, we have

$$(F|_{0,-1}T)(b) = F(T\cdot(u,v,w)) = F(u+\alpha_2, -\bar\alpha_2 u - \bar\alpha_3 w + \alpha_1 + v, w+\alpha_3).$$

Write $\alpha_1 = a_1 + b_1\rho \in \mathcal{O}$, then $\alpha_1 + \rho(\alpha_1+\bar\alpha_1) = \sqrt{-3}a_1$ and

$$-\bar\alpha_2 u - \bar\alpha_3 w + \alpha_1 = \alpha_1 - \rho(|\alpha_2|^2 + |\alpha_3|^2) = \alpha_1 + \rho Tr(\alpha_1) = \sqrt{-3}a_1.$$

Then from the first equation in (4.8), we get

$$\begin{aligned}(F|_{0,-1}T)(b) =& \frac{(u+\alpha_2)(w+\alpha_3)}{a^2} + \sum_{\alpha\in\mathcal{O}^*}\frac{X(\alpha(u+\alpha_2))X(\alpha(w+\alpha_3))}{\alpha^2}e^{\frac{2\pi N(\alpha)(-\bar\alpha_2 u - \bar\alpha_3 w + \alpha_1)}{\sqrt{3}}}q_v^{N(\alpha)}\\ =& \frac{(u+\alpha_2)(w+\alpha_3)}{a^2} + \sum_{\alpha\in\mathcal{O}^*}\frac{X(\alpha u)X(\alpha w)}{\alpha^2}e^{\frac{2\pi N(\alpha)(\alpha_1 - \rho(|\alpha_2|^2+|\alpha_3|^2))}{\sqrt{3}}}q_v^{N(\alpha)}\\ =& \frac{(u+\alpha_2)(w+\alpha_3)}{a^2} + \sum_{\alpha\in\mathcal{O}^*}\frac{X(\alpha u)X(\alpha w)}{\alpha^2}e^{\frac{2\pi N(\alpha)(\alpha_1+\rho(\alpha_1+\bar\alpha_1))}{\sqrt{3}}}q_v^{N(\alpha)}\\ =& \frac{uw}{a^2} + \sum_{\alpha\in\mathcal{O}^*}\frac{X(\alpha u)X(\alpha w)}{\alpha^2}q_v^{N(\alpha)} + \frac{1}{a^2}(\alpha_2\alpha_3 + \alpha_3 u + \alpha_2 w)\\ =& F(b) + \frac{1}{a^2}(\alpha_2\alpha_3 + \alpha_3 u + \alpha_2 w).\end{aligned}$$

This proves the lemma. □

Lemma 5.7. *Let $M = \left(\begin{smallmatrix} a & b \\ 3c & d\end{smallmatrix}\right) \in \Gamma_0(3)$ then we have $F|_{0,-1}\widetilde{M} = \chi(d)F$.*

Proof. For any $b = (u, v, w) \in B_3$, using (5.3), we get

$$\begin{aligned}
(F|_{0,-1}\widetilde{M})(b) &= j_{1,3}(\widetilde{M}, b)F(\widetilde{M} \cdot (u, v, w)) \\
&= \kappa_0^2 \sum_{l,m\geqslant 0} a_l\, a_m\, (-c\sqrt{-3}v + d)^{-6(l+m)-1}\Theta_{6(l+m)+1}\Big(\frac{av + b\sqrt{-3}}{\sqrt{-3}(-c\sqrt{-3}v + d)}\Big) u^{6l+1}w^{6m+1} \\
&= \kappa_0^2 \sum_{l,m\geqslant 0} a_l\, a_m\, \big(\Theta_{6(l+m)+1}|_{6(l+m)+1}M\big)\Big(\frac{v}{\sqrt{-3}}\Big)\, u^{6l+1}w^{6m+1} \\
&= \left(\frac{d}{3}\right) F(b)
\end{aligned}$$

where the last equality comes from the fact that Θ_k is modular form of weight k on $\Gamma_0(3)$ with character $\left(\frac{\cdot}{3}\right)$. This proves the lemma. □

Lemma 5.8. *We have $F|_{0,-1}H_3 = \det(H_3)F$.*

Proof. For any $b = (u, v, w) \in B_3$, using (5.3), we get

$$\begin{aligned}
(F|_{0,-1}H_3)(b) &= j_{1,3}(H_3, b)F(H_3 \cdot (u, v, w)) \\
&= \kappa_0^2 \sum_{l,m\geqslant 0} a_l a_m v^{-6(l+m)-1}\Theta_{6(l+m)+1}\left(\frac{1}{\sqrt{-3}v}\right) u^{6l+1}w^{6m+1} \\
&= \kappa_0^2 \sum_{l,m\geqslant 0} a_l a_m v^{-6(l+m)-1}\Theta_{6(l+m)+1}\left(\frac{-1}{3(\frac{v}{\sqrt{-3}})}\right) u^{6l+1}w^{6m+1} \\
&= -\left(\kappa_0^2 \sum_{l,m\geqslant 0} a_l a_m \Theta_{6(l+m)+1}\Big(\frac{v}{\sqrt{-3}}\Big) u^{6l+1}w^{6m+1}\right) \\
&= \det(H_3)F(b)
\end{aligned}$$

where the last but one equality comes from the transformation equation $3^{-k/2}\tau^{-k}\Theta_k(-1/3\tau) = i^{-k}\Theta_k(\tau)$ given in Section 2.

This proves the lemma. □

The last lemma we need is the following one, whose proof is trivial.

Lemma 5.9. *We have $F|_{0,-1}\iota = F$.*

From the previous lemmas, we deduce the modular behaviour of F under the generators of $\Gamma_3[\sqrt{-3}]$ as given in Theorem 4.3. For any $z \in \mathbb{C}$, let $P(z) = \frac{\rho-1}{a^2}z$ then we have

i	$1,2,11$	$3,10$	$4,9$	$5,8$	$6,7$	$12,15$	$13,14$
$F\|_{0,-1}g_i - \det(g_i)F$	0	$P(w)$	$P(u)$	$-P(u)$	$-P(w)$	$P(v-\rho)$	$-P(v-\rho)$

Theorem 5.10. *For any $\gamma \in \Gamma_3[\sqrt{-3}]$, we have*

$$(F|_{0,-1}\gamma) = \det(\gamma)\, F + P_\gamma$$

where P_γ is a polynomial, depending on γ, of degree at most 1 in u, v and w.

Proof. From the previous table, we see that $F|_{0,-1}g_i = \det(g_i)F + P_{g_i}$. Since the elements g_i for $1 \leqslant i \leqslant 15$ generate the group $\Gamma_3[\sqrt{-3}]$, we only need to prove that the set of polynomials of degree at most 1 in u, v and w with coefficients in $\mathbb{C}$ is stable under the action of weight $(0, -1)$ of $\Gamma_3[\sqrt{-3}]$, this is obvious. □

Remark 5.11. The transformation behaviour of the function F under the group $\Gamma_3[\sqrt{-3}]$ tells us that F can be interpreted as a generalisation of Eichler integrals to higher dimension domain since the polynomial appearing in Theorem 5.10 satisfies the cocycle relaltion

$$P_{\gamma_1\gamma_2} = P_{\gamma_1}|_{0,-1}\gamma_2 + \det(\gamma_1)P_{\gamma_2}$$

for any $\gamma_1, \gamma_2 \in \Gamma_3[\sqrt{-3}]$.

Remark 5.12. In the previous section, we introduced the function χ_{n+1} (see 4.13) and claim that χ_{n+1} is a modular form of weight $n+1$ on $\Gamma[\sqrt{-3}]$ with character det. The same computations as those made in the proofs of the lemmas 5.5, 5.6, 5.7, 5.8 and 5.9 lead to

$$\chi_{n+1}|_{0,n+1}D_{\varepsilon_1,\varepsilon_2,\varepsilon_3} = \frac{\varepsilon_2\,\varepsilon_1}{\varepsilon_1^{n+3}}\chi_{n+1}, \quad \chi_{n+1}|_{0,n+1}T_{\alpha_1,\alpha_2,\alpha_3} = \chi_{n+1}, \quad \chi_{n+1}|_{0,n+1}\widetilde{m} = \left(\frac{d}{3}\right)\chi_{n+1},$$
$$\chi_{n+1}|_{0,n+1}H_3 = (-1)^{n+3}\chi_{n+1}, \quad \chi_{n+1}|_{0,n+1}\iota = \chi_{n+1}$$

where $m = \left(\begin{smallmatrix} a & b \\ 3c & d \end{smallmatrix}\right) \in \Gamma_0(3)$. Using the expression of the fifteen generators of the group $\Gamma[\sqrt{-3}]$ as we given in Theorem 4.3, we get $\chi_{n+1} \in M_{n+1}(\Gamma[\sqrt{-3}], \det)$. The holomorphicity of χ_{n+1} can be proved in the way as we proved those of F. From the previous modular properties of χ_{6j+5} (we write n as $6j+4$ otherwise χ_{n+1} is identically zero) and the expression of the generators of Γ_3 as given in Theorem 4.2, we get

$$\chi_{6j+5}|_{0,6j+5}r_0 = -\chi_{6j+5},$$
$$\chi_{6j+5}|_{0,6j+5}r_i = \det r_i\,\chi_{6j+5} \ \text{ for } \ i = 1,\ldots,5.$$

So for any $\gamma \in \Gamma_3$, $\chi_{6j+5}^2|_{0,2(6j+5)}\gamma = (\chi_{6j+5}|_{0,6j+5}\gamma)^2 = \det^2(\gamma)\chi_{6j+5}^2$ and this shows that χ_{6j+5}^2 is a modular form of weight $12j+10$ on Γ_3 with character $\det^2$ for any $j \in \mathbb{Z}_{\geqslant 0}$. We can prove a bit more; χ_{6j+5}^2 is a cusp form. We know that the group Γ_3 has only one cusp and since $X(0) = 0$, the Fourier-Jacobi expansion of χ_{6j+5} at this cusp starts with

$$\begin{aligned}
\chi_{6j+5}(u,v,w) &= \sum_{n\geqslant 0}\Bigg(\sum_{\alpha\in\mathcal{O},\,\mathrm{N}(\alpha)=n} \alpha^{6j+4}X(\alpha u)X(\alpha w)\Bigg)q_v^n \\
&= \Bigg(\sum_{\varepsilon\in\mathcal{O}^\times} \varepsilon^4 X(\varepsilon u)X(\varepsilon w)\Bigg)q_v + \ldots = \Bigg(\sum_{\varepsilon\in\mathcal{O}^\times} \varepsilon^6 X(u)X(w)\Bigg)q_v + \ldots \\
&= 6X(u)X(w)q_v + \ldots
\end{aligned}$$

This proves our claim.

From Theorem 5.10, we deduce the following corollary.

Corollary 5.13. *The Hessian matrix of F is a modular form of weight $(2,1)$ on $\Gamma_3[\sqrt{-3}]$ with character* det *i.e.* $\mathrm{Hess}(F) \in M_{2,1}(\Gamma_3[\sqrt{-3}], \det)$.

Proof. We apply Lemma 4.7 to the function F and $k = -1$, we get

$$\begin{aligned}
\mathrm{Hess}(F|_{0,-1}\gamma)(b) &= j_{1,3}(\gamma,b)^{-1}\,j_{2,3}(\gamma,b)^{-1}\ \mathrm{Hess}(F)(\gamma\cdot b)\,j_{2,3}(\gamma,b)^{-t} \\
&= j_{1,3}(\gamma,b)^{-1}\,\mathrm{Sym}^2(j_{2,3}(\gamma,b)^{-1})\ \mathrm{Hess}(F)(\gamma\cdot b) \\
&= (\mathrm{Hess}(F)|_{2,1}\gamma)(b).
\end{aligned}$$

But for any $\gamma \in \Gamma_3[\sqrt{-3}]$, by Theorem 5.10, we have $F|_{0,-1}\gamma = \det(\gamma)\,F + P_\gamma$ where P_γ is a polynomial in u, v and w of degree at most 1 so

$$\mathrm{Hess}(F)|_{2,1}\gamma = \mathrm{Hess}(F|_{0,-1}\gamma) = \mathrm{Hess}(\det(\gamma)\,F + P_\gamma) = \det(\gamma)\ \mathrm{Hess}(F).$$

This proves the proposition since the function F is holomorphic on B_3, so its Hessian matrix too. ☐

Having the previous table at hands and the generators of Γ_3 (Theorem 4.2) allows us to say more about the modular behaviour of F: for any $b = (u, v, w) \in B_3$, we have

$$\begin{aligned} F|_{0,-1}r_0 &= -\det(r_0)F, & F|_{0,-1}r_i &= \det(r_i)F \quad \text{for} \quad i = 1,4,5, \\ (F|_{0,-1}r_2)(b) &= \det(r_2)F(b) - P(w/(\rho+2)), & (F|_{0,-1}r_3)(b) &= \det(r_3)F(b) - P(u/(\rho+2)), \end{aligned} \tag{5.5}$$

whose main consequence is the following theorem.

Theorem 5.14. *We have* $\mathcal{D}^{(3)}F \in S_{4,8}(\Gamma_3, \det^2)$.

Proof. We know that $\mathcal{D}^{(3)}(F|_{0,-1}\gamma) = \det(\gamma)^2(\mathcal{D}^{(3)}F)|_{4,8}\gamma$ for any $\gamma \in \Gamma_3$, see Proposition 4.9 and $\mathcal{D}^{(3)}F$ is holomorphic on B_3 since F is. The differential operator $\mathcal{D}^{(3)}$ involves partial derivatives of order at least 2 and satisfies $\mathcal{D}^{(3)}(\lambda f) = \lambda^4\mathcal{D}^{(3)}(f)$ for any constant λ, the formulas (5.5) give

$$(\mathcal{D}^{(3)}F)|_{4,8}r_i = \det(r_i)^2\mathcal{D}^{(3)}(F) \quad 0 \leqslant i \leqslant 5$$

but the elements r_i for $0 \leqslant i \leqslant 5$, generate the group Γ_3 by Theorem 4.2 so $\mathcal{D}^{(3)}F \in M_{4,8}(\Gamma_3, \det^2)$. It remains to show that $\mathcal{D}^{(3)}F$ is a cusp form. Plugging the first terms of the q_v-expansion of F, see (5.4), in the differential operator $\mathcal{D}^{(3)}$ gives

$$\begin{aligned} \mathcal{D}^{(3)}F(b) = -\frac{48\pi^2}{a^4}[&X_2^2(X_1X_1'''' - 4X_1'X_1''' + 3(X_1'')^2)q_v^2 + O(q_v^4), O(q_v^4), \ldots, O(q_v^4), \\ &X_1^2(X_2X_2'''' - 4X_2'X_2''' + 3(X_2'')^2)q_v^2 + O(q_v^4)]^t \end{aligned}$$

this shows that $\mathcal{D}^{(3)}F$ is a cusp form since the group Γ_3 has only one cusp. □

Remark 5.15. In fact the order of vanishing of $\mathcal{D}^{(3)}F$ at the unique cusp of Γ_3 is at least 4 since $XX'''' - 4X'X''' + 3(X'')^2 = 0$, see (4.9), we write this fact as $\mathcal{D}^{(3)}F(b) = O(q_v^4)^{15}$.

Other modular properties of the function F that we will need are the following ones. Via the group morphism σ, see (4.12), we embed the group $\mathfrak{S}_4$ into Γ_3. Direct computations give

$$\sigma(R_1) = H_3\, D_{-1,-1,1}, \quad \sigma(R_2) = D_{-1,1,1}, \quad \sigma(R_3) = T_{\rho^2,-1,0}$$

and by Lemmas 5.5, 5.6 and 5.8, we get for $i = 1$ or 2 and any $b \in B_3$,

$$F|_{0,-1}\sigma(R_i) = \mathrm{sign}(R_i)F \quad \text{and} \quad (F|_{0,-1}\sigma(R_3))(b) = \mathrm{sign}(R_3)F(b) - \frac{w}{a^2}. \tag{5.6}$$

The elements R_i are defined by (4.5).

5.4. Proof of ‘Theta representation’ of Lagrangian densities.

In this last paragraph we eventually prove ‘Theta representation’ of Lagrangian densities.

5.4.1. ***General Lagrangian densities.*** In this section, we prove that the function F given by (5.1) satisfies the integrability conditions (1.2), recall that this is equivalent to proving that the expression (3.17) does by Corollary 5.1. We have seen that the function $\mathcal{D}^{(3)}F$ belongs to the space $S_{4,8}(\Gamma_3, \det^2)$ which is finite dimensional so the easiest way for proving that $\mathcal{D}^{(3)}F = 0$ would have been to check the vanishing of sufficiently many coefficients in the q_v-expansion of $\mathcal{D}^{(3)}F$. But unfortunately, this dimension is not known. So to prove that $\mathcal{D}^{(3)}F = 0$, we need to proceed differently. Our strategy for proving that $\mathcal{D}^{(3)}F = 0$ is as follows: assume that $\mathcal{D}^{(3)}F \in S_{4,8}(\Gamma_3, \det^2)$ is not identically zero, then it suffices to prove that the order of vanishing of $\mathcal{D}^{(3)}F$ along $B_2 \subset B_3$ is 3 so, by Lemma 4.8, we can divide it by χ_{15} and therefore get a modular of weight $(4, -7)$ and this contradicts the corollary 4.5, so $\mathcal{D}^{(3)}F = 0$ i.e. F solves the integrability conditions (1.2). So we have to prove that the order of vanishing of $\mathcal{D}^{(3)}F$ along B_2 is 3. We apply Lemma 4.6: for any $l \in \mathbb{Z}_{\geqslant 0}$, we have

$$\left.\frac{\partial^l\mathcal{D}^{(3)}F}{\partial w^l}\right|_{B_2} \in \bigoplus_{i=0}^{4} S_{i,8+l}(\Gamma_2, \det^2) = \bigoplus_{i=0}^{4} S_{i,8+l}(\Gamma_2[\sqrt{-3}], \det^2)^{\mathfrak{S}_4\times\mu_2}.$$

Noticing that -1_3 acts by $(-1)^k \operatorname{Sym}^j(-1_2)$ on $M_{j,k}(\Gamma_2[\sqrt{-3}], \det^l)$ so it acts trivially when $j+k$ is even and using Proposition 5.1 of [7], we know that $M_{j,k}(\Gamma_2[\sqrt{-3}], \det^2) = (0)$ if $j \not\equiv k \bmod 3$, we get

$$\begin{aligned}
\mathcal{D}^{(3)}F\Big|_{B_2} &\in S_{2,8}(\Gamma_2, \det^2) = S_{2,8}(\Gamma_2[\sqrt{-3}], \det^2)^{\mathfrak{S}_4},\\
\frac{\partial \mathcal{D}^{(3)}F}{\partial w}\Big|_{B_2} &\in S_{0,9}(\Gamma_2, \det^2) \oplus S_{3,9}(\Gamma_2, \det^2) = S_{0,9}(\Gamma_2[\sqrt{-3}], \det^2)^{\mathfrak{S}_4\times\mu_2} \oplus S_{3,9}(\Gamma_2[\sqrt{-3}], \det^2)^{\mathfrak{S}_4},\\
\frac{\partial^2 \mathcal{D}^{(3)}F}{\partial w^2}\Big|_{B_2} &\in S_{1,10}(\Gamma_2, \det^2) \oplus S_{4,10}(\Gamma_2, \det^2) = S_{1,10}^{\mathfrak{S}_4\times\mu_2}(\Gamma_2[\sqrt{-3}], \det^2) \oplus S_{4,10}^{\mathfrak{S}_4}(\Gamma_2[\sqrt{-3}], \det^2).
\end{aligned}$$

By using the results of [17] in the scalar-valued case and by those of [7] in the vector-valued one, we get the decomposition into irreducible representations of $\mathfrak{S}_4$ of the spaces $S_{j,k}(\Gamma_2[\sqrt{-3}], \det^2)$ for $j = 0, 1, 2, 3$ while for $j = 4$ we use the result of [3]:

$$\begin{aligned}
S_{2,8}(\Gamma_2[\sqrt{-3}], \det^2) &= s[4] \oplus s[3,1] \oplus s[2,2],\\
S_{0,9}(\Gamma_2[\sqrt{-3}], \det^2) &= (0),\\
S_{3,9}(\Gamma_2[\sqrt{-3}], \det^2) &= s[4] \oplus 2\,s[3,1] \oplus s[2,2] \oplus s[2,1^2],\\
S_{1,10}(\Gamma_2[\sqrt{-3}], \det^2) &= s[2,1^2] \oplus s[1^4],\\
S_{4,10}(\Gamma[\sqrt{-3}], \det^2) &= 2s[4] + 3s[3,1] + 2s[2,2] + 2s[2,1^2] + s[1^4].
\end{aligned}$$

The one-dimensional irreducible representation $s[4]$ corresponds to the trivial one and its presence in one of the previous decompositions indicates the existence of a $\mathfrak{S}_4$-invariant cusp form in the corresponding space. Therefore we obtain the following dimensions:

$$\begin{aligned}
&\dim S_{2,8}(\Gamma_2, \det^2) = \dim S_{3,9}(\Gamma_2, \det^2) = 1, \quad \dim S_{0,9}(\Gamma_2, \det^2) = \dim S_{1,10}(\Gamma_2, \det^2) = 0,\\
&\dim S_{4,10}(\Gamma[\sqrt{-3}], \det^2) = 2.
\end{aligned}$$

A generator of $S_{2,8}(\Gamma_2, \det^2)$ has been constructed in [7], Example 16.7, the q_v-expansion of its last component starts with $X^2(u)q_v^2 + O(q_v^3)$ but $\mathcal{D}^{(3)}F(b) = O(q_v^4)^{15}$, see Remark 5.15, we get $\mathcal{D}^{(3)}F|_{B_2} = 0$. A generator of the space $S_{3,9}(\Gamma_2, \det^2)$ can be constructed as follows: let $E_{1,1}$ be the Eisenstein series of weight $(1,1)$ and character det on $\Gamma_2[\sqrt{-3}]$ (see 4.11) and F be the unique $\mathfrak{S}_4$-anti-invariant cusp form of weight $(2,8)$ and character det on $\Gamma_2[\sqrt{-3}]$. This form F can be constructed as follows: let $D_0 = \operatorname{Sym}^2(\Phi_0)/\varphi_1\varphi_2(\varphi_1 - \varphi_2) \in S_{2,5}(\Gamma_2[\sqrt{-3}])$ (see [7], Lemma 14.1) and set $D_1 = D_0|_{2,5}R_3$ and $D_2 = D_0|_{2,5}R_3^2$, then $\operatorname{Span}_{\mathbb{C}}(D_0, D_1, D_2) = S_{2,5}(\Gamma_2[\sqrt{-3}]) = s[3,1]$. The space $S_{2,5}(\Gamma_2[\sqrt{-3}], \det) = s[3,1]$ is in turn generated by $D_0' = \zeta(D_1 + D_2)/\varphi_0(\varphi_1 - \varphi_2)$, $D_1' = D_0'|_{2,5}R_3$ and $D_2' = D_0'|_{2,5}R_3^2$ (see [7], Section14.2). The isotypival decomposition of the space $S_{2,8}(\Gamma_2[\sqrt{-3}], \det)$ is given by $S_{2,8}(\Gamma_2[\sqrt{-3}], \det) = s[3,1] \oplus s[2,2] \oplus s[2,1^2] \oplus s[1^4]$, and the $\mathfrak{S}_4$-anti-invariant part is generated by $F = (-\varphi_0 + \varphi_1 + \varphi_2)D_0' + (\varphi_0 - \varphi_1 + \varphi_2)D_1' + (\varphi_0 + \varphi_1 - \varphi_2)D_2'$. Since the Eisenstein series $E_{1,1}$ is also $\mathfrak{S}_4$-anti-invariant, we get that $\operatorname{Sym}^2(F, E_{1,1}) \in S_{3,9}(\Gamma_2, \det^2)$ and the q_v-expansion of its last component starts with (up to a multiplicative constant) $X^2(u)q_v^2 + O(q_v^3)$, but since $\mathcal{D}^{(3)}F(b) = O(q_v^4)^{15}$, see Remark 5.15, $\frac{\partial \mathcal{D}^{(3)}F}{\partial w}|_{B_2} = 0$.

It remains to show that $\frac{\partial^2 \mathcal{D}^{(3)}F}{\partial w^2}|_{B_2} = 0$. We have seen that $\frac{\partial^2 \mathcal{D}^{(3)}F}{\partial w^2}|_{B_2}$ belongs to the 2-dimensional space $S_{4,10}(\Gamma_2, \det^2)$. A basis of this space is described in [8], Section 17 and the q_v-expansion of the last component of its generators starts with $X^2(u)q_v^2 + O(q_v^3)$ so $\frac{\partial^2 \mathcal{D}^{(3)}F}{\partial w^2}|_{B_2} = 0$ since $\mathcal{D}^{(3)}F(b) = O(q_v^4)^{15}$, see Remark 5.15. By applying Lemma 4.8, we conclude that $\mathcal{D}^{(3)}F = 0$ thus proving the main result of this section:

Theorem 5.16. *The function $F : B_3 \to \mathbb{C}$ solves the integrability conditions (1.2).*

5.4.2. ***Two-dimensional case.*** In this section we prove that the expression given by (3.12) solve the integrability conditions (3.9) by applying Proposition 4.12. Recall that this expression is given by

$$g(y,z) = \frac{y}{b} + \sum_{\alpha\in\mathcal{O}^*} \frac{\sigma(\alpha y)}{\alpha} e^{2\pi i N(\alpha) z}.$$

Let $G = a\frac{\partial F}{\partial w}(u,v,0)$ so G is holomorphic on B_2 and a direct computation shows that for any $(u,v)\in B_2$ we have (recall that $X'(0)=1/a$ and $a = b\,\beta_0$):

$$G(u,v) = a\frac{\partial F}{\partial w}(u,v,0) = \frac{u}{a} + \sum_{\alpha\in\mathcal{O}^*} \frac{X(\alpha u)}{\alpha} q_v^{N(\alpha)} = \frac{1}{\beta_0} g(u, \frac{v}{\sqrt{-3}}) = \frac{1}{\beta_0}(g|_{0,0}\gamma)(u,v)$$

where $\gamma = \mathrm{diag}(1/\sqrt{-3},1,1)\in \mathrm{GL}(3,\mathbb{C})$. This shows that g defines a holomorphic function on the domain $\left\{(y,z)\in\mathbb{C}^3 : |y|^2 < 2\sqrt{3}\mathrm{Im}(z)\right\}$ which is clearly biholomorphic to B_2. Using $\mathcal{D}^{(2)}(\lambda f) = \lambda^4\mathcal{D}^{(2)} f$ for any constant λ and applying the first statement of Proposition 4.11, mutatis mutandis the arguments in the proof of Corollary 5.1, we get $\mathcal{D}^{(2)}G = 0$ if and only if $\mathcal{D}^{(2)}g = 0$.

In order to apply Proposition 4.12, we need to show that the function F has the right Taylor expansion about $w=0$. The same computations that led to (5.3) give:

$$\begin{aligned} F(u,v,w) &= \frac{uw}{a^2} + \sum_{\alpha\in\mathcal{O}^*} \frac{X(\alpha u)X(\alpha w)}{\alpha^2} q_v^{N(\alpha)} = \sum_{n\geqslant 0} \frac{\partial^n F}{\partial w^n}(u,v,0)\frac{w^n}{n!} \\ &= \frac{1}{a}\Big(G(u,v) w + \sum_{l\geqslant 1} a_l \zeta_{6l}(u,v)\frac{w^{6l+1}}{(6l+1)!}\Big) \end{aligned}$$

for any $(u,v,w)\in B_3$ where

$$\zeta_{6n}(u,v) = \sum_{\alpha\in\mathcal{O}} \alpha^{6n-1} X(\alpha u) q_v^{N(\alpha)}.$$

Note that $\zeta = \zeta_6/6$ and $\zeta_{6n}\in S_{6n}(\Gamma_2[\sqrt{-3}],\det)$ (see [17], Proposition 2 and Corollary 2). As a corollary of Theorem 5.16, we get

Corollary 5.17. *The function $G : B_2 \to \mathbb{C}$ solves the integrability conditions (3.9).*

Remark 5.18. From the modular properties of the function F and by viewing Γ_2 as a subgroup of Γ_3 via the embedding (4.12), we immediately deduce

$$G|_{0,0}\gamma = \det(\gamma) G + c_\gamma \quad \text{for } \gamma\in\Gamma_2$$

where c_γ is a constant depending only on γ. Another interesting fact about the function G can be obtained by either Lemma 4.6 or the previous modular properties of G: the gradient of G belongs to the space $M_{1,1}(\Gamma_2[\sqrt{-3}],\det)$ and is $\mathfrak{S}_4$-anti-invariant. It is known that the space of such forms is one dimensional and generated by the already mentioned Eisenstein series $E_{1,1}$, see Section 4.1.2. The q_v-expansions of $E_{1,1}$ and ∇G are given by

$$E_{1,1}(u,v) = \begin{pmatrix} \frac{27\sqrt{3}\rho^2}{2\pi a}+\dots \\ 162\rho^2 X(u) q_v+\dots \end{pmatrix}, \quad \nabla G(u,v) = \sum_{\alpha\in\mathcal{O}} \begin{pmatrix} X'(\alpha u) \\ \frac{2\pi}{\sqrt{3}}\bar{\alpha} X(\alpha u)\end{pmatrix} q_v^{\mathrm{N}(\alpha)} = \begin{pmatrix} \frac{1}{a}+\dots \\ \frac{12\pi}{\sqrt{3}} X(u) q_v+\dots \end{pmatrix}$$

so we get

$$\nabla G = \frac{27\sqrt{3}}{2\pi}\rho^2 E_{1,1}.$$

5.4.3. ***One-dimensional case.***
In this section we prove that the expression given by (3.5) solves the differential equation (1.6) by applying Proposition 4.13. Recall that this expression is given by

$$g(z)=\sum_{(k,l)\in\mathbb{Z}^2}e^{2\pi i(k^2+kl+l^2)z}=\sum_{\alpha\in\mathcal{O}}q^{N(\alpha)}=\Theta(z)$$

so g is defined if and only if $z\in\mathfrak{H}$. In order to apply Proposition 4.13, we need to show that the function G introduced in the previous section has the right Taylor expansion about $u=0$. For any $(u,v)\in B_2$, formula (5.3) gives

$$G(u,v)=a\frac{\partial F}{\partial w}(u,v,0)=\frac{1}{a}\sum_{l\geqslant 0}a_l\Theta_{6l+1}\Big(\frac{v}{\sqrt{-3}}\Big)u^{6l+1}=\frac{1}{a}\Big(\Theta\Big(\frac{v}{\sqrt{-3}}\Big)u+\sum_{l\geqslant 1}a_l\Theta_{6l+1}\Big(\frac{v}{\sqrt{-3}}\Big)u^{6l+1}\Big).$$

Note that if $(u,v)\in B_2$ then $\frac{v}{\sqrt{-3}}\in\mathfrak{H}$ and $\Theta\Big(\frac{v}{\sqrt{-3}}\Big)=(\Theta|_1\gamma)(v)$ where $\gamma=\begin{pmatrix}1/\sqrt{-3}&0\\0&1\end{pmatrix}\in\mathrm{GL}(2,\mathbb{C})$ so by applying the second statement of Proposition 4.11 and Proposition 4.13 to G as a corollary of 5.17, we get

Corollary 5.19. *The function $\Theta:\mathfrak{H}\to\mathbb{C}$ solves the differential equation (1.6).*

5.5. **Non-degeneracy of Lagrangian densities.** In this section we give a closed formula for the non-degeneracy condition of (1.2) which is, up to the $\mathrm{GL}(4,\mathbb{C})$-action, given by the non-vanishing of $\det(\mathrm{Hess}(F))$. We will also give closed formula for non-degeneracy condition of (3.9) and (1.6). We have seen (Corollary 5.13) that

$$\mathrm{Hess}(F)|_{2,1}\gamma=j_{1,3}(\gamma,b)^{-1}\,j_{2,3}(\gamma,b)^{-1}\,\mathrm{Hess}(F)(\gamma\cdot b)\,j_{2,3}(\gamma,b)^{-t}=\det(\gamma)\,\mathrm{Hess}(F)\tag{5.7}$$

for any $\gamma\in\Gamma_3[\sqrt{-3}]$. We have seen that (see Section 4.1.1) $j_{2,3}(\gamma,b)^{-1}=j_{1,3}(\gamma,b)J(\gamma,b)^t$ and $\det(J(\gamma,b))=j_{1,3}(\gamma,b)^{-4}\det(\gamma)$ so, by taking the determinant of 5.7, we get

$$j_{1,3}(\gamma,b)^{-5}\,\det(\mathrm{Hess}(F))(\gamma\cdot b)=\det(\gamma)\det(\mathrm{Hess}(F))(b)$$

and since F is holomorphic on B_3, this shows that $\det(\mathrm{Hess}(F))\in M_5(\Gamma_3[\sqrt{-3}],\det)$. By using the expression of F as in Section 5.4.2, we get

$$\det(\mathrm{Hess}(F))(u,v,w)=-\frac{1}{a^3}\Big(t(u,v)w-\frac{a_1}{120}\zeta_6(G_{uu}G_{vv}-G_{uv}^2)(u,v)w^7+\dots\Big)$$

where $t(u,v)=(G_{uu}G_v^2-2G_uG_vG_{uv}+G_u^2G_{vv})(u,v)$. As a variant of Lemma 4.6, replace the group Γ_3 by $\Gamma_3[\sqrt{-3}]$ in the statement, we get $t\in M_6(\Gamma_2[\sqrt{-3}],\det)=\zeta\cdot\mathbb{C}$. The Fourier-Jacobi expansion of t starts with

$$t(u,v)=\frac{8\pi^2}{a^2}\left(Xq_v+9X(YZ+4a^2(XX''-(X')^2))q_v^3+\dots\right)=\frac{8\pi^2}{a^2}\left(Xq_v-27XYZq_v^3+\dots\right)=\frac{8\pi^2}{a^2}\zeta(u,v).$$

since $XX''-(X')^2=-YZ/a^2$ (see (4.10), note that $a=b\beta_0$). So the Taylor expansion of $\det(\mathrm{Hess}(F))$ about $w=0$ starts with

$$\det(\mathrm{Hess}(F))(u,v,w)=-\frac{8\pi^2}{a^5}\left(\zeta(u,v)w+O(w^7)\right).$$

From Remark 5.12, it is easily seen that the Taylor expansion of χ_5 about $w=0$ starts with

$$\chi_5(u,v,w)=\frac{6}{a}\left(\zeta(u,v)w+O(w^7)\right).$$

Moreover using the transformation formulas of F under the generators of Γ_3 as given by (5.5), we see that $\det(\mathrm{Hess}(F))^2$ belongs to the space $S_{10}(\Gamma_3,\det^2)$ like χ_5^2 so by applying Lemma 4.8, we get $\det(\mathrm{Hess}(F))=-\frac{4\pi^2}{3a^4}\chi_5$ and we deduce

Proposition 5.20. *The non-degeneracy condition of (1.2) is equivalent to the non-vanishing of the modular form χ_5.*

The non-degeneracy condition of (3.9) is equivalent to the non-vanishing of $t(u,v) = (G_{uu}G_v^2 - 2G_uG_vG_{uv} + G_u^2G_{vv})(u,v) = \frac{8\pi^2}{a^2}\zeta(u,v)$ as we just saw so we deduce:

Corollary 5.21. *The non-degeneracy condition of (3.9) is equivalent to the non-vanishing of the modular form ζ.*

The non-degeneracy condition of (1.6) is equivalent to the non-vanishing of

$$gg'' - 2g'^2 = \Theta\Theta'' - 2\Theta'^2 = \frac{1}{2}[\Theta,\Theta]_2$$

($[\Theta,\Theta]_2$ is the second the Rankin-Cohen bracket of Θ with itself, see [39], Section 5.2). Since Θ belongs to the space $M_1(\Gamma_0(3),\left(\frac{\cdot}{3}\right))$, we have $[\Theta,\Theta]_2 \in M_6(\Gamma_0(3))$ and is in fact a cusp form. It is not hard to see that $[\Theta,\Theta]_2(\tau)$ is proportional to $f_6(\tau) = \eta(\tau)^6\eta(3\tau)^6$ where η denotes the Dedekind eta-function: $\eta(\tau) = q^{1/24}\prod_{n\geqslant 1}(1-q^n)$. Note that f_6 is one of the famous eta products and plays a similar role for $\Gamma_0(3)$ as the discriminant form Δ for $\mathrm{SL}(2,\mathbb{Z})$, we deduce

Corollary 5.22. *The non-degeneracy condition of (1.6) is equivalent to the non-vanishing of the modular form f_6.*

6. Power series representation

In this section we prove that the expression (3.18) (resp. (3.13), (3.6)) satisfies the integrability conditions (1.2) (resp. (3.9), (1.6)).

Theorem 6.1. *The expression (3.18) satisfies the integrability conditions (1.2).*

Proof. Let $\gamma_3 = \begin{pmatrix} 0 & \rho & 0 & \rho^2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \in \mathrm{SL}(4,\mathcal{O})$, it is easily seen that the map

$$\begin{array}{ccc} \mathbb{D}_3 = \{(x,y,z)\in\mathbb{C}^3 \,:\, |x|^2+|y|^2+|z|^2<1\} & \to & B_3 \\ (x,y,z) & \mapsto & \gamma_3\cdot(x,y,z) = \left(\frac{x}{1-z}, \rho\frac{1+z}{1-z}, \frac{y}{1-z}\right) \end{array}$$

is a biholomorphism so, Proposition 4.9, for any $(x,y,z)\in\mathbb{D}_3$ we have

$$(\mathcal{D}^{(3)}(F|_{0,-1}\gamma_3))(x,y,z) = \det(\gamma_3)^2((\mathcal{D}^{(3)}F)|_{4,8}\gamma_3)(x,y,z) = 0$$

since we know, Theorem 5.16, that F (see (5.1)) satisfies the integrability conditions (1.2), we deduce that $F|_{0,-1}\gamma_3$ also does. Using (5.3), a quick computation shows that

$$(F|_{0,-1}\gamma_3)(x,y,z) = \kappa_0^2\sum_{l,m\geqslant 0} a_la_m(1-z)^{-6(l+m)-1}\Theta_{6(l+m)+1}\left(\frac{\tau_0-\bar\tau_0 z}{1-z}\right)x^{6l+1}y^{6m+1}$$

where $\tau_0 = \frac{1}{2}+\frac{i}{2\sqrt{3}} = \rho/\sqrt{-3}$. Since for any $(x,y,z)\in\mathbb{D}_3$ we have $|z|<1$, by applying Theorem **??**, we get

$$\begin{aligned}(F|_{0,-1}\gamma_3)(x,y,z) &= \kappa_0^2\sum_{l,m\geqslant 0} a_la_m\Big(\mu_{l+m}\sum_{n\geqslant 0}\frac{C_nC_{l+m+n}}{(6n+1)!}(\mu_1 z)^{6n+1}\Big)x^{6l+1}y^{6m+1} \\ &= \kappa_0^2\sum_{l,m,n\geqslant 0} C_lC_mC_nC_{l+m+n}\,\mu_{l+m}\,\mu_1^{6n+1}\,\lambda_1^{6(l+m)+2}\,\frac{x^{6l+1}}{(6l+1)!}\frac{y^{6m+1}}{(6m+1)!}\frac{z^{6n+1}}{(6n+1)!}\end{aligned}$$

where the last equality comes from $a_l = \frac{C_l\lambda_1^{6l+1}}{(6l+1)!}$. Using the expression of the constants λ_1, μ_1 and μ_{l+m}, we get

$$\mu_{l+m}\,\mu_1^{6n+1}\,\lambda_1^{6(l+m)+2} = \chi_0\,\chi_1^{6(l+m+n)+3} \quad \text{with} \quad \chi_0 = \frac{(2\pi)^2}{\Gamma(1/3)^3\,6^{2/3}}e^{\pi i/3} \quad \text{and} \quad \chi_1 = \frac{\Gamma(1/3)^6\,2^{1/3}\,3^{5/6}}{(2\pi)^3}$$

so

$$(F|_{0,-1}\gamma_3)(x,y,z) = \kappa_0^2\chi_0 \sum_{l,m,n\geqslant 0} C_l C_m C_n C_{l+m+n} \frac{(\chi_1 x)^{6l+1}}{(6l+1)!}\frac{(\chi_1 y)^{6m+1}}{(6m+1)!}\frac{(\chi_1 z)^{6n+1}}{(6n+1)!}.$$

Using again $\mathcal{D}^{(3)}(\lambda f) = \lambda^4 \mathcal{D}^{(3)}(f)$ for any constant λ, we see that the function

$$E(x,y,z) = \sum_{l,m,n\geqslant 0} C_l C_m C_n C_{l+m+n} \frac{(\chi_1 x)^{6l+1}}{(6l+1)!}\frac{(\chi_1 y)^{6m+1}}{(6m+1)!}\frac{(\chi_1 z)^{6n+1}}{(6n+1)!},$$

defined on $\mathbb{D}_3$ where it is holomorphic, solves the integrability conditions (1.2). Applying Proposition 4.9 to E and $\operatorname{diag}(1,\chi_1,1,1) \in \mathrm{GL}(4,\mathbb{C})$, we conclude that the expression (3.18) which is defined on $\{(x,y,z) \in \mathbb{C}^3 : |x|^2+|y|^2+|z|^2 < |\chi_1|^2\}$ satisfies the integrability conditions (1.2). ☐

Theorem 6.2. *The expression (3.13) satisfies the integrability conditions (3.9).*

Proof. Assume that the Taylor expansion about $x=0$ of a function f starts with $f(x,y,z) = f_1(y,z)x + O(x^7)$ then, same computations as for Proposition 4.12, we have

$$\mathcal{D}^{(3)} f(x,y,z) = -\Big(0,\dots,0,\mathcal{D}^{(2)} f_1(y,z)\Big)^t \, x^2 + O(x^5)^{15}.$$

We just proved that the expression (3.18), i.e.,

$$\begin{aligned} f(x,y,z) &= \sum_{l,m,n\geqslant 0} C_l C_m C_n C_{l+m+n} \frac{x^{6l+1}}{(6l+1)!}\frac{y^{6m+1}}{(6m+1)!}\frac{z^{6n+1}}{(6n+1)!} \\ &= \Big(\sum_{m,n\geqslant 0} C_m C_n C_{m+n} \frac{y^{6m+1}}{(6m+1)!}\frac{z^{6n+1}}{(6n+1)!}\Big)x + O(x^7) \\ &= g(y,z)x + O(x^7) \end{aligned}$$

(recall that $C_0 = 1$) satisfies the integrability conditions (1.2), i.e., $\mathcal{D}^{(3)} f = 0$. Uniqueness of the Taylor expansion of f about $x=0$ concludes the proof. ☐

The next theorem has already been obtained in Section 2.6 nevertheless we are going to prove it again using a similar idea which leads to the previous one.

Theorem 6.3. *The expression (3.6) satisfies the integrability conditions (1.6).*

Proof. Assume that the Taylor expansion about $y=0$ of a function f starts with $f(y,z) = f_1(z)y + O(y^7)$ then, same computations as for Proposition 4.13

$$\mathcal{D}^{(2)} f(y,z) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \mathcal{D}^{(1)} f_1(z) \end{pmatrix} y^2 + O(y^5)^5.$$

We just proved that the expression (3.13), i.e.,

$$\begin{aligned} g(y,z) &= \sum_{m,n\geqslant 0} C_m C_n C_{m+n} \frac{y^{6m+1}}{(6m+1)!}\frac{z^{6n+1}}{(6n+1)!} \\ &= \Big(\sum_{n\geqslant 0} C_n^2 \frac{z^{6n+1}}{(6n+1)!}\Big)y + O(y^7) \end{aligned}$$

(recall that $C_0 = 1$) satisfies the integrability conditions (3.9), i.e., $\mathcal{D}^{(2)} g = 0$. Uniqueness of the Taylor expansion of g about $y=0$ concludes the proof. ☐

7. Parametric representation

7.1. **Lagrangian densities $f = v_{x_1} v_{x_2} g(v_{x_3})$.** Parametric form (3.7) was first obtained in [13] based on the standard technique using GL(2)-invariance of equation (1.6) as explained in [6]. Thus, symmetry (3.2) implies the existence of a parametric representation of the form (3.7) where $\{h_1, h_2\}$ is a basis of solutions of some linear second-order ODE. The substitution into (1.6) implies that this linear ODE must coincide with the hypergeometric equation (3.3). We refer to [13] for the details of this standard computation.

Although the theta representation for g has been obtained in Section 5 from the generic case via a suitable limiting procedure, here we sketch an alternative way based on the parametric representation (3.7). The idea is to invert the period map $z = \frac{h_1(u)}{h_2(u)}$ expressing it as the ratio of two modular forms. This approach, which can be viewed as a limiting case of the Picard construction [31], was briefly outlined in [25], Section 4. It requires a suitable choice of h_1 and h_2:

$$h_1(u) = \ln(u)\, h_2(u) + \sum_{n\geqslant 1} a_n u^n,$$

$$h_2(u) = {}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; u\right)$$

where ${}_2F_1(a, b; c; u)$ is the standard hypergeometric series. Then the period map $z = \frac{h_1(u)}{h_2(u)}$ can be inverted in the form

$$u = \frac{B^3(z)}{\Theta^3(z)}$$

where B is a modular form of weight 1 defined as

$$B(z) = \frac{1}{2}(\Theta(z/3) - \Theta(z)) = 3q^{1/3}(1 + q + 2q^2 + 2q^4 + q^5 + 2q^6 + q^8 + \ldots).$$

This implies

$$g = h_2(u) = {}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; u\right) = {}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; \frac{B^3(z)}{\Theta^3(z)}\right) = \Theta(z)$$

where the last equality is due to [4], Theorem 2.3 (a).

7.2. **Lagrangian densities $f = v_{x_1} g(v_{x_2}, v_{x_3})$.** Parametric form (3.14) can be obtained by seeking solution to (3.9) in the form

$$y = \frac{h_1(u_1, u_2)}{h_3(u_1, u_2)}, \qquad z = \frac{h_2(u_1, u_2)}{h_3(u_1, u_2)}, \qquad g = F(u_1, u_2)$$

where h_1, h_2 and h_3 are three linearly independent solutions of (3.11), and F is a function to be determined. This ansatz is dictated by the symmetry (3.10); here the assumption that h_i solves (3.11) is based on a conjecture formulated in [29], Remark 7. Thus, the only unknown function to be determined is F. Direct substitution into (3.9) leads to an involutive system of five PDEs for F with a generic solution $F = G(w),\ w = \frac{u_1(u_2-1)}{u_2(u_1-1)}$ where G solves the hypergeometric equation

$$w(1-w)G'' + \frac{2}{3}(1-2w)G' = 0.$$

This can be integrated once to yield $G' = c[w(w-1)]^{-2/3}$, leading to formula (3.14). Without any loss of generality one can set $c = 1$ and take

$$G(w) = w^{1/3}{}_2F_1\left(\frac{1}{3}, \frac{2}{3}; \frac{4}{3}; w\right). \tag{7.1}$$

Note that the period map

$$y=\frac{h_1(u_1,u_2)}{h_3(u_1,u_2)},\qquad z=\frac{h_2(u_1,u_2)}{h_3(u_1,u_2)}$$

was studied by Picard in [31]. Picard's results imply that, for a suitable choice of h_1, h_2 and h_3, the inverse map is given by the formula

$$u_1=\frac{\varphi_1(y,z)}{\varphi_0(y,z)},\qquad u_2=\frac{\varphi_2(y,z)}{\varphi_0(y,z)} \tag{7.2}$$

where φ_ν as in Section 4.1.2 up to multiplicative constant. Formulas (3.14) and (7.2) lead to a simple expression of the differential dg is terms of φ_ν:

$$dg=G'(w)dw=\tfrac{\varphi_2(\varphi_0-\varphi_2)}{\zeta^2}(\varphi_0 d\varphi_1-\varphi_1 d\varphi_0)+\tfrac{\varphi_1(\varphi_1-\varphi_0)}{\zeta^2}(\varphi_0 d\varphi_2-\varphi_2 d\varphi_0) \tag{7.3}$$

where ζ satisfies the relation $\zeta^3=\varphi_0\varphi_1\varphi_2(\varphi_1-\varphi_0)(\varphi_2-\varphi_0)(\varphi_2-\varphi_1)$; up to a multiplicative constant ζ is the cusp form introduced in Section 4.1.2. Note that the expression of dg coincides, up to a multiplicative constant, with that of the Eisenstein series $E_{1,1}$, introduced in Remark 5.18, as given in Section 12 of [7].

Remark 7.1. Using (7.1) and (7.2) one can represent g in the following explicit form:

$$g={}_2F_1\left(\frac{1}{3},\frac{2}{3};\frac{4}{3};\frac{\varphi_1(\varphi_2-\varphi_0)}{\varphi_2(\varphi_1-\varphi_0)}\right)\left(\frac{\varphi_1(\varphi_2-\varphi_0)}{\varphi_2(\varphi_1-\varphi_0)}\right)^{1/3}.$$

7.3. **Generic Lagrangian densities $f(v_{x_1},v_{x_2},v_{x_3})$.** We start with a brief description of the general construction of [29] that parametrises broad classes of dispersionless integrable systems in 3D by generalised hypergeometric functions. Consider the generalised hypergeometric system of Appell's type,

$$\begin{aligned}\frac{\partial^2 h}{\partial u_i\partial u_j}=&\frac{s_i}{u_i-u_j}\frac{\partial h}{\partial u_j}+\frac{s_j}{u_j-u_i}\frac{\partial h}{\partial u_i},\quad i\neq j,\\ \frac{\partial^2 h}{\partial {u_i}^2}=&-\sigma\frac{s_i}{u_i(u_i-1)}h+\frac{s_i}{u_i(u_i-1)}\sum_{j\neq i}^n\frac{u_j(u_j-1)}{u_j-u_i}\frac{\partial h}{\partial u_j}\\ &+\Big(\sum_{j\neq i}^n\frac{s_j}{u_i-u_j}+\frac{s_i+s_{n+1}}{u_i}+\frac{s_i+s_{n+2}}{u_i-1}\Big)\frac{\partial h}{\partial u_i}.\end{aligned} \tag{7.4}$$

Here $s_1,...,s_{n+2}$ are arbitrary constants, $\sigma=1+s_1+\cdots+s_{n+2}$, and h is a function of n variables $u_1,\dots,u_n$. This system is involutive and has $n+1$ linearly independent solutions known as generalised hypergeometric functions [2, 19, 29]. Introducing the differential

$$\omega=t^{s_{n+1}}(t-1)^{s_{n+2}}(t-u_1)^{s_1}\dots(t-u_n)^{s_n}dt,$$

solutions of (7.4) can be expressed in terms of the corresponding periods, $\int_p^q\omega$; $p,q\in\{0,1,\infty,u_1,\dots,u_n\}$. Only $n+1$ of these periods are linearly independent [26, 27, 9, 22]. In low dimensions, analogous observations were made by Picard in 1883 [31]. With any generalised hypergeometric system (7.4) one can associate an integrable system in 3D having a dispersionless Lax representation [29].

We will need a particular case of this construction which leads to integrable second-order quasilinear PDEs in 3D (see example 4 of [29]). It corresponds to the case $n=3$; let $\{h_1,h_2,h_3,h_4\}$ be a basis of solutions of the corresponding system (7.4). To these data we associate a system of four PDEs for the three dependent variables u_1,u_2,u_3 viewed as functions of the auxiliary independent variables x_1,x_2,x_3:

$$(h_1/h_4)_{x_2}=(h_2/h_4)_{x_1},\qquad (h_1/h_4)_{x_3}=(h_3/h_4)_{x_1},\qquad (h_2/h_4)_{x_3}=(h_3/h_4)_{x_2}, \tag{7.5}$$

$$u_3(u_3-1)(u_1-u_2)X_1(u_2)+u_1(u_1-1)(u_2-u_3)X_2(u_3)+u_2(u_2-1)(u_3-u_1)X_3(u_1)=0; \tag{7.6}$$

where $X_i = B_{i1}\partial_{x_1} + B_{i2}\partial_{x_2} + B_{i3}\partial_{x_3}$ with B_{ij} the cofactor of h_{j,u_i} of the matrix

$$J = \begin{pmatrix} h_1 & h_2 & h_3 & h_4 \\ h_{1,u_1} & h_{2,u_1} & h_{3,u_1} & h_{4,u_1} \\ h_{1,u_2} & h_{2,u_2} & h_{3,u_2} & h_{4,u_2} \\ h_{1,u_3} & h_{2,u_3} & h_{3,u_3} & h_{4,u_3} \end{pmatrix}.$$

Parametrising equations (7.5) in terms of the auxiliary potential v,

$$h_1/h_4 = v_{x_1}, \qquad h_2/h_4 = v_{x_2}, \qquad h_3/h_4 = v_{x_3}, \tag{7.7}$$

expressing from (7.7) the parameters u_i in terms of partial derivatives v_{x_j} and substituting the result into (7.6), we obtain a single second-order PDE for the function v:

$$\sum f_{ij}(v_{x_1}, v_{x_2}, v_{x_3}) v_{x_i x_j} = 0. \tag{7.8}$$

This PDE was shown in [29] to be integrable via a dispersionless Lax representation. The classification of integrable second-order PDEs of type (7.8) was previously addressed in [5], and the construction of [29] described above gives a generic case of this classification. Setting $v_{x_i} = p_i$, with any second-order equation of type (7.8) we associate a conformal structure

$$[\mathcal{G}] = \sum f_{ij}(p_1, p_2, p_3) dp_i dp_j. \tag{7.9}$$

The space with coordinates p_1, p_2, p_3 is also endowed with a natural projective structure (see [5] for geometric aspects of PDEs of type (7.8)). It is clear that the knowledge of $[\mathcal{G}]$ allows one to reconstruct the corresponding PDE (7.8). It was shown in [5] that the requirement of integrability implies that the conformal structure $[\mathcal{G}]$ must be flat (that is, its Cotton tensor vanishes; this necessary condition is however not sufficient for integrability).

Equation (7.8) can be represented in a somewhat more explicit form. Let us view formulas (7.7) as a period map from the parameter space u_1, u_2, u_3 to the projective space $\mathbb{P}^3$ with coordinates p_1, p_2, p_3:

$$p_1 = \frac{h_1(u_1, u_2, u_3)}{h_4(u_1, u_2, u_3)}, \qquad p_2 = \frac{h_2(u_1, u_2, u_3)}{h_4(u_1, u_2, u_3)}, \qquad p_3 = \frac{h_3(u_1, u_2, u_3)}{h_4(u_1, u_2, u_3)}, \tag{7.10}$$

and introduce the inverse map by the formulas

$$u_1 = \frac{\varphi_1(p_1, p_2, p_3)}{\varphi_0(p_1, p_2, p_3)}, \qquad u_2 = \frac{\varphi_2(p_1, p_2, p_3)}{\varphi_0(p_1, p_2, p_3)}, \qquad u_3 = \frac{\varphi_3(p_1, p_2, p_3)}{\varphi_0(p_1, p_2, p_3)}. \tag{7.11}$$

Note that for certain special values of constants s_i (when the monodromy group of (7.4) is discrete) the functions φ_ν become single-valued modular forms on a 3-dimensional complex ball. Differentiating (7.11) with respect to u_i by using the chain rule we get:

$$\begin{pmatrix} 0 & 0 & 0 \\ h_4 & 0 & 0 \\ 0 & h_4 & 0 \\ 0 & 0 & h_4 \end{pmatrix} = \begin{pmatrix} h_1 & h_2 & h_3 & h_4 \\ h_{1,u_1} & h_{2,u_1} & h_{3,u_1} & h_{4,u_1} \\ h_{1,u_2} & h_{2,u_2} & h_{3,u_2} & h_{4,u_2} \\ h_{1,u_3} & h_{2,u_3} & h_{3,u_3} & h_{4,u_3} \end{pmatrix} \begin{pmatrix} (\varphi_1/\varphi_0)_{p_1} & (\varphi_2/\varphi_0)_{p_1} & (\varphi_3/\varphi_0)_{p_1} \\ (\varphi_1/\varphi_0)_{p_2} & (\varphi_2/\varphi_0)_{p_2} & (\varphi_3/\varphi_0)_{p_2} \\ (\varphi_1/\varphi_0)_{p_3} & (\varphi_2/\varphi_0)_{p_3} & (\varphi_3/\varphi_0)_{p_3} \\ H(\varphi_1/\varphi_0) & H(\varphi_2/\varphi_0) & H(\varphi_3/\varphi_0) \end{pmatrix}$$

where $H = -p_1\partial_{p_1} - p_2\partial_{p_2} - p_3\partial_{p_3}$. This shows that the cofactor matrix of J is proportional to a matrix of the form

$$\begin{pmatrix} * & * & * & * \\ (\varphi_1/\varphi_0)_{p_1} & (\varphi_1/\varphi_0)_{p_2} & (\varphi_1/\varphi_0)_{p_3} & H(\varphi_1/\varphi_0) \\ (\varphi_2/\varphi_0)_{p_1} & (\varphi_2/\varphi_0)_{p_2} & (\varphi_2/\varphi_0)_{p_3} & H(\varphi_2/\varphi_0) \\ (\varphi_3/\varphi_0)_{p_1} & (\varphi_3/\varphi_0)_{p_2} & (\varphi_3/\varphi_0)_{p_3} & H(\varphi_3/\varphi_0) \end{pmatrix},$$

where the exact values of matrix elements of the first row are not important. Thus we have $B_{11} = k(\varphi_1/\varphi_0)_{p_1}$, $B_{12} = k(\varphi_1/\varphi_0)_{p_2}$, etc. Note that the coefficient of proportionality k does not effect the equation (7.6), as well as the second-order equation (7.8). Substituting $u_i = \varphi_i/\varphi_0$ along with the values for B_{ij} into (7.6) we obtain the conformal structure of the corresponding second-order PDE (7.8) in the following explicit form:

$$\begin{gathered}[\mathcal{G}] = \varphi_3(\varphi_3 - \varphi_0)(\varphi_1 - \varphi_2)[\varphi_0, \varphi_1][\varphi_0, \varphi_2] \\ +\varphi_2(\varphi_2 - \varphi_0)(\varphi_3 - \varphi_1)[\varphi_0, \varphi_1][\varphi_0, \varphi_3] + \varphi_1(\varphi_1 - \varphi_0)(\varphi_2 - \varphi_3)[\varphi_0, \varphi_2][\varphi_0, \varphi_3].\end{gathered} \tag{7.12}$$

Here we use the notation $[\varphi_\mu,\varphi_\nu]=\varphi_\mu d\varphi_\nu-\varphi_\nu d\varphi_\mu$ where d is the standard differential. In the special cases where φ_ν are modular forms, the conformal structure $[\mathcal{G}]$ can be interpreted as a vector-valued modular form with values in degree two symmetric tensors.

Lagrangian equations correspond to the above construction for specific values of the parameters: $n=3,\ s_i=-\frac{1}{3}$. This is one of the exceptional cases where the inverse period map (7.11) can be expressed via modular forms. In this case hypergeometric system (7.4) reduces to the Picard system (3.15) and the differential ω becomes holomorphic differential on the family of genus 4 trigonal curves

$$r^3=t(t-1)(t-u_1)(t-u_2)(t-u_3),$$

namely $\omega=dt/r$. We will see that the corresponding conformal metric $[\mathcal{G}]$ becomes second symmetric differential d^2f of the Lagrangian density f. The proof of parametric representation (3.19) can be obtained as follows. Symbolically, we can represent linear systems for h and F in the form

$$h_{u_iu_j}=\Gamma^p_{ij}h_{u_p}+g_{ij}h \tag{7.13}$$

and

$$F_{u_iu_j}=\Gamma^p_{ij}F_{u_p}+g_{ij}F+\nu_{ij}, \tag{7.14}$$

where the coefficients Γ^p_{ij}, g_{ij} and ν_{ij} can be read off (3.15) and (3.16). Let us consider the period map

$$x=\frac{h_1(u_1,u_2,u_3)}{h_4(u_1,u_2,u_3)},\ y=\frac{h_2(u_1,u_2,u_3)}{h_4(u_1,u_2,u_3)},\ z=\frac{h_3(u_1,u_2,u_3)}{h_4(u_1,u_2,u_3)}. \tag{7.15}$$

Its inverse is given by the formula

$$u_1=\frac{\varphi_1(x,y,z)}{\varphi_0(x,y,z)},\ u_2=\frac{\varphi_2(x,y,z)}{\varphi_0(x,y,z)},\ u_3=\frac{\varphi_3(x,y,z)}{\varphi_0(x,y,z)} \tag{7.16}$$

where φ_ν are single-valued modular forms on a 3-dimensional complex ball. This case was studied in [23, 20, 24] where φ_ν were expressed via suitable genus 4 theta constants (note that under the identification $x=p_1,\ y=p_2,\ z=p_3$ equations (7.15), (7.16) coincide with (7.10), (7.11)). Formulae (3.19, 7.16) allow to express the second symmetric differential d^2f is terms of φ_ν. Calculating second-order partial derivatives of relations (3.19) with respect to x,y,z we obtain

$$d^2f=\frac{\nu_{ij}du_idu_j}{h_4}$$

where we have to substitute u_i from (7.16). Taking ν_{ij} from equations (3.16) and choosing h_4 to be the Lauricella series $h_4=F_D^{(3)}\left(\frac{2}{3},\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{4}{3};u_1,u_2,u_3\right)$ gives the following expression for d^2f:

$$\begin{gathered}d^2f=\tfrac{2}{U^{2/3}H}(\varphi_3(\varphi_3-\varphi_0)(\varphi_1-\varphi_2)[\varphi_0,\varphi_1][\varphi_0,\varphi_2]\\ +\varphi_2(\varphi_2-\varphi_0)(\varphi_3-\varphi_1)[\varphi_0,\varphi_1][\varphi_0,\varphi_3]+\varphi_1(\varphi_1-\varphi_0)(\varphi_2-\varphi_3)[\varphi_0,\varphi_2][\varphi_0,\varphi_3]);\end{gathered} \tag{7.17}$$

here

$$U=\varphi_0\varphi_1\varphi_2\varphi_3(\varphi_1-\varphi_0)(\varphi_2-\varphi_0)(\varphi_3-\varphi_0)(\varphi_1-\varphi_2)(\varphi_2-\varphi_3)(\varphi_3-\varphi_1)$$

and

$$H=\varphi_0^{1/3}F_D^{(3)}\left(\frac{2}{3},\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{4}{3};\frac{\varphi_1}{\varphi_0},\frac{\varphi_2}{\varphi_0},\frac{\varphi_3}{\varphi_0}\right).$$

Since d^2f given by (7.17) is proportional to the conformal structure $[\mathcal{G}]$ given by (7.12), we conclude that the Lagrangian density f specified by parametric representation (3.19) indeed defines an integrable Euler-Lagrange equation. Conformal flatness of $[\mathcal{G}]$ implies that d^2f can be viewed as a conformally flat Hessian metric. Note that (7.17) coincides, up to a multiplicative constant, with the vector-valued modular form of Corollary 5.13.

8. Hamiltonian systems and 2-dimensional Lagrangian densities

Let us first note that up to the symmetry (1.4) Lagrangian densities $f = v_{x_1} g(v_{x_2}, v_{x_3})$ are equivalent to densities of the form $f = g(\frac{v_{x_2}}{v_{x_1}}, \frac{v_{x_3}}{v_{x_1}})$: use the transformation $\tilde{x} = \frac{1}{x}$, $\tilde{y} = \frac{y}{x}$, $\tilde{z} = \frac{z}{x}$, $\tilde{f} = \frac{f}{x}$. Setting $m = \frac{v_{x_2}}{v_{x_1}}$, $n = \frac{v_{x_3}}{v_{x_1}}$ we can represent the Euler-Lagrange equation corresponding to the Lagrangian density $f = g(\frac{v_{x_2}}{v_{x_1}}, \frac{v_{x_3}}{v_{x_1}})$ as a first-order system

$$(g_m)_{x_2} + (g_n)_{x_3} - (g + mg_m + ng_n)_{x_1} = 0, \quad m_{x_3} + mn_{x_1} = n_{x_2} + nm_{x_1}. \tag{8.1}$$

We will show that system (8.1) is transformable into a Hamiltonian form investigated previously in [14], thus providing a link between integrable Hamiltonians and Picard modular forms.

Let us begin by recalling the necessary setup from paper [14] which provides a classification of integrable Hamiltonian systems of hydrodynamic type in 2+1 dimensions. One class of such systems is

$$v_t = (2vh_v + wh_w - h)_x + (vh_w)_y, \quad w_t = (wh_v)_x + (2wh_w + vh_v - h)_y, \tag{8.2}$$

where $h(v, w)$ is the Hamiltonian density. System (8.2) can be cast into Hamiltonian form

$$\begin{pmatrix} v \\ w \end{pmatrix}_t = P \begin{pmatrix} h_v \\ h_w \end{pmatrix}$$

where P is the following Hamiltonian operator:

$$P = \begin{pmatrix} 2v & w \\ w & 0 \end{pmatrix} \frac{d}{dx} + \begin{pmatrix} 0 & v \\ v & 2w \end{pmatrix} \frac{d}{dy} + \begin{pmatrix} v_x & v_y \\ w_x & w_y \end{pmatrix}.$$

It was observed that the classification of integrable Hamiltonian densities $h(v, w)$ simplifies considerably under the Legendre transformation $h(v, w) \to H(V, W)$ defined as

$$V = h_v, \quad W = h_w, \quad H = vh_v + wh_w - h, \quad H_V = v, \quad H_W = w.$$

In the new variables, equations (8.2) take the form

$$(H_V)_t = (VH_V + H)_x + (WH_V)_y, \quad (H_W)_t = (VH_W)_x + (WH_W + H)_y. \tag{8.3}$$

Our main observation is that systems (8.1) and (8.3) are equivalent under the contact transformation

$$g = H, \quad g_m = H_V, \quad g_n = WH_V, \quad m = V + \frac{H_W}{H_V} W, \quad n = -\frac{H_W}{H_V}, \tag{8.4}$$

(note the contact condition $dg - g_m dm - g_n dn = dH - H_V dV - H_W dW$), followed by relabelling the independent variables x_1, x_2, x_3. In particular, under the contact transformation (8.4) the integrability conditions for H derived in [14] transform into the integrability conditions (3.9) for g. Note that the integrability conditions for both H and g are invariant under 10-dimensional symmetry groups whose actions possess open orbits (along with several orbits of lower dimensions). The contact transformation (8.4) can be inverted in the form

$$H = g, \quad H_V = g_m, \quad H_W = -ng_m, \quad V = m + \frac{g_n}{g_m} n, \quad W = \frac{g_n}{g_m}, \tag{8.5}$$

leading to the relation

$$g(m, n) = H\left(m + \frac{g_n}{g_m} n, \ \frac{g_n}{g_m}\right).$$

In was shown in [14] that, besides the generic potential $H(V, W)$ generating an open orbit, the system of integrability conditions for H possesses four non-degenerate solutions giving rise to orbits of lower dimensions. These are

$$H = \frac{1}{Wg(V)}, \quad H = \frac{1}{WV}, \quad H = V - W \log W, \quad H = V - W^2/2,$$

(orbits of dimensions 9, 8, 8, 7, respectively). Their images under the contact transformation (8.4) are Lagrangian densities equivalent to

$$f = v_{x_1}v_{x_2}g(v_{x_3}), \quad f = v_{x_1}v_{x_2}v_{x_3}, \quad f = v_{x_1}(v_{x_3} + e^{v_{x_2}}), \quad f = v_{x_1}(v_{x_3} + v_{x_2}^2),$$

respectively. We expect that these orbits are related to a suitable compactification of the family of Picard curves $r^3 = t(t-1)(t-u_1)(t-u_2)$.

9. Concluding remarks

Attempts to generalise/specialise Theorems 3.1-3.5 lead to the following observations:

- There exists a whole variety of integrable Lagrangians whose densities f are polynomial, or can be expressed in terms of elementary functions. It would be interesting to clarify how these examples can be recovered as degenerations of the 'master-Lagrangian' constructed in Theorem 3.5, and to describe singular orbits of lower dimension. This should be related to understanding degenerations/compactifications of the moduli space of Picard curves (see discussion at the end of section 8).
- Although parametric, theta and power series representations of Theorem 3.5 possess straightforward generalisations to dimensions higher than three, the relation to integrable Lagrangians will be lost: one can show that in higher dimensions every integrable first-order Lagrangian density is necessarily of the form
$$f = \frac{Q(v_x)}{l(v_x)}$$
where Q and l are arbitrary quadratic and linear functions of the first-order derivatives of v, respectively (furthermore, the corresponding Euler-Lagrange equations are linearisable). Thus, the occurrence of modular forms is a essentially three-dimensional phenomenon.

Acknowledgements

This research was supported by the EPSRC grant EP/N031369/1. FC was also supported by Simons Foundation Award 546235 at the Institute for Computational and Experimental Research in Mathematics at Brown University. He thanks Loughborough University for hospitality during the preparation of this paper.

References

[1] D. Allcock: New complex- and quaternion-hyperbolic reflection groups. Duke Math. J. **103** (2000) 303–333

[2] P. Appell, J. Kampé de Fériet: Fonctions Hypergéométriques et Hypersphériques. Polynomes d'Hermite. Paris, Gauthier-Villars, 1926.

[3] J. Bergström, G. van der Geer: Picard modular forms and the cohomology of local systems on a Picard modular surface. arXiv:2012.07673

[4] J. M. Borwein, P. B. Borwein: A cubic counterpart of Jacobi's identity and the AGM. Trans. Amer. Math. Soc. **323** (1991) 691–701.

[5] P.A. Burovskii, E.V. Ferapontov, S.P. Tsarev: Second order quasilinear PDEs and conformal structures in projective space. International J. Math. **21**, no. 6 (2010) 799–841.

[6] P.A. Clarkson, P.J. Olver: Symmetry and the Chazy equation. J. Differential Equations **124**, no. 1 (1996) 225–246.

[7] F. Cléry, G. van der Geer: Generators for modules of vector-valued Picard modular forms. Nagoya Math. J. **212** (2013) 19–57.

[8] F. Cléry, G. van der Geer: Generating Picard modular forms by means of invariant theory. arXiv:2110.00849

[9] P. Deligne, G.D. Mostow: Monodromy of hypergeometric functions and nonlattice integral monodromy, Inst. Hautes Études Sci. Publ. Math. **63** (1986) 5–89.

[10] E.V. Ferapontov, K.R. Khusnutdinova: On integrability of (2+1)-dimensional quasilinear systems. Comm. Math. Phys. **248** (2004) 187–206.

[11] E.V. Ferapontov, K.R. Khusnutdinova, S.P. Tsarev: On a class of three-dimensional integrable Lagrangians. Comm. Math. Phys. **261**, N1 (2006) 225–243.

[12] E.V. Ferapontov, B.S. Kruglikov: Dispersionless integrable systems in 3D and Einstein-Weyl geometry. J. Diff. Geom. **97** (2014) 215–254.
[13] E.V. Ferapontov, A.V. Odesskii: Integrable Lagrangians and modular forms. J. Geom. Phys. **60**, no. 6-8 (2010) 896–906.
[14] E.V. Ferapontov, A.V. Odesskii, N.M. Stoilov: Classification of integrable two-component Hamiltonian systems of hydrodynamic type in 2+1 dimensions. Journal of Mathematical Physics, **52** (2011) 073505.
[15] J.M. Feustel: Ringe automorpher Formen auf der komplexen Einheitskugel und ihre Erzeugung durch Theta-Konstanten. Preprint Ser. Akad. Wiss. DDR, P-MATH-13, 1986.
[16] J. Feustel: Representation of Picard modular forms by theta constants. Rev. Roumaine Math. Pures Appl. **33**, no. 4 (1988) 275–281.
[17] T. Finis: Some computational results on Hecke eigenvalues of modular forms on a unitary group. Manuscripta Math. **96** (1998), 149–180.
[18] E. Freitag, R. Salvati Manni: A three dimensional ball quotient. Math. Z. **276** (2014), 345–370.
[19] I.M. Gelfand, M.I. Graev, V.S. Retakh: General hypergeometric systems of equations and series of hypergeometric type. Russian Math. Surveys **47**, no. 4 (1992) 1–88.
[20] G. González Diez: Loci of curves which are prime Galois coverings of $\mathbb{P}^1$. Proc. London Math. Soc. (3) **62**, no. 3 (1991) 469–489.
[21] E. Hofmann: Automorphic Products on Unitary Groups. Dissertation, TU Darmstadt 2011. Available at https://tuprints.ulb.tu-darmstadt.de/2540/1/ehofmann.pdf
[22] R.-P. Holzapfel: Geometry and arithmetic. Around Euler partial differential equations. Mathematische Monographien [Mathematical Monographs], 20. VEB Deutscher Verlag der Wissenschaften, Berlin, 1986. 184 pp. ISBN: 3-326-00013-8.
[23] B. Hunt: A Siegel modular 3-fold that is a Picard modular 3-fold, Compositio Math. **76**, no. 1-2 (1990) 203–242.
[24] K. Matsumoto: Theta constants associated with the cyclic triple coverings of the complex projective line branching at six points, Publ. Res. Inst. Math. Sci. **37**, no. 3 (2001) 419–440.
[25] K. Matsumoto, H. Shiga: A variant of Jacobi type formula for Picard curves. J. Math. Soc. Japan **62**, no. 1 (2010) 305–319.
[26] G.D. Mostow: Generalized Picard lattices arising from half-integral conditions, Inst. Hautes Études Sci. Publ. Math. **63** (1986) 91-106.
[27] G.D. Mostow: Existence of nonarithmetic monodromy groups, Proc. Nat. Acad. Sci. U.S.A. **78**, no. 10 (1981) Phys. Sci., 5948-5950.
[28] D. Mumford: Tata Lectures on Theta I, Progress in mathematics **28** Birkhäuser Boston, Inc., Boston, MA (1983) 235 pp.
[29] A.V. Odesskii, V.V. Sokolov: Integrable pseudopotentials related to generalized hypergeometric functions. Selecta Math. **16** (2010) 145–172.
[30] É. Picard: Sur les formes quadratiques ternaires indéfinies à indéterminées conjuguées et sur les fonctions hyperfuchsiennes correspondantes. *Acta Math.* **5** (1884), 121–182.
[31] É. Picard: Sur les fonctions de deux variables indépendantes analogues aux fonctions modulaires. *Acta Math.* **2** (1883), 114–135.
[32] B. Runge: On Picard modular forms. Math. Nachr. **184** (1997) 259–273.
[33] H. Shiga: One attempt to the K3 modular function, I. Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **6**, no. 4 (1979) 609–635.
[34] H. Shiga: One attempt to the K3 modular function, II. Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **8**, no. 1 (1981) 157–182.
[35] H. Shiga: On the representation of the Picard modular function by θ constants. I, II. Publ. Res. Inst. Math. Sci. **24**, no. 3 (1988) 311–360.
[36] F. Rodriguez Villegas, D. Zagier: Square roots of central values of Hecke L-series. In *Advances in number theory (Kingston, ON, 1991)*, Oxford Sci. Publ., Oxford Univ. Press, New York (1993), 81–99.
[37] F. Rodriguez Villegas, D. Zagier: Which primes are the sum of two cubes? Canad. Math. Soc. Conf. Proc. **15** (1995) 295–306.
[38] T. Shintani: On automorphic forms on unitary groups of order 3. Unpublished manuscript.
[39] D. Zagier, Elliptic modular forms and their applications. In *The 1-2-3 of modular forms*, by J.H. Bruinier, G. van der Geer, G. Harder, D. Zagier, Universitext, Springer-Verlag, Berlin (2008) 1–103.
[40] V.E. Zakharov: Dispersionless limit of integrable systems in $2+1$ dimensions in *Singular Limits of Dispersive Waves*, eds. N.M. Ercolani et al. (Plenum Press, NY, 1994) 165–174.

ICERM, 121 South Main Street, Providence, RI 02903, USA
Email address: cleryfabien@gmail.com

Department of Mathematical Sciences, Loughborough University, United Kingdom
Email address: E.V.Ferapontov@lboro.ac.uk

Department of Mathematics and Statistics, Brock University, St. Catharines, ON L2S 3A1, Canada
Email address: aodesski@brocku.ca

Max Planck Institute for Mathematics, Vivatsgasse 7, 53111 Bonn, Germany
Email address: dbz@mpim-bonn.mpg.de