STABLE PAIRS ON LOCAL SURFACES I: THE VERTICAL COMPONENT

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with an Appendix by Aaron Pixton and Don Zagier

Abstract. We study the full stable pair theory — with descendents — of the Calabi-Yau 3-fold $X = K_S$, where S is a surface with smooth canonical divisor C.

By both $\mathbb{C}^*$-localisation and cosection localisation we reduce to stable pairs supported on thickenings of C indexed by partitions. We show that only strict partitions contribute, and give a complete calculation for length-1 partitions. The result is a surprisingly simple closed product formula for these "vertical" thickenings.

This gives all contributions for the curve classes $[C]$ and $2[C]$ (and those which are not an integer multiple of the canonical class). Here the result is equivalent, via the descendent-MNOP correspondence, to a conjecture of Maulik-Pandharipande. We also study the relationship to the Gromov-Witten theory of S and to spin Hurwitz numbers.

Contents

1. Introduction

Let S be a smooth complex projective surface, and let $X = \mathrm{Tot}(K_S)$ be the total space of its canonical bundle K_S with its natural action of $T = \mathbb{C}^*$ on the fibres. We use the natural maps

$$X \underset{\pi}{\overset{\iota}{\leftrightarrows}} S.$$

For $\beta \in H_2(S,\mathbb{Z})$ and $\chi \in \mathbb{Z}$ we let $P_X := P_\chi(X, \iota_*\beta)$ denote the moduli space of stable pairs (F,s) on X [PT1] with curve class $[F] = \iota_*\beta$ and holomorphic Euler characteristic $\chi(F) = \chi$.

The moduli space P_X has a symmetric perfect obstruction theory [PT1], but is non-compact. The T-action induces one on P_X with compact fixed point locus P_X^T. Therefore we can define the stable pair invariants of X via T-equivariant virtual localisation [GP].[1] See Section 2 for a review of the details, and for the construction of the descendent insertions

$$\tau_\alpha(\sigma) := \pi_{P*}\big(\pi_X^*\sigma \cap \mathrm{ch}^T_{\alpha+2}(\mathbb{F})\big) \in H^*_T(P_X, \mathbb{Q})$$

for $\alpha \geq 0$. Here we use σ to denote both a class in $H^*(S,\mathbb{Q})$ and the corresponding class $\sigma \otimes 1 \in H^*_T(X,\mathbb{Q}) \cong H^*(S,\mathbb{Q}) \otimes \mathbb{Q}[t]$, where t is the equivariant parameter. The resulting descendent invariants of X live in $\mathbb{Q}[t,t^{-1}]$ and are defined by

$$P_{\chi,\beta}\big(X, \tau_{\alpha_1}(\sigma_1)\cdots\tau_{\alpha_m}(\sigma_m)\big) := \int_{[P_\chi(X,\beta)^T]^{\mathrm{vir}}} \frac{1}{e(N^{\mathrm{vir}})} \prod_{i=1}^m \tau_{\alpha_i}(\sigma_i)\big|_{P_\chi(X,\beta)^T}. \tag{1}$$

Many of these invariants vanish:

Theorem 1.1. *Suppose that S has a reduced, irreducible canonical curve. Then*

$$P_{\chi,\beta}\big(X, \tau_{\alpha_1}(\sigma_1)\cdots\tau_{\alpha_m}(\sigma_m)\big) \;=\; 0$$

unless β is an integer multiple of the canonical class k and no σ_i lies in $H^{\geq 3}(S)$.

[1] We emphasise that in this paper we are concerned with the full stable pair and Gromov-Witten invariants of X, *not* their reduced cousins computed in [KT1, KT2].

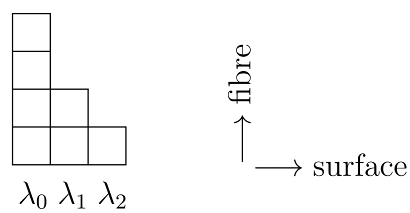
More generally one can localise the calculation of $P_{\chi,\beta}\big(X,\tau_{\alpha_1}(\sigma_1)\cdots\tau_{\alpha_m}(\sigma_m)\big)$ to (thickenings of) C. In the context of Seiberg-Witten and Gromov-Witten theory on S this goes back to ideas of Witten, Taubes and Lee-Parker [LP], formalised in algebraic geometry as Kiem-Li's cosection localisation [KL1, KL2, KL3].

So from now on we consider only S with *a smooth connected canonical divisor.*[2] Because of Theorem 1.1 we need only work with curve classes $\beta = d\mathsf{k}$, $d\in\mathbb{Z}_{>0}$, which are integer multiples of the canonical class. We use cosection localisation to further localise the T-fixed moduli space P_X^T to thickenings of C indexed by partitions $\boldsymbol{\lambda}=(\lambda_0,\lambda_1,\dots,\lambda_{l-1})$ with $\lambda_0\geq\ldots\geq\lambda_{l-1}>0$ and $|\boldsymbol{\lambda}|=\sum\lambda_i=d$. The components[3] $P^T_{\boldsymbol{\lambda}C}$ of the localised moduli space parameterise stable pairs with support $\boldsymbol{\lambda}C$ defined by the ideal sheaf

$$\mathcal{O}(-\lambda_0S)+I_C(-\lambda_1S)+I_C^2(-\lambda_2S)+\ldots+I_C^{l-1}(-\lambda_{l-1}S)+I_C^l.$$

Here $\mathcal{O}_X(-S)$ is the ideal of the zero section of K_S, and $I_C=\pi^*\mathcal{O}(-C)$ is the ideal sheaf of π^*C.

Example 1.2. The partition $\boldsymbol{\lambda}=(4,2,1)$ corresponds to the thickening $\boldsymbol{\lambda}C$ which, transverse to C, looks like



In fact only *strict* partitions $(\lambda_1>\ldots>\lambda_l)$ like this one contribute.

Theorem 1.3. *The integrals* (1) *can be localised to integrals over the moduli spaces* $P^T_{\boldsymbol{\lambda}C}\subset P_\chi(X,d\mathsf{k})$ *with* $\boldsymbol{\lambda}\vdash d$ *a* **strict** *partition of* $d=|\boldsymbol{\lambda}|$.

We form the generating function

$$\mathsf{Z}^P_{d\mathsf{k}}(X,\tau_{\alpha_1}(\sigma_1)\cdots\tau_{\alpha_m}(\sigma_m)) := \sum_{\chi\in\mathbb{Z}} P_{\chi,d\mathsf{k}}\big(X,\tau_{\alpha_1}(\sigma_1)\cdots\tau_{\alpha_m}(\sigma_m)\big)q^\chi \tag{2}$$

in $\mathbb{Q}[t,t^{-1}](\!(q)\!)$, and let

$$\mathsf{Z}^P_{d\mathsf{k}}(X,\tau_{\alpha_1}(\sigma_1)\cdots\tau_{\alpha_m}(\sigma_m))_{\mathrm{ver}} \tag{3}$$

denote the contribution from length-1 partitions $\boldsymbol{\lambda}=(d)$ in Theorem 1.3. This is a generating series of integrals over moduli spaces of stable pairs whose

[2]In fact all we require, by the deformation invariance of stable pair and Gromov-Witten invariants, is that some deformation of S should have such a divisor.

[3]By convention a component means a union of connected components.

support has ideal $I_C(-dS)$. In particular they are contained in $\pi^{-1}(C)$ and we call them "vertical". The main result of this paper is an algorithm for the computation of (3) (Remark 12.1) and a closed formula when all the insertions σ_i are H^2 classes. We let D_i denote their Poincaré dual classes (these are *any* $H_2(S,\mathbb{Q})$ classes, not necessarily divisors).

Theorem 1.4. *Suppose that S has a smooth irreducible canonical curve of genus $h=\mathsf{k}^2+1$. Set $Q:=-q$ and $|\alpha|:=\alpha_1+\ldots+\alpha_m$. Without descendents, $\mathsf{Z}^P_{d\mathsf{k}}(X)_{\mathrm{ver}}$ equals*

$$(-1)^{\chi(\mathcal{O}_S)}\left(\frac{(-1)^d}{d^{d-1}}\right)^{h-1}\left(Q^{\frac d2}-Q^{-\frac d2}\right)^{2h-2}\prod_{i=1}^{d-1}\left((d-i)Q^{\frac d2}-dQ^{\frac d2-i}+iQ^{-\frac d2}\right)^{h-1}.$$

Adding descendents, $\mathsf{Z}^P_{d\mathsf{k}}(X,\tau_{\alpha_1}(D_1)\cdots\tau_{\alpha_m}(D_m))_{\mathrm{ver}}$ equals

$$\mathsf{Z}^P_{d\mathsf{k}}(X)_{\mathrm{ver}}\,(dt)^{|\alpha|}\prod_{i=1}^m\frac{(d\mathsf{k}\cdot D_i)}{(\alpha_i+1)!}\;\frac{Q^{\frac d2(\alpha_i+1)}-Q^{-\frac d2(\alpha_i+1)}}{\left(Q^{\frac d2}-Q^{-\frac d2}\right)^{\alpha_i+1}}\,.$$

Remark 1.5. We deduce that $\mathsf{Z}^P_{d\mathsf{k}}(X,\tau_{\alpha_1}(D_1)\cdots\tau_{\alpha_m}(D_m))_{\mathrm{ver}}$ is invariant under $q\leftrightarrow q^{-1}$ up to a factor $(-1)^{|\alpha|}$. In particular we get invariance under $q\leftrightarrow q^{-1}$ for primary insertions. In the cases $d=1,2$ these expressions calculate the full generating function (2).

The MNOP correspondence [MNOP, PT1] conjectures that the Gromov-Witten and stable pairs theories of X determine one another. This has been upgraded by Pandharipande-Pixton [PP1, PP2] to a correspondence of full descendent theories. This descendent-MNOP conjecture is more complicated than the original MNOP conjecture, involving a certain non-explicit matrix $\mathsf{K}_{\mu,\nu}$. Pandharipande-Pixton have proved their conjecture in many cases, but not for the local general type surfaces of this paper. So in Sections 13, 14 we *assume* the descendent-MNOP correspondence and apply it to our results. Firstly this gives (see Theorem 13.5) the obvious vanishing result analogous to Theorem 1.1 for the Gromov-Witten generating function

$$\mathsf{Z}^{GW}_\beta(X,\tau_{\alpha_1}(\sigma_1)\cdots\tau_{\alpha_m}(\sigma_m))\;\in\;\mathbb{Q}[t,t^{-1}](\!(u)\!).\tag{4}$$

Next we consider the vertical contribution of Theorem 1.4 to the stable pairs generating function for $\beta=d\mathsf{k}$. Pushing it through the descendent-MNOP conjecture we get a contribution to the Gromov-Witten theory which we call

$$\mathsf{Z}^{GW}_{d\mathsf{k}}(X,\tau_{\alpha_1}(D_1)\cdots\tau_{\alpha_m}(D_m))_{\mathrm{ver}}\,,\tag{5}$$

which is the full generating function for $d=1,2$. Its computation has several applications:

- In Theorem 13.7 we prove that (5) is the product of the generating function without insertions $\mathsf{Z}^{GW}_{d\mathsf{k}}(X)_{\text{ver}}$ and a formal Laurent series in u depending only on d, $d\mathsf{k}\cdot D_i$ and the descendence degrees $\alpha_1,\dots,\alpha_m$.
- The lowest order term in u of (4) has coefficient the genus g descendent Gromov-Witten invariant of the surface S

$$N'_{g,\beta}(S,\tau_{\alpha_1}(\sigma_1)\cdots\tau_{\alpha_m}(\sigma_m)) \;:=\; \int_{[\overline{M}'_{g,m}(S,\beta)]^{\text{vir}}}\prod_{j=1}^m \tau_{\alpha_j}(\sigma_j), \tag{6}$$

where g is

$$g \;:=\; 1-\int_\beta c_1(S)-m+\sum_{j=1}^m\Big(\alpha_j+\frac{1}{2}\deg(\sigma_j)\Big).$$

Here $\sigma_j\in H^{\deg\sigma_j}(S,\mathbb{Q})$, and the invariant is zero if this g is not an integer. This invariant (6) satisfies the same vanishing as its 3-fold analog (Corollary 14.1). In the case of no insertions Lee-Parker [LP] proved that (6) is equal to the degree d unramified spin Hurwitz number of C with theta characteristic $K_S|_C$. This result was proved algebro-geometrically by Kiem-Li [KL1, KL2]. The unramified spin Hurwitz numbers were recently computed explicitly using TQFT by S. Gunningham [Gun]. Our vertical contribution correctly reproduces one combinatorial term of his formula (Corollary 14.2). Again this is the whole thing when $d=1,2$

- The descendent-MNOP correspondence involves a universal matrix

$$\mathsf{K}_{\mu\nu}\in\mathbb{Q}[i,w_1,w_2,w_3](\!(u)\!),$$

where μ,ν run over all partitions and $i^2=-1$. Proposition 13.6 shows that for local toric surfaces and $\deg\sigma_i\geq 2$ we only need to know the specialisation

$$\mathsf{K}_{\mu\nu}\Big|_{w_1=t,\,w_2=w_3=0}\quad\text{for } \mu,\nu \text{ of length one.} \tag{7}$$

In this case writing $\mu=(a)$, $\nu=(b)$, the specialisation (7) equals

$$t^{a-b}\cdot f_{ab}(u)\quad\text{for some } f_{ab}(u)\in\mathbb{Q}[i](\!(u)\!)$$

by [PP1]. We conjecture that $f_{ab}(u)$ is a Laurent *monomial* of degree $1-a$ (Conjecture 14.4). Assuming this we show the $f_{ab}(u)$ are uniquely determined by the fact that the GW generating function (4) starts in the correct degree. They then uniquely determine the surface invariants (6), confirming (Corollary 14.5) old conjectural formulae of Maulik-Pandharipande [MP]. Maulik-Pandharipande's formulae were first proved on the GW side by Kiem-Li [KL1, KL2] and later J. Lee in symplectic geometry [Lee]. Our calculation via descendent-MNOP requires a combinatorial identity we found experimentally in Maple and Mathematica, and which is proved in Appendix by A. Pixton and D. Zagier (Theorem A.1).

Remark 1.6. This paper only considers the vertical component of the zero locus of the cosection in P_X^T. In a sequel [KT4] we calculate the contribution of the other components in the case of *bare curves* (i.e. minimal χ, so that the stable pairs have no cokernel or "free points"). This turns out to explain part of the structure of S. Gunningham's formula [Gun] from the stable pairs point of view.

Relations to older work. The results of this paper can be seen as being precisely *orthogonal* to the earlier work [KT1, KT2] on reduced classes. There we also considered stable pair invariants on $X = K_S$ for S a surface with holomorphic 2-forms: $h^{2,0}(S) > 0$. But we worked only with effective curve classes β for which the Noether-Lefschetz locus has the *expected codimension* $h^{0,2}(S)$. The standard invariants (GW, stable pairs) therefore vanish, and we get interesting *reduced* invariants only by reducing the obstruction bundle in a canonical way.

Here we study the standard (non-reduced) GW/stable pairs invariants. These need not vanish for curve classes whose Noether-Lefschetz locus has the "wrong" codimension. We find the only classes which contribute are multiples of the canonical class k (whose Noether-Lefschetz locus has codimension 0).

The papers [KT1, KT2] also focused on the *horizontal component* of the moduli space of stable pairs. This is the only component relevant for (sufficiently ample) enumerative problems on S such as Göttsche's conjecture. Here the horizontal component does not contribute to the invariants and we study the *vertical component* instead. There we derived universality results but no closed formula. Here we obtain a closed product formula when all insertions come from H^2 classes.

Plan. We localise $[P_X]^{\mathrm{vir}}$ first to its T-fixed locus, in Section 2, then further to pairs supported on thickenings of a canonical curve C in Section 3. This will be enough to prove Theorem 1.1. The moduli space of T-fixed pairs supported on a *vertically thickened* smooth curve C is identified with a nested Hilbert scheme of C in Section 4. Section 5 expresses the virtual cycle as a cycle on this nested Hilbert scheme. This is further simplified to an expression on a single symmetric product $\mathrm{Sym}^{n_0} C$ in Section 6. In Sections 7 and 8 we see how the virtual normal bundle and descendent integrands simplify on $\mathrm{Sym}^{n_0} C$. This allows us to compute the integrals in Sections 9 and 10 and derive Theorem 1.4. The formulae are are rather lengthy and complicated at each stage, until right at the end they are summed up into a mysteriously simple closed product formula. This suggests one should work with the generating series, rather than individual invariants, from the beginning, but we have not found a way to do this. Finally Sections 13 and 14 discuss applications to the Gromov-Witten invariants of X and S respectively.

Notation. Given any map $f\colon A \to B$ we also use f for the induced map $f \times \mathrm{id}_C\colon A \times C \to B \times C$. We suppress various pullback maps for clarity of exposition. We denote the cohomology class Poincaré dual to a cycle A by $[A]$. We use $^\vee$ for derived dual of complexes, and * for the underived dual $\mathscr{H}om(\,\cdot\,, \mathcal{O})$ of coherent sheaves.

We use the standard conventions for (possibly negative) binomial coefficients. That is

$$\binom{n}{k} \text{ is defined to be } (-1)^k\binom{k-n-1}{k} \text{ when } n<0, k \geq 0, \tag{8}$$

and it is defined to be zero whenever $k < 0$ or $k > n \geq 0$. The binomial theorem $(1+x)^n = \sum_{k\geq 0}\binom{n}{k}x^k$ then holds for any $n \in \mathbb{Z}$.

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2. T-localised stable pair theory

Let $P_X := P_\chi(X, \iota_*\beta)$ denote the moduli space of stable pairs (F, s) on X. It is a quasi-projective scheme whose product with X,

$$\begin{array}{ccc} & P_X \times X & \\ {\scriptstyle \pi_P} \swarrow & & \searrow {\scriptstyle \pi_X} \\ P_X & & X, \end{array}$$

carries a universal sheaf $\mathbb{F}$, section s and universal complex

$$\mathbb{I}^\bullet = \{\mathcal{O} \longrightarrow \mathbb{F}\}.$$

The action of T on X induces one on P_X with respect to which $\mathbb{F}$ and $\mathbb{I}^\bullet$ are T-equivariant. Since the T-fixed locus

$$P_X^T \subset P_X$$

is compact we may use virtual localisation [GP] to define stable pair invariants of X via residue integrals over the virtual cycle of P_X^T.

To describe the virtual cycle, we view stable pairs (F, s) as objects $I^\bullet := \{\mathcal{O}_X \xrightarrow{s} F\}$ of $D(X)$ of trivial determinant as in [PT1]. Then P_X acquires a T-equivariant perfect symmetric obstruction theory [HT, Theorem 4.1]

$$E^\bullet := R\mathscr{H}om_{\pi_P}(\mathbb{I}^\bullet, \mathbb{I}^\bullet)_0^\vee[-1] \longrightarrow \mathbb{L}_{P_X} \tag{9}$$

with obstruction sheaf

$$\mathrm{Ob}_X := \mathscr{E}xt^2_{\pi_P}(\mathbb{I}^\bullet, \mathbb{I}^\bullet)_0.$$

Here $(\,\cdot\,)_0$ denotes trace-free part. By [GP] the T-fixed locus P_X^T inherits a perfect obstruction theory

$$\big(R\mathscr{H}om_{\pi_P}(\mathbb{I}^\bullet, \mathbb{I}^\bullet)_0^f\big)^\vee[-1] \longrightarrow \mathbb{L}_{P_X^T} \tag{10}$$

with obstruction sheaf

$$\Big(\mathrm{Ob}_X\big|_{P_X^T}\Big)^f. \tag{11}$$

Here $(\,\cdot\,)^f$ denotes the T-fixed part: the weight-0 part of the complex.

The obstruction theory (10) defines a virtual cycle on P_X^T by [BF, LT]. The T-localised invariants of X are defined by integrating insertions against the cap product of $e(N^{\mathrm{vir}})^{-1}$ with this virtual cycle. Here the virtual normal bundle $N^{\mathrm{vir}} = \{V_0 \to V_1\}$ is defined to be the part of (10) with nonzero weights and

$$e(N^{\mathrm{vir}}) := \frac{c_{\mathrm{top}}^T(V_0)}{c_{\mathrm{top}}^T(V_1)} \in H_T^*(P_X^T, \mathbb{Q}) \otimes_{\mathbb{Q}[t]} \mathbb{Q}[t, t^{-1}] \cong H^*(P_X^T, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{Q}[t, t^{-1}]$$

is its T-equivariant virtual Euler class.[4] As usual

$$t := c_1(\mathfrak{t}) \in H^*(BT, \mathbb{Q}) \cong \mathbb{Q}[t]$$

denotes the first Chern class of the standard weight-1 representation $\mathfrak{t}$ of T, the generator of the equivariant cohomology of BT.

In this paper we are interested in descendent insertions. The sheaf $\mathbb{F}$ is T-equivariant, so we can consider its T-equivariant Chern classes

$$\mathrm{ch}_i^T(\mathbb{F}) \in H_T^*(P_X, \mathbb{Q}).$$

[4]We may choose the V_i to be T-equivariant vector bundles with no weight-0 parts, so that the $c_{\mathrm{top}}^T(V_i)$ are invertible in $H_T^*(P_X^T, \mathbb{Q}) \otimes_{\mathbb{Q}[t]} \mathbb{Q}[t, t^{-1}]$.

Given any $\sigma \in H^*(S,\mathbb{Q})$, we consider it as lying in $H^*_T(X,\mathbb{Q})$ (or its localization at t) by identifying it with the element

$$\sigma \otimes 1 \in H^*(S,\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{Q}[t] \;\cong\; H^*_T(S,\mathbb{Q}) \xrightarrow[\sim]{\pi^*} H^*_T(X,\mathbb{Q}). \tag{12}$$

Then for any integer $\alpha \geq 0$, define

$$\tau_\alpha(\sigma) := \pi_{P*}\big(\pi_X^*\sigma \cap \mathrm{ch}^T_{\alpha+2}(\mathbb{F})\big) \;\in\; H^*_T(P_X,\mathbb{Q}). \tag{13}$$

The descendent invariants of X are

$$P_{\chi,\beta}\big(X,\tau_{\alpha_1}(\sigma_1)\cdots\tau_{\alpha_m}(\sigma_m)\big) := \int_{[P_\chi(X,\beta)^T]^{\mathrm{vir}}} \frac{1}{e(N^{\mathrm{vir}})} \prod_{i=1}^m \tau_{\alpha_i}(\sigma_i)\big|_{P_\chi(X,\beta)^T} \tag{14}$$

in $\mathbb{Q}[t,t^{-1}]$.

3. The cosection

Let $\theta \in H^0(K_S)$ be a nonzero holomorphic 2-form with zero divisor C. We construct a natural induced cosection of the obstruction sheaf $\big(\mathrm{Ob}_X|_{P_X^T}\big)^f$. To use Serre duality it is convenient to compactify X,

$$X \subset \overline{X} := \mathbb{P}(K_S \oplus \mathcal{O}_S),$$

and use the projections

$$P_X \xleftarrow{\;\overline{\pi}_P\;} P_X \times \overline{X} \xrightarrow{\;\pi_{\overline{X}}\;} \overline{X}.$$

The universal stable pair $\mathbb{I}^\bullet = \{\mathcal{O}_{P_X\times X} \to \mathbb{F}\}$ pushes forward to a universal stable pair

$$\mathbb{I}^\bullet_{\overline{X}} := \big\{\mathcal{O}_{P_X\times\overline{X}} \longrightarrow j_*\mathbb{F}\big\}$$

on $P_X \times \overline{X}$. Since $\mathbb{I}^\bullet_{\overline{X}}$ is isomorphic to $\mathcal{O}$ away from the support of $\mathbb{F}$ in X, and since $\omega_{\overline{X}}$ is also trivial on restriction to X, we see that

$$R\mathscr{H}om\big(\mathbb{I}^\bullet_{\overline{X}},\mathbb{I}^\bullet_{\overline{X}}\otimes\omega_{\overline{X}}\big)_0 \;\cong\; R\mathscr{H}om\big(\mathbb{I}^\bullet_{\overline{X}},\mathbb{I}^\bullet_{\overline{X}}\big)_0 \;=\; j_*\,R\mathscr{H}om\big(\mathbb{I}^\bullet,\mathbb{I}^\bullet\big)_0\,.$$

Pushing down by $\overline{\pi}_P$ gives

$$R\mathscr{H}om_{\overline{\pi}_P}\big(\mathbb{I}^\bullet_{\overline{X}},\mathbb{I}^\bullet_{\overline{X}}\otimes\omega_{\overline{X}}\big)_0 \;\cong\; R\mathscr{H}om_{\overline{\pi}_P}\big(\mathbb{I}^\bullet_{\overline{X}},\mathbb{I}^\bullet_{\overline{X}}\big)_0 \;=\; R\mathscr{H}om_{\pi_P}\big(\mathbb{I}^\bullet,\mathbb{I}^\bullet\big)_0\,. \tag{15}$$

Translation by θ up the K_S fibres of $X \to S$ defines a vector field v_θ on X, vanishing only on the preimage of $C \subset S$. Translating stable pairs by v_θ defines a vector field V_θ on P_X:

$$V_\theta \;=\; v_\theta \lrcorner \mathrm{At}(\mathbb{I}^\bullet) \;\in\; \Gamma\big(\mathscr{E}xt^1_{\pi_P}(\mathbb{I}^\bullet,\mathbb{I}^\bullet)_0\big). \tag{16}$$

Pairing with the obstruction sheaf using (15) defines a map

$$\begin{aligned} \mathscr{E}xt^2_{\pi_P}\big(\mathbb{I}^\bullet,\mathbb{I}^\bullet\big)_0 &\xrightarrow{V_\theta\otimes 1} \mathscr{E}xt^1_{\pi_P}\big(\mathbb{I}^\bullet,\mathbb{I}^\bullet\big)_0\otimes\mathscr{E}xt^2_{\pi_P}\big(\mathbb{I}^\bullet,\mathbb{I}^\bullet\big)_0 \\ &\cong \mathscr{E}xt^1_{\overline{\pi}_P}\big(\mathbb{I}^\bullet_{\overline{X}},\mathbb{I}^\bullet_{\overline{X}}\otimes\omega_{\overline{X}}\big)_0\otimes\mathscr{E}xt^2_{\overline{\pi}_P}\big(\mathbb{I}^\bullet_{\overline{X}},\mathbb{I}^\bullet_{\overline{X}}\big)_0 \\ &\xrightarrow{\cup} \mathscr{E}xt^3_{\overline{\pi}_P}\big(\mathbb{I}^\bullet_{\overline{X}},\mathbb{I}^\bullet_{\overline{X}}\otimes\omega_{\overline{X}}\big)\xrightarrow{\mathrm{tr}} R^3\overline{\pi}_{P*}\omega_{\overline{X}}\ \cong\ \mathcal{O}_{P_X}. \end{aligned} \tag{17}$$

In the last Section we localised to the fixed locus $P_X^T\subset P_X$. Restricting (17) to P_X^T and taking fixed (weight 0) parts gives a *cosection*

$$\sigma_\theta\colon\ \Big(\mathrm{Ob}_X\big|_{P_X^T}\Big)^f\longrightarrow\mathcal{O}_{P_X^T}. \tag{18}$$

Its zero locus inherits a scheme structure from the cokernel of (18).

Basechange issues[5] make it nontrivial to equate the zero scheme of the cosection σ_θ with the zero scheme of vector field V_θ (16). The correct formulation involves restricting V_θ to any subscheme $Z\subset P_X^T$ by first taking its image in the sheaf $\mathscr{E}xt^1_{\pi_P}(\mathbb{I}^\bullet,\mathbb{I}^\bullet)_0|_Z$ then further restricting to $\mathscr{E}xt^1_{\pi_P^Z}(\mathbb{I}^\bullet|_{Z\times X},\mathbb{I}^\bullet|_{Z\times X})_0$, where $\pi_P^Z\colon Z\times X\to Z$ is the restriction of $\pi_P\colon P_X\times X\to P_X$. Equivalently, but more directly, we just set

$$V_{\theta,Z}\ :=\ v_\theta\,\lrcorner\,\mathrm{At}(\mathbb{I}^\bullet|_{Z\times X})\ \in\ \Gamma\Big(\mathscr{E}xt^1_{\pi_P^Z}\big(\mathbb{I}^\bullet|_{Z\times X},\mathbb{I}^\bullet|_{Z\times X}\big)_0\Big). \tag{19}$$

It is a P_X-vector field on Z (so it need not be tangent to Z).

Lemma 3.1. *The zero locus $Z(\sigma_\theta)$ of the cosection* (18) *is the largest subscheme $Z\subset P_X^T$ for which $V_{\theta,Z}$* (19) *is identically zero.*

Proof. By basechange and the vanishing of the higher $(\mathscr{E}xt_{\pi_P})_0$s, we have

$$\mathrm{Ob}_X^f\big|_Z=\Big(\mathscr{E}xt^2_{\pi_P^Z}(\mathbb{I}^\bullet|_{Z\times X},\mathbb{I}^\bullet|_{Z\times X})_0\Big)^f, \tag{20}$$

and the restriction of the cosection (18) to Z is the map

$$\mathrm{Ob}_X^f\big|_Z\longrightarrow\mathcal{O}_Z \tag{21}$$

given by restricting (17) to Z. It follows that the zero locus $Z(\sigma_\theta)$ is the largest Z for which this map vanishes.

The map (21) is therefore the pairing with the section $V_{\theta,Z}$ (19) of

$$\Big(\mathscr{E}xt^1_{\pi_P^Z}\big(\mathbb{I}^\bullet|_{Z\times X},\mathbb{I}^\bullet|_{Z\times X}\otimes\omega_X\big)_0\Big)^f. \tag{22}$$

[Though V_θ has T-weight 1, the identification in the second line of (17) multiplies by the weight -1 trivialisation of $\omega_{\overline{X}}|_X$, giving a T-fixed section.] But this pairing makes the coherent sheaf (22) the dual $\mathscr{H}om(\mathrm{Ob}_X^f|_Z,\mathcal{O}_Z)$ of the

[5] $\mathscr{E}xt^1_{\pi_P}$ does not basechange well, but we will be able to use that fact that $\mathscr{E}xt^2_{\pi_P}$ does.

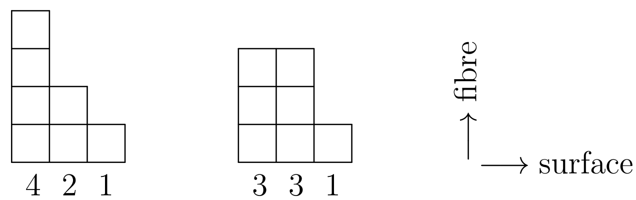
sheaf $\mathrm{Ob}^f_X|_Z$ (20), by relative Serre duality for the map $\overline{\pi}^Z_P$, its compatibility with the T-action, and the vanishing of the other $\mathscr{E}xt_0$s. Therefore $Z(\sigma_\theta)$ is the largest $Z \subset P^T_X$ for which the section $V_{\theta,Z}$ vanishes, as claimed. □

From now on we assume C is reduced and irreducible. To describe a subscheme of P^T_X containing the zero scheme of the cosection (18) we need some notation. For any (finite, 2-dimensional) partition $\boldsymbol{\lambda} = (\lambda_0 \geq \lambda_1 \geq \cdots \geq \lambda_{l-1})$, we denote by $\boldsymbol{\lambda}C \subset X$ the Cohen-Macaulay curve defined by the T-invariant ideal sheaf

$$I_{\boldsymbol{\lambda}C} := \mathcal{O}(-\lambda_0 S) + I_C(-\lambda_1 S) + I_C^2(-\lambda_2 S) + \ldots + I_C^{l-1}(-\lambda_{l-1}S) + I_C^l.$$

Here $I_C = \pi^*\mathcal{O}_S(-C)$ is the ideal sheaf of π^*C, and $\mathcal{O}(-S) \cong K_S^{-1} \otimes \mathfrak{t}^{-1}$ is the ideal sheaf of the zero section $S \subset K_S = X$.

Example 3.2. The partitions $\boldsymbol{\lambda} = (4,2,1)$ and $\boldsymbol{\lambda} = (3,3,1)$ of 7 give the following two thickenings $\boldsymbol{\lambda}C$ of total size $|\boldsymbol{\lambda}| = \sum \lambda_i = 7$.



The first is *strict*: $\lambda_0 > \lambda_1 > \cdots > \lambda_{l-1} > 0$, while the second is not.

We can slice horizontally instead of vertically. If $\boldsymbol{\lambda}^t = (\mu_0, \mu_1, \cdots)$ denotes the transpose partition, we can think of $\boldsymbol{\lambda}C$ as the curve obtained by thickening C to order μ_0 at T-weight level 0, μ_1 at T-weight level -1, etc.:

$$I_{\boldsymbol{\lambda}C} = I_C^{\mu_0} + I_C^{\mu_1}(-S) + I_C^{\mu_2}(-2S) + \ldots + I_C^{\mu_{k-1}}(-(k-1)S) + O(-kS).$$

In the above Example 3.2, the transposed partitions $\boldsymbol{\lambda}^t$ are $(3,2,1,1)$ and $(3,2,2)$ respectively.

If $\boldsymbol{\lambda}$ has size $|\boldsymbol{\lambda}| = \sum \lambda_i = d$ we write $\boldsymbol{\lambda} \vdash d$. We fix χ throughout this Section and denote by

$$P_{\boldsymbol{\lambda}C} := P_\chi(\boldsymbol{\lambda}C) \subset P_\chi(X, d[C])$$

the moduli space of stable pairs with holomorphic Euler characteristic χ whose scheme-theoretic support is precisely $\boldsymbol{\lambda}C$. Since $\boldsymbol{\lambda}C$ is T-invariant, $P_{\boldsymbol{\lambda}C}$ has a T-action and its fixed locus is a closed subscheme

$$P^T_{\boldsymbol{\lambda}C} = P_{\boldsymbol{\lambda}C} \cap P_\chi(X, d[C])^T.$$

We will find that the support of stable pairs in the zero locus $Z(\sigma_\theta)$ of the cosection have support $\boldsymbol{\lambda}C$ for $\boldsymbol{\lambda}$ *strict*.

Proposition 3.3. *The zero scheme $Z(\sigma_\theta)$ of the cosection* (18) *is nonempty only if $\beta = d[C]$ for some $d > 0$. In this case, it is a closed subscheme of*

$$\bigsqcup_{\boldsymbol{\lambda}\vdash d\text{ strict}} P^T_{\boldsymbol{\lambda}C}\,.$$

Proof. Let $Z := Z(\sigma_\theta)$ and let s denote the tautological section of π^*K_S cutting out the zero section $S \subset X$. We use T-invariance to write the ideal sheaf of the support of $\mathbb{F}|_{Z\times X}$ in the form

$$\begin{gathered}\pi^*I_0 + \pi^*I_1.s + \cdots + \pi^*I_{k-1}.s^{k-1} + (s^k),\\ I_0 \subset I_1 \subset \cdots \subset I_{k-1} \subset \mathcal{O}_{Z\times S},\end{gathered} \tag{23}$$

for some integer $k > 0$.

Let t denote the coordinate on $\mathbb{C}_t := \mathbb{C}$. Then pulling back $\mathbb{I}^\bullet|_{Z\times X}$ to $Z \times X \times \mathbb{C}_t$ and translating by tv_θ gives a new family of stable pairs over $Z\times X\times\mathbb{C}_t$ whose support is defined by the ideal

$$\pi^*I_0 + \pi^*I_1.(s - t\pi^*\theta) + \cdots + \pi^*I_{n-1}.(s - t\pi^*\theta)^{k-1} + ((s - t\pi^*\theta)^k). \tag{24}$$

Restricting to Spec $\mathbb{C}[t]/(t^2) \subset \mathbb{C}_t$ gives a flat family of stable pairs on X parameterized by $Z \times$ Spec $\mathbb{C}[t]/(t^2)$ whose support has ideal (24) mod t^2,

$$\pi^*I_0 + \pi^*I_1.(s - t\pi^*\theta) + \cdots + \pi^*I_{k-1}.(s^{k-1} - (k-1)t\pi^*\theta s^{k-2}) + (s^k - kt\pi^*\theta s^{k-1}). \tag{25}$$

The corresponding first order deformation of $\mathbb{I}^\bullet|_{Z\times X}$ is classified by its extension class

$$V_{\theta,Z} \;\in\; \operatorname{Ext}^1\big(\mathbb{I}^\bullet|_{Z\times X}, \mathbb{I}^\bullet|_{Z\times X}\big)_0 \;=\; \Gamma\big(\mathscr{E}xt^1_{\pi_P|_Z}(\mathbb{I}^\bullet|_{Z\times X}, \mathbb{I}^\bullet|_{Z\times X})_0\big)$$

of (19). By Lemma 3.1 this is zero, so the family is trivial. In particular its support is pulled back from $Z \times X$, so (25) is the same ideal as (23)$\otimes\mathbb{C}[t]/(t^2)$. That is,

$$\theta\cdot I_i \subset I_{i-1},\ \forall i = 1,\ldots,k-1, \qquad \text{and } \theta \in I_{k-1}.$$

Since C is reduced and irreducible, and each $\mathcal{O}/I_i$ is pure, this implies that each $I_i = (\theta^{\mu_i})$ for some integer μ_i, and that $\mu_i + 1 \geq \mu_{i-1}$. Thus we can write (23) as

$$(\theta^{\mu_0}) + (s\theta^{\mu_1}) + \cdots + (s^{k-1}\theta^{\mu_{k-1}}) + (s^k),$$

where we have suppressed some π^*s for clarity. Rewriting this as

$$(s^{\lambda_0}) + (\theta s^{\lambda_1}) + \cdots + (\theta^{l-1}s^{\lambda_{l-1}}) + (\theta^l),$$

where $\boldsymbol{\lambda} = (\lambda_0, \lambda_1, \ldots)$ is the transpose of the partition $(\mu_0, \mu_1, \ldots)$, the condition $\mu_i + 1 \geq \mu_{i-1}$ becomes the requirement that $\boldsymbol{\lambda}$ be strict. $\square$

So in Example 3.2 we find that $P^T_{\boldsymbol{\lambda}C}$ contains zeros of the cosection when $\boldsymbol{\lambda}=(4,2,1)$, but not when $\boldsymbol{\lambda}=(3,3,1)$.

We have only considered the effect of the cosection on the underlying Cohen-Macaulay support curve of a stable pair, showing it forces it to be of the form $\boldsymbol{\lambda}C$ with $\boldsymbol{\lambda}$ strict. The proof also shows that bare curves of this form (i.e. a stable pair isomorphic to $(\mathcal{O}_{\boldsymbol{\lambda}C},1)$ with no cokernel of "free points") lie in $Z(\sigma_\theta)$. For more general stable pairs, being in $Z(\sigma_\theta)$ also imposes conditions on its cokernel; see the sequel [KT4] for more details.

In this paper we content ourselves with a characterization of *vertical component* of $Z(\sigma_\theta)$, where $\boldsymbol{\lambda}=(d)$ has length 1. Here there is no further condition on the cokernels of stable pairs.

Corollary 3.4. *The zero scheme $Z(\sigma_\theta)$ of the cosection* (18) *on $P^T_{d[C]}$ has a component*

$$P^T_{(d)C}:=P^T_{\boldsymbol{\lambda}C},\quad \boldsymbol{\lambda}=(d).$$

Proof. The vector field v_θ vanishes on $\pi^*C\subset X$. As a consequence the vector field V_θ vanishes on $P^T_{(d)C}$ which therefore lies in the zero scheme Z of the cosection (18). By Proposition 3.3 it is a whole component of Z. (In fact we will see in Proposition 4.1 it is a disjoint union of connected components.) $\square$

Corollary 3.5. *Assume S has a reduced, irreducible canonical curve C. Then*

$$P_{\chi,\beta}\big(X,\tau_{\alpha_1}(\sigma_1)\cdots\tau_{\alpha_m}(\sigma_m)\big)=0$$

unless $\beta=d\mathsf{k}$ for some $d\in\mathbb{Z}_{>0}$ and no σ_i lies in $H^{\geq 3}(S)$.

Proof. For β not a multiple of $d\mathsf{k}$ then $Z(\sigma_\theta)$ is empty by Proposition 3.3, so the invariants vanish.

If $\sigma\in H^{\geq 3}(S)$, we can write $\sigma=[\gamma]$ for some cycle $\gamma\in H_{\leq 1}(S)$ disjoint from C. Therefore $\pi_X^*\sigma\cap \mathrm{ch}^T_{\alpha+2}(\mathbb{F})=0$ over the locus of pairs with support $\boldsymbol{\lambda}C$, so the insertions $\tau_\alpha(\sigma)$ certainly vanish over $P^T_{\boldsymbol{\lambda}C}$ for any strict $\boldsymbol{\lambda}\vdash d$. Since the virtual cycle can be cosection localised to this locus, the associated invariants vanish. This completes the proof of Theorem 1.1 in the Introduction. $\square$

4. Nested Hilbert schemes

We now begin the process of describing T-fixed stable pairs — especially those in the vertical component $P^T_{(d)C}$ of $Z(\sigma_\theta)$ — more explicitly.

4.1. **T-equivariant sheaves on X.** Given a T-equivariant coherent sheaf F on X, its pushdown by $\pi\colon X\to S$ decomposes into weight spaces:

$$\pi_*F \;=\; \bigoplus_i F_i\otimes \mathfrak{t}^i, \tag{26}$$

where F_i is T-fixed so $F_i \otimes \mathfrak{t}^i$ is the summand of weight i. For instance

$$\pi_* \mathcal{O}_X \;=\; \bigoplus_{i \geq 0} K_S^{-i} \otimes \mathfrak{t}^{-i}. \tag{27}$$

Since π is affine, the pushdown loses no information; we can recover the $\mathcal{O}_X$-module structure on F by describing the action of (27) that (26) carries. This is generated by the action of the weight -1 piece $K_S^{-1} \otimes \mathfrak{t}^{-i}$, so we find that the $\mathcal{O}_X$-module structure is determined by the map

$$\bigoplus_i F_i \otimes \mathfrak{t}^i \otimes \left(K_S^{-1} \otimes \mathfrak{t}^{-1}\right) \longrightarrow \bigoplus_i F_i \otimes \mathfrak{t}^i, \tag{28}$$

which commutes with both the actions of $\mathcal{O}_S$ and T. That is, (28) is a T-equivariant map of $\mathcal{O}_S$-modules. By T-equivariance, it is a sum of maps

$$F_i \otimes K_S^{-1} \longrightarrow F_{i-1}. \tag{29}$$

4.2. ***T*-equivariant pairs on *X*.** Having described T-equivariant coherent sheaves F on X as graded sheaves (26) on S with T-equivariant maps (29), we can generate a similar description of T-equivariant pairs (F, s) on X. Here $s \in H^0(F)^T$ is a T-equivariant section of F.

Applying π_* to $\mathcal{O}_X \xrightarrow{s} F$ gives a graded map between (27) and (26),

$$\begin{array}{ccccccccccc} & & & & \mathcal{O}_S & \oplus & \left(K_S^{-1} \otimes \mathfrak{t}^{-1}\right) & \oplus & \left(K_S^{-2} \otimes \mathfrak{t}^{-2}\right) & \oplus & \cdots \\ & & & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & F_1 & \oplus & & F_0 & \oplus & \left(F_{-1} \otimes \mathfrak{t}^{-1}\right) & \oplus & \left(F_{-2} \otimes \mathfrak{t}^{-2}\right) & \oplus & \cdots, \end{array} \tag{30}$$

which commutes with the maps (29) along the top and bottom rows. So writing

$$G_i := F_{-i} \otimes K_S^i\,, \tag{31}$$

(which is T-fixed) we find the data (F, s) on X is equivalent to the following data of sheaves and commuting maps on S:

$$\begin{array}{ccccccccc} & & \mathcal{O}_S & = & \mathcal{O}_S & = & \mathcal{O}_S & = & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots G_{-1} & \longrightarrow & G_0 & \longrightarrow & G_1 & \longrightarrow & G_2 & \longrightarrow & \cdots\ . \end{array} \tag{32}$$

4.3. ***T*-equivariant stable pairs in the vertical component.** In Section 4.2 we gave a general description of T-equivariant pairs on X. Now we restrict attention to T-equivariant *stable* pairs (F, s) whose scheme theoretic support is $\pi^{-1}C$ for some fixed *connected smooth* curve $C \subset S$. This will lead to a description of the connected component $P^T_{(d)C}$ of $Z(\sigma_\theta)$ of Corollary 3.4. We only consider pairs with proper support, which implies that there is a maximal

$d \geq 0$ such that $G_{d-1} \neq 0$ in the description (31). (This is the smallest d such that F is supported on $dS \subset X$.)

Thus F is pushed forward from $\pi^{-1}(C)$ and $\mathcal{O}_X \xrightarrow{s} F$ has finite cokernel. Thus all of the sheaves G_i in (32) are supporeted on C, the vertical maps factor through $\mathcal{O}_S \to \mathcal{O}_C$, and generically on C the induced maps from $\mathcal{O}_C$ are isomorphisms. It follows in particular that G_{-i} is 0-dimensional for $i > 0$ and so vanishes by purity of F.

The upshot is that the stable pair is equivalent to a commutative diagram

$$\begin{array}{ccccccccc} \mathcal{O}_C & = & \mathcal{O}_C & = & \mathcal{O}_C & = & \cdots & = & \mathcal{O}_C \\ \downarrow & & \downarrow & & \downarrow & & & & \downarrow \\ G_0 & \to & G_1 & \to & G_2 & \to & \cdots & \to & G_{d-1} \end{array} \tag{33}$$

of $\mathcal{O}_C$-modules, with each G_n pure 1-dimensional and each vertical map an isomorphism away from a finite number of points.

Since C is smooth, it follows that each G_i is a line bundle with section, that the horizontal maps are all injections, and the diagram is the top two rows of

$$\begin{array}{ccccccccc} \mathcal{O}_C & = & \mathcal{O}_C & = & \mathcal{O}_C & = & \cdots & = & \mathcal{O}_C \\ \downarrow & & \downarrow & & \downarrow & & & & \downarrow \\ \mathcal{O}(Z_0) & \hookrightarrow & \mathcal{O}(Z_1) & \hookrightarrow & \mathcal{O}(Z_2) & \hookrightarrow & \cdots & \hookrightarrow & \mathcal{O}(Z_{d-1}) \\ \downarrow & & \downarrow & & \downarrow & & & & \downarrow \\ \mathcal{O}_{Z_0}(Z_0) & \hookrightarrow & \mathcal{O}_{Z_1}(Z_1) & \hookrightarrow & \mathcal{O}_{Z_2}(Z_2) & \hookrightarrow & \cdots & \hookrightarrow & \mathcal{O}_{Z_{d-1}}(Z_{d-1}). \end{array} \tag{34}$$

Here the Z_i are Cartier divisors on C, and all columns are the obvious short exact sequences.

4.4. **Stable pairs and the nested Hilbert scheme.** Thus a T-equivariant stable pair (F, s) with proper support in $\pi^{-1}(C)$ is equivalent to a chain of divisors

$$Z_0 \subset Z_1 \subset Z_2 \subset \cdots \subset Z_{d-1} \subset C. \tag{35}$$

Hence it defines a point of the nested Hilbert scheme

$$C^{[\mathbf{n}]}, \qquad \mathbf{n} = (n_0, \ldots, n_{d-1}),$$

of length-n_i zero-dimensional subschemes Z_i of C satisfying the nesting condition (35). Here

$$\chi(F) \ = \ \sum_{i \geq 0} \chi(F_{-i}) \ = \ \sum_{i \geq 0} \chi(G_i \otimes K_S^{-i}|_C) \ = \ \sum_{i \geq 0} \big(\chi(K_S^{-i}|_C) + n_i\big) \tag{36}$$

determines $|\mathbf{n}| = \sum n_i$. When $C \in |K_S|$ is a canonical curve, we find that

$$\chi(F) = \sum_i \big(n_i - (i+1)\mathsf{k}^2\big), \tag{37}$$

where $\mathsf{k} := c_1(K_S)$.

Conversely, a point of the nested Hilbert scheme gives a diagram (34), which we have noted is equivalent to a T-fixed stable pair on X supported on $\pi^{-1}(C)\cap dS$. Thus we get a set-theoretic isomorphism

$$P^T_{\chi,(d)C} \;=\; \bigsqcup_{\mathbf{n}} C^{[\mathbf{n}]}, \tag{38}$$

where the disjoint union is taken over all $\mathbf{n} = (n_0, \ldots, n_{d-1})$ whose length $|\mathbf{n}|$ satisfies

$$\chi = \sum_i \big(n_i - (i+1)\mathsf{k}^2\big). \tag{39}$$

Proposition 4.1. *The bijection* (38) *is an isomorphism of schemes.*

Proof. We simply notice that the constructions of this Section work equally well for T-equivariant sheaves and stable pairs on $X \times B$, flat over any base B.

Pushing down by the affine map $\pi\colon X \times B \to S \times B$ gives a graded sheaf $\bigoplus_i F_i$ on $S \times B$. It is flat over B, therefore so are all its weight spaces F_i. The original sheaf F on $X \times B$ can be reconstructed from the maps (29). Therefore a T-equivariant stable pair (F, s) on $X \times B$, flat over B, is equivalent to the data (32) with each G_i flat over B.

When F is supported on $\pi^{-1}(C \times B)$, with C a smooth connected curve in S, we showed that each G_i is a line bundle on any closed fibre $C \times \{b\}$ (where $b \in B$). Being locally free is an open condition on sheaves, so this shows that each G_i is a line bundle on $C \times B$. Together with its nonzero section (32) we find it defines a divisor $Z_i \subset C \times B$, flat over B.

Thus we get the diagram (34) of flat sheaves and nested divisors over B. This defines a classifying morphism $B \to \bigsqcup_{\mathbf{n}} C^{[\mathbf{n}]}$.

Conversely, the universal family on $C^{[\mathbf{n}]}$ defines a diagram (34), equivalent to a T-equivariant stable pair (F, s) on $X \times B$ supported on

$$(\pi^{-1}(C) \cap dS) \times B$$

and flat over B. This defines the inverse classifying map $\bigsqcup_{\mathbf{n}} C^{[\mathbf{n}]} \to B$. □

4.5. **The dual description.** In this Section we give an explicit description of the pairs constructed in the last Section in terms of the geometry of the vertical thickening $(d)C \subset dS \subset X$. For clarity of exposition we work at a single point of moduli space, though just as in the last Section there is no difficulty in having everything vary in a flat family over a base B.

So we fix a point of $C^{[\mathbf{n}]}$, i.e. an increasing flag of effective divisors

$$Z_0 \subset Z_1 \subset Z_2 \subset \cdots \subset Z_{d-1} \subset C \tag{40}$$

as in (35). Setting $D_i := Z_{d-1} - Z_i$ gives a dual *decreasing* flag of effective divisors

$$D_0 \supset D_1 \supset D_2 \supset \cdots \supset D_{d-2}, \qquad D_{d-1} = \emptyset, \tag{41}$$

in C. These fit together to define a *Weil divisor*[6]

$$D \subset (d)C$$

in the way described in Section 4.2. That is, take $G_i = \mathcal{O}_{D_i}$ in (32) and use the following example of the diagram (33),

$$\begin{array}{ccccccccccc}
\mathcal{O}_S & = & \mathcal{O}_S & = & \mathcal{O}_S & = & \cdots & = & \mathcal{O}_S & = & \mathcal{O}_S \\
\downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow \\
\mathcal{O}_{D_0} & \to & \mathcal{O}_{D_1} & \to & \mathcal{O}_{D_2} & \to & \cdots & \to & \mathcal{O}_{D_{d-2}} & \to & 0.
\end{array}$$

All arrows are the obvious restriction maps. By the construction of Section 4.2 this is equivalent to a T-equivariant pair $\mathcal{O}_X \to G$ with no cokernel, so G must be a structure sheaf $\mathcal{O}_D$ of a subscheme $D \subset (d)C$ such that $\pi_*\mathcal{O}_D$ is

$$\bigoplus_{i=0}^{d-2} \mathcal{O}_{D_i} \otimes K_S^{-i} \otimes \mathfrak{t}^{-i}.$$

Now π^*Z_{d-1} is a Cartier divisor on $(d)C$, defining a line bundle $\mathcal{O}_{(d)C}(\pi^*Z_{d-1})$ with a canonical section vanishing on $\pi^*Z_{d-1} \supset D$. It therefore factors through the ideal sheaf I_D of $D \subset C$, defining a unique section

$$\mathcal{O}_X \xrightarrow{s} \mathcal{O}_{(d)C}(\pi^*Z_{d-1}) \otimes I_D. \tag{42}$$

This defines a T-equivariant stable pair.

Proposition 4.2. *The isomorphism of Proposition 4.1 takes the nested flag of subschemes $Z_0 \subset Z_1 \subset Z_2 \subset \cdots \subset Z_{d-1} \subset C$ to the stable pair* (42).

Proof. By Section 4.3, the stable pair (42) is described by a diagram of the form (33). By pushing down (42) we find that

$$G_i \;=\; \mathcal{O}_C(Z_{d-1}) \otimes I_{D_i},$$

which by the definition of D_i is

$$\mathcal{O}_C(Z_{d-1} - D_i) \;\cong\; \mathcal{O}_C(Z_i).$$

[6] $D \subset (d)C$ is Cartier if and only if all the D_i are empty.

Therefore for the pair (42), the diagram (33) becomes

$$\begin{array}{ccccccccc} \mathcal{O}_C & = & \mathcal{O}_C & = & \mathcal{O}_C & = & \cdots & = & \mathcal{O}_C \\ \downarrow & & \downarrow & & \downarrow & & & & \downarrow \\ \mathcal{O}(Z_0) & \hookrightarrow & \mathcal{O}(Z_1) & \hookrightarrow & \mathcal{O}(Z_2) & \hookrightarrow & \cdots & \hookrightarrow & \mathcal{O}(Z_{d-1}), \end{array}$$

with all maps the canonical ones. But this is precisely the diagram (34) corresponding to the flag $Z_0 \subset Z_1 \subset Z_2 \subset \cdots \subset Z_{d-1} \subset C$ from which we construct the T-equivariant stable pair via the isomorphism (38). □

Remark 4.3. This description of stable pairs in terms of Hilbert schemes parameterising either the subschemes Z_i (40) or the dual subschemes D_i (41) is related to, but *different from*, the description [PT3, Appendix B.2] of stable pairs on surfaces in terms of relative Hilbert schemes. The latter description is dual to the one above in a different way, involving the (derived dual) of the sheaf F and complex $I^\bullet = \{\mathcal{O}_S \to F\}$.

5. Localised virtual cycle on the vertical component

In Corollary 3.4 we showed that the contribution to $[P^T_{d\mathsf{k}}]^{\mathrm{vir}}$ of its vertical component is the push forward of a cycle on $P^T_{(d)C} \cong C^{[\mathbf{n}]}$. We denote this Kiem-Li [KL3] cosection localised virtual cycle by

$$\left[P^T_{\mathrm{ver}}\right]^{\mathrm{vir}} \in A_*(C^{[\mathbf{n}]}). \tag{43}$$

In this Section we compute it.

Denote by

$$H_{\mathsf{k}} := \mathrm{Hilb}_{\mathsf{k}}(S)$$

the Hilbert scheme of effective divisors in class $\mathsf{k} = c_1(K_S)$ on S. A result of H.-l. Chang and Y.-H. Kiem [CK] simplifies our life considerably.

Theorem 5.1. *Assume that S has a smooth irreducible canonical curve C. Then we may assume C defines a* smooth point *of H_{k} at which*

$$\dim_{\{C\}} H_{\mathsf{k}} \equiv \chi(\mathcal{O}_S) \mod 2.$$

Proof. Chang-Kiem [CK, Proposition 4.2] use a result of Green-Lazarsfeld to prove that there exists a canonical curve at which H_{k} is smooth. It follows that the smooth locus of H_{k} intersects $|K_S|$ in a non-empty Zariski open subset.

The smooth irreducible canonical curves form another Zariski open subset of $|K_S|$, and our assumption implies it is also non-empty. Since $|K_S|$ is a projective space it is in particular irreducible, so the two Zariski open subsets have non-empty intersection. Choosing C in this intersection gives the result.

Finally the parity of $\dim H_{\mathsf{k}}$ at C is also given in [CK, Proposition 4.2]. □

Using this result, we will find we are in the following situation.

Consider M a projective scheme with perfect obstruction theory, obstruction sheaf Ob and cosection vanishing on $Z \stackrel{\iota}{\hookrightarrow} M$:

$$\mathrm{Ob} \xrightarrow{\sigma} \mathcal{O}_M \longrightarrow \iota_*\mathcal{O}_Z \longrightarrow 0.$$

Suppose that Z is smooth, and that M is smooth in a neighbourhood of Z. It is then clear what the virtual cycle of M should be. Away from Z the surjection $\mathrm{Ob} \to \mathcal{O}_M$ ensures that it is zero. Fulton-MacPherson intersection theory allows us to write it as the pushforward of a class on Z which, by smoothness and the locally freeness of Ob near Z, should calculate $c_{\mathrm{top}}(\mathrm{Ob})$. To find it we use the exact sequence

$$0 \longrightarrow T_Z \longrightarrow T_M|_Z \xrightarrow{d\sigma|_Z} \mathrm{Ob}^*|_Z \longrightarrow Q \longrightarrow 0 \tag{44}$$

on Z. Here $Q \cong \mathrm{Ob}^*|_Z \big/ N_{Z/M}$ is defined to be the cokernel of $d\sigma|_Z$; this is locally free by the smoothness of $Z \subset M$. Excess intersection theory says that its top Chern class on Z, pushed forward to M, represents the top Chern class of Ob^*:

$$\iota_* c_{\mathrm{top}}(Q) = c_{\mathrm{top}}(\mathrm{Ob}^*). \tag{45}$$

Let m denote the dimension of M in the neighbourhood of Z, and let vd be the virtual dimension of the obstruction theory. Therefore $r := \mathrm{rk}(\mathrm{Ob}|_Z)$ is $m - vd$. Finally let c denote the codimension of $Z \subset M$, so that $\mathrm{rk}(Q) = r - c = m - vd - c$.

Since (45) differs from $c_{\mathrm{top}}(\mathrm{Ob})$ only by the sign $(-1)^r$, and we expect the virtual cycle to be

$$(-1)^r \iota_*\big(c_{\mathrm{top}}(Q)\big) \;=\; (-1)^{m-vd}\iota_*\big(c_{m-vd-c}(Q)\big) \;=\; (-1)^c \iota_*\big(c_{m-vd-c}(Q^*)\big).$$

By (44) this is

$$(-1)^c\big[\iota_* c(Q^*)\big]_{vd} \;=\; (-1)^c\,\iota_*\big[c(\mathrm{Ob}|_Z)s(N^*_{Z/M})\big]_{vd}\,,$$

where $c(\,\cdot\,)$ and $s(\,\cdot\,)$ denote the *total* Chern and Segre classes respectively. Unsurprisingly, the formulation of Kiem-Li gives precisely this answer.

Proposition 5.2. *In the above situation, Kiem and Li's localised virtual cycle of M is the class in $A_{vd}(Z)$ given by the vd-dimensional part of*

$$(-1)^c\big(c(\mathrm{Ob}|_Z)s(N^*_{Z/M})\big) \cap [Z].$$

Proof. In our situation Kiem and Li's recipe for their localised class is the following. Let

$$\begin{array}{c} E \subset \mathrm{Bl}_Z M \\ \big\downarrow p \\ M \end{array}$$

be the blow up of M in Z with exceptional divisor E. Then the pullback of the cosection has zero locus E, giving an exact sequence

$$0 \longrightarrow G \longrightarrow p^* \mathrm{Ob} \xrightarrow{p^*\sigma} \mathcal{O}(-E) \longrightarrow 0$$

for some vector bundle G of rank $g = r - 1 = m - vd - 1$. Kiem and Li tell us to intersect the zero section of G with itself and then with $-E$, and push the result down to Z. Since $p|_E \colon E \to Z$ is the projective bundle $\mathbb{P}(N_{Z/M}) \to Z$ of relative dimension $c - 1$, this gives

$$\begin{aligned} -(p|_E)_* c_{\mathrm{top}}(G|_E) &= -\Big[(p|_E)_* c(G|_E)\Big]_{m-1-g} \\ &= -\Big[(p|_E)_* \big(p^* c(\mathrm{Ob}|_Z) s(\mathcal{O}_E(-E))\big)\Big]_{vd} \\ &= -\Big[c(\mathrm{Ob}|_Z) \cdot (p|_E)_* s(\mathcal{O}_{\mathbb{P}(N_{Z/M})}(1))\Big]_{vd} \\ &= -\Big[c(\mathrm{Ob}|_Z) \cdot (-1)^{c-1} s(N^*_{Z/M})\Big]_{vd}, \end{aligned}$$

which gives the required result. □

We can apply this to describe the virtual cycle $[P^T_{\mathrm{ver}}]^{\mathrm{vir}}$ (43) as follows. Recall that the zero locus of cosection (18) is

$$\bigsqcup_{\mathbf{n}} C^{[\mathbf{n}]},$$

where the sum is over all $\mathbf{n}$ satisfying (39). Note that n_{d-1} is the length $l(Z_{d-1})$ of the last divisor in the flag (35) — i.e. the dimension of the nested Hilbert scheme $C^{[\mathbf{n}]}$.

Corollary 5.3. *Under the assumptions of Theorem 5.1, the Kiem-Li cosection-localised virtual cycle of the connected component $C^{[\mathbf{n}]}$ of $P^T_{(d)C}$ is*

$$(-1)^{\chi(\mathcal{O}_S)} \cdot c_{n_{d-1}-vd}\Big(\mathrm{Ob}\big|_{C^{[\mathbf{n}]}}\Big) \ \in\ A_{vd}(C^{[\mathbf{n}]}). \tag{46}$$

Therefore $[P^T_{\rm ver}]^{\rm vir}$ *is the sum of (the pushforwards of) these classes over all nonnegative integers* $n_0 \le \cdots \le n_{d-1}$ *satisfying*

$$\sum_{i=0}^{d-1}(n_i - (i+1)\mathsf{k}^2) = \chi. \tag{47}$$

Remark 5.4. We will see in (57) below that $vd = n_0$, so in the *uniformly thickened* case $n_0 = \ldots = n_{d-1}$ the localised virtual class is just $(-1)^{\chi(\mathcal{O}_S)}[P^T_{(d)C}]$.

Proof. Let U denote the smooth Zariski open neighbourhood $U \subset H_{\mathsf{k}}$ of the smooth point $\{C\}$ given to us by Theorem 5.1. The nested Hilbert scheme of the (smooth!) universal curve over U defines a neighbourhood of $P^T_{(d)C} \subset P^T_X$:

$$P_U := P^T_X\big|_U\,. \tag{48}$$

By the same working as in Section 4 this is isomorphic to the open set of P^T_X consisting of stable pairs supported on curves in U.

Since the nested Hilbert schemes of smooth curves are smooth, $P_U \to U$ is a smooth map. Therefore both $P^T_{(d)C}$ and P_U are smooth, and by Proposition 3.4 we can apply Proposition 5.2 to $P^T_{(d)C} \subset P^T_X$ in place of $Z \subset M$.

Since $N^*_{Z/M}$ is the pullback of the conormal bundle of $\{C\} \subset H_k$, it is trivial on $P^T_{(d)C}$ with Segre class 1. And the codimension of $Z \subset M$ is $c = \dim_{\{C\}} H_{\mathsf{k}} \equiv \chi(\mathcal{O}_S) \bmod 2$, which fixes the sign. Finally, the sum is over $\mathbf{n}$ satisfying (37). □

Therefore to compute we need only calculate the K-theory class of the bundle

$$\mathrm{Ob}\big|_{C^{[\mathbf{n}]}} \;=\; \mathscr{E}xt^2_{\pi_P}(\mathbb{I}^\bullet, \mathbb{I}^\bullet)^f_0\big|_{C^{[\mathbf{n}]}}.$$

We do this in the next Section. This will also determine the value of vd (which we have not yet found, notice!).

6. Obstruction bundle of the vertical component

Throughout this Section we use the notation

$$\langle\,\cdot\,,\,\cdot\,\rangle := \big[R\mathscr{H}om_{\pi_P}(\,\cdot\,,\,\cdot\,)\big],$$

where the square brackets take the T-equivariant K-theory class of an element of the equivariant derived category $D(P^T_X)^T$. We will compute the restriction to $P^T_{(d)C}$ of the (dual of the) perfect obstruction theory (9):

$$\big[E^\bullet\big]^\vee \;=\; -\langle \mathbb{I}^\bullet, \mathbb{I}^\bullet\rangle_0\,.$$

As usual the subscript denotes the trace-free part.

We work on the neighbourhood P_U (48) of $P^T_{(d)C} \subset P^T_X$. Thus we have the description of Section 4, which we now summarise. $U \subset H_{\mathsf{k}}$ is a smooth open set of smooth curves in class $\mathsf{k} = c_1(K_S)$ with universal curve

$$\mathcal{C} \xrightarrow{\ p\ } U$$

whose relative nested Hilbert scheme is isomorphic to P_U:

$$\begin{array}{ccc} \mathrm{Hilb}^{\mathbf{n}}(\mathcal{C}) & & P_U \\ \big\downarrow p & \cong & \big\downarrow p \\ U & & U. \end{array}$$

That is, over P_U the universal curve carries a universal family of nested divisors

$$\mathcal{Z}_0 \subset \mathcal{Z}_1 \subset \mathcal{Z}_2 \subset \cdots \subset \mathcal{Z}_{d-1} \subset p^*\mathcal{C} \ \subset\ X \times P_U.$$

These define the universal stable pair via (34) (or equivalently via (42)).

Therefore the universal sheaf $\mathbb{F}$ is an iterated (and equivariant) extension of the sheaves

$$\mathcal{O}_{\mathcal{C}}(\mathcal{Z}_i) \otimes K_S^{-i}, \quad i = 0, \dots, d-1, \tag{49}$$

on $X \times P_U$. (The sheaves (49) are of course pushed forward from $S \times P_U$; they are the eigensheaves of the T-action on $\pi_*\mathbb{F}$ as in Section 4.) Hence the K-theory class of the universal sheaf is

$$\big[\mathbb{F}\big] = \sum_{i=0}^{d-1} \big[\mathcal{O}_{\mathcal{C}}(\mathcal{Z}_i) \otimes K_S^{-i} \otimes \mathfrak{t}^{-i}\big]. \tag{50}$$

Similarly the class of the universal complex is

$$\big[\mathbb{I}^\bullet\big] = \big[\mathcal{O}_{X \times P_U}\big] - \big[\mathbb{F}\big],$$

from which we compute

$$\begin{aligned} -\langle \mathbb{I}^\bullet, \mathbb{I}^\bullet \rangle_0 &= \langle \mathcal{O}_{X\times P_U}, \mathbb{F}\rangle + \langle \mathbb{F}, \mathcal{O}_{X\times P_U}\rangle - \langle \mathbb{F}, \mathbb{F}\rangle \\ &= \big[R\pi_{P*}\mathbb{F}\big] - \big[R\pi_{P*}\mathbb{F}\big]^\vee \otimes \mathfrak{t} - \langle \mathbb{F}, \mathbb{F}\rangle \end{aligned}$$

by (T-equivariant) Serre duality. By (50) this is

$$-\langle \mathbb{I}^\bullet, \mathbb{I}^\bullet \rangle_0 \;=\; \sum_{i=0}^{d-1} \big[R\pi_{P*}\big(\mathcal{O}_{\mathcal{C}}(\mathcal{Z}_i)\otimes K_S^{-i}\big)\big]\mathfrak{t}^{-i} - \big[R\pi_{P*}\big(\mathcal{O}_{\mathcal{C}}(\mathcal{Z}_i)\otimes K_S^{-i}\big)\big]^\vee \mathfrak{t}^{i+1} - \langle \mathbb{F}, \mathbb{F}\rangle, \tag{51}$$

where

$$-\langle \mathbb{F}, \mathbb{F}\rangle \;=\; -\sum_{i,j=0}^{d-1} \langle \mathcal{O}_{\mathcal{C}}(\mathcal{Z}_i), \mathcal{O}_{\mathcal{C}}(\mathcal{Z}_j) \otimes K_S^{i-j}\rangle \mathfrak{t}^{i-j}. \tag{52}$$

Since $R\mathscr{H}om(\mathcal{O}_{\mathcal{C}}(\mathcal{Z}_i),\mathcal{O}_{\mathcal{C}}(\mathcal{Z}_j))$ has the same K-theory class as the alternating sum of its cohomology sheaves, a local Koszul resolution gives

$$\big[R\mathscr{H}om(\mathcal{O}_{\mathcal{C}}(\mathcal{Z}_i),\mathcal{O}_{\mathcal{C}}(\mathcal{Z}_j))\big]=\big[\big(\mathcal{O}_{\mathcal{C}}-\mathcal{O}_{\mathcal{C}}(\mathcal{C})-K_S\big|_{\mathcal{C}}\mathfrak{t}+K_S(\mathcal{C})\big|_{\mathcal{C}}\mathfrak{t}\big)(\mathcal{Z}_j-\mathcal{Z}_i)\big].$$

Substituting into (52), we find

$$\begin{aligned}-\langle\mathbb{F},\mathbb{F}\rangle \;=\; R\pi_{P*}\sum_{i,j=0}^{d-1}\Big[&K_S^{i-j+1}\big|_{\mathcal{C}}(\Delta_{ij})\mathfrak{t}^{i-j}+K_S^{i-j+1}\big|_{\mathcal{C}}(\Delta_{ij})\mathfrak{t}^{i-j+1}\\ &-K_S^{i-j}\big|_{\mathcal{C}}(\Delta_{ij})\mathfrak{t}^{i-j}-K_S^{i-j+2}\big|_{\mathcal{C}}(\Delta_{ij})\mathfrak{t}^{i-j+1}\Big],\end{aligned}\tag{53}$$

where Δ_{ij} is the divisor $\mathcal{Z}_j-\mathcal{Z}_i$ (effective if and only if $j\ge i$).

The moving part of (51) is (the K-theory class of) N^{vir}, and will be used in Section 7. For now we concentrate on the fixed part — i.e. the dual of the obstruction theory $\big[(E^\bullet)^f\big]^\vee$ of P_X^T. We also restrict to $C^{[\mathbf{n}]}\subset P^T_{(d)C}\subset P^T_X$, so $\mathcal{C}$ becomes plain C. We set

$$\Delta_i:=\Delta_{i-1,i}=\mathcal{Z}_i-\mathcal{Z}_{i-1}\quad\text{of length }\delta_i:=n_i-n_{i-1},$$

and use the standard isomorphism [Che]

$$\begin{aligned}C^{[\mathbf{n}]}&\xrightarrow{\ \sim\ } C^{[n_0]}\times C^{[\delta_1]}\times\cdots\times C^{[\delta_{d-1}]}\,,\\ (Z_0,Z_1,\dots,Z_{d-1})&\longmapsto (Z_0,\Delta_1,\dots,\Delta_{d-1}).\end{aligned}\tag{54}$$

The fixed parts of (51) and (53) give $\big[(E^\bullet)^f\big]^\vee=-\langle\mathbb{I}^\bullet,\mathbb{I}^\bullet\rangle_0^f$ as

$$R\pi_{P*}\Big[\mathcal{O}_C(\mathcal{Z}_0)+\sum_{j=1}^{d-1}\Big(K_S\big|_C+\mathcal{O}_C(\Delta_j)-\mathcal{O}_C-K_S\big|_C(\Delta_j)\Big)+K_S\big|_C-\mathcal{O}_C\Big].$$

Simplifying gives

$$\big[(E^\bullet)^f\big]^\vee \;=\; R\pi_{P*}\Big[K_S\big|_C+\mathcal{O}_{\mathcal{Z}_0}(\mathcal{Z}_0)+\sum_{i=1}^{d-1}\Big(\mathcal{O}_{\Delta_i}(\Delta_i)-K_S\big|_{\Delta_i}(\Delta_i)\Big)\Big].\tag{55}$$

The first term is the natural obstruction theory of H_{k}, and the next two give the tangent bundle of $C^{[\mathbf{n}]}$ via the isomorphism (54). Subtracting the tangent terms leaves minus the K-theory class of the obstruction bundle, so

$$\Big[\mathrm{Ob}\big|_{C^{[\mathbf{n}]}}\Big] \;=\; \Big[R^1\pi_{P*}\big(K_S\big|_C\big)\Big]+\sum_{i=1}^{d-1}\pi_{P*}\Big[K_S\big|_{\Delta_i}(\Delta_i)\Big]\,.\tag{56}$$

In particular the virtual dimension of P_X^T at any point of $C^{[\mathbf{n}]}$ is $\chi(K_S|_C)+n_0$, where n_0 is the length of Z_0. As C is in the canonical class $\beta=\mathsf{k}$ we have

$\chi(K_S|_C)=0$, so finally we obtain

$$vd=n_0. \tag{57}$$

We now substitute (56) into (46). The first term of (56) is the class of a trivial bundle over $P^T_{(d)C}$, so does not contribute. Therefore the cosection localised virtual cycle in $A_{vd}(C^{[\mathbf{n}]})$ is simply[7]

$$(-1)^{\chi(\mathcal{O}_S)}\prod_{i=1}^{d-1}c_{\mathrm{top}}\Big(\pi_{P*}\big(K_S\big|_{\Delta_i}(\Delta_i)\big)\Big)\in A_{n_0}\big(C^{[n_0]}\times C^{[\delta_1]}\times\cdots\times C^{[\delta_{d-1}]}\big).$$

This is easily calculated via relative Serre duality. Since $\Delta_i\subset C^{[\delta_i]}\times C$ is a divisor, its relative canonical bundle over $C^{[\delta_i]}$ is

$$\omega_{\Delta_i/C^{[\delta_i]}}\ \cong\ \omega_C(\Delta_i)\big|_{\Delta_i}\ \cong\ K_S^2\otimes\mathcal{O}_{\Delta_i}(\Delta_i).$$

Therefore the localised virtual cycle is

$$\begin{aligned}(-1)^{\chi(\mathcal{O}_S)}\prod_{i=1}^{d-1}c_{\delta_i}\Big(\pi_{P*}\big(K_S^{-1}\big|_{\Delta_i}\otimes\omega_{\Delta_i/C^{[\delta_i]}}\big)\Big)&=(-1)^{\chi(\mathcal{O}_S)}\prod_{i=1}^{d-1}c_{\delta_i}\Big(\big(\pi_{P*}K_S\big|_{\Delta_i}\big)^*\Big)\\&=(-1)^{\chi(\mathcal{O}_S)}\prod_{i=1}^{d-1}(-1)^{\delta_i}c_{\delta_i}\Big(\big(K_S|_C\big)^{[\delta_i]}\Big).\end{aligned}$$

Using the binomial convention (8),

$$\int_{C^{[k]}}c_k\big(L^{[k]}\big)=\binom{\deg L}{k},$$

for any line bundle L on C. This is easiest to see when L has a section s with reduced zeros $z_1,\dots,z_{\deg L}$. Then the induced section $s^{[k]}$ of $L^{[k]}$ has reduced zeros at precisely the points $(z_{i_1},\dots,z_{i_k})$, where $\{i_1,\dots,i_k\}$ is any subset of $\{1,\dots,\deg L\}$.[8] Putting it all together, we have proved the following.

Proposition 6.1. *The Kiem-Li localised virtual cycle* (46) *is*

$$(-1)^{\chi(\mathcal{O}_S)+n_{d-1}-n_0}\prod_{i=1}^{d-1}\binom{\mathsf{k}^2}{\delta_i}$$

times by the cycle

$$\big[C^{[n_0]}\big]\times[\mathrm{pt}]\times\ldots\times[\mathrm{pt}] \tag{58}$$

[7]As noted in Remark 5.4, when $n_0=n_{d-1}$ this reduces to $(-1)^{\chi(\mathcal{O}_S)}\big[C^{[n_0]}\big]$.

[8]More generally when $n\geq 2h-1$ Lemma VIII.2.5 of [ACGH] gives an expression for $c_\bullet(L^{[k]})$. Combining with (79) below gives the formula. For general n the formula follows using the "embedding trick" of Section 10.1.

in $A_{n_0}\big(C^{[n_0]} \times C^{[\delta_1]} \times \ldots \times C^{[\delta_{d-1}]}\big) = A_{n_0}(C^{[\mathbf{n}]})$. □

7. The virtual normal bundle

We want to calculate the contribution of the vertical component $[P^T_{\mathrm{ver}}]^{\mathrm{vir}}$ (43) to the invariants (1). By Proposition 6.1 we can now pull everything back to an integral on $C^{[n_0]}$. We do this first with the virtual normal bundle. We use the projections

(59)
$$\begin{array}{ccc} & C^{[\mathbf{n}]} \times C & \\ {\scriptstyle \pi}\swarrow & & \searrow{\scriptstyle p_C} \\ C^{[\mathbf{n}]} & & C \end{array}$$

and usually suppress p_C^* as before. We also use the standard notation

$$c_s(E) := 1 + s\, c_1(E) + s^2\, c_2(E) + \ldots$$

for any complex of sheaves E. When E is a vector bundle of rank r we have

(60)
$$e(E \otimes \mathfrak{t}^w) = \sum_{i=0}^{r} c_i(W)(wt)^{r-i} = (wt)^r c_{1/wt}(E) = (wt)^r c_{-1/wt}(E^\vee).$$

Therefore the same identity holds for $E = \{\cdots \to E^i \to E^{i+1} \to \cdots\}$ a finite complex of rank $r := \sum_i (-1)^i \operatorname{rk}(E^i)$. In particular, when E is a trivial bundle (or constant complex) we have

(61)
$$e(E \otimes \mathfrak{t}^w) = (wt)^r.$$

Proposition 7.1. *The pull-back of* $\frac{1}{e(N^{\mathrm{vir}})}$ *to the cycle* $C^{[n_0]}$ *of* (58) *equals*

$$A\, t^{n_0}\, c_{-1/dt}(E) \prod_{i=1}^{d-1} c_{-1/it}(F_i)$$

where

(62)
$$E = R\pi_*\Big[K_S|_C^{-(d-1)}(\mathcal{Z}_0 + \Delta_{0,d-1})\Big],$$
$$F_i = R\pi_*\Big[K_S|_C^{-(i-1)}(\mathcal{Z}_0 + \Delta_{0,i-1}) \otimes (\mathcal{O}_C - K_S|_C^{-1}(\Delta_i))\Big], \quad \text{and}$$
$$A = (-1)^{\frac{1}{2}d(d-1)\mathsf{k}^2 + \sum_{i=1}^{d-1} n_i} \left(\frac{d!}{d^d}\right)^{\mathsf{k}^2} d^{\,n_{d-1}} \prod_{i=1}^{d-1} i^{\,-\delta_i}\,.$$

Proof. Taking the moving parts of (51) and (53) we find

$$\frac{1}{e(N^{\mathrm{vir}})} = e((R\pi_*\mathcal{O}(\mathcal{Z}_0))^\vee \otimes \mathfrak{t}) \prod_{i=1}^{d-1} \frac{e((R\pi_* K_S|_C^{-i}(\mathcal{Z}_i))^\vee \otimes \mathfrak{t}^{i+1})}{e((R\pi_* K_S|_C^{-i}(\mathcal{Z}_i)) \otimes \mathfrak{t}^{-i})} \tag{63}$$

$$\times \prod_{\substack{i,j=0 \\ i\neq j}}^{d-1} \frac{e(R\pi_* K_S|_C^{i-j}(\Delta_{ij}) \otimes \mathfrak{t}^{i-j})}{e(R\pi_* K_S|_C^{i-j+1}(\Delta_{ij}) \otimes \mathfrak{t}^{i-j})} \prod_{\substack{i,j=0 \\ i+1\neq j}}^{d-1} \frac{e(R\pi_* K_S|_C^{i-j+2}(\Delta_{ij}) \otimes \mathfrak{t}^{i-j+1})}{e(R\pi_* K_S|_C^{i-j+1}(\Delta_{ij}) \otimes \mathfrak{t}^{i-j+1})}.$$

We start with the second line of (63). On our cycle $C^{[n_0]} \times \Delta_1 \times \ldots \times \Delta_{d-1}$ the divisors

$$\Delta_{ij} = \begin{cases} \sum_{k=i+1}^{j} \Delta_k, & j > i \\ \sum_{k=j+1}^{i} \Delta_k, & i > j \end{cases}$$

are fixed, since the divisors Δ_k are. Therefore each $R\pi_* K_S|_C^{i-j+l}(\Delta_{ij})$ is a constant complex $\mathcal{O}_{C^{[n_0]}}^{\oplus r}$, where

$$r = \chi\big(K_S|_C^{i-j+l}(\Delta_{ij})\big) = n_j - n_i + (i-j+l-1)\mathsf{k}^2$$

by Riemann-Roch.

So by (61) the second line of (63) is

$$\prod_{\substack{i,j=0 \\ i\neq j}}^{d-1} \frac{((i-j)t)^{n_j-n_i+(i-j-1)\mathsf{k}^2}}{((i-j)t)^{n_j-n_i+(i-j)\mathsf{k}^2}} \prod_{\substack{i,j=0 \\ i+1\neq j}}^{d-1} \frac{((i-j+1)t)^{n_j-n_i+(i-j+1)\mathsf{k}^2}}{((i-j+1)t)^{n_j-n_i+(i-j)\mathsf{k}^2}}$$

which simplifies to

$$\prod_{\substack{i,j=0 \\ i\neq j}}^{d-1} \frac{1}{((i-j)t)^{\mathsf{k}^2}} \prod_{\substack{i,j=0 \\ i+1\neq j}}^{d-1} ((i+1-j)t)^{\mathsf{k}^2}.$$

The only terms in this expression which do not cancel immediately are those with $i=0$ in the first sum and $i=d-1$ in the second sum. This gives

$$\prod_{j=1}^{d-1} \frac{1}{(-jt)^{\mathsf{k}^2}} \cdot \prod_{j=0}^{d-1} ((d-j)t)^{\mathsf{k}^2} = (-1)^{(d-1)\mathsf{k}^2} d^{\mathsf{k}^2} t^{\mathsf{k}^2}. \tag{64}$$

We now deal with the first line of (63). Applying (60) gives

(65)

$$t^{n_0-\mathsf{k}^2} c_{-1/t}(R\pi_*\mathcal{O}(\mathcal{Z}_0)) \prod_{i=1}^{d-1} \frac{((i+1)t)^{n_i-(i+1)\mathsf{k}^2}}{(-it)^{n_i-(i+1)\mathsf{k}^2}} \prod_{i=1}^{d-1} \frac{c_{-1/(i+1)t}(R\pi_* K_S|_C^{-i}(\mathcal{Z}_i))}{c_{-1/it}(R\pi_* K_S|_C^{-i}(\mathcal{Z}_i))}.$$

The first product can be simplified as

$$
\begin{aligned}
&(-1)^{\sum_{i=1}^{d-1}(n_i-(i+1)\mathsf{k}^2)}\prod_{i=1}^{d-1}\Big(\frac{i+1}{i}\Big)^{-(i+1)\mathsf{k}^2}\ \prod_{i=1}^{d-1}\Big(\frac{i+1}{i}\Big)^{n_i} \\
&= (-1)^{\left(\frac12 d(d+1)-1\right)\mathsf{k}^2+\sum_{i=1}^{d-1}n_i}\prod_{i=1}^{d-1}\Big(\frac{i}{i+1}\Big)^{(i+1)\mathsf{k}^2}\left(\prod_{i=1}^{d-1} i^{\,n_{i-1}-n_i}\right)d^{\,n_{d-1}} \qquad (66)\\
&= (-1)^{\frac12 d(d-1)\mathsf{k}^2+(d-1)\mathsf{k}^2+\sum_{i=1}^{d-1}n_i}\left(\frac{(d-1)!}{d^d}\right)^{\mathsf{k}^2} d^{\,n_{d-1}}\prod_{i=1}^{d-1} i^{-\delta_i}.
\end{aligned}
$$

Multiplying $t^{n_0-\mathsf{k}^2}$, (66) and (64) together gives $A\,t^{n_0}$, as required.

What remains in (65) is

$$
c_{-1/t}(R\pi_*\mathcal{O}(\mathcal{Z}_0))\prod_{i=1}^{d-1}\frac{c_{-1/(i+1)t}(R\pi_*K_S|_C^{-i}(\mathcal{Z}_0+\Delta_{0,i}))}{c_{-1/it}(R\pi_*K_S|_C^{-i}(\mathcal{Z}_0+\Delta_{0,i}))}\,.
$$

Reordering the product gives

$$
\prod_{i=1}^{d-1}\frac{c_{-1/it}(R\pi_*K_S|_C^{-(i-1)}(\mathcal{Z}_0+\Delta_{0,i-1}))}{c_{-1/it}(R\pi_*K_S|_C^{-i}(\mathcal{Z}_0+\Delta_{0,i}))}\cdot c_{-1/dt}(R\pi_*K_S|_C^{-(d-1)}(\mathcal{Z}_0+\Delta_{0,d-1})),
$$

which we write as

$$
\left[\prod_{i=1}^{d-1}c_{-1/it}\Big(R\pi_*\Big(K_S|_C^{-(i-1)}(\mathcal{Z}_0+\Delta_{0,i-1})-K_S|_C^{-i}(\mathcal{Z}_0+\Delta_{0,i})\Big)\Big)\right]c_{-1/dt}(E).
$$

This is $c_{-1/dt}(E)\prod_{i=1}^{d-1}c_{-1/it}(F_i)$ as claimed. □

8. Descendent insertions

Recall from Section 2 that given a cohomology class $\sigma\in H^*(S,\mathbb{Q})$ and a nonnegative integer α, we defined in (13) the descendent insertion

$$
\tau_\alpha(\sigma)\ \in\ H^*_T(P_X,\mathbb{Q}).
$$

We have localised the vertical component $[P^T_{\rm ver}]^{\rm vir}$ of the virtual cycle $[P^T_X]^{\rm vir}$ to

$$
P^T_{(d)C}=\bigsqcup_{\mathbf{n}}C^{[\mathbf{n}]}.
$$

We next restrict the descendents to $C^{[\mathbf{n}]}$. We use the projections (59) and the universal divisors $\mathcal{Z}_0\subset\cdots\subset\mathcal{Z}_{d-1}\subset C^{[\mathbf{n}]}\times C$.

Proposition 8.1. *Let* $E(x) := \frac{1-e^{-x}}{x} = 1 - x/2! + x^2/3! - \dots$ *. The restriction of* $\tau_\alpha(\sigma)$ *to* $C^{[\mathbf{n}]} \subset P_X$ *is the degree*[9] $2\alpha + \deg\sigma - 2$ *part of*

$$\pi_*\left[p_C^*\left(\sigma|_C - \frac{\sigma.\mathsf{k}}{2}\Big|_C\right) E(\mathsf{k}+t) \sum_{j=0}^{d-1} e^{[\mathcal{Z}_j]-j(\mathsf{k}+t)}\right].$$

Proof. As in (50), the K-theory class of the restriction of the universal sheaf $\mathbb{F}$ to $C^{[\mathbf{n}]} \times X$ is

$$\big[\mathbb{F}\big] = \sum_{j=0}^{d-1} \big[\mathcal{O}_{C^{[\mathbf{n}]}\times C}(\mathcal{Z}_j) \otimes K_S^{-j} \otimes \mathfrak{t}^{-j}\big].$$

Let i denote both the inclusion $C \hookrightarrow X$ and its basechange $C^{[\mathbf{n}]} \times C \hookrightarrow C^{[\mathbf{n}]} \times X$. It has normal bundle $\nu_C = K_S|_C \oplus K_S|_C \otimes \mathfrak{t}$, so by T-equivariant Grothendieck-Riemann-Roch [EG],

$$\begin{aligned} \mathrm{ch}^T(\mathbb{F}) &= \sum_j \mathrm{ch}(i_*\mathcal{O}(\mathcal{Z}_j)) e^{-j(\mathsf{k}+t)} \\ &= \sum_j i_*\left(\mathrm{ch}(\mathcal{O}(\mathcal{Z}_j))\mathrm{td}^{-1}(\nu_C)\right) e^{-j(\mathsf{k}+t)} \\ &= i_* \sum_j e^{[\mathcal{Z}_j]}\, E(\mathsf{k})\, E(\mathsf{k}+t)\, e^{-j(\mathsf{k}+t)}. \end{aligned} \tag{67}$$

If we write this as i_*A then, again on restriction to $C^{[\mathbf{n}]} \subset P_X$ we find

$$\tau_\alpha(\sigma) = \pi_{P*}\big(\pi_X^*\sigma \cap [i_*A]_{2\alpha+4}\big) = \pi_*\big(i^*\pi_X^*\sigma \cap [A]_{2\alpha}\big)$$

because $\pi = \pi_P \circ i$. Recalling the identification (12), we also see that $i^*\pi_X^*\sigma = p_C^*(\sigma|_C)$. Substituting into (67) gives

$$\pi_*\left[p_C^*(\sigma|_C)E(\mathsf{k})E(\mathsf{k}+t)\sum_{j=0}^{d-1} e^{[\mathcal{Z}_j]-j(\mathsf{k}+t)}\right]_{2\alpha+\deg\sigma},$$

which simplifies to the required formula. ☐

Corollary 8.2. *Let* $D \in H^2(S)$*. Then on restriction to the cycle* $C^{[n_0]}$ *of* (58) *we find that* $\tau_\alpha(\sigma)$ *is the degree* 2α *part of*

$$(\mathsf{k}\cdot D)E(t)\sum_{j=0}^{d-1} e^{\omega - jt},$$

where ω *is the class of the divisor* $\mathcal{Z}_0 \subset C^{[n_0]} \times C$ *restricted to* $C^{[n_0]} \times \{c_0\}$ *(and* $c_0 \in C$ *is any basepoint).*

[9] This is the *real* cohomological degree; twice the complex degree.

In the formula of Theorem 1.4 we only consider insertions $\tau_{\alpha_i}(D_i)$ coming from $D_i \in H^2(S)$. Expanding as a polynomial in ω,

$$\prod_{i=1}^m \tau_{\alpha_i}(D_i) \;=\; \prod_{i=1}^m (\mathsf{k}\cdot D_i) \prod_{i=1}^m \left[E(t)\sum_{j=0}^{d-1} e^{\omega - jt}\right]_{2\alpha_i} \;=\; \prod_{i=1}^m (\mathsf{k}\cdot D_i) \sum_{a=0}^{\infty} \gamma_a\,\omega^a, \tag{68}$$

for some $\gamma_a \in \mathbb{Q}[t]$ whose precise form we do not need. Here $[\,\cdot\,]_{2\alpha_i}$ denotes the degree $2\alpha_i$ part in the degree 2 variables ω and t.

Therefore by Propositions 6.1 and 7.1, $\mathsf{Z}^P_{d\mathsf{k}}(X, \tau_{\alpha_1}(D_1)\cdots\tau_{\alpha_m}(D_m))_{\mathrm{ver}}$ equals

$$B\prod_{i=1}^m(\mathsf{k}\cdot D_i) \sum_{0\le n_0\le\cdots\le n_{d-1};\; a\ge 0} q^{\chi} t^{n_0}\gamma_a \int_{C^{[n_0]}} \omega^a c_{-1/dt}(E)\prod_{i=1}^{d-1} c_{-1/it}(F_i), \tag{69}$$

where E, F_i are defined in (62), $\chi = \sum_{i=0}^{d-1}(n_i - (i+1)\mathsf{k}^2)$ by (47), and

$$\begin{aligned} B \;&=\; (-1)^{\chi(\mathcal{O}_S)+n_{d-1}-n_0} A \prod_{i=1}^{d-1}\binom{\mathsf{k}^2}{\delta_i} \\ &=\; (-1)^{\frac12 d(d-1)\mathsf{k}^2 + \chi(\mathcal{O}_S) + \sum_{i=0}^{d-1} n_i} \left(\frac{d!}{d^d}\right)^{\mathsf{k}^2} (-d)^{n_{d-1}} \prod_{i=1}^{d-1}\left[i^{-\delta_i}\binom{\mathsf{k}^2}{\delta_i}\right]. \end{aligned}$$

9. Expression in terms of tautological classes

We now write the integrand of (69) in terms of tautological classes on the symmetric product $C^{[n_0]}$. For now we assume, for simplicity, that

$$n_0 > 2h - 2,$$

where $h = \mathsf{k}^2 + 1$ is the canonical genus; later we will explain how to remove this assumption. Therefore the Abel-Jacobi map

$$\begin{aligned} \mathrm{AJ}\colon C^{[n_0]} &\longrightarrow \mathrm{Pic}^{n_0}(C), \\ Z_0 &\longmapsto \mathcal{O}_C(Z_0), \end{aligned}$$

is a projective bundle. In fact, using the notation

$$\begin{array}{ccc} C^{[n_0]}\times C & \xrightarrow{\mathrm{AJ}\times 1} & \mathrm{Pic}^{n_0}(C)\times C \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi_2} \\ C^{[n_0]} & \xrightarrow{\mathrm{AJ}} & \mathrm{Pic}^{n_0}(C) \end{array}$$

and letting $\mathcal{P}$ be a Poincaré line bundle on $\mathrm{Pic}^{n_0}(C)\times C$, we have

$$C^{[n_0]} = \mathbb{P}(\pi_{2*}\mathcal{P}).$$

We normalise $\mathcal{P}$ by fixing

$$\mathcal{P}\big|_{\mathrm{Pic}^{n_0}(C)\times\{c_0\}} \cong \mathcal{O}_{\mathrm{Pic}^{n_0}(C)}\,, \tag{70}$$

by tensoring it with $\pi_2^*\mathcal{P}^{-1}|_{\mathrm{Pic}^{n_0}(C)\times\{c_0\}}$ if necessary. This fixes a tautological line bundle

$$\mathcal{O}(-1) \subset \mathrm{AJ}^*\pi_{2*}\mathcal{P} \tag{71}$$

on $C^{[n_0]}$, and so the tautological class

$$\omega := c_1(\mathcal{O}(1)) \in H^2(C^{[n_0]},\mathbb{Z}). \tag{72}$$

The map $\pi_2^*\pi_{2*}\mathcal{P} \to \mathcal{P}$, pulled back along $\mathrm{AJ}\times 1$ and composed with the inclusion (71), gives a canonical section of $(\mathrm{AJ}\times 1)^*\mathcal{P}(1)$ vanishing on the universal divisor $\mathcal{Z}_0 \subset C^{[n_0]}\times C$. Therefore

$$(\mathrm{AJ}\times 1)^*\mathcal{P}(1) \cong \mathcal{O}(\mathcal{Z}_0) \quad \text{and so} \quad \mathcal{O}(1)\big|_{C^{[n_0]}\times\{c_0\}} \cong \mathcal{O}(\mathcal{Z}_0)\big|_{C^{[n_0]}\times\{c_0\}} \tag{73}$$

by the normalisation condition (70). In particular the ω of (72) is the divisor class $\big[\mathcal{Z}_0|_{C^{[n_0]}\times\{c_0\}}\big]$, and so is the same ω as appears in Corollary 8.2.

The second tautological class we use is the pullback of the class of the theta divisor on $\mathrm{Pic}^{n_0}(C)$,

$$\theta \in H^2(\mathrm{Pic}^{n_0}(C),\mathbb{Z}) \cong \mathrm{Hom}(\Lambda^2 H^1(C,\mathbb{Z}),\mathbb{Z})$$

which takes $\alpha,\beta \in H^1(C,\mathbb{Z})$ to $\int_C \alpha\wedge\beta$. We denote its pullback $\mathrm{AJ}^*\theta$ to $C^{[n_0]}$ by θ also.

Proposition 9.1. *The integrand $\omega^a c_{-1/dt}(E)\prod_{i=1}^{d-1} c_{-1/it}(F_i)$ in (69) can be written in terms of the tautological classes ω, θ as*

$$\omega^a \sum_{k=0}^{\infty}\Big(1-\frac{\omega}{dt}\Big)^{n_{d-1}-d\mathsf{k}^2-k}\ \frac{(\theta/dt)^k}{k!}\ \prod_{i=1}^{d-1}\Big(1-\frac{\omega}{it}\Big)^{\mathsf{k}^2-\delta_i}. \tag{74}$$

Proof. By (73) we see the complex E (62) satisfies

$$E(-1) \;=\; R\pi_{1*}\big((\mathrm{AJ}\times 1)^*\mathcal{P}\otimes K_S|_C^{-(d-1)}(\Delta_{0,d-1})\big). \tag{75}$$

We begin by computing the Chern *character* of this. By Grothendieck-Riemann-Roch we get

$$\begin{aligned}
&\pi_{1*}\big[\,\mathrm{ch}\,\big((\mathrm{AJ}\times 1)^*\mathcal{P}\otimes K_S|_C^{-(d-1)}(\Delta_{0,d-1})\big)\mathrm{td}(C)\big]\\
=\;& \pi_{1*}\Big[\exp\big((\mathrm{AJ}\times 1)^*c_1(\mathcal{P})-(d-1)\mathsf{k}+[\Delta_{0,d-1}]\big)(1+c_1(T_C)/2)\Big]\\
=\;& n_0-(d-1)\mathsf{k}^2+(n_{d-1}-n_0)-\mathsf{k}^2-\theta\\
=\;& n_{d-1}-d\,\mathsf{k}^2-\theta.
\end{aligned}$$

Here we have identified $c_1(\mathcal{P})$ with

$$\begin{aligned}\big(0,\mathrm{id},n_0[c_0]\big) \;&\in\; H^2(\mathrm{Pic}^{n_0}(C)) \;\oplus\; \big(H^1(C)^*\otimes H^1(C)\big) \;\oplus\; H^2(C)\\ &=\; H^2\big(\,\mathrm{Pic}^{n_0}(C)\times C\big)\end{aligned}$$

using the normalisation condition (70), and used the (pullback by AJ $\times$ 1 of) the standard identity (section VIII.2 of [ACGH])

$$\frac{1}{2}\pi_{2*}\big(\,\mathrm{id}^{\wedge 2}\,\big) \;=\; -\theta.$$

Therefore (75) has rank $n_{d-1}-d\,\mathsf{k}^2$, first Chern class $-\theta$, and all higher Chern *characters* vanish. From this we deduce that

$$c_k \;=\; \frac{(-\theta)^k}{k!} \quad \text{for all } k>0. \tag{76}$$

So now applying the identity

$$c_s(V(1)) = \sum_{k=0}^{\infty}(1+\omega s)^{\mathrm{rk}(V)-k}c_k(V)s^k \tag{77}$$

to (76) we obtain

$$c_{-1/dt}(E) \;=\; \sum_{k=0}^{\infty}\Big(1-\frac{\omega}{dt}\Big)^{n_{d-1}-d\mathsf{k}^2-k}\,\frac{(-\theta)^k}{k!}\,\Big(\frac{-1}{dt}\Big)^k. \tag{78}$$

This gives the first term of the integrand. The second is easier. By (73) again,

$$F_i(-1) \;=\; R\pi_{1*}\big[K_S|_C^{-(i-1)}((\mathrm{AJ}\times 1)^*\mathcal{P}+\Delta_{0,i-1})\otimes(\mathcal{O}_C-K_S|_C^{-1}(\Delta_i))\big],$$

which is a *constant* complex by the normalisation condition (70). It has rank $\deg(K_S|_C(-\Delta_i))=\mathsf{k}^2-\delta_i$, so by (77) we find

$$c_{-1/it}(F_i) = \Big(1-\frac{\omega}{it}\Big)^{\mathsf{k}^2-\delta_i}. \qquad \square$$

10. Evaluation of the integral

Still working under the assumption $n_0>2h-2$ for the time being, we can now compute the integral in (69).

Proposition 10.1. *The integral of* (74) *over* $C^{[n_0]}$ *is*

$$\sum(dt)^{-n_0+a}\binom{n_0-n_{d-1}+(d+1)\mathsf{k}^2-a-|\mathbf{j}|}{n_0-a-|\mathbf{j}|}\prod_{i=1}^{d-1}\Big(\frac{-d}{i}\Big)^{j_i}\binom{\mathsf{k}^2-\delta_i}{j_i},$$

where the sum is over all $j_1,\dots,j_{d-1}\ge 0$, *and we set* $|\mathbf{j}|:=j_1+\dots+j_{d-1}$.

Proof. Expanding (74) by the binomial theorem using the convention (8) gives the sum over all $k,l,j_1,\dots,j_{d-1}\geq 0$ of

$$\left[\Big(\frac{1}{dt}\Big)^k\Big(\frac{-1}{dt}\Big)^l\binom{n_{d-1}-d\,\mathsf{k}^2-k}{l}\prod_{i=1}^{d-1}\Big(\frac{-1}{it}\Big)^{j_i}\binom{\mathsf{k}^2-\delta_i}{j_i}\right]\frac{\theta^k}{k!}\,\omega^{a+l+|\mathbf{j}|}.$$

We can now integrate over $C^{[n_0]}$ using (section VIII.3 of [ACGH])

$$\int_{C^{[n_0]}}\frac{\theta^k}{k!}\,\omega^{n_0-k} \;=\; \binom{h}{k},\ \text{ for all } k\in[0,n_0] \tag{79}$$

where $h=\mathsf{k}^2+1$ is the genus of C. This gives the sum over all $j_1,\dots,j_{d-1}\geq 0$ and $k\in[0,n_0]$ of

$$\Big(\frac{1}{dt}\Big)^k\Big(\frac{-1}{dt}\Big)^{n_0-a-k-|\mathbf{j}|}\binom{h}{k}\binom{n_{d-1}-d\,\mathsf{k}^2-k}{n_0-a-k-|\mathbf{j}|}\prod_{i=1}^{d-1}\Big(\frac{-1}{it}\Big)^{j_i}\binom{\mathsf{k}^2-\delta_i}{j_i}.$$

We can sum over all $k\geq 0$ since $\binom{h}{k}=0$ for $k>n_0\geq 2h-1\geq h$ when $h\geq 1$ (and when $h=0$ it is also clear we can sum over all $k\geq 0$). So using

$$\binom{a}{b} \;=\; (-1)^b\binom{b-a-1}{b}$$

we get the sum over all $k,j_1,\dots,j_{d-1}\geq 0$ of

$$(dt)^{-n_0+a+|\mathbf{j}|}\binom{h}{k}\binom{n_0-a-|\mathbf{j}|-n_{d-1}+d\mathsf{k}^2-1}{n_0-a-k-|\mathbf{j}|}\prod_{i=1}^{d-1}\Big(\frac{-1}{it}\Big)^{j_i}\binom{\mathsf{k}^2-\delta_i}{j_i}.$$

Summing over k using the Chu-Vandermonde identity

$$\sum_{k=0}^{\infty}\binom{a}{c-k}\binom{b}{k}=\binom{a+b}{c}$$

gives the claimed formula. $\square$

10.1. **Extension to all n_0.** We established Proposition 10.1 assuming $n_0>2h-2$. However the answer holds for *any* n_0. For general n_0, pick $N>n_0$ such that $N>2h-2$. Then $C^{[N]}\cong\mathbb{P}(\pi_{2*}\mathcal{Q})$, where $\mathcal{Q}$ is the normalised Poincaré bundle on $\operatorname{Pic}^N(C)\times C$. We can embed[10]

$$\begin{aligned} C^{[n_0]} &\;\hookrightarrow\; C^{[N]},\\ Z_0 &\;\longmapsto\; Z_0+(N-n_0)c_0. \end{aligned}$$

[10] The method described here was used in the case of the Hilbert scheme of curves on surfaces in [DKO] and also [KT2].

Denote the universal divisor on $C^{[N]} \times C$ by $\mathcal{W}$, and let $s \in H^0(\mathcal{O}(\mathcal{W}))$ be the section cutting it out. Then $C^{[n_0]} \subset C^{[N]}$ is the locus of effective divisors containing $(N-n_0)c_0$; i.e. it is the locus where s vanishes on restriction to the Artinian thickened point $(N-n_0)c_0$. Denote the restriction to $C^{[N]} \times (N-n_0)c_0$ of $\pi_2 : C^{[N]} \times C \to C^{[N]}$ by π_2 as well. Then $\pi_{*2}(s|_{C^{[N]}\times(N-n_0)c_0})$ defines a section of the locally free sheaf

$$F := \pi_{2*}\big(\mathcal{O}(\mathcal{W})|_{C^{[N]}\times(N-n_0)c_0}\big)$$

which cuts out $C^{[n_0]}$. The rank of F is the codimension of $C^{[n_0]}$, so it is a regular section and we can identify the normal bundle

$$N_{C^{[n_0]}/C^{[N]}} \;\cong\; F|_{C^{[n_0]}}$$

and the cycle class

$$\big[C^{[n_0]}\big] \;=\; \big[c_{N-n_0}(F)\big] \in H_{2(N-n_0)}\big(C^{[N]}\big). \tag{80}$$

All results of Sections 9 and 10 can be obtained by pushing forward to $C^{[N]}$ and then pushing down AJ using the commutative diagram

$$\begin{array}{ccc} C^{[n_0]} & \xrightarrow{+(N-n_0)c_0} & C^{[N]} \\ \downarrow{\scriptstyle \mathrm{AJ}} & & \downarrow{\scriptstyle \mathrm{AJ}} \\ \mathrm{Pic}^{n_0}(C) & \xrightarrow[\simeq]{\otimes\mathcal{O}((N-n_0)c_0)} & \mathrm{Pic}^N(C). \end{array}$$

Pushing forward to $C^{[N]}$ introduces the class (80), while $\mathcal{P}$ gets replaced by $\mathcal{Q}(-(N-n_0)c_0)$ and $\mathcal{Z}_0$ gets replaced by $\mathcal{W}-(N-n_0)c_0$. The calculation proceeds in exactly the same manner except for one difference: the usual relation $\mathrm{AJ}_*\omega^{i+n_0-h} = \theta^i/i!$ that goes into the Poincaré formula (79) for $n_0 > 2h-2$ is replaced by the identity

$$\mathrm{AJ}_*\big(c_{N-n_0}(F)\omega^{i+n_0-h}\big) \;=\; \begin{cases} \frac{\theta^i}{i!} & \text{if } i \geq 0, \\ 0 & \text{otherwise;} \end{cases} \tag{81}$$

see for instance [KT2, Lemma 4.3].[11] This removes the extra class (80) and produces the same formulae as for $n_0 > 2h-2$.

[11] Although [KT2, Lemma 4.3] is derived for the Hilbert scheme of curves on a surface the same formula holds in the (easier) setting of the Hilbert scheme of points on a curve.

11. Final formula without descendents

Plugging Proposition 10.1 into (69) evaluates $\mathsf{Z}_{d\mathsf{k}}(X,\tau_{\alpha_1}(D_1)\cdots\tau_{\alpha_m}(D_m))$ as the sum over all $a,\ j_1,\dots,j_{d-1}\ge 0$ and all $0\le n_0\le\dots\le n_{d-1}$ of

$$(-1)^{\chi(\mathcal{O}_S)+\frac12 d(d-1)\mathsf{k}^2+\sum_{i=0}^{d-1}n_i}\left(\frac{d!}{d^d}\right)^{\mathsf{k}^2}(-d)^{n_{d-1}}\prod_{i=1}^{d-1}\left[i^{-\delta_i}\binom{\mathsf{k}^2}{\delta_i}\right]\,t^{n_0}\prod_{i=1}^m(\mathsf{k}\cdot D_i)$$
$$\times\, q^{\chi}\,\gamma_a(dt)^{-n_0+a}\binom{n_0-n_{d-1}+(d+1)\mathsf{k}^2-a-|\mathbf{j}|}{n_0-a-|\mathbf{j}|}\prod_{i=1}^{d-1}\left(\frac{-d}{i}\right)^{j_i}\binom{\mathsf{k}^2-\delta_i}{j_i}.$$

Here the exponent of q is

$$\chi\ =\ \sum_{i=0}^{d-1}\big(n_i-(i+1)\mathsf{k}^2\big)\ =\ dn_0+\sum_{i=1}^{d-1}(d-i)\delta_i-\frac12 d(d+1)\mathsf{k}^2.$$

We combine the first and third products, collect powers of d and t, and write each n_i as $n_0+\delta_1+\ldots+\delta_i$. The result is the sum over $a,n_0\ge 0$ and all $j_i,\delta_i\ge 0$ of

$$(-1)^{\chi(\mathcal{O}_S)+\frac12 d(d-1)\mathsf{k}^2}\left(\frac{d!}{d^d}\right)^{\mathsf{k}^2}\left[\prod_{i=1}^{d-1}\binom{\mathsf{k}^2}{\delta_i}\binom{\mathsf{k}^2-\delta_i}{j_i}\left(\frac{-d}{i}\right)^{\delta_i+j_i}(-q)^{(d-i)\delta_i}\right]$$
$$\times\, q^{-\frac12 d(d+1)\mathsf{k}^2}\left[\prod_{i=1}^m(\mathsf{k}\cdot D_i)\right]\gamma_a(dt)^a\binom{(d+1)\mathsf{k}^2-|\delta|-a-|\mathbf{j}|}{n_0-a-|\mathbf{j}|}\big(-(-q)^d\big)^{n_0},$$

where we have used $|\delta|$ to denote $\delta_1+\ldots+\delta_{d-1}=n_{d-1}-n_0$.

Remarkably this horrible-looking expression can be summed. The sum over n_0 only involves the last 2 terms; using our convention (8) it takes the form

$$C\sum_{n_0\ge 0}\binom{r}{n_0-s}x^{n_0}\ =\ Cx^s(1+x)^r.$$

This replaces the last two terms with

$$\big(-(-q)^d\big)^{a+|\mathbf{j}|}\big(1-(-q)^d\big)^{(d+1)\mathsf{k}^2-|\delta|-a-|\mathbf{j}|}.$$

Setting $Q:=-q$, we write this as

$$\big(-Q^d\big)^a(1-Q^d)^{2\mathsf{k}^2-a}\prod_{i=1}^{d-1}(-Q^d)^{j_i}(1-Q^d)^{\mathsf{k}^2-\delta_i-j_i}.$$

Combining with the $\binom{\mathsf{k}^2-\delta_i}{j_i}\left(\frac{-d}{i}\right)^{j_i}$ term we can now sum over $j_i \geq 0$ using the binomial theorem again to give

$$(-1)^{\chi(\mathcal{O}_S)+\frac{1}{2}d(d-1)\mathsf{k}^2}\left(\frac{d!}{d^d}\right)^{\mathsf{k}^2}\left[\prod_{i=1}^{d-1}\binom{\mathsf{k}^2}{\delta_i}\left(\frac{-d}{i}\right)^{\delta_i}\left((1-Q^d)+\frac{dQ^d}{i}\right)^{\mathsf{k}^2-\delta_i}Q^{(d-i)\delta_i}\right]$$
$$\times(-Q)^{-\frac{1}{2}d(d+1)\mathsf{k}^2}\left[\prod_{i=1}^{m}(\mathsf{k}\cdot D_i)\right]\gamma_a(dt)^a\big(-Q^d\big)^a(1-Q^d)^{2\mathsf{k}^2-a}.$$

Moving $\big((d-1)!\big)^{\mathsf{k}^2}=\prod_{i=1}^{d-1}i^{\,\mathsf{k}^2}$ inside the product gives

$$(-1)^{\chi(\mathcal{O}_S)+d\,\mathsf{k}^2}\left(\frac{d}{d^d}\right)^{\mathsf{k}^2}\left[\prod_{i=1}^{d-1}\binom{\mathsf{k}^2}{\delta_i}\big(i(1-Q^d)+dQ^d\big)^{\mathsf{k}^2-\delta_i}\big(-dQ^{(d-i)}\big)^{\delta_i}\right]$$
$$\times Q^{-\frac{1}{2}d(d+1)\mathsf{k}^2}\left[\prod_{i=1}^{m}(\mathsf{k}\cdot D_i)\right]\gamma_a(dt)^a\big(-Q^d\big)^a(1-Q^d)^{2\mathsf{k}^2-a}.$$

So now we can sum over all $\delta_i \geq 0$ (by the binomial theorem again) and $a \geq 0$ to give the full expression:

$$(-1)^{\chi(\mathcal{O}_S)+d\,\mathsf{k}^2}\left(\frac{1}{d}\right)^{(d-1)\mathsf{k}^2}\left[\prod_{i=1}^{d-1}\big(i(1-Q^d)+dQ^d-dQ^{d-i}\big)^{\mathsf{k}^2}\right]$$
$$\times\left[\prod_{i=1}^{d-1}Q^{-\frac{1}{2}d\mathsf{k}^2}\right]Q^{-d\,\mathsf{k}^2}(1-Q^d)^{2\mathsf{k}^2}\left[\prod_{i=1}^{m}(\mathsf{k}\cdot D_i)\right]\sum_{a\geq 0}\gamma_a(dt)^a\left(\frac{-Q^d}{1-Q^d}\right)^a.$$

Combining the first two products gives

$$(-1)^{\chi(\mathcal{O}_S)+d\,\mathsf{k}^2}\left(\frac{1}{d}\right)^{(d-1)\mathsf{k}^2}\big(Q^{-d/2}-Q^{d/2}\big)^{2\mathsf{k}^2}\prod_{i=1}^{d-1}\Big((d-i)Q^{d/2}-dQ^{d/2-i}+iQ^{-d/2}\Big)^{\mathsf{k}^2}$$
$$\times\left[\prod_{i=1}^{m}(\mathsf{k}\cdot D_i)\right]\sum_{a\geq 0}\gamma_a\left(\frac{dtQ^d}{Q^d-1}\right)^a. \tag{82}$$

When there are no insertions the second line is 1 and we have determined $\mathsf{Z}^P_{d\mathsf{k}}(X)$. There is no t-dependence, of course, because the virtual dimension is already 0. This proves the first half of Theorem 1.4.

12. Final formula with descendents

Finally we compute the insertion term in (82). We recall the definition of the coefficients γ_a (68),

$$\begin{aligned}\sum_{a=0}^{\infty}\gamma_a X^a &= \prod_{i=1}^{m}\Big[E(t)\sum_{j=0}^{d-1}e^{X-jt}\Big]_{\alpha_i} \\ &= \prod_{i=1}^{m}\sum_{j=0}^{d-1}\sum_{k=0}^{\alpha_i}\frac{(-t)^k}{(k+1)!}\Big[e^{X-jt}\Big]_{\alpha_i-k} \\ &= \prod_{i=1}^{m}\sum_{j=0}^{d-1}\sum_{k=0}^{\alpha_i}\frac{(-t)^k}{(k+1)!}\frac{1}{(\alpha_i-k)!}(X-jt)^{\alpha_i-k} \\ &= -\prod_{i=1}^{m}\frac{t^{\alpha_i}}{(\alpha_i+1)!}\sum_{j=0}^{d-1}\sum_{k=0}^{\alpha_i}\binom{\alpha_i+1}{\alpha_i-k}(-1)^{k+1}(Xt^{-1}-j)^{\alpha_i-k} \\ &= -\prod_{i=1}^{m}\frac{t^{\alpha_i}}{(\alpha_i+1)!}\sum_{j=0}^{d-1}\Big[(Xt^{-1}-j-1)^{\alpha_i+1}-(Xt^{-1}-j)^{\alpha_i+1}\Big]\end{aligned}$$

by the binomial theorem. All terms of the sum cancel except for $j=0,d-1$, leaving

$$\sum_{a=0}^{\infty}\gamma_a X^a = t^{|\alpha|}\prod_{i=1}^{m}\frac{(Xt^{-1})^{\alpha_i+1}-(Xt^{-1}-d)^{\alpha_i+1}}{(\alpha_i+1)!}.$$

Substituting

$$X = \frac{dtQ^d}{Q^d-1} = \frac{dtQ^{d/2}}{Q^{d/2}-Q^{-d/2}}$$

from the second line of (82) gives

$$\sum_{a=0}^{\infty}\gamma_a X^a = t^{|\alpha|}\prod_{i=1}^{m}\frac{d^{\alpha_i+1}}{(\alpha_i+1)!}\;\frac{Q^{d(\alpha_i+1)/2}-Q^{-d(\alpha_i+1)/2}}{(Q^{d/2}-Q^{-d/2})^{\alpha_i+1}}.$$

Substituting this into (82) gives the proof of the second half of Theorem 1.4.

Remark 12.1. Consider (3) for any insertion of the form

$$\prod_{i=1}^{m_1}\tau_{\alpha_i}(D_i)\prod_{i=1}^{m_2}\tau_{\beta_i}([1]),$$

where $D_1,\ldots,D_{m_1}\in H^2(S)$ and $[1]\in H^0(S)$ denotes the Poincaré dual of S. Denote by $\pi_1 : C^{[n_0]}\times C\to C^{[n_0]}$ and $\pi_2 : \mathrm{Pic}^{n_0}(C)\times C\to\mathrm{Pic}^{n_0}(C)$ projections as at the start of Section 9. Expanding the explicit expression for the descendent integrands in Proposition 8.1 reduces (3) to a linear combinations of integrals of the form

$$\int_{C^{[n_0]}} \frac{1}{e(N^{\mathrm{vir}})} c_1\big(\mathcal{O}\big(\mathcal{Z}_0|_{C^{[n_0]}\times\{c_0\}}\big)\big)^a \cdot \big[\pi_{1*}c_1(\mathcal{O}(\mathcal{Z}_0))\big]^b,$$

for some $a, b \geq 0$. Using $\mathcal{O}(\mathcal{Z}_0) \cong (\mathrm{AJ}\times\mathrm{id})^*\mathcal{P}(1)$ gives

$$\begin{aligned} &At^{n_0}\int_{C^{[n_0]}} (74)\cdot\big[\pi_{*1}(c_1(\mathrm{AJ}^*\pi_{2*}\mathcal{P})+\omega)\big]^b \\ &\qquad\qquad = \ At^{n_0}\int_{C^{[n_0]}} (74)\bigg(\omega^b + bn_0\omega^{b-1} - b(b-1)\omega^{b-2}\theta\bigg), \end{aligned}$$

where (74) is the same integrand as before, and A is the constant defined in (62). These integrals can be performed using the Poincaré formula as before. In this generality we are unable to resum the resulting expression to a closed formula.

13. Links to Gromov-Witten theory of X

New. Let $X, \beta, \sigma_j, \alpha_j$ be as in the introduction and consider the Gromov-Witten generating function

(83)
$$\mathsf{Z}^{GW}_\beta(X, \tau_{\alpha_1}(\sigma_1)\cdots\tau_{\alpha_m}(\sigma_m)) := \sum_{g=0}^\infty \Bigg(\int_{[\overline{M}'_{g,m}(S,\beta)]^{\mathrm{vir}}} \frac{1}{e(N^{\mathrm{vir}})}\prod_{j=1}^m \tau_{\alpha_j}(\sigma_j)\Bigg)u^{2g-2}.$$

Here $\overline{M}'_{g,m}(S,\beta)$ is the moduli space of stable maps of genus g curves in class β to S with m marked points. The prime indicates we allow stable maps with possibly disconnected domain curve but without contracted connected components. The moduli space $\overline{M}'_{g,m}(S,\beta)$ is the $\mathbb{C}^*$-fixed locus of the (non-compact) 3-fold stable map space $\overline{M}'_{g,m}(X,\beta)$. On $\overline{M}'_{g,m}(S,\beta)$, the fixed part of the 3-fold GW complex is the surface GW complex (e.g. [KT1, Proposition 3.2]) and the moving part of the 3-fold GW complex is N^{vir}. The descendent insertions are defined using ψ-classes and the evaluation maps

$$\tau_{\alpha_j}(\sigma_j) := \psi_j^{\alpha_j}\,\mathrm{ev}_j^*\,\sigma_j.$$

In this section we apply the descendent-MNOP correspondence [PP1, PP2] to (83). In the case all descendence degrees are zero, the correspondence amount to the variable substitution $-q = e^{iu}$ [MNOP, PT1]. Clearly we have:

Lemma 13.1. *Suppose S has a reduced irreducible canonical curve. If the MNOP correspondence holds for $X=\mathrm{Tot}(K_S)$, then*

$$\mathsf{Z}^{GW}_\beta(X,\tau_0(\sigma_1)\cdots\tau_0(\sigma_m))=0,$$

unless β is an integer multiple of the canonical class k and no σ_j lies in $H^{\geq 3}(S)$.

Recall from the introduction that we denote by

$$\mathsf{Z}^{GW}_{d\mathsf{k}}(X,\tau_{\alpha_1}(D_1)\cdots\tau_{\alpha_m}(D_m))_{\mathrm{ver}}$$

the contribution to $\mathsf{Z}^{GW}_{d\mathsf{k}}(X,\tau_{\alpha_1}(D_1)\cdots\tau_{\alpha_m}(D_m))$ of all vertical components on the stable pair side after applying the descendent-MNOP correspondence.

Lemma 13.2. *Suppose S has a smooth irreducible canonical curve of genus $h=\mathsf{k}^2+1$, and the MNOP correspondence holds for $X=\mathrm{Tot}(K_S)$. Then $\mathsf{Z}^{GW}_{d\mathsf{k}}(X)_{\mathrm{ver}}$ equals*

$$(-1)^{\chi(\mathcal{O}_S)}(-d)^{(h-1)(1-d)}\left[2\sin\left(\frac{du}{2}\right)\right]^{2h-2}\times$$

$$\prod_{j=1}^{\frac{d-1}{2}}2^{h-1}\Big[d^2+j^2-jd+j(d-j)\cos(du)-d(d-j)\cos(ju)-jd\cos((d-j)u)\Big]^{h-1}$$

for d odd and

$$(-1)^{\chi(\mathcal{O}_S)}(-d)^{(h-1)(1-d)}\left[2\sin\left(\frac{du}{2}\right)\right]^{2h-2}\left[d\cos\left(\frac{du}{2}\right)-d\right]^{h-1}\times$$

$$\prod_{j=1}^{\frac{d-2}{2}}2^{h-1}\Big[d^2+j^2-jd+j(d-j)\cos(du)-d(d-j)\cos(ju)-jd\cos((d-j)u)\Big]^{h-1}$$

for d even. For $d=1,2$ these are the complete 3-fold generating functions.

Proof. The generating function $\mathsf{Z}^{P}_{d\mathsf{k}}(X)_{\mathrm{ver}}$ of Theorem 1.4 is invariant under $q\leftrightarrow q^{-1}$. More precisely: in $\prod_{j=1}^{d-1}(\cdots)$ the terms $(\cdots)$ for j and $d-j$ together form an expression invariant under $q\leftrightarrow q^{-1}$. According to the correspondence the generating function $\mathsf{Z}^{GW}_{d\mathsf{k}}(X)$ is obtained by setting $-q=e^{iu}$. The formula follows from a straight-forward computation. □

In the general case the descendent-MNOP correspondence of Pandharipande-Pixton [PP1, PP2] uses universal matrices

$$\tilde{\mathsf{K}}_{\mu\nu},\mathsf{K}_{\mu\nu}\in\mathbb{Q}[i,w_1,w_2,w_3]((u)),$$

indexed by all 2D partitions μ,ν. The matrix $\tilde{\mathsf{K}}_{\mu\nu}$ is defined in terms of $\mathsf{K}_{\mu\nu}$. The correspondence takes the following form:

$$\mathsf{Z}^{P}_{d\mathsf{k}}(X,\tau_{\alpha_1}(\sigma_1)\cdots\tau_{\alpha_m}(\sigma_m))=\mathsf{Z}^{GW}_{d\mathsf{k}}\big(X,\overline{\tau_{\alpha_1}(\sigma_1)\cdots\tau_{\alpha_m}(\sigma_m)}\big),$$

for any $\sigma_1, \ldots, \sigma_m \in H_T^*(X)$ and where

$$\overline{\tau_{\alpha_1}(\sigma_1) \cdots \tau_{\alpha_m}(\sigma_m)} := \sum_P \prod_{S \in P} \sum_\nu \tilde{\mathsf{K}}_{\mu_S, \nu} \, \tau_\nu(\sigma_S). \tag{84}$$

Here the first sum is over all set partitions P of $\{1, \ldots, m\}$, the second sum is over all 2D partitions ν, μ_S denotes the subpartition of μ defined by S, and

$$\sigma_S := \prod_{j \in S} \sigma_j.$$

Moreover for any partition ν of length l and any class $\sigma \in H_T^*(X, \mathbb{Q})$, the insertion $\tau_\nu(\sigma)$ is defined as follows: let

$$\Delta_*^X \sigma = \sum_{j_1, \ldots, j_l} \theta_{j_1}^\sigma \otimes \cdots \otimes \theta_{j_l}^\sigma$$

be the Künneth decomposition of $\Delta_*^X \sigma$ in X^l, then

$$\tau_\nu(\sigma) := \sum_{j_1, \ldots, j_l} \tau_{\nu_1 - 1}(\theta_{j_1}^\sigma) \cdots \tau_{\nu_l - 1}(\theta_{j_l}^\sigma).$$

Furthermore, for any partitions μ, ν, Pandharipande-Pixton prove $\mathsf{K}_{\mu\nu} = 0$ unless $\mu = \nu$ or $\mu \neq \nu$ and

$$|\nu| < |\mu| - |\ell(\mu) - \ell(\nu)|, \tag{85}$$

where $\ell(\nu)$ denotes the length of the partition ν [PP1, Thm. 2]. The following lemma allows us to relate Künneth decompositions on S^l, X^l.

Lemma 13.3. *Let $\pi : X \to S$ be projection and let $\Delta^S : S \subset S^l$, and $\Delta^X : X \subset X^l$ be inclusions of the small diagonals. Then*

$$\Delta_*^X \, \pi^* \, \sigma = (\pi \times \cdots \times \pi)^* \, \Delta_*^S \, (\mathsf{k}^{l-1} \cdot \sigma),$$

for all $\sigma \in H^*(S)$.

Proof. Let $\iota : S \subset X$ denote inclusion of the zero section and assume $\sigma \in H^*(S)$ is Poincaré dual to a cycle in $A_*(S)$. Consider the diagram

> MK: Need to come from Chow? You have better proof in mind?

$$\begin{array}{ccc} S & \xhookrightarrow{\iota} & X \\ {\scriptstyle \Delta^S} \big\downarrow & & \big\downarrow {\scriptstyle \Delta^X} \\ S^l & \xhookrightarrow[\iota \times \cdots \times \iota]{} & X^l. \end{array}$$

We use the refined Gysin map [Ful, Thm. 6.2]

$$(\iota \times \cdots \times \iota)^! : A_k(X) \to A_{k-l}(S)$$

and conclude

$$(\iota\times\cdots\times\iota)^*\,\Delta^X_*\,\pi^*\sigma=\Delta^S_*\,(\iota\times\cdots\times\iota)^!\,\pi^*\,\sigma.$$

By the excess intersection formula [Ful, Thm. 6.3]

$$(\iota\times\cdots\times\iota)^!\,\pi^*\,\sigma=c_1(\Delta^{S*}N_{S^l/X^l}/N_{S/X})\cdot\iota^*\,\pi^*\sigma=k^{l-1}\cdot\sigma. \qquad \square$$

Lemma 13.4. *Let $\sigma_1,\dots,\sigma_m\in H^*(S)$ and view them as classes in $H^*_T(X)$ via the usual identification* (12). *Then only ν with $\ell(\nu)=1$ in* (84) *contribute.*

Proof. **Step 1:** For $\sigma_1,\dots,\sigma_m\in H^{\geq 3}(S)$ the result follows immediately from Lemma 13.3. Later in the proof we need an intermediate vanishing result which we now derive. For $\sigma_1,\dots,\sigma_m\in H^{\geq 3}(S)$ and $m>0$, only the set partition $P=\{1\}\cup\cdots\cup\{m\}$ contributes in (84). Denote by (a) the partition of length 1 and size a. Then

$$\begin{aligned}\overline{\tau_{\alpha_1}(\sigma_1)\cdots\tau_{\alpha_m}(\sigma_m)}&=\prod_{j=1}^m\sum_b\tilde{\mathsf{K}}_{(\alpha_j+1),(b)}\,\tau_{b-1}(\sigma_j)\\&=\prod_{j=1}^m\sum_b\mathsf{K}_{(\alpha_j+1),(b)}\,\tau_{b-1}(\sigma_j)\\&=\sum_{b_1,\dots,b_m}\prod_{j=1}^m\mathsf{K}_{(\alpha_j+1),(b_j)}\,\tau_{b_j-1}(\sigma_j),\end{aligned}$$

where in the second line we used that $\tilde{\mathsf{K}}_{(a),\nu}=\mathsf{K}_{(a),\nu}$ for any partition ν [PP1, Sect. 7]. We conclude

$$\sum_{b_1,\dots,b_m}\prod_{j=1}^m\mathsf{K}_{(\alpha_j+1),(b_j)}\mathsf{Z}^{GW}_\beta(X,\tau_{b_1-1}(\sigma_1)\cdots\tau_{b_m-1}(\sigma_m))=\mathsf{Z}^P_\beta(X,\tau_{\alpha_1}(\sigma_1)\cdots\tau_{\alpha_m}(\sigma_m)),$$

for all $\alpha_1,\dots,\alpha_m\geq 0$. Inverting this relation and using Theorem 1.1 implies

$$\mathsf{Z}^{GW}_\beta(X,\tau_{\alpha_1}(\sigma_1)\cdots\tau_{\alpha_m}(\sigma_m))=0, \tag{86}$$

for all $\sigma_1,\dots,\sigma_m\in H^{\geq 3}(S)$ and $\alpha_1,\dots,\alpha_m\geq 0$.

Step 2: Assume $\sigma_1,\dots,\sigma_m\in H^*(S)$. Suppose $\tau_\nu(\sigma_S)$ in (84) contributes for some $S\in P$ and ν with $\ell(\nu)>1$. Then $\sigma:=\sigma_S\in H^k(S)$ for $k=0,1,2$ by Lemma 13.3. There are three cases:

(1) $k=2$. Then $\ell(\nu)=2$ and $\Delta^X_*\,\pi^*\,\sigma=(\pi\times\pi)^*\,\Delta^S_*\,(\mathsf{k}\cdot\sigma)$ with
$$\Delta^S_*\,(\mathsf{k}\cdot\sigma)\in H^8(S\times S),$$
by Lemma 13.3. Therefore the Künneth components of $\Delta^S_*\,(\mathsf{k}\cdot\sigma)$ have degrees $(4,4)$. This does not contribute by (86).

(2) $k = 1$. Then $\ell(\nu) = 2$ and $\Delta_*^X \pi^* \sigma = (\pi \times \pi)^* \Delta_*^S (\mathsf{k} \cdot \sigma)$ with

$$\Delta_*^S (\mathsf{k} \cdot \sigma) \in H^7(S \times S),$$

by Lemma 13.3. Hence the Künneth components of $\Delta_*^S (\mathsf{k} \cdot \sigma)$ have degrees $(4, 3)$, $(3, 4)$. This does not contribute by (86).

(3) $k = 0$. Then $\ell(\nu) = 2$ or $\ell(\nu) = 3$. If $\ell(\nu) = 2$ then $\Delta_*^X \pi^* \sigma = (\pi \times \pi)^* \Delta_*^S (\mathsf{k} \cdot \sigma)$ with

$$\Delta_*^S (\mathsf{k} \cdot \sigma) \in H^6(S \times S),$$

and the Künneth components of $\Delta_*^S (\mathsf{k} \cdot \sigma)$ have degrees $(4, 2)$, $(2, 4)$, $(3, 3)$. This does not contribute by (86). If $\ell(\nu) = 3$ then $\Delta_*^X \pi^* \sigma = (\pi \times \pi \times \pi)^* \Delta_*^S (\mathsf{k} \cdot \sigma)$ with

$$\Delta_*^S (\mathsf{k} \cdot \sigma) \in H^{10}(S \times S \times S),$$

and the Künneth components of $\Delta_*^S (\mathsf{k} \cdot \sigma)$ have degrees $(4, 4, 2)$, $(4, 3, 3)$ and permutations thereof. This does not contribute by (86). □

Theorem 13.5. *Suppose S has a reduced irreducible canonical curve. If the descendent-MNOP correspondence holds for $X = \mathrm{Tot}(K_S)$, then*

$$\mathsf{Z}_\beta^{GW}(X, \tau_{\alpha_1}(\sigma_1) \cdots \tau_{\alpha_m}(\sigma_m)) = 0,$$

unless β is an integer multiple of the canonical class k and no σ_j lies in $H^{\geq 3}(S)$.

Proof. **Step 1:** For $\sigma_1, \ldots, \sigma_m \in H^{\geq 3}(S)$ the result was established in the proof of Lemma 13.4 (Step 1: equation (86)).

Step 2: We now consider the case $\sigma_1, \ldots, \sigma_m \in H^{\geq 2}(S)$. Suppose β is not a multiples of k or at least one class lies in $H^{\geq 3}(S)$. We proceed by induction on the number of classes N in $H^2(S)$. For any N, only partitions ν with $\ell(\nu) = 1$ contribute in (84) by Lemma 13.4. For $N = 1$ only the set partition $P = \{1\} \sqcup \cdots \sqcup \{m\}$ contributes in (84) and we deduce

(87)
$$\sum_{b_1,\ldots,b_m} \prod_{j=1}^m \mathsf{K}_{(\alpha_j+1),(b_j)} \mathsf{Z}_\beta^{GW}(X, \tau_{b_1-1}(\sigma_1) \cdots \tau_{b_m-1}(\sigma_m)) = \mathsf{Z}_\beta^P(X, \tau_{\alpha_1}(\sigma_1) \cdots \tau_{\alpha_m}(\sigma_m)).$$

Inverting and using Theorem 1.1 proves the case $N = 1$. Let $N > 1$ and suppose the cases $1, \ldots, N-1$ are established. Any set partition $P \neq \{1\} \sqcup \cdots \sqcup \{m\}$ in (84) does not contribute by the induction hypothesis. Again we obtain (87) and vanishing follows by inverting and using Theorem 1.1.

Step 3: We now consider the case $\sigma_1, \ldots, \sigma_m \in H^{\geq 1}(S)$. Suppose β is not a multiples of k or at least one class lies in $H^{\geq 3}(S)$. We use induction on the number of classes N in $H^1(S)$. Suppose $N = 1$ and let M be the number of classes in $H^2(S)$. For $M = 0$ only the set partition $P = \{1\} \sqcup \cdots \sqcup \{m\}$ contributes in (84) (cupping the single H^1 class with a class in H^3 does not contribute by previous steps). We obtain (87) and vanishing follows by inverting and using Theorem 1.1. As in the previous step we can establish all cases $(N, M) = (1, M)$ by induction on M and all cases (N, M) by induction on N.

Step 4: We finally consider the case $\sigma_1, \ldots, \sigma_m \in H^*(S)$. Suppose β is not a multiples of k or at least one class lies in $H^{\geq 3}(S)$. This case follows by a triple induction on (N, M, K), where N is the number of classes in $H^0(S)$, M is the number of classes in $H^1(S)$ and K is the number of classes in $H^2(S)$. $\square$

In the previous proof we showed that if $\sigma_1, \ldots, \sigma_m \in H^{\geq 2}(S)$, then only the set partition $P = \{1\} \sqcup \cdots \sqcup \{m\}$ contributes in (84) and therefore

$$\sum_{b_1, \ldots, b_m} \prod_{j=1}^{m} \mathsf{K}_{(\alpha_j+1),(b_j)} \mathsf{Z}_{\beta}^{GW}(X, \tau_{b_1-1}(\sigma_1) \cdots \tau_{b_m-1}(\sigma_m)) = \mathsf{Z}_{\beta}^{P}(X, \tau_{\alpha_1}(\sigma_1) \cdots \tau_{\alpha_m}(\sigma_m)),$$

or equivalently

$$\sum_{b_1, \ldots, b_m} \prod_{j=1}^{m} \mathsf{K}^{-1}_{(\alpha_j+1),(b_j)} \mathsf{Z}_{\beta}^{P}(X, \tau_{b_1-1}(\sigma_1) \cdots \tau_{b_m-1}(\sigma_m)) = \mathsf{Z}_{\beta}^{GW}(X, \tau_{\alpha_1}(\sigma_1) \cdots \tau_{\alpha_m}(\sigma_m)).$$

The correspondence requires us to substitute $w_i = c_i^T(T_X)$ for $i = 1, 2, 3$. Using $T_X|_S = T_S \oplus K_S \otimes \mathfrak{t}$, we see that $w_1 = t$, $w_2 = c_2(S) - \mathsf{k}^2 - \mathsf{k}\, t$, and $w_3 = c_2(S)t$. Any occurrence of w_2 or w_3 contributes zero by Theorem 13.5. We summarize:

Proposition 13.6. *For any $\sigma_1, \ldots, \sigma_m \in H^{\geq 2}(S)$, the generating function $\mathsf{Z}_{d\mathsf{k}}^{GW}(X, \tau_{\alpha_1}(\sigma_1) \cdots \tau_{\alpha_m}(\sigma_m))$ is equal to*

$$\sum_{\substack{1 \leq b_1 \leq \alpha_1 + 1 \\ \cdots \\ 1 \leq b_m \leq \alpha_m + 1}} \prod_{j=1}^{m} \mathsf{K}^{-1}_{(\alpha_j+1),(b_j)} \Big|_{w_1 = t, w_2 = w_3 = 0} \cdot \mathsf{Z}_{d\mathsf{k}}^{P}(X, \tau_{b_1-1}(\sigma_1) \cdots \tau_{b_m-1}(\sigma_m)),$$

where

$$\mathsf{K}_{(a),(b)} \Big|_{w_1 = t,\, w_2 = w_3 = 0} \in \mathbb{Q}[i, t]((u)).$$

Theorem 13.7. *Suppose S has a smooth irreducible canonical curve of genus $h = \mathsf{k}^2 + 1$. Suppose the descendent-MNOP correspondence holds for $X =$*

$\mathrm{Tot}(K_S)$. *Then* $\mathsf{Z}^{GW}_{d\mathsf{k}}(X, \tau_{\alpha_1}(D_1)\cdots\tau_{\alpha_m}(D_m))_{\mathrm{ver}}$ *equals*

$$t^{|\alpha|}\, \mathsf{Z}^{GW}_{d\mathsf{k}}(X)_{\mathrm{ver}} \prod_{i=1}^{m} (d\mathsf{k}\cdot D_i)$$

times by

$$\sum_{\substack{1\le b_1\le \alpha_1+1\\ \cdots \\ 1\le b_m\le \alpha_m+1}} \prod_{j=1}^{m} \mathsf{K}^{-1}_{(\alpha_j+1),(b_j)}\Big|_{w_1=t, w_2=w_3=0} \cdot \frac{1}{b_j!}\left(\frac{-id}{2}\right)^{b_j-1} \frac{\sin(b_j\, du/2)}{\sin^{b_j}(du/2)}.$$

For $d = 1, 2$ these are the complete 3-fold generating functions.

Proof. The part without descendents was done in Lemma 13.2. The descendent term of Theorem 1.4 is invariant under $q \leftrightarrow q^{-1}$ up to a sign $(-1)^{|\alpha|}$. Setting $-q = e^{iu}$ this term becomes

$$t^{|\alpha|} \prod_{j=1}^{m} (\mathsf{k}\cdot D_j)(-i/2)^{\alpha_j} \frac{d^{\alpha_j+1}}{(\alpha_j+1)!} \frac{\sin((\alpha_j+1)\, du/2)}{\sin^{\alpha_j+1}(du/2)}.$$

Combining with Proposition 13.6 gives the desired result. □

14. Links to Gromov-Witten theory of S

New. Consider the 3-fold GW generating function of the previous Section

$$\mathsf{Z}^{GW}_{\beta}(X, \tau_{\alpha_1}(\sigma_1)\cdots\tau_{\alpha_m}(\sigma_m)).$$

The virtual dimension of $[\overline{M}'_{g,m}(S,\beta)]^{\mathrm{vir}}$ is equal to

$$g - 1 + \int_\beta c_1(S) + m.$$

Next *define* g by the equation

$$g - 1 + \int_\beta c_1(S) + m = \sum_{j=1}^{m} \Big(\alpha_j + \frac{1}{2}\deg(\sigma_j)\Big), \tag{88}$$

where $\sigma_j \in H^{\deg \sigma_j}(S, \mathbb{Q})$.

The 3-fold GW generating function of the previous Section is a Laurent series in u and its leading coefficient is a surface GW invariant. This can be seen as follows. Denote the universal curve by

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{f} & S \\ \downarrow{\scriptstyle \pi} & & \\ \overline{M}'_{g,m}(S,\beta). & & \end{array}$$

Then by [KT1, Proposition 3.2]

$$N^{\mathrm{vir}} = R\pi_* f^* K_S \otimes \mathfrak{t}.$$

Using (60), this implies (see also Section 3.1 of [KT1])

$$\frac{1}{e(N^{\mathrm{vir}})} = t^r + \cdots ,$$

$$r := -\operatorname{rk}(N^{\mathrm{vir}}) = g - 1 + \int_\beta c_1(S) = \sum_{j=1}^m \Big(\alpha_j + \frac{1}{2}\deg\sigma_j - 1\Big),$$

where $\cdots$ refers to terms of order $< r$ in t with coefficient in $H^{>0}(\overline{M}'_{g,m}(S,\beta))$. We obtain

(89)

$$\mathsf{Z}_\beta^{GW}(X, \tau_{\alpha_1}(\sigma_1)\cdots\tau_{\alpha_m}(\sigma_m)) = \Big(t^r \int_{[\overline{M}'_{g,m}(S,\beta)]^{\mathrm{vir}}} \prod_{j=1}^m \tau_{\alpha_j}(\sigma_j)\Big) u^{2g-2} + O(u^{2g-1}),$$

where $O(u^{2g-1})$ stands for terms of order $u^{>2g-2}$. Hence we can read off the surface GW invariants from the leading coefficient of the 3-fold GW generating function. Consequently Theorem 13.5 immediately implies the following well-known vanishing:

Corollary 14.1. *Suppose S has a reduced irreducible canonical curve and let g be defined by* (88). *If the descendent-MNOP correspondence holds for $X = \operatorname{Tot}(K_S)$, then*

$$\int_{[\overline{M}'_{g,m}(S,\beta)]^{\mathrm{vir}}} \prod_{j=1}^m \tau_{\alpha_j}(\sigma_j) = 0,$$

unless β is an integer multiple of the canonical class k and no σ_j lies in $H^{\geq 3}(S)$.

This vanishing result is originally due to Lee-Parker [LP] (of course not assuming MNOP). See also [MP] and [KL1, KL2] for a discussion in the algebro-geometric setup.

Next we assume S has a connected smooth canonical curve and we consider the GW invariants

MK: MP, KL always take S general type and minimal. Why?

$$\int_{[\overline{M}'_{g,m}(S,d\mathsf{k})]^{\mathrm{vir}}} \prod_{j=1}^{m} \tau_{\alpha_j}(D_j). \tag{90}$$

We start with the case of no insertions $m = 0$. Then Lee-Parker prove

$$\int_{[\overline{M}'_{g,0}(S,d\mathsf{k})]^{\mathrm{vir}}} 1 = \sum_u \frac{(-1)^{h^0(u^*K_S|_C)}}{|\operatorname{Aut}(u)|}, \tag{91}$$

where the sum is over all degree d étale maps $u : \Sigma \to C$ and $K_S|_C$ should be viewed as a theta characteristic of C, i.e. a square root of the canonical bundle K_C. This was proved in the algebro-geometric setup by Kiem-Li [KL1, KL2]. These "signed" counts of étale covers of C are known as unramified spin Hurwitz numbers. They have been calculated by Gunningham using a TQFT formalism [Gun].

Corollary 14.2. *Suppose that S has a smooth irreducible canonical curve of genus $h = \mathsf{k}^2 + 1$. Suppose the MNOP correspondence holds for $X = \operatorname{Tot}(K_S)$. Then the vertical contribution to the unramified spin Hurwitz number* (91) *is*

$$(-1)^{\chi(\mathcal{O}_S)} \left(\frac{2^{\frac{d-1}{2}}}{d!} \right)^{2-2h}.$$

For $d = 1, 2$ this is the entire unramified spin Hurwitz number (91).

Remark 14.3. Gunningham's formula for (91) is of the form [Gun]

$$\sum_{\substack{\mu \,\vdash\, d \\ \mu \ \text{strict}}} (-1)^{\chi(\mathcal{O}_S)\, \ell(\mu)}\, (d_\mu)^{2-2h},$$

where the sum is over all partitions $\mu = (\mu_1 \geq \mu_2 \geq \cdots)$ of d, strict means $\mu_1 > \mu_2 > \cdots$, and $\ell(\mu)$ denotes the length of the partition, i.e. the biggest i for which $\mu_i \neq 0$. Moreover d_μ is some explicit combinatorial number associated to μ and related to representations of the Sergeev algebra. Our vertical contribution correctly reproduces the term corresponding to $\mu = (d)$. We study further relations between stable pairs on $X = \operatorname{Tot}(K_S)$ and Gunningham's formula in a sequel [KT4].

Proof of Corollary 14.2. By (89), it suffices to extract the leading term in u of the 3-fold generating function of Lemma 13.2. For both d odd or even, the first line of the formula for $\mathsf{Z}^{GW}_{d\mathsf{k}}(X)_{\mathrm{ver}}$ is easy and we only need to expand to lowest non-zero order in u using

$$\sin(x) = x + O(x^2),\ \cos(x) = 1 + O(x).$$

The terms in the second line for $\mathsf{Z}^{GW}_{d\mathsf{k}}(X)_{\mathrm{ver}}$ only contain cosines. Interestingly the lowest order contribution of each term in the product comes from a Taylor expansion up to order four (all the terms of order < 4 cancel), so we use

$$\cos(x) = 1 - \frac{x^2}{2} + \frac{x^4}{24} + O(x^5).$$

Computing up to this order gives the following leading term

$$(-1)^{\chi(\mathcal{O}_S)} \left(\frac{2^{\frac{d-1}{2}}}{d!} \right)^{2-2h} u^{d(2h-2)}. \qquad \square$$

MK: Note the subtle $(-2)^{-\alpha}$ versus $(-2)^{\alpha}$ for $d = 1, 2$. We indeed get this too!

Finally we consider (90) with descendents so $m > 0$. For $d = 1, 2$, Maulik-Pandharipande [MP] conjectured the following formulae

(92)
$$\int_{[\overline{M}'_{g,m}(S,\mathsf{k})]^{\mathrm{vir}}} \prod_{j=1}^m \tau_{\alpha_j}(D_j) = (-1)^{\chi(\mathcal{O}_S)} \prod_{j=1}^m (\mathsf{k} \cdot D_j) \frac{\alpha_j!}{(2\alpha_j+1)!} (-2)^{-\alpha_j},$$
$$\int_{[\overline{M}'_{g,m}(S,2\mathsf{k})]^{\mathrm{vir}}} \prod_{j=1}^m \tau_{\alpha_j}(D_j) = (-1)^{\chi(\mathcal{O}_S)} 2^{h-1} \prod_{j=1}^m 2(\mathsf{k} \cdot D_j) \frac{\alpha_j!}{(2\alpha_j+1)!} (-2)^{\alpha_j},$$

These formulae were proved by Kiem-Li [KL1, KL2] using cosection localisation on the moduli space of stable maps and later by Lee in symplectic geometry [Lee].

The leading term of the generating function $\mathsf{Z}^{GW}_{d\mathsf{k}}(X, \tau_{\alpha_1}(D_1) \cdots \tau_{\alpha_m}(D_m))$ has order u^{2g-2}, where

$$2g - 2 = d(2h-2) + 2|\alpha|,$$

by (88). Fixing $d = 1$ or $d = 2$, Theorem 13.7 implies

(93)
$$\sum_{\substack{1 \le b_1 \le \alpha_1 + 1 \\ \cdots \\ 1 \le b_m \le \alpha_m + 1}} \prod_{j=1}^m \mathsf{K}^{-1}_{(\alpha_j+1),(b_j)}\Big|_{w_1=t, w_2=w_3=0} \cdot \frac{1}{b_j!} \left(\frac{-id}{2} \right)^{b_j - 1} \frac{\sin(b_j\, du/2)}{\sin^{b_j}(du/2)} = O(u^{2|\alpha|}).$$

Observation: equation (93) for $m = 1$ implies equation (93) for all $m \ge 1$. A priori all we know is that $\mathsf{K}_{(a),(b)}|_{w_1=t,w_2=w_3=0}$ is nonzero unless $b \le a$ in which case

$$\mathsf{K}_{(a),(b)}\Big|_{w_1=t,w_2=w_3=0} = t^{a-b} f_{ab}(u),$$

for some $f_{ab}(u) \in \mathbb{Q}[i]((u))$ and $f_{aa} = (iu)^{1-a}$ [PP1, Thm. 2, Thm. 3]. Since $f_{ab}(u)$ could have many terms, (93) does not determine $f_{ab}(u)$.

Conjecture 14.4. *For each $a \geq b \geq 1$, we have*

$$\mathsf{K}_{(a),(b)}\Big|_{w_1=t, w_2=w_3=0} = t^{a-b} K_{ab}\, u^{1-a},$$

for some $K_{ab} \in \mathbb{Q}[i]$.

The direct evidence for this is little: it is known that this is true for $a = b$ in which case $K_{aa} = i^{1-a}$ [PP1, Thm. 2] and $K_{21} = i^{-1}$ [PP1, Sect. 2.5]. Note that Conjecture 14.4 is equivalent to

$$\mathsf{K}^{-1}_{(a),(b)}\Big|_{w_1=t, w_2=w_3=0} = t^{a-b} L_{ab}\, u^{b-1}, \tag{94}$$

where $L_{ab} \in \mathbb{Q}[i]$ is inverse to K_{ab}. Our motivation for Conjecture 14.4 is:

Theorem 14.5. *Assume Conjecture 14.4 and the descendent-MNOP correspondence holds for $X = K_S$ where S has a smooth connected canonical curve. Let $\sigma_1, \ldots, \sigma_m \in H^2(S)$. Then the vertical contribution to $\int_{[\overline{M}'_{g,m}(S,\beta)]^{\mathrm{vir}}} \prod_{j=1}^m \tau_{\alpha_j}(D_j)$ equals*

$$(-1)^{\chi(\mathcal{O}_S)} \left(\frac{2^{\frac{d-1}{2}}}{d!} \right)^{2-2h} \prod_{j=1}^{m} (d\mathsf{k} \cdot D_j) \frac{\alpha_j!}{(2\alpha_j+1)!} (-2)^{-\alpha_j} d^{2\alpha_j}.$$

In particular, Maulik-Pandharipande's formulae for $d = 1, 2$ are true.

Proof of Theorem 14.5. By the correspondence, we have (93) for $d = 1, 2$ and any $\alpha_1, \ldots, \alpha_m$. By Conjecture 14.4, the matrix entries $\mathsf{K}^{-1}_{(a),(b)}|_{w_1=t,w_2=w_3=0}$ are entirely determined by coefficients $K_{ab} \in \mathbb{Q}[i]$, or equivalently $L_{ab} \in \mathbb{Q}[i]$. By the uniqueness statement of Theorem A.1 in the Appendix, the coefficients L_{ab} are *uniquely* determined by (93) for $d = 1$, $m = 1$. From the second part of Theorem A.1, we deduce Maulik-Pandharipande's formula for $d = 1$, $m = 1$. The multiplicative nature of (93) implies the result for $d = 1, 2$ and any $m \geq 1$. $\square$

Remark 14.6. The proof of Theorem A.1 actually gives a formula for $c_n(\alpha)$ (see (97) in Appendix A) and hence the matrix coefficients $\mathsf{K}^{-1}_{(a),(b)}|_{w_1=t,w_2=w_3=0} = t^{a-b} L_{ab} u^{b-1}$, namely

$$L_{ab} = i^{b-1}(-1)^{a+1} \sum_{j=1}^{b} (-1)^{b-j} \binom{b}{j} j^{b-a}.$$

Using equation (98) of Appendix A we deduce the following formula for the vertical contribution of all descendent GW invariants

$$\frac{\sum_{\alpha_1,\ldots,\alpha_m\geq 0}\mathsf{Z}^{GW}_{\mathsf{dk}}(X,\tau_{\alpha_1}(D_1)\cdots\tau_{\alpha_m}(D_m))_{\mathrm{ver}}\prod_{j=1}^m v_j^{\alpha_j}}{\mathsf{Z}^{GW}_{\mathsf{dk}}(X,\tau_0(D_1)\cdots\tau_0(D_m))_{\mathrm{ver}}}$$
$$=\prod_{j=1}^m\sum_{n\geq 1}\frac{\sin(n\,du/2)\,(du/2)^{n-1}}{\sin^n(du/2)}\frac{(tv_j)^n}{(tv_j)(tv_j+1)\cdots(tv_j+n)},$$

where t is the equivariant parameter and $v_1,\ldots,v_m$ are formal variables.

Appendix A.
Combinatorial identity
by Aaron Pixton and Don Zagier

Theorem A.1. *For each $\alpha\in\mathbb{Z}_{>0}$, there exist unique $\{c_n(\alpha)\}_{n=1}^{\alpha}$ with $c_\alpha(\alpha)=\frac{(-1)^{\alpha-1}}{\alpha!}$ such that*

$$\sum_{n=1}^{\alpha}c_n(\alpha)\frac{x^n\sin(nx)}{\sin^n x} \;=\; A_\alpha x^{2\alpha-1}+O(x^{2\alpha}), \tag{95}$$

for some $A_\alpha\in\mathbb{R}$. Moreover, the leading coefficient A_α is then

$$A_\alpha \;=\; \frac{1}{(2\alpha-1)!!} \;=\; \frac{1}{(2\alpha-1)(2\alpha-3)\cdots 1}\,. \tag{96}$$

Proof. **Existence**. We first show that one solution to (95) is given by

$$c_n(\alpha):=\sum_{k=1}^{n}\frac{(-1)^{n-k}k^{n-\alpha}}{k!(n-k)!}\,. \tag{97}$$

Notice that these $c_n(\alpha)$ equal the nth difference $\Delta_1^n\big(x^{n-\alpha}/n!\big)$ at $x=0$ when $n>\alpha>0$. Therefore they vanish in this range, and their generating series $L_\alpha(y):=\sum_{n=1}^\infty c_n(\alpha)\,y^n$ is a *polynomial* in y. So we may substitute $y=\frac{xe^{ix}}{\sin x}$ and split into real and imaginary parts:

$$F_\alpha(x):=L_\alpha\left(\frac{xe^{ix}}{\sin x}\right) \;=\; E_\alpha(x)+iO_\alpha(x).$$

We see that $E_\alpha(x)\in\mathbb{Q}[\![x^2]\!]$ is even, while $O_\alpha(x)\in x\,\mathbb{Q}[\![x^2]\!]$ is odd and equals the left hand side of (95).

Since $nc_n(\alpha)=c_n(\alpha-1)-c_{n-1}(\alpha-1)$ we have

$$L'_\alpha(y)=\frac{1-y}{y}L_{\alpha-1}(y)\quad\text{and so}\quad F'_\alpha(x)=f(x)F_{\alpha-1}(x),$$

where $f(x) = x + (1 - x\cot(x))^2/x \ \in \ x\,\mathbb{Q}[\![x^2]\!]$. Taking odd parts,

$$O'_\alpha(x) = f(x) O_{\alpha-1}(x).$$

Equation (95) with $A_\alpha = 1/(2\alpha-1)!!$ now follows by induction on α and the fact that $f(x) = x + O(x^3)$.

Uniqueness. This actually follows from existence. Rewrite (95) as

$$\sum_{n=1}^{\alpha-1} c_n(\alpha) \frac{x^n \sin(nx)}{\sin^n x} \ = \ \frac{(-1)^\alpha}{\alpha!} \frac{x^\alpha \sin(\alpha x)}{\sin^\alpha x} \ + \ O(x^{2\alpha-1}).$$

Expanding both sides in odd powers of x and letting M_{ij} be the coefficient of x^{2i-1} in $x^j \sin(jx)/\sin^j x$, it is equivalent to the linear equations

$$M_{\le \alpha-1}\,\boldsymbol{c} \ = \ \boldsymbol{b}.$$

Here $M_{\le n} := (M_{ij})_{1 \le i,j \le n}$ is the $n \times n$ truncation of M, the vector of unknowns is $\boldsymbol{c} = \big(c_1(\alpha), \dots, c_{\alpha-1}(\alpha)\big)^t$ and $\boldsymbol{b} \in \mathbb{R}^{\alpha-1}$ is given. Therefore uniqueness is equivalent to showing that $\det(M_{\le\alpha-1}) \ne 0 \ \ \forall \alpha$.

Expanding (95) in odd powers of x, we see our existence result gives an infinite upper triangular matrix $C = (c_i(j))_{i \le j}$ such that MC is a lower triangular matrix with diagonal entries $A_\alpha = 1/(2\alpha-1)!!$. Since these are non-zero, $(MC)_{\le\alpha-1} = M_{\le\alpha-1} C_{\le\alpha-1}$ has non-zero determinant and so does $M_{\le\alpha-1}$. □

Remark A.2. For the application to GW theory the following formula for the generating series of the left hand sides of (95) is useful:

(98)
$$\sum_{\alpha=1}^{\infty} (-v)^{\alpha-1} \sum_{n=1}^{\alpha} c_n(\alpha) \frac{x^n \sin(nx)}{\sin^n x} \ = \ \sum_{n=1}^{\infty} \frac{x^n \sin(nx)}{\sin^n x} \frac{v^n}{v(v+1)\cdots(v+n)}\,.$$

It can be proved by making a geometric series expansion of each $1/(v+k)$ on the right hand side, collecting all $v^{\alpha-1}\frac{x^n \sin(nx)}{\sin^n x}$ terms, and then showing (by induction on α) that the coefficient is $(-1)^{\alpha-1}$ times by the explicit formula (97) for $c_n(\alpha)$.

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