Simple proofs of Ramanujan's congruences for the partition function

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Let $p(n)$ denote the number of partitions of n. Ramanujan proved that $p(n)$ is divisible by ℓ whenever $\ell \in \{5,\,7,\,11\}$ and $24n-1$ is divisible by ℓ. Many proofs of these congruences are now known. The one here is inspired by those given in a recent paper by M. Hirschhorn (JNT **139**, 2014) and is substantially the same as his for $\ell = 5$ or $\ell = 7$, but considerably simpler when $\ell = 11$.

Theorem (Ramanujan). *If $\ell \in \{5,\,7,\,11\}$ and $24\,n \equiv 1 \pmod{\ell}$, then $p(n) \equiv 0 \pmod{\ell}$.*

Proof. For each of these primes (and, by a famous theorem of Serre, for no other primes $\ell > 3$) the $(\ell-1)$st power of the Dedekind eta-function can be represented as a binary theta series. Choosing one such representation in each case, we find

$$\ell = 5: \quad \eta(\tau)^4 \;=\; \sum_{\substack{a\equiv 1 \pmod 6\\ b\equiv 1 \pmod 4}} (-1)^{[a/6]+[b/4]}\, b\, Q^{a^2+3b^2} \;\equiv\; \sum Q^{(\mathrm{R\ or\ 0})\,+\,\mathrm{N}} \;=\; \sum Q^{\not\equiv 0}\,,$$

$$\ell = 7: \quad \eta(\tau)^6 \;=\; \sum_{\substack{a\equiv 1 \pmod 4\\ b\equiv 1 \pmod 4}} (-1)^{[a/4]+[b/4]}\, ab\, Q^{3a^2+3b^2} \;\equiv\; \sum Q^{\mathrm{N}\,+\,\mathrm{N}} \;=\; \sum Q^{\not\equiv 0}\,,$$

$$\ell = 11: \quad \eta(\tau)^{10} \;=\; \sum_{\substack{a\equiv 2 \pmod 6\\ b\equiv 1 \pmod 6}} \frac{ab(a^2-b^2)}{6}\, Q^{2a^2+2b^2} \;\equiv\; \sum Q^{\mathrm{N}\,+\,\mathrm{N}} \;=\; \sum Q^{\not\equiv 0}\,,$$

where $Q = e^{\pi i\tau/12}$ and where R (resp. N, 0, $\equiv$) denotes quadratic residues (resp. non-residues, zero, congruence) modulo ℓ. Hence in all three cases we have

$$\sum_{n\ge 0} p(n)\, Q^{24n-1} \;=\; \frac{1}{\eta(\tau)} \;\equiv\; \frac{\eta(\tau)^{\ell-1}}{\eta(\ell\tau)} \;\equiv\; \frac{\sum Q^{\not\equiv 0 \pmod{\ell}}}{\sum Q^{\equiv 0 \pmod{\ell}}} \;=\; \sum Q^{\not\equiv 0 \pmod{\ell}}\,. \qquad \square$$