NU-INVARIANTS OF EXTRA-TWISTED CONNECTED SUMS

SEBASTIAN GOETTE, JOHANNES NORDSTRÖM, AND DON ZAGIER

Abstract. We analyse the possible ways of gluing twisted products of circles with asymptotically cylindrical Calabi-Yau manifolds to produce manifolds with holonomy G_2, thus generalising the twisted connected sum construction of Kovalev and Corti, Haskins, Nordström, Pacini. We then express the extended ν-invariant of Crowley, Goette, and Nordström in terms of fixpoint and gluing contributions, which include different types of (generalised) Dedekind sums. Surprisingly, the calculations involve some non-trivial number-theoretical arguments connected with special values of the Dedekind eta-function and the theory of complex multiplication. One consequence of our computations is that there exist compact G_2-manifolds that are not G_2-nullbordant.

Though compact Riemannian manifolds with holonomy G_2, or equivalently compact manifolds with a torsion-free G_2-structure, have been constructed more than 25 years ago, we still do not know very much about them. On the one hand, only a few obstructions against G_2-metrics on a given compact 7-manifold are known (see Joyce [24, §10.2]). On the other hand, our current supply of examples is much smaller than allowed by these obstructions. It is therefore still interesting to explore new invariants of G_2-manifolds in the hope to discover new obstructions, and to find new examples on which these invariants can be tested.

The extended ν-invariant was introduced in [14] to exhibit 2-connected 7-manifolds with a disconnected moduli space of G_2-metrics. In the present paper, we apply it to a larger class of examples to find the first examples of G_2-manifolds whose G_2-bordism class can be shown to be nontrivial. We compute the η-invariants that appear in the definition of the extended ν-invariant using gluing formulas as well as variation and adiabatic limit formulas for manifolds with boundary. The details may be of interest to index theorists because in contrast to [14], we cannot rely on spectral symmetry here.

Extra-twisted connected sums. There are currently two major sources of compact G_2-manifolds, that is, compact Riemannian manifolds whose holonomy group is isomorphic to G_2. The first is Joyce's Kummer construction [24], based on resolution of singularities in flat orbifolds. It has recently been generalised by Joyce and Karigiannis to more complicated spaces [25]. The second is the twisted connected sum construction pioneered by Kovalev [27] and systematically studied by Corti, Haskins, Nordström and Pacini [12, 13].

For the latter construction, one starts with two asymptotically cylindrical Calabi-Yau manifolds $V_\pm$. The cross-sections of their ends approach the products of a K3 surface $\Sigma_\pm$ and a circle $S^1_{\zeta_\pm}$ that we wish to call the *interior* circle. In the classical set up, one takes the products of $V_\pm$ and an *exterior* circle $S^1_{\xi_\pm}$ of length $\xi_\pm = \zeta_\mp$. Then one glues truncated copies $M_\pm$ of $V_\pm \times S^1_{\xi_\pm}$ along their ends, in a way that swaps the roles of interior and exterior circles.

In this paper, we assume in addition that some finite cyclic groups $\Gamma_\pm \cong \mathbb{Z}/k_\pm$ act on $V_\pm$, preserving the Calabi-Yau structure. We also assume that the induced actions on the K3 factors $\Sigma_\pm$ are trivial, and that the induced actions on the interior circles $S^1_{\zeta_\pm}$ are free. We

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consider twisted products $M_\pm \cong (V_\pm \times S^1_{\xi_\pm})/\Gamma_\pm$, where $\Gamma_\pm$ acts diagonally, and freely on the exterior circle factors. The manifolds M_+ and M_- have ends whose asymptotic cross-sections are isometric to products of $\Sigma_\pm$ with a 2-torus, but these 2-tori need now not be isometric to products of two circles. Often one can arrange that the two 2-tori are isometric, and glue M_+ to M_- with an "extra" twist. The cases where $k_\pm \leq 2$ have already been considered in [14, 31].

The known supply of asymptotically cylindrical Calabi-Yau manifolds with nontrivial symmetries that fix the K3 surface Σ is very limited. In the present paper, we use examples constructed from Fano threefolds of higher index (that is, closed complex 3-folds Y whose anticanonical bundle $-K_Y$ is ample and divisible by an integer > 1), and from hypersurfaces in weighted projective spaces. We only consider examples whose Picard rank $b_2(Y)$ is 1 for simplicity. The possible groups obtained this way are $\Gamma \cong \mathbb{Z}/k$ with $k \leq 6$; see Table 1.

But even though we use only a few asymptotically cylindrical Calabi-Yau manifolds with nontrivial symmetry group, there are typically several different ways to glue two given twisted products $M_\pm$ by changing the size of the exterior circles and the isometry between the torus factors in the asymptotic cross sections. Some of the G_2-manifolds we construct this way will not be simply connected—we obtain cyclic fundamental groups of order up to 21; see Proposition 1.11 and example 250 of Table 2—but the universal covers of these examples will again be extra-twisted connected sums; see Proposition 3.9.

Apart from the choice of V_+, V_- and Γ_+, Γ_-, an extra-twisted connected sum is described by two square matrices encoding the gluing of the tori and of the K3 surfaces, and a pair units $\varepsilon_+ \in \mathbb{Z}/k_+$, $\varepsilon_- \in \mathbb{Z}/k_-$ that fix the actions of $\Gamma_\pm$. These data have to satisfy the conditions described in Proposition 1.8. There are typically several ways to obtain the same extra-twisted connected sum up to isometries and orientation reversal; see Proposition 3.5. In addition, passing to dual tori often leads to non-isometric extra-twisted connected sums built from the same blocks with the same K3-matching and the same gluing angle. We thus have a kind of partial t-duality for extra-twisted connected sums; see Proposition 3.7 and Example 3.12.

Our combinatorial description of extra-twisted connected sums allows us to find all possible combinations (among the asymptotically cylindrical Calabi-Yau 3-folds used in this paper) by a small computer program. Table 2 lists 255 examples of extra-twisted connected sums, 192 of which are not contained in [31]. Of all the examples, 125 are simply connected, representing at least 106 different G_2-deformation types, that is, classes of G_2-manifolds related by diffeomorphisms and deformations through torsion-free G_2-structures.

The ν-invariant and its analytic refinement. The ν-invariant of G_2-structures on closed 7-manifolds was introduced in [16]. It takes values in $\mathbb{Z}/48$, and its parity is determined by the Betti numbers of the underlying manifold. The examples in this paper show that all odd elements of $\mathbb{Z}/48$ appear as ν-invariants of torsion-free G_2-structures on extra-twisted connected sums; see Table 2. To obtain more examples with even ν from our construction we would need to use blocks with Picard rank higher than 1. As we explain below, such blocks exist, and we expect that all even elements of $\mathbb{Z}/48$ can be realised as well. The total number of extra-twisted connected sums we can currently construct is much less than the number of twisted connected sums constructed in [13]. Nevertheless, the present method is more efficient at constructing different G_2-deformation types that we can distinguish.

To date, only very few obstructions against the existence of a metric with full holonomy G_2 on a seven-manifold M are known. It is clear that M must be spin and have a finite fundamental group, and the de Rham class of the three-form φ that defines the torsion-free

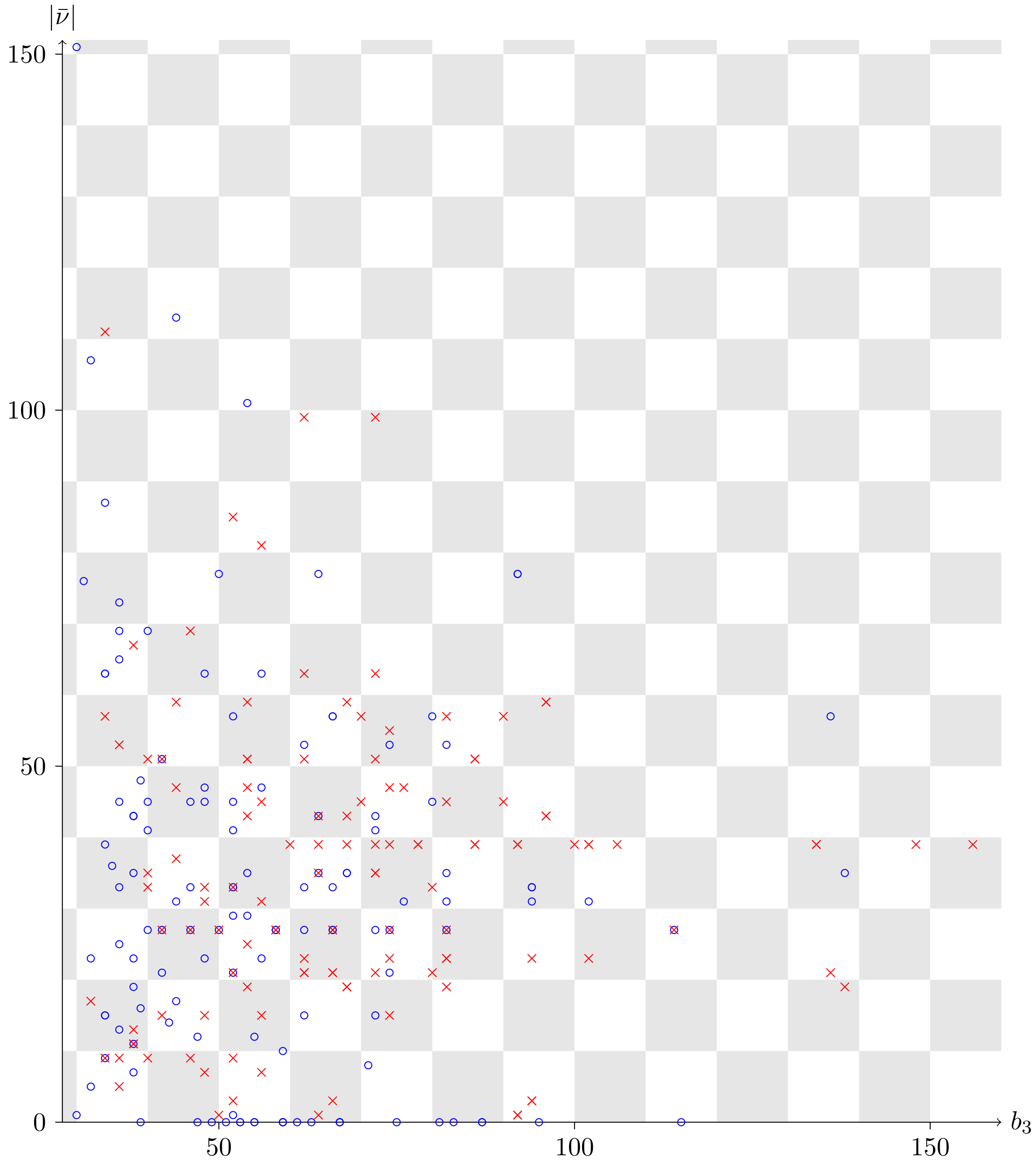

FIGURE 1. The landscape of examples in Table 2

G_2-structure must satisfy certain cohomological inequalities (see Joyce [24, §10.2]). When Crowley and Nordström discovered the ν-invariant in [16], it was hoped that the ν-invariant could be an obstruction against deforming a given topological G_2-structure into a torsion-free one, defining a G_2-holonomy metric. While computations in [14] showed that G_2-manifolds can have non-zero ν-invariants, all examples considered there (and also in [31]) have a ν-invariant that is divisible by 3. In particular, those examples are all G_2-nullbordant (in the sense of being a boundary of a compact 8-manifold whose tangent bundle has a G_2-structure restricting to the torsion-free one on the boundary), as a consequence of the result of Schelling [35] that $3 \mid \nu(M)$ if and only if M is G_2-nullbordant. However, we now find that G_2-bordism does not present any obstruction to G_2 holonomy metrics.

Observation 1. *There exist compact G_2-manifolds that are not G_2-nullbordant.*

To compute the ν-invariant, we use the method described in [14]. That is, we consider an integer-valued invariant

$$\bar{\nu}(M) = 3\eta(B) - 24\eta(D) \quad \text{such that} \quad \nu(M) \equiv \bar{\nu}(M) - 24(1 + b_1(M)) \mod 48 \ . \tag{0.1}$$

Here, B is the signature operator and D the spin Dirac operator on M; see Section 2. In Example 2.16, we exhibit an extra-twisted connected sum M with $\bar{\nu}(M) = -11$; see also Example 4.16. Hence, $\nu(M) \equiv 13 \bmod 48$. It follows from [35] that M is not G_2-nullbordant, and we have an example for Observation 1.

Our results are summarized in Table 2 and plotted in Figure 1, where crosses stand for simply-connected (in fact, two-connected) examples and circles for examples with nontrivial fundamental group. Because orientation reversal changes the sign of $\bar{\nu}$, we used the absolute value of $\bar{\nu}$ here. To get all values of $\bar{\nu}$, the whole picture should thus be extended by reflexion along the horizontal axis. In comparison with [14], we get some examples where the absolute value of $\bar{\nu}$ is much larger, for example $\bar{\nu} = -151$ in line 254 of Table 2 with fundamental group $\mathbb{Z}/3$, or example 240 with $\bar{\nu} = -111$, which is simply connected. All this seems to indicate that there could still be a wealth of unknown examples of G_2-manifolds.

Approaching the computation of $\bar{\nu}$. We still follow the route of computation for $\bar{\nu}$ that we laid out in [14]. That is, we first write $M = M_+ \cup M_-$ and use the gluing formula of Bunke [9] and Kirk, Lesch [26] to write $\bar{\nu}(M, g)$ as a sum of the contributions $\bar{\nu}(M_\pm, g)$ from the two pieces and a gluing term; see Theorem 2.2.

The gluing term consists of two pieces. The first is an integer contribution $3m_\rho(L; N_+, N_-)$ that depends on the relative positions of the images of $H^2(V_\pm)$ in $H^2(\Sigma)$, where Σ is the K3 surface that appears as a factor of the cross-section at infinity of $M_\pm$. If $\vartheta \in (0, \pi)$ denotes the oriented angle between the exterior circles, which we call the *gluing angle*, then the second contribution is $-72\frac{\rho}{\pi}$, where $\rho = \pi - 2\vartheta$. If the gluing angle is an irrational multiple of π, then this term will be irrational, too.

In [14], restricting attention to the case when both $\#\Gamma_\pm \leq 2$ ensured that both pieces have spectral symmetry, so that $\bar{\nu}(M_\pm, g) = 0$. Moreover, the gluing angle was forced to be an integer multiple of $\frac{\pi}{12}$, see also Remark 1.9. However, if $M_\pm$ is a twisted product $(V_\pm \times S^1)/\Gamma_\pm$ and $\#\Gamma_\pm \geq 3$, then the spectra of the relevant Dirac operators on $M_\pm$ with respect to the appropriate boundary conditions are no longer necessarily symmetric.

To compute the contribution $\bar{\nu}(M_\pm, g)$, we consider an adiabatic limit by scaling the exterior circle factor to 0. In Theorem 6.1, we state an adiabatic limit formula generalising both Dai's theorem for manifolds with boundary [19] and the formula for adiabatic limits of Seifert fibrations [21]. We see that the isolated fixpoints of the $\Gamma_\pm$-action on $V_\pm$ contribute by generalised Dedekind sums $D_\gamma(V) \in \mathbb{Q}$ introduced in Definition 2.4 (note that involutions on Calabi-Yau 3-folds cannot have isolated fixed points, consistent with the claim that the η-invariants vanish when $\#\Gamma_\pm \leq 2$).

When passing to the adiabatic limit, we also modify the metric on the boundary of $M_\pm$. Hence, we also have to consider the variational formula for η-invariants on manifolds with boundary due to Cheeger [11], Bismut and Cheeger [4], and Dai and Freed [20]. The relevant boundary contribution can be computed as a line integral of a universal η-form $\tilde{\eta}(\mathbb{A})$ for families of two-dimensional tori described in Proposition 2.10. The relevant η-form integrals can be expressed in terms of the logarithm $\mathcal{L}$ of the Dedekind η-function using results from

Section 7. The action of the group $\Gamma_\pm$ on T^2 is described by an invertible element $\varepsilon_\pm \in \mathbb{Z}/k_\pm$ if $k_\pm \geq 2$. In the following and throughout the paper, we represent its inverse by $\varepsilon^*_\pm \in \mathbb{Z}$. The precise choice of $\varepsilon^*_\pm$ does not matter in the following result. Also, the ratio of the lengths of the exterior and the interior circle is denoted $\kappa_\pm = \frac{\xi_\pm}{\zeta_\pm}$.

Theorem 2. *For all extra-twisted connected sums M, the extended ν-invariant is given by*

$$\bar\nu(M) = \bar\nu(M_+, g) + \bar\nu(M_-, g) - 72\frac{\rho}{\pi} + 3m_\rho(L; N_+, N_-)\ , \tag{0.2a}$$

$$\text{where} \qquad \bar\nu(M_\pm, g) = D_{\gamma_\pm}(V_\pm) - \frac{288}{\pi}\operatorname{Im}\mathcal{L}\biggl(\frac{\kappa_\pm^{-1} i - \varepsilon^*_\pm}{k_\pm}\biggr) - 24\,\frac{\varepsilon^*_\pm}{k_\pm}\ . \tag{0.2b}$$

This is proved in Section 2.5 (assuming Proposition 7.1). The occurrence of the Dedekind η-function can also be motivated by regarding $\tilde\eta(\mathbb{A})$ as a connection form of the Chern connection on a holomorphic determinant line bundle; see Remark 2.15. From the theory of complex multiplication one knows that the values of $\operatorname{Im}\mathcal{L}$ in (0.2b) can be expressed in terms of logarithms of algebraic numbers; for the values used in this paper, this is done explicitly in section 7.2. The linear combinations that appear in (0.2a) can be worked out from the functional equation (7.1) of $\mathcal{L}$ (see Proposition 7.3), giving one proof of Theorem 3 below.

Evaluating $\bar\nu$ via elementary hyperbolic geometry. In Section 4, we follow a different path to rewrite the right hand side of (0.2a) in terms of Dedekind sums. The universal η-form $\tilde\eta(\mathbb{A})$ for bundles of flat tori can be understood as a 1-form on the upper half plane, whose exterior derivative turns out to be a constant multiple of the hyperbolic area form. The relevant path of integration consists of two sides of some ideal hyperbolic polygon P, depending on the gluing data. The remaining sides can be chosen such that $\tilde\eta(\mathbb{A})$ vanishes along them. To apply Stokes' Theorem, it remains to determine the area and the contribution from the cusps. At this point, we use a strict version of an adiabatic limit formula for η-forms that was proved by Bunke, Ma [10] and Liu [29] modulo exact forms; see also Proposition 4.12.

We determine the remaining corners of the polygon P using continued fractions. Here, we also need the entries of the gluing matrix $\left(\begin{smallmatrix} m & p \\ n & q\end{smallmatrix}\right)$ that encodes the matching of the tori as described in (1.6). At this point, one can already finish the computation of $\bar\nu(M)$ for any particular example by hand. However, one can simplify these computations by observing that the contributions from the cusps and the hyperbolic area formula add up to a classical Dedekind sum $S(k,n)$ given for integers $n > 0$ and k by

$$S(k,n) = \sum_{j=1}^{n-1} \biggl(\biggl(\frac{j}{n}\biggr)\biggr)\biggl(\biggl(\frac{jk}{n}\biggr)\biggr) \in \frac{1}{6n}\mathbb{Z}\ , \quad \text{where } (\!(x)\!) = \begin{cases} 0 & \text{for } x \in \mathbb{Z} \\ x - \lfloor x \rfloor - \frac12 & \text{for } x \notin \mathbb{Z}\end{cases} \tag{0.3}$$

denotes a sawtooth function. For more background on the Dedekind η-function, Dedekind sums, and their appearance in topology we refer the reader to [23]. We may use Proposition 3.5 to make sure that $m \geq 0$ and $n > 0$.

Theorem 3. *Assume that $n > 0$. Then $A = \frac{m - \varepsilon^*_+ n}{k_+}$ is an integer and*

$$\bar\nu(M) = D_{\gamma_+}(V_+) + D_{\gamma_-}(V_-) + 3\,m_\rho(L; N_+, N_-) + 24\,\Bigl(\frac{q}{k_- n} - \frac{m}{k_+ n} + 12\,S(A,n)\Bigr)\ . \tag{0.4}$$

The proof outlined above can be found in Section 4.7. We see that all possibly irrational terms in Theorem 2 have been subsumed into the last parenthesis, which is always rational. Hence, this presentation makes it easier to check that $\bar\nu(M)$ is an integer; see also Remark 4.15.

All in all, our way to a tractable formula for the ν-invariant consists of many small steps, but in each of our examples, the sum of the various contributions is an integer. This, together with the fact that the two completely different approaches described above give the same expression, could be seen as a sanity check for our results presented here.

Scope. To keep this article within reasonable size, we had to leave out some aspects of the construction.

(i) We only consider examples built from blocks of Picard rank 1. These examples automatically have matchings of pure angle in the sense of Remark 1.18. On the other hand, by (5.2) most examples obtained this way have even $b_3(M)$, and hence odd $\bar{\nu}(M)$.

There exist building blocks of Picard rank 2 with automorphism group $\mathbb{Z}/k$ for each $k = 1, \ldots, 6$. Thus we expect that there are examples of extra-twisted connected sums that realise all 24 even values of $\nu(M) \in \mathbb{Z}/48$ as well.

(ii) All examples constructed in Section 5 are either two-connected or have two-connected universal cover. We distinguish them only by their extended ν-invariants and their third Betti number. See Figure 1 for a plot of all possible pairs of these invariants. We do not attempt to compute the torsion part of their fourth cohomology or the divisibility of their first Pontryagin classes. That would be needed in order to apply diffeomorphism classification results [17] to exhibit examples of 7-manifolds where the moduli space of G_2-metrics is disconnected, but such examples have already been seen in [14].

(iii) We might consider asymptotically cylindrical Calabi-Yau manifolds with arbitrary automorphisms that act on the asymptotic cylinder $\Sigma \times T^2 \times \mathbb{R}$ as a product but not necessarily fixing the Σ factor. But first of all, it looks more difficult to construct matchings in this situation. And worse, these examples would never be simply connected. Instead, their universal covers would again be extra-twisted connected sums of the type considered here. This has been explained in [15, Remark 1.12].

(iv) It is not clear how to define a ν-invariant for non-compact or singular G_2-spaces. Theorem 2.2 contains a possible definition for G_2-manifolds with an asymptotically cylindrical end. However, it is also not clear to us how to interpret the resulting numbers in (0.2b). From Dai and Freed's point of view in [20], the invariant $e^{2\pi i \frac{\nu(M_\pm, g)}{6}}$ should take values in a certain determinant line associated to the cross-section at infinity. The invariant $\bar{\nu}(M)$ would therefore take values in a "logarithm" of this determinant line.

Organisation. In Section 1, we recall the extra-twisted connected sum construction. We discuss the matching problem for tori in Subsection 1.3, and for K3 surfaces in Subsection 1.4.

Theorem 2 is proved in Section 2. The fixpoint contributions are computed in Subsection 2.3, the variational formula is discussed in 2.4, and a direct computation of the η-form integrals can be found in 2.5.

We discuss the combinatorics of torus matchings in Section 3.

Theorem 3 is proved by elementary hyperbolic geometry in Section 4. Adiabatic deformations of tori are identified with hyperbolic geodesics in Subsection 4.3, and the contribution from the cusps is explained in 4.6. In Subsection 4.5, we use continued fractions to construct ideal polygons, and in 4.7, we rewrite the sum of the cusp contributions as a Dedekind sum.

Section 5 contains more details about the construction of examples, in particular we describe some building blocks with group actions in Subsection 5.2, and possible K3 matchings in Subsection 5.3.

Section 6 contains the proofs of some technical results used in Sections 2 and 4. Section 7 contains the evaluation of η-form integrals in terms of the Dedekind η-function, as well as explicit formulas for the values that we use in this paper.

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1. Extra-twisted connected sums

We generalise the twisted connected sum construction of [27, 13] by allowing twisted products of asymptotically cylindrical Calabi-Yau manifolds with circles. This approach was already employed in [14], where we considered products twisted by an involution (on one or both sides). Here, we allow twists by more general finite cyclic groups.

1.1. The gluing construction.

Let $V_\pm$ be asymptotically cylindrical Calabi-Yau manifolds of complex dimension 3, and assume that their ends are asymptotic to $\Sigma_\pm \times S^1_{\zeta_\pm} \times (0,\infty)$, where $\Sigma_\pm$ are K3 surfaces, and $S^1_{\zeta_\pm} = \mathbb{R}/\zeta_\pm\mathbb{Z}$. The Calabi-Yau structure on $V_\pm$ can be described in terms of a pair $(\Omega_\pm, \omega_\pm)$, where $\Omega_\pm$ is a complex 3-form (holomorphic with respect to the complex structure) and $\omega_\pm$ is a Kähler form, normalised so that $8\omega_\pm^3 = 6\Omega_\pm \wedge \overline{\Omega}_\pm$. Along the cylindrical end $\Sigma_\pm \times S^1_{\zeta_\pm} \times (0,\infty)$, the asymptotic limits of $\Omega_\pm$ and $\omega_\pm$ are of the form

$$\Omega_\pm := (du_\pm - i dt_\pm) \wedge (\omega_\pm^J + i\omega_\pm^K), \qquad \omega := dt_\pm \wedge du_\pm + \omega_\pm^I$$

respectively, where $t_\pm$ is the coordinate on the $(0,\infty)$ factor, $u_\pm$ is the coordinate on $S^1_{\zeta_\pm} = \mathbb{R}/\zeta_\pm\mathbb{Z}$, and the triple $(\omega_\pm^I, \omega_\pm^J, \omega_\pm^K)$ defines a hyper-Kähler structure on $\Sigma_\pm$. Such a Calabi-Yau structure induces a metric $g_{V_\pm}$ of holonomy $SU(3)$ whose asymptotic limit is of the form $dt_\pm^2 + du_\pm^2 + g_{\Sigma_\pm}$, where $g_{\Sigma_\pm}$ is a metric of holonomy $SU(2)$ induced by the hyper-Kähler structure. Note in particular that the circumference of the circle factor in the asymptotic cylinder is $\zeta_\pm$.

Remark 1.1. The condition on the cylindrical ends forces $V_\pm$ to be simply-connected by [22, Theorem A].

Letting $S^1_{\xi_\pm} = \mathbb{R}/\xi_\pm\mathbb{Z}$ and denoting its coordinate by $v_\pm$, we can define a product torsion-free G_2-structure on $V_\pm \times S^1_{\xi_\pm}$ by

$$\varphi_\pm = \operatorname{Re}\Omega_\pm + dv_\pm \wedge \omega_\pm .$$

Then $\varphi_\pm$ defines the metric $dv_\pm^2 + g_{V_\pm}$, with holonomy contained in G_2. Note that the exterior circle factor has circumference $\xi_\pm$. The asymptotic limit of $\varphi_\pm$ has the form

$$dv_\pm \wedge dt_\pm \wedge du_\pm + dv_\pm \wedge \omega_\pm^I + du_\pm \wedge \omega_\pm^J + dt_\pm \wedge \omega_\pm^K . \tag{1.1}$$

Now assume further that two groups $\Gamma_\pm \cong \mathbb{Z}/k_\pm$ act on $V_\pm$, preserving the Calabi-Yau structures (and in particular the metrics), such that the actions on the end are products of the trivial actions on $\Sigma_\pm \times (0,\infty)$ and free actions on $S^1_{\zeta_\pm}$. We extend the $\Gamma_\pm$-action diagonally to $\widetilde{M}_\pm = V_\pm \times S^1_{\xi_\pm}$, such that $\Gamma_\pm$ acts isometrically and freely on $S^1_{\xi_\pm}$. Then the torsion-free G_2-structure $\varphi_\pm$ descends to the quotient $M_\pm = \widetilde{M}_\pm/\Gamma_\pm$, which we can thus regard as an asymptotically cylindrical G_2-manifold. The cross-section of the asymptotic cylinder is the product $X_\pm$ of $\Sigma_\pm$ with a torus $(S^1_{\zeta_\pm} \times S^1_{\xi_\pm})/\Gamma_\pm$.

We now suppose that we have a suitable isometry between the cross-sections. This isometry will necessarily be a product of isometries

$$\mathtt{t} : (S^1_{\zeta_+} \times S^1_{\xi_+})/\Gamma_+ \to (S^1_{\zeta_-} \times S^1_{\xi_-})/\Gamma_-$$

and

$$\mathtt{r} : \Sigma_+ \to \Sigma_-.$$

We require that the isometry

$$\begin{aligned} \Sigma_+ \times (S^1_{\xi_+} \times S^1_{\zeta_+})/\Gamma_+ \times \mathbb{R} \;&\to\; \Sigma_- \times (S^1_{\xi_-} \times S^1_{\zeta_-})/\Gamma_- \times \mathbb{R} \\ (x, z, t) &\mapsto (\mathtt{r}(x), \mathtt{t}(z), -t) \end{aligned} \tag{1.2}$$

identifies the asymptotic limits (1.1). In particular, exactly one of $\mathtt{t}$ and $\mathtt{r}$ is orientation-preserving. Our convention is to require $\mathtt{t}$ to be orientation-reversing. We will refer to $\mathtt{t}$ as a *torus matching* and to $\mathtt{r}$ as a *hyper-Kähler rotation*.

A key feature of the construction is how the torus matching aligns the external circle directions. As above, we denote by

$$u_\pm \in \mathbb{R}/\zeta_\pm\mathbb{Z}, \qquad v_\pm \in \mathbb{R}/\xi_\pm\mathbb{Z}$$

the coordinates in the direction of the interior and exterior circles respectively. In [14, (11)], the *gluing angle* ϑ was introduced as the directed angle between the exterior circles under $\mathtt{t}$, so

$$\begin{aligned} \partial_{v_-} &= \cos\vartheta\, \partial_{v_+} + \sin\vartheta\, \partial_{u_+} \,, \\ \partial_{u_-} &= \sin\vartheta\, \partial_{v_+} - \cos\vartheta\, \partial_{u_+} \,. \end{aligned} \tag{1.3}$$

The condition that (1.2) preserves the asymptotic limits of the G_2-structures is now equivalent to the following condition; see [31, §1.2].

Definition 1.2. Let $\Sigma_\pm$ be K3 surfaces with hyper-Kähler structures $(\omega^I_\pm, \omega^J_\pm, \omega^K_\pm)$. Call a diffeomorphism $\mathtt{r} : \Sigma_+ \to \Sigma_-$ a *hyper-Kähler rotation* with angle ϑ if

$$\begin{aligned} \mathtt{r}^*\omega^K_- &= -\omega^K_+ \\ \mathtt{r}^*(\omega^I_- + i\omega^J_-) &= e^{i\vartheta}(\omega^I_+ - i\omega^J_+). \end{aligned} \tag{1.4}$$

Extend the cylindrical coordinate $t_\pm$ to a smooth function on all $V_\pm$ (taking negative values away from the cylindrical end), and let $V_{\pm,\ell}$ be the truncation $\{\, x \in V_\pm \mid t_\pm(x) \le 2\ell \,\}$. Let $\widetilde{M}_{\pm,\ell} = V_{\pm,\ell} \times S^1_{\xi_\pm}$ and $M_{\pm,\ell} = \widetilde{M}_{\pm,\ell}/\Gamma_\pm$ and let $\tilde{X}_\pm \cong \Sigma_\pm \times S^1_{\zeta_\pm} \times S^1_{\xi_\pm}$ and $X_\pm = \tilde{X}_\pm/\Gamma_\pm$ denote their boundaries. For sufficiently large ℓ, it is possible to obtain a new closed G_2-manifold M_ℓ (with an approximately cylindrical neck region of length 4ℓ) by gluing $M_{+,\ell}$ and $M_{-,\ell}$ along their boundaries using a diffeomorphism $X_+ \cong X_-$. This procedure is described in detail in [31] following the ideas in [27, 13]. Let us summarise.

Theorem 1.3. *Given a pair of ACyl Calabi-Yau 3-folds $V_\pm$ with asymptotic cross-sections $\Sigma_\pm \times S^1_{\zeta_\pm}$ and automorphisms $\Gamma_\pm$, a torus matching $\mathtt{t} : (S^1_{\zeta_+} \times S^1_{\xi_+})/\Gamma_+ \to (S^1_{\zeta_-} \times S^1_{\xi_-})/\Gamma_-$ and a hyper-Kähler rotation $\mathtt{r} : \Sigma_+ \to \Sigma_-$ with angle ϑ equal to the gluing angle of $\mathtt{t}$, the manifold M_ℓ above admits torsion-free G_2-structures. Moreover, the holonomy G_2 metrics in M_ℓ can be taken to be close to the metrics on the two halves in the sense explained in §2.1.1.*

Note that after gluing, M_+ and M_- induce opposite orientations on the 2-torus. Note that ϑ is invariant under swapping the roles of M_+ and M_-. The angle between the interior circles is $\pi - \vartheta$; see Figure 2.

For further discussion of how extra-twisted connected sums using different data can be essentially the same see Proposition 3.5.

Remark 1.4. The G_2-structure on M defines a unique spin structure on M that we need for the analytic description of the ν-invariant. Its restriction to $M_\pm$ is the spin structure induced by the $SU(3)$-structure, and hence by the Calabi-Yau structure on $V_\pm$.

Because $\Gamma_\pm$ preserves the Calabi-Yau structure on $V_\pm$, it acts canonically on the associated complex spinor bundle $SV_\pm \cong \Lambda^{0,\bullet}T^*V_\pm$. The spinor bundle on $M_\pm$ that is induced by the G_2-structure then satisfies $SM_\pm \cong p^*SV_\pm/\Gamma_\pm$. On the cylinder $\Sigma_\pm \times (0,\infty) \times (S^1_{\zeta_\pm} \times S^1_{\xi_\pm})/\Gamma_\pm$, it is isomorphic to the pullback of the direct sum of two copies of $S\Sigma_\pm$.

1.2. **Setting up the matching problem.** Understanding the possible torus matchings $\mathsf{t} : (S^1_{\zeta_+} \times S^1_{\xi_+})/\Gamma_+ \to (S^1_{\zeta_-} \times S^1_{\xi_-})/\Gamma_-$ for given values of $k_\pm = \#\Gamma_\pm$—and in particular the possible gluing angles ϑ—is essentially a combinatorial problem, which will be discussed in the next subsection and in Subsection 3.1. Given a torus matching, Theorem 1.3 raises the question of how to find pairs of ACyl Calabi-Yau 3-folds with automorphisms and a hyper-Kähler rotation of the correct angle ϑ between the K3 surfaces in the asymptotic cross-section. We now explain how this question can be reduced to complex algebraic geometry, as in [13, §6] and [31, §6].

Definition 1.5. Let Z be a non-singular algebraic 3-fold and $\Sigma \subset Z$ a non-singular K3 surface. Let N be the image of $H^2(Z) \to H^2(\Sigma)$. We call (Z, Σ) a *building block* if

(i) the class in $H^2(Z)$ of the anticanonical line bundle $-K_Z$ is indivisible,
(ii) $\Sigma \in |-K_Z|$ (*i.e.* Σ is an anticanonical divisor), and there is a projective morphism $f\colon Z \to \mathbb{P}^1$ with $\Sigma = f^\star(\infty)$,
(iii) The inclusion $N \hookrightarrow H^2(\Sigma)$ is primitive, that is, $H^2(\Sigma)/N$ is torsion-free.
(iv) The group $H^3(Z)$—and thus also $H^4(Z)$—is torsion-free.

We call N, equipped with the restriction of the intersection form on $H^2(\Sigma)$, the *polarising lattice* of the block. (Because $H^{2,0}(Z)$ is automatically trivial, $N \subseteq H^{1,1}(\Sigma)$ [13, Lemma 3.6], so that Σ is 'N-polarised'.)

If Γ is a group acting faithfully on Z by biholomorphisms that fix Σ pointwise then we call (Z, Σ, Γ) a *building block with automorphisms.* (Γ is then necessarily cyclic.)

Given such a (Z, Σ), [22, Theorem D] gives the existence of ACyl Calabi-Yau structures on $V := Z \setminus \Sigma$, and it is easy to see that Γ restricts to isomorphisms of these structures.

Rather than to look for a hyper-Kähler rotation for a given pair of ACyl Calabi-Yau structures, it is easier to first choose the diffeomorphism $\mathsf{r} : \Sigma_+ \to \Sigma_-$ satisfying obvious necessary conditions in terms of cohomology classes and then find Calabi-Yau structures that make r a hyper-Kähler rotation. Recall that the *period* of a complex K3 surface Σ is the positive-definite 2-plane $\Pi \subset H^2(\Sigma; \mathbb{R})$ spanned by the real and imaginary parts of elements of $H^{2,0}(\Sigma; \mathbb{C})$.

Definition 1.6. Let $(Z_\pm, \Sigma_\pm)$ be a pair of building blocks, and let $\Pi_\pm \subset H^2(\Sigma_\pm; \mathbb{R})$ be the periods. Call a diffeomorphism $\mathsf{r} : \Sigma_+ \to \Sigma_-$ a *K3 matching* with angle ϑ if there are Kähler classes $\mathsf{k}_\pm \in H^2(Z_\pm; \mathbb{R})$ such that (with respect to the intersection form) the angle between $\mathsf{r}^*(\mathsf{k}_-)$ and Π_+ is ϑ, the angle between $(\mathsf{r}^{-1})^*(\mathsf{k}_+)$ and Π_- is ϑ and $\Pi_+ \cap \mathsf{r}^*\Pi_-$ is non-trivial.

It is easy to see that the ACyl Calabi-Yau structures of [22, Theorem D] can be chosen to ensure that a given K3 matching is a hyper-Kähler rotation; see [31, Theorem 1.1 and Lemma 6.2].

Theorem 1.7. *Given $\zeta_\pm > 0$, blocks $(Z_\pm, \Sigma_\pm)$ and a K3 matching $\mathfrak{r} : \Sigma_+ \to \Sigma_-$ with angle ϑ, there exist ACyl Calabi-Yau structures $(\Omega_\pm, \omega_\pm)$ on $V_\pm := Z_\pm \setminus \Sigma_\pm$ with asymptotic limit*

$$\big((du_\pm - i dt_\pm) \wedge (\omega^J_\pm + i\omega^K_\pm), dt_\pm \wedge du_\pm + \omega^I_\pm\big)$$

on $\Sigma_\pm \times S^1_{\zeta_\pm}$ (in particular, the circumference of the S^1 factor with respect to the induced metric is $\zeta_\pm$), such that $\mathfrak{r}$ is an angle ϑ hyper-Kähler rotation of the hyper-Kähler structures $(\omega^I_\pm, \omega^J_\pm, \omega^K_\pm)$.

Thus given a K3 matching of two building blocks with automorphism and a torus matching with the corresponding k_+, k_- and gluing angle ϑ, we can find ACyl Calabi-Yau structures so that Theorem 1.3 can be applied to build a G_2-manifold.

Given a pair of blocks, there is no reason to expect to be able to find any K3 matchings at all. However, if we consider the sets of complex deformations of a pair of blocks, one can in many cases guarantee that there exist some elements of each of the two sets that admit a K3 matching, and moreover control the topology of the resulting G_2-manifold. This will be discussed further in §1.4.

1.3. **Isometries of quotients of rectangular tori.** In this section, we analyse how to find torus matchings in the sense of §1.1.

We consider $M_+ = \widetilde{M}_+/\Gamma_+$ with covering space $\widetilde{M}_+ = V_+ \times S^1_{\xi_+}$ and $\Gamma_+ = \mathbb{Z}/k_+$. The asymptotic cross-section of the covering space is isometric to a product

$$\partial\widetilde{M}_+ \cong \Sigma_+ \times \widetilde{T}_+ \qquad \text{with} \qquad \widetilde{T}_+ \cong S^1_{\zeta_+} \times S^1_{\xi_+} ,$$

where Σ_+ is a K3 surface and ζ_+, ξ_+ are the lengths of the interior and exterior circle, respectively.

By a *torus matching* we refer to the following data: numbers $k_\pm \geq 1$, actions of $\Gamma_\pm = \mathbb{Z}/k_\pm$ on $\widetilde{T}_\pm = S^1_{\zeta_\pm} \times S^1_{\xi_\pm}$ that are free on both factors, and an orientation-reversing isomorphism $\mathtt{t}\colon \widetilde{T}_+/\Gamma_+ \to \widetilde{T}_-/\Gamma_-$ of flat tori, such that there exist lengths ζ_+, ξ_+, ζ_-, $\xi_- > 0$ for which $\mathtt{t}$ becomes an isometry. We consider two torus matchings to be equivalent if there exist (linear) isomorphisms of the respective tori that map exterior circles to exterior circles, interior circles to interior circles, and that intertwine the actions of $\Gamma_\pm$ and $\mathtt{t}$ (we consider other symmetries in Proposition 3.5). It is clear that using torus matchings that are equivalent in this sense in Theorem 1.3 yields G_2 metrics related by deformation.

Equip $\mathbb{R}^2 \cong \mathbb{C}$ with the standard Euclidean metric. We choose ζ_+, ξ_+, ζ_-, $\xi_- > 0$ as above and represent the torus $\widetilde{T}_+$ isometrically as $\mathbb{C}/\tilde{\Lambda}_+$, where $\tilde{\Lambda}_+ \subset \mathbb{C}$ is the lattice with orthogonal basis $(\mu_+, \lambda_+) = (i\xi_+, \zeta_+)$.

We assume that $\Gamma_+ \cong \mathbb{Z}/k_+$ acts on $\widetilde{T}_+ = \mathbb{C}/\tilde{\Lambda}_+$ such that the action on both circles is free. (If the action on the exterior circle was not free, then the quotient M_+ would be an orbifold. If the action on the interior circle had a kernel Γ_{+0}, then we could reduce the exterior circle to the quotient $S^1_{\xi_+}/\Gamma_{+0}$ without changing M_+, so we do not have to consider this situation.) We fix a generator that rotates the exterior circle by the angle $\frac{2\pi}{k_+}$. If $k_+ \geq 2$, its action on the interior circle is given by $\frac{2\pi\varepsilon_+}{k_+}$ for some $\varepsilon_+ \in \mathbb{Z}$. Really we only care about the residue $\varepsilon_+ \in \mathbb{Z}/k_+$, which is uniquely defined. The requirement that the action on the interior circle is free means ε_+ is coprime to k_+, in other words $\gcd(\varepsilon_+, k_+) = 1$.

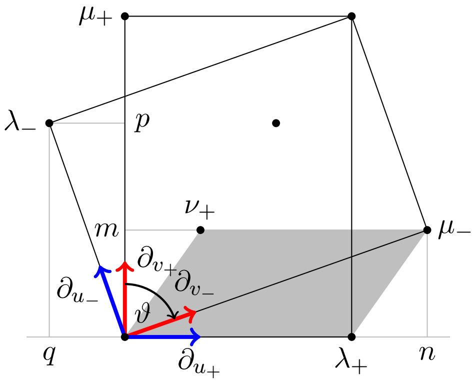


FIGURE 2. Fundamental domains of T and $\widetilde{T}_\pm$.

We represent $T = \widetilde{T}_+/\Gamma_+$ by the lattice Λ with basis

$$(\nu_+, \lambda_+) = (\mu_+, \lambda_+) \cdot \begin{pmatrix} \frac{1}{k_+} & 0 \\ \frac{\varepsilon_+}{k_+} & 1 \end{pmatrix} = \left(\frac{\varepsilon_+\zeta_+ + i\xi_+}{k_+}, \zeta_+ \right). \tag{1.5}$$

This is sketched in Figure 2 for $k_+ = 3$ and $\varepsilon_+ = 1$. A fundamental domain for Λ is shaded.

Represent $T_- = \widetilde{T}_-/\Gamma_-$ similarly, and define $\varepsilon_- \in \mathbb{Z}/k_-$ with $\gcd(\varepsilon_-, k_-) = 1$ analogously. The isometry $\widetilde{T}_+/\Gamma_+ \to \widetilde{T}_-/\Gamma_-$ determines a sublattice $\tilde{\Lambda}_- \subset \Lambda$ such that $\widetilde{T}_- \cong \mathbb{C}/\tilde{\Lambda}_-$. Let (μ_-, λ_-) denote a basis of $\tilde{\Lambda}_-$, where λ_- and μ_- correspond to the interior and exterior circle as above. We represent this basis as

$$(\mu_-, \lambda_-) = \frac{1}{k_+} \cdot (\mu_+, \lambda_+) \cdot \begin{pmatrix} m & p \\ n & q \end{pmatrix} \tag{1.6}$$

where $\left(\begin{smallmatrix} m & p \\ n & q \end{smallmatrix}\right) \in M_2(\mathbb{Z})$, and call $G = \left(\begin{smallmatrix} m & p \\ n & q \end{smallmatrix}\right)$ the *gluing matrix*. In Figure 2, we have $k_- = 3$, $\varepsilon_- = 1$, and the gluing matrix is $\left(\begin{smallmatrix} 1 & 2 \\ 4 & -1 \end{smallmatrix}\right)$; see entry 209 in Table 2.

In summary, we can associate to a torus matching the following data that is clearly invariant under our notion of equivalence

- k_+ and k_-
- the gluing matrix G
- $\varepsilon_+ \in \mathbb{Z}/k_+$ and $\varepsilon_- \in \mathbb{Z}/k_-$ with $\gcd(\varepsilon_+, k_+) = \gcd(\varepsilon_-, k_-) = 1$.

For the construction, we also need the more geometric data

- the angle ϑ between the exterior circle directions
- the ratios $\frac{\xi_+}{\xi_-}$ and $\kappa_\pm = \frac{\xi_\pm}{\zeta_\pm}$

which are not obviously invariant. However, among the (selection of) compatibility conditions that we now show, we see that the angle ϑ is in fact also determined by the equivalence class of the torus matching, and that if $\vartheta \notin \frac{\pi}{2}\mathbb{Z}$, then the aspect ratios $\frac{\xi_\pm}{\zeta_\pm}$ and $\frac{\xi_+}{\xi_-}$ are as well.

Proposition 1.8. (i) *The data of a torus matching satisfies the following relations.*

$$\det \begin{pmatrix} m & p \\ n & q \end{pmatrix} = -k_-k_+, \tag{1.7}$$

$$\varepsilon_+ m - n \equiv \varepsilon_+ p - q \equiv 0 \mod k_+ \tag{1.8a}$$

$$\varepsilon_- p + m \equiv \varepsilon_- q + n \equiv 0 \mod k_- \tag{1.8b}$$

$$\gcd\Big(\frac{n-\varepsilon_+m}{k_+},m\Big)=\gcd\Big(\frac{q-\varepsilon_+p}{k_+},p\Big)=\gcd(\varepsilon_+,k_+)=1, \tag{1.9a}$$

$$\gcd\Big(\frac{m+\varepsilon_-p}{k_-},p\Big)=\gcd\Big(\frac{n+\varepsilon_-q}{k_-},q\Big)=\gcd(\varepsilon_-,k_-)=1, \tag{1.9b}$$

(ii) *Either* $m=q=0$, *or* $n=p=0$, *or* $\frac{nq}{mp}<0$ *and* $\kappa_+=\frac{\xi_+}{\zeta_+}=\sqrt{-\frac{nq}{mp}}$. *In the latter case, we also have* $\zeta_-=\sqrt{-\frac{qk_-}{mk_+}}\,\zeta_+$, $\xi_-=\sqrt{\frac{nk_-}{pk_+}}\,\zeta_+$, *and* $\kappa_-=\frac{\xi_-}{\zeta_-}=\sqrt{-\frac{mn}{pq}}$.

(iii) *The gluing angle* ϑ *is given as*

$$\vartheta=\arg\big(m\kappa_++in\big)\in(-\pi,\pi]\;.$$

In particular, $\vartheta\in(0,\pi)$ *if and only if* $n>0$, *and* $\cos\vartheta=\operatorname{sign}(m)\sqrt{-\frac{mq}{k_+k_-}}$.

Proof. For (1.7), we note that the bases (μ_-,λ_-) and (μ_+,λ_+) induce opposite orientations, and compute

$$-\frac{1}{k_+^2}\det\begin{pmatrix}m&p\\n&q\end{pmatrix}=\frac{\operatorname{vol}(\widetilde{T}_-)}{\operatorname{vol}(\widetilde{T}_+)}=\frac{k_-\operatorname{vol}(T)}{k_+\operatorname{vol}(T)}=\frac{k_-}{k_+}\;.$$

We represent λ_- and μ_- in the basis (1.5) of Λ. By (1.6), we get

$$\begin{aligned}(\mu_-,\lambda_-)&=\frac{1}{k_+}\cdot(\mu_+,\lambda_+)\cdot\begin{pmatrix}\frac{1}{k_+}&0\\\frac{\varepsilon_+}{k_+}&1\end{pmatrix}\cdot\begin{pmatrix}k_+&0\\-\varepsilon_+&1\end{pmatrix}\cdot\begin{pmatrix}m&p\\n&q\end{pmatrix}\\&=(\nu_+,\lambda_+)\cdot\begin{pmatrix}m&p\\\frac{n-\varepsilon_+m}{k_+}&\frac{q-\varepsilon_+p}{k_+}\end{pmatrix}\;.\end{aligned} \tag{1.10}$$

Because $\tilde\Lambda_-\subset\Lambda$, the coefficients are integers, and (1.8a) follows.

The group $\Gamma_-\cong\Lambda/\tilde\Lambda_-$ again acts freely by rotations on both the interior and the exterior circle of $\widetilde{T}_-$. Equivalently, the elements λ_- and μ_- corresponding to the factors $S^1_{\zeta_-}$ and $S^1_{\xi_-}$ are primitive in Λ, which gives the first two gcd conditions in 1.9a. The last condition in 1.9a holds if and only if Γ_+ acts freely on $\widetilde{T}_+$.

Using (1.7), we can invert the gluing matrix. Then equation (1.6) is equivalent to

$$(\mu_+,\lambda_+)=\frac{1}{k_-}\cdot(\mu_-,\lambda_-)\cdot\begin{pmatrix}-q&p\\n&-m\end{pmatrix}\;. \tag{1.11}$$

Now, the same arguments as above give (1.8b) and (1.9b).

The vectors λ_-,μ_- in (1.6) are perpendicular with respect to the standard metric on $\mathbb{C}\cong\mathbb{R}^2$ if and only if

$$0=\frac{mp\xi_+^2+nq\zeta_+^2}{k_+^2}=\big(mp\kappa_+^2+nq\big)\cdot\frac{\zeta_+^2}{k_+^2}\;,$$

and the condition on $\left(\begin{smallmatrix}m&p\\n&q\end{smallmatrix}\right)$ and κ_+ follows. The remaining claims in (ii) follow because

$$\zeta_-=|\lambda_-|=\frac{|q+ip\kappa_+|\,\zeta_+}{k_+}=\frac{\sqrt{q^2+p^2\kappa_+^2}}{k_+}\,\zeta_+=\sqrt{\frac{q(mq-np)}{m}}\,\frac{\zeta_+}{k_+}=\sqrt{-\frac{qk_-}{mk_+}}\,\zeta_+\;,$$

$$\xi_-=|\mu_-|=\frac{\sqrt{n^2+m^2\kappa_+^2}}{k_+}\,\zeta_+=\sqrt{\frac{n(np-mq)}{p}}\,\frac{\zeta_+}{k_+}=\sqrt{\frac{nk_-}{pk_+}}\,\zeta_+\;.$$

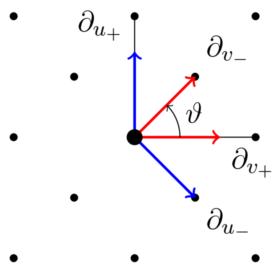


FIGURE 3. $G = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \vartheta = \frac{\pi}{4}$

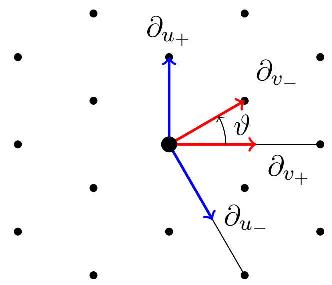


FIGURE 4. $G = \begin{pmatrix} 1 & 1 \\ 1 & -3 \end{pmatrix}, \vartheta = \frac{\pi}{6}$

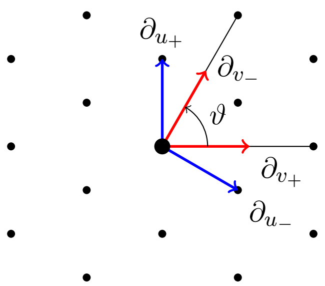


FIGURE 5. $G = \begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix}, \vartheta = \frac{\pi}{3}$

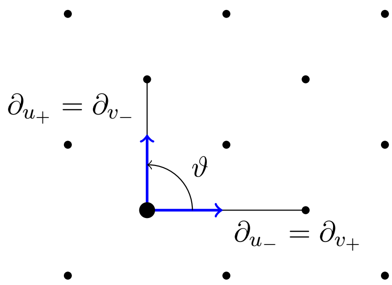


FIGURE 6. $G = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}, \vartheta = \frac{\pi}{2}$

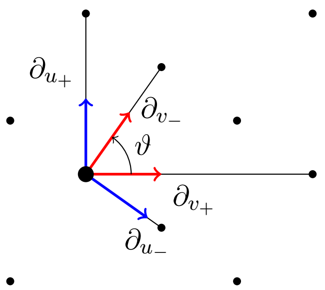


FIGURE 7. $G = \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}, \vartheta = \arccos \frac{1}{\sqrt{3}}$

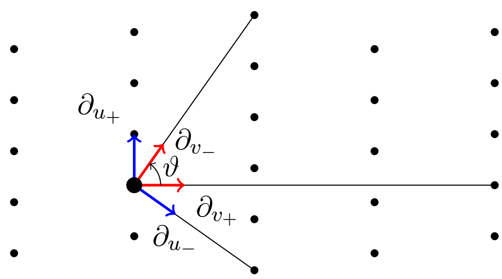


FIGURE 8. $G = \begin{pmatrix} 1 & 1 \\ 10 & -5 \end{pmatrix}, \vartheta = \arccos \frac{1}{\sqrt{3}}$

In [14], the gluing angle $\vartheta \in (-\pi, \pi]$ has been defined as the directed angle between μ_- and μ_+, see also (1.3). We have $\vartheta \in (0, \pi)$ if and only if the scalar product $\langle \mu_-, \lambda_+ \rangle = \frac{n}{k_+} |\zeta_+|^2$ is positive. Hence, we get (iii) by

$$\vartheta = \arg \frac{\mu_+}{\mu_-} = \arg \frac{ik_+\xi_+}{n\zeta_+ + im\xi_+} = \arg \frac{k_+(m\xi_+^2 + in\xi_+\zeta_+)}{n^2\zeta_+^2 + m^2\xi_+^2} = \arg\bigl(m\kappa_+ + in\bigr) . \qquad \square$$

Remark 1.9. When $k_\pm \leq 2$, the only possibilities (up to swapping M_+ and M_-) with m, n, $q \geq 0$ and $p = 1$ are the ones already studied in [31, 14], illustrated in Figures 3–5. If we allow $p \geq 1$, there are two more with gluing matrices $\begin{pmatrix} 1 & 3 \\ 1 & -1 \end{pmatrix}$ and $\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$; the latter is depicted in Figure 6. Notice that $\vartheta = \frac{\pi}{2}$ in this example, so the radii $\xi_+ = \zeta_-$ and $\xi_- = \zeta_+$ can be chosen independently.

Once we allow k_+ or k_- to be larger than 2, there are many more possibilities. Figure 7 illustrates a torus matching with $k_+ = 1$ and $k_- = 3$, where $\kappa_+ = \sqrt{2}$ and $\kappa_- = \frac{1}{\sqrt{2}}$ (so the tori have proportions of A4 paper). We consider this further in Section 3. Let us for now give a single more complicated example that we will refer to in the course of our calculations.

Example 1.10. For $k_+ = 3$ and $k_- = 5$, one valid gluing matrix is

$$\begin{pmatrix} 1 & 1 \\ 10 & -5 \end{pmatrix}$$

with $\varepsilon_+ = 1$ and $\varepsilon_- = -1$. The torus matching is illustrated in Figure 8. The aspect ratios are $\kappa_+ = 5\sqrt{2}$ and $\kappa_- = \sqrt{2}$, and the gluing angle is $\vartheta = \arg(1 + \sqrt{2}\,i) = \arccos \frac{1}{\sqrt{3}}$. One example with this gluing matrix may be found in Table 2, no. 228.

Generalising the computations from [31, §1.3], the gluing matrix also determines the fundamental group of the extra-twisted connected sum.

Proposition 1.11. *An extra-twisted connected sum M with gluing matrix $G = \left(\begin{smallmatrix} m & p \\ n & q \end{smallmatrix}\right)$ has fundamental group isomorphic to $\mathbb{Z}/p$.*

Proof. Let $\iota_\pm \colon T^2 \to M_\pm$ denote the inclusion map and note that $\pi_1(T^2) \cong \pi_1(X) \cong \mathbb{Z}^2$. Since $\pi_1 V_\pm = 1$ by Remark 1.1, we also have $\pi_1(M_\pm) \cong \mathbb{Z}$, and the interior circle $S^1_{\zeta_\pm}$ is null-homotopic in $M_\pm$, and we have a short exact sequence

$$0 \longrightarrow \pi_1(S^1_{\zeta_\pm}) \longrightarrow \pi_1(T^2) \xrightarrow{\iota_{\pm *}} \pi_1(M_\pm) \longrightarrow 0\ .$$

Because $\iota_{\pm *}$ is surjective, it follows from the Seifert-van Kampen theorem that

$$\pi_1(M) \cong \pi_1(T^2)/\big(\ker(\iota_{+*}) + \ker(\iota_{-*})\big)\ .$$

As basis of $\Lambda = \pi_1(T^2)$, we choose the vectors $\nu_+ = \frac{\mu_+ + \varepsilon_+ \lambda_+}{k_+}$ and λ_+ as in (1.5). Dividing out $\pi_1(S^1_{\zeta_+}) = \ker(\iota_+)$, we are left with a cyclic group generated by ν_+. Modulo $\pi_1(S^1_{\zeta_+})$, the group $\pi_1(S^1_{\zeta_-}) = \ker \iota_-$ is generated by $p\nu_+$, so $\pi_1(M) \cong \mathbb{Z}/p$. ☐

We will discuss covering spaces in Proposition 3.9. Some examples of non-simply connected extra-twisted connected sums will be given in Examples 3.1 (i) and 3.12.

1.4. **Matchings and polarising lattices.** In Theorem 1.3 we set up our gluing construction, using a torus matching $\mathfrak{t}$ and a hyper-Kähler rotation $\mathfrak{r}$. We studied the torus matchings in §1.3, while Theorem 1.7 reduced the problem of finding hyper-Kähler rotations to the less metric problem of finding K3 matchings. For the final piece of the machine, we review from [31, §6] how to find K3 matchings between building blocks.

The properties of the G_2-manifolds produced by Theorem 1.3 can clearly depend not just on the choices of building blocks and torus matching but also on the choice of hyper-Kähler rotation. However, the topological properties that we care about depend on the hyper-Kähler rotation only via what we term its associated "configuration" of the polarising lattices of the building blocks.

Recall from Definition 1.5 that the polarising lattice of a block (Z, Σ) refers to the image N of $H^2(Z)$ in $H^2(\Sigma)$, equipped with the intersection form. We use L to denote a fixed even unimodular lattice with signature $(3, 19)$, so that $H^2(\Sigma)$ is isometric to L for any K3 surface Σ.

Definition 1.12 ([31, Definition 6.3])**.** A *configuration* of polarising lattices N_+, N_- is a pair of primitive embeddings $N_\pm \hookrightarrow L$. Two configurations are equivalent if they are related by the action of the isometry group $O(L)$.

Clearly we can associated a configuration to any hyper-Kähler rotation $\mathfrak{r}$. Given the claim that the topology depends mainly on the blocks and the configuration, it is natural to phrase the matching problem as follows.

Question 1.13. Given $\vartheta \in \mathbb{R}$ and a pair of sets of building blocks $(Z_\pm, \Sigma_\pm)$ (each family with fixed topology, and in particular with fixed polarising lattice $N_\pm$), which configurations of N_+ and N_- are realised by a ϑ-hyper-Kähler rotation of some elements of the families?

For any lattice Λ, let $\Lambda(\mathbb{R}) := \Lambda \otimes_\mathbb{Z} \mathbb{R}$. Given a configuration, let $\pi_\pm : L(\mathbb{R}) \to N_\pm(\mathbb{R})$ denote the orthogonal projection, and let $N_\pm(\mathbb{R})^\mu$ denote the $(\cos\mu)^2$-eigenspace of the self-adjoint endomorphism $\pi_\pm\pi_\mp : N_\pm(\mathbb{R}) \to N_\pm(\mathbb{R})$, and let $N_\pm(\mathbb{R})^{\neq\mu}$ denote the orthogonal complement to $N_\pm(\mathbb{R})^\mu$ in $N_\pm(\mathbb{R})$.

Remark 1.14. By Proposition 1.8(iii), the gluing angle ϑ of any torus matching has $(\cos\vartheta)^2$ rational. Therefore $N_\pm(\mathbb{R})^\vartheta$ and $N_\pm(\mathbb{R})^{\neq\vartheta}$ are both spanned by their respective subsets $N_\pm^\vartheta$ and $N_\pm^{\neq\vartheta}$ of integral points.

The condition (1.4) implies that if there exists a ϑ-hyper-Kähler rotation compatible with a given configuration then there are positive classes $[\omega_\pm^I], [\omega_\pm^J] \in N_\pm(\mathbb{R})$ and $[\omega_\pm^K] \in N_\pm^\perp(\mathbb{R})$ such that

$$[\omega_-^K] = -[\omega_+^K]$$
$$([\omega_-^I] + i[\omega_-^J]) = e^{i\vartheta}([\omega_+^I] - i[\omega_+^J]).$$

From this we can deduce the following necessary conditions for realising a given configuration (see [31, §6.3] for explanation) by a ϑ-hyper-Kähler rotation of some (Z_+, Σ_-) and (Z_-, Σ_-).

(i) $N_+ + N_-$ is non-degenerate of signature $(2, \mathrm{rk} - 2)$.
(ii) $N_\pm^\vartheta$ contains the restriction of some Kähler class from $Z_\pm$; in particular $N_\pm^\vartheta$ is non-trivial.
(iii) The Picard lattice $\operatorname{Pic}\Sigma_\pm := H^2(\Sigma_\pm;\mathbb{Z}) \cap H^{1,1}(\Sigma_\pm;\mathbb{C})$ contains both $N_\pm$ and $N_\mp^{\neq\vartheta}$.

Let $\Lambda_\pm$ be the set of integral points in $N_\pm(\mathbb{R}) + N_\mp(\mathbb{R})^{\neq\vartheta}$. Then $\Lambda_\pm$ is primitive in L, in the sense that $L/\Lambda_\pm$ is torsion-free, and could also be described as the "primitive hull" of the sublattice $N_\pm + N_\mp^{\neq\vartheta} \subset L$ *i.e.* its smallest overlattice that is primitive. It is a non-degenerate lattice of signature $(1, \mathrm{rk}\,\Lambda_\pm - 1)$, and (iii) means that $\operatorname{Pic}\Sigma_\pm$ contains $\Lambda_\pm$, so that $\Sigma_\pm$ is $\Lambda_\pm$-*polarised*.

On the other hand, it turns out to be possible to express a sufficient condition for being able to match some elements from a pair of families in terms of those families containing suitably generic $\Lambda_\pm$-polarised K3 surfaces. For completeness, let us describe the notion of genericity that we need, even though we will not use any of the technical details. Recall that marked K3 surfaces whose Picard lattice contains a fixed primitive lattice $\Lambda \subset L$ of signature $(1, \mathrm{rk}\,\Lambda - 1)$ can be parametrised by their periods, which belong to the Griffiths domain

$$D_\Lambda = \{\text{oriented positive-definite planes } \Pi \subset \Lambda^\perp(\mathbb{R})\} \cong \{\Pi \in \mathbb{P}(\Lambda^\perp(\mathbb{C})) : \Pi^2 = 0,\ \Pi\overline{\Pi} > 0\},$$

where the second description gives rise to a complex analytic structure.

Definition 1.15 ([31, Definition 2.27])**.** Let $N \subset L$ be a primitive sublattice, $\Lambda \subset L$ a primitive overlattice of N, and $\mathrm{Amp}_\mathcal{Z}$ an open subcone of the positive cone in $N(\mathbb{R})$. We say that a set of building blocks $\mathcal{Z}$ with polarising lattice N is $(\Lambda, \mathrm{Amp}_\mathcal{Z})$-*generic* if there is a subset $U_\mathcal{Z}$ of the Griffiths domain D_Λ with complement a countable union of complex analytic submanifolds of positive codimension with the property that: for any $\Pi \in U_\mathcal{Z}$ and $\mathrm{k} \in \mathrm{Amp}_\mathcal{Z}$ there is a building block $(Z, \Sigma) \in \mathcal{Z}$ and a marking $h : L \to H^2(\Sigma;\mathbb{Z})$ such that $h(\Pi) = H^{2,0}(\Sigma)$, and $h(\mathrm{k})$ is the image of the restriction to Σ of a Kähler class on Z.

All that matters for the purposes of this paper is that the conditions in the definition make the following proposition work.

Proposition 1.16 ([31, Theorem 6.10])**.** *Let $\mathcal{Z}_\pm$ be a pair of sets of building blocks with polarising lattices $N_\pm$, and $\vartheta \in \mathbb{R}$ such that $(\cos\vartheta)^2$ is rational* [1] *. Let $N_\pm \hookrightarrow L$ be a configuration of the polarising lattices, and let $\Lambda_\pm \subset L$ be the lattice of integral points in $N_\pm(\mathbb{R}) + N_\mp(\mathbb{R})^{\neq\vartheta}$. Suppose that the set $\mathcal{Z}_\pm$ is $(\Lambda_\pm, \mathrm{Amp}_{\mathcal{Z}_\pm})$-generic. If*

$$\cos\vartheta \neq 0 \text{ and } (\operatorname{sign}\cos\vartheta)\pi_-(N_+(\mathbb{R})^\vartheta \cap \mathrm{Amp}_{\mathcal{Z}_+}) \cap \mathrm{Amp}_{\mathcal{Z}_-} \neq \emptyset. \tag{1.12}$$

or

$$\cos\vartheta = 0 \text{ and } N_+(\mathbb{R})^{\frac{\pi}{2}} \cap \mathrm{Amp}_{\mathcal{Z}_+} \neq \emptyset \text{ and } N_-(\mathbb{R})^{\frac{\pi}{2}} \cap \mathrm{Amp}_{\mathcal{Z}_-} \neq \emptyset \tag{1.13}$$

then there exist $(Z_\pm, \Sigma_\pm) \in \mathcal{Z}_\pm$ with an angle ϑ K3 matching $\mathsf{r} : \Sigma_+ \to \Sigma_-$ with the prescribed configuration.

In [14] we found that the following invariants of a configuration plays a key role in the calculation of ν (see Theorem 2.2).

Definition 1.17. Given a configuration $N_+, N_- \subset L$, let $A_\pm : L(\mathbb{R}) \to L(\mathbb{R})$ denote the reflection of $L(\mathbb{R}) := L \otimes \mathbb{R}$ in $N_\pm$ (with respect to the intersection form of $L(\mathbb{R})$; this is well-defined since $N_\pm$ is non-degenerate). Suppose $A_+ \circ A_-$ preserves some decomposition $L(\mathbb{R}) = L^+ \oplus L^-$ as a sum of positive and negative-definite subspaces. Then the *configuration angles* are the arguments $\alpha_1^+, \alpha_2^+, \alpha_3^+$ and $\alpha_1^-, \ldots, \alpha_{19}^-$ of the eigenvalues of the restrictions $A_+ \circ A_- : L^+ \to L^+$ and $A_+ \circ A_- : L^- \to L^-$ respectively.

Remark 1.18. Since $\Lambda_\pm$ is always at least as big as $N_\pm$, the genericity results required to apply Proposition 1.16 are the weakest possible when $\Lambda_\pm = N_\pm$. This happens in particular if $N_\pm^\vartheta = N_\pm$, that is, if $\pi_\pm \circ \pi_\mp|_{N_\pm} = (\cos\vartheta)^2 \,\mathrm{id}_{N_\pm}$. In that case we will say that N_+ and N_- meet *at pure angle ϑ*.

Unless $\vartheta = \pm\frac{\pi}{2}$, meeting at pure angle ϑ implies that $\operatorname{rk} N_+ = \operatorname{rk} N_- =$ multiplicity of $\pm 2\vartheta$ as configuration angles, while the remaining configuration angles are all 0.

Even with Proposition 1.16 in hand, it is still hard in general to completely answer Question 1.13 concerning which configurations can be realised by matching; in some situations it is hard to prove any genericity result of the type required (roughly, this becomes harder the larger Λ is), and even if one does, it may be hard to know whether one has found the "best possible" choice of Amp (or whether some blocks in the family have a bigger Kähler cone than the generic members).

However, all examples we know of building blocks do in fact have the property that they come in families that are (N, Amp) generic, for N the polarising lattice, and Amp some open cone in $N(\mathbb{R})$ (in particular, Proposition 5.11 asserts this for the examples in this paper). Finding all matchings of a pair of blocks where the configurations are at pure angle ϑ and the Kähler classes are required to be in a particular cone Amp is only a lattice-arithmetic problem. That can certainly be solved by a brute force algorithm, though not very easily by hand if the ranks of the polarising lattices are greater than 1.

In this paper, we will restrict attention to blocks where the polarising lattices have rank 1, which makes it possible to answer Question 1.13 decisively. Condition (ii) then automatically requires the configurations to be of pure angle, and there is no ambiguity in the choice of Amp.

[1] The hypothesis on $\cos\vartheta$ is missing from the statement [31, Theorem 6.10], but the proof there implicitly assumes that $N_\pm(\mathbb{R}) + N_\mp(\mathbb{R})^{\neq\vartheta}$ is spanned by $\Lambda_\pm$. As per Remark 1.14, this always holds in the context where we apply the result.

If the generators of the polarising lattices have square-norms n_+ and n_-, then the bilinear form on $N_+ + N_-$ imposed by the configuration will be defined by a matrix

$$\begin{pmatrix} n_+ & h \\ h & n_- \end{pmatrix},$$

and the gluing angle is determined by

$$(\cos\vartheta)^2 = \frac{h^2}{n_+ n_-}. \tag{1.14}$$

Thus there exists a matching of the blocks with gluing angle ϑ only if $\cos\vartheta\sqrt{n_+n_-}$ is an integer. By Nikulin [30, Theorem 1.12.4], this is also sufficient.

We give here one example that we will refer to while developing the calculations in Sections 2 and 4; see Subsection 5.3 for further examples of matchings.

Example 1.19. Consider two building blocks Z_+, Z_- of rank 1 with polarising lattices (6) and (2), respectively. We consider the configuration such that the restriction of the intersection form to $N_+ + N_-$ is defined by

$$\begin{pmatrix} 6 & 2 \\ 2 & 2 \end{pmatrix},$$

which has pure angle $\vartheta = \arccos\frac{1}{\sqrt{3}}$. We will combine this configuration with the gluing data of Example 1.10, using a $\mathbb{Z}/3$-block from Example 5.5 as Z_+ and a $\mathbb{Z}/5$-block from Example 5.10 as Z_-, see Table 2, no. 228. The configuration angles are

$$\alpha_1^+ = -\alpha_2^+ = 2\arccos\frac{1}{\sqrt{3}} \qquad \text{and} \qquad \alpha_3^+ = \alpha_1^- = \cdots = \alpha_{19}^- = 0\,.$$

2. Computing the extended ν-invariant

We prove Theorem 2, following the path outlined in the introduction. We recall the adapted Dirac operator (Section 2.1) and the gluing formula (Section 2.2) from [14]. The contributions from both halves consist of an adiabatic limit (Section 2.3) and a variational term (Section 2.4). To complete the computation, we rewrite the variational term in Section 2.5, using Proposition 7.1. In Section 4, we will present an alternative approach to the computation of the variational terms that leads to Theorem 3.

2.1. **A modification of the spin Dirac operator.**

The extended ν-invariant of a G_2-manifold is defined in (0.1) using the η-invariant of the signature operator B and the spin Dirac operator D. For computations, it is much more comfortable to work with a Riemannian metric that is of product type in the gluing region and sufficiently close to some G_2-metric. However, the η-invariant of the spin Dirac operator of such a gluing metric typically differs from the one in the G_2-case both by a small local contribution and by a $\mathbb{Z}$-valued spectral flow. To avoid the latter, we modified the spin Dirac operator in [14]. Because all our following considerations rely on the gluing metric and the modified Dirac operator, we take the time to introduce them now.

2.1.1. Let $(M_\ell, \bar g_\ell)$ denote the G_2-manifold produced by Theorem 1.3. For $\ell \gg 1$, it is close to a Riemannian manifold (M_ℓ, g_ℓ) produced by naive gluing, in a sense we want to make precise.

Recall from Section 1.1 that $(V_\pm, g^{V_\pm})$ are Calabi-Yau manifolds with one end each that is asymptotic to a cylinder $\Sigma_\pm \times S^1_{\zeta_\pm} \times (0,\infty)$. We extend $t_\pm$ to smooth functions on $X_\pm$ that are nonpositive outside the cylindrical region. Then we first choose new metrics $g_\ell^{V_\pm}$ that agree

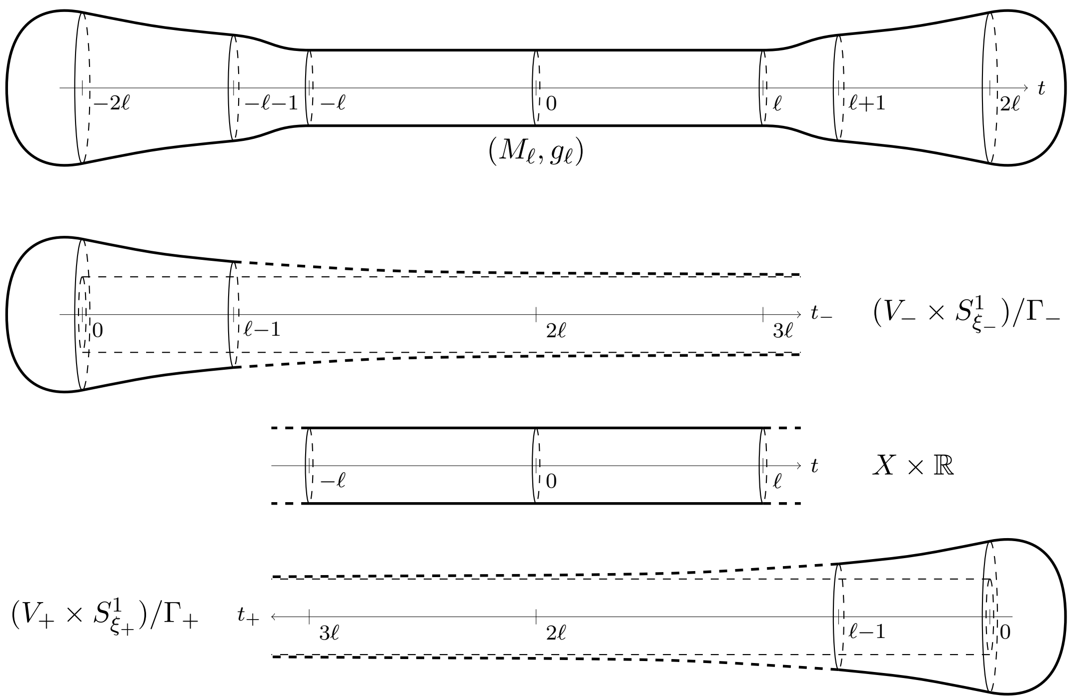


FIGURE 9. The gluing metric g_ℓ on M_ℓ

with the original Calabi-Yau metrics $g^{V_\pm}$ on $\{\, x \in V_\pm \bigm| t_\pm(x) \le \ell - 1 \,\}$ and with the cylindrical metrics on $\{\, x \in V_\pm \bigm| t_\pm(x) \ge \ell \,\}$. This can be done such that $\bigl\| g_\ell^{V_\pm} - g^{V_\pm} \bigr\|_{C^k} = O(e^{-c\ell})$ for a fixed $c > 0$ and all k.

We consider the twisted products $M_\pm = (V_\pm \times S^1_{\xi_\pm})/\Gamma_\pm$ of $(V_\pm, g_\ell^{V_\pm})$ and a circle $S^1_{\xi_\pm}$ of length $\xi_\pm$. Let us regard $t_\pm$ as functions on $M_\pm$. The cross sections $t_\pm^{-1}(s)$ for $s \ge \ell$ are isometric to $X = \Sigma \times T^2$ with $T^2 \cong (S^1_{\zeta_\pm} \times S^1_{\xi_\pm})/\Gamma_\pm$ by the construction in Section 1, but with different orientations. Hence, there is an orientation preserving isometry between the cylindrical regions $\{\, x \in M_\pm \bigm| t_\pm(x) \in [\ell, 3\ell] \,\}$ that identifies t_+ with $4\ell - t_-$. This allows us to glue M_+ to M_- after chopping off the ends $\{\, x \in M_\pm \bigm| t_\pm(x) > 3\ell \,\}$, see Figure 9. The resulting manifold will be denoted (M_ℓ, g_ℓ), and we refer to g_ℓ as the *gluing metric*. Let t be the function on M that agrees with $t_- - 2\ell$ and $2\ell - t_+$ wherever those are defined.

By [14, Rem 4.5], the resulting metric g_ℓ has the following properties.

(i) The restriction of g_ℓ to $\{\, x \in M_\ell \mid \pm t(x) \ge -\ell \,\}$ is isometric to the region $t_\pm \le 3\ell$ in the twisted product $(V_\pm \times S^1_{\xi_\pm})/\Gamma_\pm$. of $(V_\pm, g_\ell^{V_\pm})$ and a circle $S^1_{\xi_\pm}$ of length $\xi_\pm$.

(ii) The restriction of the metric $g_\ell^{V_\pm}$ to $\{\, x \in V_\pm \mid t_\pm(x) \le \ell - 1 \,\}$ agrees with the original asymptotically cylindrical Calabi-Yau metric $g^{V_\pm}$.

(iii) The manifold $t^{-1}((-\ell, \ell)) \subset M_\ell$ is the Riemannian product of the K3 surface Σ, the torus T^2 and the interval $(-\ell, \ell)$ of length 2ℓ.

(iv) We have $\bigl\| g_\ell|_{X \times (\pm[\ell, \ell+1])} - g^X \oplus dt^2 \bigr\|_{C^k} = O(e^{-c\ell})$ for all k.

(v) There exists $c > 0$ such that for all k, we have

$$\| g_\ell - \bar g_\ell \|_{C^k} = O\bigl(e^{-c\ell}\bigr) \, .$$

It follows from (ii) and (iii) that the metric g_ℓ has local holonomy contained in G_2 except over the set $X \times ([-\ell-1,-\ell] \cup [\ell,\ell+1])$, where it is controlled by (i) and (iv). The gluing region contains pieces $X \times \pm[\ell+1,2\ell]$ that have the geometry of a product of a circle and an asymptotically cylindrical Calabi-Yau manifold. If they are long enough, the G_2-structure on (M_ℓ, g_ℓ) has sufficiently small torsion. The piece $X \times [-\ell,\ell]$ is a straight cylinder. If it is long enough, we can control the kernel of the Dirac operator $D_{M,\ell}$, see section 2.1.3 below. In [14, Section 5], we have seen that the lengths of both pieces can be chosen on the same scale.

We now consider the two halves separately. Let $V_{\pm,\ell} = V_\pm \setminus ((2\ell,\infty) \times S^1_{\zeta_\pm} \times \Sigma)$ as before. For $a > 0$, put

$$\widetilde{M}_{\pm,a} = V_{\pm,\ell} \times S^1_{a\zeta_\pm} \qquad \text{and} \qquad M_{\pm,a} = \widetilde{M}_{\pm,a}/\Gamma_\pm \ , \tag{2.1}$$

where $S^1_{a\zeta_\pm}$ denotes an exterior circle of length $\xi_\pm = a\zeta_\pm$ and $V_\pm$ carries the metric $g_\ell^{V_\pm}$ introduced in (i) above. Then the new metric $g_{\pm,a}$ on $M_{\pm,a}$ satisfies properties analogous to (i)–(iv) above. For $a = \kappa_\pm$ as in Proposition 1.8 (ii), we recover the restriction of the metric g_ℓ. We consider the odd signature operator $B_{M_{\pm,a}}$ for the new metric.

2.1.2. By Remark 1.4 and property (i) of the metrics $g_{\pm,a}$, we may describe the spinor bundle on $M_{\pm,a}$ with Hermitian metric and Clifford connection $\nabla^{SM} = \nabla^{SM_{\pm,a}}$ as

$$SM_{\pm,a} = S\widetilde{M}_{\pm,a}/\Gamma_\pm \qquad \text{with} \qquad S\widetilde{M}_{\pm,a} = p^* SV_{\pm,\ell} \ , \tag{2.2}$$

where $p\colon V_{\pm,\ell} \times S^1_{a\zeta_\pm} \to V_{\pm,\ell}$ is the projection. Let $\partial_{v_\pm}$ denote the unit tangent vector to the exterior circle factor in the twisted Riemannian product $M_{\pm,a}$ for all $a > 0$. As in [14, Section 5.1], we construct a unit spinor s on $M_{\pm,a}$ ($s_{\ell,1}$ in the notation of [14]) such that

(i) the spinor s is pulled back from a $\Gamma_\pm$-invariant unit spinor on $V_{\pm,\ell}$ independent of a,
(ii) its derivative $\nabla^{SM} s$ is supported on $X \times (\pm[\ell,\ell+1])$,
(iii) there exists $c > 0$ such that $\|\nabla^{SM} s\| = O(e^{-c\ell})$,
(iv) we have $\nabla^{SM}_{\partial_{v_\pm}} s = 0$.

In [14], we also identify $SM_{\pm,\ell}$ with the the spinor bundle for the metric $\bar g_\ell|_{M_{\pm,\ell}}$ in such a way that s corresponds to the restriction of the parallel spinor on the G_2-manifold $(M, \bar g_\ell)$.

2.1.3. Let $D'_{M_{\pm,a}}$ denote the geometric spin Dirac operator of $M_{\pm,a}$, and let $c_{v_\pm}$ denote Clifford multiplication by $\partial_{v_\pm}$. Decomposing $D'_{M_{\pm,a}} s$ using (2.2) and the properties of $g_{\pm,a}$ and s above, we find functions $f_\pm$, $h_\pm$ on $V_\pm$ and a spinor $r_\pm \in \Gamma(SM)$ that is pulled back from a $\Gamma_\pm$-invariant spinor on $V_{\pm,\ell}$, all independent of a, such that

$$D'_{M_{\pm,a}} s = f_\pm \cdot s + h_\pm \cdot c_{v_\pm} s + r_\pm \ , \tag{2.3}$$

and such that $r_\pm$ is perpendicular to s and $c_{v_\pm} s$ everywhere. As in [14, (38) & (39)], put

$$\begin{aligned} D_{M_{\pm,a}} = D'_{M_{\pm,a}} &- \langle \,\cdot\,, s\rangle \big(f_\pm s + h_\pm c_{v_\pm} s + r_\pm\big) - \langle \,\cdot\,, r_\pm\rangle\, s \\ &- \langle \,\cdot\,, c_{v_\pm} s\rangle \big(h_\pm s - f_\pm c_{v_\pm} s - c_{v_\pm} r_\pm\big) + \langle \,\cdot\,, c_{v_\pm} r_\pm\rangle\, c_{v_\pm} s \ . \end{aligned} \tag{2.4}$$

Then $\ker D_{M_{\pm,a}}$ contains the parallel unit spinors s and $c_{v_\pm} s$ for all ℓ and all $a > 0$.

In the special case $a = \kappa_\pm$, the operators above combine to an operator $D_{M,\ell}$ on M_ℓ.

(i) On $M_\ell \setminus \big(X \times ([-\ell-1,-\ell] \cup [\ell,\ell+1])\big)$, the operator $D_{M,\ell}$ agrees with the geometric spin Dirac operator of the gluing metric g_ℓ described above.
(ii) By [14, Prop 5.5], the kernel of $D_{M,\ell}$ is spanned by the nowhere vanishing section s.
(iii) We have $D_{M,\ell}|_{M_\pm}(c_{v_\pm} s) = 0$ by (2.3) and (2.4), see [14, (41)].

(iv) By [14, Prop 5.7], there is a constant $c > 0$ such that for $\ell \gg 1$, we have

$$\eta(D_{(M,\bar g_\ell)}) = \eta(D_{M,\ell}) + O\big(e^{-c\ell}\big) ,$$

where $D_{(M,\bar g_\ell)}$ is the geometric Dirac operator of the G_2-manifold $(M, \bar g_\ell)$.

In particular, there is no spectral flow if we deform $D_{(M,\bar g_\ell)}$ into $D_{M,\ell}$. From 2.1.1 (v) and 2.1.3 (iv), we conclude that

$$\bar\nu(M) = \lim_{\ell\to\infty}\big(3\eta(B_{M,\ell}) - 24\eta(D_{M,\ell})\big) . \tag{2.5}$$

Note that the linear combination of η-invariants on the right hand in principle differs from the extended ν-invariant of $(M, \bar g_\ell)$ by a Mathai-Quillen term and two differences of η-invariants. These terms tend to 0 as $\ell \to \infty$ by our construction (In fact, it turns out that we already get $\bar\nu(M) = 3\eta(B_{M,\ell}) - 24\eta(D_{M,\ell})$ for any sufficiently large ℓ, *cf.* Remark 2.1).

2.2. **The gluing formula.** In the following, we cut (M, g_ℓ) into two halves $M_{\pm,\kappa_\pm}$ with common boundary $\Sigma \times T^2 \times \{0\}$. We identify $\Lambda^\bullet T^*(\Sigma \times T^2)$ with the restriction of $\Lambda^{\mathrm{ev}} T^*M$, let the boundary operator A of the odd signature operator B act on $\Omega^\bullet(X)$ as in [14, (28)], and put

$$L_{B_\pm} = \mathrm{im}\big(H^\bullet(M_\pm) \to H^\bullet(X)\big) \subset H^\bullet(X) , \tag{2.6}$$

then $L_{B_\pm}$ are Lagrangian subspaces of $H^\bullet(X) \cong \ker A$. As in [14, sec 4.1], let $\eta(B_{M_{\pm,a}}; L_{B_\pm})$ denote the η-invariant of $B_{M_{\pm,a}}$ with respect to APS boundary conditions modified by $L_{B_\pm}$; in particular, the forms in the domain of $B_{M_{\pm,a}}$ orthogonally project to 0 on the Lagrangian in $\Omega^\bullet(X)$ given as the direct sum of $L_{B_\pm}$ with the sum of all eigenspaces of A of eigenvalues of sign $\pm$.

For the operator $D_{M_{\pm,a}}$, we define similar boundary conditions as in [14, (44)]. We identify the spinor bundle of $\Sigma \times T^2$ with the restriction of the spinor bundle of M and write

$$D_{M_{\pm,a}}|_{X\times(-\ell,\ell)} = c_t\left(\frac{\partial}{\partial t} + A_{\pm,a}\right) ,$$

where c_t denotes Clifford multiplication with $\frac{\partial}{\partial t}$ and $A_{\pm,a}$ now denotes the boundary operator of the modified spin Dirac operator $D_{M_{\pm,a}}$. Then $\ker A_{\pm,a} \cong H^{0,\bullet}(X) \cong \mathbb{C}^4$ is independent of a.

Together with the L^2-metric on spinors, c_t introduces a symplectic structure on $\ker A_{\pm,a}$. Let s span $\ker(D_{M,\ell})$ as in 2.1.3 (ii) above, then by 2.1.3 (iii),

$$L_{D_-} = \mathrm{span}\{s, c_{v_-}s\} \qquad \text{and} \qquad L_{D_+} = \mathrm{span}\{s, c_{v_+}s\} \tag{2.7}$$

are subspaces of $\ker A_{\pm,a}$. As explained in [14, Section 5.2], they are exactly the Lagrangian subspaces of $A_{\pm,a}$-harmonic spinors on X that extend to $D_{M_{\pm,a}}$-harmonic spinors on $M_{\pm,a}$ for each a. Define $\eta(D_{M_{\pm,a}}; L_{D_\pm})$ as above. In particular, the spinors in the domain of $D_{M_{\pm,a}}$ project orthogonally to 0 on the Lagrangian in $\Gamma(SX)$ given as the direct sum of $L_{D_\pm}$ with the sum of all eigenspaces of $A_{\pm,a}$ of eigenvalues of sign $\pm$.

We define

$$\bar\nu(M_{\pm,a}) = \lim_{\ell\to\infty}\big(3\eta(B_{M_{\pm,a}}; L_{B_\pm}) - 24\eta(D_{M_{\pm,a}}; L_{D_\pm})\big). \tag{2.8}$$

Remark 2.1. While $\eta(B_{M_{\pm,a}}; L_{B_\pm})$ and $\eta(D_{M_{\pm,a}}; L_{D_\pm})$ can depend on ℓ, it turns out in Theorems 2.5 and 2.13 that $3\eta(B_{M_{\pm,a}}; L_{B_\pm}) - 24\eta(D_{M_{\pm,a}}; L_{D_\pm})$ does not.

We recall the gluing angle ϑ from (1.3). In Definition 1.17, we have introduced the configuration angles α_1^+, α_2^+, α_3^+, α_1^-, …, $\alpha_{19}^- \in (-\pi,\pi]$. For $\rho\in\mathbb{R}$, define

$$\begin{aligned} m_\rho(L;N_+,N_-) &= \operatorname{sign}\rho\left(\#\left\{\,j\;\middle|\;\alpha_j^-\in\{\pi-|\rho|\,,\pi\}\,\right\}-1\right)\\ &\qquad+2\operatorname{sign}\rho\,\#\left\{\,j\;\middle|\;\alpha_j^-\in(\pi-|\rho|\,,\pi)\,\right\}\qquad\in\mathbb{Z}\,, \end{aligned}\tag{2.9}$$

Theorem 2.2 ([14, Theorem 1]). *Let M be an extra-twisted connected sum with $\vartheta\notin\pi\mathbb{Z}$, and put $\rho=\pi-2\vartheta$. Then the extended ν-invariant of M is given by*

$$\bar\nu(M,g)=\bar\nu(M_{+,\kappa_+})+\bar\nu(M_{-,\kappa_-})-72\frac{\rho}{\pi}+3m_\rho(L;N_+,N_-)\;. \tag{2.10}$$

Note that in the examples in [14], we had $\Gamma_\pm=\mathbb{Z}/k_\pm$ with $k_\pm\in\{1,2\}$. In these cases, one could find orientation reversing isometries of $M_\pm$ that anticommute with $B_{M_\pm}$, $D_{M_\pm}$ and preserve the boundary conditions, leading to $\eta(B_{M_\pm};L_{B_\pm})=\eta(D_{M_\pm};L_{D_\pm})=0$. Here, we want to deal with examples where this is no longer the case. We have examples where $\vartheta\notin\mathbb{Q}\pi$, so that $\frac{\rho}{\pi}\notin\mathbb{Q}$. In these cases at least one of the invariants $\bar\nu(M_{\pm,\kappa_\pm})$ above must be irrational. In particular, they can no longer both vanish.

Example 2.3. With the configuration angles of Example 1.19, we get $\rho=\pi-2\arccos\frac{1}{\sqrt3}>0$ and hence

$$-72\frac{\rho}{\pi}+3m_\rho(L;N_+,N_-)=-72+\frac{144}{\pi}\,\arccos\frac{1}{\sqrt3}-3\;.$$

2.3. **The adiabatic limit of η-invariants.** To compute $\bar\nu(M_{\pm,\kappa_\pm})$ in Theorem 2.2 above, we consider the limits $\bar\nu(M_{\pm,a})$ for $a\to0$ in this subsection. These limits differ from $\bar\nu(M_{\pm,\kappa_\pm})$, and the difference is described as an integral over a variational term in the next subsection. A direct computation of this contribution is given in Section 2.5, completing the proof of Theorem 2. Two technical intermediate results have been postponed to Section 6 for better readability.

We still work on the manifolds $M_{\pm,a}$, which are twisted Riemannian products by property 2.1.1 (i) above. We also still consider the modification $D_{M_{\pm,a}}$ of the spin Dirac operator considered in (2.4).

We write

$$\bar\nu\big(M_\pm\big)=\bar\nu\big(M_{\pm,\kappa_\pm}\big)=\lim_{a\to0}\bar\nu\big(M_{\pm,a}\big)+\int_0^{\kappa_\pm}\frac{d}{da}\,\bar\nu\big(M_{\pm,a}\big)\,da\;. \tag{2.11}$$

We consider $W_\pm=V_{\pm,\ell}/\Gamma_\pm$ as an orbifold with boundary, where the boundary itself is a manifold by assumption. Let $\Lambda W_\pm$ denote its inertia orbifold. The orbifold $\hat A$-form on $\Lambda W_\pm$ is defined in [21, (1.6)]. We will also need the orbifold $\hat L$-form; see [21, Cor 1.10]. Let $\mathbb{A}$ denote the Bismut superconnection of the fibrewise spin Dirac operator for the map $p\colon M_{\pm,a}\to W_\pm$ with respect to the fibrewise trivial spin structure; see Remark 1.4. Let $\eta_{\Lambda W_\pm}(\mathbb{A})\in\Omega^\bullet(\Lambda W_\pm)$ denote the orbifold η-form as in [21, Def 1.7].

To compute the adiabatic limit, we need to combine Dai's result for manifolds with boundary in [19] with the version for Seifert fibrations without boundary in [21]. This in done in Theorem 6.1, which we have postponed because its proof is technical and independent from the problem at hand. It implies that

$$\lim_{a\to0}\bar\nu\big(M_{\pm,a}\big)=\int_{\Lambda W_\pm\setminus W_\pm}\Big(3\hat L_{\Lambda W_\pm}\big(TW_\pm,\nabla^{TW_\pm}\big)-24\hat A_{\Lambda W_\pm}\big(TW_\pm,\nabla^{TW_\pm}\big)\Big)\,2\eta_{\Lambda W_\pm}(\mathbb{A})\;. \tag{2.12}$$

Because $W_\pm$ is even-dimensional, there is no contribution from η-invariants on $W_\pm$. Moreover, there are no very small eigenvalues in our situation. We remark that the circle orbibundle $M_{\pm,a}\to W_\pm$ is flat by construction, so the integral above localises at the orbifold singularities of $W_\pm$, and there is no contribution from the principal stratum. We are in a local product situation, so the orbifold η-forms all reduce to equivariant η-invariants.

The action of $\Gamma = \mathbb{Z}/k\mathbb{Z}$ on V is faithful because it is free on ∂V. At each fixpoint $p \in V^\gamma$ of $\gamma \in \Gamma$, the tangent space T_pV splits as a sum of complex eigenspaces of the differential of γ with eigenvalues $e^{i\alpha_\ell}$, with $\alpha_1, \alpha_2, \alpha_3 \in \frac{2\pi}{k}\,\mathbb{Z}$. Because γ preserves the holomorphic volume form, the angles α_ℓ add up to a multiple of 2π. Hence, the complex codimension of the fixpoint set has to be at least 2. If the fixpoints are not isolated, then $V^\gamma \subset V$ is totally geodesic, and the eigenspaces locally form bundles over V^γ. The tangent bundle TV^γ corresponds to $\alpha_\ell = 0$. Let $\nu_\gamma \to V^\gamma$ denote the normal bundle.

We assume that the coordinate $v \in \mathbb{R}/\xi\mathbb{Z}$ on the exterior circle has been chosen such that inserting ∂_v into the G_2-form φ gives the Kähler form on V; see [14, (8)]. Then let $\gamma \in \Gamma$ be the generator that acts on the exterior circle by sending v to $v + \frac{\xi}{k}$. We start by defining a *generalised Dedekind sum* as in [21]. Note that it depends on the particular choice of generator γ.

Definition 2.4. Let $\gamma \in \Gamma \cong \mathbb{Z}/k\mathbb{Z}$ be a generator. For $0 < j < k$, let $V^{0,j}$ denote the set of isolated fixpoints of γ^j, and for each $p \in V^{0,j}$, let $\alpha_{j,1}(p)$, $\alpha_{j,2}(p)$, $\alpha_{j,3}(p)$ denote the angles of the action of γ^j on the 3-dimensional complex vector space T_pV, chosen such that $\alpha_{j,1}(p) + \alpha_{j,2}(p) + \alpha_{j,3}(p) \in 4\pi\mathbb{Z}$. Then define

$$D_\gamma(V) = \frac{3}{k}\sum_{j=1}^{k-1}\cot\frac{\pi j}{k}\sum_{p\in V^{0,j}}\frac{\cos\frac{\alpha_{j,1}(p)}{2}\cos\frac{\alpha_{j,2}(p)}{2}\cos\frac{\alpha_{j,3}(p)}{2}-1}{\sin\frac{\alpha_{j,1}(p)}{2}\sin\frac{\alpha_{j,2}(p)}{2}\sin\frac{\alpha_{j,3}(p)}{2}}\;,$$

Theorem 2.5. *Let $\gamma_\pm \in \Gamma_\pm \cong \mathbb{Z}/k_\pm\mathbb{Z}$ be the generator that acts on the exterior circle $\mathbb{R}/\xi_\pm\mathbb{Z}$ by sending $v_\pm$ to $v_\pm + \frac{\xi_\pm}{k}$. Define $D_{\gamma_\pm}(V_\pm)$ as above, then*

$$\lim_{a\to 0}\bar\nu(M_{\pm,a}) = D_{\gamma_\pm}(V_\pm)\;.$$

In particular, non-isolated fixpoints do not contribute to the adiabatic limit of the extended ν-invariant. We have shown in [14] that $\bar\nu(M_\pm) = 0$ if $k_\pm = 1$ or $k_\pm = 2$. This is consistent because involutions of odd-dimensional Calabi-Yau manifolds cannot have isolated fixpoints.

Proof. Being a Calabi-Yau manifold, V has a spin structure with spinor bundle $\Lambda^{0,\bullet}T^*V$. The Kähler metric identifies $\Lambda^{0,1}T^*V$ with TV with its natural complex structure. Let $\gamma \in \Gamma$. Because V^γ is at most one-dimensional, we can split T_pV into one-dimensional eigenspaces that are also invariant under the curvature tensor $F \in \Lambda^{1,1}\operatorname{End}(T_pV)$. This allows us to decompose the action of $\gamma e^{-\frac{F}{2\pi i}}$ on the spinor space $\Lambda^{0,\bullet}T^*V|_{V^\gamma}$ as

$$\gamma e^{-\frac{F}{2\pi i}}|_{\Lambda^{0,\bullet}T_p^*V} \cong \bigotimes_{j=1}^{3}\begin{pmatrix}1 & \\ & e^{i\alpha_\ell}(1+\beta_\ell)\end{pmatrix}\;,$$

where $\beta_\ell \in \Lambda^{1,1}T^*V^\gamma$ are real differential forms that represent the Chern roots of the subbundle of $TV|_{V^\gamma}$ corresponding to the eigenvalue $e^{i\alpha_\ell}$. We assume that $\alpha_1 + \alpha_2 + \alpha_3 = 0$. Because V is Ricci-flat, we know that $\beta_1 + \beta_2 + \beta_3 = 0$. This allows us to twist each tensor factor above with a line L_ℓ on which $\gamma e^{-\frac{F}{2\pi i}}$ acts as $e^{-i\frac{\alpha_\ell}{2}}\left(1 - \frac{\beta_\ell}{2}\right)$, and we get

$$\gamma e^{-\frac{F}{2\pi i}}|_{\Lambda^{0,\bullet}T_p^*V} \cong \bigotimes_{\ell=1}^{3}\begin{pmatrix}e^{-\frac{i\alpha_\ell}{2}}\left(1-\frac{\beta_\ell}{2}\right) & \\ & e^{\frac{i\alpha_\ell}{2}}\left(1+\frac{\beta_\ell}{2}\right)\end{pmatrix}\;. \tag{2.13}$$

Finally, we note that the Seifert fibration $M_\pm \to V_\pm/\Gamma_\pm$ is locally of product geometry. Therefore, the equivariant η-form $\tilde\eta_{\gamma^j}(\mathbb{A})$ reduces to half the equivariant η-invariant. If $\gamma \in \Gamma$ denotes the preferred generator, then

$$\eta_{\gamma^j}(D_{S^1}) = \eta_{\gamma^j}(B_{S^1}) = -i\cot\frac{\pi j}{k} \qquad \in \Omega^0(V) \tag{2.14}$$

with respect to the preferred orientations.

Complex one-dimensional fixpoint sets. Assume that $C \subset V^{\gamma^j}$ is a connected component of the fixpoint set of γ^j with $\dim_{\mathbb{C}} C = 1$, and with normal bundle $\nu_C \to C$ in V. Along C, we have $\alpha_2 = -\alpha_1$ and $\alpha_3 = 0$. To compute the orbifold $\hat A$-class following [21, (1.6) & (1.7)] and [3, sect. 6.4], we need

$$\begin{aligned}\operatorname{ch}\bigl(\gamma^j, \Lambda^{0,\mathrm{ev}}\nu_C^* - \Lambda^{0,\mathrm{odd}}\nu_C^*\bigr)|_{(C,\gamma^j)} &= \prod_{\ell=1}^{2} \operatorname{str}\begin{pmatrix} e^{-\frac{i\alpha_\ell}{2}}\Bigl(1-\frac{\beta_\ell}{2}\Bigr) & \\ & e^{\frac{i\alpha_\ell}{2}}\Bigl(1+\frac{\beta_\ell}{2}\Bigr)\end{pmatrix}\\ &= 4\sin^2\frac{\alpha_1}{2} - 2i(\beta_1-\beta_2)\sin\frac{\alpha_1}{2}\cos\frac{\alpha_1}{2}\ ,\end{aligned}$$

which follows from (2.13). For dimension reasons, $\hat A(TC) = 1$, so β_3 cannot contribute. Then by (2.14), the whole contribution of $(C, \gamma^j) \in \Lambda V$ to the untwisted η-invariant is

$$\begin{aligned}&\frac{(-1)^{\mathrm{rk}_{\mathbb{C}}\nu_C}\hat A(TC)}{k\operatorname{ch}(\gamma^j, \Lambda^{0,\mathrm{ev}}\nu_C^* - \Lambda^{0,\mathrm{odd}}\nu_C^*)}[C]\cdot\eta_{\gamma^j}(D_{S^1})\\ &\qquad = \frac{1}{4\sin^2\frac{\alpha_1}{2} - 2i(\beta_1-\beta_2)\sin\frac{\alpha_1}{2}\cos\frac{\alpha_1}{2}}[C]\cdot\frac{\eta_{\gamma^j}(D_{S^1})}{k}\\ &\qquad = \frac{i\cos\frac{\alpha_1}{2}}{8k\sin^3\frac{\alpha_1}{2}}\cdot(\beta_1-\beta_2)[C]\cdot\eta_{\gamma^j}(D_{S^1})\ .\end{aligned} \tag{2.15}$$

For the signature η-invariant, we have to compute the equivariant twist Chern character following [3, Def 6.15]. The spinor bundle $\Lambda^{0,\bullet}T_p^*C$ of $T_pC \subset T_pV$ contributes only by its rank. By (2.13), we have

$$\begin{aligned}\operatorname{ch}\bigl(\gamma^j, \Lambda^{0,\bullet}T_p^*V\bigr)|_{(C,\gamma^j)} &= 2\prod_{\ell=1}^{2}\operatorname{tr}\begin{pmatrix} e^{-\frac{i\alpha_\ell}{2}}\Bigl(1-\frac{\beta_\ell}{2}\Bigr) & \\ & e^{\frac{i\alpha_\ell}{2}}\Bigl(1+\frac{\beta_\ell}{2}\Bigr)\end{pmatrix}\\ &= 8\cos^2\frac{\alpha_1}{2} + 4i(\beta_1-\beta_2)\cos\frac{\alpha_1}{2}\sin\frac{\alpha_1}{2}\ .\end{aligned}$$

By (2.14) and the above, the whole contribution of V^{γ^j} to the signature η-invariant is

$$\begin{aligned}&\frac{(-1)^{\mathrm{rk}_{\mathbb{C}}\nu_C}\hat A(TC)\operatorname{ch}\bigl(\gamma^j, \Lambda^{0,\bullet}T_p^*V\bigr)}{k\operatorname{ch}(\gamma^j, \Lambda^{0,\mathrm{ev}}\nu_C^* - \Lambda^{0,\mathrm{odd}}\nu_C^*)}[C]\cdot\eta_{\gamma^j}(B_{S^1})\\ &\qquad = \frac{8\cos^2\frac{\alpha_1}{2} + 4i(\beta_1-\beta_2)\cos\frac{\alpha_1}{2}\sin\frac{\alpha_1}{2}}{4\sin^2\frac{\alpha_1}{2} - 2i(\beta_1-\beta_2)\cos\frac{\alpha_1}{2}\sin\frac{\alpha_1}{2}}[C]\cdot\eta_{\gamma^j}(D_{S^1})\\ &\qquad = \frac{i\cos\frac{\alpha_1}{2}}{k\sin^3\frac{\alpha_1}{2}}\cdot(\beta_1-\beta_2)[C]\cdot\eta_{\gamma^j}(D_{S^1})\ .\end{aligned} \tag{2.16}$$

From (2.12), (2.15) and (2.16), we see that (V^{γ^j}, γ^j) does not contribute to $\lim_{a\to 0}\bar\nu(M_a)$.

Isolated fixpoints. At an isolated fixpoint p of $\gamma=\gamma^j$, we have $\nu_p=T_pV$. The action of γ is determined by three nonzero angles $\alpha_\ell=\alpha_{j,\ell}(p)$ for $\ell=1$, 2, 3 that add up to 0. If necessary, we add a multiple of 2π to one of the angles.

The contribution to the orbifold $\hat{A}$-form is the number

$$\operatorname{ch}\bigl(\gamma,\Lambda^{0,\mathrm{ev}}T_p^*V-\Lambda^{0,\mathrm{odd}}T_p^*V\bigr)|_{(p,\gamma)}=\prod_{\ell=1}^3\operatorname{str}\begin{pmatrix}e^{-\frac{i\alpha_\ell}{2}}&\\&e^{\frac{i\alpha_\ell}{2}}\end{pmatrix}=8i\,\sin\frac{\alpha_1}{2}\sin\frac{\alpha_2}{2}\sin\frac{\alpha_3}{2}\;.$$

By (2.14), the contribution to the untwisted η-invariant is

$$\frac{(-1)^{\operatorname{rk}_{\mathbb{C}}T_pV}}{k\operatorname{ch}(\gamma,\Lambda^{0,\mathrm{ev}}T_p^*V-\Lambda^{0,\mathrm{odd}}T_p^*V)}[p]\cdot\eta_{\gamma^j}(D_{S^1})=\frac{\cot\frac{\pi j}{k}}{8k\sin\frac{\alpha_1}{2}\sin\frac{\alpha_2}{2}\sin\frac{\alpha_3}{2}}\;. \tag{2.17}$$

For the signature η-invariant, we multiply the above with the equivariant Chern character

$$\operatorname{ch}\bigl(\gamma,\Lambda^{0,\bullet}T_p^*V\bigr)=\prod_{\ell=1}^3\operatorname{tr}\begin{pmatrix}e^{-\frac{i\alpha_\ell}{2}}&\\&e^{\frac{i\alpha_\ell}{2}}\end{pmatrix}=8\cos\frac{\alpha_1}{2}\cos\frac{\alpha_2}{2}\cos\frac{\alpha_3}{2}$$

and obtain the contribution to the signature η-invariant

$$\frac{(-1)^{\operatorname{rk}_{\mathbb{C}}T_pV}\operatorname{ch}\bigl(\gamma,\Lambda^{0,\bullet}T_p^*V\bigr)}{k\operatorname{ch}(\gamma,\Lambda^{0,\mathrm{ev}}T_p^*V-\Lambda^{0,\mathrm{odd}}T_p^*V)}[p]\cdot\eta_{\gamma^j}(B_{S^1})=\frac{1}{k}\cot\frac{\alpha_1}{2}\cot\frac{\alpha_2}{2}\cot\frac{\alpha_3}{2}\cot\frac{\pi j}{k}\;. \tag{2.18}$$

From (2.12), (2.17) and (2.18), we obtain the Theorem. □

Remark 2.6. When considering examples, it is more convenient to fix the generator τ that acts on the interior circle $S^1_\zeta\cong\mathbb{R}/\zeta\mathbb{Z}$ by $u\mapsto u+\frac{\zeta}{k}$. If a unit $\varepsilon\in\mathbb{Z}/k$ is chosen as in equation (1.5), then the generator chosen above is $\gamma=\tau^\varepsilon$, and the contribution of the isolated fixpoints to the extended ν-invariant is given by $D_{\tau^\varepsilon}(V)$.

Now assume that Z is a building block with a Γ-action that fixes an anticanonical divisor Σ of Z pointwise, and that $V\cong Z\setminus\Sigma$. The orientation convention of [13, equation (3.2)] says that the complex structure rotates the outward cylindrical direction into the positive direction of the interior circle. If we identify the asymptotic cylinder with the normal bundle to Σ in Z, then the outward cylindrical direction becomes the inward normal direction. This means that the interior circle through $v\in\nu_\Sigma$ is oriented by $-iv\in T_v\nu_\Sigma$. Hence $\tau\in\Gamma$ should act on ν_Σ by $e^{-\frac{2\pi i}{k}}$.

Example 2.7. In Example 5.10, we describe a building block with $\mathbb{Z}/5$-symmetry. It has one isolated fixpoint p. Let $\zeta_5=e^{\frac{2\pi i}{5}}$, then there is a generator τ of $\mathbb{Z}/5$ that acts on T_pZ as $\operatorname{diag}(\zeta_5,\zeta_5,\zeta_5^{-2})$ and on ν_Σ by $\zeta_5^{-1}=e^{-\frac{2\pi i}{5}}$. Hence, the generator γ in Theorem 2.5 corresponds to τ^ε, for $\varepsilon\not\equiv 0$ mod 5. Then γ acts on T_pZ as $\operatorname{diag}(\zeta_5^\varepsilon,\zeta_5^\varepsilon,\zeta_5^{-2\varepsilon})$, so

$$\alpha_{j,1}(p)=\alpha_{j,2}(p)=\frac{2\varepsilon\pi}{5}\qquad\text{and}\qquad\alpha_{j,3}(p)=-\frac{4\varepsilon\pi}{5}\;.$$

If we represent $\varepsilon\in\mathbb{Z}/5\setminus\{0\}$ by an element of $\{-2,-1,1,2\}$, we get

$$D_\gamma(V)=D_{\tau^\varepsilon}(V)=\lim_{r\to0}\bar\nu\bigl((V\times S^1_r)/\Gamma\bigr)=\frac{24}{5\varepsilon}\;.$$

We can use this block as Z_- in Example 1.10, with $\varepsilon_-=-1$ and hence $D_{\gamma_-}(V^-)=-\frac{24}{5}$.

2.4. **The variation of η-invariants.** In this section, we apply a variation formula for η-invariants on manifolds with boundary by Dai and Freed [20, Theorem 1.9]. Similar formulas in the case where the boundary operator is invertible have been established by Cheeger [11, Section 8] and Bismut and Cheeger [4, Theorem 6.36]. Dai and Freed actually interpret the reduced η-invariant in $\mathbb{R}/\mathbb{Z}$ with respect to a certain class of possible boundary conditions as a section of the dual of the determinant line bundle of the fibrewise boundary operators, equipped with the Quillen metric and the Bismut-Freed connection. Because we have fixed the boundary condition in Section 2.2, we recover the reduced η-invariant as an $\mathbb{R}/\mathbb{Z}$-valued function.

We now consider the family $\mathcal{M}_\pm = M_\pm \times (0,\infty) \to (0,\infty)$ with fibre $M_{\pm,a}$ over $a \in (0,\infty)$. Recall that the $M_{\pm,a}$ implicitly depends on a choice of parameter ℓ; this parameter is fixed throughout this subsection. We choose the trivial connection $T^H\mathcal{M}_\pm \subset T\mathcal{M}_\pm$, and the fibrewise metric is induced from the metric $g_\ell^{V_\pm} \oplus a^2\zeta_\pm^2 g^{S^1}$ on $\widetilde{M}_\pm$. Using these data, Bismut and Freed [6, (1.7)] construct a connection $\tilde{\nabla}^u$ on the infinite-dimensional vector bundle $\Omega^\bullet(\mathcal{M}_\pm/(0,\infty)) \to (0,\infty)$ of fibrewise exterior differential forms that is unitary with respect to the fibrewise L^2-metrics.

We similarly obtain a unitary connection on $\Omega^\bullet(\partial\mathcal{M}_\pm/(0,\infty)) \to (0,\infty)$, where $\partial\mathcal{M}_\pm$ denotes the collection of fibrewise boundaries. In our situation, it is not hard to see that the subbundle of fibrewise harmonic forms and its subbundle representing the subspaces $L_{B_\pm}$ of (2.6) in each fibre are parallel with respect to $\tilde{\nabla}^u$. Dai and Freed regard $L_{B_\pm}$ as graphs of isometries $H^+(X) \to H^-(X)$ whose determinants define a section of unit length of the determinant line bundle $\det H^\bullet(X) = \Lambda^{\max} H^+(X)^* \otimes \Lambda^{\max} H^-(X)$. This section is again parallel with respect to the connection induced by $\tilde{\nabla}^u$ on $\det H^\bullet(X)$.

The variational formula is typically phrased in terms of the Quillen metric and the Bismut-Freed connection on the determinant line bundle over $(0,\infty)$ (which preserves the Quillen metric). However, if the kernels of the boundary operators form a bundle over the base, as in the case at hand, it is easier to work with the L^2-metric above. A simple fibrewise rescaling of the determinant line bundle transforms one metric into the other, as in [20, Prop. 2.15]. It is shown in [20, (3.8)] that the Bismut-Freed connection becomes a unitary connection with respect to the L^2-metric. It is given by

$$\tilde{\nabla}^u - 2\pi i\,\tilde{\eta}(\mathbb{B})\,, \qquad \text{where} \qquad \tilde{\eta}(\mathbb{B}) = -\frac{1}{4\pi i}\int_0^\infty \operatorname{str}\Big(A_X\,\big[\tilde{\nabla}^u, A_X\big]\,e^{-tA_X^2}\Big)\,dt \tag{2.19}$$

is the η-form of the family of boundary operators A_X, and $\mathbb{B}_t = \sqrt{t}\,A_X + \tilde{\nabla}^u$ is the corresponding Bismut superconnection. Note that we do not need to specify the degree 1 component of $\tilde{\eta}(\mathbb{B})$ explicitly because our base space is one-dimensional here and the degree 0 component vanishes.

The situation for the spin Dirac operators is completely analogous. However, we need to check that there is no spectral flow for $a \in (0,\kappa_\pm)$ under our boundary conditions.

Proposition 2.8. *If ℓ is sufficiently large and $a > 0$, then the operator $D_{M_{\pm,a}}$ has trivial kernel under APS-boundary conditions modified by the Lagrangian subspace $L_{D_\pm} \subset \ker A_{\pm,a}$.*

Proof. Let s denote the $D_{M_{\pm,a}}$-harmonic spinor constructed in 2.1.2. Recall that $\ker(A_{\pm,a})$ is spanned by the restrictions of s, $c_{v_\pm}s$, $c_t s$, and $c_t c_{v_\pm} s$. If $u \in \ker D_{M_{\pm,a}}$, the divergence theorem implies that

$$\langle u, c_t s\rangle_{L^2(\partial M_{\pm,a})} = \langle u, D'_{M_{\pm,a}} s\rangle_{L^2(M_{\pm,a})} - \langle D'_{M_{\pm,a}} u, s\rangle_{L^2(M_{\pm,a})}\,,$$

where $D'_{M_{\pm,a}}$ denotes the geometric spin Dirac operator on $M_{\pm,a}$. Because $D'_{M_{\pm,a}} - D_{M_{\pm,a}}$ is self-adjoint by (2.4), the right hand side vanishes, and the restriction of u to the boundary is L^2-perpendicular to $c_t s$. For the same reason, using 2.1.3 (iii), it is also L^2-perpendicular to $c_t c_{v_\pm} s$.

If u moreover satisfies the modified APS boundary of Section 2.2, then u is also L^2-perpendicular to s and $c_{v_\pm} s$ at the boundary, and hence to the whole $\ker(A_{\pm,a})$. The claim now follows from [14, Proposition 5.4]. □

Hence, fixing APS boundary conditions modified by the Lagrangian of (2.6), (2.7) as before, the variational formulas in the version of [20, Theorem 3.3] in our situation read

$$\begin{aligned} d\eta(B_{M_{\pm,a}}) &= \int_{\mathcal{M}_\pm/(0,\infty)} 2\hat{L}\big(\nabla^{T(\mathcal{M}_\pm/(0,\infty))}\big) - 2\tilde{\eta}(\mathbb{B}) \quad \in \Omega^1((0,\infty))\;, \\ d\eta(D_{M_{\pm,a}}) &= \int_{\mathcal{M}_\pm/(0,\infty)} 2\hat{A}\big(\nabla^{T(\mathcal{M}_\pm/(0,\infty))}\big) - 2\tilde{\eta}(\mathbb{D}) \quad \in \Omega^1((0,\infty))\;, \end{aligned} \tag{2.20}$$

where $\int_{\mathcal{M}_\pm/(0,\infty)}$ denotes integration along the fibres. The first term is the usual local variation formula for η-invariants on closed manifolds. The second term is the boundary contribution. In the second line, $\mathbb{D}$ is the superconnection on the bundle of fibrewise spinors associated to the boundary operators $C_{X_{\pm,a}}$ corresponding to $D_{M_{\pm,a}}$.

Proposition 2.9. *The local variation terms vanish, that is*

$$\int_{\mathcal{M}_\pm/(0,\infty)} \hat{L}\big(\nabla^{T(\mathcal{M}_\pm/(0,\infty))}\big) = \int_{\mathcal{M}_\pm/(0,\infty)} \hat{A}\big(\nabla^{T(\mathcal{M}_\pm/(0,\infty))}\big) = 0\;.$$

Proof. We split the vertical tangent bundle

$$T(\mathcal{M}_\pm/(0,\infty)) \cong p_{V_\pm}^* TV_\pm \oplus \underline{\mathbb{R}}\;,$$

where $p_{V_\pm} \colon \mathcal{M}_\pm \to V_\pm$ denotes obvious projection. This splitting is parallel with respect to the Bismut connection on the vertical tangent bundle. Because the metric on $V_\pm$ is unchanged, the Bismut connection on $p_{V_\pm}^* TV_\pm$ is pulled back from $V_\pm$. The connection on $\underline{\mathbb{R}}$ is Euclidean and therefore flat. We conclude that

$$\hat{A}\big(\nabla^{T\mathcal{M}_\pm/(0,\infty)}\big) = \hat{A}\big(\nabla^{p_{V_\pm}^* TV_\pm}\big) \cdot \hat{A}\big(\nabla^{\underline{\mathbb{R}}}\big) = p_{V_\pm}^* \hat{A}\big(\nabla^{TV_\pm}\big)\;.$$

Because this expression is of horizontal degree 0 and the fibres are odd-dimensional, the integral in the proposition vanishes. The same holds for the $\hat{L}$-form integral above. □

We now consider the η-forms $\eta(\mathbb{B})$ and $\eta(\mathbb{D})$ in (2.20). We write the family $\partial\mathcal{M}_\pm$ as a product $\Sigma_\pm \times E_\pm$, where $E_\pm \to (0,\infty)$ denotes the family of tori $E_{\pm,a} = (S^1_{\zeta_\pm} \times S^1_{a\zeta_\pm})/\Gamma_\pm$ for $a \in (0,\infty)$. Let $\mathbb{A}$ denote the superconnection associated to the fibrewise spin Dirac operator for this family, equipped with the trivial spin structure.

Proposition 2.10. *The variation of $\bar{\nu}(M_{\pm,a})$ is given by*

$$d\bar{\nu}(\mathcal{M}_\pm/(0,\infty)) = 288\tilde{\eta}(\mathbb{A})\;. \tag{2.21}$$

Proof. By the definition of $\bar{\nu}(M_{\pm,a})$ in Theorem 2.2, equation (2.20) and Proposition 2.9, we are left with

$$d\bar{\nu}(\mathcal{M}_\pm/(0,\infty)) = 48\tilde{\eta}(\mathbb{D}) - 6\tilde{\eta}(\mathbb{B})\;.$$

Let B_Σ and D_Σ denote the signature operator and the untwisted Dirac operator on the K3 surface Σ. Then $\operatorname{ind}(B_\Sigma) = -16$ and $\operatorname{ind}(D_\Sigma) = 2$. The proposition follows because

$$\tilde\eta(\mathbb{B}) = 2\operatorname{ind}(B_\Sigma)\,\tilde\eta(\mathbb{A}) \;, \tag{2.22a}$$

$$\tilde\eta(\mathbb{D}) = \operatorname{ind}(D_\Sigma)\,\tilde\eta(\mathbb{A}) \;. \tag{2.22b}$$

These equations will be proved in Section 6.2. □

2.5. **A direct computation of the η-form integral.** We rewrite the η-form integral directly in terms of the eigenvalues of the Dirac operator on the family of flat tori over $\mathcal{H}$. Bismut and Cheeger did similar computations in [5]. In Section 7, we exhibit another way to compute the contribution from the variational formula to the ν-invariant in terms of logarithms of Dedekind η-functions.

Because η-forms are invariant under rescaling, we may assume that $\zeta = 1$ and consider a family of tori $(S^1 \times S^1_a)/\Gamma$ for $a \in (0,\infty)$ as in Proposition 2.10. With ε relatively prime to k as in Section 1.3, we have $(S^1 \times S^1_a)/\Gamma \cong \mathbb{R}^2/\Lambda_a$, where

$$\Lambda_a = \begin{pmatrix}1\\0\end{pmatrix}\mathbb{Z} \oplus \begin{pmatrix}\varepsilon\\a\end{pmatrix}\frac{1}{k}\mathbb{Z} \quad\subset\quad \mathbb{R}^2 \,.$$

Let us denote the total space of this family by E and the fibres by Z. Consider a flat connection on $\underline{\mathbb{R}^2} \to (0,\infty)$ given by

$$\nabla = d - \begin{pmatrix}0&0\\0&1\end{pmatrix}\frac{da}{a} \;,$$

then Λ is parallel with respect to ∇. This connection induces a splitting $TE = TZ \oplus T^HE$. A horizontal lift of $V = \frac{\partial}{\partial a}$ at a point $(x,y,a) \in \mathbb{R}^2 \times (0,\infty)$ is given as

$$\bar V_{(x,y,a)} = \frac{y}{a}\frac{\partial}{\partial y} + \frac{\partial}{\partial a} \;.$$

We equip $\underline{\mathbb{R}^2} \to (0,\infty)$ and E with the fibrewise metric g^{TZ} induced from the standard metric on $\mathbb{R}^2$. The Levi-Civita connection on E induces a Euclidean connection ∇^{TZ} on $TZ \cong \pi^*\underline{\mathbb{R}^2} \to E$ that coincides with the pullback of the trivial connection d. The mean curvature of the fibres is given as

$$h = -\frac{1}{2}\operatorname{tr}\bigl((g^{TZ})^{-1}\,\mathcal{L}_{\bar V} g^{TZ}\bigr)\,da = -\frac{da}{a} \;.$$

We consider the fibrewise product spin structure, so $S^+ \cong S^- \cong \underline{\mathbb{C}} \to E$. Let c_1, c_2 denote Clifford multiplication with the standard orthonormal basis vectors e_1, e_2 on $S^+ \oplus S^-$, then the complex Clifford volume element ic_1c_2 acts by ± 1 on $S^\pm$. The Levi-Civita connection induces the trivial connection on $S^\pm$. For the Bismut superconnection, we have to consider a connection on $\pi_*S^\pm$ of the form

$$\tilde\nabla^u_{\frac{\partial}{\partial a}} f = \Bigl(\nabla^{S^\pm} - \frac{h}{2}\Bigr)_{\bar V} f = \Bigl(\frac{y}{a}\frac{\partial}{\partial y} + \frac{\partial}{\partial a} + \frac{1}{2a}\Bigr) f$$

under the natural identification $\Gamma(\pi_*S^\pm) \cong \Gamma(S^\pm)$. Then the Bismut superconnection is given by

$$\mathbb{A}_t = \sqrt{t}\,A_{T^2} + \tilde\nabla^u = \sqrt{t}\,\Bigl(c_1\frac{\partial}{\partial x} + c_2\frac{\partial}{\partial y}\Bigr) + da\,\Bigl(\frac{y}{a}\frac{\partial}{\partial y} + \frac{\partial}{\partial a} + \frac{1}{2a}\Bigr) \,. \tag{2.23}$$

Starting from (2.19), we compute

$$\tilde\eta(\mathbb{A}) = -\frac{1}{4\pi}\,\frac{da}{a}\int_0^\infty \operatorname{tr}_{\pi_*S}\Bigl(\frac{\partial^2}{\partial x\,\partial y}\,e^{t\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)}\Bigr)\,dt \,. \tag{2.24}$$

We have used the definition $\operatorname{str}(\,\cdot\,) = \operatorname{tr}(ic_1c_2\,\cdot\,)$ of the supertrace. Also $\operatorname{str}(1) = \operatorname{tr}(ic_1c_2) = 0$.

With respect to the standard Euclidean metric, the lattice dual to Λ_a is given by

$$\Lambda_a^* = \left\{\, \mu \in \mathbb{C} \;\middle|\; \langle \lambda, \mu \rangle \in \mathbb{Z} \text{ for all } \lambda \in \Lambda \,\right\} = \left\{ \begin{pmatrix} n \\ m/a \end{pmatrix} \;\middle|\; \varepsilon n + m \equiv 0 \mod k \right\}.$$

For m, n as above, we consider sections

$$\varphi^{\pm}_{m,n}(x,y,a) = \frac{1}{\sqrt{a}}\, e^{2\pi i (nx + my/a)} \in \Gamma(S^{\pm}) \cong C^\infty(E;\mathbb{C})$$

of L^2-norm 1. They are parallel under $\tilde{\nabla}^u$, and they are eigensections of the fibrewise Laplacian for the eigenvalue $4\pi^2\left(n^2 + \frac{m^2}{a^2}\right)$. Note that each admissible pair (m,n) appears twice (once for S^+ and once for S^-), hence (2.24) becomes

$$\begin{aligned} \tilde{\eta}(\mathbb{A}) &= -\frac{1}{2\pi}\,\frac{da}{a} \int_0^\infty \sum_{m+\varepsilon n \equiv 0 \mod k} \left(-4\pi^2\,\frac{mn}{a}\right) e^{-4\pi^2 t\left(n^2+\frac{m^2}{a^2}\right)}\, dt \\ &= \frac{da}{2\pi} \int_0^\infty \sum_{m+\varepsilon n \equiv 0 \mod k} mn\, e^{-t\,(m^2+a^2n^2)}\, dt\,. \end{aligned}$$

In the definition below, we substitute $-m$ for m.

Definition 2.11. For each ε relatively prime to k, we define a function $F_{k,\varepsilon}\colon (0,\infty) \to \mathbb{R}$ by

$$F_{k,\varepsilon}(s) = \int_0^\infty \int_0^s \sum_{m \equiv \varepsilon n \mod k} mn\, e^{-t(m^2+n^2a^2)}\, da\, dt\,. \tag{2.25}$$

Proposition 2.12. *Consider the family $E \to (0,\infty)$ above. Then*

$$\int_{[0,s]} \tilde{\eta}(\mathbb{A}) = -\frac{1}{2\pi}\, F_{k,\varepsilon}(s)\,. \quad \square$$

Theorem 2.13. *The variation of $\bar{\nu}(M_{\pm,a})$ is given by*

$$\bar{\nu}\big(M_{\pm,\kappa_\pm}\big) - \lim_{a\to 0} \bar{\nu}\big(M_{\pm,a}\big) = -\frac{144}{\pi}\, F_{k_\pm,\varepsilon_\pm}(\kappa_\pm)\,.$$

Proof. This follows from Propositions 2.10 and 2.12. □

We can now give a formula for the extended ν-invariant.

Theorem 2.14. *The extended ν-invariant of an extra-twisted connected sum is given as*

$$\begin{aligned} \bar{\nu}(M) = D_{\gamma_+}(V_+) + D_{\gamma_-}(V_-) \\ - \frac{144}{\pi}\left(F_{k_+,\varepsilon_+}(\kappa_+) + F_{k_-,\varepsilon_-}(\kappa_-)\right) - 72\frac{\rho}{\pi} + 3m_\rho(L;N_+,N_-)\,. \end{aligned} \tag{2.26}$$

Proof. This follows from Theorems 2.2, 2.5 and 2.13. □

Proof of Theorem 2. Combine Theorem 2.14 with Proposition 7.1. □

Remark 2.15. Using ideas and results of Atiyah [1], Bismut and Freed [6], and Ray and Singer [32], we can motivate the appearance of the Dedekind η-function. We consider the universal family $p\colon E \to \mathcal{H}$ of flat tori over the upper half plane that we will describe in more detail in Section 4. There exists a Kähler structure on E whose restriction to each fibre $p^{-1}(\tau)$ induces the flat Riemannian metric of volume 1 with the conformal structure induced by $\tau \in \mathcal{H}$. The fibrewise canonical bundle of p is holomorphically trivial, so we may regard the bundle of fibrewise antiholomorphic forms as a model for the fibrewise spinor bundle on E.

Following [6], the η-form $\tilde{\eta}(\mathbb{A})$ describes a natural connection on the determinant line bundle of the fibrewise Dirac operator. Atiyah explains that this connection agrees with the Chern connection on the determinant line bundle with respect to the Quillen metric. Using results of Ray and Singer [32, Theorem 4.1], he shows that the determinant line bundle admits a holomorphic section whose norm can be written in terms of the Dedekind η-function, see the discussion before [1, (5.19)]. This implies that the η-form itself can be described by the logarithmic derivative of the Dedekind η-function.

Example 2.16. We consider the gluing data from Examples 1.10, then $\kappa_- = \sqrt{2}$, $\kappa_+ = 5\sqrt{2}$ and $\vartheta = \arccos \frac{1}{\sqrt{3}}$. From Theorem 2 and Examples 2.3, 2.7, we get

$$\begin{aligned}\bar{\nu}(M) = -\frac{24}{5} + \frac{144}{\pi}\left(\arccos\frac{1}{\sqrt{3}} - \frac{1}{2}\right) - 3 \\ - \frac{144}{\pi}\left(2\operatorname{Im}\mathcal{L}\left(\frac{\sqrt{2}\,i - 10}{30}\right) + \frac{\pi}{18} + 2\operatorname{Im}\mathcal{L}\left(\frac{\sqrt{2}\,i + 2}{10}\right) - \frac{\pi}{30}\right) .\end{aligned}$$

The functional equation (7.1) for $\mathcal{L}$ allows us to conclude that

$$\begin{aligned}2\operatorname{Im}\mathcal{L}\left(\frac{\sqrt{2}\,i - 10}{30}\right) + \frac{\pi}{18} + 2\operatorname{Im}\mathcal{L}\left(\frac{\sqrt{2}\,i + 2}{10}\right) - \frac{\pi}{30} + \frac{1}{2} - \arccos\frac{1}{\sqrt{3}} \\ = \frac{\pi}{6}\left(\frac{1}{30} + \frac{1}{10} - 12\,S(3,10)\right) ,\end{aligned}$$

see Proposition 7.3. Because $3^2 \equiv -1 \mod 10$, we have $S(3,10) = 0$, and hence we confirm entry 228 of Table 2 by computing

$$\bar{\nu}(M) = -\frac{24}{5} - 3 - 24\left(\frac{1}{30} + \frac{1}{10}\right) = -11 .$$

Theorem 3 would of course give the same result. In Example 4.16, we explain how to regroup the terms in formula (0.4) to obtain a sum of integers.

3. Torus matchings

We now pick up the thread from §1.3 and discuss the combinatorics of gluing matrices and torus isometries more systematically. While this is slightly tangential to the main narrative of the paper, it will help us in Section 5 to give an accurate count of the extra twisted connected sums that we can construct from our supply of building blocks. Moreover, we find an additional divisibility property (3.6) of the gluing data that we use when proving Theorem 3 in Section 4.

3.1. **Combinatorics of torus isometries.** We analyse the conditions necessary to reconstruct a torus matching from given gluing data.

Example 3.1. We start with some examples and non-examples of torus matchings.

(i) Let M be a twisted connected sum as in [13, 27]. Assume that the group $\Gamma \cong \mathbb{Z}/k$ acts by isomorphisms on the two ACyl Calabi-Yau manifolds $V_\pm$ used in the construction of M such that the induced action on the cross-section acts trivially on the K3 factor and freely on the interior circle $S^1_{\zeta_\pm}$. Then $(\mathbb{Z}/k)^2$ acts on M, where each factor $\mathbb{Z}/k$ acts on the ACyl Calabi-Yau manifold $Y_\pm$ on one side and on the exterior circle on the other. For each $\varepsilon_+ \in \mathbb{Z}/k$ with $\gcd(\varepsilon_+, k) = 1$, we obtain a free $\mathbb{Z}/k$-action on M where

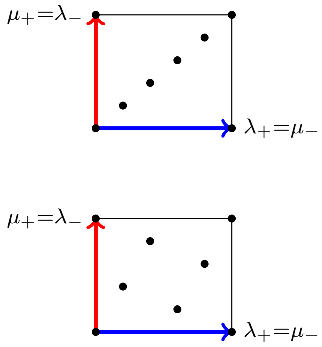


FIGURE 10.

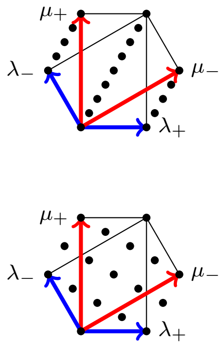


FIGURE 11.

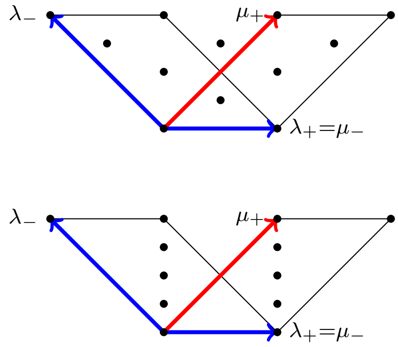


FIGURE 12.

a generator acts as $(1, \varepsilon_+) \in (\mathbb{Z}/k)^2$, see Figure 10 for $k = 5$ and $\varepsilon_+ = 1, 3$. The points of the lattice Λ are indicated by dots.

The corresponding torus matching has $k_+ = k_- = k$, gluing matrix $\left(\begin{smallmatrix} 0 & k \\ k & 0 \end{smallmatrix}\right)$, and $\varepsilon_+\varepsilon_- \equiv 1 \bmod k$. The gluing angle is $\vartheta = \pm\frac{\pi}{2}$, and we have $\xi_+ = \zeta_-$ and $\zeta_+ = \xi_-$, but the ratio $\kappa_+ = \kappa_-^{-1}$ can be chosen arbitrarily.

(ii) There are also examples with gluing angle $\vartheta \notin \frac{\pi}{2}\mathbb{Z}$ where the gluing matrix together with the numbers k_+, k_- does not determine the torus matching completely. As an example, consider $k_+ = k_- = 8$ and the gluing matrix $\left(\begin{smallmatrix} 4 & 4 \\ 12 & -4 \end{smallmatrix}\right)$. This determines $\kappa_+ = \kappa_- = \sqrt{3}$ and $\vartheta = \arctan\sqrt{3}$. We can either pick $\varepsilon_+ = \varepsilon_- = 1$ or $\varepsilon_+ = \varepsilon_- = -3$, see Figure 11.

(iii) If we want to construct a torus matching from a gluing matrix $\left(\begin{smallmatrix} m & p \\ n & q \end{smallmatrix}\right)$, numbers $k_\pm$ and $\varepsilon_\pm$, it is not quite enough to satisfy only the conditions listed in Proposition 1.8 (i). If we set $k_+ = k_- = 4$ and pick the gluing matrix $\left(\begin{smallmatrix} 0 & 4 \\ 4 & -8 \end{smallmatrix}\right)$, we may choose $\varepsilon_+ = -\varepsilon_- = \pm 1$. Then all conclusions in Proposition 1.8 (i) hold, but equation (3.2) is violated, which is part of the conclusions in Proposition 1.8 (ii). Hence we cannot have a matching of quotients of rectangular tori, see Figure 12.

To search systematically for torus matchings with fixed k_+ and k_-, it is helpful to first note the following formal consequences of (1.8) and Proposition 1.8 (ii).

$$\gcd(m,n) = \gcd(m,k_+) = \gcd(n,k_+) \text{ and } \gcd(p,q) = \gcd(p,k_+) = \gcd(q,k_+) \tag{3.1a}$$

$$\gcd(m,p) = \gcd(m,k_-) = \gcd(p,k_-) \text{ and } \gcd(n,q) = \gcd(n,k_-) = \gcd(q,k_-) \tag{3.1b}$$

$$np \cdot mq \le 0\,, \qquad \text{and if } np \cdot mq = 0, \text{ then either } n = p = 0 \text{ or } m = q = 0\,. \tag{3.2}$$

Proposition 3.2. *Let $k_+ > 0$, $\varepsilon_+ \in (\mathbb{Z}/k_+)^*$, and let $\left(\begin{smallmatrix} m & p \\ n & q \end{smallmatrix}\right)$ be a gluing matrix with $\det\left(\begin{smallmatrix} m & p \\ n & q \end{smallmatrix}\right)$ negative and divisible by k_+. Suppose that (1.8a), (1.9a) and (3.2) are satisfied. Then there exists a unique torus matching with these data. If one chooses a, $b \in \mathbb{Z}$ such that*

$$1 = bp - a\frac{q - \varepsilon_+ p}{k_+}\,, \tag{3.3a}$$

$$\textit{then} \qquad \varepsilon_- \equiv a\frac{n - \varepsilon_+ m}{k_+} - bm \mod k_-\,. \tag{3.3b}$$

Proof. Let k_+, $\left(\begin{smallmatrix} m & p \\ n & q \end{smallmatrix}\right)$ and ε_+ be given as above. Let $\tilde\Lambda_+ \subset \mathbb{C}$ be a lattice with basis $(\mu_+, \lambda_+) = (i\xi_+, \zeta_+)$ as in Section 1.3, then we can construct a sublattice $\Lambda \subset \frac{1}{k_+}\tilde\Lambda_+$ of index k_+ with basis (ν_+, λ_+) given by (1.5), and $\tilde\Lambda_+ \subset \Lambda$ is also a sublattice of index k_+. By assumption (1.8a), the gluing matrix then determines a sublattice $\tilde{\Lambda}_- \subset \Lambda$ of index $k_- = -\det\left(\begin{smallmatrix} m & p \\ n & q \end{smallmatrix}\right)/k_+$ with basis (μ_-, λ_-) determined by (1.10).

We conclude that $\Lambda \subset \frac{1}{k_-}\tilde{\Lambda}_-$ is again a sublattice of index k_-. For $c, d \in \{0, \dots, k_- - 1\}$, let $\lambda(c,d) = \frac{c}{k_-}\mu_- + \frac{d}{k_-}\lambda_- \in \frac{1}{k_-}\tilde{\Lambda}_-$. Suppose that $\lambda(c,d) \in \Lambda$. The vectors λ_- and μ_- are primitive in Λ by (1.9a) and (1.10), so $c = 0$ if and only if $d = 0$. Similarly, for each c there can at most be one d in the given range such that $\lambda(c,d) \in \Lambda$ and vice versa. Because there are exactly k_- elements of Λ with coordinates c, d in the given range, for each c there is exactly one d such that $\lambda(c,d) \in \Lambda$, and vice versa. Specifying $c = 1$, we hence get a unique $\varepsilon_- = d \in \mathbb{Z}/k_-$. Moreover, $\gcd(\varepsilon_-, k_-) = 1$, and Λ is an extension of $\tilde{\Lambda}_-$ by a cyclic group $\Gamma_- \cong \mathbb{Z}/k_-$. This proves existence and uniqueness of the gluing data.

We set $k_- = -\frac{1}{k_+}\det\left(\begin{smallmatrix} m & p \\ n & q \end{smallmatrix}\right)$. To determine ε_-, we fix the basis $\left(\frac{\lambda_-}{k_-}, \frac{\mu_-}{k_-}\right)$ of $\mathbb{R}^2$. With respect to this basis, we have $\lambda_+ = \left(\begin{smallmatrix} -m \\ p \end{smallmatrix}\right)$ and $\mu_+ = \left(\begin{smallmatrix} n \\ -q \end{smallmatrix}\right)$ by (1.11). The lattice Λ is spanned by the vectors λ_+ and

$$\nu_+ = \frac{\mu_+ + \varepsilon_+\lambda_+}{k_+} = \frac{1}{k_+}\begin{pmatrix} n - \varepsilon_+ m \\ \varepsilon_+ p - q \end{pmatrix} ,$$

which has integer coordinates by (1.8a). By (1.9a), we can find a and b satisfying (3.3a). Then the vector $\nu_- = a\nu_+ + b\lambda_+$ has second coordinate 1, hence together with λ_-, it also spans Λ. Its first coordinate is given by the right hand side of (3.3b), which therefore agrees with ε_- modulo k_-. $\square$

Remark 3.3. As a consequence of (3.3), we check that

$$n - \varepsilon_+ m + \varepsilon_- q - \varepsilon_+\varepsilon_- p = n - \varepsilon_+ m + \left(a\,\frac{n - \varepsilon_+ m}{k_+} - bm\right)(q - \varepsilon_+ p) = bk_+k_- , \tag{3.4}$$

in particular, the left hand side is always divisible by k_+k_-. If $\varepsilon_\pm^*$ is inverse to $\varepsilon_\pm$ modulo $k_\pm$, we can deduce from the above that similarly

$$k_+k_- \,\big|\, p - \varepsilon_+^* q + \varepsilon_-^* m - \varepsilon_+^*\varepsilon_-^* n . \tag{3.5}$$

We can now prove a claim used in Remark 4.14 below by computing

$$-\frac{m - \varepsilon_+^* n}{k_+} \cdot \frac{q + \varepsilon_-^* n}{k_-} = \frac{np - mq - n\big(p - \varepsilon_+^* q + \varepsilon_-^* m - \varepsilon_+^*\varepsilon_-^* n\big)}{k_+k_-} \equiv 1 \mod n . \tag{3.6}$$

For given positive integers k_- and k_+, there are only finitely many matrices $\left(\begin{smallmatrix} m & p \\ n & q \end{smallmatrix}\right) \in M_2(\mathbb{Z})$ that satisfy conditions (1.7) and (3.2). For

$$np - mq = k_-k_+ > 0$$

and (3.2) imply that

$$np \in \{0, \dots, k_-k_+\} \qquad \text{and} \qquad mq \in \{-k_-k_+, \dots, 0\} . \tag{3.7}$$

By Remark 3.6 below, we may assume in addition $\vartheta \in (0, \pi)$. Then $n > 0$ by Proposition 1.8 (iii), and therefore also $p > 0$. In other words, $\left(\begin{smallmatrix} m & p \\ n & q \end{smallmatrix}\right)$ can be chosen to be either off-diagonal with non-negative entries, or with exactly three positive entries and one negative entry, which can only be m or q.

For small k_+ and k_- it is now easy even by hand to enumerate all $\left(\begin{smallmatrix} m & p \\ n & q \end{smallmatrix}\right)$ that satisfy (3.1) in addition. Most of those have $\gcd\left(\begin{smallmatrix} m & p \\ n & q \end{smallmatrix}\right) = 1$ and thus give rise to a unique torus matching, while for the few remaining ones it is easy to enumerate any ε_+ that satisfy (1.8a) and (1.9a).

Remark 3.4. Assume that we are given gluing data $k_\pm \in \mathbb{Z}$, $\varepsilon_\pm \in \mathbb{Z}/k_\pm$ and $\left(\begin{smallmatrix} m & p \\ n & q \end{smallmatrix}\right) \in M_2(\mathbb{Z})$. Then the lattice Λ is a sublattice of the lattice $\tilde\Lambda_+ + \tilde\Lambda_-$ spanned by $\tilde\Lambda_+$ and $\tilde\Lambda_-$. Because we have assumed the groups $\Gamma_\pm = \mathbb{Z}/k_\pm$ act freely on the interior and on the exterior circles, we can compute the index of Λ in $\tilde\Lambda_+ + \tilde\Lambda_-$ in four different ways:

$$\left[\tilde\Lambda_+ + \tilde\Lambda_- : \Lambda\right] = \gcd(m, p, k_+) = \gcd(n, q, k_+) = \gcd(m, n, k_-) = \gcd(p, q, k_-) \;.$$

For the first equation, we choose the vectors $\frac{\lambda_\pm}{k_+}$ and $\frac{\mu_\pm}{k_+}$ as a basis for $\mathbb{R}^2$; see Figure 2. Then the smallest positive second coordinate of an element of Λ is 1, because $\Gamma_+ \cong \mathbb{Z}/k_+$ acts freely on the exterior circle of $\widetilde{M}_+$. On the other hand, the smallest positive second coordinate of an element of $\tilde\Lambda_+ + \tilde\Lambda_-$ is $\gcd(m, p, k_+)$. The other equations follow similarly.

We note that we have not used the numbers $\varepsilon_\pm$ in the argument above. If $\tilde\Lambda_+ + \tilde\Lambda_- = \Lambda$, then Λ is uniquely determined by $k_\pm$ and the gluing matrix, and so are $\varepsilon_\pm$. This is the case in most examples with $\vartheta \neq \frac{\pi}{2}$. in Table 2. If $\vartheta = \frac{\pi}{2}$ and $k_+ = k_- \geq 3$, then there are at least two possible choices for ε_+. On the other hand, in examples 237, 238, 254 and 255 in Table 2, we have $[\tilde\Lambda_+ + \tilde\Lambda_- : \Lambda] > 1$, but nevertheless, Λ and $\varepsilon_\pm$ are uniquely determined.

3.2. **New extra-twisted connected sums from old.** Having discussed how to enumerate torus matchings, we now move on to discuss relations between them. We will find several ways to describe isometric extra-twisted connected sums, but we also discuss covering spaces and a kind of “t-duality”.

Proposition 3.5. *Let M be an extra-twisted connected sum constructed from asymptotically cylindrical Calabi-Yau manifolds $V_\pm$ with gluing data $k_\pm \in \mathbb{Z}$, $\varepsilon_\pm \in \mathbb{Z}/k_\pm$ and $\left(\begin{smallmatrix} m & p \\ n & q \end{smallmatrix}\right) \in M_2(\mathbb{Z})$, and with gluing angle ϑ. Then the following gluing data describe an isometric extra-twisted connected sum M', possibly with opposite orientation:*

$$\begin{pmatrix} m' & p' \\ n' & q' \end{pmatrix} = \begin{pmatrix} -q & p \\ n & -m \end{pmatrix}, \quad \begin{matrix} k'_+ = k_- \,, \\ k'_- = k_+ \,, \end{matrix} \quad \begin{matrix} \varepsilon'_+ = \varepsilon_- \,, \\ \varepsilon'_- = \varepsilon_+ \,, \end{matrix} \quad \text{and} \quad \vartheta' = \vartheta \;; \tag{3.8}$$

$$\begin{pmatrix} m' & p' \\ n' & q' \end{pmatrix} = \begin{pmatrix} m & -p \\ -n & q \end{pmatrix}, \quad \begin{matrix} k'_+ = k_+ \,, \\ k'_- = k_- \,, \end{matrix} \quad \begin{matrix} \varepsilon'_+ = -\varepsilon_- \,, \\ \varepsilon'_- = -\varepsilon_+ \,, \end{matrix} \quad \text{and} \quad \vartheta' = -\vartheta \;; \tag{3.9}$$

$$\begin{pmatrix} m' & p' \\ n' & q' \end{pmatrix} = \begin{pmatrix} -m & -p \\ -n & -q \end{pmatrix}, \quad \begin{matrix} k'_+ = k_+ \,, \\ k'_- = k_- \,, \end{matrix} \quad \begin{matrix} \varepsilon'_+ = \varepsilon_- \,, \\ \varepsilon'_- = \varepsilon_+ \,, \end{matrix} \quad \text{and} \quad \vartheta' = \vartheta \pm \pi \,. \tag{3.10}$$

In (3.8)*, we have to swap the roles of V_+ and V_-. In* (3.9)*, we change the orientation of M. In* (3.10)*, we pass to the opposite Calabi-Yau structure on one side.*

The three elements above generate a group $H \cong (\mathbb{Z}/2)^3$ that acts on the set of gluing data describing a given deformation family up to isomorphism.

Proof. We obtain (3.8) by exchanging the roles of V_+ and V_-, see equation (1.11). Because the definition of the gluing angle is symmetric, it does not change.

For (3.9), we change the orientation of M by swapping the orientations of the two exterior circles. This changes the sign of the gluing angle and of $\varepsilon_\pm$. The new gluing matrix arises by conjugating with $\left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$.

In (3.10), we rotate one of the two sides, say M_+, by π, which leads to the new gluing angle. This has the effect of changing the orientations of both V_+ and the exterior circle, and hence the gluing matrix is multiplied by $-\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. If ω_+ and Ω_+ describe the old Calabi-Yau structure on V_+, the new one carries the opposite complex structure and is given by $-\omega_+$ and $-\bar\Omega_+$. ☐

Remark 3.6. The subgroup spanned by (3.9) and (3.10) is rich enough to make sure that we can always assume m, n, $p \geq 0$ and $q \leq 0$. Moreover, if we use the same building block and the same finite group $\Gamma \cong \mathbb{Z}/k$ for M_+ and M_-, we may apply (3.8) to get $m + q \leq 0$. In Table 2, we only list gluing data satisfying these conventions.

Recall that $\Sigma \times T \cong \Sigma_\pm \times T_\pm$ denote the isometric cross-sections at infinity of the asymptotically cylindrical G_2-manifolds $M_\pm$ used in Theorem 1.3 to construct M. If Λ^* denotes the dual of the lattice $\Lambda \in \mathbb{C} \cong \mathbb{R}^2$ with respect to the standard metric, we write $T^* = \mathbb{R}^2/\Lambda^*$ for the dual of the torus $T = \mathbb{R}^2/\Lambda$.

Proposition 3.7. *Let M be an extra-twisted connected sum glued along $\Sigma \times T$ with gluing data $k_\pm \in \mathbb{Z}$, $\varepsilon_\pm \in \mathbb{Z}/k_\pm$ and $\left(\begin{smallmatrix} m & p \\ n & q \end{smallmatrix}\right) \in M_2(\mathbb{Z})$ and with gluing angle ϑ. Then there exists a deformation family of extra-twisted connected sums M' with isomorphic asymptotically flat Calabi-Yau manifolds $V_\pm$, glued along $\Sigma \times T^*$ with gluing data*

$$\begin{pmatrix} m' & p' \\ n' & q' \end{pmatrix} = \begin{pmatrix} -q & n \\ p & -m \end{pmatrix} , \qquad \begin{matrix} k'_+ = k_+ \ , & \varepsilon'_+ = -\varepsilon^*_+ \ , \\ k'_- = k_- \ , & \varepsilon'_- = -\varepsilon^*_- \ , \end{matrix} \qquad \text{and} \qquad \vartheta' = \vartheta \ . \tag{3.11}$$

Together with the group H in Proposition 3.5, this transformation generates a group isomorphic to $(\mathbb{Z}/2)^4$ that acts on the possible gluing data compatible with a given K3 matching.

Thus, the new extra-twisted connected sum is in a certain sense "t-dual" to the original one, but can in general not be deformed into the original one. We remark that any torsion free G_2-structure close to an extra twisted connected sum is again an extra twisted connected sum. Hence the duality described here can be used to identify small open subsets of the moduli space of G_2-metrics.

Proof. We recall the generators for the involved lattices in $\mathbb{C}$ from Section 1.3:

$$\begin{aligned} \tilde{\Lambda}_+ &= \langle \zeta_+, i\xi_+ \rangle \ , \\ \Lambda &= \left\langle \zeta_+, \frac{\varepsilon_+\zeta_+ + i\xi_+}{k_+} \right\rangle \ , \\ \text{and} \qquad \tilde{\Lambda}_- &= \left\langle \frac{q\zeta_+ + ip\xi_+}{k_+}, \frac{n\zeta_+ + im\xi_+}{k_+} \right\rangle \ . \end{aligned}$$

Because T is a $k_\pm$-fold quotient of $T_\pm$, the dual torus T^* is a $k_\pm$-fold covering of $\widetilde{T}^*_\pm$, or equivalently, a $k_\pm$-fold quotient of $k_\pm\widetilde{T}^*_\pm$. With respect to the standard Euclidean metric on $\mathbb{C} \cong \mathbb{R}^2$, we have generators for the dual lattices, where we take ε^*_+ in $\mathbb{Z}/k_+$ as above:

$$\begin{aligned} k_+\tilde{\Lambda}^*_+ &= \left\langle \frac{k_+}{\zeta_+}, \frac{ik_+}{\xi_+} \right\rangle \ , \\ \Lambda^* &= \left\langle \frac{k_+}{\zeta_+}, -\frac{\varepsilon^*_+}{\zeta_+} + \frac{i}{\xi_+} \right\rangle \ , \\ \text{and} \qquad k_-\tilde{\Lambda}^*_- &= \left\langle -\frac{m}{\zeta_+} + \frac{in}{\xi_+}, \frac{p}{\zeta_+} - \frac{iq}{\xi_+} \right\rangle \ . \end{aligned}$$

The gluing data (3.11) can be read off from this description. And because the gluing angle does not change, the K3 matching used to construct M also works for M'. □

Alternatively, one can rotate both tori $\widetilde{T}_\pm$ by a right angle, thus swapping the role of exterior and interior circles. This leads to the same gluing data as above.

Remark 3.8. For the sake of completeness, let us add the following observations.

(i) Given gluing data $k_\pm$, $\varepsilon_\pm$ and $\left(\begin{smallmatrix} m & p \\ n & q \end{smallmatrix}\right)$ as above with gluing angle $\vartheta \notin \frac{\pi}{2} + \pi\mathbb{Z}$. Then we can write valid gluing data of the form

$$\begin{pmatrix} m' & p' \\ n' & q' \end{pmatrix} = \begin{pmatrix} p & m \\ -q & -n \end{pmatrix}, \qquad \begin{matrix} k'_+ = k_+ , & \varepsilon'_+ = -\varepsilon_+ , \\ k'_- = k_- , & \varepsilon'_- = \ \ \varepsilon^*_- , \end{matrix} \quad \text{and} \quad \vartheta' = \pm\frac{\pi}{2} - \vartheta . \tag{3.12}$$

One can check that this transformation together with those of Propositions 3.5 and 3.7 generates a group isomorphic to $D_4 \ltimes (\mathbb{Z}/2)^2$ that acts on the set of valid gluing data.

However, to construct an extra-twisted connected sum, we need a K3 matching that is compatible with the gluing angle. If both blocks have rank 1, the compatibility condition is given by (1.14). The new gluing data above will in general not be compatible with the K3 matching used for the original extra-twisted connected sum M. And it is not hard to find examples in Table 2 where the new gluing angle is not compatible with any possible K3 matching of rank 1 blocks.

(ii) Let us now consider matchings with gluing angle $\vartheta = \pm\frac{\pi}{2}$ as in Example 3.1 (i). We recall that $k_+ = k_- = k$, $\varepsilon_- = \varepsilon^*_+ \in \mathbb{Z}/k$, and the gluing matrix takes the form $\left(\begin{smallmatrix} 0 & k \\ k & 0 \end{smallmatrix}\right)$ by Example 3.1 (i). This implies that the transformation (3.11) of Proposition 3.7 acts exactly as the composition of the three elements (3.8)–(3.10) of H. Hence, we are reduced to an action of the group H in this special case. This is not surprising because (3.11) mainly affects the ratio of circle lengths, which is not specified by the gluing data if $\vartheta = \frac{\pi}{2}$, see the discussion before Proposition 1.8. Exploiting the action of H, we may assume in Table 2 that $p = n > 0$ and that $\varepsilon_+ > 0$.

One may note that in all our examples of this type in Table 2 we have $\varepsilon^*_+ = \pm\varepsilon_+$, so in none of these examples the group H acts effectively.

3.3. **Covering spaces.** By Proposition 1.11, an extra-twisted connected sum with gluing matrix $\left(\begin{smallmatrix} m & p \\ n & q \end{smallmatrix}\right)$ is simply-connected if and only if $p = 1$. Let us enumerate its connected covering spaces if $p > 1$.

Proposition 3.9. *Assume that M is an extra-twisted connected sum with gluing data given by $k_\pm$, $\varepsilon_\pm \in \mathbb{Z}/k_\pm$ and gluing matrix $\left(\begin{smallmatrix} m & p \\ n & q \end{smallmatrix}\right)$. Assume that $p > 1$ and that $\ell \mid p$. Then there exists a unique connected ℓ-fold covering space $\widetilde{M}$. It is an extra-twisted connected sum constructed from the same building blocks as M with the same gluing angle ϑ, and with gluing data*

$$\tilde{k}_\pm = \frac{k_\pm}{\gcd(\ell, k_\pm)} , \qquad \tilde{\varepsilon}_\pm = \frac{\ell}{\gcd(\ell, k_\pm)} \varepsilon_\pm , \tag{3.13a}$$

$$\begin{pmatrix} \tilde{m} & \tilde{p} \\ \tilde{n} & \tilde{q} \end{pmatrix} = \begin{pmatrix} \frac{m}{\gcd(\ell, k_-)} & \frac{p}{\ell} \\ \frac{n\ell}{\gcd(\ell, k_+)\gcd(\ell, k_-)} & \frac{q}{\gcd(\ell, k_+)} \end{pmatrix} \in M_2(\mathbb{Z}) . \tag{3.13b}$$

If $\tilde{k}_\pm = k_\pm$, then the covering is constructed using the same finite groups $\Gamma_\pm$. Otherwise, one has to pass to a proper subgroup of one or both of these groups.

Proof. Because $\pi_1(M)$ is cyclic by Proposition 1.11, there is a unique connected ℓ-fold covering $\pi\colon \widetilde{M} \to M$. Let $\widetilde{M}_\pm \to M_\pm$ denote its restriction to the two halves of M. Because $\pi_*\colon \pi_1(M_\pm) \to \pi_1(M)$ is surjective, we see that $\widetilde{M}_\pm \to M_\pm$ are also connected ℓ-fold coverings, which are uniquely determined by ℓ up to isomorphism since $\pi_1(M_\pm) \cong \mathbb{Z}$. It follows that $\widetilde{M}$ is an extra-twisted sum, glued from $\widetilde{M}_+$ and $\widetilde{M}_-$.

Let $\tilde{X} \to X$ denote the restriction of π to $X = M_+ \cap M_-$. The corresponding sublattice $\tilde{\Lambda} \subset \Lambda = \pi_1(X)$ is spanned by the vectors $\lambda_\pm$ corresponding to the interior circles, and by

$$\tilde{\nu}_\pm = \ell\nu_\pm = \frac{\ell}{k_\pm}(\mu_\pm + \varepsilon_\pm\lambda_\pm) \,.$$

The smallest multiples of $\mu_\pm$ inside $\tilde{\Lambda}$ are

$$\tilde{\mu}_\pm = \frac{k_+}{\gcd(\ell, k_\pm)} \cdot \tilde{\nu}_\pm - \frac{\ell\varepsilon_\pm}{\gcd(\ell, k_\pm)}\,\lambda_\pm = \frac{\ell}{\gcd(\ell, k_\pm)}\mu_\pm \,,$$

and $\tilde{\Lambda}$ is an extension of the lattice spanned by $\lambda_\pm$ and $\tilde{\mu}_\pm$ by a finite cyclic group $\tilde{\Gamma}_\pm \cong \mathbb{Z}/\tilde{k}_\pm$. With respect to the new bases $(\tilde{\mu}_\pm, \lambda_\pm)$, $\tilde{k}_\pm$ and $\tilde{\varepsilon}_\pm$ are given by (3.13a), and the new gluing matrix takes the form of (3.13b), which has determinant $-\tilde{k}_+\tilde{k}_-$. ☐

Let us reverse the construction above. In fact, the next result is dual to the one above in the sense of Proposition 3.7.

Proposition 3.10. *Assume that M is an extra-twisted connected sum with gluing data given by $k_\pm$, $\varepsilon_\pm \in \mathbb{Z}/k_\pm$ and gluing matrix $\left(\begin{smallmatrix} m & p \\ n & q \end{smallmatrix}\right)$. Consider the lattice $\Lambda' \subset \mathbb{C}$ of elements that act on the internal circle $S^1_{\zeta_+}$ by an element of $\mathbb{Z}/k_+$ and on $S^1_{\zeta_-}$ by an element of $\mathbb{Z}/k_-$ Then the group Λ'/Λ is isomorphic to $\mathbb{Z}/n$ and acts effectively on M. If $\ell \mid n$ then the subgroup of order ℓ acts freely on M if and only if* $\gcd(\ell, k_+) = \gcd(\ell, k_-) = 1$.

Note that there can be more automorphisms of M, for example some that do not project to the identity of the K3 surface in the neck.

Proof. From its description above, it is clear that elements of Λ' act on $V_\pm$ by elements of $\Gamma_\pm$, and on the external circles $S^1_{\xi_\pm}$ by rotation. Elements of Λ act trivially. Hence we get an effective action of Λ'/Λ on M that preserves the closed G_2-structure one obtains by gluing, see Section 1.1. From the proof of [24, Theorem G2], it is clear that it also preserves the resulting torsion free G_2-structure on M.

Now let $\alpha \in \Lambda'$. Because α projects to an element of Γ_- and the projection map $\Lambda \to \Gamma_-$ is surjective, we can add an element of Λ to make sure that α projects to the identity of the internal circle $\zeta_-\mathbb{R} \subset \mathbb{C}$. Hence, α is a multiple of ξ_-. The projection to $\zeta_+\mathbb{R} \subset \mathbb{C}$ will be a multiple of $\frac{1}{k_+}\zeta_+$. From the definition of the gluing matrix in (1.6), we infer that $\alpha = \frac{a}{n}\xi_-$, and $\frac{a}{n}\xi_- \in \Lambda$ if and only if $n \mid a$ because ξ_- is a primitive element of Λ, see also Figure 2. Hence we conclude that Λ'/Λ is a cyclic group of order n.

An element $\alpha \in \Lambda'$ will act with fixpoints on M_- if and only if it projects to the identity of the external circle $S^1_{\xi_-}$, that is, if and only if it fixes V_- as a set. If we represent α by $\frac{a}{n}\xi_-$, we can add an element $\lambda \in \Lambda$ such that $\alpha + \lambda \in \zeta_-\mathbb{R}$ if and only if $\frac{a}{n}$ can be written as $\frac{b}{k_-}$. Such $a \not\equiv 0 \bmod n$ exist if and only if $\gcd(n, k_-) > 1$. If we restrict to a subgroup of order ℓ, this can happen if and only if $\gcd(\ell, k_-) > 1$. Similarly, there will be fixpoints in M_+ if and only if $\gcd(\ell, k_+) > 1$. ☐

Remark 3.11. If the group Λ'/Λ acts freely on M, the quotient will again be an extra twisted connected sum, and we have reversed the covering space construction of Proposition 3.9 in the special case where $\tilde{k}_\pm = k_\pm$. If there are fixpoints, we obtain an orbifold as a quotient. The fixpoint sets will be a union of products of compact subsets of $V_\pm$ and $S^1_{\xi_\pm}$. Suppose that we find a crepant resolution of the corresponding orbifold quotients of $V_\pm$. Then we can form an extra twisted connected sum using the resolved asymptotically cylindrical manifolds $\tilde{V}_\pm$. The gluing data will now be given by

$$\tilde{k}_\pm = \frac{k_\pm}{\gcd(\ell, k_\pm)}\ , \qquad \tilde{\varepsilon}_\pm = \frac{\gcd(\ell, k_\pm)}{\ell}\varepsilon_\pm\ , \tag{3.14a}$$

$$\begin{pmatrix} \tilde{m} & \tilde{p} \\ \tilde{n} & \tilde{q} \end{pmatrix} = \begin{pmatrix} \frac{m}{\gcd(\ell,k_+)} & \frac{p\ell}{\gcd(\ell,k_+)\gcd(\ell,k_-)} \\ \frac{n}{\ell} & \frac{q}{\gcd(\ell,k_-)} \end{pmatrix} \in M_2(\mathbb{Z})\ . \tag{3.14b}$$

This construction is dual to (3.13) in the sense of Proposition 3.7.

To invert the covering space construction of Proposition 3.9, we need to allow multiples $k'_\pm$ of $k_\pm$ such that the $\mathbb{Z}/k_\pm$-actions on $V_\pm$ can be extended to $\mathbb{Z}/k'_\pm$-actions. Then similar constructions as above are possible. We leave the details to the reader.

Table 2 describes 255 deformation families of extra-twisted connected sums, 125 of which are simply connected. Among the remaining examples, there are 64 where taking the universal cover implies passing to subgroups of Γ_+ or Γ_-.

Among the examples in Table 2, the one with largest fundamental group $\pi_1(M) \cong \mathbb{Z}/21$ is entry 250, which has $k_+ = 4$ and $k_- = 6$. The universal cover has $\tilde{k}_+ = 4$ and $\tilde{k}_- = 2$. It can be found as entry 174 with M_+ and M_- swapped. Entries 175 and 248 are the two intermediate covering spaces.

Example 3.12. Consider the Example 1.10 (228), where numbers in parentheses refer to Table 2, possibly up to the isometry (3.8). Applying the duality (3.11), we get the gluing matrix

$$\begin{pmatrix} 5 & 10 \\ 1 & -1 \end{pmatrix} \quad \text{with} \quad \begin{matrix} k_+ = 3\ , \\ k_- = 5\ , \end{matrix} \qquad \begin{matrix} \varepsilon_+ = -1\ , \\ \varepsilon_- = 1\ , \end{matrix} \qquad \bar{\nu} = -43 \qquad (231)\ .$$

By Proposition 1.11, the corresponding extra-twisted connected sum M' is not simply connected. By Proposition 3.9, its universal covering and the intermediate covering spaces have gluing data and $\bar{\nu}$-invariant

$$\begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix} \quad \text{with} \quad \begin{matrix} k_+ = 3\ , \\ k_- = 1\ , \end{matrix} \qquad \varepsilon_- = -1\ , \qquad \bar{\nu} = -19 \qquad (21)\ ,$$

$$\begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix} \quad \text{with} \quad \begin{matrix} k_+ = 3\ , \\ k_- = 1\ , \end{matrix} \qquad \varepsilon_- = 1\ , \qquad \bar{\nu} = -35 \qquad (23)\ ,$$

$$\begin{pmatrix} 5 & 5 \\ 2 & -1 \end{pmatrix} \quad \text{with} \quad \begin{matrix} k_+ = 3\ , \\ k_- = 5\ , \end{matrix} \qquad \begin{matrix} \varepsilon_+ = 1\ , \\ \varepsilon_- = 2\ , \end{matrix} \qquad \bar{\nu} = -23 \qquad (230)\ .$$

Note that the universal covering is different from the original manifold M from Example 1.10 (228). In particular, we have forgotten the $\mathbb{Z}/5$-action on block 12 of Table 1, thus obtaining block 9. The first two lines above are again related by (3.11). Applying (3.11) to the last line gives an extra-twisted connected sum with fundamental group $\mathbb{Z}/2$, whose universal cover is the original manifold M, and which is described by

$$\begin{pmatrix} 1 & 2 \\ 5 & -5 \end{pmatrix} \quad \text{with} \quad \begin{matrix} k_+ = 3\ , \\ k_- = 5\ , \end{matrix} \qquad \begin{matrix} \varepsilon_+ = -1\ , \\ \varepsilon_- = 2\ , \end{matrix} \qquad \bar{\nu} = -7 \qquad (229)\ .$$

4. Hyperbolic geometry and η-forms

In Section 2, we have proved Theorem 2. The special values of the function $\mathcal{L}$ appearing there are hard to describe, see 7.2, however, the linear combinations needed in Theorem 2 have a much easier description by Proposition 7.3, leading to the closed formula for $\bar{\nu}(M)$ in Theorem 3. Here, we will pursue a more geometric approach to derive this formula. We will consider the η-form appearing in Proposition 2.10 as the η-form of the tautological family of flat tori over the upper half plane $\mathcal{H} \subset \mathbb{C}$. It is a primitive of the hyperbolic area form. We will then use elementary hyperbolic geometry and an adiabatic limit formula for η-forms as in [10, 29] to complete the computation.

Throughout this section, we will assume that $m \geq 0$, n, $p > 0$, whereas $q \leq 0$. This is no loss of generality by Remark 3.6. As a consequence, we always have $\vartheta \in (0, \frac{\pi}{2}]$ and $\rho \geq 0$.

4.1. **A universal family of flat tori.** We extend the η-form $\tilde{\eta}(\mathbb{A})$ that we introduced in Section 2.4 to $\mathcal{H} \times (0, \infty)$, which we regard as the moduli space of flat tori. We then apply the so-called transgression formula

$$d\tilde{\eta}(\mathbb{A}) = \int_{E/\mathcal{H}} \hat{A}(T(E/\mathcal{H})) - \operatorname{ch}(\operatorname{ind}(A_{T^2})) , \tag{4.1}$$

see [3, Theorem 10.32], where $\mathbb{A}$ extends the Bismut superconnection (2.23). We will see that the integral of the fibrewise $\hat{A}$-form over the fibres vanishes. The index bundle of the fibrewise operators A_{T^2} consists of fibrewise parallel sections of the spinor bundle. This will allow us to give a simple formula for its Chern character form, and hence for $d\tilde{\eta}(\mathbb{A})$. We follow Bismut and Cheeger [5], but consider tori of varying area.

We consider a universal family $p\colon E \to \mathcal{H} \times (0, \infty)$ of flat tori by setting

$$E_{(\tau,r)} = \mathbb{C} \; / \; \sqrt{r/y}\, \operatorname{span}_{\mathbb{Z}}\{\tau, 1\} \qquad \text{for } \tau = x + iy \in \mathcal{H} \text{ and } r \in (0, \infty) ,$$

such that $E_{(\tau,r)}$ has area r with respect to the standard Euclidean metric on $\mathbb{C} \cong \mathbb{R}^2$. The group $SL(2, \mathbb{Z})$ acts on $\mathcal{H}$ by Möbius transformations, and we lift this action to an action on E by fibrewise isometries for $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2, \mathbb{Z})$ by

$$E_{(\tau,r)} \longrightarrow E_{\left(\frac{a\tau+b}{c\tau+d}, r\right)} \qquad \text{with} \qquad [z] \longmapsto \left[\frac{|c\tau + d|}{c\tau + d} \cdot z \right]$$

We can extend this action to $GL(2, \mathbb{Z})$ by $\tau \mapsto \frac{a\bar{\tau}+b}{c\bar{\tau}+d}$ and

$$E_{(\tau,r)} \longrightarrow E_{\left(\frac{a\bar{\tau}+b}{c\bar{\tau}+d}, r\right)} \qquad \text{with} \qquad [z] \longmapsto \left[\frac{|c\bar{\tau} + d|}{c\bar{\tau} + d} \cdot \bar{z} \right]$$

if $\det \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = -1$. This action corresponds to multiplication with the inverse transpose of $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ on $\mathbb{R}^2/\mathbb{Z}^2$ if we trivialise E by identifying $\mathcal{H} \times (0, \infty) \times \mathbb{R}^2/\mathbb{Z}^2$ with $E_{(\tau,r)}$ via

$$\big(\tau, r; [(u, v)]\big) \longmapsto \big[\sqrt{r/y}(u\tau + v)\big] .$$

Let $W = \mathcal{H} \times (0, \infty) \times \mathbb{R}^2 \to \mathcal{H} \times (0, \infty)$ be the fibrewise universal covering of E, regarded as a trivial oriented Euclidean vector bundle with the standard metric. There is a unique connection ∇^W on W such that the sections $\sqrt{r/y}\, \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$ and $\sqrt{r/y}\, \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)$ are parallel, and ∇^W is clearly $GL(2, \mathbb{Z})$-invariant. Let $T^H W \subset TW$ denote the induced horizontal subbundle of TW, and let $T^H E$ denote the induced horizontal subbundle of TE. It induces a metric connection

$${}^0\nabla = \frac{1}{2} \left(\nabla^W + (\nabla^W)^* \right)$$

on W, where $(\nabla^W)^*$ denotes the adjoint of ∇^W with respect to the standard metric. These connections are given by

$$\nabla^W = d - \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \frac{dx}{y} + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \frac{dy}{2y} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \frac{dr}{2r} , \tag{4.2a}$$

$$(\nabla^W)^* = d + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \frac{dx}{y} - \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \frac{dy}{2y} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \frac{dr}{2r} , \tag{4.2b}$$

$$\text{and} \qquad {}^0\nabla = d + \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \frac{dx}{2y} . \tag{4.2c}$$

As in [5, Prop 2.1], the vertical tangent bundle $T(E/\mathcal{H})$ of $E \to \mathcal{H}$ together with its natural connection is isomorphic to the pullback of $(W, {}^0\nabla)$. In particular, the integral of the form $\hat{A}(T(E/\mathcal{H}))$ over the fibres in (4.1) vanishes.

Because W is trivial, there is an associated spinor bundle $S = S^+ \oplus S^- \to \mathcal{H} \times (0,\infty)$. Let $\widetilde{GL}(2,\mathbb{R})$ denote the double cover of $GL(2,\mathbb{R})$ that is nontrivial over both connected components, and let $\widetilde{GL}(2,\mathbb{Z}) \subset \widetilde{GL}(2,\mathbb{R})$ denote the preimage of $GL(2,\mathbb{Z})$. Then the induced action on E and W lifts to S in a way that is compatible with Clifford multiplication. Let $\widetilde{SL}(2,\mathbb{Z})$ denote the preimage of $SL(2,\mathbb{Z})$ in $\widetilde{GL}(2,\mathbb{Z})$, then elements of $\widetilde{GL}(2,\mathbb{Z}) \setminus \widetilde{SL}(2,\mathbb{Z})$ swap the bundles S^+ and S^-. Because the vertical tangent bundle is isomorphic to p^*W, the bundle p^*S becomes a fibrewise spinor bundle on E. Moreover, the kernel of the fibrewise Dirac operator consists of fibrewise parallel spinors. Therefore, the index bundle $\operatorname{ind}(A_{T^2})$ in (4.1) is isomorphic to S, and the L^2-unitary connection $\tilde{\nabla}^u$ and the connection ${}^0\nabla$ induce the same connection on the index bundle.

From equation (4.2c) for ${}^0\nabla$, we can now compute the curvature ${}^0\nabla^2$ and the Euler class

$$\operatorname{Pf}\left(\frac{{}^0\nabla^2}{2\pi}\right) = \frac{dx\,dy}{4\pi y^2} = \frac{1}{4\pi}\, dA_{\text{hyp}} , \tag{4.3}$$

where A_{hyp} denotes the hyperbolic area form. Here, we have used that the hyperbolic metric on $\mathcal{H}$ is given by $g^{\text{hyp}} = \frac{1}{y^2}\, g^{\text{Eucl}}$.

4.2. **The η-form.** We collect some properties of the η-form $\eta(\mathbb{A}) \in \Omega^\bullet(\mathcal{H} \times (0,\infty))$.

The data considered above suffice to define the Bismut superconnection $\mathbb{A}$ for the spinor bundle S on $E \to \mathcal{H} \times (0,\infty)$, which extends the superconnection $\mathbb{A}$ introduced in Proposition 2.10; see also (2.23). It follows that $\mathbb{A}$ is $\widetilde{GL}(2,\mathbb{Z})$-invariant.

Proposition 4.1. *The η-form $\tilde{\eta}(\mathbb{A}) \in \Omega^\bullet\big(\mathcal{H} \times (0,\infty)\big)$ equals the pullback of its restriction to $\mathcal{H} \times \{1\}$ along the product projection.*

Proof. We identify sections of $p_*S \to \mathcal{H} \times (0,\infty)$ with sections of E. Then the unitary connection $\tilde{\nabla}^u$ on p_*S equals ${}^0\nabla + \frac{dr}{2r}$. Let $D_{(\tau,r)}$ denote the fibrewise Dirac operator acting on sections of $S|_{E_{(\tau,r)}}$. Then the Bismut superconnection $\mathbb{A}$ is given by

$$\mathbb{A}_t = \sqrt{t}\, D_{(\tau,r)} + {}^0\nabla + \frac{dr}{2r} = \sqrt{t/r}\, D_{(\tau,1)} + {}^0\nabla + \frac{dr}{2r} .$$

For $r = 1$, this is explained in [5] after Definition 2.14, in particular, there is no term of horizontal degree 2. For the general definition of $\mathbb{A}_t$, see [3, Proposition 10.15].

Introducing $t \in (0,\infty)$ as an additional parameter, we may define a superconnection $\overline{\mathbb{A}}$ on the pullback of p_*S to $\mathcal{H} \times (0,\infty)^2$ by

$$\overline{\mathbb{A}} = \sqrt{t/r}\, D_{(\tau,1)} + {}^0\overline{\nabla} + \frac{dr}{2r} - \frac{dt}{2t} ,$$

where ${}^0\overline{\nabla} = {}^0\nabla + \frac{\partial}{\partial t}\,dt$. Let $(2\pi i)^{\frac{1-N}{2}}$ denote the operator that multiplies a k-form by $(2\pi i)^{\frac{1-k}{2}}$. Then the η-form on $\mathcal{H} \times (0,\infty)$ can be defined as

$$\begin{aligned}\tilde{\eta}(\mathbb{A}) &= -(2\pi i)^{\frac{1-N}{2}} \int_0^\infty \operatorname{str}\Big(\frac{\partial \mathbb{A}_t}{\partial t}\, e^{-\mathbb{A}_t^2}\Big)\,dt \\ &= -\int_{\mathcal{H}\times(0,\infty)^2/\mathcal{H}\times(0,\infty)} (2\pi i)^{-\frac{N}{2}} \operatorname{str}\Big(e^{-\overline{\mathbb{A}}^2}\Big)\,.\end{aligned} \tag{4.4}$$

The component of degree 1 on $\mathcal{H} \times (0,\infty)$ is described by (2.19). Note that $\frac{dr}{2r} - \frac{dt}{2t}$ squares to 0 and supercommutes with the rest of $\overline{\mathbb{A}}$, and hence contributes neither to $\overline{\mathbb{A}}^2$ nor to $\tilde{\eta}(\mathbb{A})$.

We observe that $\overline{\mathbb{A}}$ can be pulled back from $\mathcal{H} \times (0,\infty)$ by the map $(\tau, r, t) \mapsto (\tau, u)$ with $u = t/r$. It follows from (4.4) that $\tilde{\eta}(\mathbb{A})$ does not involve the exterior variable dr. By substituting u for $\frac{t}{r}$ in the integral, one obtains a formula that no longer depends on r. □

In other words, $\tilde{\eta}(\mathbb{A})$ is independent of $r \in (0,\infty)$ and does not contain the exterior variable dr. From now on, we regard $\tilde{\eta}(\mathbb{A})$ as a form on $\mathcal{H}$. Following Bismut and Cheeger, we get an explicit expression for the right hand side of (4.1) even if the area of the fibres is not constant.

Theorem 4.2 ([5, Theorem 2.22])**.** *The η-form $\tilde{\eta}(\mathbb{A})$ on $\mathcal{H}$ has the exterior derivative*

$$d\tilde{\eta}(\mathbb{A}) = -(-1)^{\frac{\operatorname{rk} W}{2}} \operatorname{Pf}\Big(\frac{{}^0\nabla^2}{2\pi}\Big)\,\hat{A}^{-1}\Big(\frac{{}^0\nabla^2}{2\pi}\Big)\,.$$

Hence, in our setting, by (4.3), we have

$$d\tilde{\eta}(\mathbb{A}) = \frac{1}{4\pi}\,dA_{\mathrm{hyp}}\,. \tag{4.5}$$

Remark 4.3. The η-form is not exact on $\mathcal{H}$. This does not contradict Proposition 2.10. If we were to leave the path in $\mathcal{H}$ given by the adiabatic limit construction in section 2.4, the vertical tangent bundle of the family $E_\pm$ would no longer split as in the proof of Proposition 2.9, so the local variation terms in (2.20) would no longer vanish and contribute to $d\bar{\nu}$ as well.

Lemma 4.4. *The spinorial η-form $\tilde{\eta}(\mathbb{A})$ is $PGL(2,\mathbb{Z})$-equivariant, more precisely, for $g \in PGL(2,\mathbb{Z})$ acting on $\mathcal{H}$ by Möbius transformations, we have*

$$g^*\tilde{\eta}(\mathbb{A}) = \det g \cdot \tilde{\eta}(\mathbb{A})\,.$$

Proof. The η-form is invariant under orientation preserving spin isometries. We know that each $g \in PGL(2,\mathbb{Z})$ has two possible lifts to $GL(2,\mathbb{Z})$ that act fibrewise on E over the given action on $\mathcal{H}$. Each of these lifts has two lifts to $\widetilde{GL}(2,\mathbb{Z})$ that also act on the spinor bundle $S \to \mathcal{H}$ and therefore also on the fibrewise spinor bundle p^*S over E.

If $g \in PSL(2,\mathbb{Z})$, all four lifts preserve the superconnection $\mathbb{A}$ and the subbundles $S^\pm \subset S \to \mathcal{H}$. Therefore $g^*\tilde{\eta}(\mathbb{A}) = \tilde{\eta}(\mathbb{A})$ in this case.

If $g \in PGL(2,\mathbb{Z}) \setminus PSL(2,\mathbb{Z})$, then all four lifts of g to $\widetilde{GL}(2,\mathbb{Z})$ preserve the superconnection $\mathbb{A}$, but swap the bundles S^+ and S^-. This reverses the sign of the supertrace in the definition of the η-form, so we now have $g^*\tilde{\eta}(\mathbb{A}) = -\tilde{\eta}(\mathbb{A})$. □

4.3. **Adiabatic limits and hyperbolic geodesics.** We represent the two isometric tori $T_\pm = \widetilde{T}_\pm/\Gamma_\pm$ by points in the upper halfplane $\mathcal{H}$. By Proposition 4.1, we may rescale all tori to area 1 without changing the contribution to the ν-invariant. When we consider adiabatic limits of $M_\pm$, the points corresponding to $X_{\pm,a}$ trace out geodesic arcs in $\mathcal{H}$. These arcs will be used to compute the sum of integrals of the η-form $\tilde{\eta}(\mathbb{A})$ of Proposition 2.10.

We represent $T = \widetilde{T}_+/\Gamma_+$ by the basis (1.5). In equation (2.1) we have considered families of metrics on $M_\pm$. These induce two families of metrics on T_+. We write $X_{+,a} = \partial M_{+,a} = \Sigma_+ \times T_{+,a}$. For the second one, we consider the isomorphism $T \cong \widetilde{T}_-/\Gamma_-$ and write $X_{-,a} = \partial M_{-,a} = \Sigma_- \times T_{-,a}$.

We let $\overline{\mathcal{H}} = \mathcal{H} \cup \mathbb{R} \cup \{\infty\}$ denote the closure of $\mathcal{H}$ in $\mathbb{C}P^1$ and write $\partial_\infty\mathcal{H} = \mathbb{R} \cup \infty$. We extend the action of $GL(2,\mathbb{Z})$ by Möbius transformations to $\overline{\mathcal{H}}$. For $(z,w) \in \mathbb{C}^2$ we put

$$\tau = \begin{cases} \infty & \text{if } w = 0, \\ z/w & \text{if } \mathrm{Im}(z/w) \geq 0, \text{ and} \\ \bar{z}/\bar{w} & \text{if } \mathrm{Im}(z/w) < 0. \end{cases}$$

If (z,w) span an integral lattice Λ in $\mathbb{C}$, then $\mathbb{C}/\Lambda$ is isometric to $E_{(\tau,r)}$ for $r = |\mathrm{Im}(\bar{z}w)|$.

Lemma 4.5. *Consider the families of flat tori $T_{+,a}$ and $T_{-,a}$ as above, assuming m, $n \geq 0$.*

(i) *The family $T_{+,a}$ is represented in $\mathcal{H}$ by a vertical line γ_+ with real part $\frac{\varepsilon_+}{k_+}$. The adiabatic limit $a \to 0$ corresponds to $\frac{\varepsilon_+}{k_+} \in \partial_\infty\mathcal{H}$.*

(ii) *The family $T_{-,a}$ is represented in $\mathcal{H}$ by a hyperbolic geodesic γ_- between $\frac{\varepsilon_+}{k_+} - \frac{n}{k_+m}$ and $\frac{\varepsilon_+}{k_+} - \frac{q}{k_+p} \in \partial_\infty\mathcal{H}$. The adiabatic limit $a \to 0$ corresponds to $\frac{\varepsilon_+}{k_+} - \frac{n}{k_+m}$.*

(iii) *The geodesics γ_+ and γ_- intersect at $\frac{\varepsilon_+ + i\kappa_+}{k_+}$ with unoriented angle*

$$\left|\measuredangle_{\frac{\varepsilon_+ + i\kappa_+}{k_+}}\left(\frac{\varepsilon_+}{k_+}\,,\ \frac{\varepsilon_+}{k_+} - \frac{n}{k_+m}\right)\right| = 2\vartheta\,.$$

In the case of Example 3.1 (i), we have $2\vartheta = \pi$, so both γ_- and γ_+ agree with the vertical line with real part $\frac{\varepsilon_+}{k_+}$.

Proof. We write $T_{+,a} = \mathbb{C}/\Lambda_{+,a}$, where the lattice $\Lambda_{+,a}$ is generated by $\frac{\varepsilon_+ + ia}{k_+}\zeta_+$ and $\zeta_+ \in \mathbb{C}$, so $T_{+,a}$ is represented by the point $\frac{\varepsilon_+ + ia}{k_+} \in \mathcal{H}$ on the hyperbolic geodesic from $\frac{\varepsilon_+}{k_+}$ to ∞.

Analogously, the torus $T_{-,a} = \mathbb{C}/\Lambda_{-,a}$ can be represented by $\frac{\varepsilon_- + ia}{k_-}$ on the hyperbolic geodesic from $\frac{\varepsilon_-}{k_-}$ to ∞. We now consider the matrix

$$\begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} \frac{\varepsilon_+ p - q}{k_+} & \frac{n - \varepsilon_+ m + \varepsilon_- q - \varepsilon_- \varepsilon_+ p}{k_- k_+} \\ p & -\frac{\varepsilon_- p + m}{k_-} \end{pmatrix}$$

of determinant $\frac{mq-np}{k_+k_-} = -1$ by (1.7), which is integral by (1.8) and (3.4). Assertion (ii) follows because the antiholomorphic Möbius transformation $\tau \mapsto \frac{e\bar{\tau}+f}{g\bar{\tau}+h}$ maps

$$\frac{\varepsilon_-}{k_-} \longmapsto \frac{\varepsilon_+}{k_+} - \frac{n}{k_+m}\,, \qquad \infty \longmapsto \frac{\varepsilon_+}{k_+} - \frac{q}{k_+p}\,, \qquad \text{and} \qquad \frac{\varepsilon_- + i\kappa_-}{k_-} \longmapsto \frac{\varepsilon_+ + i\kappa_+}{k_+}\,.$$

In the last step, we have used the formulas for $\kappa_\pm$ in Proposition 1.8 (ii).

We compute the angle in (iii) using Figure 13. We note that the hyperbolic upper half plane is conformal to the Euclidean half plane, so we may compute the angle using Euclidean geometry. Let c denote the Euclidean center of the circle through the points $\frac{\varepsilon_+}{k_+} - \frac{n}{k_+m}$, $\frac{\varepsilon_+}{k_+} - \frac{q}{k_+p}$ and $\frac{\varepsilon_+ + is}{k_+}$. The angle between the two hyperbolic geodesic arcs from $\frac{\varepsilon_+}{k_+}$ and $\frac{\varepsilon_+}{k_+} - \frac{n}{k_+m}$ to $\frac{\varepsilon_+ + is}{k_+}$ equals the central angle subtending the arc from $\frac{\varepsilon_+}{k_+} - \frac{n}{k_+m}$ to $\frac{\varepsilon_+ + is}{k_+}$. It is therefore twice the inscribed angle at $\frac{\varepsilon_+}{k_+} - \frac{q}{k_+p}$, which we recognise as the gluing angle $\vartheta \in \left(0, \frac{\pi}{2}\right]$ from

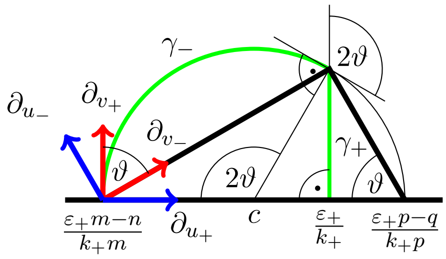


FIGURE 13. The hyperbolic angle between γ_+ and γ_-.

Proposition 1.8 (iii). This is easiest to see by considering the line from $\frac{\varepsilon_+ m-n}{k_+ m}$ to $\frac{\varepsilon_+ + i\kappa_+}{k_+}$ with direction $n + im\kappa_+$, see also Figure 2. Here we have used our assumption that m, $n \geq 0$. $\square$

4.4. **Axes of hyperbolic reflections.** The isometry group of the upper half plane with its hyperbolic metric is isomorphic to $PGL(2,\mathbb{R})$, acting by Möbius transformations. Each orientation reversing isometry of $\mathcal{H}$ is a hyperbolic glide reflection along a hyperbolic geodesic. By Lemma 4.4, the η-form $\tilde{\eta}(\mathbb{A})$ changes sign under pullback by elements g of $GL(2,\mathbb{Z}) \setminus SL(2,\mathbb{Z})$. If g is a reflection, then the restriction of $\tilde{\eta}(\mathbb{A})$ to its axis vanishes. If g describes a reflection about γ_g, then hgh^{-1} is a reflection about $h(\gamma_g)$.

Remark 4.6. Let $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in GL(2,\mathbb{Z}) \setminus SL(2,\mathbb{Z})$ represent the map $\tau \mapsto \frac{a\bar{\tau}+b}{c\bar{\tau}+d}$. It describes a reflection if and only if the trace $a+d$ vanishes. The line of reflection is vertical if and only if $c = 0$. In this case, the corresponding reflections in $PSL(2,\mathbb{Z})$ are of the form $\left(\begin{smallmatrix} -1 & k \\ 0 & 1 \end{smallmatrix}\right)$, and the fixed line has real part $\frac{k}{2}$. This is implies that all possible axes of reflections in $GL(2,\mathbb{Z}) \setminus SL(2,\mathbb{Z})$ are of the form $h(\gamma)$ for $h \in SL(2,\mathbb{Z})$ and γ a vertical geodesic with real part in $\frac{1}{2}\mathbb{Z}$.

Lemma 4.7. *Let $\frac{a}{b}$ and $\frac{c}{d} \in \partial_\infty\mathcal{H} = \mathbb{Q} \cup \{\infty\}$ be represented by reduced fractions. If $ad - bc = \pm 1$, then there exists a hyperbolic reflection, represented by an element $g \in GL(2,\mathbb{Z}) \setminus SL(2,\mathbb{Z})$, that fixes the hyperbolic geodesic γ from $\frac{a}{b}$ to $\frac{c}{d}$.*

Proof. Assume $ad - bc = 1$. The element $h = \left(\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right) \in SL(2,\mathbb{Z})$ maps $\infty = \frac{1}{0}$ to $\frac{a}{b}$ and $0 = \frac{0}{1}$ to $\frac{c}{d}$. Hence, it maps the y-axis to γ. This implies that $g = h \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right) h^{-1} \in GL(2,\mathbb{Z}) \setminus SL(2,\mathbb{Z})$ is a reflection that fixes γ. $\square$

4.5. **Continued Fractions and Hyperbolic Polygons.** Let $M_\pm = \widetilde{M}_\pm/\Gamma_\pm$ be $\mathbb{Z}/k_\pm$-blocks with boundary $X_\pm \cong \Sigma_\pm \times T_\pm$ and $T_\pm = \widetilde{T}_\pm/\Gamma_\pm$ as before. Define ε_+, $\kappa_\pm$ as in Section 1.3, and let $\gamma_\pm$ be the hyperbolic geodesics of Lemma 4.5. For simplicity, we assume $n > 0$ and $m \geq 0$. To compute the difference of $\bar{\nu}(M_{+,\kappa_+})$ and the adiabatic limit $\lim_{a\to 0} \bar{\nu}(M_{+,a})$ by formula (2.21) for M_+, we integrate $\tilde{\eta}(\mathbb{A})$ along the vertical line γ_+ from $\frac{\varepsilon_+}{k_+}$ to $\frac{\varepsilon_+ + i\kappa_+}{k_+}$. For M_-, we note that the orientation of the boundary was reversed during gluing. Hence, we integrate $\tilde{\eta}(\mathbb{A})$ along the geodesic ray γ_- from $\frac{\varepsilon_+ + is}{k_+}$ to $\frac{\varepsilon_+}{k_+} - \frac{n}{k_+ m} \in \partial_\infty\mathcal{H}$. Note that this last point is ∞ in case $m = 0$ and $\vartheta = \frac{\pi}{2}$. It follows from Theorem 2.5 and Proposition 2.10 that the integrals exist.

We will now complete the two geodesic rays above to an ideal hyperbolic polygon P with one finite corner $\frac{\varepsilon_+ + i\kappa_+}{k_+}$ and further corners represented by reduced fractions $\frac{\varepsilon_+}{k_+} - \frac{n}{k_+ m} = \frac{a_0}{b_0}$, $\ldots$, $\frac{a_\ell}{b_\ell} = \frac{\varepsilon_+}{k_+} \in \mathbb{Q} \cup \{\infty\} \subset \partial_\infty\mathcal{H}$ such that $a_j b_{j-1} - b_j a_{j-1} = 1$ for $j = 1, \ldots, \ell$. Then $\tilde{\eta}(\mathbb{A})$ vanishes along the geodesics joining $\frac{a_{j-1}}{b_{j-1}}$ and $\frac{a_j}{b_j}$ by Lemma 4.7.

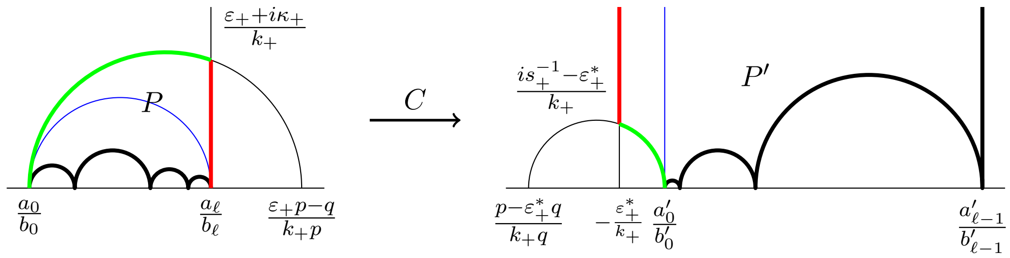


FIGURE 14. The hyperbolic polygons P and P'

It turns out to be easier to construct the image P' of P under C, where $C = \begin{pmatrix} \varepsilon_+^* & -r \\ -k_+ & \varepsilon_+ \end{pmatrix} \in SL(2, \mathbb{Z})$ act as a Möbius transformation, where the integer r is defined by $\varepsilon_+ \varepsilon_+^* = k_+ r + 1$. We note that

$$C\left(\frac{\varepsilon_+}{k_+}\right) = \infty\ , \quad C\left(\frac{\varepsilon_+ m - n}{k_+ m}\right) = \frac{m - \varepsilon_+^* n}{k_+ n}\ , \quad \text{and} \quad C\left(\frac{\varepsilon_+ p - q}{k_+ p}\right) = \frac{p - \varepsilon_+^* q}{k_+ q}\ . \tag{4.6}$$

We let $a_0' = \frac{m - \varepsilon_+^* n}{k_+}$, $b_0' = n \in \mathbb{Z}$ and represent $\frac{a_0'}{b_0'}$ as a continued fraction with minus signs,

$$\frac{a_0'}{b_0'} = \frac{m - \varepsilon_+^* n}{k_+ n} = c_1 - \cfrac{1}{c_2 - \ddots_{\,\frac{1}{c_\ell}}}\ .$$

This way, we obtain a sequence of integers $c_1, \ldots, c_\ell$ with $c_2, \ldots, c_\ell \geq 2$ and reduced fractions

$$\frac{a_0'}{b_0'} \quad < \quad \frac{a_1'}{b_1'} = c_1 - \cfrac{1}{c_2 - \ddots_{\,\frac{1}{c_{\ell-1}}}} \quad < \quad \cdots \quad < \quad \frac{a_{\ell-1}'}{b_{\ell-1}'} = \frac{c_1}{1} \quad \text{and} \quad \frac{a_\ell'}{b_\ell'} = \frac{1}{0}\ .$$

As explained in [36, §V], the numbers a_j', b_j' and c_j are related by the formula

$$\begin{pmatrix} a_j' & -a_{j+1}' \\ b_j' & -b_{j+1}' \end{pmatrix} = \begin{pmatrix} c_1 & -1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} c_{\ell-j} & -1 \\ 1 & 0 \end{pmatrix} \in SL(2, \mathbb{Z}) \tag{4.7}$$

for $j = 0, \ldots, \ell - 1$, which also shows that $a_{j+1}' b_j' - a_j' b_{j+1}' = 1$.

Remark 4.8. For later use, we note that

(i) because $a_1' b_0' - a_0' b_1' = 1$, the number $-b_1'$ is inverse to $a_0' = \frac{m - \varepsilon_+^* n}{k_+}$ modulo $b_0' = n$,
(ii) because $-\frac{q + \varepsilon_-^* n}{k_-}$ is also inverse to a_0' by equation (3.6), we have

$$b_1' \equiv \frac{q + \varepsilon_-^* n}{k_-} \mod n\ .$$

Let $\frac{a_j}{b_j}$ denote the preimage of $\frac{a_j'}{b_j'}$ under the Möbius transformation C for $0 < j < \ell$, the cases $j = 0, \ell$ being settled by (4.6). Then $a_j b_{j-1} - a_{j-1} b_j = 1$. The finite corner at $\frac{\varepsilon_+ + i\kappa_+}{k_+}$ gets mapped to $\frac{i\kappa_+^{-1} - \varepsilon_+^*}{k_+}$. Thus we have completed the construction of P and P'; see Figure 14.

4.6. **The contribution from the cusps.** Thanks to Theorem 4.2, we can in principle evaluate the integral of $\eta(\mathbb{A})$ over the arcs γ_+ and γ_- from Lemma 4.5 by computing the area of the polygon P above. However, the polygon P has cusps, that is, rational points in the boundary $\partial_\infty\mathcal{H} = \mathbb{R}\cup\{\infty\}$ at infinity. In this section, we compute the contribution from a cusp as a limit of the integral of the η-form over certain horocyclic arcs that escape to infinity.

Adiabatic limit formulas for η-forms have been proved by Bunke, Ma [10] and Liu [29], but only modulo exact forms. Here, we have to integrate the η-form over an interval, so we need an adiabatic limit formula that holds "on the nose". We will prove such a formula in Section 6.3, but only for the simple special case at hand.

We want to define a distance between two hyperbolic geodesics ending in a cusp point $\frac{e}{f}$. To move $\frac{e}{f}$ to $\infty = \frac{1}{0}$, assume that the fraction $\frac{e}{f}$ is reduced and find a and $b \in \mathbb{Z}$ such that

$$ae + bf = 1\ . \tag{4.8}$$

Then $\left(\begin{smallmatrix} a & b \\ -f & e\end{smallmatrix}\right) \in SL(2,\mathbb{Z})$, and the Möbius transformation $z \mapsto \frac{az+b}{-fz+e}$ rotates the cusp $\frac{e}{f}$ into ∞. Now consider hyperbolic geodesics starting at x, $y \in \mathbb{R}\cup\{\infty\}$ and ending in $\frac{e}{f}$. They get rotated by the matrix above to vertical lines with real parts $\frac{ax+b}{-fx+e}$ and $\frac{ay+b}{-fy+e}$. Because of (4.8), the difference is

$$\frac{ax+b}{-fx+e} - \frac{ay+b}{-fy+e} = \frac{x-y}{(fy-e)(fx-e)}\ .$$

Definition 4.9. The *cusp angle* between two hyperbolic geodesics going from x, $y \in \mathbb{R}\cup\{\infty\} = \partial_\infty\mathcal{H}$ to a cusp point represented by a reduced fraction $\frac{e}{f} \in \mathbb{Q}\cup\{\infty\}$, with $x \neq \frac{e}{f} \neq y$, is defined as

$$\measuredangle_{\frac{e}{f}}(x,y) = \frac{x-y}{(fx-e)(fy-e)} \in \mathbb{R}$$

if x, $y \in \mathbb{R}$, and by the obvious extension of this formula if one of the points is ∞.

Note that $\measuredangle_{\frac{e}{f}}(x,y)$ is a geometric notion for the covering map $\mathcal{H} \to \mathcal{H}/SL(2,\mathbb{Z})$. If measures how often a line joining the two geodesics above in the universal covering space $\mathcal{H}$ winds around the cusp in $\mathcal{H}/SL(2,\mathbb{Z})$. The sign is chosen such that oriented ideal triangles have positive cusp angles. In particular, cusp angles are $SL(2,\mathbb{Z})$-invariant.

From (4.7), we see that $\left(\begin{smallmatrix} a'_j \\ b'_j\end{smallmatrix}\right) = C_j\left(\begin{smallmatrix} c_{\ell-j} \\ 1\end{smallmatrix}\right)$, $\left(\begin{smallmatrix} a'_{j+1} \\ b'_{j+1}\end{smallmatrix}\right) = C_j\left(\begin{smallmatrix} 1 \\ 0\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} a'_{j+2} \\ b'_{j+2}\end{smallmatrix}\right) = C_j\left(\begin{smallmatrix} 0 \\ -1\end{smallmatrix}\right)$, with $C_j = \left(\begin{smallmatrix} a'_{j+1} & -a'_{j+2} \\ b'_{j+1} & -b'_{j+2}\end{smallmatrix}\right) \in SL(2,\mathbb{Z})$. Because cusp angles are $SL(2,\mathbb{Z})$-invariant, we can now compute the cusp angles of P. For $j = 0, \dots, \ell-2$, we obtain

$$\measuredangle_{\frac{a_{j+1}}{b_{j+1}}}\left(\frac{a_j}{b_j}, \frac{a_{j+2}}{b_{j+2}}\right) = \measuredangle_{\frac{a'_{j+1}}{b'_{j+1}}}\left(\frac{a'_j}{b'_j}, \frac{a'_{j+2}}{b'_{j+2}}\right) = \measuredangle_\infty(c_{\ell-j}, 0) = c_{\ell-j}\ , \tag{4.9a}$$

$$\measuredangle_{\frac{a_0}{b_0}}\left(\frac{\varepsilon_+ + i\kappa_+}{k_+}, \frac{a_1}{b_1}\right) = \measuredangle_{\frac{a'_0}{b'_0}}\left(\frac{p-\varepsilon_+^* q}{k_+ p}, \infty\right) + \measuredangle_{\frac{a'_0}{b'_0}}\left(\infty, \frac{a'_1}{b'_1}\right) = \frac{-q}{k_- n} + \frac{b'_1}{b'_0}\ , \tag{4.9b}$$

$$\measuredangle_{\frac{a_\ell}{b_\ell}}\left(\frac{a_{\ell-1}}{b_{\ell-1}}, \frac{\varepsilon_+ + i\kappa_+}{k_+}\right) = \measuredangle_\infty\left(c_1, -\frac{\varepsilon_+^*}{k_+}\right) = c_1 + \frac{\varepsilon_+^*}{k_+}\ . \tag{4.9c}$$

For (4.9b), we have used Lemma 4.5 (ii) and equation (4.6).

Proposition 4.10. *Let $\frac{e}{f} \in \mathbb{Q}\cup\{\infty\} \subset \partial_\infty\mathcal{H}$ be a cusp point, and let x, $y \in \partial_\infty\mathcal{H}$. Assume that α_r is a family of horocyclic arcs centered at $\frac{e}{f}$ from the geodesic from $\frac{e}{f}$ to x to the*

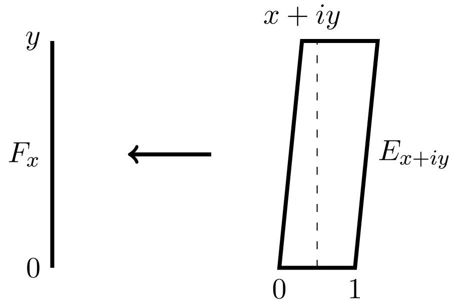


FIGURE 15. The torus E_{x+iy} as a bundle.

geodesic from $\frac{e}{f}$ to y that converges to $\frac{e}{f}$ as $r \to \infty$. Then

$$\lim_{r\to\infty} \int_{\alpha_r} \tilde\eta(\mathbb{A}) = \frac{\measuredangle_{e/f}(x,y)}{12} \; .$$

Proof. By the considerations above, we can rotate the cusp point $\frac{e}{f}$ to ∞. Hence we consider the horocycle $x \mapsto x + iy$ for large fixed y and prove that

$$\lim_{y\to\infty} \tilde\eta(\mathbb{A})|_{\mathbb{R}+iy} = -\frac{dx}{12} \; . \tag{4.10}$$

We pull the universal family $E \to \mathcal{H} \times (0,\infty)$ from Section 4.1 back to $\mathbb{R}$ by $x \mapsto (x+iy, y)$. Let us write E_{x+iy} for $E_{(x+iy,y)} = \mathbb{C}/\operatorname{span}_{\mathbb{Z}}\{x + iy, 1\}$ and $F_x = \mathbb{R}/y\mathbb{Z}$, and consider the map $E_{x+iy} \to F_x$ given by the imaginary part. Then we obtain a family of fibred tori,

$$\begin{array}{ccccc} S^1_1 & \longrightarrow & E_{x+iy} & \longrightarrow & E \\ & & {\scriptstyle\mathrm{Im}}\big\downarrow & & \big\downarrow \\ & & F_x & \longrightarrow & F \\ & & & & \big\downarrow \\ & & & & \mathbb{R} \ni x \; . \end{array} \tag{4.11}$$

Here, the interior circles form the fibres of E_{x+iy} of length 1, and the base F_y can be identified with the exterior circles in $\widetilde{T}_+$. In particular, the orientation of E_{x+iy} agrees with the one in [10, 29]. Let $T^H E$ denote the horizontal subbundle for the fibration $E \to \mathbb{R}$ induced by the pullback of the connection ∇^W from (4.2a). The Euclidean metric on E_{x+iy} defines a connection for the fibration $E \to F$ with holonomy $-x \in \mathbb{R}/\mathbb{Z}$. In Figure 15, the dashed line is horizontal with respect to this connection.

Let $L \to F$ denote a Hermitian line bundle with a fibrewise flat connection that contains $E \to F$ as a circle bundle. Then over F_x, the bundle L has holonomy $e^{-2\pi i x}$. Let φ be a coordinate on $F_x = \mathbb{R}/y\mathbb{Z}$. The bundle L carries a Hermitian connection ∇^L that we can describe as

$$\nabla^L = d + \frac{2\pi i x}{y}\, d\varphi$$

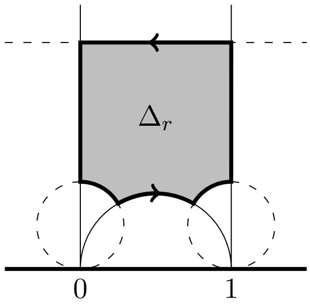


FIGURE 16. An ideal triangle, truncated by horocyclic arcs.

in a trivialisation over a neighbourhood of F_x in F. We compute curvature and first Chern form of (L, ∇^L) on F as

$$(\nabla^L)^2 = \frac{2\pi i}{y}\, dx\, d\varphi \qquad \text{and} \qquad c_1(\nabla^L) = -\frac{1}{y}\, dx\, d\varphi\ .$$

By Proposition 4.12 below, we have

$$\lim_{y\to\infty} \tilde\eta(\mathbb{A})|_{\mathbb{R}+iy} = \int_{F/\mathbb{R}} \tilde\eta(\mathbb{A}')\ , \tag{4.12}$$

where now $\mathbb{A}'$ is the superconnection of the fibrewise spinor bundle over the circle bundle $E \to F$. The corresponding η-form of $E \to F$ was computed by Zhang in [37, Theorem 1.7]. Here, it only has a component of degree 2, given by

$$\tilde\eta(\mathbb{A}') = \frac{1}{12}\, c_1\big(\nabla^L\big) = -\frac{1}{12y}\, dx\, d\varphi\ .$$

Integration over the fibre of $F \to \mathbb{R}$ of length y proves (4.10). □

Example 4.11. Consider the ideal triangle Δ of area π with corners 0, 1 and $\infty \in \partial_\infty\mathcal{H}$. By Remark 4.6 and Lemma 4.7, the form $\tilde\eta(\mathbb{A})$ vanishes on its sides. All cusp angels equal 1. Let Δ_r denote truncations of Δ by horocyclic arcs centered at the corners that converge to the full triangle Δ as $r \to \infty$; see Figure 16. Then by Theorem 4.2 and Proposition 4.10, we check that

$$\lim_{r\to\infty} \int_{\partial\Delta_r} \tilde\eta(\mathbb{A}) = \frac{\measuredangle_0(\infty,1) + \measuredangle_1(0,\infty) + \measuredangle_\infty(1,0)}{12} = \frac{1}{4} = \frac{A_{\mathrm{hyp}}(\Delta)}{4\pi} = \int_\Delta d\tilde\eta(\mathbb{A})\ .$$

The following Proposition is inspired by a result [10, Theorem 5.11] of Bunke and Ma; see also Liu in [29, Theorem 1.3], where the following equality is proved up to exact forms. Moreover, Liu assumes that the fibrewise Dirac operator of the fibration $F \to \mathbb{R}$ is invertible [29, Ass 3.1], which is not the case here.

Proposition 4.12. *In the situation above, let $\mathbb{A}'$ denote the superconnection associated with the fibrewise spin Dirac operator on the bundle $E \to F$. Then*

$$\lim_{y\to\infty} \tilde\eta(\mathbb{A})|_{\mathbb{R}+iy} = \int_{F/\mathbb{R}} \tilde\eta(\mathbb{A}')\ .$$

We postpone the proof to Section 6.3.

4.7. **Evaluation of the η-Form Integrals.** With all preliminaries understood, we can now prove Theorem 3. We start by integrating the η-form from Proposition 2.10 along the geodesic rays γ_+ and γ_-. These integrals exist by Theorem 2.5 and Proposition 2.10.

Theorem 4.13. *Assume that $m \geq 0$, $n > 0$. Then we have*

$$\bar\nu(M_{+,\kappa_+}) + \bar\nu(M_{-,\kappa_-}) - \lim_{r\to 0}\big(\bar\nu(M_{+,r}) + \bar\nu(M_{-,r})\big) = 72\,\frac{\rho}{\pi} + 24\left(\frac{q}{k_-n} - \frac{m}{k_+n} + 12\,S\Big(\frac{m-\varepsilon_+^*n}{k_+},n\Big)\right). \tag{4.13}$$

It will follow from the proof below that the first three terms on the right hand side stem from the area of the triangle spanned by γ_+ and γ_-. The Dedekind sum comes from the polygon we get by omitting the finite corner. The two regions are separated by the blue geodesic in Figure 14.

Proof. By Propositions 2.10, 4.10 and the discussion at the beginning of subsection 4.5, we have

$$\begin{aligned}\bar\nu(M_{+,\kappa_+}) + \bar\nu(M_{-,\kappa_-}) - \lim_{r\to 0}\big(\bar\nu(M_{+,r}) + \bar\nu(M_{-,r})\big) = 288\int_{\gamma_+\cup\gamma_-}\tilde\eta(\mathbb{A}) = 288\int_P d\tilde\eta(\mathbb{A})\\ - 24\measuredangle_{\frac{a_0}{b_0}}\left(\frac{\varepsilon_+ + i\kappa_+}{k_+},\frac{a_1}{b_1}\right) - 24\measuredangle_{\frac{a_\ell}{b_\ell}}\left(\frac{a_{\ell-1}}{b_{\ell-1}},\frac{\varepsilon_+ + i\kappa_+}{k_+}\right) - 24\sum_{j=0}^{\ell-2}\measuredangle_{\frac{a_{j+1}}{b_{j+1}}}\left(\frac{a_j}{b_j},\frac{a_{j+2}}{b_{j+2}}\right).\end{aligned} \tag{4.14}$$

From (4.5), Lemma 4.5 (iii) and the hyperbolic area formula, we get

$$288\int_P d\tilde\eta(\mathbb{A}) = \frac{72}{\pi}\,A_{\mathrm{hyp}}(P) = 72\,\ell - 144\frac{\vartheta}{\pi} = 72\,(\ell-1) + 72\,\frac{\rho}{\pi}\,. \tag{4.15}$$

From Definition 4.9 and equations (4.9), we get

$$\begin{aligned}\measuredangle_{\frac{a_0}{b_0}}\left(\frac{\varepsilon_+ + i\kappa_+}{k_+},\frac{a_1}{b_1}\right) + \measuredangle_{\frac{a_\ell}{b_\ell}}\left(\frac{a_{\ell-1}}{b_{\ell-1}},\frac{\varepsilon_+ + i\kappa_+}{k_+}\right) + \sum_{j=0}^{\ell-2}\measuredangle_{\frac{a_{j+1}}{b_{j+1}}}\left(\frac{a_j}{b_j},\frac{a_{j+2}}{b_{j+2}}\right)\\ = \left(\frac{-q}{k_-n} + \frac{b_1'}{b_0'}\right) + \left(\frac{\varepsilon_+^*}{k_+} + c_1\right) + \sum_{j=2}^{\ell} c_j\,.\end{aligned} \tag{4.16}$$

This number can be interpreted along the lines of [36, §V]. The product of the matrices on the right hand side of (4.7) for $j = 0$ is given by

$$A = \begin{pmatrix} c_1 & -1\\ 1 & 0\end{pmatrix}\cdots\begin{pmatrix} c_\ell & -1\\ 1 & 0\end{pmatrix} = \begin{pmatrix} a_0' & -a_1'\\ b_0' & -b_1'\end{pmatrix}\,.$$

Now it follows from $a'_0 = \frac{m-\varepsilon_+^* n}{k_+}$, $b'_0 = n$ and [36, equations (6), (25) & (26)] that

$$\begin{aligned}\left(\frac{-q}{k_-n}+\frac{b'_1}{b'_0}\right)+\left(\frac{\varepsilon_+^*}{k_+}+c_1\right)+\sum_{j=2}^{\ell} c_j \\ &= \frac{-q}{k_-n}+\frac{b'_1}{b'_0}+\frac{\varepsilon_+^*}{k_+}+3(\ell-1)+N(A) \\ &= \frac{-q}{k_-n}+\frac{b'_1}{b'_0}+\frac{\varepsilon_+^*}{k_+}+3(\ell-1)+\frac{m-\varepsilon_+^* n}{k_+n}-\frac{b'_1}{b'_0}-12\,S(-b'_1,b'_0) \\ &= 3(\ell-1)+\frac{m}{k_+n}-\frac{q}{k_-n}-12\,S(-b'_1,n)\;,\end{aligned} \tag{4.17}$$

where the Dedekind sum $S(-b'_1, n)$ is defined in (0.3), and $N\colon SL(2,\mathbb{Z}) \to \mathbb{Z}$ is introduced in (7.1), see also [36, (3)].

The Dedekind sum $S(k, n)$ is odd and n-periodic in k, and it does not change if k is replaced by its inverse modulo n. Our claim (4.13) follows from Remark 4.8 (i) and (4.14)–(4.17). □

Proof of Theorem 3. We may assume that $n > 0$. If $m \geq 0$, the theorem follows from Theorems 2.2, 2.5 and 4.13. If $m < 0$, we additionally use Proposition 3.5, combining (3.9) and (3.10) to reduce to the case $m > 0$, $n > 0$. □

Remark 4.14. The formula in Theorem 3 is indeed symmetric in the two halves of the twisted connected sum. Swapping the two halves amounts to exchanging m and $-q$, and $\varepsilon_\pm$ and $k_\pm$; see Proposition 3.5, in particular (3.8). By equation (3.6), the number $\frac{m-\varepsilon_+^* n}{k_+}$ is inverse to $-\frac{q+\varepsilon_-^* n}{k_-}$ modulo n, so the Dedekind sum above is the same in both cases.

Remark 4.15. We can evaluate $3n(P) - \ell(P) \bmod \mathbb{Z}$ in a slightly different way. By Remark 4.8 (ii), we have $\frac{-q}{k_-n} + \frac{b'_1}{b'_0} \equiv \frac{\varepsilon_-^*}{k_-} \bmod \mathbb{Z}$, so instead of (4.17), we get

$$\left(\frac{-q}{k_-n}+\frac{b'_1}{b'_0}\right)+\left(\frac{\varepsilon_+^*}{k_+}+c_1\right)+\sum_{j=2}^{\ell} c_j \equiv \frac{\varepsilon_+^*}{k_+}+\frac{\varepsilon_-^*}{k_-} \mod \mathbb{Z}\;.$$

Following the proofs of Theorems 4.13 and 3 above, we see that

$$\nu(M,g) \equiv D_{\gamma_+}(V_+) + D_{\gamma_-}(V_-) + 3\,m_\rho(L;N_+,N_-) - 24\left(\frac{\varepsilon_+^*}{k_+}+\frac{\varepsilon_-^*}{k_-}\right) \mod 24\mathbb{Z}\;. \tag{4.18}$$

The terms $\frac{\varepsilon_\pm}{k_\pm}$ and $D_{\gamma_\pm}(V_\pm)$ depend on the $\Gamma_\pm$-action on $\widetilde{M}_\pm$ only, and one can check that in all our examples

$$D_{\gamma_\pm}(V_\pm) - 24\frac{\varepsilon_\pm^*}{k_\pm} \in \mathbb{Z}\;, \tag{4.19}$$

as one would expect from the formula above. In particular for $k_\pm = 5$, we only found examples where the $\Gamma_\pm$-action on $V_\pm$ has isolated fixpoints, and (4.19) holds for all choices of $\varepsilon_\pm$.

Example 4.16. Let us illustrate the remark above using our standard example. We start with the $\mathbb{Z}/5$-block from Example 5.10, whose fixpoint contribution was computed in Example 2.7. We check that modulo 24,

$$\frac{24}{5\varepsilon} - 24\frac{\varepsilon^*}{5} \equiv \begin{cases} 0 & \text{if } \varepsilon \equiv \pm 1, \text{ and} \\ 12 & \text{if } \varepsilon \equiv \pm 2. \end{cases}$$

The $\mathbb{Z}/3$-block from Example 5.5 has no isolated fixpoints, so the contribution to ν mod 24 is simply $-24\frac{\varepsilon^*}{3} = -8\varepsilon$. Together with $m_\rho(L; N_+, N_-) = -1$ from Example 2.3, we find that $\nu(M) \equiv -11$ mod 24, which confirms our computations in Example 2.16.

Remark 4.17. One can check that we recover the formula for $\bar{\nu}(M)$ in [14] in the case where k_+, $k_- \in \{1, 2\}$. Involutive isomorphisms of Calabi-Yau manifolds cannot have isolated fixpoints; see Section 2.3, so the first two terms on the right side of (0.4) vanish.

We start with a rectangular twisted connected sum as in [27, 13], so $k_+ = k_- = 1$. Because $m = q = 0 = \rho = A$, we have $\bar{\nu}(M) = 0$ by (0.4).

Next, consider [14, Example 3.11] with $k_- = 1$, $k_+ = 2$ and gluing matrix $\left(\begin{smallmatrix} 1 & 1 \\ 1 & -1 \end{smallmatrix}\right)$, so $\vartheta = \frac{\pi}{4}$. By (0.4), we get

$$\bar{\nu}(M) = -3 + 24\left(-1 - \frac{1}{2} + 12\,S(0,1)\right) = -39\,,$$

see entries 1–18 in Table 2.

In [14, Example 3.11], we considered a simply connected examples with $k_- = k_+ = 2$, $\left(\begin{smallmatrix} -1 & 1 \\ 1 & 3 \end{smallmatrix}\right)$ and $\vartheta = \frac{\pi}{6}$. We found

$$\bar{\nu}(M) = -3 + 24\left(-\frac{3}{2} - \frac{1}{2} + 12\,S(0,1)\right) = -51\,,$$

see entries 110, 111 in Table 2. The other examples in [14] involve building blocks of rank ≥ 2.

5. Examples

In this section, we generate examples of extra-twisted connected sums and compute their $\bar{\nu}$-invariants. To do this we will define some building blocks whose polarising lattice has rank 1. We will also describe the topology of those blocks, and compute some parts of the topology of the resulting extra-twisted connected sums.

5.1. **The cohomology of an extra-twisted connected sum.** We have previously explained how to compute the fundamental group of an extra-twisted connected sum M from the gluing matrix, in Proposition 1.11. We now compute some other basic topological features. In particular we show that all our examples have $H_2(M) = 0$ (so those that have $\pi_1 M = 0$ are in fact 2-connected), and give a formula (5.2) for $b_3(M)$.

Remark 5.1. The most important topological properties that we do not compute are the torsion in $H^4(M)$, and the Pontryagin class $p_1(M) \in H^4(M)$. In general, the torsion in $H^4(M)$ can have contributions both from the action of $\Gamma_\pm$ on the two halves, as well as from the gluing. In [31], attention was focussed on blocks with involution such that the former contribution vanishes, and on certain matchings (namely ones with gluing matrix $\left(\begin{smallmatrix} 1 & 1 \\ 1 & -1 \end{smallmatrix}\right)$ or $\left(\begin{smallmatrix} 1 & 1 \\ 1 & -3 \end{smallmatrix}\right)$) where the torsion in $H^4(M)$ can then be determined in a simple way from the configuration $N_+ + N_-$. We do not attempt here to generalise those arguments.

Topology of $V_\pm$. Let us first recall from [12, §5] some relevant facts about the topology of the ACyl Calabi-Yau manifold $V := Z \setminus \Sigma$ constructed from a building block (Z, Σ), and make some observations about the action on cohomology of an automorphism group Γ. Let us assume that the kernel of the restriction map $H^2(Z) \to H^2(\Sigma)$ is generated by $[\Sigma]$. Note that this implies that the action of Γ on $H^2(Z)$ is trivial.

We identify $\Sigma \subset Z$ with a standard K3 surface and denote the image of $H^2(Z)$ in $L = H^2(\Sigma)$ by N. Then $N \subset L$ is primitive, but typically not unimodular. Let $T = N^\perp \subset L$, and let N^* be the dual of N, such that we have a short exact sequence

$$0 \longrightarrow T \longrightarrow L \longrightarrow N^* \longrightarrow 0\,.$$

By [12, Lemma 5.2], Z and V are simply connected. Using excision and suspension, we have

$$H^k(Z,V) \cong H^k(\Sigma \times D^2, \Sigma \times S^1) \cong H^{k-2}(\Sigma) .$$

The long exact sequence of the pair (Z,V) becomes

$$\cdots \longrightarrow H^{k-2}(\Sigma) \xrightarrow{\iota_!} H^k(Z) \xrightarrow{j^*} H^k(V) \xrightarrow{\delta} H^{k-1}(\Sigma) \longrightarrow \cdots . \tag{5.1}$$

According to [12, Lemma 5.4], we have $H^1(V) = 0 = H^5(V)$, and the exact sequence (5.1) gives rise to split short exact sequences

$$\begin{aligned} 0 \longrightarrow \mathbb{Z} \longrightarrow H^2(Z) \longrightarrow H^2(V) \longrightarrow 0 , \\ 0 \longrightarrow H^3(Z) \longrightarrow H^3(V) \longrightarrow T \longrightarrow 0 , \\ 0 \longrightarrow N^* \longrightarrow H^4(Z) \longrightarrow H^4(V) \longrightarrow 0 . \end{aligned}$$

It follows that $H^\bullet(V)$ is torsion free if $H^\bullet(Z)$ is torsion free. The inclusion $\mathbb{Z} \hookrightarrow H^2(Z)$ maps 1 to the cohomology class $\iota_! 1$ induced by $\iota \colon \Sigma \to Z$. It is easy to see that the sequences above are Γ-equivariant. In particular, Γ acts trivially on $H^2(V)$, while the Γ-invariant part of $H^3(V)$ is the direct sum of T and $H^3(Z)^\Gamma$.

The map δ in (5.1) involves restriction to $\Sigma \times S^1_\zeta$ followed by integration over S^1_ζ. Write

$$H^k(\Sigma \times S^1_\zeta) = H^k(\Sigma) \oplus H^{k-1}(\Sigma)\mathbf{u} ,$$

where $\mathbf{u} \in H^1(S^1_\zeta)$ is the generator. The restriction map $\iota^* \colon H^\bullet(V) \to H^\bullet(\Sigma \times S^1_\zeta)$ is described in [12, Cor 5.5]. We have in particular that $H^2(V)$ maps isomorphically to $N \subset H^2(\Sigma)$, while the image of $H^3(V)$ is $T\mathbf{u} \subset H^2(\Sigma)\mathbf{u}$.

Topology of $M_\pm$. We may regard $M_\pm = (V_\pm \times S^1_{\xi_\pm})/\Gamma_\pm$ as the mapping torus of a generator $\gamma_0 \in \Gamma_\pm \cong \mathbb{Z}/k_\pm$. Generalising the discussion of blocks with involution from [31, §2.2], we can use excision and the Thom isomorphism to see that

$$H^\bullet(M_\pm, V_\pm) \cong H^{\bullet-1}(V_\pm) .$$

From the long exact sequence of the pair $(M_\pm, V_\pm)$, we get

$$\cdots \longrightarrow H^{\ell-1}(V_\pm) \longrightarrow H^\ell(M_\pm) \xrightarrow{\iota_V^*} H^\ell(V_\pm) \xrightarrow{\gamma_0^* - \mathrm{id}} H^\ell(V_\pm) \longrightarrow \cdots ,$$

where $\iota_V \colon V_\pm \to M_\pm$ is the inclusion of V as fibre of the obvious projection $M_\pm \to S^1_{\xi_\pm/k_\pm}$.

Since $H^1(V) = 0$ while $H^2(V)$ is Γ-invariant, it is immediate that $H^2(M_\pm) \cong H^2(V_\pm)$. Since $H^3(V_\pm)^\Gamma$ is torsion-free, we also have a splitting $H^3(M_\pm) \cong H^2(V_\pm) \oplus H^3(V_\pm)^\Gamma$, and $H^3(M_\pm)$ is torsion-free too. While the splitting is not natural with $\mathbb{Z}$ coefficients, it is natural with $\mathbb{Q}$ coefficients.

We can treat the cross-section $\Sigma \times T^2$ similarly. Now $H^2(\Sigma \times T^2) = H^2(\Sigma) \oplus H^2(T^2)$, and $H^2(M_\pm) \to H^2(\Sigma \times T^2)$ maps isomorphically to $N_\pm$.

Meanwhile, the pull-back $H^1(T^2) \to H^1(S^1_\xi \times S^1_\zeta)$ of the quotient map is injective (with image of index $k_\pm$). We abuse notation slightly to use $\mathbf{u}_\pm, \mathbf{v}_\pm$ to denote not only the generators of $H^1(S^1_{\zeta_\pm} \times S^1_{\xi_\pm})$ obtained by pulling back the generators of the two factors, but also their unique pre-images in $H^1(T^2; \mathbb{Q})$. (In terms of de Rham cohomology, these classes are represented by the 1-forms $\frac{1}{\zeta_+} du_+$ and $\frac{1}{\xi_\pm} dv_\pm$.) Switching to rational coefficients, we then have the splitting $H^3(\Sigma \times T^2; \mathbb{Q}) = H^2(\Sigma; \mathbb{Q})\mathbf{u}_\pm \oplus H^2(\Sigma; \mathbb{Q})\mathbf{v}_\pm$. In the splitting

$$H^3(M_\pm; \mathbb{Q}) = H^2(V_\pm)\mathbf{v}_\pm \oplus H^3(V_\pm; \mathbb{Q})^\Gamma,$$

the first term has image exactly $N_\pm \mathbf{v}_\pm$, the second term has image exactly $T_\pm \mathbf{u}_\pm$, and the kernel of $H^3(M_\pm;\mathbb{Q}) \to H^3(\Sigma\times T^2;\mathbb{Q})$ is the $H^3(Z_\pm;\mathbb{Q})^\Gamma$ component in $H^3(V_\pm;\mathbb{Q})^\Gamma$.

The topology of M. Generalising the discussion from [31, §7.1] of extra-twisted connected sums that involve only involutions, we can now apply the Mayer-Vietoris sequence to compute some basic features of the topology of an extra-twisted connected sum.

Proposition 5.2. *Let M be an extra-twisted connected sum of building blocks (Z,Σ) such that $H^2(Z) \to H^2(\Sigma)$ is generated by $[\Sigma]$, with configuration of polarising lattices $N_+, N_- \hookrightarrow L$. Let $\rho_\pm$ be the ranks of the polarising lattices (so $\rho_\pm = b_2(Z_\pm)-1$). If $\cos\vartheta \neq 0$ let d_ϑ be the rank of $N_+^\vartheta \cong N_-^\vartheta$ defined in §1.4, otherwise let $d_\vartheta = \operatorname{rk} N_+^{\frac{\pi}{2}} + \operatorname{rk} N_-^{\frac{\pi}{2}}$.*

(i) *The free part of $H^2(M)$ is isomorphic to $N_+ \cap N_- \subset L$,*
(ii) *The torsion in $H^3(M)$ is isomorphic to the cotorsion of $N_+ + N_-$ in L.*
(iii) $b_3(M) = b_2(M) + 23 - \rho_+ - \rho_- + d_\vartheta + b_3(Z_+)^{\Gamma_+} + b_3(Z_-)^{\Gamma_-}$.

The examples considered in this paper use configurations where N_+ is transverse to N_-, and $N_+ + N_-$ is embedded primitively in L. Thus the proposition implies that our examples have $H_2(M) = 0$, and those examples that are simply-connected are in fact 2-connected. Moreover, all our examples have $\rho_+ = \rho_- = 1$, and hence $\operatorname{rk} N_+^\vartheta = \operatorname{rk} N_-^\vartheta = 1$. Thus if $\vartheta \notin \frac{\pi}{2}\mathbb{Z}$ we have $d_\vartheta = 1$ and

$$b_3(M) = 22 + b_3^\Gamma(Z_+) + b_3^\Gamma(Z_-)\,, \tag{5.2}$$

while if $\vartheta \in \frac{\pi}{2}\mathbb{Z} \setminus \pi\mathbb{Z}$ then $d_\vartheta = 2$ and

$$b_3(M) = 23 + b_3^\Gamma(Z_+) + b_3^\Gamma(Z_-)\,. \tag{5.3}$$

Proof. We have a Mayer-Vietoris sequence

$$\cdots \longrightarrow H^k(M) \longrightarrow H^k(M_+) \oplus H^k(M_-) \longrightarrow H^k(\Sigma\times T^2) \longrightarrow H^{k+1}(M) \longrightarrow \cdots\,.$$

The image of $H^1(M_+) \oplus H^1(M_-)$ in $H^1(\Sigma\times T^2)$ has finite index, and indeed it is dual to the fundamental group computed in Proposition 1.11. Since $H^2(M_\pm)$ maps isomorphically to $N_\pm \subset H^2(\Sigma\times T^2)$, the image of $H^2(M)$ in $H^2(M_+)\oplus H^2(M_-)$ is isomorphic to $N_+\cap N_- \subset L$ (as determined by the configuration).

For $H^3(M)$ we get a short exact sequence

$$0 \longrightarrow L/(N_+ + N_-) \oplus \mathbb{Z} \longrightarrow H^3(M) \longrightarrow \ker\bigl(H^3(M_+)\oplus H^3(M_-) \to H^3(\Sigma\times T^2)\bigr) \longrightarrow 0$$

Since the last term is torsion-free, the sequence splits, and the torsion of $H^3(M)$ equals the torsion of $L/(N_+ + N_-)$.

Finally we want to determine $b_3(M)$. The contribution from $L/(N_+ + N_-)\oplus\mathbb{Z}$ equals $23 - \rho_+ - \rho_- + b_2(M)$. The other term we describe as the sum of the kernels of $H^3(M_\pm) \to H^3(\Sigma\times T^2)$, which by the above have rank $b_3(Z_\pm)^\Gamma$, and the intersection of the images in $H^3(\Sigma\times T^2)$. Points in the intersection of the images are those that can be written as both $n_+\mathbf{v}_+ + t_+\mathbf{u}_+$ and $n_-\mathbf{v}_- + t_-\mathbf{u}_-$, with $n_\pm \in N_\pm$ and $t_\pm \in T_\pm$. Now, the gluing identifies the tori in such a way that

$$\xi_+\mathbf{v}_+ = \cos\vartheta\,\xi_-\mathbf{v}_- + \sin\vartheta\,\zeta_-\mathbf{u}_-, \qquad \zeta_+\mathbf{u}_+ = \sin\vartheta\,\xi_-\mathbf{v}_- - \cos\vartheta\,\zeta_-\mathbf{u}_-\,.$$

If $\cos\vartheta \neq 0$ then n_+ and n_- determine each other, because the orthogonal projection of $\frac{1}{\zeta_-}n_-$ to N_+ must be $\frac{\cos\vartheta}{\zeta_+}n_+$ and vice versa. Thus, in the notation of §1.4, in fact $n_\pm \in N_\pm^\vartheta$, so the intersection of the images is isomorphic to $N_+^\vartheta \cong N_-^\vartheta$. On the other hand, if $\cos\vartheta = 0$ then we can simply take each $n_\pm$ freely in $N_\pm^{\frac{\pi}{2}}$ (*i.e.* in the orthogonal complement to $N_\mp$ in $N_\pm$).

Either way, the contribution to $b_3(M)$ from the intersection of the images in $H^3(\Sigma \times T^2)$ is what we denoted as d_ϑ. Adding that to the other contributions proves (iii). □

5.2. **Examples of building blocks.** We now give examples of building blocks. For simplicity, we restrict attention to blocks whose polarising lattice N has rank 1. We list the relevant data for the examples in Table 1.

Each family of blocks Z is obtained by blowing up Fano 3-folds Y of Picard rank 1. We list the index r of Y, the anticanonical degree $-K_Y^3$, the norm-square of the generator of the Picard lattice N of Y (which is isometric to the polarising lattice of Z), the third Betti number $b_3(Y)$, and the result of evaluating $c_2(Z)$ on the pull-back $H \in H^2(Z)$ of the generator $-\frac{1}{r}K_Y \in H^2(Y)$ (the latter number is needed to compute the Pontryagin class of the extra-twisted connected sums built from the block, although we do not do that in this paper).

Recall that the Picard lattice of a Fano 3-fold Y is $H^2(Y)$ equipped with the non-degenerate symmetric bilinear form $(A, B) \mapsto A.B.(-K_Y)$. For rank 1 Fanos, the norm-square of the generator $-\frac{1}{r}K_Y$ is thus simply computed as $\frac{1}{r^2}(-K_Y)^3$.

Example 5.3. If we ignore the desire for automorphisms, then we can simply take the list of rank 1 blocks from [12, Table 1]. These are obtained by blowing up a Fano 3-fold Y of Picard rank 1 along the transverse intersection C of two smooth anticanonical divisors. As explained in [12, §5.2], the resulting building block Z has $b_3(Z) = b_3(Y) + b_1(C) = b_3(Y) + (-K_Y)^3 + 2$. For the final piece of data we wish to include in Table 1, [18, (4.4)] gives $c_2(Z)H = \frac{24-K_Y^3}{r}$.

All other examples we consider will in fact be subfamilies of the families from Example 5.3 that admit automorphisms. For each suitable automorphism that we have found on some elements of the family, we list in Table 1 its order k and the rank $b_3^\Gamma(Z)$ of the invariant part of $H^3(Z)$ (so the number against $k = 1$ is $b_3(Z)$), and the number of isolated fixed points (among all elements of Γ). The formula (5.2) for the third Betti number of an extra-twisted connected sum involves $b_3^\Gamma(Z)$, while the computation of the $\bar{\nu}$-invariant in Theorem 2.5 relies on some further details about the fixed points that is not included in the table, but only in the descriptions of the individual examples.

The pattern is that we consider special elements Y of the given family of Fano 3-folds that admit a group of automorphisms Γ, whose fixed set is a union of a K3 divisor Σ and (possibly) some isolated fixed points. After blowing up a curve $C \subset \Sigma$ like in Example 5.3, one obtains a building block Z with automorphisms whose fixed sets are the proper transforms of the union of the fixed set of the corresponding automorphism on Y and a copy $\widetilde{C} \subset V \subset Z$ of C; it is a section of the exceptional set, which is a trivial $\mathbb{P}^1$-bundle over C.

In all but one of our examples (Example 5.9), the fixed set Z^γ is the same for all non-identity elements $\gamma \in \Gamma$. Because the cohomology of Z is Γ-invariant except in degree 3 (so that *e.g.* $b_2(Z/\Gamma) = b_2(Z) = 2$) we can in those cases easily compute $b_3^\Gamma(Z)$ from

$$\chi(Z/\Gamma) = \frac{1}{k}\chi(Z) + \frac{k-1}{k}\chi(Z^\gamma).$$

Example 5.4. Blocks with involutions were already considered in [31]. In some sense, the simplest way to obtain examples is to start from a Fano 3-fold X with even anti-canonical class $-K_X$ (*i.e.* $\mathbb{P}^3$ or a del Pezzo 3-fold), and consider a double cover Y of X branched over an anticanonical divisor; see [31, Examples 3.24 & 3.25]. Blowing up the double cover Y along a transverse intersection C of the ramification divisor and another anticanonical divisor yields a block Z with involution.

	Y	r	$-K_Y^3$	N	$b_3(Y)$	$c_2(Z)H$	Ex	k	$b_3^\Gamma(Z)$	#fix
1	$\mathbb{P}^3$	4	64	4	0	22	5.3	1	66	
2	Q	3	54	6	0	26	5.3	1	56	
3	V_1	2	8	2	42	16	5.3	1	52	
4	V_2	2	16	4	20	20	5.3	1	38	
5							5.4	2	18	
6	V_3	2	24	6	10	24	5.3	1	36	
7	V_4	2	32	8	4	28	5.3	1	38	
8	V_5	2	40	10	0	32	5.3	1	42	
9		1	2	2	104	26	5.3	1	108	
10							5.4	2	46	
11							5.9	3	24	2
12							5.10	5	8	1
13							5.9	6	4	2
14		1	4	4	60	28	5.3	1	66	
15							5.4	2	26	
16							5.8	3	12	1
17							5.6	4	6	
18		1	6	6	40	30	5.3	1	48	
19							5.4	2	18	
20							5.5	3	8	
21		1	8	8	28	32	5.3	1	38	
22							5.4	2	14	
23		1	10	10	20	34	5.3	1	32	
24							5.4	2	12	
25		1	12	12	14	36	5.3	1	28	
26		1	14	14	10	38	5.3	1	26	
27		1	16	16	6	40	5.3	1	24	
28		1	18	18	4	42	5.3	1	24	
29		1	22	22	0	46	5.3	1	24	

TABLE 1. Rank 1 building blocks

Similarly to Example 5.4, we can take X to be one of the two Fano 3-folds with index $r > 2$. Then the r-fold branched cover Y of X branched over an anticanonical divisor can be blown up along the intersection of the ramification locus with another anticanonical divisor of Y to give a building block Z with an automorphism of order r.

Example 5.5. For $X = Q \subset \mathbb{P}^4$ the quadric 3-fold (which has $r = 3$), Y is isomorphic to a complete intersection of a quadric and a cubic in $\mathbb{P}^5$ of the forms $X_1^2 + \cdots + X_5^2$ and $X_0^3 + F(X_1, \ldots, X_5)$, and the branch switching automorphisms correspond to multiplying the homogeneous coordinate X_0 by cube roots of unity. The fixed set is the anticanonical divisor $\{X_0 = 0\}$, which is smooth for a generic F. Blowing up a transverse intersection C with another anticanonical divisor yields a block Z with an automorphism group of order 3 (number 20 in Table 1); these building blocks are then special elements of the family 18 in the table, which was obtained in Example 5.3.

If we let $\tau \in \Gamma$ be the generator that multiplies X_0 by $\zeta^{-1} = e^{-\frac{2\pi i}{3}}$, then the fixed set $Z^\tau \subset Z$ consists of the proper transform Σ of the ramification locus and a section $\widetilde{C}$ of the exceptional set. Clearly τ acts on the normal bundle of Σ as multiplication by ζ^{-1}.

Example 5.6. For $X = \mathbb{P}^3$ (which has $r = 4$), Y is isomorphic to a quartic in $\mathbb{P}^4$ with defining equation of the form $X_0^4 + F(X_1, X_2, X_3, X_4)$, giving entry 17 in Table 1. These are special elements of the family 14 of blocks obtained in Example 5.3.

Further, we can of course also consider this as a family of blocks with involution; then we recover a subfamily of family 15, which already came up in Example 5.4.

Before producing some examples with isolated fixpoints, let us recall that we need to find a generator τ of Γ that acts on the normal bundle ν_Σ by $\zeta^{-1} = e^{-\frac{2\pi i}{k}}$. Then by Remark 2.6, the contribution of the fixpoint set to the extended ν-invariant is given by $D_{\tau^\varepsilon}(Z)$ with ε as in equation (1.5), where for brevity, we write $D_\gamma(Z)$ instead of $D_\gamma(V)$ as in Definition 2.4.

Remark 5.7. While the action of Γ on the normal bundle of the fixed curve $\widetilde{C}$ does not affect the $\bar{\nu}$-invariant by Theorem 2.5, let us mention that it is easy to describe in a uniform way in all our examples.

The exceptional divisor E in Z is biholomorphic to $C \times \mathbb{P}^1$. We can choose the identification so that $C \times \{(1 : 0)\}$ is the intersection $E \cap \Sigma$, while $C \times \{(0 : 1)\}$ is the 1-dimensional component $\widetilde{C}$ of the fixed set of Γ in Z. The action of $\tau \in \Gamma$ on E is trivial on the C factor, and can be identified with $(Y_0 : Y_1) \mapsto (\zeta Y_0 : Y_1)$ on the $\mathbb{P}^1$ factor, for ζ such that τ acts on ν_Σ by ζ^{-1}.

Then τ acts on the normal bundle of $\tilde{C}$ in E (which is trivial) as multiplication by ζ. If we write the normal bundle of $\widetilde{C}$ in Z as a direct sum of this trivial bundle and another line bundle, then (because $\widetilde{C}$ is contained in V which has a Calabi-Yau structure preserved by Γ) the second summand must be isomorphic to $T^*\widetilde{C}$, and τ must act on it as multiplication by ζ^{-1}.

Example 5.8. Consider a smooth quartic $Y \subset \mathbb{P}^4$ of the form $X_0^3 X_1 + F(X_1, X_2, X_3, X_4)$. Multiplying X_0 by a primitive third root of unity, say $\zeta^{-1} = e^{-\frac{2\pi i}{3}}$, defines an automorphism τ of order 3. Its fixed set is the union of the K3 surface $\Sigma = \{X_0 = 0\}$ and the isolated point $(1 : 0 : 0 : 0 : 0)$.

Blowing up Y along the intersection C (a quartic plane curve) of Σ with another anti-canonical divisor yields a building block Z with automorphism group $\Gamma \cong \mathbb{Z}/3$, number 16 in Table 1. It is a different subfamily of family 14 than the one considered in Example 5.6.

The block Z has $\chi(Z) = -60$. The fixed set Z^τ of τ is the union of the proper transform of Σ, a section $\widetilde{C}$ of the exceptional divisor and the isolated fixed point, so its Euler characteristic is 21. Thus $\chi(Z/\Gamma) = -\frac{1}{3}60 + \frac{2}{3}21 = -6$, and hence $b_3^\Gamma(Z) = 12$.

Clearly, τ acts on the normal bundle of Σ as multiplication by ζ^{-1}. Meanwhile, in the affine chart $(z_1, \ldots, z_4) \mapsto (1 : z_1 : \cdots : z_4)$, the action of τ is represented by multiplication with ζ. Hence, the action of τ on the tangent space at $(1 : 0 : 0 : 0 : 0)$ is diagonal with eigenvalue ζ. We can now compute $D_{\tau^\varepsilon}(Z) = 2\varepsilon$ for $\varepsilon = \pm 1$.

Example 5.9. Consider a smooth sextic Y in the weighted projective space $\mathbb{P}^4(1^4, 3)$ of the form $X_0^6 + F(X_1, X_2, X_3) + X_4^2$. Multiplying X_0 by a primitive 6th root of unity, say $\zeta^{-1} = e^{-\frac{\pi i}{3}}$, defines an automorphism τ of order 6. Its fixed set is the K3 surface $\Sigma = \{X_0 = 0\}$. In addition, τ^2 has two isolated fixed points, at $x_\pm = (1 : 0 : 0 : 0 : \pm i)$, which are swapped by τ.

Blowing up Y along the intersection C of Σ with another anticanonical divisor that is stable under τ as a set yields a building block Z with automorphism group $\Gamma \cong \mathbb{Z}/6$, line 13 in

Table 1. It can be considered as a more special subfamily of family 9 appearing in Example 5.3 or of family 10 of involution blocks from Example 5.4. But if we consider it as a block with automorphism group of order 3 (number 11 in the table), then that is distinct from the previous examples.

Clearly, τ acts on the normal bundle of Σ as multiplication by ζ^{-1}. Meanwhile, in the affine chart $(z_1, \dots, z_4) \mapsto (1 : z_1 : \dots : z_4)$, the action of τ is represented by $(z_1, \dots, z_4) \mapsto (\zeta z_1, \zeta z_2, \zeta z_3, \zeta^3 z_4)$. The isolated fixed points correspond to $(0, 0, 0, \pm i)$, and have tangent space $z_4 = 0$. Thus the action of τ^2 on the tangent spaces is diagonal with eigenvalue ζ^2. Again, we find $D_{\tau^\varepsilon}(Z) = 2\varepsilon$ for the automorphism group $\mathbb{Z}/6$, and $D_{\tau^\varepsilon} = 4\varepsilon$ if we restrict to the automorphism group $\mathbb{Z}/3$.

We have $\chi(Z) = -102$, while the fixed set Z^{τ^2} of τ^2 is the union of the proper transform of Σ, a copy $\widetilde{C}$ of C and the two isolated fixed points, so has Euler characteristic 24. In the case where we consider the automorphism group $\Gamma' \cong \mathbb{Z}/3$ generated by τ^2, we thus find $\chi(Z/\Gamma') = -\frac{1}{3}102 + \frac{2}{3}24 = -18$, so $b_3^{\Gamma'}(Z) = 24$.

In turn, we can consider Z/Γ as a $\mathbb{Z}/2$ quotient of Z/Γ' with fixed set of Euler characteristic $\chi(\Sigma) + \chi(\widetilde{C}) = 22$. Thus $\chi(Z/\Gamma) = -\frac{1}{2}18 + \frac{1}{2}22 = 2$, and $b_3^{\Gamma}(Z) = 4$.

Example 5.10. Consider again a smooth sextic Y in the weighted projective space $\mathbb{P}^4(1^4, 3)$, but now of the form $X_0^5 X_1 + F(X_1, \dots, X_3) + X_4^2$. Multiplying X_0 by a primitive 5th root of unity, say $\zeta^{-1} = e^{-\frac{2\pi i}{5}}$, defines an automorphism τ of order 5. Its fixed set is the union of the K3 surface $\Sigma = \{X_0 = 0\}$ and the isolated point $(1 : 0 : 0 : 0 : 0)$.

Blowing up Y along the intersection C of Σ with another anticanonical divisor yields a building block Z with automorphism group $\Gamma \cong \mathbb{Z}/5$. It can be considered as another more special subfamily (entry 12 in Table 1) of the family 9 that we already considered in Example 5.9.

Clearly, τ acts on the normal bundle of Σ as multiplication by ζ^{-1}. Meanwhile, in the affine chart $(z_1, \dots, z_4) \mapsto (1 : z_1 : \dots : z_4)$, the action of τ is represented by $(z_1, \dots, z_4) \mapsto (\zeta z_1, \zeta z_2, \zeta z_3, \zeta^3 z_4)$. The tangent space at the fixed point is $z_1 = 0$, so the eigenvalues of τ on the tangent space are ζ, ζ and ζ^3. In Example 2.7, we have computed $D_{\tau^\varepsilon}(Z) = -\frac{24}{5\varepsilon}$ for $\varepsilon \in \{\pm 1, \pm 2\}$.

We have $\chi(Z) = -102$, while the fixed set Z^τ of τ is the union of the proper transform of Σ, a copy $\widetilde{C}$ of C, and the isolated fixed point. Thus Z^τ has Euler characteristic 23, so we find $\chi(Z/\Gamma) = -\frac{1}{5}102 + \frac{4}{5}23 = -2$, and $b_3^{\Gamma}(Z) = 8$.

In order to construct extra-twisted connected sums from our examples of blocks, we need to note that they have a genericity property described in Definition 1.15.

Proposition 5.11. *Each family $\mathcal{Z}$ of blocks above is (N, Amp)-polarised, where N is the polarising lattice of the family, and $\mathrm{Amp} \subset N(\mathbb{R})$ is one of the two open half-lines.*

Proof. For the families of blocks without automorphism in Example 5.3, this is just an instance of [12, Proposition 6.9], which is a consequence of the results of Beauville [2] on anticanonical divisors in Fano 3-folds. The same argument applies to the families of blocks with involution that are obtained in Examples 5.4, 5.5 or 5.6 from a cover of a Fano 3-fold X branched over an anticanonical divisor Σ, since Σ can be any smooth anticanonical divisor in X.

In Example 5.8, the K3 divisor Σ is a hypersurface in $\mathbb{P}^3$ defined by the quartic polynomial F. Clearly F can be chosen to be any smooth quartic, so a generic K3 surface with Picard lattice containing an ample primitive class of norm-square 4 will appear this way. Indeed, in particular any K3 surface with Picard lattice exactly $\langle 4 \rangle$ can embedded as a quartic in $\mathbb{P}^3$ (see Saint-Donat [34, Theorem 6.1]).

Similarly, we see directly that any K3 that is a double cover of $\mathbb{P}^2$ branched over a smooth sextic curve can appear as the K3 divisor in blocks of the classes from Examples 5.9 and 5.10, and a generic K3 surfaces whose Picard lattice contains an ample class of norm-square 2 can be presented that way (see Reid [33, Theorem 3.8(d)]). □

5.3. **Examples of matchings.** We now study the matchings that can be produced from the blocks in the previous section. Table 2 lists all extra-twisted connected sums that can be made from the blocks in Table 1, except those where both blocks have trivial automorphism group, which were studied in [13, 27]. Note that some examples with $k_\pm \leq 2$ were already considered in [13, 14, 31], in particular tables 4 and 5 in [31] contain some of the examples with $k_+ \leq 2$, $k_- = 2$ with additional information on $p_1(TM)$ and the torsion in $H^4(M)$. Table 2 contains 192 examples where $k_- \geq 3$; these are genuinely new.

We explained in §3.1 one way to find all gluing matrices for a given pair of orders (k_+, k_-) of automorphism groups. The gluing matrix determines the gluing angle ϑ, and for each pair of rank 1 blocks one can then determine whether there is a corresponding configuration as explained in §1.4. However, we find it convenient here to do these steps in the opposite order, and first enumerate all possible configurations involving the blocks from Table 1.

Given k_+ and k_- and a configuration, there may be several different gluing matrices G that have the right gluing angle and $\det G = -k_+k_-$, and several different choices of blocks Z_+, Z_- with the right polarising lattices and automorphism groups of order $k_\pm$. Each such choice yields a family of extra-twisted connected sums M by application of Proposition 1.16. The fundamental group $\pi_1(M)$ depends only on the gluing matrix, while $b_3(M)$ depends on the choices of Z_+ and Z_-. The invariant $\bar{\nu}(M)$ depends on the gluing matrix together with data for the isolated fixed points of the automorphisms on $Z_\pm$. It turns out that for those pairs of configuration and gluing matrix where there is more than one choice of (Z_+, Z_-), there is never any ambiguity in the isolated fixed point data, so in practice, $\bar{\nu}(M)$ only depends on the gluing matrix.

We therefore organise Table 2 with the data about the extra-twisted connected sums from blocks with polarising lattices of rank 1 as follows. For each extra-twisted connected sum we first list the orders $k_\pm$ of $\Gamma_\pm$, the even lattice describing the configuration of the K3 matching and $(\cos\vartheta)^2$ of the gluing angle ϑ as determined by (1.14). Then follow the building blocks Z_+ and Z_- (the numbers referring to the entries in Table 1) and the third Betti number of the extra-twisted connected sum, the gluing matrix $G = \left(\begin{smallmatrix} m & p \\ n & q \end{smallmatrix}\right)$, and the parameters $\varepsilon_\pm$ (see Proposition 1.8). By Remark 3.6, we always assume that m, n, $p > 0$ and $q < 0$. Moreover, if $k_+ = k_-$ we may swap the blocks if necessary to make sure that $m + q \leq 0$. Finally, we list the value of the $\bar{\nu}$-invariant. Where there are several different choices of Z_+ with the same k_+, k_- and configuration they are separated by commas, as are the corresponding values of $b_3(M)$, while the different choices of the gluing matrix G (and the corresponding $\varepsilon_+, \varepsilon_-$ and $\bar{\nu}(M)$) are listed on separate rows. The number at the very left is the running number of the first example in the line, for example, the third line contains examples 5, 6 and 7.

Remark 5.12. If a non simply-connected extra-twisted connected sum has a nontrivial covering constructed with the same groups Γ_+, Γ_-, see Proposition 3.9, then it is listed a few lines above, *e.g.* entry 20 is the universal cover of entry 22. If one needs to pass to a subgroup of at least one of the groups Γ_+, Γ_-, then one should determine $\tilde{k}_+$, $\tilde{k}_-$ and the gluing matrix by (3.13b) and find the covering in a different section of the table, possibly with the roles of Z_+ and Z_- swapped, except if $\vartheta = \frac{\pi}{2}$. In the latter case, the universal cover is an ordinary twisted sum of the type discussed in [13, 27], and therefore not listed here.

	k_+	k_-	$N_+ + N_-$	$\cos^2\vartheta$	Z_+	Z_-	$b_3(M)$	G	ε_+	ε_-	$\bar\nu(M)$
1	1	2	$\begin{pmatrix}2&2\\2&4\end{pmatrix}$	1/2	3, 9	5	92, 148	$\begin{pmatrix}1&1\\1&-1\end{pmatrix}$		1	−39
						15	100, 156				
5			$\begin{pmatrix}4&2\\2&2\end{pmatrix}$	1/2	1, 4, 14	10	134, 106, 134	$\begin{pmatrix}1&1\\1&-1\end{pmatrix}$		1	−39
8			$\begin{pmatrix}4&4\\4&8\end{pmatrix}$	1/2	1, 4, 14	22	102, 74, 102	$\begin{pmatrix}1&1\\1&-1\end{pmatrix}$		1	−39
11			$\begin{pmatrix}8&4\\4&4\end{pmatrix}$	1/2	7, 21	5	78, 78	$\begin{pmatrix}1&1\\1&-1\end{pmatrix}$		1	−39
						15	86, 86				
15			$\begin{pmatrix}12&6\\6&6\end{pmatrix}$	1/2	25	19	68	$\begin{pmatrix}1&1\\1&-1\end{pmatrix}$		1	−39
16			$\begin{pmatrix}16&4\\4&2\end{pmatrix}$	1/2	27	10	92	$\begin{pmatrix}1&1\\1&-1\end{pmatrix}$		1	−39
17			$\begin{pmatrix}16&8\\8&8\end{pmatrix}$	1/2	27	22	60	$\begin{pmatrix}1&1\\1&-1\end{pmatrix}$		1	−39
18			$\begin{pmatrix}18&6\\6&4\end{pmatrix}$	1/2	28	5, 15	64, 72	$\begin{pmatrix}1&1\\1&-1\end{pmatrix}$		1	−39
20	1	3	$\begin{pmatrix}2&2\\2&6\end{pmatrix}$	1/3	3, 9	20	82, 138	$\begin{pmatrix}1&1\\2&-1\end{pmatrix}$		−1	−19
22								$\begin{pmatrix}1&2\\1&-1\end{pmatrix}$		1	−35
24			$\begin{pmatrix}4&4\\4&6\end{pmatrix}$	2/3	1, 4, 14	20	96, 68, 96	$\begin{pmatrix}1&1\\1&-2\end{pmatrix}$		−1	−43
27								$\begin{pmatrix}2&1\\1&-1\end{pmatrix}$		1	−59
30			$\begin{pmatrix}6&2\\2&2\end{pmatrix}$	1/3	2, 6, 18	11	102, 82, 94	$\begin{pmatrix}1&1\\2&-1\end{pmatrix}$		−1	−23
33								$\begin{pmatrix}1&2\\1&-1\end{pmatrix}$		1	−31
36			$\begin{pmatrix}6&4\\4&4\end{pmatrix}$	2/3	2, 6, 18	16	90, 70, 82	$\begin{pmatrix}1&1\\1&-2\end{pmatrix}$		−1	−45
39								$\begin{pmatrix}2&1\\1&-1\end{pmatrix}$		1	−57
42			$\begin{pmatrix}8&4\\4&6\end{pmatrix}$	1/3	7, 21	20	68, 68	$\begin{pmatrix}1&1\\2&-1\end{pmatrix}$		−1	−19
44								$\begin{pmatrix}1&2\\1&-1\end{pmatrix}$		1	−35
46			$\begin{pmatrix}12&4\\4&2\end{pmatrix}$	2/3	25	11	74	$\begin{pmatrix}1&1\\1&-2\end{pmatrix}$		−1	−47
47								$\begin{pmatrix}2&1\\1&-1\end{pmatrix}$		1	−55
48			$\begin{pmatrix}12&4\\4&4\end{pmatrix}$	1/3	25	16	62	$\begin{pmatrix}1&1\\2&-1\end{pmatrix}$		−1	−21
49								$\begin{pmatrix}1&2\\1&-1\end{pmatrix}$		1	−33
50			$\begin{pmatrix}16&8\\8&6\end{pmatrix}$	2/3	27	20	54	$\begin{pmatrix}1&1\\1&-2\end{pmatrix}$		−1	−43
51								$\begin{pmatrix}2&1\\1&-1\end{pmatrix}$		1	−59
52			$\begin{pmatrix}18&6\\6&6\end{pmatrix}$	1/3	28	20	54	$\begin{pmatrix}1&1\\2&-1\end{pmatrix}$		−1	−19
53								$\begin{pmatrix}1&2\\1&-1\end{pmatrix}$		1	−35
54	1	4	$\begin{pmatrix}2&2\\2&4\end{pmatrix}$	1/2	3, 9	17	80, 136	$\begin{pmatrix}1&1\\2&-2\end{pmatrix}$		−1	−21
56								$\begin{pmatrix}2&2\\1&-1\end{pmatrix}$		1	−57
58			$\begin{pmatrix}4&2\\2&4\end{pmatrix}$	1/4	1, 4, 14	17	94, 66, 94	$\begin{pmatrix}1&1\\3&-1\end{pmatrix}$		−1	3
61								$\begin{pmatrix}1&3\\1&-1\end{pmatrix}$		1	−33
64			$\begin{pmatrix}8&4\\4&4\end{pmatrix}$	1/2	7, 21	17	66, 66	$\begin{pmatrix}1&1\\2&-2\end{pmatrix}$		−1	−21
66								$\begin{pmatrix}2&2\\1&-1\end{pmatrix}$		1	−57
68			$\begin{pmatrix}12&6\\6&4\end{pmatrix}$	3/4	25	17	56	$\begin{pmatrix}1&1\\1&-3\end{pmatrix}$		−1	−45
69								$\begin{pmatrix}3&1\\1&-1\end{pmatrix}$		1	−81
70			$\begin{pmatrix}16&4\\4&4\end{pmatrix}$	1/4	27	17	52	$\begin{pmatrix}1&1\\3&-1\end{pmatrix}$		−1	3
71								$\begin{pmatrix}1&3\\1&-1\end{pmatrix}$		1	−33

Table 2: Examples of extra-twisted connected sums

	k_+	k_-	$N_+{+}N_-$	$\cos^2\vartheta$	Z_+	Z_-	$b_3(M)$	G	ε_+	ε_-	$\bar\nu(M)$
72			$\left(\begin{smallmatrix}18&6\\6&4\end{smallmatrix}\right)$	1/2	28	17	52	$\left(\begin{smallmatrix}1&1\\2&-2\end{smallmatrix}\right)$		-1	-21
73								$\left(\begin{smallmatrix}2&2\\1&-1\end{smallmatrix}\right)$		1	-57
74	1	5	$\left(\begin{smallmatrix}10&2\\2&2\end{smallmatrix}\right)$	1/5	8, 23	12	72, 62	$\left(\begin{smallmatrix}1&1\\4&-1\end{smallmatrix}\right)$		-1	21
76								$\left(\begin{smallmatrix}1&2\\2&-1\end{smallmatrix}\right)$		2	-15
78								$\left(\begin{smallmatrix}1&4\\1&-1\end{smallmatrix}\right)$		1	-27
80			$\left(\begin{smallmatrix}10&4\\4&2\end{smallmatrix}\right)$	4/5	8, 23	12	72, 62	$\left(\begin{smallmatrix}1&1\\1&-4\end{smallmatrix}\right)$		-1	-51
82								$\left(\begin{smallmatrix}2&1\\1&-2\end{smallmatrix}\right)$		-2	-63
84								$\left(\begin{smallmatrix}4&1\\1&-1\end{smallmatrix}\right)$		1	-99
86	1	6	$\left(\begin{smallmatrix}4&2\\2&2\end{smallmatrix}\right)$	1/2	1, 4, 14	13	92, 64, 92	$\left(\begin{smallmatrix}1&1\\3&-3\end{smallmatrix}\right)$		-1	-1
89								$\left(\begin{smallmatrix}3&3\\1&-1\end{smallmatrix}\right)$		1	-77
92			$\left(\begin{smallmatrix}6&2\\2&2\end{smallmatrix}\right)$	1/3	2, 6, 18	13	82, 62, 74	$\left(\begin{smallmatrix}1&1\\4&-2\end{smallmatrix}\right)$		-1	23
95								$\left(\begin{smallmatrix}2&4\\1&-1\end{smallmatrix}\right)$		1	-53
98			$\left(\begin{smallmatrix}12&2\\2&2\end{smallmatrix}\right)$	1/6	25	13	54	$\left(\begin{smallmatrix}1&1\\5&-1\end{smallmatrix}\right)$		-1	47
99								$\left(\begin{smallmatrix}1&5\\1&-1\end{smallmatrix}\right)$		1	-29
100			$\left(\begin{smallmatrix}12&4\\4&2\end{smallmatrix}\right)$	2/3	25	13	54	$\left(\begin{smallmatrix}1&1\\2&-4\end{smallmatrix}\right)$		-1	-25
101								$\left(\begin{smallmatrix}4&2\\1&-1\end{smallmatrix}\right)$		1	-101
102			$\left(\begin{smallmatrix}16&4\\4&2\end{smallmatrix}\right)$	1/2	27	13	50	$\left(\begin{smallmatrix}1&1\\3&-3\end{smallmatrix}\right)$		-1	-1
103								$\left(\begin{smallmatrix}3&3\\1&-1\end{smallmatrix}\right)$		1	-77
104	2	2	$\left(\begin{smallmatrix}2&0\\0&2\end{smallmatrix}\right)$	0	10	10	115	$\left(\begin{smallmatrix}0&2\\2&0\end{smallmatrix}\right)$	1	1	0
105			$\left(\begin{smallmatrix}2&1\\1&2\end{smallmatrix}\right)$	1/4	10	10	114	$\left(\begin{smallmatrix}1&1\\3&-1\end{smallmatrix}\right)$	1	1	-27
106								$\left(\begin{smallmatrix}1&3\\1&-1\end{smallmatrix}\right)$	1	1	-27
107			$\left(\begin{smallmatrix}2&0\\0&4\end{smallmatrix}\right)$	0	10	5, 15	87, 95	$\left(\begin{smallmatrix}0&2\\2&0\end{smallmatrix}\right)$	1	1	0
109			$\left(\begin{smallmatrix}2&0\\0&6\end{smallmatrix}\right)$	0	10	19	87	$\left(\begin{smallmatrix}0&2\\2&0\end{smallmatrix}\right)$	1	1	0
110			$\left(\begin{smallmatrix}2&3\\3&6\end{smallmatrix}\right)$	3/4	10	19	86	$\left(\begin{smallmatrix}1&1\\1&-3\end{smallmatrix}\right)$	1	1	-51
111								$\left(\begin{smallmatrix}3&1\\1&-1\end{smallmatrix}\right)$	1	1	-51
112			$\left(\begin{smallmatrix}2&0\\0&8\end{smallmatrix}\right)$	0	10	22	83	$\left(\begin{smallmatrix}0&2\\2&0\end{smallmatrix}\right)$	1	1	0
113			$\left(\begin{smallmatrix}2&2\\2&8\end{smallmatrix}\right)$	1/4	10	22	82	$\left(\begin{smallmatrix}1&1\\3&-1\end{smallmatrix}\right)$	1	1	-27
114								$\left(\begin{smallmatrix}1&3\\1&-1\end{smallmatrix}\right)$	1	1	-27
115			$\left(\begin{smallmatrix}2&0\\0&10\end{smallmatrix}\right)$	0	10	24	81	$\left(\begin{smallmatrix}0&2\\2&0\end{smallmatrix}\right)$	1	1	0
116			$\left(\begin{smallmatrix}4&0\\0&4\end{smallmatrix}\right)$	0	5, 15	5	59, 67	$\left(\begin{smallmatrix}0&2\\2&0\end{smallmatrix}\right)$	1	1	0
						15	67, 75				
120			$\left(\begin{smallmatrix}4&2\\2&4\end{smallmatrix}\right)$	1/4	5, 15	5	58, 66	$\left(\begin{smallmatrix}1&1\\3&-1\end{smallmatrix}\right)$	1	1	-27
						15	66, 74				
124								$\left(\begin{smallmatrix}1&3\\1&-1\end{smallmatrix}\right)$	1	1	-27
128			$\left(\begin{smallmatrix}4&0\\0&6\end{smallmatrix}\right)$	0	5, 15	19	59, 67	$\left(\begin{smallmatrix}0&2\\2&0\end{smallmatrix}\right)$	1	1	0
130			$\left(\begin{smallmatrix}4&0\\0&8\end{smallmatrix}\right)$	0	5, 15	22	55, 63	$\left(\begin{smallmatrix}0&2\\2&0\end{smallmatrix}\right)$	1	1	0
132			$\left(\begin{smallmatrix}4&0\\0&10\end{smallmatrix}\right)$	0	5, 15	24	53, 61	$\left(\begin{smallmatrix}0&2\\2&0\end{smallmatrix}\right)$	1	1	0
134			$\left(\begin{smallmatrix}6&0\\0&6\end{smallmatrix}\right)$	0	19	19	59	$\left(\begin{smallmatrix}0&2\\2&0\end{smallmatrix}\right)$	1	1	0

Table 2: Examples of extra-twisted connected sums

	k_+	k_-	$N_+ + N_-$	$\cos^2\vartheta$	Z_+	Z_-	$b_3(M)$	G	ε_+	ε_-	$\bar\nu(M)$
135			$\begin{pmatrix}6&3\\3&6\end{pmatrix}$	1/4	19	19	58	$\begin{pmatrix}1&1\\3&-1\end{pmatrix}$	1	1	−27
136								$\begin{pmatrix}1&3\\1&-1\end{pmatrix}$	1	1	−27
137			$\begin{pmatrix}6&0\\0&8\end{pmatrix}$	0	19	22	55	$\begin{pmatrix}0&2\\2&0\end{pmatrix}$	1	1	0
138			$\begin{pmatrix}6&6\\6&8\end{pmatrix}$	3/4	19	22	54	$\begin{pmatrix}1&1\\1&-3\end{pmatrix}$	1	1	−51
139								$\begin{pmatrix}3&1\\1&-1\end{pmatrix}$	1	1	−51
140			$\begin{pmatrix}6&0\\0&10\end{pmatrix}$	0	19	24	53	$\begin{pmatrix}0&2\\2&0\end{pmatrix}$	1	1	0
141			$\begin{pmatrix}8&0\\0&8\end{pmatrix}$	0	22	22	51	$\begin{pmatrix}0&2\\2&0\end{pmatrix}$	1	1	0
142			$\begin{pmatrix}8&4\\4&8\end{pmatrix}$	1/4	22	22	50	$\begin{pmatrix}1&1\\3&-1\end{pmatrix}$	1	1	−27
143								$\begin{pmatrix}1&3\\1&-1\end{pmatrix}$	1	1	−27
144			$\begin{pmatrix}8&0\\0&10\end{pmatrix}$	0	22	24	49	$\begin{pmatrix}0&2\\2&0\end{pmatrix}$	1	1	0
145			$\begin{pmatrix}10&0\\0&10\end{pmatrix}$	0	24	24	47	$\begin{pmatrix}0&2\\2&0\end{pmatrix}$	1	1	0
146			$\begin{pmatrix}10&5\\5&10\end{pmatrix}$	1/4	24	24	46	$\begin{pmatrix}1&1\\3&-1\end{pmatrix}$	1	1	−27
147								$\begin{pmatrix}1&3\\1&-1\end{pmatrix}$	1	1	−27
148	2	3	$\begin{pmatrix}2&2\\2&4\end{pmatrix}$	1/2	10	16	80	$\begin{pmatrix}1&1\\3&-3\end{pmatrix}$	1	−1	−33
149								$\begin{pmatrix}3&3\\1&-1\end{pmatrix}$	1	1	−45
150			$\begin{pmatrix}2&2\\2&6\end{pmatrix}$	1/3	10	20	76	$\begin{pmatrix}2&1\\4&-1\end{pmatrix}$	1	1	−47
151								$\begin{pmatrix}1&4\\1&-2\end{pmatrix}$	1	−1	−31
152			$\begin{pmatrix}4&2\\2&2\end{pmatrix}$	1/2	5, 15	11	64, 72	$\begin{pmatrix}1&1\\3&-3\end{pmatrix}$	1	−1	−35
154								$\begin{pmatrix}3&3\\1&-1\end{pmatrix}$	1	1	−43
156			$\begin{pmatrix}4&2\\2&6\end{pmatrix}$	1/6	5, 15	20	48, 56	$\begin{pmatrix}1&1\\5&-1\end{pmatrix}$	1	−1	−7
158								$\begin{pmatrix}1&5\\1&-1\end{pmatrix}$	1	1	−23
160			$\begin{pmatrix}4&4\\4&6\end{pmatrix}$	2/3	5, 15	20	48, 56	$\begin{pmatrix}4&1\\2&-1\end{pmatrix}$	1	−1	−31
162								$\begin{pmatrix}1&2\\1&-4\end{pmatrix}$	1	1	−47
164			$\begin{pmatrix}6&2\\2&2\end{pmatrix}$	1/3	19	11	64	$\begin{pmatrix}2&1\\4&-1\end{pmatrix}$	1	1	−43
165								$\begin{pmatrix}1&4\\1&-2\end{pmatrix}$	1	−1	−35
166			$\begin{pmatrix}6&2\\2&4\end{pmatrix}$	1/6	19	16	52	$\begin{pmatrix}1&1\\5&-1\end{pmatrix}$	1	−1	−9
167								$\begin{pmatrix}1&5\\1&-1\end{pmatrix}$	1	1	−21
168			$\begin{pmatrix}6&4\\4&4\end{pmatrix}$	2/3	19	16	52	$\begin{pmatrix}4&1\\2&-1\end{pmatrix}$	1	−1	−33
169								$\begin{pmatrix}1&2\\1&-4\end{pmatrix}$	1	1	−45
170			$\begin{pmatrix}8&4\\4&4\end{pmatrix}$	1/2	22	16	48	$\begin{pmatrix}1&1\\3&-3\end{pmatrix}$	1	−1	−33
171								$\begin{pmatrix}3&3\\1&-1\end{pmatrix}$	1	1	−45
172			$\begin{pmatrix}8&4\\4&6\end{pmatrix}$	1/3	22	20	44	$\begin{pmatrix}2&1\\4&-1\end{pmatrix}$	1	1	−47
173								$\begin{pmatrix}1&4\\1&-2\end{pmatrix}$	1	−1	−31
174	2	4	$\begin{pmatrix}2&1\\1&4\end{pmatrix}$	1/8	10	17	74	$\begin{pmatrix}1&1\\7&-1\end{pmatrix}$	1	−1	15
175								$\begin{pmatrix}1&7\\1&-1\end{pmatrix}$	1	1	−21
176			$\begin{pmatrix}6&3\\3&4\end{pmatrix}$	3/8	19	17	46	$\begin{pmatrix}1&1\\5&-3\end{pmatrix}$	1	−1	−9
177								$\begin{pmatrix}3&1\\5&-1\end{pmatrix}$	1	1	−69
178								$\begin{pmatrix}1&5\\1&-3\end{pmatrix}$	1	−1	−33

Table 2: Examples of extra-twisted connected sums

	k_+	k_-	$N_+ + N_-$	$\cos^2\vartheta$	Z_+	Z_-	$b_3(M)$	G	ε_+	ε_-	$\bar\nu(M)$
179								$\begin{pmatrix} 3 & 5 \\ 1 & -1 \end{pmatrix}$	1	1	-45
180			$\begin{pmatrix} 8 & 2 \\ 2 & 4 \end{pmatrix}$	1/8	22	17	42	$\begin{pmatrix} 1 & 1 \\ 7 & -1 \end{pmatrix}$	1	-1	15
181								$\begin{pmatrix} 1 & 7 \\ 1 & -1 \end{pmatrix}$	1	1	-21
182			$\begin{pmatrix} 10 & 5 \\ 5 & 4 \end{pmatrix}$	5/8	24	17	40	$\begin{pmatrix} 1 & 1 \\ 3 & -5 \end{pmatrix}$	1	-1	-33
183								$\begin{pmatrix} 5 & 1 \\ 3 & -1 \end{pmatrix}$	1	-1	-9
184								$\begin{pmatrix} 1 & 3 \\ 1 & -5 \end{pmatrix}$	1	1	-45
185								$\begin{pmatrix} 5 & 3 \\ 1 & -1 \end{pmatrix}$	1	1	-69
186	2	5	$\begin{pmatrix} 4 & 2 \\ 2 & 2 \end{pmatrix}$	1/2	5, 15	12	48, 56	$\begin{pmatrix} 1 & 1 \\ 5 & -5 \end{pmatrix}$	1	-1	-15
188								$\begin{pmatrix} 5 & 5 \\ 1 & -1 \end{pmatrix}$	1	1	-63
190			$\begin{pmatrix} 10 & 2 \\ 2 & 2 \end{pmatrix}$	1/5	24	12	42	$\begin{pmatrix} 2 & 1 \\ 8 & -1 \end{pmatrix}$	1	-2	-27
191								$\begin{pmatrix} 1 & 8 \\ 1 & -2 \end{pmatrix}$	1	-2	-27
192			$\begin{pmatrix} 10 & 4 \\ 4 & 2 \end{pmatrix}$	4/5	24	12	42	$\begin{pmatrix} 8 & 1 \\ 2 & -1 \end{pmatrix}$	1	2	-51
193								$\begin{pmatrix} 1 & 2 \\ 1 & -8 \end{pmatrix}$	1	2	-51
194	2	6	$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$	1/4	10	13	72	$\begin{pmatrix} 1 & 1 \\ 9 & -3 \end{pmatrix}$	1	-1	35
195								$\begin{pmatrix} 3 & 9 \\ 1 & -1 \end{pmatrix}$	1	1	-41
196			$\begin{pmatrix} 6 & 1 \\ 1 & 2 \end{pmatrix}$	1/12	19	13	44	$\begin{pmatrix} 1 & 1 \\ 11 & -1 \end{pmatrix}$	1	-1	59
197								$\begin{pmatrix} 1 & 11 \\ 1 & -1 \end{pmatrix}$	1	1	-17
198			$\begin{pmatrix} 6 & 3 \\ 3 & 2 \end{pmatrix}$	3/4	19	13	44	$\begin{pmatrix} 1 & 1 \\ 3 & -9 \end{pmatrix}$	1	-1	-37
199								$\begin{pmatrix} 9 & 3 \\ 1 & -1 \end{pmatrix}$	1	1	-113
200			$\begin{pmatrix} 8 & 2 \\ 2 & 2 \end{pmatrix}$	1/4	22	13	40	$\begin{pmatrix} 1 & 1 \\ 9 & -3 \end{pmatrix}$	1	-1	35
201								$\begin{pmatrix} 3 & 9 \\ 1 & -1 \end{pmatrix}$	1	1	-41
202	3	3	$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$	0	11	11	71	$\begin{pmatrix} 0 & 3 \\ 3 & 0 \end{pmatrix}$	1	1	-8
203			$\begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}$	0	11	16	59	$\begin{pmatrix} 0 & 3 \\ 3 & 0 \end{pmatrix}$	1	1	-10
204			$\begin{pmatrix} 2 & 0 \\ 0 & 6 \end{pmatrix}$	0	11	20	55	$\begin{pmatrix} 0 & 3 \\ 3 & 0 \end{pmatrix}$	1	1	-12
205			$\begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$	0	16	16	47	$\begin{pmatrix} 0 & 3 \\ 3 & 0 \end{pmatrix}$	1	1	-12
206			$\begin{pmatrix} 4 & 0 \\ 0 & 6 \end{pmatrix}$	0	16	20	43	$\begin{pmatrix} 0 & 3 \\ 3 & 0 \end{pmatrix}$	1	1	-14
207			$\begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix}$	0	20	20	39	$\begin{pmatrix} 0 & 3 \\ 3 & 0 \end{pmatrix}$	1	1	-16
208			$\begin{pmatrix} 6 & 2 \\ 2 & 6 \end{pmatrix}$	1/9	20	20	38	$\begin{pmatrix} 1 & 1 \\ 8 & -1 \end{pmatrix}$	-1	-1	13
209								$\begin{pmatrix} 1 & 2 \\ 4 & -1 \end{pmatrix}$	1	1	-43
210								$\begin{pmatrix} 1 & 4 \\ 2 & -1 \end{pmatrix}$	-1	-1	-11
211								$\begin{pmatrix} 1 & 8 \\ 1 & -1 \end{pmatrix}$	1	1	-19
212			$\begin{pmatrix} 6 & 4 \\ 4 & 6 \end{pmatrix}$	4/9	20	20	38	$\begin{pmatrix} 1 & 1 \\ 5 & -4 \end{pmatrix}$	-1	-1	-11
213								$\begin{pmatrix} 2 & 1 \\ 5 & -2 \end{pmatrix}$	1	1	-67
214								$\begin{pmatrix} 1 & 5 \\ 1 & -4 \end{pmatrix}$	1	1	-43
215								$\begin{pmatrix} 2 & 5 \\ 1 & -2 \end{pmatrix}$	-1	-1	-35
216	3	4	$\begin{pmatrix} 2 & 2 \\ 2 & 4 \end{pmatrix}$	1/2	11	17	52	$\begin{pmatrix} 3 & 1 \\ 6 & -2 \end{pmatrix}$	1	1	-85
217								$\begin{pmatrix} 6 & 2 \\ 3 & -1 \end{pmatrix}$	1	-1	-1
218								$\begin{pmatrix} 1 & 3 \\ 2 & -6 \end{pmatrix}$	-1	1	-29

Table 2: Examples of extra-twisted connected sums

	k_+	k_-	$N_+{+}N_-$	$\cos^2\vartheta$	Z_+	Z_-	$b_3(M)$	G	ε_+	ε_-	$\bar\nu(M)$
219								$\left(\begin{smallmatrix}2&6\\1&-3\end{smallmatrix}\right)$	-1	-1	-41
220			$\left(\begin{smallmatrix}4&2\\2&4\end{smallmatrix}\right)$	1/4	16	17	40	$\left(\begin{smallmatrix}3&1\\9&-1\end{smallmatrix}\right)$	-1	1	-51
221								$\left(\begin{smallmatrix}1&9\\1&-3\end{smallmatrix}\right)$	1	-1	-27
222			$\left(\begin{smallmatrix}6&2\\2&4\end{smallmatrix}\right)$	1/6	20	17	36	$\left(\begin{smallmatrix}1&1\\10&-2\end{smallmatrix}\right)$	1	-1	-5
223								$\left(\begin{smallmatrix}2&2\\5&-1\end{smallmatrix}\right)$	1	1	-65
224								$\left(\begin{smallmatrix}1&5\\2&-2\end{smallmatrix}\right)$	-1	-1	-13
225								$\left(\begin{smallmatrix}2&10\\1&-1\end{smallmatrix}\right)$	-1	1	-25
226			$\left(\begin{smallmatrix}6&4\\4&4\end{smallmatrix}\right)$	2/3	20	17	36	$\left(\begin{smallmatrix}1&1\\4&-8\end{smallmatrix}\right)$	1	-1	-53
227								$\left(\begin{smallmatrix}8&4\\1&-1\end{smallmatrix}\right)$	-1	1	-73
228	3	5	$\left(\begin{smallmatrix}6&2\\2&2\end{smallmatrix}\right)$	1/3	20	12	38	$\left(\begin{smallmatrix}1&1\\10&-5\end{smallmatrix}\right)$	1	-1	-11
229								$\left(\begin{smallmatrix}1&2\\5&-5\end{smallmatrix}\right)$	-1	2	-7
230								$\left(\begin{smallmatrix}5&5\\2&-1\end{smallmatrix}\right)$	1	2	-23
231								$\left(\begin{smallmatrix}5&10\\1&-1\end{smallmatrix}\right)$	-1	1	-43
232	4	4	$\left(\begin{smallmatrix}4&0\\0&4\end{smallmatrix}\right)$	0	17	17	35	$\left(\begin{smallmatrix}0&4\\4&0\end{smallmatrix}\right)$	1	1	-36
233			$\left(\begin{smallmatrix}4&1\\1&4\end{smallmatrix}\right)$	1/16	17	17	34	$\left(\begin{smallmatrix}1&1\\15&-1\end{smallmatrix}\right)$	-1	-1	57
234								$\left(\begin{smallmatrix}1&3\\5&-1\end{smallmatrix}\right)$	1	1	-63
235								$\left(\begin{smallmatrix}1&5\\3&-1\end{smallmatrix}\right)$	-1	-1	9
236								$\left(\begin{smallmatrix}1&15\\1&-1\end{smallmatrix}\right)$	1	1	-15
237			$\left(\begin{smallmatrix}4&2\\2&4\end{smallmatrix}\right)$	1/4	17	17	34	$\left(\begin{smallmatrix}2&2\\6&-2\end{smallmatrix}\right)$	1	1	-87
238								$\left(\begin{smallmatrix}2&6\\2&-2\end{smallmatrix}\right)$	-1	-1	-15
239			$\left(\begin{smallmatrix}4&3\\3&4\end{smallmatrix}\right)$	9/16	17	17	34	$\left(\begin{smallmatrix}1&1\\7&-9\end{smallmatrix}\right)$	-1	-1	9
240								$\left(\begin{smallmatrix}3&1\\7&-3\end{smallmatrix}\right)$	1	1	-111
241								$\left(\begin{smallmatrix}1&7\\1&-9\end{smallmatrix}\right)$	1	1	-63
242								$\left(\begin{smallmatrix}3&7\\1&-3\end{smallmatrix}\right)$	-1	-1	-39
243	4	5	$\left(\begin{smallmatrix}4&2\\2&2\end{smallmatrix}\right)$	1/2	17	12	36	$\left(\begin{smallmatrix}2&1\\10&-5\end{smallmatrix}\right)$	-1	-2	-9
244								$\left(\begin{smallmatrix}1&2\\5&-10\end{smallmatrix}\right)$	1	2	-69
245								$\left(\begin{smallmatrix}10&5\\2&-1\end{smallmatrix}\right)$	-1	2	-33
246								$\left(\begin{smallmatrix}5&10\\1&-2\end{smallmatrix}\right)$	1	-2	-45
247	4	6	$\left(\begin{smallmatrix}4&1\\1&2\end{smallmatrix}\right)$	1/8	17	13	32	$\left(\begin{smallmatrix}1&1\\21&-3\end{smallmatrix}\right)$	1	-1	17
248								$\left(\begin{smallmatrix}3&3\\7&-1\end{smallmatrix}\right)$	1	1	-107
249								$\left(\begin{smallmatrix}1&7\\3&-3\end{smallmatrix}\right)$	-1	-1	5
250								$\left(\begin{smallmatrix}3&21\\1&-1\end{smallmatrix}\right)$	-1	1	-23
251	5	5	$\left(\begin{smallmatrix}2&0\\0&2\end{smallmatrix}\right)$	0	12	12	39	$\left(\begin{smallmatrix}0&5\\5&0\end{smallmatrix}\right)$	1	1	-48
252								$\left(\begin{smallmatrix}0&5\\5&0\end{smallmatrix}\right)$	2	-2	0
253	6	6	$\left(\begin{smallmatrix}2&0\\0&2\end{smallmatrix}\right)$	0	13	13	31	$\left(\begin{smallmatrix}0&6\\6&0\end{smallmatrix}\right)$	1	1	-76
254			$\left(\begin{smallmatrix}2&1\\1&2\end{smallmatrix}\right)$	1/4	13	13	30	$\left(\begin{smallmatrix}3&3\\9&-3\end{smallmatrix}\right)$	1	1	-151
255								$\left(\begin{smallmatrix}3&9\\3&-3\end{smallmatrix}\right)$	-1	-1	1

Table 2: Examples of extra-twisted connected sums

6. Proofs of some intermediate results

For completeness, we give short proofs of the claims (2.12), (2.22) and (4.12). This section does not attempt to be self-contained. Instead, we will state the analogue of statements in existing proofs, and add explanations only where we deviate from those.

6.1. **Adiabatic limits of twisted products.** Let $\Gamma \cong \mathbb{Z}/k$ be a finite group that acts effectively and isometrically on an even-dimensional manifold V with boundary ∂V. We assume that V has product geometry near ∂V. We consider $W = V/\Gamma$ as an orbifold with inertia orbifold ΛW.

Consider $S^1 \cong \mathbb{R}/\mathbb{Z}$ and let a generator $\gamma_0 \in \Gamma$ act by sending $[v] \in \mathbb{R}/\xi\mathbb{Z}$ to $\big[v + \frac{\xi}{k}\big]$. Then we will consider the Seifert fibration

$$p\colon M = \big(V \times S^1\big)/\Gamma \longrightarrow W\ , \tag{6.1}$$

where Γ acts diagonally on $V \times S^1$. We split $TM = TW \oplus TS^1$ by abuse of notation and consider a family of metrics

$$g^{TM}_\varepsilon = \varepsilon^{-2}\, g^{TW} \oplus g^{TS^1}$$

for $\varepsilon > 0$. The example we have in mind is of course $M_\pm$ with metric $g^{TM}_\varepsilon = \frac{1}{\varepsilon\zeta_\pm}\, g_{\pm,\varepsilon}$; see paragraph 2.1.1.

By a *Dirac bundle* we mean a Hermitian vector bundle with Hermitian connection and a compatible Clifford multiplication; see [28, def II.5.2]. We assume that V is equipped with a fixed Dirac bundle $E_V \to V$, on which Γ acts, preserving its structure. On M, we consider the Dirac bundle

$$E = p^* E_V/\Gamma \longrightarrow M\ . \tag{6.2}$$

We let $(e_1, \dots, e_m)$ denote a local orthonormal frame of TM for g^{TM}_1 such that e_1 is vertical and $e_2, \dots, e_m$ are horizontal. Clifford multiplication with e_i will be denoted c_i. Here, we assume that c_1 acts as $-i^{\frac{m+1}{2}} c_2 \cdots c_m$ on p^*E_V, so that the Clifford volume element $i^{\frac{m+1}{2}} c_1 \cdots c_m$ acts as 1. The examples we have in mind are the spinor bundle SV and the bundle of exterior forms $\Lambda^\bullet T^*V$ on V, leading to the spinor bundle and the bundle of even forms on M.

We consider a Dirac-type operator on E of the form

$$D_{M,\varepsilon} = D_{S^1} + \varepsilon D_W \tag{6.3}$$

as in [21, (2.3)], where $D_{S^1} = c_1 \nabla^E_{e_1}$ is the fibrewise Dirac operator. In the case of the odd signature operator on M, $D_W = B_W$ is the signature operator on the orbifold W.

In the case of the modified spin Dirac operator on M, we assume that SV admits a Γ-invariant spinor s such that ∇^{SV} is supported away from ∂V. If $D'_{M,\varepsilon}$ and D'_W denote the geometric Dirac operators on M and on the orbifold W with respect to the metrics above, equation (2.3) becomes

$$D'_{M,\varepsilon}(p^*s) = p^*(\varepsilon D'_W s) = \varepsilon\, p^*(f\, s + h\, c_1 s + r)\ ,$$

with f, h and r as before. We now consider the operator

$$\begin{aligned} D_W = D'_W &- \langle\,\cdot\,, s\rangle\,(f\, s + h\, c_1 s + r) - \langle\,\cdot\,, r\rangle\, s \\ &- \langle\,\cdot\,, c_1 s\rangle\,(h\, s - f\, c_1 s - c_1 r) + \langle\,\cdot\,, c_1 r\rangle\, c_1 s\ . \end{aligned} \tag{6.4}$$

Then (6.3) and (6.4) are equivalent to (2.4). We also conclude that $D_W - D'_W$ is supported away from ∂W by Property 2.1.3 (i) and self-adjoint.

The situation here is simpler than in [21] because D_W is independent of ε and

$$D_{S^1} D_W + D_W D_{S^1} = 0\ . \tag{6.5}$$

In the case of the modified spin Dirac operator, this follows from (6.4) because f, h, s and r all have vanishing vertical derivative.

Because we are in a local product situation, the space $L^2(E)$ splits into eigenspaces of $D^2_{S^1}$ which we may regard as spaces of L^2-sections of orbibundles over W. These spaces are invariant under D_W by (6.5). In particular, $H = \ker D_{S^1} \subset p_*E$ is isomorphic to the original E_V, and the connection ∇^E induces a unitary connection $\nabla^{p_*E} = \nabla^H \oplus \nabla^{H^\perp}$.

To avoid a clash of notation later, we write u for the inward normal coordinate on M and W near their respective boundary. Then let $e_2 = \frac{\partial}{\partial u}$ be the inward normal unit vector to ∂M with respect to g_1, extended parallelly over the cylindrical neighbourhood $u \in [0,1]$ of ∂M. The boundary operator $D_{\partial M,\varepsilon}$ splits in the same manner as $D_{M,\varepsilon}$, and in that cylindrical neighbourhood of ∂M, we have

$$D_{M,\varepsilon} = c_2 \left(\varepsilon \frac{\partial}{\partial u} + D^\partial_{S^1} + \varepsilon D_{\partial W} \right) ,$$

where $D^\partial_{S^1} = -c_2 D_{S^1}$ denotes the fibrewise boundary operator. By (6.5), the operators $D^\partial_{S^1}$ and $D_{\partial W}$ anticommute as well. Both respect the splitting of $(p|_{\partial M})_*E$ into $H|_{\partial W}$ and $H^\perp|_{\partial W}$.

Let $\Pi_{+,\varepsilon}$ denote the spectral projection onto the subspace of $L^2(\partial M; E)$ spanned by the eigenspinors of $D^\partial_{S^1} + \varepsilon D_{\partial W}$ with positive eigenvalues. Then $\Pi_{+,\varepsilon}$ respects the splitting into $H|_{\partial W}$ and its orthogonal complement, so

$$\Pi_{+,\varepsilon} = \Pi^H_+ \oplus \Pi^\perp_{+,\varepsilon} .$$

Moreover, $\ker(D^\partial_{S^1} + \varepsilon D_{\partial W})$ in the sections of $H|_{\partial W}$ and equals the kernel of the restriction of $D_{\partial W}$ to $H|_{\partial W}$. The relevant symplectic structure on $\ker(D^\partial_{S^1} + \varepsilon D_{\partial W})$ is induced by c_2.

We denote the restriction of D_W to H by $D_{W,1}$ in analogy with [21]. It is a Dirac operator in the case of the odd signature operator, and a Dirac operator modified by (6.4) in the case of the modified spin Dirac operator. Clifford multiplication with the global vertical tangent vector field e_1 still acts on H and anticommutes with $D_{W,1}$. Because c_1 and c_2 anticommute, the Clifford multiplication c_1 commutes with the boundary operator $D_{\partial W}$, so the projection Π^H_+ commutes with c_1 as well. The Lagrangian subspaces L_D and L_B of (2.6) and (2.7) are also invariant under c_1. We immediately conclude that

$$\eta_{\mathrm{APS}}(D_{W,1}; L_D) = 0 . \tag{6.6}$$

Recall that $\eta(\mathbb{A}) \in \Omega^\bullet(\Lambda W)$ denotes the orbifold η-form of the Bismut superconnection of the fibrewise spin Dirac operator with respect to the fibrewise trivial spin structure. We may regard the signature operator as a Dirac operator twisted by the pullback of the spinor bundle on the base; for this reason, the η-form $\eta(\mathbb{A})$ occurs in both formulas in the theorem below.

Theorem 6.1 (Compare Dai [19, Theorem 1.1], see also [21, Theorem 0.1])**.** *With the assumptions and notations above,*

$$\lim_{\varepsilon\to 0} \eta\big(D_{M,\varepsilon}; L_D\big) = \int_{\Lambda W\setminus W} \hat{A}_{\Lambda W}\big(TW, \nabla^{TW}\big)\, 2\eta_{\Lambda W}(\mathbb{A}) ,$$

$$\lim_{\varepsilon\to 0} \eta\big(B_{M,\varepsilon}; L_B\big) = \int_{\Lambda W\setminus W} \hat{L}_{\Lambda W}\big(TW, \nabla^{TW}\big)\, 2\eta_{\Lambda W}(\mathbb{A}) .$$

This result is not covered by [19] because the fibrewise operator is allowed to have a kernel, and because $p\colon M \to V$ is a Seifert fibration. It is not covered by [21] because the base orbifold is allowed to have a boundary. But of course, the Seifert fibration is locally a twisted product, and hence the situation here is more specialised than in the two references above. A little extra complication comes from the construction (2.4) of the modified spin Dirac operator. Our proof below relies crucially on the fact that the operators $D_{M,\varepsilon}$ and $D_{\partial M,\varepsilon}$ both respect the splitting of the bundle $p_*E \to B$ of fibrewise sections into the fibrewise harmonic spinors, which form a bundle $H \to B$ by assumption, and its orthogonal complement $H^\perp$. We believe that with a little extra work, this proof extends to totally geodesic Seifert fibrations. Probably Dai's proof also extends to totally geodesic fibre bundles because [19, Prop 5.2] then holds for sections of $H^\perp$.

Proof. We follow the proof in [21] as far as possible. We will view D_W as a differential operator on p_*E and D_{S^1} as an endomorphism of p_*E. We consider the restriction of $D_{M,\varepsilon}$ to $H^\perp$. In accordance with [21, sect 2.c], we denote it by $D_{M,\varepsilon,4} = D_{S^1} + \varepsilon D_{W,4}$, where $D_{W,4}$ describes the action of D_W on sections of p_*E. Note that in our setting, $D_{W,2} = D_{W,3} = 0$. Let $\langle\,\cdot\,,\,\cdot\,\rangle_{M/W}$ denote the fibrewise L^2-product, let div_W denote the divergence of a vector field or a one-form on W, let $\Delta^{H^\perp}$ denote the horizontal Laplacian on $H^\perp \to W$, and let c^W denote Clifford multiplication by horizontal vectors. Then

$$\begin{aligned}
\operatorname{div}_W \big\langle \nabla^{H^\perp}\sigma, \tau\big\rangle_{M/W} &= -\big\langle \Delta^{H^\perp}\sigma, \tau\big\rangle_{M/W} + \big\langle \nabla^{H^\perp}\sigma, \nabla^{H^\perp}\tau\big\rangle_{M/W} \,, \\
\operatorname{div}_W \big\langle c^W\sigma, \tau\big\rangle_{M/W} &= \big\langle D_W\sigma, \tau\big\rangle_{M/W} - \big\langle \sigma, D_W\tau\big\rangle_{M/W} \,.
\end{aligned}$$

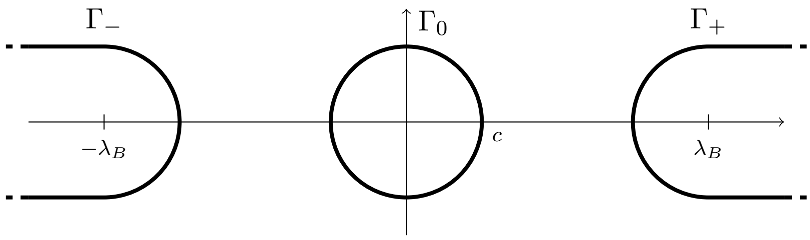
Because D'_W has been modified to D_W by a self-adjoint operator of order 0 supported away from the boundary, and because $\frac{\partial}{\partial r}$ is the inward normal direction, we conclude that

$$\begin{aligned}
&\big\|\big(i - \varepsilon^{-1}D_{M,\varepsilon,4}\big)\sigma\big\|^2_{L^2(W;H^\perp)} - \big\|\nabla^{H^\perp}\sigma\big\|^2_{L^2(W;H^\perp)} \\
&\qquad = \big\langle\big(1 + \varepsilon^{-2}D^2_{S^1} + D^2_{W,4} - \Delta^{H^\perp}\big)\sigma, \sigma\big\rangle_M - \langle ic_2\sigma, \sigma\rangle_{\partial M} - \varepsilon^{-1}\langle D_{\partial M,\varepsilon,4}\sigma, \sigma\rangle_{\partial M} \,. \qquad (6.7)
\end{aligned}$$

Let $H^1(W, H^\perp; \Pi^\perp_{+,\varepsilon})$ denote the subspace of the first Sobolev space generated by sections that satisfy the APS boundary condition. If $\sigma \in H^1(W, H^\perp; \Pi^\perp_{+,\varepsilon})$, then $\langle c_2\sigma, \sigma\rangle = 0$ because c_2 anticommutes with $D_{\partial M,\varepsilon,4}$, and $\langle D_{\partial M,\varepsilon,4}\sigma, \sigma\rangle \le 0$, so

$$\big\|\big(i - \varepsilon^{-1}D_{M,\varepsilon,4}\big)\sigma\big\|^2_{L^2(W;H^\perp)} \ge \big\|\nabla^{H^\perp}\sigma\big\|^2_{L^2(W;H^\perp)} + \big\langle\big(1 + \varepsilon^{-2}D^2_{S^1} + D^2_{W,4} - \Delta^{H^\perp}\big)\sigma, \sigma\big\rangle_M \,. \qquad (6.8)$$

Let λ_B denote the smallest absolute value of a nonzero eigenvalue of the effective horizontal operator $D_{W,1}$ with respect to the given boundary conditions, and let $0 < c < \frac{\lambda_B}{2}$. Let $\Gamma = \Gamma_+ \dot\cup \Gamma_0 \dot\cup \Gamma_-$ denote a contour in $\mathbb{C}$, where $\Gamma_\pm$ goes around $\pm[\lambda_B, +\infty]$ with distance c, and Γ_0 is a circle around 0 with radius c.



Assume that λ is not in the spectrum of $D_{M,\varepsilon,4}$ with APS boundary conditions given by $\Pi^\perp_{+,\varepsilon}$. Using parametrices on $\partial W \times [0,\infty)$ and on the double of W, one can construct a resolvent

$$R_\varepsilon(\lambda)\colon L^2(W; H^\perp) \to H^1(W, H^\perp; \Pi^\perp_{+,\varepsilon})$$

of $D_{M,\varepsilon,4}$. We define the family of Schatten norms of operators A acting on $L^2(M;E) \cong L^2(W;p_*E)$ by

$$\|A\|_p = \operatorname{tr}\Big((A^*A)^{\frac{p}{2}}\Big)$$

for $1 \leq p < \infty$, and let A_∞ denote the operator norm. Because $D_W'^2 - \Delta^{p_*E}$ is a bundle endomorphism on $E \to M$, we can use the inequality (6.8) above to prove the analogue of [21, Prop 2.7]. In particular, there exists a constant $\varepsilon_0 > 0$ such that for all $p > \dim M$, all $\varepsilon \in (0, \varepsilon_0)$ and all $\lambda \in \Gamma$, one has

$$\|R_\varepsilon(\lambda)\| = O(1, \varepsilon\,|\lambda|) \qquad \text{and} \qquad \|R_\varepsilon(\lambda)\| = O(|\lambda|)\;. \tag{6.9}$$

Let $H^1(W, H; \Pi^H_{+,\Lambda})$ denote the subspace of the first Sobolev space spanned by sections satisfying the Lagrangian APS boundary condition fixed above. Then we consider the resolvent

$$(\lambda - D_{W,1})^{-1} \colon L^2(W;H) \longrightarrow H^1(W, H; \Pi^H_{+,\Lambda})\;.$$

Obviously $(\lambda - D_{M,\varepsilon})^{-1} = (\lambda - D_{W,1})^{-1} \oplus R_\varepsilon(\lambda)$. Because $D_{W,1}$ is the effective horizontal operator, Proposition 2.8 in [21] reduces to

$$\big\|(\lambda - D_{W,1})^{-1}\big\|_\infty = O(1) \qquad \text{and} \qquad \big\|(\lambda - D_{W,1})^{-1}\big\|_p = O(|\lambda|) \tag{6.10}$$

for all $\lambda \in \Gamma$, which can be proved in the same way as (6.7). As an analogue of [21, Prop 2.9], we get

$$\big\|(\lambda - \varepsilon^{-1} D_{M,\varepsilon})^{-1} - (\lambda - D_{W,1})^{-1}\big\|_\infty = O(\varepsilon\,|\lambda|) \tag{6.11}$$

for all $\lambda \in \Gamma$.

Because $D_{M,\varepsilon} = \varepsilon\, D_{W,1} \oplus D_{M,\varepsilon,4}$, the spectral projection P_ε in [21, sect 2.f] coincides with the spectral projection onto $\ker(D_{W,1}) = \ker(D_{M,\varepsilon})$ independent of ε. Using (6.6), (6.9–6.11), we can adapt the proof of [21, Prop 2.10] to show that there exists a small $\alpha > 0$ such that

$$\begin{aligned}
&\lim_{\varepsilon\to 0} \int_{\varepsilon^{\alpha-2}}^\infty \frac{1}{\sqrt{\pi t}} \operatorname{tr}\Big(D_{M,\varepsilon} e^{-tD^2_{M,\varepsilon}}\Big)\, dt \\
&\qquad = \lim_{\varepsilon\to 0} \int_{\varepsilon^{\alpha-2}}^\infty \frac{1}{\sqrt{\pi t}} \operatorname{tr}\Big((1 - P_\varepsilon)\Big(D_{M,\varepsilon} e^{-tD^2_{M,\varepsilon}}\Big)(1 - P_\varepsilon)\Big)\, dt = \eta(D_{W,1}) = 0\;.
\end{aligned} \tag{6.12}$$

Note that the orbifold η-form $\eta_{\Lambda W}(\mathbb{A})$ vanishes on the principal stratum $W \subset \Lambda W$ because the Seifert fibration $M \to V$ is a twisted product and the fibrewise operator D_{S^1} has symmetric spectrum. The additional divergent terms in the heat asymptotics of the supertrace of e^{-tD_W} caused by the non-geometric terms introduced in (2.4) and (6.4) do not cause extra complications here because they are supported on the regular stratum (and away from the boundary). Because the singular stratum does not extend to the boundary ∂W, the right hand side of the expression in the theorem vanishes near the boundary.

We can now use finite propagation speed to obtain the analogue of [21, Prop 2.12], which says that

$$\lim_{\varepsilon\to 0} \int_0^{\varepsilon^{\alpha-2}} \frac{1}{\sqrt{\pi t}} \operatorname{tr}\Big(D_{M,\varepsilon} e^{-tD^2_{M,\varepsilon}}\Big)\, dt$$

equals the right hand side of the expression in the theorem. Together with (6.12), this finishes the proof. $\square$

6.2. **A product formula for η-forms.** Recall the family $\partial\mathcal{M}_\pm = \Sigma_\pm \times E_\pm \to (0,\infty)$, where $\Sigma_\pm$ denotes a fixed K3 surface and $E_\pm \to (0,\infty)$ denotes the family of tori $E_{\pm,a} = (S^1_{\xi_\pm} \times S^1_{a\zeta_\pm})/\Gamma_\pm$ for $a \in (0,\infty)$. We now consider the η-forms $\eta(\mathbb{B})$ and $\eta(\mathbb{D})$ associated with the fibrewise signature and Dirac operators on $\partial\mathcal{M}_\pm \to (0,\infty)$.

We start with the signature operator. Because $X_{\pm,a} = \Sigma \times E_{\pm,a}$, we may split

$$\Omega^\bullet(\mathcal{M}_\pm,(0,\infty)) \cong \Omega^\bullet(\Sigma) \,\hat\otimes\, \Omega^\bullet(E_\pm/(0,\infty)) \;.$$

Let $\mathbb{B}_E$ denote the superconnection associated to the fibrewise signature operator for the family $E_\pm \to (0,\infty)$, equipped with the trivial spin structure. Let B_Σ denote the signature operator on Σ. With respect to the splitting above, we have

$$\mathbb{B}_t = \sqrt{t}B_\Sigma \,\hat\otimes\, \mathrm{id} + \mathrm{id} \,\hat\otimes\, \mathbb{B}_{E,t} = \sqrt{t}B_\Sigma \,\hat\otimes\, \mathrm{id} + \mathrm{id} \,\hat\otimes\, \sqrt{t}B_E + \tilde\nabla^u \;.$$

Because B_Σ is independent of a, we have $[\tilde\nabla^u, B_\Sigma] = 0$. Also Σ is even-dimensional, so the signature operator B_Σ has symmetric spectrum. We conclude

$$\begin{aligned}\mathrm{str}_{\Omega^\bullet(\partial\mathcal{M}/(0,\infty))}\biggl(\frac{\partial\mathbb{B}_t}{\partial t}\,e^{-\mathbb{B}_t^2}\biggr) &= \mathrm{str}_{\Omega^\bullet(\Sigma)}\biggl(\frac{1}{\sqrt{4t}}\,B_\Sigma\,e^{-tB_\Sigma^2}\biggr)\cdot \mathrm{str}_{\Omega^\bullet(E/(0,\infty))}\bigl(e^{-\mathbb{B}_{E,t}^2}\bigr)\\ &\qquad + \mathrm{str}_{\Omega^\bullet(\Sigma)}\Bigl(e^{-tB_\Sigma^2}\Bigr)\cdot \mathrm{str}_{\Omega^\bullet(E/(0,\infty))}\biggl(\frac{\partial\mathbb{B}_{E,t}}{\partial t}\,e^{-\mathbb{B}_{E,t}^2}\biggr)\\ &= \mathrm{ind}(B_\Sigma)\cdot\frac12\,\mathrm{str}_{\Omega^\bullet(E/(0,\infty))}\Bigl(B_E\,\bigl[\tilde\nabla^u, B_E\bigr]\,e^{-tB_E^2}\Bigr)\;,\end{aligned} \tag{6.13}$$

where we have used the McKean-Singer formula for $\mathrm{ind}(B_\Sigma)$ in the last step.

We now want to relate $\mathbb{B}_{E,t} = \sqrt{t}\,B_E + \tilde\nabla^u$ to the superconnection $\mathbb{A}$ of the fibrewise Dirac operator on $E_\pm \to (0,\infty)$ with respect to the trivial spin structure. We regard the fibrewise spinor bundle as the pullback of a bundle $S = S^+ \oplus S^-$ on the base $(0,\infty)$. The signature operator acts on the fibrewise spinor bundle twisted by itself. We therefore have

$$\Omega^\bullet(E_\pm/(0,\infty)) \cong p_*p^*(S\otimes S)\;,$$

and the corresponding superconnection is now given by $\mathbb{B}_{E,t} = \mathbb{A}_t \otimes \mathrm{id} + \mathrm{id}\otimes\nabla^S$. The two terms above supercommute, so we have

$$\mathrm{str}_{\Omega^\bullet(E/(0,\infty))}\biggl(\frac{\partial\mathbb{B}_{E,t}}{\partial t}\,e^{-\mathbb{B}_{E,t}^2}\biggr) = \mathrm{str}_{p_*p^*S}\biggl(\frac{\partial\mathbb{A}_t}{\partial t}\,e^{-\mathbb{A}_t^2}\biggr)\cdot \mathrm{tr}_S\Bigl(e^{-(\nabla^S)^2}\Bigr)\;.$$

Integrating over t, we get

$$\tilde\eta(\mathbb{B}_E) = \tilde\eta(\mathbb{A})\;\mathrm{ch}(\nabla^S)$$

In degree 1, this equals twice the spinorial η-form because $\mathrm{rk}\,S = 2$. Equivalently, the reader is invited to compare Bismut and Cheeger's results for the universal spinorial η-form and the signature η-form of bundles of flat tori in [5, Thms 2.22, 2.25]. Together with (6.13), we see that $\tilde\eta(\mathbb{B}) = 2\,\mathrm{ind}(B_\Sigma)\,\tilde\eta(\mathbb{A})$, which is exactly (2.22a).

Similarly, the fibrewise spinor bundle of $X_{\pm,a}$ is the exterior tensor product of the spinor bundles of Σ and $E_{\pm,a}$. Proceeding as in (6.13), we prove (2.22b).

6.3. **Adiabatic limits of families of flat tori.** We consider a family of fibred manifolds $E \to F \to \mathbb{R}$ as in Section 4.6, diagram (4.11). We will prove Proposition 4.12, which is a special case of the adiabatic limit formula for η-forms of Bunke, Ma [10] and Liu [29], but as a strict equation of forms, not as an equation modulo exact forms. To this end, we will simply compute both sides of the equation. We believe that under suitable conditions, the adiabatic limit formula for η-forms holds in this strict sense for more general iterated fibre bundles.

We fix $y \in \mathbb{R}$; later we will consider the limit $y \to \infty$. For $x \in \mathbb{R}$, we identify

$$E_x = \mathbb{C}/(\mathbb{Z} + (x+iy)\mathbb{Z}) \qquad \text{and} \qquad F_x = \mathbb{R}/y\mathbb{Z} \;.$$

The fibration $E \to F$ is formed by taking the imaginary part. The standard Euclidean metric on $\mathbb{C}$ induces a fibrewise metric on $E \to \mathbb{R}$. The group S^1 acts isometrically by translation in the real direction in $\mathbb{C}$.

On the total space of E, we consider the fibrewise orthonormal base induced by $e_1 = 1$ and $e_2 = i \in \mathbb{C}$. The connection ∇^W in (4.2a) induces a horizontal subspace $T^H E \subset TE$ for the fibration $E \to \mathbb{R}$. It is spanned by the vector field e_3 induced from the vector field

$$\mathbb{C} \times \mathbb{R} \longrightarrow \mathbb{C} \times \mathbb{R} \qquad \text{with} \qquad (u+iv, x) \longmapsto (v/y; 1) \;,$$

which is invariant under the x-dependent action of $\mathbb{Z}^2$ on $\mathbb{C}$. Obviously,

$$[e_1, e_2] = [e_1, e_3] = 0 \qquad \text{and} \qquad [e_2, e_3] = \frac{1}{y}\, e_1 \;. \tag{6.14}$$

The connection $\nabla^{T(E/\mathbb{R})}$ on the vertical tangent bundle is given by

$$\nabla^{T(E/\mathbb{R})} e_1 = \frac{1}{2y} e_2 \, dx \qquad \text{and} \qquad \nabla^{T(E/\mathbb{R})} e_2 = -\frac{1}{2y} e_1 \, dx \;,$$

see (4.2c). Because $y \in \mathbb{R}$ is constant, this connection is flat.

We identify the fibrewise spinor bundles $S(E/\mathbb{R}) = S^+(E/\mathbb{R}) \oplus S^-(E/\mathbb{R}) \to E$ with $\underline{\mathbb{C}} \oplus \underline{\mathbb{C}}$. If d denotes the trivial connection on the spinor bundle, then $\nabla^{T(E/\mathbb{R})}$ induces the connection

$$\nabla^{S(E/\mathbb{R})} = d + \frac{1}{4y}\, c_1 c_2 \, dx \;.$$

Let $W = p_* S(E/\mathbb{R})$ denote the infinite-dimensional vector bundle over $\mathbb{R}$ whose fibres are the spaces of sections of $S(E/\mathbb{R})|_{E_x}$. We identify sections of $p_* S(E/\mathbb{R})$ with sections of $S(E/\mathbb{R})$. Because the fibres of p have vanishing mean curvature, the induced connection takes the form

$$\nabla^{p_* S(E/\mathbb{R})} s = \nabla^{S(E/\mathbb{R})}_{e_3} s \, dx \;.$$

Let D_x denote the fibrewise Dirac operator over $x \in \mathbb{R}$. Then the Bismut superconnection for the fibration $E \to \mathbb{R}$ takes the form

$$\mathbb{B}_t = \sqrt{t}\, D_x + \nabla^{p_* S(E/\mathbb{R})} \;.$$

Because

$$\begin{aligned}
[\nabla^{p_* S(E/\mathbb{R})}, D_x] &= -c_1 [\nabla^0_{e_3}, \nabla^0_{e_1}]\, dx - c_2 [\nabla^0_{e_3}, \nabla^0_{e_2}]\, dx - [c_1 c_2, c_1] \nabla^0_{e_1} \frac{dx}{4y} - [c_1 c_2, c_2] \nabla^0_{e_2} \frac{dx}{4y} \\
&= c_2 \nabla^0_{e_1} \frac{dx}{y} - c_2 \nabla^0_{e_1} \frac{dx}{2y} + c_1 \nabla^0_{e_2} \frac{dx}{2y} = c_2 \nabla^S_{e_1} \frac{dx}{2y} + c_1 \nabla^S_{e_2} \frac{dx}{2y} \;,
\end{aligned}$$

by (2.19), the η-form for bundles with even-dimensional fibres is given by

$$\begin{aligned}
\tilde{\eta}(\mathbb{B}) &= -\frac{1}{8y\pi i} \int_0^\infty \operatorname{tr}\Bigl(i c_1 c_2 \left(c_1 \nabla^S_{e_1} + c_2 \nabla^S_{e_2} \right) \left(c_1 \nabla^S_{e_2} + c_2 \nabla^S_{e_1} \right) dx \, e^{-tD_x^2} \Bigr)\, dt \\
&= \frac{1}{8y\pi} \int_0^\infty \operatorname{tr}\Bigl(\left((\nabla^S_{e_1})^2 - (\nabla^S_{e_2})^2 \right) e^{-tD_x^2} \Bigr)\, dt \, dx \;.
\end{aligned} \tag{6.15}$$

The space of vertical sections is spanned by sections of the form $\varphi_{m,n} s_\pm$, where

$$\varphi_{m,n}(u,v) = e^{2\pi i \left(m(u - \frac{x}{y} v) + n \frac{v}{y} \right)}$$

for m, $n \in \mathbb{Z}$ and $s_\pm$ is a fibrewise parallel section of $S^\pm(E/\mathbb{R})$. The vertical Laplacian takes the form $-\partial_u^2$, and its kernel is spanned by the functions $\varphi_{0,n}s^\pm$. Because $S(E/\mathbb{R})$ has rank 2, we can therefore rewrite the η-form as

$$\tilde{\eta}(\mathbb{B}) = \frac{dx}{4\pi y} \int_0^\infty \sum_{m,n\in\mathbb{Z}} 4\pi^2 \biggl(\Bigl(\frac{n-mx}{y}\Bigr)^2 - m^2\biggr) e^{-4\pi^2 t\left(\left(\frac{n-mx}{y}\right)^2 + m^2\right)}\, dt\,. \tag{6.16}$$

For fixed m, the sum over n describes the spectrum of a Dirac operator on a circle S^1_y of length y with coefficients in a flat vector bundle. Approximating the heat kernel on S^1_y by the Euclidean heat kernel gives

$$\begin{aligned}\sum_n 4\pi^2 \biggl(\Bigl(\frac{n-mx}{y}\Bigr)^2 - m^2\biggr) e^{-4\pi^2 t\left(\frac{n-mx}{y}\right)^2} &= -\biggl(4\pi^2 m^2 + \frac{\partial}{\partial t}\biggr) \sum_{n\in\mathbb{Z}} e^{-4\pi^2 t\left(\frac{n-mx}{y}\right)^2}\\ &= -\biggl(4\pi^2 m^2 - \frac{1}{2t}\biggr) \frac{y}{\sqrt{4\pi t}} + O\Bigl((1+m^2)e^{-\frac{y^2-c}{4t}}\Bigr)\end{aligned}$$

for each small $c > 0$, uniformly in m. For $\alpha > 0$ small, we compute

$$\begin{aligned}&\frac{dx}{4\pi y} \int_0^{y^{2-\alpha}} \sum_{m,n\in\mathbb{Z}} 4\pi^2 \biggl(\Bigl(\frac{n-mx}{y}\Bigr)^2 - m^2\biggr) e^{-4\pi^2 t\left(\left(\frac{n-mx}{y}\right)^2 + m^2\right)}\, dt\\ &\qquad = -\frac{dx}{4\pi} \int_0^{y^{2-\alpha}} \sum_{m\in\mathbb{Z}} \biggl(4\pi^2 m^2 - \frac{1}{2t}\biggr) \frac{1}{\sqrt{4\pi t}}\, e^{-4\pi^2 m^2 t}\, dt \end{aligned}\tag{6.17}$$

For $t \geq y^{2-\alpha}$, we only need to study the contribution from $\ker D^{E/F}$, which can be written as

$$\begin{aligned}\frac{1}{2\pi i} \int_{y^{2-a}}^\infty \operatorname{str}\biggl(P^{\ker D^{E/F}} \frac{\partial \mathbb{B}_t}{\partial t}\, e^{-\mathbb{B}_t^2}\biggr)\, dt &= \frac{dx}{2y\pi} \int_{y^{2-\alpha}}^\infty \sum_n \frac{4\pi^2 n^2}{y^2}\, e^{-\frac{4\pi^2 n^2 t}{y^2}}\, dt\\ &= \frac{dx}{4y\pi} \sum_n e^{-4\pi^2 y^{-\alpha} n^2}\, dx\,.\end{aligned}$$

This sum converges to 0 as $y \to \infty$ for $\alpha < 1$ because

$$\frac{1}{y} \sum_n e^{-4\pi^2 y^{-\alpha} n^2} \leq \frac{2}{y} \sum_{n=0}^\infty e^{-4\pi^2 y^{-\alpha} n} = \frac{2}{y} \cdot \frac{1}{1 - e^{-4\pi^2 y^{-\alpha}}} = \frac{2}{4\pi^2 y^{1-\alpha}} + o\bigl(4\pi^2 y^{-\alpha}\bigr)\,.$$

In general, one would expect here the η-form of the effective fibrewise operator on $F \to \mathbb{R}$, acting on sections of $\ker D^{E/F}$, and some extra terms in the case that there are very small eigenvalues. Because the kernel bundle is trivial here, it is not surprising that this form vanishes in our situation. Combining this with the computations above, we finally see that

$$\lim_{y\to\infty} \tilde{\eta}(\mathbb{B}) = \frac{dx}{4\pi} \int_0^\infty \sum_{m\in\mathbb{Z}} \biggl(\frac{1}{2t} - 4\pi^2 m^2\biggr) e^{-4\pi^2 m^2 t} \frac{dt}{\sqrt{4\pi t}} \tag{6.18}$$

We now consider the fibration $E \to F$. We choose the horizontal bundle spanned by the vectors e_2 and e_3 above. We identify the spinor bundle $S(E/F) \to E$ with $\underline{\mathbb{C}}$. From equation (6.14), we get the superconnection $\mathbb{A}_t$ for the family $E \to F$ as

$$\mathbb{A}_t = \sqrt{t}\, c_1 \nabla^S_{e_1} + \nabla^{p_* S(E/\mathbb{R})} + \frac{1}{4y\sqrt{t}}\, c_1\, dv\, dx\,.$$

Its curvature is given by

$$\mathbb{A}_t^2 = -t(\nabla^0_{e_1})^2 + \frac{1}{2y}\,dv\,dx\,.$$

Assuming that the Clifford volume element ic_1 acts as 1, the η-form of the bundle $E \to F$ with odd-dimensional fibres takes the form

$$\begin{aligned}\tilde\eta(\mathbb{A}) &= (2\pi i)^{-\frac{N^F}{2}} \int_0^\infty \operatorname{tr}\Bigl(\frac{\partial \mathbb{A}_t}{\partial t}\,e^{-\mathbb{A}_t^2}\Bigr)\frac{dt}{\sqrt{\pi}}\\ &= \int_0^\infty \operatorname{tr}\biggl(c_1\Bigl(\nabla^S_{e_1} - \frac{1}{8\pi iyt}\,dv\,dx\Bigr)\Bigl(1-\frac{1}{4\pi iy}\,dv\,dx\,\nabla^S_{e_1}\Bigr)e^{t(\nabla^S_{e_1})^2}\biggr)\frac{dt}{\sqrt{4\pi t}}\\ &= \int_0^\infty \operatorname{tr}\biggl(\Bigl(-i\nabla^S_{e_1} + \frac{dv\,dx}{8\pi yt}\bigl(1+2t(\nabla^S_{e_1})^2\bigr)\Bigr)e^{t(\nabla^S_{e_1})^2}\biggr)\frac{dt}{\sqrt{4\pi t}}\\ &= \int_0^\infty \operatorname{tr}\biggl(\Bigl(-i\nabla^S_{e_1} + \frac{dv\,dx}{8\pi yt}\Bigl(1+2t\frac{\partial}{\partial t}\Bigr)\Bigr)e^{t(\nabla^S_{e_1})^2}\biggr)\frac{dt}{\sqrt{4\pi t}}\,.\end{aligned} \tag{6.19}$$

The space of vertical sections is spanned by sections of the form φ_m for $m \in \mathbb{Z}$, where

$$\varphi_m(u) = e^{2\pi imu}\,.$$

We can now compute the integral of $\tilde\eta(\mathbb{A})$ over the fibres of $F \to \mathbb{R}$ as

$$\begin{aligned}\int_{F/\mathbb{R}} \tilde A(\mathbb{A}) &= \frac{dx}{4\pi}\int_0^\infty \sum_{m\in\mathbb{Z}}\Bigl(\frac{1}{2t}+\frac{\partial}{\partial t}\Bigr)e^{-4\pi^2m^2t}\,\frac{dt}{\sqrt{4\pi t}}\\ &= \frac{dx}{4\pi}\int_0^\infty \sum_{m\in\mathbb{Z}}\Bigl(\frac{1}{2t}-4\pi^2m^2\Bigr)\,e^{-4\pi^2m^2t}\,\frac{dt}{\sqrt{4\pi t}}\,.\end{aligned} \tag{6.20}$$

Proof of Proposition 4.12. The Proposition follows by comparing (6.18) and (6.20). □

7. On the values of the function $F_{k,\varepsilon}(s)$

In Section 2 of this paper (Proposition 2.12, Theorem 2.13, Theorem 2.14) it was shown that the ν-invariants of extra twisted connected sums can be computed in terms of values of the analytic function $F_{k,\varepsilon}\colon (0,\infty)\to\mathbb{R}$ defined for each $k\in\mathbb{N}$ and integer ε prime to k by

$$F_{k,\varepsilon}(s) \;=\; \int_0^\infty\int_0^s \sum_{m\,\equiv\,\varepsilon n\ (\mathrm{mod}\,k)} mn\,e^{-t(m^2+n^2a^2)}\,da\,dt$$

(Definition 2.11). In this section we will give a closed formula for $F_{k,\varepsilon}(s)$ in terms of the Dedekind eta-function, show that it is equal to the arccosine (or arcsine, or arctangent) of a computable algebraic number whenever s^2 is rational, and show that the specific combinations of $F_{k,\varepsilon}$-values occurring in Theorems 2 and 3 can be evaluated in terms of Dedekind sums.

7.1. Evaluation of $F_{k,\varepsilon}(s)$ in terms of the Dedekind eta-function.

For τ in the upper half-plane $\mathcal{H}$ we denote by $\eta(\tau)$ and $\mathcal{L}(\tau)$ the Dedekind η-function and the principal branch (real on the positive imaginary axis) of its logarithm, given explicitly by

$$\eta(\tau) \;=\; e^{\pi i\tau/12}\prod_{n=1}^\infty\bigl(1-e^{2\pi in\tau}\bigr)\,,\qquad \mathcal{L}(\tau) \;=\; \frac{\pi i\tau}{12} - \sum_{n=1}^\infty \frac{\sigma(n)}{n}\,e^{2\pi in\tau}\,,$$

where $\sigma(n)$ denotes the sum of the positive divisors of n. The fact that $\eta(\tau)^{24}$ is a modular form of weight 12 on $SL_2(\mathbb{Z})$ implies that L satisfies the transformation equation

$$\mathcal{L}\Big(\frac{a\tau+b}{c\tau+d}\Big) \;=\; \mathcal{L}(\tau) \,+\, \frac{1}{4}\,\mathrm{Log}\big(-(c\tau+d)^2\big) \,+\, \frac{\pi i}{12}N(a,b,c,d)\,, \tag{7.1}$$

for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$, where Log denotes the principal branch (real on the positive real axis) of the logarithm on $\mathbb{C}\setminus(-\infty,0]$ and $N(a,b,c,d)$ is an integer given by $N(a,b,c,d) = b/d$ $(=\pm b)$ if $c=0$ and by $N(a,b,c,d) = \frac{a+d}{c} - 12S(d,c)$ if $c \neq 0$, where the Dedekind sum $S(d,c)$ is defined in (0.3). Our first result is:

Proposition 7.1. *The value of $F_{k,\varepsilon}(s)$ for any $k \in \mathbb{N}$, integer ε prime to k, and positive real number s is given by*

$$F_{k,\varepsilon}(s) \;=\; 2\,\mathrm{Im}\,\mathcal{L}\Big(\frac{-\varepsilon^*+is^{-1}}{k}\Big) \,+\, \frac{\pi\varepsilon^*}{6k}\,, \tag{7.2}$$

where $\varepsilon^* \in \mathbb{Z}$ is any solution of $\varepsilon\varepsilon^* \equiv 1 \pmod k$.

Proof. We first rewrite the definition of $F_{k,\varepsilon}$ as

$$s\,\frac{d}{ds}\,F_{k,\varepsilon}(s) \;=\; \frac{\pi}{k}\int_0^\infty \Theta_{k,\varepsilon}(s,t)\,dt \qquad \text{and} \qquad F_{k,\varepsilon}(0)=0\,, \tag{7.3}$$

where $\Theta_{k,\varepsilon}(s,t)$ is defined for $s,\,t>0$ by

$$\Theta_{k,\varepsilon}(s,t) \;=\; \sum_{m\,\equiv\,\varepsilon n\ (\mathrm{mod}\,k)} mn\,e^{-\pi t(m^2/s+n^2 s)/k}\,.$$

This theta series satisfies the functional equations

$$\Theta_{k,\varepsilon}(s,t) \;=\; -\Theta_{k,-\varepsilon}(s,t) \;=\; \Theta_{k,\varepsilon^*}(s^{-1},t) \;=\; t^{-3}\,\Theta_{k,\varepsilon}(s,t^{-1})\,, \tag{7.4}$$

(where $\varepsilon^* \equiv \varepsilon^{-1} \pmod k$ as above), as we see by changing the sign of m, interchanging m and n, or applying the Poisson summation formula with respect to both m and n. If instead we apply Poisson summation with respect to m only, we obtain the stronger identity

$$\Theta_{k,\varepsilon}(s,t) \;=\; \frac{(s/t)^{3/2}}{i\sqrt{k}} \sum_{m,n\in\mathbb{Z}} mn\,\zeta_k^{mn\varepsilon}\,e^{-\pi s(m^2/t+n^2t)/k} \qquad \big(\zeta_k := e^{2\pi i/k}\big)\,, \tag{7.5}$$

which also makes it clear that the integral in (7.3) converges, since it shows that $\Theta_{k,\varepsilon}(s,t)$ is exponentially small as t tends to either 0 or ∞. Inserting (7.5) into (7.3) and applying the elementary formula

$$\int_0^\infty e^{-c_1t-c_2/t}\,t^{-3/2}\,dt \;=\; = \;\sqrt{\frac{\pi}{c_2}}\,e^{-2\sqrt{c_1c_2}} \qquad (c_1,\,c_2>0)$$

with $c_1 = \pi s n^2/k$, $c_2 = \pi s m^2/k$, we find

$$\frac{ik}{2\pi}\,F'_{k,\varepsilon}(s) \;=\; \sum_{m,n>0} n\left(\zeta_k^{\varepsilon mn} - \zeta_k^{-\varepsilon mn}\right)e^{-2\pi mns/k} \;=\; \sum_{n=1}^\infty \sigma(n)\left(\zeta_k^{\varepsilon n} - \zeta_k^{-\varepsilon n}\right)e^{-2\pi ns/k}\,,$$

and this can be integrated immediately using the definition of $\mathcal{L}$ to give the formula

$$F_{k,\varepsilon}(s) \;=\; 2\,\mathrm{Im}\,\mathcal{L}\Big(\frac{\varepsilon+is}{k}\Big) \,+\, c_{k,\varepsilon} \tag{7.6}$$

for some constant $c_{k,\varepsilon}$ depending only on k and ε. We then use the modularity property (7.1) and the fact that $\frac{\pm\varepsilon+is}{k} = \gamma\big(\frac{\mp\varepsilon^*+is^{-1}}{k}\big)$ with $\gamma = \left(\begin{smallmatrix} \pm\varepsilon & * \\ k & \pm\varepsilon^* \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$ to deduce (7.2) from (7.6), up to determining the value of the constant $c_{k,\varepsilon}$. That is determined immediately from the property $F_{k,\varepsilon}(0)=0$, because $\mathcal{L}(\tau)=\pi i\tau/12+\mathrm{o}(1)$ for $\mathrm{Im}(\tau)\to\infty$. $\square$

Using the transformation law (7.1) again, we can evaluate the constant $c_{k,\varepsilon}$ in (7.6) to get

$$F_{k,\varepsilon}(s) \;=\; 2\,\mathrm{Im}\,\mathcal{L}\Bigl(\frac{\varepsilon+is}{k}\Bigr)\;+\;2\pi S(\varepsilon,k)-\frac{\pi\varepsilon}{6k}\,. \tag{7.7}$$

giving an alternative formula for the function $F_{k,\varepsilon}(s)$. In some cases this same transformation law can be used to give a complete formula for $F_{k,\varepsilon}(s)$ in terms of Dedekind sums. This happens whenever one (and hence both) of the two $SL_2(\mathbb{Z})$-equivalent numbers $\frac{\pm\varepsilon+is}{k}$ and $\frac{\mp\varepsilon^*+is}{k}$ is $SL_2(\mathbb{Z})$-equivalent to its negative conjugate. An easy calculation shows that the equation $\gamma\tau=-\overline{\tau}$ for $\gamma=\bigl(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\bigr)\in SL_2(\mathbb{Z})$ and τ in the upper half-plane holds if and only if $a=d$ and $|c\tau+a|=1$, which in our situation says that $s^2=\frac{1}{c^2}-\bigl(\frac{\varepsilon}{k}+\frac{a}{c}\bigr)^2$ for some integers a and c with $a^2\equiv 1 \pmod c$. In all such cases, the number $F_{k,\varepsilon}(s)$ is the sum of a rational multiple of π and the arctangent of the square-root of a positive rational number. Concrete examples where this happens and where the Dedekind sum occurring can be evaluated in closed form are the special values

$$F_{k,1}\Bigl(\frac{1}{\sqrt{k^2-1}}\Bigr)\;=\;-\,F_{k,1}\bigl(\sqrt{k^2-1}\bigr)\;-\;\frac{(k-1)(k-2)}{6k}\,\pi\;=\;\arctan\sqrt{\frac{k+1}{k-1}}\;-\;\frac{3k+2}{12k}\,\pi$$

for integers $k>1$ and

$$F_{k,1}\Bigl(\sqrt{\frac{m}{n}}\Bigr)\;=\;\arctan\sqrt{\frac{m}{n}}\;-\;\frac{km+2}{12k}\,\pi$$

for positive integers m and n with $m+n=2k$. We omit the details.

7.2. **Algebraic values.** Except in the cases just mentioned, there is in general no simple closed formula for the values of $F_{k,\varepsilon}(s)$. However, if the square of the argument s is a rational number, as is the case for all of the special values needed in this paper, one has the following general result.

Proposition 7.2. *If $s>0$ is the square-root of a rational number, then the value of $F_{k,\varepsilon}(s)$ for any k and ε is i times the logarithm of a computable algebraic number.*

Proof. It is known from the theory of complex multiplication that the ratio of the values of the Dedekind eta-function at any two arguments belonging to the same imaginary quadratic field is a computable algebraic number. (More precisely, the value of $\eta(\tau)$ for τ belonging to any imaginary quadratic field is an algebraic multiple of a certain product of gamma-values, the so-called Chowla-Selberg number, that depends only on the field. See [8, Part 1, Section 6] for further details.) Since both $\frac{\varepsilon^*+i/s}{k}$ and $\frac{-\varepsilon^*+i/s}{k}$ belong to the imaginary quadratic field $\mathbb{Q}(is)$ when s^2 is rational, this proposition is an immediate corollary of Proposition 7.1. $\square$

We do not describe here the algorithm for computing special eta-values at CM points, since it is standard in principle but is quite complicated. We limit ourselves instead to giving the values of $F_{k,\varepsilon}(s)$ for the specific triples (k,ε,s) that are used in this paper. These values are given by

$$F_{k,\varepsilon}(s)=\pi\,\bigl(S(\varepsilon,k)+b\bigr)+\tfrac{\sigma}{2}\,\arccos(c)\,,$$
$$F_{k,\varepsilon^*}(1/s)=\pi\,\bigl(S(\varepsilon,k)-b\bigr)-\tfrac{\sigma}{2}\,\arccos(c)\,.$$

k	ε	s	$S(\varepsilon,k)$	b	σ	c
3	1	1	1/18	0	± 1	1
4	1	1	1/8	0	± 1	1
4	1	$\sqrt{3}$	1/8	1/12	± 1	1
5	1	1	1/5	0	± 1	1
6	1	1	5/18	0	± 1	1
6	1	$\sqrt{3}$	5/18	1/6	± 1	1
3	1	$\sqrt{2}$	1/18	$-1/6$	1	1/3
3	1	$\sqrt{5}$	1/18	$-1/12$	1	2/3
3	1	$2\sqrt{2}$	1/18	1/4	-1	1/3
4	1	$\sqrt{7}$	1/8	0	1	3/4
4	1	$\sqrt{15}$	1/8	$-1/6$	1	$-1/4$
4	1	$\sqrt{5/3}$	1/8	$-1/6$	1	1/4
5	1	2	1/5	0	1	3/5
5	2	1	0	1/10	-1	3/5
5	2	4	0	1/10	-1	4/5
6	1	$\sqrt{2}$	5/18	$-1/12$	1	1/3
6	1	$\sqrt{5}$	5/18	1/12	1	2/3
6	1	$\sqrt{11}$	5/18	1/6	1	5/6
3	1	2	1/18	1/6	-1	$\sqrt{3}-1$
4	1	$\sqrt{2}$	1/8	$-1/8$	1	$\sqrt{2}-1$
4	1	$\sqrt{5}$	1/8	$-1/4$	1	$\frac{1}{2}\left(1-\sqrt{5}\right)$
4	1	3	1/8	0	1	$\sqrt{3}-1$
4	1	5	1/8	0	1	$3\sqrt{5}-6$
5	2	2	0	1/10	-1	$3\sqrt{5}-6$
6	1	$\sqrt{7}$	5/18	2/3	-1	$\frac{1}{4}\left(1-\sqrt{21}\right)$
3	1	$\sqrt{3}$	1/18	$-1/6$	1	$\sqrt[3]{2}-1$
4	1	$3\sqrt{3}$	1/8	$-1/12$	1	$\sqrt[3]{2}-1$
3	1	$2\sqrt{5}$	1/18	$-1/6$	1	$\frac{1}{3}\left(1-\sqrt{5}+\sqrt{5(\sqrt{5}-1)/2}\right)$
3	1	$4\sqrt{2}$	1/18	$-1/12$	1	$\frac{1}{6}\left(6-5\sqrt{2}+(4\sqrt{2}+2)\sqrt{\sqrt{2}-1}\right)$
3	1	$\sqrt{5}/2$	1/18	0	1	$\frac{1}{3}\left(\sqrt{5}-1+\sqrt{5(\sqrt{5}-1)/2}\right)$
4	1	$3\sqrt{7}$	1/8	0	1	$\frac{1}{16}\left(9+\sqrt{21}-\sqrt{26\sqrt{21}-114}\right)$
4	1	$3/\sqrt{7}$	1/8	0	1	$\frac{1}{16}\left(9+\sqrt{21}+\sqrt{26\sqrt{21}-114}\right)$
3	1	$5\sqrt{2}$	1/18	1/6	-1	$c=0.766\cdots,\ P(3c)=0$
3	1	$5/\sqrt{2}$	1/18	0	1	$c=0.940\cdots,\ P(-3c)=0$
5	1	$\sqrt{2}$	1/5	0	1	$c=0.861\cdots,\ Q(c)=0$
5	2	$\sqrt{2}$	0	1/10	-1	$c=0.634\cdots,\ Q(-c)=0$

TABLE 3. Data needed to compute $F_{k,\varepsilon}$

with $b \in \mathbb{Q}$, $\sigma \in \{\pm 1\}$ and $c \in \overline{\mathbb{Q}}$ as in Table 3. The functions P and Q appearing in the last four lines of the table are the two sextic polynomials

$$P(X) = 16\,X^6 - 416\,X^5 + 2440\,X^4 + 4880\,X^3 - 12615\,X^2 - 1826\,X - 32159\,,$$
$$Q(X) = 16\,X^6 - 32\,X^5 + 200\,X^4 + 560\,X^3 + 105\,X^2 - 402\,X - 191\,.$$

7.3. **Evaluation of $A(k_+, \varepsilon_+; k_-, \varepsilon_-; G)$ in terms of the Dedekind sums.** In this final subsection we place ourselves in the situation of Theorem 3. Specifically, this means that we have two pairs of coprime numbers $(k_\pm, \varepsilon_\pm)$ with $k_\pm$ positive and a 2×2 "gluing matrix" $G = \left(\begin{smallmatrix} m & p \\ n & q \end{smallmatrix}\right) \in \mathrm{M}_2(\mathbb{Z})$ satisfying conditions (1.7)–(1.9). Equivalently, $\det G = -k_+k_-$ and

$$m - \varepsilon_+^* n = Ak_+, \quad p - \varepsilon_+^* q = Bk_+, \quad p + \varepsilon_-^* m = Ck_-, \quad q + \varepsilon_-^* n = Dk_- \tag{7.8}$$

for some integers A, B, C, D with $(A, n) = (B, q) = (C, m) = (D, n) = 1$. We further assume that $n > 0$, $mnpq < 0$ and set $s_+ = \sqrt{-nq/mp}$, $s_- = \sqrt{-mn/pq}$, and $\rho = \pi - 2\arg(ms_+ + in)$. Then the invariant we want to compute is the combination of $F_{k,\varepsilon}$-values defined by

$$\mathcal{F}(k_+, \varepsilon_+; k_-, \varepsilon_-; G) \;:=\; \frac{1}{\pi}\left(F_{k_+,\varepsilon_+}(s_+) \;+\; F_{k_-,\varepsilon_-}(s_-) \;+\; \frac{\rho}{2}\right).$$

Proposition 7.3. *The number $\mathcal{F}(k_+, \varepsilon_+; k_-, \varepsilon_-; G)$ is always rational and is given by*

$$\mathcal{F}(k_+, \varepsilon_+; k_-, \varepsilon_-; G) \;=\; \frac{1}{6}\Big(\frac{m}{k_+n} - \frac{q}{k_-n} - 12\,S(A, n)\Big)\,,$$

where $S(A, n)$ is the Dedekind sum as defined in (0.3).

Proof. Set $\lambda = \frac{\varepsilon_-^* A + B}{k_-} = \frac{C - \varepsilon_+^* D}{k_+}$, which is an integer by (3.5). The equations (7.8) can be rewritten as

$$\left(\begin{smallmatrix} q & p \\ -n & -m \end{smallmatrix}\right) \;=\; \left(\begin{smallmatrix} k_- & \varepsilon_-^* \\ 0 & 1 \end{smallmatrix}\right)\gamma\left(\begin{smallmatrix} 1 & \varepsilon_+^* \\ 0 & k_+ \end{smallmatrix}\right) \qquad \text{with} \qquad \gamma = \left(\begin{smallmatrix} D & \lambda \\ -n & -A \end{smallmatrix}\right) \in SL_2(\mathbb{Z})\,.$$

It is easily checked that γ maps $\tau_+ = \frac{\varepsilon_+^* + is_+^{-1}}{k_+}$ to $\tau_- = \frac{-\varepsilon_-^* + is_-^{-1}}{k_-}$. From the transformation law (7.1) of $\mathcal{L}$, we get

$$\mathcal{L}(\tau_-) - \mathcal{L}(\tau_+) = \frac{1}{4}\,\mathrm{Log}\bigg(-\Big(\frac{m + ins_+^{-1}}{k_+}\Big)^2\bigg) + \frac{\pi i}{12}\left(\frac{A - D}{n} - 12S(A, n)\right).$$

Because $\mathcal{L}(-\bar{z}) = \overline{\mathcal{L}(z)}$, Proposition 7.1 gives

$$\begin{aligned}\mathcal{F}(k_+, \varepsilon_+; k_-, \varepsilon_-; G) &= \frac{2}{\pi}\left(\operatorname{Im}\mathcal{L}(\tau_-) - \operatorname{Im}\mathcal{L}(\tau_+)\right) + \frac{\rho}{2\pi} + \frac{\varepsilon_+^*}{6k_+} + \frac{\varepsilon_-^*}{6k_-} \\ &= \frac{1}{6}\bigg(\frac{m}{k_+n} - \frac{q}{k_-n} - 12\,S(A, n)\bigg)\,. \end{aligned}$$

□

We observe that (0.2) and Proposition 7.3 give an alternative proof of Theorem 3.

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Math. Inst. Univ. Freiburg, Ernst-Zermelo-Str. 1, 79104 Freiburg, Germany
Email address: `sebastian.goette@math.uni-freiburg.de`

Department of Mathematical Sciences, University of Bath, Bath BA2 7AY, UK
Email address: `j.nordstrom@bath.ac.uk`

Max Planck Institute for Mathematics, 53111 Bonn, Germany
International Centre for Theoretical Physics, Trieste, Italy
Email address: `dbz@mpim-bonn.mpg.de`