MODULAR LINEAR DIFFERENTIAL OPERATORS AND GENERALIZED RANKIN-COHEN BRACKETS

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Abstract. The aim of this paper is to give expressions for modular linear differential operators of any order. In particular, we show that they can all be described in terms of Rankin-Cohen brackets and a modified Rankin-Cohen bracket found by Kaneko and Koike. We also give more uniform descriptions of MLDOs in terms of canonically defined higher Serre derivatives and an extension of Rankin-Cohen brackets, as well as in terms of quasimodular forms and almost holomorphic modular forms. The last of these descriptions involves the holomorphic projection map. The paper also includes some general results on the theory of quasimodular forms on both cocompact and non-cocompact subgroups of $SL_2(\mathbb{R})$, as well as a slight sharpening of a theorem of Martin and Royer on Rankin-Cohen brackets of quasimodular forms.

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1. Introduction

Modular linear differential operators (MLDOs) and the corresponding modular linear differential equations (MLDEs) have appeared in recent years in a variety of contexts, ranging from the study of supersingular j-invariants to the classification of vertex operator algebras in terms of the spaces of modular forms spanned by the characters of their irreducible modules. In this paper we study and describe these operators and these differential equations in several different ways:

- as ordinary differential operators $L = \sum_{r=0}^{n} a_r(\tau)\, D^r$, where D is the normalized differentiation operator (see below) and the $a_r(\tau)$ are quasimodular forms that have to satisfy certain auxiliary conditions in order to make the operator L modular;
- as operators of the form $L = \sum_{r=0}^{n} b_r(\tau)\, \mathfrak{d}^r$, where $\mathfrak{d}^r$ is the rth iterate of the Serre derivative $\mathfrak{d}$ (see below) and the $b_r(\tau)$ are modular forms;
- as linear combinations of two other types of higher-order Serre derivatives;
- in terms of Rankin-Cohen brackets and the Kaneko-Koike operator (defined below);
- uniformly in terms of extended Rankin-Cohen brackets;
- in terms of quasimodular forms and the action of $\mathfrak{sl}_2$ on the space of quasimodular forms;
- in terms of almost holomorphic modular forms and the holomorphic projection operator.

Each of these approaches leads to a complete description of all MLDEs, but the representations obtained are quite different and are related to one another in non-obvious ways. The most uniform of these are two canonical bijections between the spaces of MLDOs and of quasimodular forms on any lattice in $SL_2(\mathbb{R})$, respecting both the weights and the natural filtrations on these two spaces, that are given in Section 8 and in Section 13 (or, in terms of almost holomorphic modular forms, in Section 14).

The paper falls naturally into two parts. In Section 2–6 we will recall some basic definitions, give the precise definition of MLDEs and simple properties of their solution spaces, describe a number of concrete examples of MLDOs including the Rankin-Cohen brackets and three kinds of higher-order Serre derivatives, and state the main structure theorems describing all MLDOs in the case of the full modular group, as well as giving examples coming from characters of modules over vertex operator algebras. The remaining sections give refinements, proofs, and extensions to other lattices (with a fairly detailed discussion of the different forms that the theory takes for cocompact and non-cocompact lattices) and also more conceptual descriptions of the various isomorphisms in terms of "extended Rankin-Cohen brackets" and of almost holomorphic modular forms. We also give as an application a result saying that one of the three kinds of higher-order Serre derivatives can be modified slightly to act on the space of quasimodular forms with a given upper bound on their depth (Theorem 11), and as a corollary of this a slight sharpening of a result of Martin and Royer [23] saying that a similar modification of Rankin-Cohen brackets also acts on pairs of quasimodular forms without increasing their combined depths.

2. Review of basic definitions

In this preliminary section we review some basic notions, including modular forms, quasimodular forms for the full modular group, and the Serre derivative. This material will be familiar to most readers but is included anyway for completeness and to fix notations.

By *lattice* we will mean a discrete subgroup Γ of finite covolume in $SL_2(\mathbb{R})$, acting in the complex upper half-plane $\mathfrak{H}$ by fractional transformations, i.e., $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ sends $\tau \in \mathfrak{H}$ to $\gamma\tau = \frac{a\tau+b}{c\tau+d}$. The case of most interest is when Γ is the full modular group $SL_2(\mathbb{Z})$, which we will denote by Γ_1, but in the latter sections of the paper will consider other lattices as well, including cocompact ones whose fundamental domains are compact in $\mathfrak{H}$ and for which there are no cusps, no Eisenstein series, and no Serre derivatives. If Γ is an arbitrary lattice, then for $k \in \mathbb{Z}$ we denote by $M_k(\Gamma)$ the space of holomorphic modular forms of weight k on Γ, meaning holomorphic functions f on $\mathfrak{H}$ satisfying $f|_k\gamma = f$ for all $\gamma \in \Gamma$. Here $f \mapsto f|_k\gamma$ is the operation ("slash operator") defined for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{R})$ by

$$\big(f|_k\gamma\big)(\tau) \;=\; (c\tau+d)^{-k} f\Big(\frac{a\tau+b}{c\tau+b}\Big) \tag{1}$$

and the word "holomorphic" includes a growth condition at cusps that is standard and will not be repeated here. (For $\Gamma = \Gamma_1$ it is just the condition that $f(\tau)$ is bounded as $\Im(\tau) \to \infty$.) The space $M_k(\Gamma)$ is finite-dimensional over $\mathbb{C}$ for each k and the direct sum $M_*(\Gamma) := \bigoplus_k M_k(\Gamma)$ is a finitely generated graded $\mathbb{C}$-algebra. When $\Gamma = \Gamma_1$ we usually omit it from the notations, writing simply M_* for $M_*(\Gamma_1)$. It is given explicitly by $M_* = \mathbb{C}[E_4, E_6]$, where E_k is the standard normalized Eisenstein series

$$E_k(\tau) \;=\; 1 \,+\, C_k \sum_{n=1}^{\infty} n^{k-1}\, \frac{q^n}{1-q^n} \qquad (k = 2,\, 4,\, 6,\, \dots)\,. \tag{2}$$

(Here and from now on $q = e^{2\pi i\tau}$, and the C_k are well-known rational numbers with $C_2 = -24$, $C_4 = 240$, and $C_6 = -504$.) The form E_2 is not modular, but belongs to the larger $\mathbb{C}$-algebra $\widetilde{M}_* = \mathbb{C}[E_2, E_4, E_6]$ of quasimodular forms on Γ_1, and for the moment we simply take this as the definition of $\widetilde{M}_*$. (The definition and properties of quasimodular forms for arbitrary lattices Γ will be given in Section 8.) The important point is that the derivative of a modular form of strictly positive weight is never modular, but is always quasimodular. In fact the ring $\widetilde{M}_*(\Gamma)$ (for any Γ) is closed under differentiation for any Γ, as one sees explicitly in the case of the full modular group from Ramanujan's famous formulas

$$E_2' \;=\; \frac{E_2^2 - E_4}{12}\,, \qquad E_4' \;=\; \frac{E_2E_4 - E_6}{3}\,, \qquad E_6' \;=\; \frac{E_2E_6 - E_4^2}{2}\,. \tag{3}$$

(Here and from now on we denote by $f'(\tau)$ or by $Df(\tau)$ the renormalized derivative $(2\pi i)^{-1}\, d/d\tau$, where the factor $(2\pi i)^{-1}$ is included so that the operator $D = q\, d/dq$ preserves the space of power series with rational coefficients in q.) These formulas show that the derivative $Df = f'$ of a quasimodular form of weight k has weight $k+2$ in this case, and the same holds for quasimodular forms on any lattice. Another consequence of (3), which will play an important role in this paper, is that the *Serre derivative*

$$\mathfrak{d}_k f(\tau) \;:=\; f'(\tau) \,-\, \frac{k}{12}\, E_2(\tau)\, f(\tau)\,, \tag{4}$$

maps M_k to M_{k+2}. Since a modular form has a well-defined weight, we will often omit the index k. In particular, we will often write $\mathfrak{d}^n$ (or by abuse of notation $\mathfrak{d}_k^n$) for the iterated Serre derivative $\mathfrak{d}_{k+2n-2}\cdots\mathfrak{d}_{k+2}\mathfrak{d}_k$, which maps modular forms of weight k to modular forms of weight $k+2n$.

3. Modular linear differential equations and their solution spaces

We can now formulate the key definition. A *modular linear differential operator* (MLDO) of *weight* K and *type* $(k, k+K)$ on Γ is a linear differential operator L of finite order, with holomorphic coefficients (also at the cusps, in the usual sense), satisfying

$$L\big(f|_k\gamma\big) \;=\; L(f)|_{k+K}\,\gamma \tag{5}$$

for all holomorphic functions f in the upper half-plane and all $\gamma \in \Gamma$. The corresponding *modular linear differential equation* (MLDE) is then the linear differential equation $Lf = 0$. Notice that in this definition, k can be positive or negative and in fact need not even be an integer, but K, as we will see, will always be a non-negative integer if the operator L is non-zero, and in fact always strictly positive except in the uninteresting case when L is multiplication by a constant.

Equation (5) implies in particular that L maps $M_k(\Gamma)$ to $M_{k+K}(\Gamma)$. But often we will apply the operator L to some space of holomorphic or meromorphic modular forms of weight k on some smaller subgroup $\Gamma' \subset \Gamma$. The important remark is that the kernel of L is a finite-dimensional space (of dimension equal to the order n of L) and invariant under the action of Γ in weight k. Conversely, any vector space V having these two properties is the solution space of some MLDE. This statement is well-known to experts and can be found in the literature (e.g. in [27] or [24]), but since the proof is enlightening and very short we give it here. Choose a basis $f_1,\, f_2,\, \dots,\, f_n$ of V and define a differential operator Δ_V of order n (which up to a scalar factor depends only on V and not on the chosen basis) by

$$\Delta_V(f) \;=\; \begin{vmatrix} f & \mathfrak{d}_k f & \cdots & \mathfrak{d}_k^n f \\ f_1 & \mathfrak{d}_k f_1 & \cdots & \mathfrak{d}_k^n f_1 \\ \vdots & \vdots & & \vdots \\ f_n & \mathfrak{d}_k f_n & \cdots & \mathfrak{d}_k^n f_n \end{vmatrix} \;=\; \begin{vmatrix} f & Df & \cdots & D^n f \\ f_1 & Df_1 & \cdots & D^n f_1 \\ \vdots & \vdots & & \vdots \\ f_n & Df_n & \cdots & D^n f_n \end{vmatrix},$$

where $\mathfrak{d}_k^r = \mathfrak{d}_{k+2r-2} \cdots \mathfrak{d}_{k+2} \mathfrak{d}_k$ is the rth iterate of the Serre derivative (4) and where the equality of the two determinants follows from the inductively proved fact that each operator $\mathfrak{d}_k^r$ is a the sum of D^r and a linear combination of D^p with $p < r$ and with quasimodular coefficients (of weight $2r - 2p$) depending only on k, r and p. Expanding the determinants by their first rows, we obtain two expressions $\Delta_V = \sum_{r=0}^n a_r(\tau)\, D^r$ and $\Delta_V = \sum_{r=0}^n b_r(\tau)\, \mathfrak{d}_k^r$ for the operator Δ_V in which each a_r is a quasimodular form of weight $n(k+n+1) - 2r$ and depth $\le n - r$, while the b_r are modular forms of the same weights. The second expression together with the transformation property $\mathfrak{d}_k^r(f|_k\gamma) = (\mathfrak{d}_k^r f)|_{k+2r}\gamma$ shows that Δ_V is indeed an MLDO of type $(k, n(k+n+1))$.

We make several remarks about this construction. First of all, if V is the solution space of $Lf = 0$ for some MLDO L, then L and Δ_V needn't agree up to a scalar factor, but may differ by a non-constant left factor, and indeed this often happens. An example is given by the space V spanned by the Rogers-Ramanujan functions defined by (9). As we will discuss in the next section, V is the solution space of the operator $L_{2,1/5}$ defined in (6), but when we calculate Δ_V with respect to the basis (G_0, G_1) we find that Δ_V is equal to $\frac{1}{5}\eta(\tau)^4$ times $L_{2,1/5}$, where $\eta(\tau) = \Delta(\tau)^{1/24} = q^{1/24} \prod(1 - q^n)$ as usual. One way to normalize the MLDO having a given space V as its solution space is to fix the top coefficient (e.g. taking it to be 1, which is the case of *monic* MLDOs discussed below, although making this normalization for a general MLDO with holomorphic coefficients may lead to an MLDO with only meromorphic coefficients). Another is to normalize L so that the quasimodular or modular forms occurring as coefficients in its D- or $\mathfrak{d}$-expansion are holomorphic and have no common factor of positive weight. In ideal cases, like for the operator (6) below, these agree. But in general one should be aware that giving a modular linear differential equation is the same as giving its solution space, but is not quite the same as giving a modular linear differential operator, since the opeators L and hL for any function $h(\tau)$ give the same differential equation $Lf = 0$, and for some purposes it is important to keep track of this distinction.

4. Examples of modular linear differential operators of small orders

The simplest non-trivial example of an MLDO is the Serre derivative $\mathfrak{d}_k$ as defined in (4). It has order 1, weight 2, and type $(k, k+2)$ and up to a constant factor is the only MLDO with these parameters. Notice that k here can be positive or negative, and that for $k = 0$ the Serre derivative reduces to just D, which is indeed an MLDO of type $(0, 2)$.

The next example, which has order 2 and weight 4, arose in [17] in connection with the study of supersingular j-invariants in characteristic $p > 3$. (The definition of supersingular plays no role here and will not be recalled.) If k is a positive even integer, then the composition $\mathfrak{d}_k^2$ of $\mathfrak{d}_k$ with $\mathfrak{d}_{k+2}$ goes from M_k to M_{k+4}. If $k+4$ is not divisible by 3 (which is always true in the application to supersingular j-values, where $k = p-1$ with $p > 3$ prime), then $M_{k+4} = M_k \cdot E_4$ and we obtain an endomorphism $E_4^{-1}\mathfrak{d}_k^2 = E_4^{-1}\mathfrak{d}_{k+2}\mathfrak{d}_k$ of the finite-dimensional space M_k. Its eigenvalues turn out to be λ_{k-12n} for $0 \le n \le k/12$, where $\lambda_k = \frac{k(k+2)}{144}$. The eigenfunction $F_k(\tau)$ with eigenvalue λ_k is unique up to a scalar and can be given explicitly in terms of the Euler-Gauss hypergeometric function ${}_2F_1(a, b; c; x)$ by

$$F_k(\tau) \;=\; E_4(\tau)^{k/4}\,{}_2F_1\Big(-\frac{k}{12}, -\frac{k-4}{12}; -\frac{k-5}{6}; \frac{1728}{j(\tau)}\Big),$$

where $j(\tau) = E_4(\tau)^3/\Delta(\tau)$ is the usual modular invariant (here $\Delta(\tau) = q\prod_{n\geq 1}(1-q^n)^{24} = (E_4(\tau)^3 - E_6(\tau)^2)/1728 \in M_{12}$, the Ramanujan discriminant function), and the eigenfunction with eigenvalue λ_{k-12n} is then $\Delta(\tau)^n F_{k-12n}(\tau)$. In particular the function $F_k(\tau)$ is a solution (and in fact the unique solution in M_k up to a scalar) of the MLDE $L_{2,k}(f) = 0$, where $L_{2,k}$ is the MLDO

$$L_{2,k} \;=\; \mathfrak{d}_k^2 \;-\; \lambda_k\, E_4 \tag{6}$$

of order 2 and type $(k, k+4)$. Written out explicitly in terms of D rather than $\mathfrak{d}$, this differential equation takes the form

$$f''(\tau) \;-\; \frac{k+1}{6}\, E_2(\tau) f'(\tau) \;+\; \frac{k(k+1)}{12}\, E_2'(\tau)\, f(\tau) \;=\; 0\,. \tag{7}$$

Our third example has order 3. Recall that the "Thetanullwerte" $\theta_j(\tau)$ $(j = 2,3,4)$ are the values at $z=0$ of the Jacobi theta functions $\theta_j(\tau,z)$ (the function $\theta_1(\tau,z)$ vanishes at $z=0$), given explicitly as $\sum q^{n^2/2}$, $\sum q^{(n+\frac12)^2/2}$, and $\sum(-1)^n q^{n^2/2}$, respectively, and are individually modular forms of weight $1/2$ on the congruence subgroup $\Gamma(2)$ of Γ_1 but are permuted (up to roots of unity) by the action of Γ_1. The space of modular forms of weight $k = n/2$ spanned by θ_2^n, θ_3^n and θ_4^n is therefore $SL_2(\mathbb{Z})$-invariant for any positive value of n and is 3-dimensional for $n \neq 4$. It follows from the considerations given in Section 3 that this space is the solution space of an MLDE of order 3, and by looking at the first few coefficients of the q-expansions we find that this equation is $\mathfrak{d}_k^3 f - \frac{3k^2-6k+8}{144} E_4\, \mathfrak{d}_k f - \frac{k^2(k-6)}{864} E_6 f = 0$, which in terms of ordinary derivatives becomes

$$f''' \;-\; \frac{k+2}{4} E_2 f'' \;+\; \Big(\frac{(k+1)(k+2)}{4} E_2' + \frac{k}{8} E_4\Big) f' \;-\; \Big(\frac{k(k+1)(k+2)}{24} E_2'' \;+\; \frac{k^2}{32} E_4'\Big) f \;=\; 0\,. \tag{8}$$

Finally, as already mentioned in the introduction, there are many examples of interesting MLDEs coming from the theory of vertex operator algebras and their characters. Roughly speaking, for a wide class of vertex operator algebras there are finitely many irreducible modules, each of which has a character (a power series in q whose coefficients are the dimensions of its graded pieces), and a well-known theorem of Zhu [36] says that under appropriate assumptions the vector space spanned by these characters is invariant under the action of the modular group and hence is precisely the solution space of some MLDE. We omit all definitions here, since this is not our main subject, but simply refer to [27], [28], [11], [16], [1], [25], [3], [32], [2], [12], [29] and their bibliographies for more details and examples. A very simple case is the so-called (2,5) minimal model, whose two characters are the famous Rogers-Ramanujan modular functions

$$G_0(\tau) \;=\; \sum_{n=0}^{\infty} \frac{q^{n^2-\frac{1}{60}}}{(1-q)(1-q^2)\cdots(1-q^n)}\,, \qquad G_1(\tau) \;=\; \sum_{n=0}^{\infty} \frac{q^{n^2+n+\frac{11}{60}}}{(1-q)(1-q^2)\cdots(1-q^n)}\,. \tag{9}$$

These are individually only invariant under the congruence subgroup $\Gamma(5)$ of $SL_2(\mathbb{Z})$ but together span the solution space of the MLDE $\mathfrak{d}^2 f - \frac{11}{3600} E_4 f = 0$, which can be written simply as $L_{2,1/5} f = 0$ where $L_{2,k}$ is defined as in (6), but now with k being $\frac15$ rather than an even integer. (See [27], which of course predates [17], and [26], as well as [11] and [12] for further discussion of the Kaneko-Zagier equation and its solutions from this point of view.) Similarly the third-order MLDO annihilating the nth powers of the Thetanullwerte arises in connection with the space of characters of the lattice VOA associated to the root lattice D_n. (See [28] or §3 of [25].) Sometimes one can also use the modular transformation properties of solutions of an MLDE "in reverse" to show the *non*-existence of VOAs of certain types. For instance, the non-existence of a certain apparently possible simple VOA with central charge $c = 164/5$ was proven in [2] using the third order MLDE

$$f''' \;-\; \frac12\, E_2\, f'' \;+\; \Big(\frac12\, E_2' - \frac{169}{100} E_4\Big)\, f' \;+\; \frac{1271}{1080}\, E_6\, f \;=\; 0\,, \tag{10}$$

whose solution space is spanned by three explicit homogeneous polynomials of degree 82 in the Rogers-Ramanujan functions G_0 and G_1 with huge coefficients. As a more complicated example, the 5th order MLDE

$$\begin{aligned} D^5(f) \;-\; \frac{5}{3}E_2D^4(f) + \Bigl(10E_2' + \frac{83}{99}E_4\Bigr)D^3(f) \;-\; \Bigl(10E_2'' + \frac{83}{66}E_4' + \frac{427}{3267}E_6\Bigr)D^2(f) \\ +\; \Bigl(\frac{5}{3}E_2''' + \frac{83}{330}E_4'' + \frac{427}{9801}E_6' + \frac{202}{107811}E_8\Bigr)D(f) \;+\; \frac{7888}{39135393}E_{10}f \;=\; 0 \end{aligned} \tag{11}$$

is used in [29] in connection with the minimal model of minimal model of type (2,11) and central charge $c = -232/11$.

Remark. Not all solutions of modular linear differential equations are necessarily modular forms. For instance, for the Kaneko-Zagier equation, one solution is always modular, and if the parameter is $\frac{1}{5}$ there are two independent modular solutions (namely, the two Rogers-Ramanujan functions multipllied by a power of $\eta(\tau)$), but in general the second element of a basis of solutions is a linear combination of products of modular forms and of Eichler integrals of modular forms of weight 2, as discussed in [14] and [13]. This second type of solution was described in [13] as a mixed mock modular form, and this is correct but is somewhat misleading since the mock modular forms occurring are of weight 0, so have shadows of weight 2, and the non-holomorphic Eichler integral of a modular form of weight 2 is simply the complex conjugate of the ordinary Eichler integral of a different modular form of weight 2, so that the notions of "mock" and "mixed mock" are not needed at all. (In fact, it seems probable that truly mixed mock modular forms, i.e., sums of products of modular forms and of mock modular forms of weight different from zero, can never be the solutions of any MLDE.) Perhaps the best way to see these non-modular solutions is as modular forms of second or higher order in the sense introduced in [4] and [19], the latter in connection with a specific second order differential equation coming from percolation theory.

5. Higher order examples: Rankin-Cohen brackets and higher Serre derivatives

The examples given above all had specified and quite small orders. Here we consider three families of examples of MLDOs of arbitrary order. The first and most important is given by the *Rankin-Cohen bracket*

$$[f,\,g]_n^{(k,\,\ell)} \;=\; \sum_{i=0}^{n}(-1)^i\binom{n+k-1}{n-i}\binom{n+\ell-1}{i}\,D^i(f)\,D^{n-i}(g) \qquad \bigl(n\in\mathbb{Z}_{\ge 0}\bigr)\,, \tag{12}$$

which belongs to $M_{k+\ell+2n}(\Gamma)$ if $f \in M_k(\Gamma)$ and $g \in M_\ell(\Gamma)$ for any group Γ. These bilinear operations will play a key role in the whole paper and will be discussed and generalized in Section 9. The map $f \mapsto [f,g]_n^{(k,\ell)}$ for fixed $g \in M_\ell(\Gamma)$ and any $n \in \mathbb{Z}_{\ge 0}$ is an MLDO of order n and type $(k, k+\ell+2n)$. As with the Serre derivative, we will often omit the superscripts and write simply $[f,g]_n$ for $[f,g]_n^{(k,\ell)}$ when f and g are modular forms of weights k and ℓ.

If $g = E_2$, then the Rankin-Cohen bracket $[\,\cdot\,,g]_n^{(\cdot,2)}$ no longer sends modular forms to modular forms, but it was discovered by Kaneko and Koike [15, p. 467][1] that the modified bracket

$$\Theta_k^n(f) \;:=\; D^n(f) \;-\; \frac{k+n-1}{12}\,\bigl[f,E_2\bigr]_{n-1}^{(k,2)} \tag{13}$$

sends modular forms of weight k to modular forms of weight $k+2n$ for arbitrary k and for all non-negative integers n. This is an MLDO of type $(k, k+2n)$ that reduces to the Serre derivative for $n = 1$ and to the Kaneko-Zagier operator $L_{2,k}$ for $n = 2$ and that will play an important role in the sequel. We will give a proof of the modularity and an interpretation of $\Theta_k^n(f)$ for $f \in M_k$ as an "extended Rankin-Cohen bracket" of f with a constant function in Section 9. We should also mention that the Kaneko-Koike operator was subsequently found at least two more times, by Henri Cohen and Frederik Strömberg [7] (Prop. 5.3.27, p. 164) in the above form and by Xuanzhong Dai

[1] Actually they rediscovered this statement, which was already in the original paper [6] of Cohen and was cited as such in [20] (p. 214, (ii)), but was then forgotten, and we will retain the name "Kaneko-Koike operator."

([9], eq. (1.7)) in the form of a generalized Rankin-Cohen bracket (slightly different from ours) with 1.

Finally, we have the *canonical higher Serre derivatives* $\mathfrak{d}_k^{[n]}$, which, like both the iterated Serre derivatives $\mathfrak{d}_k^n$ and the Kaneko-Koike operators Θ_k^n, are a family of monic MLDOs of order n and weight $2n$ that reduce to the Serre derivative if $n=1$, but which have much nicer properties than either of these other two families. These operators were first used in [31] in connection with the calculation of special values of L-functions of Hecke grossencharacters (whose definition again plays no role here and will be omitted) and are discussed in detail in Section 5 of the textbook [35]. They are defined recursively by

$$\mathfrak{d}_k^{[n+1]}(f) \;=\; \mathfrak{d}_{k+2n}\big(\mathfrak{d}_k^{[n]}(f)\big) \;-\; \frac{n(n+k-1)}{144}\,E_4\,\mathfrak{d}_k^{[n-1]}(f) \qquad (n\ge 1) \tag{14}$$

with the initial values $\mathfrak{d}_k^{[0]}(f)=f$, $\mathfrak{d}_k^{[1]}(f)=\mathfrak{d}_k(f)$, and have the attractive property that the Rankin-Cohen brackets are given by the *same* formula

$$[f,g]_n^{(k,\ell)} \;=\; \sum_{i=0}^{n}(-1)^i\binom{n+k-1}{n-i}\binom{n+\ell-1}{i}\,\mathfrak{d}_k^{[i]}(f)\,\mathfrak{d}_\ell^{[n-i]}(g) \qquad (n\in\mathbb{Z}_{\ge 0}) \tag{15}$$

in terms of the higher Serre derivatives as in terms of the ordinary ones, but with the difference that now each individual term of the sum defining $[f,g]_n$ is modular and not just quasimodular.

As already mentioned, the higher Serre derivatives $\mathfrak{d}_k^{[n]}$ actually form a much simpler family of higher order generalizations of $\mathfrak{d}_k$ than the more obvious iterated Serre derivatives $\mathfrak{d}_k^n$, because they have a very simple expansion in terms of the usual derivatives D^r (equation (16) below), whereas no corresponding explicit form for the iterated derivatives $\mathfrak{d}_k^n$ in terms of the D^r is known. For the Kaneko-Koike operators Θ_k^n, which as already mentioned can be seen as yet a third type of higher Serre derivatives, the situation is intermediate, since they are given by a relatively explicit closed formula, but as linear combinations of the $\mathfrak{d}_k^{[r]}$ rather than of the D^r, as stated in the following theorem.

Theorem 1. *The canonical higher Serre derivatives of a holomorphic function f in $\mathfrak{H}$ are related to the usual derivatives by*

$$\mathfrak{d}_k^{[n]}(f) \;=\; \sum_{r=0}^{n}\binom{n}{r}\,(k+r)_{n-r}\,\Big(-\frac{E_2}{12}\Big)^{n-r}\,D^r(f) \qquad (k,n\in\mathbb{Z}_{\ge 0}), \tag{16}$$

and the Kaneko-Koike derivatives are related to the canonical higher Serre derivatives by

$$\Theta_k^n(f) \;=\; \sum_{m=0}^{n}\binom{n}{m}\binom{k+n-1}{m}\,\omega_m\,\mathfrak{d}_k^{[n-m]}(f) \qquad (k,n\in\mathbb{Z}_{\ge 0}), \tag{17}$$

where the ω_m are modular forms in $M_{2m}(\Gamma_1)$ depending only on m, the first few being given by

m	0	1	2	3	4	5	6	7	8
ω_m	1	0	$-\frac{E_4}{72}$	$-\frac{E_6}{144}$	$-\frac{E_4^2}{288}$	$-\frac{5E_4E_6}{2592}$	$-\frac{9E_4^3+16E_6^2}{20736}$	$-\frac{35E_4^2E_6}{41472}$	$-\frac{117E_4^4+128E_4E_6^2}{373248}$

.

Here and from now on, $(x)_m$ denotes the “shifted factorial” (sometimes also called “rising factorial” or “ascending Pochhammer symbol”) $x(x+1)\cdots(x+m-1)$.

The proof of Theorem 1, and a description of the forms ω_m, will be given in Section 9.

6. Structure theorems

We now come to our central subject, the description of all MLDEs for a given lattice Γ. We denote by $\mathsf{MLDO}_{k,k+K}(\Gamma)$ the space of all modular linear differential operators of type $(k,k+K)$

on Γ and by $\mathsf{MLDO}^{(\le n)}_{k,k+K}(\Gamma)$ $(n \ge 0)$ the subspace of operators of order $\le n$, often omitting Γ from the notations when it is equal to Γ_1. We write the expansion of an element $L \in \mathsf{MLDO}^{(\le n)}_{k,k+K}(\Gamma)$ in the form

$$L \;=\; \sum_{r=0}^{n} a_r(\tau)\, D^r \tag{18}$$

where $a_0, \ldots, a_n$ are holomorphic functions in the upper half-plane. (At first sight it might seem more natural to write an operator L of order n as $a_0D^n + \cdots + a_n$ rather than $a_0 + \cdots + a_nD^n$, but this numbering of the coefficients would not respect the filtration of $\mathsf{MLDO}_{k,k+K}$ or the addition of operators of different orders, and would also make our later formulas much more complicated.) It is easy to see—and can be seen clearly in the examples (7), (8) (10) or (11) given in the last two sectlions—that each a_r must be a quasimodular form of weight $K - 2r$ and depth $\le n - r$ (where the *depth* of a quasimodular form will be discussed in Section 8 for general lattices, but for Γ_1 is just the degree of the form as a polynomial in E_2). In particular, since the weight of a holomorphic quasimodular form cannot be negative, we see that the order of any $L \in \mathsf{MLDO}_{k,k+K}(\Gamma)$ is at most $K/2$ and that we have an injective map

$$\begin{aligned} \mathsf{MLDO}^{(\le n)}_{k,k+K}(\Gamma) \quad &\hookrightarrow \quad \bigoplus_{r=0}^{n} \widetilde{M}^{(\le n-r)}_{K-2r}(\Gamma) \\ a_0 + a_1D + \cdots + a_nD^n \quad &\mapsto \quad (a_0, a_1, \ldots, a_n)\,, \end{aligned} \tag{19}$$

where $\widetilde{M}^{(\le p)}_{k}(\Gamma)$ denotes the space of quasimodular forms of weight k and depth $\le p$ on the group Γ. In particular we have the dimension estimate

$$\dim\bigl(\mathsf{MLDO}^{(\le n)}_{k,k+K}(\Gamma)\bigr) \;\le\; \sum_{r=0}^{n} \dim\bigl(\widetilde{M}^{(\le n-r)}_{K-2r}(\Gamma)\bigr) \;\le\; \sum_{r=0}^{n} \dim\bigl(\widetilde{M}_{K-2r}(\Gamma)\bigr) \tag{20}$$

(independent of k), but of course the actual dimension is much smaller, because the differential operator L defined by (18) with arbitrary quasimodular forms $a_r(\tau)$ of weight $K - 2r$ and depth $\le n - r$ as coefficients would in general be only a *quasimodular* linear differential operator of weight K, and in particular would send $M_k(\Gamma)$ for any k to $\widetilde{M}_{k+K}(\Gamma)$, but not in general to $M_{k+K}(\Gamma)$. There are therefore two natural questions:

Question 1. What conditions must quasimodular forms $a_r \in \widetilde{M}_{K-2r}(\Gamma)$ $(0 \le r \le n)$ satisfy in order that the differential operator defined by (18) is modular?

Question 2. What is the exact dimension of $\mathsf{MLDO}^{(\le n)}_{k,k+K}(\Gamma)$?

The first question will be answered in two ways in Section 7. The second is easier and will be answered here, although for the moment only for $\Gamma = \Gamma_1$.

Theorem 2. *An MLDO of weight K on the full modular group has order at most $K/2$ and is a linear combination of Rankin-Cohen brackets with modular forms of positive weight, together with the $(K/2)$-nd Kaneko-Koike operator if the order is equal to $K/2$.*

Corollary. *The dimension of $\mathsf{MLDO}^{(\le n)}_{k,k+K} = \mathsf{MLDO}^{(\le n)}_{k,k+K}(\Gamma_1)$ is given for any k, K and n by*

$$\dim\bigl(\mathsf{MLDO}^{(\le n)}_{k,k+K}\bigr) \;=\; \sum_{r=0}^{n} \dim\bigl(M_{K-2r}\bigr)\,. \tag{21}$$

Proof. We have already seen the first statement of the theorem, that the order of any L in $\mathsf{MLDO}_{k,k+K}$ is at most $K/2$. If it is exactly $K/2$, then the top coefficient $a_n(\tau)$ in (18) is a (holomorphic) quasimodular form of weight 0 and hence constant. Since the nth Kaneko-Koike operator (13) begins with D^n, we can simply subtract $a_n\, \Theta^n_k$ from L to reduce it to an operator of lower order. Now if the order is $n < K/2$, then the top coefficient $a_n(\tau)$ in (18) is a quasimodular form of weight $K-2n$ and depth 0, and hence actually a modular form. Then, since the Rankin-Cohen bracket (12) of a modular form f of weight k and a modular form g of weight $\ell > 0$ has an expansion beginning with a non-zero multiple of $f^{(n)}g$, we see that by subtracting from L a suitable non-zero multiple of $\big[\,\cdot\,, a_n\big]_n^{(k,K-2n)}$ we can again reduce its order by at least one. Continuing by induction, we obtain the theorem. The corollary is then immediate since $\dim M_0 = 1$ and since the $(K/2)$th Kaneko-Koike operator and the rth Rankin-Cohen brackets with modular forms of strictly positive weight $K-2r$ are easily seen to be linearly independent. □

Theorem 2 gives us canonical isomorphisms

$$\mathsf{MLDO}_{k,k+K} \;\cong\; \bigoplus_{r\ge 0} M_{K-2r}\,, \qquad \mathsf{MLDO}_{k,k+K}^{(\le n)} \;\cong\; \bigoplus_{r=0}^{n} M_{K-2r}\,, \tag{22}$$

defined (from right to left) by mapping $1 \in M_0$ to the Kaneko-Koike operator (13) (or to any chosen multiple of it) and $g \in M_{K-2r}$ with $K-2r > 0$ to the rth Rankin-Cohen bracket $[\,\cdot\,, g]_r^{k,K-2r}$ (or again to any chosen multiple of it, where the multiples in each case can be arbitrary non-zero numbers depending only on k, K and r). But three further descriptions can be obtained by noticing that any MLDO, as well as the expansion (18) in terms of ordinary higher derivatives, has three further expansions

$$L \;=\; \sum_{r=0}^{n} b_r(\tau)\,\mathfrak{d}_k^r \;=\; \sum_{r=0}^{n} c_r(\tau)\,\Theta_k^r \;=\; \sum_{r=0}^{n} d_r(\tau)\,\mathfrak{d}_k^{[r]} \tag{23}$$

in terms of the three different types of higher-order Serre derivatives introduced in Section 5, and unlike the situation for the original expansion (18) the condition on the coefficients here is trivial: since each of $\mathfrak{d}_k^r$, Θ_k^r, and $\mathfrak{d}_k^{[r]}$ preserves modularity and increases the weight by $2r$, we simply need that each of the three coefficients b_r, c_r and d_r belongs to M_{K-2r} for each r. This gives

Theorem 3. *We have three isomorphisms (22) given by mapping an operator $L \in \mathsf{MLDO}_{k,k+K}^{(\le n)}$ with the three expansions (23) to one of the vectors $(b_0,\dots,b_n)$, $(c_0,\dots,c_n)$, or $(d_0,\dots,d_n)$.*

Thus, although the final statement of the isomorphism of $\mathsf{MLDO}_{k,k+K}$ or $\mathsf{MLDO}_{k,k+K}^{(\le n)}$ to a known space is the same in Theorem 3 as in Theorem 2, the actual isomorphisms are completely different: whereas in Theorem 2 we mapped a modular form $g \in M_{K-2r}$ to (a multiple of) $[\,\cdot\,, g]_r$ or $\Theta_k^r(\,\cdot\,)$ (depending whether $2r < K$ or $2r = K$), which is a linear combination of derivatives of g times powers of D, we now map the same g to an operator that again begins with gD^r but is now simply a multiple of g, no longer containing any of its higher derivatives.

We can summarize the whole discussion of this section formally in terms of the filtration and its splittings. For any lattice $\Gamma \subset SL_2(\mathbb{R})$ and any integer $n \ge 0$ we have a canonical map σ from $\mathsf{MLDO}_{k,k+K}^{(\le n)}(\Gamma)$ to $M_{K-2n}(\Gamma)$ assigning to an operator L with the D-expansion (18) its top coefficient a_n, which we call the *symbol* of L. For $\Gamma = \Gamma_1$ this gives a short exact sequence

$$0 \longrightarrow \mathsf{MLDO}_{k,k+K}^{(\le n-1)} \longrightarrow \mathsf{MLDO}_{k,k+K}^{(\le n)} \xrightarrow{\;\sigma\;} M_{K-2n} \longrightarrow 0\,, \tag{24}$$

where the exactness at all places except the last (i.e., the surjectivity of σ) is clear and the last follows there from any of the four splittings of σ that we have given (mapping $g \in M_{K-2n}$ to the nth Rankin-Cohen bracket with g if $K-2n > 0$ and to the nth Kaneko-Koike operator if $g = 1$, or else multiplying it with any of the three Serre derivatives of order n defined in Section 5), corresponding to the four isomorphisms between MLDOs and vectors of modular forms described in Theorems 2 and 3. The space $\mathsf{MLDO}_{k,k+K}$ is filtered by the order and the exact sequence (24) gives a canonical identification of the graded vector space $\bigoplus_{n\ge 0}\mathsf{MLDO}_{k,k+K}^{(\le n)}(\Gamma)/\mathsf{MLDO}_{k,k+K}^{(\le n-1)}(\Gamma)$ associated to this filtration with $\bigoplus_{n\ge 0} M_{K-2n}$, with each of the four splittings giving an isomorphism of $\mathsf{MLDO}_{k,k+K}$ with the same space. Since all four isomorphisms induce the *same* isomorphism of graded as opposed to filtered vector spaces (because the exact sequence (24), unlike its splittings, is canonical), the passage from any one to any other can be described in terms of block unipotent matrices (blocks of size $\dim M_{K-2r}$, with zeros below the diagonal and identity matrices on the diagonal), but not all of these can be made explicit. In fact, the only passage between the different isomorphisms that we know how to describe in closed form is that between the isomorphism induced by the Kaneko-Koike operators Θ_k^r and the canonical higher Serre derivatives $\mathfrak{d}_k^{[r]}$, because of Theorem 1.

We end this section by giving examples of all four isomorphisms we have described for some of the explicit examples of MLDOs introduced in Section 4. For the Kaneko-Zagier operator $L_{2,k}$ this is almost trivial: it was already given as a combination of D^r and of $\mathfrak{d}_r$ in equations (7) and (6); its expression as a linear combination of Θ_k^2 and a Rankin-Cohen bracket or as a linear combination with modular coefficients of Θ_k^r is simply Θ_k^2 itself, and in terms of the higher Serre derivatives it equals $\mathfrak{d}_k^{[2]} - \frac{k(k+1)}{144}E_4$. For the third-order MLDO $L_{3,k}^\theta$ of type $(k, k+6)$ defined by the left-hand side of (8), the expression corresponding to Theorem 2 is

$$L_{3,k}^\theta f \;=\; \Theta_k^3(f) \,-\, \frac{k}{32}\,[f,E_4]_1^{(k,4)}$$

and those corresponding to Theorem 3 are

$$\begin{aligned} L_{3,k}^\theta f \;&=\; \mathfrak{d}_k^3(f) \,-\, \frac{3k^2-6k+8}{144} E_4 \mathfrak{d}_k(f) \,-\, \frac{k^2(k-6)}{864}\,E_6\,f \\ &=\; \Theta_k^3(f) \,+\, \frac{k}{8}\,E_4\,\Theta_k^1(f) \,+\, \frac{k^2}{96}\,E_6\,f \\ &=\; \mathfrak{d}_k^{[3]}(f) \,-\, \frac{(k-1)(k-2)}{48}\,E_4\,\mathfrak{d}_k(f) \,-\, \frac{k(k^2-6k+2)}{864}\,E_6\,f\,, \end{aligned} \tag{25}$$

while for the third-order MLDO of type (0,6) defined by $L_3 = D^3 - \frac{1}{2}E_2D^2 + (\frac{1}{2}E_2' - \frac{169}{100}E_4)D$ (i.e., the left-hand side of equation (10) without the final term, which is modular anyway) the corresponding expressions, which are easier here because k is 0 rather than a variable, are

$$\begin{aligned} L_3(f) \;&=\; \Theta_0^3(f) \,+\, \frac{169}{400}\,[f,E_4]_1^{(0,4)} \\ &=\; \mathfrak{d}_0^3(f) \,-\, \frac{1571}{900}\,E_4\,\mathfrak{d}_0(f) \;=\; \Theta_0^3(f) \,-\, \frac{169}{100}\,E_4\,\Theta_0^1(f) \;=\; \mathfrak{d}_0^{[3]}(f) \,-\, \frac{1039}{600}\,E_4\,\mathfrak{d}_0(f)\,. \end{aligned} \tag{26}$$

Similarly, writing L_5 for the operator in (11) without its last term, we find the four representations

$$\begin{aligned} L_5(f) \;&=\; \Theta_0^5(f) \,-\, \frac{83}{1980}\,[f,E_4]_3^{(0,4)} \,-\, \frac{61}{9801}\,[f,E_6]_2^{(0,6)} \,-\, \frac{101}{431244}\,[f,E_8]_1^{(0,8)} \\ &=\; \mathfrak{d}_0^5(f) \,-\, \frac{53}{396}\,E_4\,\mathfrak{d}_0^3(f) \,+\, \frac{295}{8712}\,E_6\,\mathfrak{d}_0^2(f) \,-\, \frac{6151}{1724976}\,E_8\,\mathfrak{d}_0(f) \\ &=\; \Theta_0^5(f) \,+\, \frac{83}{99}\,E_4\,\Theta_0^3(f) \,+\, \frac{1885}{6534}\,E_6\,\Theta_0^2(f) \,+\, \frac{7181}{143748}\,E_8\,\Theta_0(f) \\ &=\; \mathfrak{d}_0^{[5]}(f) \,+\, \frac{1}{198}\,E_4\,\mathfrak{d}_0^{[3]}(f) \,+\, \frac{35}{3267}\,E_6\,\mathfrak{d}_0^{[2]}(f) \,-\, \frac{2689}{1149984}\,E_8\,\mathfrak{d}_0(f)\,. \end{aligned} \tag{27}$$

7. The expansion coefficients of modular linear differential operators

We now turn to the first question posed in the last section, namely, the determination of the conditions that must be satisfied by the coefficients a_r in (18) in order that the differential operator defined there is modular.

Of course, from one point of view the answer to this question is almost tautological: we define a different series of coefficients b_s by requiring the first expansion in (23) to hold; these are computable combinations of the a's and their derivatives and hence are automatically quasimodular of the correct weights, and L is an MLDO if and only if they are actually modular. For small orders n we can carry this process out by hand, but for general n we cannot, because the expansion of $\mathfrak{d}_k^s$

as a polynomial in D with quasimodular coefficients is not known in closed form. We can, in fact, do this if we use the third expansion in (23) together with equation (16) and its easy inversion, in which case we find the following result, whose proof will be given in Section 9 when we prove (16):

Theorem 4. *The operator L defined by (18) is modular of type $(k, k+K)$ if and only if the function*

$$d_r(\tau) \;:=\; \sum_{j\ge 0}\binom{r+j}{j}\,(k+r+j)_j\,\Bigl(\frac{E_2(\tau)}{12}\Bigr)^j\,a_{r+j}(\tau) \tag{28}$$

is a modular form for $r=0,\dots,n$, in which case L is given by the final expression in (23).

(Recall that $(x)_m$ denotes the shifted factorial as defined at the end of Section 5.)

It is nevertheless interesting to give a direct description of the conditions that the original coefficients a_r must fulfill in order for L to be modular. We will do this first in terms of the modular transformation properties that they must satisfy and then give a second characterization by constructing certain explicit linear combinations of the a_r and their derivatives whose modularity is a necessary and sufficient condition for that of the operator L.

To study the transformation properties of the a_r, we will need the following lemma.

Lemma 1. *For any holomorphic function f, $\gamma=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in SL_2(\mathbb{R})$, $k\in\mathbb{Z}$ and $m\ge 0$ we have*

$$D^m\bigl((f\big|_k\gamma)(\tau)\bigr) \;=\; \sum_{r=0}^{m}\binom{m}{r}\,(k+r)_{m-r}\,\Bigl(-\frac{1}{2\pi i}\,\frac{c}{c\tau+d}\Bigr)^{m-r}\,\bigl((D^r f)\big|_{k+2r}\gamma\bigr)(\tau)\,. \tag{29}$$

Proof. This follows by induction on m, using $D(\gamma\tau)=\frac{1}{2\pi i}\frac{1}{(c\tau+d)^2}$ and the elementary identity $(k+m+r)\binom{m}{r}(k+r)_{m-r}+\binom{m}{r-1}(k+r-1)_{m-r+1}=\binom{m+1}{r}(k+r)_{m+1-r}$. □

For completeness and for later use, we also give the counterpart to this lemma in the other direction (first differentiate, then apply a modular transformation rather than vice versa), namely

$$\bigl((D^m f)|_{k+2m}\gamma\bigr)(\tau) \;=\; \sum_{r=0}^{m}\binom{m}{r}\,(k+r)_{m-r}\,\Bigl(\frac{1}{2\pi i}\,\frac{c}{c\tau+d}\Bigr)^{m-r}\,\bigl(D^r(f|_k\gamma)\bigr)(\tau)\,. \tag{30}$$

The proof of this formula (which is also given, for instance, at the top of p. 54 of [31], although only stated explicitly there for the case when $f\in M_k(\Gamma)$) is similar to that of (29) and will be omitted here, but indicated briefly in Section 9.

Using the lemma we can easily determine the modular transformation property of the vector of coefficients $a_r(\tau)$ in (18) needed to make the operator L modular.

Theorem 5. *Let $a_r(\tau)$ $(0\le r\le n)$ be holomorphic functions in the upper half-plane. Then the differential operator L defined by (18) is an MLDO of type $(k,k+K)$ with respect to a lattice Γ if and only if the $a_r(\tau)$ transform by*

$$\bigl(a_r\big|_{K-2r}\gamma\bigr)(\tau) \;=\; \sum_{j=0}^{n-r}\binom{r+j}{j}\,(k+r)_j\,\Bigl(-\frac{1}{2\pi i}\,\frac{c}{c\tau+d}\Bigr)^j\,a_{r+j}(\tau) \tag{31}$$

for all $0\le r\le n$, all $\tau\in\mathfrak{H}$, and all $\gamma=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\Gamma$.

Proof. Multiplying both sides of (29) by $a_m(\tau)$ and summing over m, we find

$$L(f\big|_k\gamma)(\tau) \;=\; \sum_{0\le r\le m\le n} a_m(\tau)\binom{m}{r}\,(k+r)_{m-r}\,\Bigl(-\frac{1}{2\pi i}\,\frac{c}{c\tau+d}\Bigr)^{m-r}\,\bigl((D^r f)\big|_{k+2r}\gamma\bigr)(\tau)\,.$$

On the other hand, from the definition of the slash operation it follows that

$$\big(L(f)|_{k+K}\gamma\big)(\tau) \;=\; \sum_{r=0}^{n}\big(a_r\big|_{K-2r}\gamma\big)(\tau)\,\big(D^r(f)\big|_{k+2r}\gamma\big)(\tau)\,.$$

Comparing the coefficients of $D^r(f)|_{k+2r}\gamma$, we see that (5) is equivalent to (31). □

Theorem 5 gives a complete answer to the first question posed in the previous section, since it gives a necessary and sufficient criterion for the coefficients in (18) to make L modular, but it is not a very easy one to check in practice. We would prefer to replace this description by one in terms of actual modular forms, in which case no further transformation properties are needed. To do this, we look at the first few cases of the transformation equation (31). If $r = n$, the top coefficient, it says

$$(a_n\big|_{K-2n}\gamma)(\tau) \;=\; a_n(\tau) \tag{32}$$

for all $\gamma \in \Gamma$, i.e., that a_n is a modular form of weight $K - 2n$, which of course we knew already. For the next case $r = n - 1$ it says

$$(a_{n-1}\big|_{K-2n+2}\gamma)(\tau) \;=\; a_{n-1}(\tau) \;-\; n(k+n-1)\,\frac{1}{2\pi i}\,\frac{c}{c\tau+d}\,a_n(\tau)\,. \tag{33}$$

But differentiating (32) with respect to τ gives

$$(a_n'\big|_{K-2n+2}\gamma)(\tau) \;=\; a_n'(\tau) \;+\; (K-2n)\,\frac{1}{2\pi i}\,\frac{c}{c\tau+d}\,a_n(\tau)\,, \tag{34}$$

and taking a suitable linear combination of this and (33) we find that the linear combination

$$h_{n-1}(\tau) \;:=\; a_{n-1}(\tau) \;+\; \frac{n(k+n-1)}{K-2n}\,a_n'(\tau) \tag{35}$$

transforms like a modular form of weight $K - 2n + 2$. Similarly, for the next case we find that

$$h_{n-2}(\tau) \;:=\; a_{n-2}(\tau) \;+\; \frac{(n-1)(k+n-2)}{K-2n+2}\,a_{n-1}'(\tau) \;+\; \frac{n(n-1)(k+n-2)(k+n-1)}{2(K-2n+1)(K-2n+2)}\,a_n''(\tau)$$

is modular of weight $K-2n+4$, and looking also at the next case we easily guess that the general statement should be that the function

$$h_m(\tau) \;:=\; \sum_{s\ge 0}\binom{m+s}{s}\,\frac{(k+m)_s}{(K-2m-s-1)_s}\,a_{m+s}^{(s)}(\tau) \tag{36}$$

(where the sum stops at $s = n-m$ because the later a_{m+s} vanish) is modular of weight $K-2m$ for $0 \le m \le n$. Since we know that $K \ge 2n$, we see that the denominators in each of these formulas are non-zero except in the case of (35) when $K = 2n$, in which case a_n is a constant and the second term becomes $0/0$. We will return to this point in a moment, but for now will simply assume that the order of our differential operator is strictly less than $K/2$.

Theorem 6. *Let $a_r(\tau)$ $(0 \le r \le n)$ be holomorphic functions in the upper half-plane. Then the differential operator L defined by (18), where $n < K/2$, is an MLDO of type $(k, k+K)$ with respect to a lattice Γ if and only if the function h_m defined by (36) belongs to $M_{K-2m}(\Gamma)$ for all $0 \le m \le n$. If this is the case, then the expansion of L as a linear combination of Rankin-Cohen brackets is given by*

$$L(f) \;=\; \sum_{m=0}^{n}\binom{K-m-1}{m}^{-1}\,\big[h_m, f\big]_m^{(K-2m,k)}\,. \tag{37}$$

Proof. If L is modular, then by induction on s, starting with Theorem 5 for $s = 0$, we find

$$a_r^{(s)}\big|_{K-2r+2s}\gamma \;=\; \sum_{\substack{j\ge 0\\ 0\le t\le s}}\binom{r+j}{j}\binom{s}{t}\,(k+r)_j\,(j-K+2r-s+1)_t\,U^{j+t}\,a_{r+j}^{(s-t)}$$

for all $s \geq 0$, where we have set $U = U_\gamma(\tau) = -\frac{1}{2\pi i}\frac{c}{c\tau+d}$ and omitted the variable τ for simplicity. Inserting this into (36) gives

$$h_m\big|_{K-2m}\gamma \;=\; \sum_{\ell,p\geq 0} \frac{(m+p+\ell)!}{m!\,p!\,\ell!}\,\frac{(k+m)_{p+\ell}\,U^\ell}{(K-2m-p-1)_p}\,a^{(p)}_{m+\ell+p}\sum_{t=0}^{\ell}(-1)^t\binom{\ell}{t}\frac{(K-2m-t-p-\ell)_t}{(K-2m-t-p-1)_t}$$

and this equals h_m because the interior sum vanishes for $\ell > 0$, since it is the ℓth difference of a polynomial of degree $\ell-1$. This proves the first assertion. For the second, we note that the inversion of (36), expressing the a's as linear combinations of the derivatives of the h's, is given by

$$a_r \;=\; \sum_{j\geq 0}(-1)^j\binom{r+j}{j}\frac{(k+r)_j}{(K-2r-2j)_j}\,h^{(j)}_{r+j}\,, \tag{38}$$

To see this, insert (36) into (38) and set $X = K-2r-1$ and $r+j = m$ to find

$$\text{RHS of (38)} \;=\; \sum_{m\geq 0}\frac{(r+1)_m\,(k+r)_m}{(X-2m)_{m+1}}\,a^{(m)}_{r+m}\sum_{j=0}^{m}(X-2j)\binom{X}{j}\binom{2m-X}{m-j}\,,$$

which equals a_r because the inner sum is equal to $X\big[\binom{2m}{m}-2\binom{2m-1}{m-1}\big] = 0$ for $m \geq 1$. Now insert equation (38) into (18) and use the definition of the Rankin-Cohen brackets to obtain (37). □

We now consider the case $K = 2n$ omitted so far, assuming that $\Gamma = \Gamma_1$. (General lattices will be treated in Section 12.) In that case the function a_n is a non-zero constant, which we can assume without loss of generality to be 1, so that $L = D^n$+(lower order terms) is monic. Recall that the problem here only concerned the function h_{n-1}, which cannot be defined by (35). But (33) with $a_n = 1$ now reduces to $(a_{n-1}\big|_2\gamma)(\tau) = a_{n-1}(\tau) - \frac{n(k+n-1)}{2\pi i}\frac{c}{c\tau+d}$, and comparing this with the transformation law of the weight 2 quasimodular form E_2, which will be recalled in the next section, we see that $a_{n-1} - \frac{n(k+n-1)}{12}E_2$ is a modular form of weight 2 on Γ_1 and hence is equal to 0. A computation similar to that of Theorem 6, though slightly more complicated, then gives the following theorem, which together with Theorem 6 gives the desired explicit version of Theorem 2. We omit the somewhat messy direct combinatorial proof of this theorem since we will find smoother approaches in Section 8 using quasimodular forms and in Section 9 using extended Rankin-Cohen brackets.

Theorem 7. *Let L be a monic MLDO of order n and type $(k, k+2n)$ on the full modular group. Then the function h_m defined by (36) is modular of weight $2n-2m$ for $0 \leq m \leq n-2$ and*

$$L(f) \;=\; \Theta^n_k(f) \,+\, \sum_{m=0}^{n-2}\binom{2n-m-1}{m}^{-1}\,\big[h_m, f\big]^{(K-2m,k)}_m\,. \tag{39}$$

8. Quasimodular forms and modular linear differential operators

In Section 6 we gave four different descriptions of MLDOs in terms of modular forms, using Rankin-Cohen brackets and higher-order Serre derivatives, the final result in each case being an isomorphism of filtered vector spaces as in (22). However, there is another space, much more familiar than the space of MLDOs, that is also canonically isomorphic (this time in two rather than four different canonical ways) to the same direct sum of spaces of modular forms, namely the space of all *quasimodular* forms of weight K and depth $\leq n$. In this section we show that there is a direct and extremely simple correspondence between quasimodular forms and MLDOs for any lattice, and that combining this correspondence with the two isomorphisms between quasimodular forms and vectors of ordinary modular forms gives two of the four descriptions of MLDOs given in Section 6 for $\Gamma = \Gamma_1$. We will postpone to Section 12 the discussion of the extent to which these latter results hold for other lattices (roughly speaking, in the cocompact case there is only one natural description of quasimodular forms in terms of modular forms and only one of the four isomorphisms in question generalizes, but in the non-cocompact case they all go through if one modifies the definitions appropriately), but we will give the definition and properties of quasimodular forms for arbitrary Γ already here.

The key structure on quasimodular forms is that they form a module over the Lie algebra $\mathfrak{sl}_2$. More explicitly, for any lattice Γ there are three natural derivations D, W and δ on the space $\widetilde{M}_*(\Gamma)$ of quasimodular forms on Γ that satisfy the commutation relations

$$[W,D] \;=\; 2D, \qquad [W,\delta] \;=\; -2\delta, \qquad [\delta,D] \;=\; W\,. \tag{40}$$

Here D is the renormalized differentiation operator $(2\pi i)^{-1}d/d\tau$ as above and W is the weight operator multiplying an element of $\widetilde{M}_k(\Gamma)$ by k (so that the first two commutation relations in (40) just say that D and δ increase and decrease the weight by 2, respectively). The definition of the operator δ is more complicated. For the case of the full modular group it is simply $12\,\partial/\partial E_2$ if we identify $\widetilde{M}_*(\Gamma_1)$ with $\mathbb{C}[E_2,E_4,E_6]$, the commutation relations (40) then being direct consequences of the Ramanujan formulas (3). To define it for general lattices, we must first review the definition and basic properties of quasimodular forms. We refer to §5.3 of [35] (where W is denoted E) for more details and for all proofs omitted here, none of which are difficult.

In fact there are two different definitions of quasimodular forms. The one used in [17], where the term was first introduced, was in terms of almost holomorphic modular forms and will be reviewed in the last section of this paper. The other, which was suggested subsequently by Werner Nahm, is more direct and more algebraic and will be discussed here.

The starting point for this definition is the observation that if we differentiate the transformation law $f\bigl(\frac{a\tau+b}{c\tau+d}\bigr)=(c\tau+d)^kf(\tau)$ of a modular form of weight k on some group $\Gamma\subset SL_2(\mathbb{R})$, we find after multiplying through by $(c\tau+d)^2$ that

$$f'\Bigl(\frac{a\tau+b}{c\tau+d}\Bigr) \;=\; (c\tau+d)^{k+2}f'(\tau) \,+\, \frac{k}{2\pi i}\,c(c\tau+d)^{k+1}f(\tau) \qquad \text{for } \begin{pmatrix} a & b\\ c & d\end{pmatrix}\in\Gamma \tag{41}$$

(recall that $f'=(2\pi i)^{-1}\,df/d\tau$), which describes precisely the extent to which f' fails to be a modular form of weight $k+2$ on Γ. (We already used this identity in the previous section when passing from equation (32) to equation (34).) Similarly, the weight 2 Eisenstein series defined in (2) (with $C_2=-\frac{1}{24}$), which is quasimodular of weight 2 and depth 1, satisfies (cf. [33])

$$E_2\Bigl(\frac{a\tau+b}{c\tau+d}\Bigr) \;=\; (c\tau+d)^2E_2(\tau)+\frac{6}{\pi i}\,c(c\tau+d) \qquad \text{for } \begin{pmatrix} a & b\\ c & d\end{pmatrix}\in SL_2(\mathbb{Z})\,. \tag{42}$$

We can rewrite (41) and (42) using the slash operator (1) as $\bigl(f'\big|_{k+2}\gamma\bigr)(\tau)=f'(\tau)+\frac{kf(\tau)}{2\pi i}\,\frac{c}{c\tau+d}$ and $\bigl(E_2\big|_2\gamma\bigr)(\tau)=E_2(\tau)+\frac{6}{\pi i}\frac{c}{c\tau+d}$, respectively. These two examples motivate the following general definition: A *quasimodular form* of weight K on a lattice $\Gamma\subset SL_2(\mathbb{R})$ is a holomorphic function $F:\mathfrak{H}\to\mathbb{C}$ of moderate growth such that the map $\gamma\mapsto(F|_K\gamma)(\tau)$ from Γ to $\mathbb{C}$ is a polynomial in $\frac{c}{c\tau+d}$ (where $\gamma=\bigl(\begin{smallmatrix} a&b\\c&d\end{smallmatrix}\bigr)$) for any fixed value of $\tau\in\mathfrak{H}$. If n is the maximum degree of these polynomials for all $\tau\in\mathfrak{H}$, we say that F is of *depth* n. Thus a quasimodular form of weight K and depth n has the transformation behavior

$$\bigl(F|_K\gamma\bigr)(\tau) \;=\; \sum_{r=0}^{n}F_r(\tau)\,\Bigl(\frac{1}{2\pi i}\,\frac{c}{c\tau+d}\Bigr)^r \tag{43}$$

for all $\gamma=\bigl(\begin{smallmatrix} a&b\\c&d\end{smallmatrix}\bigr)\in\Gamma$, where $F_0,\dots,F_n$ are holomorphic functions (of moderate growth, as usual) in the upper half-plane that are independent of γ. Taking γ to be the identity, we see that $F_0=F$, and we can now define δ by setting $\delta(F)=F_1$. It is then not very difficult to prove that $F_r=\delta^rF/r!$ for all $r\ge 0$, so that (43) takes the form of a "modified Taylor expansion"

$$\bigl(F|_K\gamma\bigr)(\tau) \;=\; \sum_{r=0}^{\infty}\frac{\delta^rF(\tau)}{r!}\,\Bigl(\frac{1}{2\pi i}\,\frac{c}{c\tau+d}\Bigr)^r\,. \tag{44}$$

Note that the infinite series always terminates, because $\delta^r F$ has weight $K-2r$ and hence vanishes for $2r > K$. The depth of F is by definition the largest value of r for which $\delta^r F$ is non-zero.

We can now combine these ideas with the results in the previous section to obtain a description of all MLDOs on an arbitrary lattice in terms of quasimodular forms. Specifically, by comparing (44) with equation (31) (Theorem 5), we find immediately that each coefficient a_r $(0 \le r \le n)$ of L in (18) is quasimodular of weight $K-2r$ and depth $n-r$, as mentioned earlier, and that the action of powers of δ on this coefficient is given by

$$\delta^j a_r \;=\; (-1)^j \,(r+1)_j \,(k+r)_j \, a_{r+j} \tag{45}$$

for all $j \ge 0$. These equations are actually overdetermined, since already the case $j=1$ tells us that $\delta a_{r-1} = -r(k+r-1)a_r$ and hence by induction on r that each a_r has the form $(-1)^r \delta^r a_0 / r!(k)_r$, which is what we would get directly from (45) with $r=0$ and which in turn implies (45) for all r and j. This establishes the following bijection between MLDOs and quasimodular forms:

Theorem 8. *Let F be a quasimodular form of weight K and depth n on an arbitrary lattice Γ. Then for every positive integer k the operator*

$$L_{F,k} \;:=\; \sum_{r=0}^{n} \frac{(-1)^r}{r!\,(k)_r}\, \delta^r(F)\, D^r \tag{46}$$

is a modular linear differential operator of order n and type $(k, k+K)$. For fixed $k>0$, the map $F \mapsto L_{F,k}$ gives an isomorphism

$$\widetilde{M}_*(\Gamma) \;\xrightarrow{\sim}\; \mathsf{MLDO}_{k,k+*}(\Gamma) \tag{47}$$

of filtered graded $M_*(\Gamma)$-modules, where $\widetilde{M}_*(\Gamma)$ is filtered by depth and $\mathsf{MLDO}_{k,k+*}(\Gamma)$ by order, with the inverse isomorphism mapping the differential operator (18) to its "constant term" $a_0(\tau)$.

As a simple example of the first statement of the theorem, if we take for F the quasimodular form $(-\frac{1}{12}E_2)^n$ of weight $2n$ and depth n, then $L_{F,k}(f)$ equals $\mathfrak{d}^{[n]}(f)/(k)_n$ with $\mathfrak{d}^{[n]}(f)$ as in equation (16). We should also remark that the weaker statement that $L_{F,k}(f)$ is a modular form of weight $k+K$ for any $F \in \widetilde{M}_K(\Gamma)$ and $f \in M_k(\Gamma)$ can be seen without the rather complicated calculations of Section 7 simply from the $\mathfrak{sl}_2$-commutation relations (40), since if f is a modular form of weight k then it satisfies $W(f) = kf$ and $\delta(f) = 0$ and then by a standard and easy induction also $\delta(D^r f) = r(k+r-1)D^{r-1} f$ (cf. equation (77) below) and therefore $\delta(L_{F,k}(f)) = 0$. The last statement of the theorem, which asserts that $L_{Fh,k} = hL_{f,k}$ for any modular form h, is true because $\delta^r(Fh) = \delta^r(F)\,h$ since δ is a derivation and annihilates h.

Theorem 8 makes sense and remains true if k is negative and non-integral, but if k is 0 or a negative integer then the right-hand side becomes meaningless. We will see one way to remedy this in the case $k=0$ in Section 10 by identifying $L_{F,k}(f)$ for $f \in M_k$ with the value $\{f, F\}$ of a pairing between modular and non-modular forms that makes sense also when $k=0$ and $f=1$. However, to understand MLDOs we do not merely want their action on modular forms, but on arbitrary differentiable functions, and here this is quite easy to do just by rescaling. For instance, if we multiply (46) by k and set $k=0$, we get the operator $\sum_{r\ge1} \frac{(-1)^r}{r!(r-1)!}\delta^r(F)\,D^r$, which is indeed an MLDO of type $(0,K)$ (equal, for example, to D if $F = -E_2/12$, $K=2$). More generally, if we divide (46) by $(k-1)!$, then this does not change any of the statements of the theorem as long as k is a strictly positive integer, but gives a new operator $\sum \frac{(-1)^r}{r!(k+r-1)!}\delta^r(F)\,D^r$ that is well-defined also for $k \in \mathbb{Z}_{\ge 0}$ if we interpret the terms having the factorial of a strictly negative integer in their denominator as 0. Then for Γ non-cocompact all of the statements of the theorem except for the statement about the inverse isomorphism remain true (for instance, for $\Gamma = \Gamma_1$, $k=0$ and $K=2$ the spaces $\mathsf{MLDO}_{0,2}(\Gamma) = \mathbb{C}\cdot D$ and $\widetilde{M}_2(\Gamma) = \mathbb{C}\cdot E_2$ are isomorphic, but we cannot see this by taking the "constant term"), whereas for cocompact lattices (which will be treated in Section 12) the map (46) still exists but is no longer always an isomorphism.

Theorem 8 gives a more conceptual explanation of the expansions discussed in Section 7. To see the relation to the various structure theorems in Section 6, we need first to know the structure of $\widetilde{M}_*(\Gamma)$. For the moment we do this only for $\Gamma = \Gamma_1$; the results for general lattices, which differ in several respects, will be discussed in Section 12. In this case there are two quite different descriptions of quasimodular forms in terms of modular forms, as already mentioned at the beginning of the section. One of them, which is an obvious consequence of the expressions for the rings $M_* = M_*(\Gamma_1)$ and $\widetilde{M}_* = \widetilde{M}_*(\Gamma_1)$ as $\mathbb{C}[E_4, E_6]$ and $\mathbb{C}[E_2, E_4, E_6]$, respectively, is that every quasimodular form on the full modular group can be written uniquely as a polynomial in E_2 with modular coefficients. This gives a canonical isomorphism

$$\bigoplus_{r=0}^{n} M_{K-2r} \;\simeq\; \widetilde{M}_K^{(\leq n)}\,, \qquad (g_0,\dots,g_n) \;\longleftrightarrow\; F \;=\; \sum_{j=0}^{n} g_j\, E_2^j\,. \tag{48}$$

The second description, which is equally easy to establish (see Proposition 3 for a proof in a slightly more general situation), is that every quasimodular form of positive weight can be expressed uniquely as a linear combination of derivatives of modular forms and of E_2. This means that if we define spaces $M_k^+ = M_k^+(\Gamma_1)$ by $M_0^+ = \{0\}$, $M_2^+ = \mathbb{C}\,E_2$, and $M_k^+ = M_k$ for $k > 2$, then for every $K > 0$ we have a second canonical isomorphism

$$\bigoplus_{r=0}^{n} M_{K-2r}^+ \;\simeq\; \widetilde{M}_K^{(\leq n)}\,, \qquad (h_0,\dots,h_n) \;\longleftrightarrow\; F \;=\; \sum_{j=0}^{n} D^j(h_j) \tag{49}$$

between $\widetilde{M}_K^{(\leq n)}$ and a space of the same dimension as before, but in which the 1-dimensional space $\mathbb{C}\cdot 1$ has been replaced by the 1-dimensional space $\mathbb{C}\cdot E_2$. (We will see a natural interpretation of this in Section 10 when we interpret the quasimodular form $E_2/12$ as a modified derivative of the modular form 1.)

If we now combine the two isomorphisms (48) and (49) with the isomorphism given in Theorem 8 between quasimodular forms and MLDOs, we get two of the four descriptions of MLDOs given in Theorems 2 and 3 of Section 6: if F has the form given in (48), then the associated MLDO $L_{F,k}$ is given by the third isomorphism in (23) with $d_r = 12^r g_r/(k)_r$, while if F has the form given in (49), then $L_{F,k}(f)$ is a linear combination of the Rankin-Cohen brackets $[h_j, f]_j^{(K-2j,k)}$ for $j < (K-2)/2$ and $\Theta_k^j(f)$ for $j = (K-2)/2$, with easily determined coefficients. (See Section 10.)

9. Cohen-Kuznetsov series, Rankin-Cohen brackets, and higher Serre derivatives

The best way to understand the relation between the derivatives and the modular transformation properties of functions in the upper half-plane is through the generating function of their derivatives. The right generating series to use, as was discovered independently by Henri Cohen [6] and Nikolai Kuznetsov [22] in 1975, is defined by the formula

$$\Phi_{f,k}(\tau,X) \;=\; \sum_{n=0}^{\infty} \frac{D^n(f)}{n!\,(k)_n}\, X^n \tag{50}$$

for any holomorphic function f in $\mathfrak{H}$ and $k\in\mathbb{Z}$ and satisfies the transformation property

$$\Phi_{f,k}\Bigl(\frac{a\tau+b}{c\tau+d},\frac{X}{(c\tau+d)^2}\Bigr) \;=\; (c\tau+d)^k\,\exp\Bigl(\frac{c}{c\tau+d}\,\frac{X}{2\pi i}\Bigr)\,\Phi_{f|_k\gamma,k}(\tau,X) \tag{51}$$

for any $\gamma = \bigl(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\bigr) \in SL_2(\mathbb{R})$. This is an easy consequence of the transformation formula (29) (Lemma 1) and is in turn equivalent to the inverse transformation formula (30). If $f \in M_k(\Gamma)$ for some lattice $\Gamma \subset SL_2(\mathbb{R})$, then (51) says that the function $\Phi_{f,k}(\tau,X)$ is invariant up to a simple automorphy factor under the action $(\tau,X) \mapsto \bigl(\frac{a\tau+b}{c\tau+d},\frac{X}{(c\tau+d)^2}\bigr)$ of Γ. In this case we usually omit the weight, writing simply $\Phi_f(\tau,X)$ for $\Phi_{f,k}(\tau,X)$.

The Cohen-Kuznetsov series are related to the Rankin-Cohen brackets defined in (12) by

$$\sum_{n=0}^{\infty} \frac{[f,g]_n^{(k,\ell)}(\tau)}{(k)_n\,(\ell)_n}\,X^n \;=\; \Phi_{f,k}(\tau,-X)\,\Phi_{g,\ell}(\tau,X)\,, \tag{52}$$

so that the transformation property (51) immediately implies[2] the transformation property

$$\bigl[f|_k\gamma,\, g|_\ell\gamma\bigr]_n^{(k,\,\ell)} \;=\; \bigl[f,g\bigr]_n^{(k,\,\ell)}\bigl|_{k+\ell+2n}\gamma \qquad (\gamma\in SL_2(\mathbb{R}))$$

of Rankin-Cohen brackets and hence that $[M_k(\Gamma),M_\ell(\Gamma)]_n^{(k,\ell)}\subseteq M_{k+\ell+2n}(\Gamma)$. But they are also the key to understanding the higher Serre derivatives $\mathfrak{d}_k^{[n]}(f)$ and $\Theta_k^n(f)$ introduced in Section 5.

For this purpose, we introduce the three modified Cohen-Kuznetsov series

$$\Phi_{f,k}^{(\widehat{D})}(\tau,X) \;=\; \sum_{n=0}^{\infty}\frac{\widehat{D}^n(f)}{n!\,(k)_n}\,X^n,\quad \Phi_{f,k}^{(\mathfrak{d})}(\tau,X) \;=\; \sum_{n=0}^{\infty}\frac{\mathfrak{d}_k^{[n]}(f)}{n!\,(k)_n}\,X^n,\quad \Phi_{f,k}^{(\Theta)}(\tau,X) \;=\; \sum_{n=0}^{\infty}\frac{\Theta_k^n(f)}{n!\,(k)_n}\,X^n$$

obtained by replacing $D^n(f)$ in (50) by $\widehat{D}^n(f)$, $\mathfrak{d}_k^{[n]}(f)$ or $\Theta_k^n(f)$, respectively, where again the index k will usually be omitted if f is modular of weight k. Here $\widehat{D}_k$ is the non-holomorphic derivative $\widehat{D}_kf(\tau)=Df(\tau)-\frac{k}{4\pi y}f(\tau)$ $(y:=\Im(\tau))$ and $\widehat{D}_k^n$ denotes the composition $\widehat{D}_{k+2n-2}\cdots\widehat{D}_{k+2}\widehat{D}_k$. It is well known, and easily checked, that $\widehat{D}_k(f|_k\gamma)=\widehat{D}_k(f)|_{k+2}\gamma$, so that $\widehat{D}_k^n$ preserves modularity (but not holomorphy). The first two of these modified series were studied in [35], where it was shown (pp. 54–55) that they are related to the original Cohen-Kuznetsov series $\Phi_{f,k}=\Phi_{f,k}^{(D)}$ by

$$\Phi_f^{(D)}(\tau,X) \;=\; e^{X/4\pi y}\,\Phi_f^{(\widehat{D})}(\tau,X) \;=\; e^{XE_2(\tau)/12}\,\Phi_f^{(\mathfrak{d})}(\tau,X)\,, \tag{53}$$

where E_2 is the non-modular Eisenstein series of weight 2 defined in Section 2. (We should mention here that our notations differ in several respects from those of [35]: the derivations denoted here by W, $\mathfrak{d}=\mathfrak{d}_k$ and $\widehat{D}$ are denoted there by E, $\vartheta=\vartheta_k$ and $\partial=\partial_k$, or inadvertently on p. 60 by $\vartheta=\vartheta_k$, the Cohen-Kuznetsov series denoted here by Φ_f, $\Phi_f^{(\widehat{D})}$ and $\Phi_f^{(\mathfrak{d})}$ are denoted in [35] by $\widetilde{f}_D$, $\widetilde{f}_\partial$ and $\widetilde{f}_{\mathfrak{d}}$, and the argument $\tau\in\mathfrak{H}$ is denoted there by z.) The second two of these relations give

$$\Phi^{(\mathfrak{d})}(\tau,X) \;=\; e^{-X\widehat{E}_2(\tau)/12}\,\Phi^{(\widehat{D})}(\tau,X)\,, \tag{54}$$

where $\widehat{E}_2(\tau)$ denotes the non-holomorphic "completion" $E_2(\tau)-\frac{3}{\pi y}$ of $E_2(\tau)$, which transforms like a modular form of weight 2 because of the transformation equation (42), so that the fact that $\mathfrak{d}_k^{[n]}$ maps modular forms of weight k on Γ_1 to modular forms of weight $k+2n$ is a direct consequence of the fact that $\widehat{D}_k^n$ preserves modularity. The equality between the first and third terms of (53), on the other hand, immediately gives equation (16), the first part of Theorem 1.

For the proof of the second part of Theorem 1 concerning the Kaneko-Koike derivatives, we introduce a Cohen-Kuznetsov series $\Phi_1=\Phi_{1,0}$ for the constant function $1\in M_0(\Gamma_1)$. The original definition does not make any sense in this case, because all but the first terms of (50) have vanishing numerator and denominator. The correct definition is given by the following proposition, which was also found by Dai ([9], eq. (6.7)).

Proposition 1. *The Cohen-Kuznetsov series $\Phi_1=\Phi_{1,0}$ defined by*

$$\Phi_{1,0}(\tau,X) \;=\; 1+\frac{X}{12}\,\Phi_{E_2,2}(\tau,X) \;=\; 1+\frac{1}{12}\sum_{n=1}^{\infty}\frac{D^{n-1}E_2(\tau)}{n!\,(n-1)!}\,X^n \tag{55}$$

[2] at least if k and ℓ are positive. If we allow the modular forms f and g to be meromorphic, then their weights can be negative; in this case Lemma 1 still implies that $[f,g]_n^{(k,\ell)}$ is modular of weight $k+\ell+2n$, but this cannot be proved using (52) since the definition of $\Phi_{f,k}$ does not make sense when k is a non-positive integer.

transforms under the action of $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_1$ *by*

$$\Phi_1\Bigl(\frac{a\tau+b}{c\tau+d},\, \frac{X}{(c\tau+d)^2}\Bigr) \;=\; \exp\Bigl(\frac{c}{c\tau+d}\,\frac{X}{2\pi i}\Bigr)\,\Phi_1(\tau,X)\,. \tag{56}$$

The modified Cohen-Kuznetsov series $\Phi_1^{(\mathfrak{d})} = \Phi_{1,0}^{(\mathfrak{d})}$ *defined by*

$$\Phi_1^{(\mathfrak{d})}(\tau,X) \;=\; e^{-XE_2(\tau)/12}\,\Phi_1(\tau,X) \tag{57}$$

is invariant under $(\tau,X) \mapsto \bigl(\frac{a\tau+b}{c\tau+d}, \frac{X}{(c\tau+d)^2}\bigr)$.

Proof. We can prove this by imitating any of the three proofs given in [35] (p. 54) for equation (51) when the weight k is positive. One of them is the one we gave above using equation (30), and we can apply this also in weight 0 by using equation (30) for $f = E_2$ and $k = 2$ together with the transformation formula (42) to deduce that $\Phi_{E_2,2}$ has precisely the transformation properties under Γ_1 needed to imply equation (56) for $\Phi_{1,0}$. Another proof in [35] used the fact that $\Phi_f(\tau,X)$ for $k>0$ is the unique solution of the differential equation $\bigl(X\frac{\partial^2}{\partial X^2} + k\frac{\partial}{\partial X} - D\bigr)\Phi_f = 0$ with constant term f and that the function $(c\tau+d)^{-k}e^{-cX/2\pi i(c\tau+d)}\Phi_f\bigl(\frac{a\tau+b}{c\tau+d}, \frac{X}{(c\tau+d)^2}\bigr)$ has the same properties. For $f=1$ and $k=0$ the first statement is no longer true, but all solutions of the differential equation $\bigl(X\frac{\partial^2}{\partial X^2} - D\bigr)\Phi_f = 0$ with constant term 1 have the form $1 + \sum_{n\ge1}\frac{D^{n-1}(F)}{n!(n-1)!}X^{n-1}$ for some function F, which is arbitrary, and therefore for any F there is a unique solution beginning $1 + FX + \mathrm{O}(X^2)$. This function therefore satisfies the transformation equation (56) if (and only if) F satisfies $F|_2\gamma = F + \frac{1}{2\pi i}\frac{c}{c\tau+d}$, which is the case for $F = \frac{1}{12}E_2$ by virtue of (42) (and for no other holomorphic function F of bounded growth at infinity because there are no non-zero modular forms of weight 2 on Γ_1). The second statement of the proposition follows immediately from the first and the transformation property (42) of $E_2(\tau)$. We observe that the definition (57) is just the relation given in (53) between $\Phi_f^{(\mathfrak{d})}$ and Φ_f for modular forms f of positive weight, but now decreed to hold also for $f=1$. □

By combining the definition (13) of the Kaneko-Koike operator with equations (52) and (55), we can rewrite that definition in terms of generating series as

$$\Phi_{f,k}^{(\Theta)}(\tau,X) \;=\; \Phi_{f,k}^{(D)}(\tau,X)\,\Phi_{1,0}(\tau,-X)\,. \tag{58}$$

Comparing this with equation (52), we see that the Kaneko-Koike derivatives $\Theta^n(f)$ can be interpreted in some sense simply as the Rankin-Cohen brackets of f with the constant function 1. (We will make more precise sense of this statement in the next section.) In any case, equation (58) together with the transformation property (56) of Φ_1 immediately implies the fact that Θ_k^n is an MLDO of type $(k, k+2n)$ and hence sends modular forms of weight k to modular forms of weight $k+2n$.

Equation (58) tells us that the fourth of our modified Cohen-Kuznetsov series $\Phi_{f,k}^{(\Theta)}$ is related to the other three by universal factors, not depending on the form f or its weight k, but no longer purely exponential in X as was the case for the relationships among the other three series. We can combine it with the previous statements (53) and (54) into a single diagram

$$\begin{array}{ccccc} & & \boxed{\Phi_f^{(\widehat{D})}(\tau,X)} & & \\ & \nearrow \cdot\, e^{X/4\pi y} & & \searrow \cdot\, e^{-X\widehat{E}_2(\tau)/12} & \\ \boxed{\Phi_f^{(D)}(\tau,X)} & & \xrightarrow{\ \cdot\, e^{-XE_2(\tau)/12}\ } & & \boxed{\Phi_f^{(\mathfrak{d})}(\tau,X)} \\ & \searrow \cdot\, \Phi_1^{(D)}(\tau,-X) & & \swarrow \cdot\, \Phi_1^{(\mathfrak{d})}(\tau,-X) & \\ & & \boxed{\Phi_f^{(\Theta)}(\tau,X)} & & \end{array} \tag{59}$$

showing the relationships between all four series. The last arrow of this diagram also proves equation (17) of Theorem 1, with the modular forms ω_m defined by the generating function

$$\sum_{m=0}^{\infty} \frac{\omega_m(\tau)}{m!^2}\,(-X)^m \;=\; \Phi_1^{(\mathfrak{d})}(\tau,X)\,. \tag{60}$$

The first terms of this series are then easily calculated to be the ones tabulated in Theorem 1.

10. Modified derivatives and extended Rankin-Cohen brackets

In this section we introduce a new and very convenient notion that will make many statements of the theory simpler and more uniform, namely, modified higher derivatives $D^{\langle n\rangle}(f)$ that are simply a renormalization of the usual derivatives $D^n(f)$ when f has positive weght but which are non-trivial also for $f=1$.

Specifically, if f is a modular form of positive weight we define

$$D^{\langle n\rangle}(f) \;=\; D_k^{\langle n\rangle}(f) \;=\; \frac{D^n(f)}{(k)_n} \qquad (f\in M_k(\Gamma),\ k>0)\,, \tag{61}$$

where $(k)_n = k(k+1)\cdots(k+n-1)$ as before. This renormalization seems unnecessary and somewhat strange at first sight but turns out to be very natural. In particular, in terms of this notation the Cohen-Kuznetsov series (50) can be written very simply as

$$\Phi_f(\tau,X) \;=\; \sum_{n=0}^{\infty} \frac{D^{\langle n\rangle}(f)}{n!}\,X^n \tag{62}$$

or symbolically as

$$\Phi_f(\tau,X) \;=\; e^{\langle XD\rangle} f(\tau)\,. \tag{63}$$

The real point, however, is that we can now define higher derivatives of the constant function 1 in a useful way. The definition (61) makes no sense in this case for $n>0$ (for $n=0$ it of course just gives $D^{\langle 0\rangle}(1)=1$), because both the numerator and the denominator of the fraction vanish. But if we recall that the Serre derivative $\mathfrak{d}_k(f)=D(f)-\frac{k}{12}E_2 f$ of a modular form of weight k is modular of weight $k+2$, and that the function $\frac{\mathfrak{d}_k(f)}{k} = D^{\langle 1\rangle}(f) - \frac{E_2}{12}f$ is therefore also modular of this weight, we see that $D^{\langle 1\rangle}(1)$ should be defined as the sum of $\frac{1}{12}E_2$ and a modular form of weight 2, and hence must be taken to be $\frac{1}{12}E_2$. For the higher derivatives, there is then no problem, since equation (61) in the positive weight case gives the inductive formula $D^{\langle n+1\rangle}f = D(D^{\langle n\rangle}(f))/(k+n)$, which makes sense even for $k=0$ as soon as $n>0$. We therefore define $D^{\langle n\rangle}(1)$ for all $n\ge 0$ by

$$D^{\langle n\rangle}(1) \;=\; D_0^{\langle n\rangle}(1) \;=\; \begin{cases} 1 & \text{if } n=0, \\ \frac{D^{n-1}(E_2)}{12\,(n-1)!} & \text{if } n\ge 1, \end{cases} \tag{64}$$

and discover that formula (62) for the Cohen-Kuznetsov series of a modular form of positive weight is still valid for $f=1$ and $k=0$ if we use the definition of Φ_1 given in Proposition 1. In fact, we could have simply used equations (62) and (55) as the motivation of the definition (64). We can also give $D^{\langle n\rangle}(1)$ by the explicit formula

$$D^{\langle n\rangle}(1) \;=\; \sum_{m=0}^{n}(-1)^m\binom{n}{m}\frac{\omega_m}{m!}\left(\frac{E_2}{12}\right)^{n-m} \tag{65}$$

with ω_m as in Theorem 1. (The proof will be given later.)

So far this is purely formal and a matter of introducing new notations for objects that we already knew. But in fact it immediately leads to many simplifications in the formulas that we have studied so far, as well as in those given in the rest of this section and the two following ones. The first concerns the Rankin-Cohen bracket. If we rewrite the definition (12) in terms of the modified derivatives (61) for f and g of positive weights k and ℓ, we find that

$$[f,\,g]_n^{(k,\,\ell)} \;=\; \frac{(k)_n\,(\ell)_n}{n!}\sum_{i=0}^{n}(-1)^i\binom{n}{i}D^{\langle i\rangle}(f)\,D^{\langle n-i\rangle}(g)\,. \tag{66}$$

This equation still holds if k or ℓ is zero, but is then not interesting since both sides vanish for $n\ge 1$, because the factor in front of the sum is then equal to 0. But the sum itself is in general non-zero and therefore give an interesting modified Rankin-Cohen bracket even in the case of forms of weight 0. We therefore define the nth *extended Rankin-Cohen bracket* $\langle f,g\rangle_n$ of any two modular forms f and g and any $n\ge 0$ by

$$\bigl\langle f,\,g\bigr\rangle_n \;:=\; \sum_{i=0}^{n}(-1)^i\binom{n}{i}D^{\langle i\rangle}(f)\,D^{\langle n-i\rangle}(g)\,. \tag{67}$$

We then have

Theorem 9. *If f and g are modular forms of weight k and ℓ, respectively, then the extended Rankin-Cohen bracket (67) is a modular form of weight $k+\ell+2n$ for all $n\ge 0$, and is given in terms of the usual Rankin-Cohen bracket, the Kaneko-Koike derivative, and the modular forms ω_m defined in Theorem 1 by*

$$\bigl\langle f,\,g\bigr\rangle_n \;=\; \begin{cases} \frac{n!}{(k)_n(\ell)_n}\,[f,g]_n^{(k,\ell)} & \text{if } k,\,\ell>0,\\ \frac{1}{(\ell)_n}\,\Theta_\ell^n(g) & \text{if } f=1,\ \ell>0,\\ \frac{1}{n!}\,\sum_{m=0}^{n}(-1)^m\binom{n}{m}^2\omega_m\,\omega_{n-m} & \text{if } f=g=1. \end{cases} \tag{68}$$

Proof. The statement about modularity follows immediately from the transformation properties of $\Phi_f(\tau,X)$ for modular forms of positive weight (equation (51)) or weight 0 (equation (56)) together with equation (52), which in the new notation and in view of equation (62) takes on the simpler form

$$\Phi_f(\tau,-X)\,\Phi_g(\tau,X) \;=\; \sum_{n=0}^{\infty}\langle f,g\rangle_n\,\frac{X^n}{n!}\,. \tag{69}$$

The proportionality of $\langle f,g\rangle_n$ with $[f,g]_n$ for k and ℓ positive follows from equation (66), which says that $[f,g]_n=\frac{(k)_n(\ell)_n}{n!}\langle f,g\rangle_n$ in this case. For g of positive weight ℓ and $f=1$, equations (69) and (58) immediately imply that $\langle 1,g\rangle_n=\frac{1}{(\ell)_n}\Theta_\ell^n(g)$, so that the Kaneko-Koike derivatives can be seen as a special case of the Rankin-Cohen bracket once the latter has been extended. Finally, the explicit formula for $\langle 1,1\rangle_n$ follows immediately from (69) and (60). $\square$

For completeness, we give a small table of values of $\langle 1,1\rangle_n$ for n even (the odd values vanish because in that case the bracket is antisymmetric), with $\Delta=(E_4^3-E_6^2)/1728$ as usual:

n	0	2	4	6	8	10	12
$n!\,\langle 1,1\rangle_n$	1	$-\frac{1}{36}E_4$	0	36Δ	$\frac{352}{3}E_4\Delta$	$260E_4^2\Delta$	$480E_4^3\Delta+1259136\Delta^2$

.

We emphasize that the point of the extended bracket is not to extend the definition of the original Rankin-Cohen bracket from positive weight to weight 0, since (12) already makes perfectly good sense for modular forms of arbitrary positive, zero, or negative integral weights (and for that matter even half-integral or arbitrary rational weights) and always sends a pair of modular forms of weights k and ℓ to a modular form of weight $k+\ell+2n$, simply because the transformation laws (29) and (30) hold for any value of k. But if one is interested in studying holomorphic modular forms only, then there are no non-zero forms of negative weights and the only forms of weight 0 are constants, and in the case when f or g is constant the original Rankin-Cohen bracket $[f,g]_n$ vanishes, whereas the new bracket $\langle f,g\rangle_n$ does not. However, the real point of introducing $\langle f,g\rangle_n$ is not even this extension to weight 0 but rather that all formulas involving Rankin-Cohen brackets become much simpler even in positive weight if we use the extended ones instead, as will see repeatedly in the rest of this paper.

An immediate consequence of the above definition and theorem is that Theorem 2, which in its original formulation involved an unaesthetic case distinction depending whether the order of the MLDO in question was less than half its weight (in which case it was a combination of Rankin-Cohen brackets) or equal to half its weight (in which case one needed the Kaneko-Koike operators as well), can now be given in a more uniform and much simpler form:

Theorem 10 (= uniform restatement of Theorem 2)**.** *Every MLDO on the full modular group can be written uniquely as a sum of extended Rankin-Cohen brackets.*

Explicitly, if $L\in\mathsf{MLDO}^{(\le n)}_{k,k+K}$, then $L(f)=\sum_{r=0}^{n}\langle h_r,f\rangle_r$ for some modular forms $h_r\in M_{K-2r}$.

A different use that we can make of the modified derivatives is to give a cleaner and more uniform description of quasimodular forms for the full modular group (and in fact also for other non-cocompact lattices, as we will see in the next section), which then also has applications to the description of MLDOs. In Section 8 we saw that the space $M_K^{(\le n)}(\Gamma_1)$ of quasimodular forms of strictly positive weight K and depth $\le n$ is canonically isomorphic to $\bigoplus_{0\le r\le n}M_{K-2r}(\Gamma_1)$ in two different ways: on the one hand, by writing its elements as polynomials in E_2 (equation (48)), and on the other, by writing them as linear combinations of rth derivatives of elements of spaces $M_k^+(\Gamma_1)$ (where $K=k+2r$) that are somewhat artificially defined as $M_k(\Gamma_1)$ for $k>2$ and as $\widetilde{M}_2(\Gamma_1)=\mathbb{C}E_2$ for $k=2$ (equation (49)). But using the renormalized derivatives $D^{\langle n\rangle}$ and their extension (64) to weight 0, we can write the latter isomorphism much more naturally by saying that each quasimodular form of depth $\le n$ is a linear combination of modified derivatives of modular forms of weight ≥ 0, namely,

$$\bigoplus_{r=0}^{n}M_{K-2r}\;\simeq\;\widetilde{M}_K^{(\le n)}\,,\qquad (h_0,\dots,h_n)\;\longleftrightarrow\;F\;=\;\sum_{r=0}^{n}D^{\langle r\rangle}(h_r) \tag{70}$$

instead of (49). Of course the content of this isomorphism is the same as that of (49), but the new one is better in several ways. One is that we now have two isomorphisms between the ring $\widetilde{M}_*$ of quasimodular forms on Γ_1 and the *same* ring $M_*[X]$ of polynomials in one variable over the ring M_* of modular forms on Γ_1, one by sending the polynomial $\sum h_rX^r$ to the quasimodular form $\sum D^{\langle r\rangle}(h_r)$ and one by sending the polynomial $\sum g_rX^r$ to the quasimodular form $\sum g_r(E_2/12)^r$. (Note that it is more natural here and in all other formulas to use $E_2/12$ rather than E_2 itself as the generator of $\widetilde{M}_*(\Gamma_1)$ over $M_*(\Gamma_1)$, because $E_2/12$ is mapped to 1 under the derivation δ on $\widetilde{M}_*$. This comment will become more important in the next section, when we replace $E_2/12$ by a choice of element $\mathcal{E}\in\widetilde{M}_2(\Gamma)$ with $\delta(\mathcal{E})=1$ for sublattices Γ of Γ_1 or more general lattices Γ with cusps.) If we compose one of these isomorphisms with the inverse of the other, we get an endomorphism of $M_*[X]=\bigoplus M_*X^r$ that is represented with respect to the basis $\{X^r\}$ by a triangular matrix with 1's on the diagonal (i.e., the coefficient of $(E_2/12)^n$ in the expansion of $D^{\langle n\rangle}(f)$ as a polynomial in $E_2/12$ is equal to f, with coefficient 1). In fact, we can now describe this whole matrix completely explicitly by using the generating series identities (53) and (57), which imply the formula

$$D^{\langle n\rangle}(f) \;=\; \sum_{r=0}^{n}\binom{n}{r}\,\mathfrak{d}^{\langle r\rangle}(f)\,\Big(\frac{E_2}{12}\Big)^{n-r} \tag{71}$$

(equivalent to (16) when the weight of f is positive, but now considerably simpler) for any $f\in M_*$. Here we have defined $\mathfrak{d}^{\langle r\rangle}(f)$ by modifying the canonical higher Serre derivatives defined in Section 5 in the same way as we did for the ordinary derivatives, i.e., by setting

$$\mathfrak{d}^{\langle n\rangle}(f) \;=\; \mathfrak{d}_k^{\langle n\rangle}(f) \;=\; \frac{\mathfrak{d}^{[n]}(f)}{(k)_n}\quad (f\in M_k(\Gamma),\ k>0),\qquad \mathfrak{d}^{\langle n\rangle}(1) \;=\; \frac{(-1)^n}{n!}\,\omega_n\,, \tag{72}$$

so that $\Phi_f^{(\mathfrak{d})}(\tau,X) = \sum \mathfrak{d}^{\langle n\rangle}(f)\,\frac{X^n}{n!}$ in all cases. Note that, just as the unnormalized canonical higher Serre derivatives can be used instead of ordinary derivatives to write the Rankin-Cohen brackets as a sum of terms that are individually modular (equation (15)), the modified ones can be used to write the extended Rankin-Cohen bracket $\langle f,g\rangle_n$ as a sum of $n+1$ terms each of which is modular:

$$\big\langle f,\,g\big\rangle_n \;=\; \sum_{r=0}^{n}(-1)^r\binom{n}{r}\,\mathfrak{d}^{\langle r\rangle}(f)\,\mathfrak{d}^{\langle n-r\rangle}(g)\,. \tag{73}$$

Here, just as with (12) and (15), the coefficients are the same as in (67).

Finally, the discussion above can also be used to give a new interpretation of the operator (46) defined in Theorem 8 in the case of its action on modular forms, and at the same time to extend it to the case when $k=0$. Specifically, we define a *pairing*

$$\{\ ,\ \}:\, M_*\otimes\widetilde{M}_*\to M_*\,,\qquad f\otimes F\;\mapsto\;\big\{f,F\big\} \;=\; \sum_{r=0}^{\infty}\frac{(-1)^r}{r!}\,\delta^r(F)\,D^{\langle r\rangle}(f)\,. \tag{74}$$

between quasimodular and modular forms. The bracket $\{f,F\}$ coincides with $L_{F,k}(f)$ as defined in Theorem 8 when k is strictly positive, and is therefore modular (of weight $k+K$) because $L_{F,k}$ is an MLDO of type $(k,k+K)$, but it now makes sense also in weight 0 and is still modular in that case. To see this, and to understand the pairing better, we observe that

$$\big\{f\,,\,D^{\langle n\rangle}(g)\big\} \;=\; \big\langle f,\,g\big\rangle_n \tag{75}$$

for any modular forms f and g and any integer $n\ge 0$ (this follows immediately from equation (79) in the next section), so that the pairing $f\otimes F\mapsto\{f,F\}$ is related to the decomposition of quasimodular forms given in equation (70) by

$$F \;=\; \sum_{r=0}^{n}D^{\langle r\rangle}(g_r)\quad\Longrightarrow\quad \big\{f,\,F\big\} \;=\; \sum_{r=0}^{n}\big\langle f,\,g_r\big\rangle_r\,. \tag{76}$$

This also gives a proof of the formula (65), because under the correspondence (47) between quasimodular forms and modular linear differential operators the MLDOs Θ_k^n and $\mathfrak{d}_k^{[n]}$ correspond to the quasimodular forms $(-1)^n(k)_nD^{\langle n\rangle}(1)$ and $(k)_n\big(-\frac{1}{12}E_2\big)^n$, respectively, so that (65) is just a consequence of (17), which was proved in the last section.

As a final remark, we observe that in equations (62), (67), (73) and (74) we were able to omit the index k that was needed in the corresponding earlier equations (50), (12), (15) and (46) because it is already incorporated into the definition of the modified derivative $D^{\langle r\rangle}$. A consequence of this is that these formulas can be written symbolically in a very simple form. This was already done in the first case in equation (63), and the other three can be written as

$$\big\langle f,\,g\big\rangle_n \;=\; \mathsf{m}\big((1\otimes D\,-\,D\otimes 1)^{\langle n\rangle}\,(f\otimes g)\big) \;=\; \mathsf{m}\big((1\otimes\mathfrak{d}\,-\,\mathfrak{d}\otimes 1)^{\langle n\rangle}\,(f\otimes g)\big)$$

and

$$\big\{f,F\big\} \;=\; \mathsf{m}\big(e^{\langle -D\otimes\delta\rangle}(f\otimes F)\big)\,,$$

where m denotes multiplication. Alternatively, we can also express the right-hand sides of these last three equations as the restrictions to the diagonal $\tau_1 = \tau_2 = \tau$ of $(D_2 - D_1)^{\langle n\rangle}(f(\tau_1)g(\tau_2))$, $(\mathfrak{d}_2 - \mathfrak{d}_1)^{\langle n\rangle}(f(\tau_1)g(\tau_2))$ or $e^{\langle -\delta_1 D_2\rangle}(f(\tau_1)g(\tau_2))$, respectively, where the subscripts on the differential operators indicate which variable τ_i they act on.

11. Application: higher Serre derivatives of quasimodular forms

The theme of this section is a non-trivial extension of the Rankin-Cohen bracket from modular to quasimodular forms that was discovered by François Martin and Emmanuel Royer and in a different form by Youngju Choie and Min Ho Lee (and that had also been found by the third author, but never published). Of course the Rankin-Cohen bracket $[f,g]_n^{(k,\ell)}$ can be defined for any two holomorphic functions f and g, any non-negative integer n, and any integers (or for that matter, even complex numbers) k and ℓ, but the original point of the specific complicated-looking bilinear combination of derivatives in its definition was that if f and g are modular forms of weights k and ℓ on some lattice, then $[f,g]_n^{(k,\ell)}$ is also a modular form, of weight $k+\ell+2n$. At first sight this statement seems impossible to generalize in an interesting way to quasimodular forms, since the ring of quasimodular forms is closed under differentiation anyway, so that each term of (12) is quasimodular if f and g are. But if we remember that modular forms are simply quasimodular forms of depth 0, then we can ask if there is a way to make a bracket of two quasimodular forms whose depth is at most the sum of the two individual depths, independent of n. (The individual terms of (12) in general have depths equal to the sum of the two individual depths plus n.) The result of Martin and Royer is that this *is* possible, and that all one has to do is to replace the upper indices k and ℓ of the bracket by $k-p$ and $\ell-q$, where p and q are the depths (or upper bounds on the depths) of f and g. We will prove this in a somewhat more general form by replacing the original Rankin-Cohen brackets with the extended ones that were defined in the last section (eq. (67) and Theorem 9), in which case the theorem still makes sense (and still is true) even if one of f or g has weight 0 and hence is constant. (If both have weight 0, then there is nothing to prove, since the depths are then also 0 and we are simply back to the modularity of the extended Rankin-Cohen brackets $\langle 1,1\rangle_n^{(0,0)}$.) The only point that has to be made here is that when we apply the definition of extended brackets to quasimodular forms that are not modular, then we have to specify the weight that is being used to define the modified derivatives in (67). So for quasimodular forms f and g, we will write $\langle f,g\rangle^{(K,L)}$ to be the bracket defined by (67) with $D^{\langle i\rangle}(f)$ and $D^{\langle n-i\rangle}(g)$ defined as $D^i(f)/(K)_i$ and $D^{n-i}(g)/(L)_{n-i}$, respectively, even if K and L are not the actual weights of f and g. This makes sense if K and L are positive or if one of them, say K, is 0 and f is a constant, in which case we use (64) instead.

Actually, as well as this extension of the Martin-Royer theorem to include forms of weight 0, we will give a strictly stronger result of which that theorem is an immediate corollary. This is the statement that the action on quasimodular forms of canonical higher Serre derivatives of arbitrary orders preserves their depth if one chooses the index of the Serre derivative to be the weight of the form minus its depth, rather than just its weight as in the modular case. Because of the expression (15) for Rankin-Cohen brackets as bilinear combinations of canonical higher Serre derivatives, this immediately gives the Martin-Royer theorem as well, but it is both stronger and simpler, since it applies to individual quasimodular forms rather than to pairs.

We now formulate the two results just described more quantitatively.

Theorem 11. *If f is a quasimodular form of weight k and depth $\le p$, then the quasimodular form $\mathfrak{d}_{k-p}^{[n]}(f)$ has depth $\le p$ for all integers $n \ge 0$.*

Corollary (Generalized Martin-Royer theorem). *If f and g belong to $\widetilde{M}_k^{(\le p)}$ and $\widetilde{M}_\ell^{(\le q)}$, respectively, then the extended Rankin-Cohen bracket $\langle f,g\rangle_n^{(k-p,\ell-q)}$ belongs to $\widetilde{M}_{k+\ell+2n}^{(\le p+q)}$ for all $n \ge 0$.*

For the proofs we will need the following lemma, which is a general statement about the action of the Lie algebra $\mathfrak{sl}_2$ that has many applications in the theory of quasimodular forms.

Lemma 2. *In the universal enveloping algebra of* $\mathfrak{sl}_2$ *we have the identities*

$$\delta D^n \;=\; D^n\delta \,+\, n\,D^{n-1}(W+n-1) \tag{77}$$

for all $n\geq 0$ *and more generally*

$$\delta^{\,r} D^n \;=\; \sum_{j=0}^{r}\binom{r}{j}\,(n-j+1)_j\,D^{n-j}\delta^{\,r-j}(W+n-r)_j \tag{78}$$

for all $r,\,n\geq 0$*, where* $(n-j+1)_j\,D^{n-j}$ *is to be interpreted as* 0 *if* $j>n$*.*

Corollary. *If* f *is a modular form of weight* k*, then we have* $\delta D^n(f)=n(k+n-1)D^{n-1}(f)$ *for all* $n\geq 0$ *and more generally* $\delta^{\,r}D^n(f)=(n-r+1)_r(k+n-r)_r\,D^{n-r}(f)$ *for all* $n,r\geq 0$*.*

The proof of the lemma (using induction on n for the first statement and then on s for the second) is straightforward and well known, so we will omit it, and the corollary follows immediately since modular forms are annihilated by δ. In terms of the modified derivatives $D^{\langle n\rangle}(f)$ introduced in the last section, we can rewrite the second statement of the corollary as

$$\frac{\delta^{\,r}(D^{\langle n\rangle}(f))}{r!} \;=\; \binom{n}{r}\,D^{\langle n-r\rangle}(f) \qquad (n,r\geq 0,\;\, f\in M_*(\Gamma)) \tag{79}$$

if the weight of f is positive. Note that this equation remains true also for $f=1$, as one sees by the following calculation using the definition (64) and equation (78) applied to $\mathcal{E}=E_2/12$:

$$\begin{aligned}(n-1)!\,\delta^r(D^{\langle n\rangle}(1)) \;&=\; \delta^r(D^{n-1}(\mathcal{E})) \;=\; \sum_{j=0}^{r}\binom{r}{j}\,(n-j)_j\,(n-r+1)_j\,D^{n-1-j}(\delta^{\,r-j}(\mathcal{E}))\\ &=\; (n-r)_r\,(n-r+1)_r\,D^{n-r-1}(\mathcal{E}) \,+\, r\,(n-r+1)_{r-1}\,(n-r+1)_{r-1}\,D^{n-r}(1)\\ &=\; \frac{(n-1)!\,n!}{(n-r-1)!\,(n-r)!}\,D^{n-r-1}(\mathcal{E}) \,+\, r\,\frac{(n-1)!^2}{(n-r)!^2}\,\delta_{n,r} \;=\; \frac{(n-1)!\,n!}{(n-r)!}\,D^{\langle n-r\rangle}(1)\,.\end{aligned}$$

(Actually, to prove (79) it is enough to give the calculation for $r=1$, which is slightly simpler, and then use induction on r, but we gave the calculation in general because it is not much longer and gives a nice illustration of the properties of the modified derivative $D^{\langle n\rangle}$.) We already used equation (79) in Section 10 to get the formula (75) relating extended Rankin-Cohen brackets to the pairing (74).

We now proceed to the proof of Theorem 11, after which, as already mentioned, the corollary follows immediately as a consequence of equation (15) (or of its extension (73) to extended Rankin-Cohen brackets in the case when one of the forms has weight 0). To do this, we consider the Cohen-Kuznetsov series of f with index $k-p$, where $f\in\widetilde{M}_k^{(\leq p)}$ as in the theorem. More specifically, we consider both the $\mathfrak{d}$- and the D-versions of this series, defined and related by

$$\Phi^{(\mathfrak{d})}_{f,k-p}(X) \;=\; \sum_{n=0}^{\infty}\frac{\mathfrak{d}^{[n]}_{k-p}(f)}{(k-p)_n\,n!}\,X^n \;=\; e^{-X\mathcal{E}}\,\Phi_{f,k-p}(X) \;=\; e^{-X\mathcal{E}}\sum_{n=0}^{\infty}\frac{D^n(f)}{(k-p)_n\,n!}\,X^n$$

with $\mathcal{E}=E_2/12$ and $\mathfrak{d}^{\langle n\rangle}_{k-p}(f)$ defined as in equation (72). (Here and for the rest of the proof we omit the argument τ for notational simplicity.) Notice, by the way, that we can assume that $k>0$, since if $k=0$ then p is also 0 and there is nothing to prove, and then $p\leq k/2<k$, so that the factors $(k-p)_n$ in the denominators of the above formulas never vanish. To prove the theorem, we have to show that the series on the left is annhilated by δ^{p+1}, or equivalently that all

of the coefficients of its image under δ^p are modular rather than merely quasimodular forms. By Leibniz's formula and the fact that $\delta(\mathcal{E}) = 1$ we have

$$e^{X\mathcal{E}}\delta^m\big(\Phi^{(\mathfrak{d})}_{f,k-p}(X)\big) \;=\; e^{X\mathcal{E}}\delta^m\big(e^{-X\mathcal{E}}\,\Phi_{f,k-p}(X)\big) \;=\; \sum_{s=0}^{m}\binom{m}{s}(-X)^{m-s}\,\delta^s\big(\Phi_{f,k-p}(X)\big) \;=\; (\delta - X)^m\big(\Phi_{f,k-p}(X)\big)$$

for every $m \ge 0$, and we want to show that this expression vanishes for $m > p$.

In fact, we will give two different arguments, omitting a few of the details of the calculation in the second case. For the first argument we use the fact that every quasimodular form is a linear combination of modified derivatives $D^{\langle r\rangle}(h)$ of modular forms (equation (70)). Since such a derivative has depth exactly r, we can assume that the quasimodular form f in Theorem 11 has the form $D^{\langle r\rangle}(h)$ for some $r \le p$ and some modular form h of weight $k - 2r$. Then we have

$$\Phi_{f,k-p}(X) \;=\; \sum_{n=0}^{\infty}\frac{D^n(D^{\langle r\rangle}(h))}{(k-p)_n\,n!}\,X^n \;=\; \frac{1}{(k-p)_{p-r}}\sum_{n=0}^{\infty}(k+n-p)_{p-r}\,D^{\langle n+r\rangle}(h)\,\frac{X^n}{n!}$$

and hence

$$(k-p)_{p-r}\,e^{X\mathcal{E}}\,\delta^m\big(\Phi^{(\mathfrak{d})}_{f,k-p}(X)\big) \;=\; \sum_{s=0}^{m}\binom{m}{s}(-X)^{m-s}\,\delta^s\Bigg(\sum_{n=0}^{\infty}(k+n-p)_{p-r}\,D^{\langle n+r\rangle}(h)\,\frac{X^n}{n!}\Bigg)$$
$$=\; X^{m-r}\sum_{\ell=0}^{\infty}\Bigg[\sum_{s=0}^{m}\binom{m}{s}(-1)^{m-s}\,(k-p+s+\ell-r)_{p-r}(s+\ell-r+1)_r\Bigg]\,D^{\langle \ell\rangle}(h)\,\frac{X^\ell}{\ell!}$$

for any $m \ge r$, where to obtain the second line we have used equation (79). If $m > p$ then the expression in square brackets vanishes for every ℓ because it is the mth difference of a polynomial in ℓ of degree p. This proves the theorem. We also get an explicit formula for the pth derivative of $\mathfrak{d}^{[n]}_{k-p}(f)$ as a modular form for every n, since the pth derivative of a monic polynomial of degree p is $p!$ and hence

$$(k-p)_{p-r}\,\delta^p\big(\Phi^{(\mathfrak{d})}_{f,k-p}(X)\big) \;=\; p!\,X^{p-r}e^{-X\mathcal{E}}\sum_{h=0}^{\infty}D^{\langle h\rangle}(f)\,\frac{X^h}{h!} \;=\; p!\,X^{p-r}\,\Phi^{(\mathfrak{d})}_{f}(X)\,,$$

so that $\delta^p(\mathfrak{d}^{[n]}(f))$ vanishes if $n < p - r$ and is a simple multiple of $\mathfrak{d}^{[n+r-p]}(h)$ if $n \ge p - r$.

For the second argument, which we only sketch, we work directly with the action of $\mathfrak{sl}_2$ on the various Cohen-Kuznetsov series involved. This approach involves slightly more calculation but has the advantages that it does not use the decomposition (70) or the special quasimodular form $\mathcal{E} = E_2/12$. For any quasimodular form $f \in \widetilde{M}_k$ and any positive integer K we have

$$\begin{aligned}(\delta - X)\,\Phi_{f,K}(X) \;&=\; \sum_{n=0}^{\infty}\frac{\delta(D^n(f)) \,-\, n(K+n-1)D^{n-1}(f)}{n!\,(K)_n}\,X^n\\ &=\; \sum_{n=0}^{\infty}\frac{D^n(\delta(f)) \,+\, n(k-K)\,D^{n-1}(f)}{n!\,(K)_n}\,X^n \qquad \text{(by eq. (77))}\\ &=\; \Phi_{\delta(f),K}(X) \,+\, \frac{k-K}{K}\,X\,\Phi_{f,K+1}(X)\end{aligned}$$

and hence by induction on m

$$(\delta - X)^m\Phi_{f,K}(X) \;=\; \sum_{s=0}^{m}\binom{m}{s}\frac{(k-K-m+1)_{m-s}}{(K)_{m-s}}\,X^{m-s}\,\Phi_{\delta^s(f),K+m-s}(X)$$

for every integer $m \ge 0$. (We omit the details of this step, which are slightly messy.) Now if we take $K = k - p$ for f of depth $\le p$, and if $m > p$, then the terms with $s \le p$ vanish because

$(p-m+s)_{m-s}=0$ and those with $s>p$ vanish because $\delta^s(f)=0$, so we again find that $(\delta-X)^m$ annihilates $\Phi_{f,K}(X)$ and therefore that δ^m annihilates $\Phi^{(\partial)}_{f,K}(X)$ as desired. □

12. Cocompact and non-cocompact lattices

In this section we discuss a basic dichotomy between the structure of the rings of quasimodular forms for non-cocompact and cocompact lattices Γ in $SL_2(\mathbb{R})$. In the former case, exemplified by the full modular group Γ_1, there is always a quasimodular but not modular form of weight 2, and then all theorems of the previous sections for Γ_1 still hold with this form in place of E_2. In the latter case, exemplified by Shimura curves, there is no such quasimodular form and the structure theorems are somewhat different. In particular, here there is no analogue of the Serre derivative or the Kaneko-Koike operator, and all MLDOs are linear combinations of Rankin-Cohen brackets.

For $\Gamma=\Gamma_1=SL_2(\mathbb{Z})$ we have already seen that the ring $\widetilde{M}_*(\Gamma)$ of quasimodular forms is simply $\mathbb{C}[E_2,E_4,E_6]$, with the derivations δ and D given by $12\,\partial/\partial E_2$ and by Ramanujan's formulas (3), respectively. If Γ is a subgroup of $SL_2(\mathbb{Z})$ of finite index, then the algebra $M_*(\Gamma)$ of modular forms on Γ is in general no longer freely generated, but we still have $\widetilde{M}_*(\Gamma)=M_*(\Gamma)[E_2]$ and $\delta=12\,\partial/\partial E_2$. The following proposition shows that a similar statement holds for any non-cocompact lattice $\Gamma\subset SL_2(\mathbb{R})$. Recall that for any lattice Γ, cocompact or not, we have derivations D, W and δ on $\widetilde{M}_*(\Gamma)$ satisfying (40), where W is the weight operator and $\ker(\delta)=M_*(\Gamma)$. In particular, δ maps $\widetilde{M}_2(\Gamma)$ to $M_0(\Gamma)=\mathbb{C}$, so it must be either 0 or surjective.

Proposition 2. *Let $\Gamma\subset SL_2(\mathbb{R})$ be an arbitrary lattice.*
(a) *If Γ is cocompact, then $M_2(\Gamma)=\widetilde{M}_2(\Gamma)$.*
(b) *If Γ is non-cocompact, then the sequence $0\to M_2(\Gamma)\to\widetilde{M}_2(\Gamma)\xrightarrow{\delta}\mathbb{C}\to 0$ is exact.*
(c) *In the non-cocompact case, $\widetilde{M}_*(\Gamma)=M_*(\Gamma)[\mathcal{E}]$ for any $\mathcal{E}\in\widetilde{M}_2(\Gamma)$ with $\delta(\mathcal{E})\neq 0$.*

Proof. Since the result is certainly known to experts, we only sketch the proof here. For (a), we observe that if $\mathcal{E}$ is a quasimodular form of weight 2 on Γ that is not modular, then $\delta\mathcal{E}=C$ for some $C\neq 0$, in which case the same argument as was given for $\Gamma=\Gamma_1$ in Section 9 shows that the "completion" defined by $\widehat{\mathcal{E}}(\tau)=\mathcal{E}(\tau)-\frac{C}{4\pi y}$ (where again $y=\Im(\tau)$) transforms under Γ like a modular form of weight 2. Then the non-holomorphic 1-form $\omega=\widehat{\mathcal{E}}(\tau)\,d\tau$ is Γ-invariant and its derivative $d\omega$ is a non-zero multiple of the volume form $y^{-2}d\tau\,d\bar{\tau}$ on $\mathfrak{H}$, so the integral of $d\omega$ over $\mathfrak{H}/\Gamma$ is non-zero, which contradicts Stokes's theorem if Γ is cocompact since then $\mathfrak{H}/\Gamma$ is closed. For (b), we observe that if Γ is non-cocompact, then it has at least one cusp, which we can assume after conjugating by an element of $SL_2(\mathbb{R})$ to be at ∞. Then the non-holomorphic weight 2 Eisenstein series defined by "Hecke's trick" as $\lim_{\epsilon\to 0}\sum(c\tau+d)^{-2}|c\tau+d|^{-\epsilon}$, where $\left(\begin{smallmatrix}\cdot&\cdot\\ c&d\end{smallmatrix}\right)$ runs over the left cosets of the stabilizer of ∞ in Γ, has the form $\mathcal{E}(\tau)-\frac{C}{4\pi y}$ for some holomorphic function $\mathcal{E}$ and constant $C\neq 0$, in which case $\mathcal{E}$ belongs to $\widetilde{M}_2(\Gamma)$ but not to $M_2(\Gamma)$. Finally, part (c) of the proposition follows by an easy induction, since if F is a quasimodular form of weight k and depth $p>0$, so that the function $G=\delta^p(F)$ is a non-zero modular form of weight $k-2p$, then the difference between F and some multiple of $G\mathcal{E}^p$ is easily checked to have weight k and depth $<p$, so that by induction on p we see that F is a polynomial in $\mathcal{E}$ with modular coefficients. □

We call a choice of $\mathcal{E}$ in case (b) a *splitting*, since it splits $\widetilde{M}_2(\Gamma)$ as $M_2(\Gamma)\oplus\mathbb{C}\mathcal{E}$. We can, and from now on will, normalize $\mathcal{E}$ multiplicatively by requiring that $\delta(\mathcal{E})=1$, but we then still have the freedom of replacing $\mathcal{E}$ by $\mathcal{E}+h$ for an arbitrary element $h\in M_2(\Gamma)$. For $\Gamma=SL_2(\mathbb{Z})$ we have $M_2(\Gamma)=\{0\}$, so in that case $\mathcal{E}=\frac{1}{12}E_2$ is unique. In all non-cocompact cases, if we identify $\widetilde{M}_*(\Gamma)$ with $M_*(\Gamma)[\mathcal{E}]$, then δ corresponds simply to $\partial/\partial\mathcal{E}$.

Proposition 2 gives a complete "multiplicative" description of the ring of quasimodular forms for all non-cocompact groups Γ as polynomials in a single function $\mathcal{E}\in\widetilde{M}_2(\Gamma)$ with modular forms as coefficients. However, this does not work for cocompact groups, since there is no function $\mathcal{E}$. However, there is also an "additive" description of $\widetilde{M}_*(\Gamma)$ which works for both cocompact and non-cocompact groups Γ, but which is a little different in the two cases.

Let $\Gamma \subset SL_2(\mathbb{R})$ be an arbitrary lattice and define $M_*^D(\Gamma)$ as the closure of $M_*(\Gamma)$ with respect to D, i.e., as the smallest vector space containing $M_*(\Gamma)$ and closed under differentiation. The space $M_*^D(\Gamma)$ has the additive structure $\mathbb{C} \oplus M_{>0}^D(\Gamma)$, where $M_{>0}^D(\Gamma)$ is the subspace

$$M_{>0}^D(\Gamma) \;=\; \mathbb{C}[D]\big(M_{>0}(\Gamma)\big) \;=\; \mathbb{C}[D] \otimes_{\mathbb{C}} M_{>0}(\Gamma) \;=\; \bigoplus_{n\geq 0,\, k>0} D^n(M_k(\Gamma))\,,$$

since $D^n : M_k(\Gamma) \to M_{>0}^D(\Gamma)$ is injective for all $n \geq 0$, $k > 0$. Clearly $M_*(\Gamma) \subset M_*^D(\Gamma) \subseteq \widetilde{M}_*(\Gamma)$. The additive description of the space of quasimodular forms on Γ, generalizing equation (49) for the case of the full modular group, is then as follows.

Proposition 3. (a) *If Γ is cocompact, then $\widetilde{M}_*(\Gamma) = M_*^D(\Gamma)$.*
(b) *If Γ is non-cocompact, then $\widetilde{M}_*(\Gamma) = M_*^D(\Gamma) \oplus \mathbb{C}[D]\mathcal{E}$, where $\mathcal{E}$ as in Proposition 2 (b) is any element in $\widetilde{M}_2(\Gamma) \smallsetminus M_2(\Gamma)$. Thus*

$$\widetilde{M}_k(\Gamma) \;=\; \begin{cases} M_k^D(\Gamma) & \text{if } k = 0 \text{ or } k \text{ is odd,} \\ M_k^D(\Gamma) \,\oplus\, \mathbb{C}\cdot D^n(\mathcal{E}) & \text{if } k = 2n+2,\ n \geq 0\,. \end{cases}$$

Proof. If F is quasimodular of weight $k > 0$ and of depth n, then the final coefficient F_n in the development (43) is quasimodular of depth 0 and hence is modular, of weight $k - 2n$. If $n < k/2$, then $D^n(F_n)$ is quasimodular of the same weight and depth as F, so subtracting a multiple of it from F reduces the depth of F and hence proves the result by induction. If Γ is cocompact, then n is always less than $k/2$, because $F_n = \delta(F_{n-1})/n$ and the map $\delta : \widetilde{M}_2(\Gamma) \to \widetilde{M}_0(\Gamma) = \mathbb{C}$ is identically 0. This proves part (a). If Γ is non-cocompact, then n can be equal to $k/2$, but in that case F_{n-1} belongs to $\widetilde{M}_2(\Gamma) = M_2(\Gamma) \oplus \mathbb{C}\cdot\mathcal{E}$, so we can reduce the depth of F by subtracting from it a multiple of $D^{n-1}(F_{n-1}) \in M_k^D(\Gamma) \oplus \mathbb{C}\cdot D^{n-1}(\mathcal{E})$ and proceed as before. □

Further properties of the subspace $M_*^D(\Gamma)$ are summarized in the following proposition.

Proposition 4. *The vector space $M_*^D(\Gamma)$ is an $\mathfrak{sl}_2$-submodule of $\widetilde{M}_*(\Gamma)$ and is an ideal of the algebra $\widetilde{M}_*(\Gamma)$. In particular, $M_*^D(\Gamma)$ is closed under multiplication.*

Proof. It follows easily from the definition of the $\mathfrak{sl}_2$-action on $\widetilde{M}_*(\Gamma)$ and from Lemma 2 that $D(D^nF) = D^{n+1}F$, $W(D^nF) = (k+2n)D^nF$ and $\delta(D^nF) = n(k+n-1)D^{n-1}F$ for each $F \in \widetilde{M}_k(\Gamma)$ and $n \geq 0$. Therefore the subspace $M_*^D(\Gamma)$ is an $\mathfrak{sl}_2$-submodule of $\widetilde{M}_*(\Gamma)$. This proves the first statement. For the second, we define a map $\mu : \widetilde{M}_*(\Gamma) \to \mathbb{C}[T]$ by $\mu(F) = 0$ if $F \in \widetilde{M}_k(\Gamma)$ with k odd and $\mu(F) = \delta^{\,p}(F)\,T^p/p!$ if $k = 2p$. It follows from the Leibniz rule that the map $\mu : \widetilde{M}_*(\Gamma) \to \mathbb{C}$ is an algebra homomorphism, so its kernel is an ideal. We claim that this kernel is $M_*^D(\Gamma)$. The inclusion $M_*^D(\Gamma) \subseteq \mathrm{Ker}(\mu)$ is obvious because $M_k^D(\Gamma) = \bigoplus_{0\leq n<k/2} D^n M_{k-2n}(\Gamma)$ for $k > 0$ (since $D^n M_0(\Gamma) = 0$ for $n > 0$) and hence $\delta^{k/2}$ acts trivially on $M_k^D(\Gamma)$ for k even. The reverse inclusion then follows from the decomposition $\widetilde{M}_k = M_k^D \oplus \mathbb{C}\cdot D^{p-1}(\mathcal{E})$ and the easily verified fact that $\mu(D^{p-1}\mathcal{E}) \neq 0$. □

The fact that $M_*^D(\Gamma)$ is closed under multiplication means that there must be a formula for any product $D^r(f)D^s(g)$ $(r,\, s \geq 0,\ f,\, g \in M_*(\Gamma))$ as a linear combination of derivatives of modular forms, a simple example being $fg' = \frac{\ell}{k+\ell}\,(fg)' + \frac{1}{k+\ell}\,[f,g]_1$ for $f \in M_k$ and $g \in M_\ell$, in which the right-hand side contains only modular forms and their derivatives but no derivatives of $\mathcal{E}$. Similarly, in the non-cocompact case, the fact that $M_*^D(\Gamma)$ is an ideal of $\widetilde{M}_*(\Gamma) = M_*^D(\Gamma) \oplus \bigoplus_s \mathbb{C}\cdot D^s(\mathcal{E})$ means that there must also be an expression for $D^r(f)D^s(\mathcal{E})$ as a linear combination of derivatives of modular forms, and the fact that $\widetilde{M}_*(\Gamma)$ is a ring means that there must also be an expression for any product $D^r(\mathcal{E})D^s(\mathcal{E})$ as a linear combination of derivatives of both modular forms and $\mathcal{E}$. These formulas, which are quite complicated, are given in [30] and will not be repeated here.

Now using this discussion of the structure of quasimodular forms for arbitrary lattices, we can easily see how all of the theorems proved in this paper have to be modified when the full modular group Γ_1 is replaced by some other lattice Γ. In particular, if Γ is non-cocompact and we have chosen a splitting, then every result discussed or proved so far remains true *mutatis mutandis.* We state this informally as the following theorem.

Theorem 12. *If $\Gamma \subset SL_2(\mathbb{R})$ is a non-cocompact lattice with a given splitting $\mathcal{E}$, then all results of Theorems 1–11 remain true with Γ_1 replaced by Γ and E_2 by $12\,\mathcal{E}$.*

Let us discuss briefly what this statement means in each case. The first point is that given the lattice Γ and the splitting $\mathcal{E}$ we have a well-defined Serre derivative $\mathfrak{d}_k = \mathfrak{d}_{k,\mathcal{E}}$ mapping $M_k(\Gamma)$ to $M_{k+2}(\Gamma)$ for every $k \geq 0$, defined by the same formula (4) as before but with $E_2(\tau)/12$ replaced by $\mathcal{E}$, or in symbolic notation, by $\mathfrak{d}_{\mathcal{E}} = D - \mathcal{E}\,W$. We will usually drop the subscript $\mathcal{E}$ for convenience, but one should not forget that in the general case the new Serre derivative is not intrinsic to Γ as it was for the full modular group, but depends on a choice of splitting. Replacing $\mathcal{E}$ by $\mathcal{E}^* = \mathcal{E} + h$ with $h \in M_2(\Gamma)$ changes $\mathfrak{d}_{\mathcal{E}}$ by $h\,W$, i.e., it changes $\mathfrak{d}_k$ to $\mathfrak{d}_k^*(f) = \mathfrak{d}_k(f) + khf$. Similarly, we have a new Kaneko-Koike operator $\Theta_k^n = \Theta_{k,\mathcal{E}}^n(f)$ from $M_k(\Gamma)$ to $M_{k+2n}(\Gamma)$ given by equation (13) with $E_2/12$ replaced by $\mathcal{E}$, and also new canonical higher Serre derivatives $\mathfrak{d}_k^{[n]} = \mathfrak{d}_{k,\mathcal{E}}^{[n]}$ defined by equation (14), but with $E_4/144$ replaced by the "curvature" $\Omega = \mathcal{E}^2 - \mathcal{E}'$, which always belongs to $M_4(\Gamma)$. Formula (15) still remains true with these new higher Serre derivatives, and shows that the entire structure of $M_*(\Gamma)$ as a Rankin-Cohen algebra is determined by just the multiplication, the new Serre derivative, and the curvature, in accordance with the main result of [34] (where Ω was denoted by $-\Phi$, and a Rankin-Cohen algebra defined by formulas (14) and (15) from an underlying graded algebra together with a derivation $\mathfrak{d}$ of weight $+2$ and an element Ω of weight 4 was called a *canonical* Rankin-Cohen algebra). With these definitions, the meaning of the generalizations of Theorem 1 (apart from the specific formulas for the values of $\omega_m = \omega_{m,\mathcal{E}} \in M_{2m}(\Gamma)$, which will of course depend on Γ and $\mathcal{E}$), Theorems 2, 3, 4, 7, 10, 11 and its corollary, and Propositions 1 and 9 are all clear. In particular, we have a notion of extended Rankin-Cohen brackets for any non-cocompact group together with a choice of splitting.

Finally, we should say a few words about the cocompact case, even though for most of the applications (in particular, in the theory of VOAs) we only care about the full modular group or its subgroups. The proofs of Theorems 5, 6 and 8 did not depend in any way on the special properties of Γ_1 and were already stated for arbitrary lattices. The description of all modular linear differential operators in terms of Rankin-Cohen brackets as given in Theorem 2 is in principle still valid for any lattice Γ, cocompact or not, but with the proviso that in the cocompact case all MLDOs have order strictly smaller than half their weight and there is no analogue of the Kaneko-Koike operator, so that we only need Rankin-Cohen brackets with modular forms of strictly positive weight. There is, however, one exception. The generalized Serre derivative $\mathfrak{d}_{\mathcal{E}}(f) = f' - k\mathcal{E}f$ does not involve $\mathcal{E}$ if $k = 0$, and similarly $\Theta_{k,\mathcal{E}}^n(f)$ does not contain $\mathcal{E}$ if $k = 1 - n$, so that the corresponding operators D and D^n are defined even in the cocompact case when no splitting exists. This corresponds to "Bol's identity" $D^n(f|_{1-n}g) = (D^nf)|_{1+n}g$ for any $f \in \mathsf{Hol}(\mathfrak{H})$ and $g \in SL_2(\mathbb{R})$, so that $D^n \in \mathsf{MLDO}_{1-n,1-n}(\Gamma)$ for any lattice Γ and any $n \geq 0$. Apart from this, however, all MLDOs for non-cocompact lattices are combinations of Rankin-Cohen brackets with forms of positive weights, there are no monic MLDOs with holomorphic quasimodular forms as coefficients, and the dimension of $\mathsf{MLDO}_{k,k+K}(\Gamma)$ is independent of k and equal to $\dim \widetilde{M}_K(\Gamma)$.

13. Primitive projection and modular linear differential operators

In Section 8 we defined an isomorphism $F \mapsto L_F$ between quasimodular forms and MLDOs, and in Section 10 the operation of L_F on modular forms was interpreted as a pairing (74) between quasimodular and modular forms. Here we will define a second isomorphism $F \mapsto \mathbf{L}_F$ between quasimodular forms and MLDOs in terms of a certain projection operator. This will be done in the opposite order from before, giving first a new pairing between quasimodular and modular forms in terms of the action of $\mathfrak{sl}_2$ on the space of quasimodular forms and the primitive projection operator for $\mathfrak{sl}_2$-modules, and then generalizing this operator to non-modular arguments. A reinterpretation of the new isomorphism in terms of almost holomorphic modular forms and the holomorphic projection operator will be given in the next section.

Theorem 13. *Let F be a quasimodular form of weight K and depth n on an arbitrary lattice Γ. Then for every positive integer k the operator*

$$\mathbf{L}_{F,k} \;:=\; \sum_{r=0}^{n} \frac{1}{r!} \left(\sum_{m=r}^{n} \frac{(-1)^m D^{m-r}(\delta^m F)}{(m-r)!\,(k+K-m-1)_m} \right) D^r \tag{80}$$

is a modular linear differential operator of order n and type $(k, k+K)$, and the map $F \mapsto \mathbf{L}_{F,k}$ gives an isomorphism from $\widetilde{M}_K^{(\leq n)}(\Gamma)$ to $\mathsf{MLDO}_{k,k+K}^{(\leq n)}(\Gamma)$.

As examples, if F is modular of weight K (so that $\delta^m(F) = 0$ for $m > 0$), then $\mathbf{L}_{F,k}$ is just multiplication by F, while if $F = E_2$ and $k > 0$ then $\mathbf{L}_{F,k}$ is just a multiple of the Serre derivative $\mathfrak{d}_k$. Note that the complicated-looking formula (80) can be written more simply as

$$\mathbf{L}_{F,k}(f) \;=\; \sum_{m=0}^{n} \frac{(-1)^m\, D^m(\delta^m(F)\, f)}{m!\,(k+K-m-1)_m}\,, \tag{81}$$

but in (80) we have expanded by Leibniz's rule to give $\mathbf{L}_{F,k}$ explicitly as a differential operator.

The origin of this new isomorphism is that there is a canonical way to project any non-negatively graded $\mathfrak{sl}_2$-module (with grading given by W) onto its "primitive" part (kernel of δ). Applied to the ring of quasimodular forms, this gives a collection of canonical projection maps π_k from $\widetilde{M}_k(\Gamma)$ to $M_k(\Gamma)$ for all $k \geq 0$ and any lattice $\Gamma \subset SL_2(\mathbb{R})$. Rather than just stating the formula as a proposition and checking that it works, we indicate how it can be derived. We make the Ansatz that π_k is an element of the universal enveloping algebra of $\mathfrak{sl}_2$, i.e., that it can be represented as a polynomial in the three generators D, W, δ of $\mathfrak{sl}_2$. These elements of course do not commute, but in view of the commutation relations (40) we can write any non-commutative polynomial in them as a linear combination of monomials $D^n \delta^m W^\ell$ with integers ℓ, m, $n \geq 0$. When we apply them to quasimodular forms of a fixed weight k, the factor W^ℓ just acts as a known scalar, so we can write the sought-for operator π_k as a linear combination of monomials $D^n \delta^m$ with complex coefficients depending only on m, n and k. Moreover, since we want π_k to preserve the weight, and since D and δ increase and decrease the weight by 2, respectively, we must have $m = n$, so our Ansatz becomes

$$\pi_k \;=\; \sum_{m\geq 0} c_m\, D^m\, \delta^m \tag{82}$$

with some as yet undetermined coefficients c_m depending on k. Here the summation can be replaced by one over just $0 \leq m \leq k/2$, because $\delta^m(f)$ vanishes for $m > k/2$. Since we want π_k to be a projection operator onto the subspace M_k of $\widetilde{M}_k$, and since δ annihilates M_k, we must have $c_0 = 1$. Furthermore, since we want the image of π_k to be contained in $M_k = \ker(\delta)$ we must have $\delta\pi_k(f) = 0$ for every $f \in \widetilde{M}_k$. Using (77) and noting that $\delta^m(f)$ has weight $k - 2m$, we calculate

$$\delta\bigl(\pi_k(f)\bigr) \;=\; \sum_{m\geq 0} c_m \bigl(D^m\delta + m(k-m-1)D^{m-1}\bigr)\,\delta^m(f) \;=\; \sum_{m\geq 1} \bigl(c_{m-1} + m(k-m-1)c_m\bigr)\, D^{m-1}\delta^m(f)\,.$$

Equating this to 0 gives the recursion $c_m = -c_{m-1}/m(k-m-1)$ for all $m \geq 1$, which together with the initial condition $c_0 = 1$ determines c_m uniquely as $(-1)^m/m!(k-m-1)_n$. If $k > 2$, then this number is finite for all $0 \leq m \leq k/2$, since $m \leq k/2 \leq k-1$ and therefore $(k-m-1)_m \neq 0$. Conversely, this calculation shows that the operator π_k defined by

$$\pi_k \;:=\; \sum_{0\leq m\leq k/2} \frac{(-1)^m}{m!\,(k-m-1)_m}\, D^m\delta^m \;=\; 1 - \frac{D\delta}{k-2} + \frac{D^2\delta^2}{2(k-2)(k-3)} - \cdots \tag{83}$$

has the required properties for every $k > 2$, establishing the following result.

Proposition 5. *For every lattice Γ and every integer $k > 2$ the operator (83) gives a projection from quasimodular forms of weight k on Γ to the subspace of modular forms of weight k on Γ.*

We make two remarks. The first is that Proposition 5 fails if $k = 2$ and we have to define $\pi_2(F)$ instead as $F - \delta(F)\mathcal{E}$ with $\mathcal{E}$ as in Section 12 in the non-cocompact case, while in the cocompact case we can simply take π_2 to be the identity. This will not be important to us since in the application given here the weight will be larger than 2 anyway. The second is that the operator π_k defined in (83) satisfies the identity $\pi_k \circ D = 0$ as well as $\delta \circ \pi_k = 0$ by a calculation exactly similar to the one above. (More specifically, the equation $\pi_k \circ D = 0$ together with the Ansatz (82) leads to the same recursion $m(k-m+1)\,c_m = -c_{m-1}$ as before and hence determines π_k uniquely up to a constant.) This means that we can give an alternative definition of the map $\pi_k : \widetilde{M}_k \to M_k$ as the projection onto the first factor in the direct sum decomposition $\widetilde{M}_k = M_k \oplus D(\widetilde{M}_{k-2})$ for $k > 2$.

Comparing the formulas (83) and (81), we find that the effect of $\mathbf{L}_{F,k}$ on modular forms of weight k is given simply by

$$\mathbf{L}_{F,k}(f) \;=\; \pi_{k+K}(fF) \qquad \bigl(F \in \widetilde{M}_K(\Gamma),\ f \in M_k(\Gamma)\bigr), \tag{84}$$

because $\delta^m(fF) = \delta^m(F)f$ for f modular. Of course the same formula holds also if f belongs to $M_k^!(\Gamma)$ (modular forms that are holomorphic in $\mathfrak{H}$ but can have poles at the cusps) or even to $M_k^{\mathrm{mer}}(\Gamma)$ (meromorphic modular forms of weight k), and this then proves the modularity of the operator $\mathbf{L}_{F,k}$ and determines it completely, since the map from $\mathsf{MLDO}_{k,k+K}(\Gamma)$ to either $\mathrm{Hom}(M_k^!(\Gamma), M_{k+K}^!(\Gamma))$ or $\mathrm{Hom}(M_k^{\mathrm{mer}}(\Gamma), M_{k+K}^{\mathrm{mer}}(\Gamma))$ is injective. To see that $F \mapsto \mathbf{L}_{F,k}$ is an isomorphism from $\widetilde{M}_K(\Gamma)$ to $\mathsf{MLDO}_{k,k+K}(\Gamma)$, at least for $k > 0$ as in the theorem, we calculate the effect on derivatives. If $F = D^n(g)$ with $g \in M_\ell$ and $K = \ell + 2n$, then equations (68) and (75) tell us that $L_{F,k}$ is given by

$$L_{D^n(g),k}(f) \;=\; \binom{k+n-1}{n}^{-1} \bigl[f,g\bigr]_n \tag{85}$$

for any $f \in M_k$, and a simple calculation that is left to the reader shows that $\mathbf{L}_{F,k}$ is given by the rather similar formula

$$\mathbf{L}_{D^n(g),k}(f) \;=\; \binom{k+\ell+2n-2}{n}^{-1} \bigl[f,g\bigr]_n\,. \tag{86}$$

for $f \in M_k$ and $g \in M_\ell$ and any $n > 0$. In view of Theorem 2, which says that all MLDOs are given by Rankin-Cohen brackets and the Kaneko-Koike operator, together with the fact that all (holomorphic) quasimodular forms are linear combinations of derivatives of modular forms or extended derivatives of the constant function 1, this completes the proof of Theorem 13 except in the case when $\ell = 0$ and $g = 1$, where we have to modify (86) to write $\mathbf{L}_{D^{\langle n\rangle}(g),k}(f)$ as a multiple of the extended Rankin-Cohen bracket $\langle f, 1\rangle_n$ if $k > 0$. If k is 0 or negative, then we have to renormalize $\mathbf{L}_{F,k}$ in a suitable way as already discussed in the case of $\mathbf{L}_{F,k}$ in the remarks following Theorem 8. A further remark is that, in virtue of the identity $\pi_k \circ D = 0$ noted above, we find that (86) can be generalized by repeated "integration by parts" to

$$\binom{k+\ell+2n-2}{n}^{-1} \bigl[f,g\bigr]_n \;=\; (-1)^r \pi_{k+\ell+2n}\bigl(D^r(f)\,D^{n-r}(g)\bigr) \qquad \bigl(0 \leq r \leq n\bigr)\,,$$

so that the nth Rankin-Cohen brackets can be seen as the result of applying the projection operator $\pi_{k,K}$ to *any* product $D^r(f)D^s(g)$ with $r+s=n$.

14. Holomorphic projection and modular linear differential operators

In this final section we recall the bijection between quasimodular forms and almost holomorphic modular forms and use it to rewrite Theorem 13 as a statement involving a holomorphic projection operator defined for real-analytic functions in the upper half-plane. This gives a more conceptual description of the map from quasimodular forms to MLDOs. The final result, Theorem 14, provides perhaps the simplest description of MLDOs of all the ones given in this paper.

By definition, an *almost holomorphic modular form* of weight k on a lattice $\Gamma \subset SL_2(\mathbb{R})$ is a function $\Phi : \mathfrak{H} \to \mathbb{C}$ that is a polynomial in $1/\Im(\tau)$ with coefficients that are holomorphic functions of moderate growth and that transforms like a modular form of weight k under the operation of Γ on $\mathfrak{H}$. The degree of this polynomial is called the *depth* of Φ. There is an isomorphism from the space $\widehat{M}_k(\Gamma)$ of all almost holomorphic modular forms of weight k on Γ to the space $\widetilde{M}_k(\Gamma)$ of quasimodular forms of weight k on Γ given by associating to each function $\Phi(\tau) \in \widehat{M}_k(\Gamma)$ its constant term with respect to $1/\Im(\tau)$. (In fact this was the original definition of quasimodular forms in [17], as already mentioned in Section 8.) The inverse isomorphism, which is less obvious, maps $F \in \widetilde{M}_k(\Gamma)$ to its "completion"

$$\widehat{F}(\tau) = \sum_{r \ge 0} \frac{F_r(\tau)}{(2\pi i(\tau - \bar{\tau}))^r}\,, \tag{87}$$

where the functions F_r are defined by (43) or in terms of δ as $F_r = \delta^r F/r!$. Then the action of the Lie algebra $\mathfrak{sl}_2$ on $\widetilde{M}_*(\Gamma)$ described in Section 8 translates into an action on $\widehat{M}_*(\Gamma)$ by new operators (denoted $\widehat{D}$, $\widehat{W}$ and $\widehat{\delta}$ to distinguish them from the corresponding operators on quasimodular forms) defined by

$$\widehat{D}\,\Phi(\tau) \;=\; \frac{1}{2\pi i}\left(\frac{\partial \Phi(\tau)}{\partial \tau} + k\,\frac{\Phi(\tau)}{\tau - \bar{\tau}}\right), \qquad \widehat{W}\,\Phi(\tau) \;=\; k\,\Phi(\tau)\,, \qquad \widehat{\delta}\,\Phi(\tau) \;=\; 2\pi i(\tau - \bar{\tau})^2\,\frac{\partial \Phi(\tau)}{\partial \bar{\tau}} \tag{88}$$

for $\Phi \in \widehat{M}_k(\Gamma)$. For a more complete discussion of all of this material we refer the reader to Section 5.3 of [35], but with the warning that the notations there are somewhat different (in particular $\widehat{D}$, $\widehat{W}$ and $\widehat{\delta}$ are denoted ∂, E and δ^*, respectively) and that there are a few misprints.

We can now use this isomorphism to transfer the projection operator π_k as defined by (83) to a projection operator $\widehat{\pi}_k$ from almost holomorphic modular forms to modular forms of the same weight simply by replacing D and δ by $\widehat{D}$ and $\widehat{\delta}$ in (83). Then (84) translates into the statement that the operator $\mathbf{L}_{F,k}(f)$ defined in Theorem 13 is equal to $\widehat{\pi}_{k+K}(f\widehat{F})$ if f is a modular form of weight k. However, this is still not quite what we want because to obtain $\mathbf{L}_{F,k}$ as an MLDO we need to know its operation on arbitrary differentiable functions in the upper half-plane, not only on modular forms. For this we use a *holomorphic projection operator* π_k^{hol} that maps real-analytic (or just differentiable) functions in the upper half-plane to holomorphic functions and that projects differentiable or real-analytic modular forms of weight $k \ge 2$ on some lattice $\Gamma \subset SL_2(\mathbb{R})$ to holomorphic modular forms of the same weight and on the same group. There are several different versions of this operator, depending on the space of functions to which it is applied. The most special one, and the one that is used most frequently in the theory of modular forms, is defined directly on the space of differentiable or real-analytic modular forms Φ of weight k on Γ that are sufficiently small at infinity and projects them to holomorphic cusp forms of weight k by the requirement that the Petersson scalar product of $\pi_k^{\mathrm{hol}}(\Phi)$ with any holomorphic cusp form h of weight k on Γ is the same as the Petersson scalar product of Φ with h. This defines $\pi_k^{\mathrm{hol}}(\Phi)$ uniquely because the Petersson scalar product on the space of cusp forms is non-degenerate. At the next level, there is a projection operator defined for arbitrary 1-periodic functions on $\mathfrak{H}$ (i.e., functions on $\mathbb{Z}\backslash\mathfrak{H}$) in terms of their Fourier expansions by the formula

$$\pi_k^{\mathrm{hol}}:\ \Phi(x+iy) \;=\; \sum_{n\in\mathbb{Z}} A_n(y)\,e^{2\pi i n x} \;\mapsto\; f(z) \;=\; \sum_{n>0} a_n\,e^{2\pi i n\tau}\,,$$

$$a_n \;:=\; \frac{(4\pi n)^{k-1}}{(k-2)!}\int_0^\infty A_n(y)\,e^{-2\pi n y}\,y^{k-2}\,dy\,,$$

which is checked by a straightforward calculation to agree with the previous definition when Γ contains the matrix $\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$ and Φ is modular of weight k on Γ. (Apply the formula $(\pi_k^{\mathrm{hol}}(\Phi),h)=(\Phi,h)$ to the nth Poincaré series $h(\tau)=P_{k,n}(\tau)=\sum_{\gamma\in\mathbb{Z}\backslash\Gamma}e^{2\pi i n\tau}|_k\gamma$ and use a standard unfolding argument on both sides.) Finally, both cases can be seen as specializations of a yet more general operator that projects *arbitrary* smooth functions in $\mathfrak{H}$ satisfying a suitable growth condition near the boundary to holomorphic functions and that is equivariant with respect to the action of $SL_2(\mathbb{R})$ and hence sends modular forms to modular forms and 1-periodic functions to 1-periodic functions. This operator is defined by integrating against the *Bergman kernel function* and maps the function Φ to the function $\pi_k^{\mathrm{hol}}(\Phi)$ defined by

$$\pi_k^{\mathrm{hol}}(\Phi)(\tau) \;:=\; \frac{k-1}{4\pi}\iint_{\mathfrak{H}}\Phi(z)\left(\frac{z-\bar z}{\tau-\bar z}\right)^k dV \qquad (\tau\in\mathfrak{H})\,, \tag{89}$$

where dV denotes the invariant volume element $y^{-2}\,dx\,dy$ (with $z=x+iy$ as usual) and the integral is absolutely convergent by virtue of the growth assumption on Φ. The fact that the map π_k^{hol} is $SL_2(\mathbb{R})$-equivariant in weight k follows directly from the transformation property

$$\frac{gz\,-\,\overline{gz}}{g\tau\,-\,\overline{gz}} \;=\; \frac{c\tau+d}{cz+d}\,\frac{z-\bar z}{\tau-\bar z} \qquad \left(z,\,\tau\in\mathfrak{H},\ g=\begin{pmatrix}a&b\\c&d\end{pmatrix}\in SL_2(\mathbb{R})\right),$$

while the fact that it is the identity on holomorphic functions (and therefore is a projection operator as claimed, since $\pi_k^{\mathrm{hol}}(\Phi)$ is obviously always holomorphic in τ) is proved by an argument involving Cauchy's theorem and integration by parts which we generalize in the following lemma.

Lemma 3. *If $h(\tau)$ is a holomorphic function in the upper half-plane satisfying suitable growth conditions, then for any integer m with $0\le m\le k-2$ we have*

$$\pi_k^{\mathrm{hol}}\left(\frac{h(\tau)}{(\tau-\bar\tau)^m}\right) \;=\; \frac{(-1)^m}{(k-m-1)_m}\,\frac{d^m h(\tau)}{d\tau^m}\,. \tag{90}$$

Proof. We can compute the integral over $\mathfrak{H}$ in (89) for $\Phi(\tau)=h(\tau)/(\tau-\bar\tau)^m$ as

$$\int_0^\infty\left(\int_{\mathbb{R}+iy}\frac{(2i)^{k-m}\,h(z)\,dz}{(\tau-z+2iy)^k}\right)y^{k-m-2}\,dy \;=\; 2\pi i\,\frac{(-1)^k(2i)^{k-m}}{(k-1)!}\int_0^\infty\frac{d^{k-1}h}{dz^{k-1}}\bigl(\tau+2iy\bigr)\,y^{k-m-2}\,dy\,.$$

where we have used the trick that $\bar z=z-2iy$ becomes a *holomorphic* function of z when we restrict to the line $\Im(z)=y$, so that we can apply Cauchy's theorem to write the inner integral as $2\pi i$ times the residue of its integrand at the unique pole $z=\tau+2iy$. The lemma now follows by $(k-m-2)$-fold integration by parts. □

We can now use Lemma 3 to extend the domain of definition of π_k^{hol}, which was previously defined on the space of differentiable functions in the upper half-plane of sufficiently small growth at infinity, to the (not direct!) sum of this space with the space $\mathsf{AHol}(\mathfrak{H})$ of "almost holomorphic functions" in the upper half-plane. Here "almost holomorphic" means a polynomial in $1/\Im(\tau)$ with holomorphic coefficients, just as it did in the case of almost holomorphic modular forms, and we no longer need to require any growth condition, because the lemma allows us to define the weight k holomorphic projection of any almost holomorphic function in the upper half-plane simply by writing it as a finite linear combination of functions $(\tau-\bar\tau)^{-m}h(\tau)$ with h holomorphic and applying formula (90), and also tells us that this definition agrees with the one given by the

Bergman integral whenever the function being projected is small at infinity as well as being almost holomorphic. This extended definition is applicable in particular to almost holomorphic modular forms, and we have:

Proposition 6. *The projection maps* π_k *and* π_k^{hol} *from* $\widehat{M}_k(\Gamma)$ *to* $M_k(\Gamma)$ *agree.*

Proof. This follows directly by comparing formulas (83) and (90), using (87) and remembering that $D^m h$ is equal to $(2\pi i)^{-m}\, d^m h/d\tau^m$. □

Proposition 6, combined with equation (84) and the bijection $F \mapsto \widehat{F}$ between quasimodular forms and almost holomorphic modular forms, tells us that $\mathbf{L}_{F,k}(f) = \pi_{k+K}^{\mathrm{hol}}(f\widehat{F})$ for $f \in M_k(\Gamma)$ and $F \in \widetilde{M}_K(\Gamma)$. (Note that the restriction $m \le k-2$ in Lemma 3 is not a problem here, since the weight here is $k+K$ and m is bounded by the depth of F and hence by $K/2$, so that the condition is always satisfied when k is positive, and in fact also for $k=0$ except in the case $K=2$ and $\delta(F)\neq 0$, where $\mathbf{L}_{F,k}$ has to be renormalized by a factor k and then is an MLDO even in this case by the remarks following Theorem 13.) But now the discussion above immediately allows us to extend this statement to arbitrary holomorphic functions f: the right-hand side makes sense because $f\widehat{F}$ is an almost holomorphic function, and the maps coincide because both are differentiable operators and agree for all modular forms of weight k. Putting everything together, we get the following theorem.

Theorem 14. *For each positive integer* k *there is an isomorphism*

$$\widetilde{M}_*(\Gamma) \xrightarrow{\sim} \mathsf{MLDO}_{k,k+*}(\Gamma) \tag{91}$$

given by associating to $F \in \widetilde{M}_K(\Gamma)$ *the modular linear differential operator* $\mathbf{L}_{F,k}$ *defined by*

$$\mathbf{L}_{F,k}(f) \;=\; \pi_{k+K}^{\mathrm{hol}}(f\widehat{F})\,, \tag{92}$$

where π_{k+K}^{hol} *is the extended holomorphic projection operator defined above.*

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