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ON THE CRITICAL VALUES OF HECKE L-SERIES

by

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Let E be the elliptic curve over \mathbb{Q} with minimal model

$$y^2 + xy = x^3 - x^2 - 2x - 1 .$$

The modular invariant, discriminant, and conductor of E are given by

$$j = -3^3 \cdot 5^3$$

$$\Delta = -7^3$$

$$N = (7^2) .$$

Let Ω denote the fundamental real period of the Néron differential $\omega = \frac{dx}{2y+x}$
on E :

$$\Omega = \int_{E(\mathbb{R})} \omega = 1.93331170561681\dots$$

Over the field $K = \mathbb{Q}(\sqrt{-7})$, E has complex multiplication by
 $\mathcal{O} = \mathbb{Z}\left[\frac{1+\sqrt{-7}}{2}\right]$. Hence Ω can be determined explicitly, using an identity of

Chowla and Selberg [1] :

$$\Omega = \frac{\Gamma(1/7)\Gamma(2/7)\Gamma(4/7)}{\sqrt{-7} \cdot 2\pi} .$$

Similarly, the L-series of E is equal to the L-series of a Hecke character χ of K. The conductor of χ is the ideal $(\sqrt{-7})$; for \mathcal{O} an ideal of K which is prime to 7 :

$$\chi(\mathcal{O}) = \alpha \text{ where } \mathcal{O} = (\alpha) , \alpha^3 \equiv 1 \pmod{\sqrt{-7}}$$

We have calculated the central critical values of the Hecke L-series which are associated to odd powers of the character χ . Let $n \geq 1$ be an integer ; the Dirichlet series

$$L(\chi^{2n-1}, s) = \sum \frac{\chi^{2n-1}(\mathcal{O})}{N\mathcal{O}^s}$$

converges absolutely in the right half-plane $\text{Re}(s) > n + \frac{1}{2}$. It extends to a holomorphic function on the entire complex plane : the modified function $\Lambda(\chi^{2n-1}, s) = (7/2\pi)^s \Gamma(s) L(\chi^{2n-1}, s)$ satisfies Hecke's functional equation :

$$\Lambda(\chi^{2n-1}, s) = (-1)^{n+1} \Lambda(\chi^{2n-1}, 2n-s) .$$

It follows that the value of $L(\chi^{2n-1}, s)$ at $s = n$, the center of the critical strip, vanishes when n is even.

When n is odd, define a_n by

$$L(\chi^{2n-1}, n) = \frac{\Omega^{2n-1}}{(2\pi i/\sqrt{-7})^{n-1}} \frac{a_n}{(n-1)!} .$$

(We found this normalization by trial and error ; it is consistent with the work of Katz on the interpolation of real analytic Eisenstein series [2].)

The values of a_n for $1 \leq n \leq 33$ are listed in Table 1.

HECKE L-SERIES

Table 1

n	a_n
1	$1/2$
3	2
5	2
7	$2(3)^2$
9	$2(7)^2$
11	$2(3^2 \cdot 5 \cdot 7)^2$
13	$2(3 \cdot 7 \cdot 29)^2$
15	$2(3 \cdot 7 \cdot 103)^2$
17	$2(3 \cdot 5 \cdot 7 \cdot 607)^2$
19	$2(3^3 \cdot 7 \cdot 4793)^2$
21	$2(3^2 \cdot 5 \cdot 7 \cdot 29 \cdot 2399)^2$
23	$2(3^3 \cdot 5 \cdot 7^2 \cdot 10091)^2$
25	$2(3^2 \cdot 7^2 \cdot 29 \cdot 61717)^2$
27	$2(3^2 \cdot 5^2 \cdot 7^2 \cdot 13 \cdot 53^2 \cdot 79)^2$
29	$2(3^4 \cdot 5^2 \cdot 7^2 \cdot 113 \cdot 127033)^2$
31	$2(3^5 \cdot 5 \cdot 7^2 \cdot 71 \cdot 1690651)^2$
33	$2(3^4 \cdot 5 \cdot 7^2 \cdot 1291 \cdot 1747169)^2$

Let $p \equiv 1 \pmod{4}$ be a prime and let χ_p be the Hecke character

$$\chi_p(\mathcal{O}) = \left(\frac{\mathbb{N}\mathcal{O}}{p}\right) \chi(\mathcal{O}) .$$

The Hecke L-series $L(\chi_p, s)$ is equal to the L-series of an elliptic curve E_p/\mathbb{Q} which becomes isomorphic to E over $\mathbb{Q}(\sqrt{p})$. Let $\Omega_p = \Omega/\sqrt{p}$ and define $a_n^{(p)}$ by

$$L(\chi_p^{2n-1}, n) = \frac{\Omega_p^{2n-1}}{(2\pi i/\sqrt{-7})^{n-1}} \frac{a_n^{(p)}}{(n-1)!}.$$

Again, $a_n^{(p)}$ vanishes when n is even. For n odd and $p = 5, 13, 17, 29, 53$

we found the values listed in Table 2.

Table 2

$n \backslash p$	5	13	17	29	53
1	1	1	1	2	0 Frank $E_{53}(\mathbb{Q}) = 2J$
3	$(2^2 \cdot 5)^2$	$(2^2 \cdot 3)^2$	$(2 \cdot 3 \cdot 13)^2$	$2(2^2 \cdot 3)^2$	$2(2^3 \cdot 7)^2$
5	$(2^2 \cdot 5)^2$	$(2^2 \cdot 157)^2$	$(2 \cdot 271)^2$	$2(2^2 \cdot 5 \cdot 37)^2$	$2(2^4 \cdot 3 \cdot 5^2 \cdot 7)^2$
7	$(2^2 \cdot 3 \cdot 5 \cdot 61)^2$	$(2^2 \cdot 3^2 \cdot 1847)^2$	$(2 \cdot 3^2 \cdot 61)^2$	$2(2^2 \cdot 3^2 \cdot 11 \cdot 29 \cdot 61)^2$	$2(2^3 \cdot 3 \cdot 5^2 \cdot 7^2 \cdot 13 \cdot 17)^2$
9	$(2^2 \cdot 5 \cdot 7 \cdot 199)^2$	$(2^2 \cdot 7 \cdot 13 \cdot 5813)^2$	$(2 \cdot 7 \cdot 266977)^2$	$2(2^2 \cdot 7 \cdot 17 \cdot 80779)^2$	$2(2^5 \cdot 3 \cdot 7^2 \cdot 19 \cdot 2699)^2$
11	$(2^2 \cdot 5^2 \cdot 271)^2$	$(2^2 \cdot 3^3 \cdot 5 \cdot 7 \cdot 13 \cdot 1021)^2$	$(2 \cdot 3^4 \cdot 5 \cdot 7 \cdot 17 \cdot 2081)^2$	$2(2^2 \cdot 3^3 \cdot 5^3 \cdot 7 \cdot 13 \cdot 29 \cdot 79)^2$	$2(2^3 \cdot 3 \cdot 5^2 \cdot 7^2 \cdot 11 \cdot 129893)^2$
13	$(2^2 \cdot 3 \cdot 5^2 \cdot 7 \cdot 3767)^2$	$(2^2 \cdot 3 \cdot 7 \cdot 13 \cdot 3747629)^2$			
15	$(2^2 \cdot 3 \cdot 5^2 \cdot 7 \cdot 89 \cdot 13687)^2$	$(2^2 \cdot 3^2 \cdot 7 \cdot 13 \cdot 101 \cdot 317 \cdot 15307)^2$			
17	$(2^2 \cdot 3 \cdot 5^4 \cdot 7 \cdot 26737)^2$	$(2^2 \cdot 3 \cdot 5 \cdot 7 \cdot 13 \cdot 4877 \cdot 6510011)^2$			

Note : In all the cases we computed, $a_n^{(p)}$ is either a square or twice a square, depending on whether $\left(\frac{p}{n}\right)$ is -1 or $+1$. Can one prove this is general? Are there "higher Tate-Shafarevitch groups" associated to the abelian varieties $(E_p)^{2n-1}$ whose orders can be conjecturally related to $a_n^{(p)}$? Do these groups carry a natural alternating pairing?

Bibliography :

- [1] CHOWLA S., and SELBERG A., On Epstein's Zeta Function. J. Crelle 227 (1967), 96-110.

- [2] KATZ N., p-adic interpolation of real analytic Eisenstein series. Annals Math. 104 (1976), 459-571.

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