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ON THE CRITICAL VALUES OF HECKE L-SERIES

by

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Let E be the elliptic curve over $\mathbb{Q}$ with minimal model

$$y^2 + xy = x^3 - x^2 - 2x - 1 \quad .$$

The modular invariant, discriminant, and conductor of E are given by

$$j = -3^3 \cdot 5^3$$

$$\Delta = -7^3$$

$$N = (7^2) \quad .$$

Let Ω denote the fundamental real period of the Néron differential $\omega = \frac{dx}{2y+x}$ on E :

$$\Omega = \int_{E(\mathbb{R})} \omega = 1.93331170561681\ldots$$

Over the field $K = \mathbb{Q}(\sqrt{-7})$, E has complex multiplication by $\mathcal{O} = \mathbb{Z}[\frac{1+\sqrt{-7}}{2}]$. Hence Ω can be determined explicitly, using an identity of Chowla and Selberg [1] :

$$\Omega = \frac{\Gamma(1/7)\Gamma(2/7)\Gamma(4/7)}{\sqrt{-7}\ .\ 2\pi i} .$$

Similarly, the L-series of E is equal to the L-series of a Hecke character χ of K. The conductor of χ is the ideal $(\sqrt{-7})$; for $\mathfrak{a}$ an ideal of K which is prime to 7 :

$$\chi(\mathfrak{a}) = \alpha \quad \text{where} \quad \mathfrak{a} = (\alpha) , \alpha^3 \equiv 1 (\bmod \sqrt{-7})$$

We have calculated the central critical values of the Hecke L-series which are associated to odd powers of the character χ. Let $n \geq 1$ be an integer ; the Dirichlet series

$$L(\chi^{2n-1},s) = \Sigma \frac{\chi^{2n-1}(\mathfrak{a})}{\mathbb{N}\mathfrak{a}^s}$$

converges absolutely in the right half-plane $Re(s) > n + \frac{1}{2}$. It extends to a holomorphic function on the entire complex plane : the modified function $\Lambda(\chi^{2n-1},s) = (7/2\pi)^s\ \Gamma(s)\ L(\chi^{2n-1},s)$ satisfies Hecke's functional equation :

$$\Lambda(\chi^{2n-1},s) = (-1)^{n+1}\ \Lambda(\chi^{2n-1},2n-s) .$$

It follows that the value of $L(\chi^{2n-1},s)$ at $s = n$, the center of the critical strip, vanishes when n is even.

When n is odd, define a_n by

$$L(\chi^{2n-1},n) = \frac{\Omega^{2n-1}}{(2\pi i/\sqrt{-7})^{n-1}} \frac{a_n}{(n-1)!} .$$

(We found this normalization by trial and error ; it is consistent with the work of Katz on the interpolation of real analytic Eisenstein series [2].)

The values of a_n for $1 \leq n \leq 33$ are listed in Table 1.

Table 1

n	a_n
1	$1/2$
3	2
5	2
7	$2(3)^2$
9	$2(7)^2$
11	$2(3^2.5.7)^2$
13	$2(3.7.29)^2$
15	$2(3.7.103)^2$
17	$2(3.5.7.607)^2$
19	$2(3^3.7.4793)^2$
21	$2(3^2.5.7.29.2399)^2$
23	$2(3^3.5.7^2.10091)^2$
25	$2(3^2.7^2.29.61717)^2$
27	$2(3^2.5^2.7^2.13.53^2.79)^2$
29	$2(3^4.5^2.7^2.113.127033)^2$
31	$2(3^5.5.7^2.71.1690651)^2$
33	$2(3^4.5.7^2.1291.1747169)^2$

Let $p \equiv 1 (\mathrm{mod}\ 4)$ be a prime and let χ_p be the Hecke character

$$\chi_p(\mathfrak{a}) = \left(\frac{\mathbb{N}\mathfrak{a}}{p}\right) \chi(\mathfrak{a}) .$$

The Hecke L-series $L(\chi_p, s)$ is equal to the L-series of an elliptic curve $E_p/\mathbb{Q}$ which becomes isomorphic to E over $\mathbb{Q}(\sqrt{p})$. Let $\Omega_p = \Omega/\sqrt{p}$ and define $a_n^{(p)}$ by

$$L(\chi_p^{2n-1},n) = \frac{\Omega_p^{2n-1}}{(2\pi i/\sqrt{-7})^{n-1}}\,\frac{a_n^{(p)}}{(n-1)!}\ .$$

Again, $a_n^{(p)}$ vanishes when n is even. For n odd and $p = 5, 13, 17, 29, 53$ we found the values listed in Table 2.

Table 2

n \ p	5	13	17	29	53
1	1	1	1	2	0 [rank $E_{53}(\mathbb{Q}) = 2$]
3	$(2^2)^2$	$(2^2 . 3)^2$	$(2 . 3 . 13)^2$	$2(2^2 . 3)^2$	$2(2^3 . 7)^2$
5	$(2^2 . 5)^2$	$(2^2 . 157)^2$	$(2 . 271)^2$	$2(2^2 . 5 . 37)^2$	$2(2^4.3.5^2.7)^2$
7	$(2^2.3.5.61)^2$	$(2^2.3^2.1847)^2$	$(2.3^2.61)^2$	$2(2^2.3^2.11.29.61)^2$	$2(2^3.3.5^2.7^2.13.17)^2$
9	$(2^2.5.7.199)^2$	$(2^2.7.13.5813)^2$	$(2.7.266977)^2$	$2(2^2.7.17.80779)^2$	$2(2^5.3.7^2.19.2699)^2$
11	$(2^2.5^2.271)^2$	$(2^2.3^3.5.7.13.1021)^2$	$(2.3^4.5.7.17.2081)^2$	$2(2^2.3^3.5^3.7.13.29.79)^2$	$2(2^3.3^2.5^2.7^2.11.129893)^2$
13	$(2^2.3.5^2.7.3767)^2$	$(2^2.3.7.13.3747629)^2$			
15	$(2^2.3.5^2.7.89.13687)^2$	$(2^2.3^2.7.13.101.317.15307)^2$			
17	$(2^2.3.5^4.7.26737)^2$	$(2^2.3.5.7.13.4877.6510011)^2$			

Note : In all the cases we computed, $a_n^{(p)}$ is either a square or twice a square, depending on whether $(\frac{p}{7})$ is -1 or $+1$. Can one prove this is general ? Are there "higher Tate-Shafarevitch groups" associated to the abelian varieties $(E_p)^{2n-1}$ whose orders can be conjecturally related to $a_n^{(p)}$? Do these groups carry a natural alternating pairing ?

Bibliography :

[1] CHOWLA S., and SELBERG A., On Epstein's Zeta Function. J. Crelle 227 (1967), 96-110.

[2] KATZ N., p-adic interpolation of real analytic Eisenstein series. Annals Math. 104 (1976), 459-571.

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