Square roots of central values of Hecke L-series

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§1. Introduction

In [2] numerical examples were produced suggesting that the "algebraic" part of central values of certain Hecke L-series are perfect squares. More precisely, let ψ_1 be the grossencharacter of $\mathbf{Q}(\sqrt{-7})$ defined by

$$\psi_1(\mathfrak{a}) = \left(\frac{m}{7}\right)\alpha \qquad \text{if} \quad \mathfrak{a} = (\alpha), \quad \alpha = \frac{m+n\sqrt{-7}}{2} \in \mathbf{Z}\left[\frac{1+\sqrt{-7}}{2}\right]$$

and consider the central value $L(\psi_1^{2k-1}, k)$ of the L-series associated to an odd power of ψ_1. This value vanishes for k even by virtue of the functional equation, but for k odd one has

$$(1) \qquad L(\psi_1^{2k-1}, k) = 2\,\frac{(2\pi/\sqrt{7})^k\,\Omega^{2k-1}}{(k-1)!}\,A(k)\,, \qquad \Omega = \frac{\Gamma(\frac{1}{7})\Gamma(\frac{2}{7})\Gamma(\frac{4}{7})}{4\pi^2}\,,$$

with $A(1) = 1/4$, $A(3) = A(5) = 1$, $A(7) = 9$, $A(9) = 49$, …, $A(33) = 44762286327255^2$, suggesting the conjecture

$$(2) \qquad L(\psi_1^{2k-1}, k) \stackrel{?}{=} 2\,\frac{(2\pi/\sqrt{7})^k\,\Omega^{2k-1}}{(k-1)!}\,B(k)^2\,,$$

with $B(1) = 1/2$ and $B(k) \in \mathbf{Z}$ for all $k \geq 3$.

In this paper we will prove this conjecture and analogous results for other grossencharacters. The method will be a modification of the method of [8], where it was shown that the central values of weight one Hecke L-series are essentially the squares of certain sums of values of weight 1/2 theta series at CM points. The new ingredient for higher weight is that we have to use (non-holomorphic) *derivatives* of modular forms. For instance, the value of $B(k)$ in (2) will turn out to be essentially the value of a certain derivative of

$$(3) \qquad \theta_{1/2}(z) = \sum_{\substack{n\geq 1\\ n \text{ odd}}} e^{\pi i n^2 z/4} \qquad (z \in \mathcal{H} = \text{ upper half-plane})$$

at $z = (1+i\sqrt{7})/2$. Because the derivatives of modular forms can be computed recursively, this leads to *a simple recursive formula* for the special values of Hecke L-series and their square roots. In particular, for the example above the result is:

THEOREM. *Define sequences of polynomials $a_{2n}(x)$, $b_n(x)$ by the recursions*

$$a_{n+1}(x) = \sqrt{(1+x)(1-27x)}\,\Big(x\,\frac{d}{dx} - \frac{2n+1}{3}\Big)\,a_n(x) - \frac{n^2}{9}\,(1-5x)\,a_{n-1}(x) \tag{4}$$

$$\begin{aligned} 21\,b_{n+1}(x) &= \Big((32nx - 56n + 42) \;-\; (x-7)(64x-7)\,\frac{d}{dx}\Big)\,b_n(x) \\ &\quad - \;2n(2n-1)\,(11x+7)\,b_{n-1}(x) \end{aligned} \tag{5}$$

with initial conditions $a_0(x) = 1$, $a_1(x) = -\frac{1}{3}\sqrt{(1-x)(1+27x)}$, $b_0(x) = 1/2$, $b_1(x) = 1$. Then the values $A(2n+1) = a_{2n}(-1)/4$ and $B(2n+1) = b_n(0)$ satisfy equations (1) and (2).

For numerical values of the first few $a_{2n}(x)$, $b_n(x)$, and $B(k)$, see §7. Surprisingly, there seems to be no simple direct proof of the identity $a_{2n}(-1) = 4\,b_n(0)^2$!

We mention briefly several applications of the theorem. First, it gives us a specific choice of the square root of the numbers $A(k)$ occurring in (1). Now it is well-known that the existence of a p-adic L-function imply that the squares $A(k)$ satisfy congruences modulo certain prime powers, and it has been conjectured [4] that there should be analogous p-adic interpolation properties for appropriately chosen square roots. Testing this on the square roots produced by the theorem, we do indeed find that these satisfy certain congruences of the desired type, e.g.

$$B(k) \equiv -k \pmod 4, \qquad B(k+10) \equiv 7B(k) \pmod{11}, \qquad (k \ge 3) \tag{6}$$

This is the topic of a forthcoming thesis by A. Sofer. Notice, by the way, that either of the congruences (6) implies the non-vanishing of $L(\psi_1^{2k-1}, k)$, which is not a priori obvious.

The second "application" is that one can compute the numbers $A(k)$ and $B(k)$ much more easily than was previously possible. The method of computation in [4] was to compute $L(\psi_1^{2k-1}, k)$ as $2\sum_{n=1}^{\infty} a_n^{(k)}\,\big(\sum_{j=0}^{k}\frac{1}{j!}\,(\frac{2\pi n}{7})^j\big)\,e^{-2\pi n/7}$, where $L(\psi_1^{2k-1}, s) = \sum_{n=1}^{\infty} a_n^{(k)}\,n^{-s}$, but this becomes unmanageable for large k. The recurrences are easier to work with and can also be used to compute the numbers in question modulo high powers of a prime p, without computing the numbers themselves, which grow very rapidly. This is useful both for testing the above-mentioned conjectures of Koblitz and in connection with a beautiful recent result of Rubin [6] (proved by him modulo the Birch–Swinnerton-Dyer conjecture) which gives a transcendental construction of points on certain elliptic curves via p-adic interpolation of values of Hecke L-series.

The third application is that the central values of the L-series under consideration are always non-negative, which in turn by a remark of Greenberg implies a result on their average value (cf. Corollary in §5 and the following comments).

In a different direction, it was also found in [2] that the values of the *twisted* L-series $L(\psi_1^{2k-1}, (\frac{\cdot}{p}), s) = \sum_{n=1}^{\infty} (\frac{n}{p})\,a_n^{(k)}\,n^{-s}$ ($p \equiv 1 \pmod 4$ prime) at $s = k$ were again essentially perfect squares:

$$L(\psi_1^{2k-1}, (\tfrac{\cdot}{p}), k) \stackrel{?}{=} \frac{3 + (\frac{p}{7})}{2}\,\frac{(\sqrt{7}/2\pi)^{k-1}\,(\Omega/\sqrt{p})^{2k-1}}{(k-1)!}\,B(k,p)^2\,; \tag{7}$$

The well-known theorem of Waldspurger, of course, tells us that the central values of the twists of the L-series of a given modular (Hecke eigen) form f are essentially square multiples of one another, the square-roots being proportional to the Fourier coefficients of the half-integral weight form attached to f by the Shimura lifting, but it does not tell us the values themselves. In §8 we propose a formula for the numbers $B(k,p)$ which when combined with Waldspurger's theorem may eventually give an explicit formula for the coefficients of the Shimura lifting of a modular form attached to a grossencharacter.

The main result of this paper (like that of [8]) involves a "factorization formula" which expresses the value (or derivative) of a weight one theta series at a CM point as a product of values (or derivatives) of weight 1/2 theta series at other CM points. This formula will be proved in §4, while §5 gives the application to grossencharacters and §§6–7 describe recurrence relations like the ones in the sample theorem above.

§2. Derivatives of modular forms

The differential operator

$$D = \frac{1}{2\pi i}\frac{d}{dz} = \frac{1}{q}\frac{d}{dq} \qquad (q = e^{2\pi i z})$$

maps holomorphic functions to holomorphic functions and functions with a Fourier expansion $\sum a(n)\,q^n$ to functions with a Fourier expansion $\sum na(n)\,q^n$ with Fourier coefficients in the same field, but it destroys the property of being a modular form. As is well-known, this can be corrected by introducing the modified differentiation operator

$$\partial_k = D - \frac{k}{4\pi y}$$

which satisfies $\partial_k(f|_k\gamma) = (\partial_k f)|_{k+2}\gamma$ for all $\gamma \in SL_2(\mathbf{R})$, where as usual $f|_k\bigl(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\bigr)(z) = (cz+d)^{-k} f\bigl(\frac{az+b}{cz+d}\bigr)$. In particular, if $\Gamma \subset SL_2(\mathbf{R})$ is some modular group and $M_k^*(\Gamma)$ denotes the space of differentiable modular forms of weight k on Γ, possibly with some character or multiplier system v (i.e., of f satisfying $f|_k\gamma = v(\gamma)f$ for all $\gamma \in \Gamma$), then $\partial_k f$ belongs to $M_{k+2}^*(\Gamma)$ and more generally $\partial_k^h f$ to $M_{k+2h}^*(\Gamma)$, where

$$\partial_k^h = \partial_{k+2h-2} \circ \partial_{k+2h-4} \circ \ldots \circ \partial_{k+2} \circ \partial_k\ .$$

In this situation we will often drop the subscript and write simply $\partial^h f$ for $\partial_k^h f$, since f determines its weight k uniquely.

An easy induction shows that

$$\partial_k^h = \sum_{j=0}^{h} \binom{h}{j} \frac{\Gamma(h+k)}{\Gamma(j+k)} \Bigl(\frac{-1}{4\pi y}\Bigr)^{h-j} D^j\ . \tag{8}$$

In particular,

$$\partial_k^h\Bigl(\sum_{n=0}^{\infty} a(n)\,e^{2\pi i n z}\Bigr) = \frac{(-1)^h h!}{(4\pi y)^h} \sum_{n=0}^{\infty} a(n)\,L_h^{k-1}(4\pi n y)\,e^{2\pi i n z}\ , \tag{9}$$

where

$$L_h^\alpha(z) = \sum_{j=0}^{h} \binom{h+\alpha}{h-j} \frac{(-z)^j}{j!} \qquad (h \in \mathbf{Z}_{\geq 0}, \alpha \in \mathbf{C}) \tag{10}$$

denotes the *h-th generalized Laguerre polynomial.* In the special case $k = 1/2$ we have the identity $L_h^{-1/2}(z) = (-1/4)^h H_{2h}(\sqrt{z})/h!$, where

$$H_p(z) = \sum_{0 \leq j \leq p/2} \frac{p!}{j!(p-2j)!} (-1)^j (2z)^{p-2j} \qquad (p \in \mathbf{Z}_{\geq 0}) \tag{11}$$

is the *pth Hermite polynomial.* In particular, the non-holomorphic derivatives of the weight 1/2 theta series $\theta_{1/2}$ defined in (3) are given by

$$\partial^h \theta_{1/2}(z) = \frac{1}{(16\pi y)^h} \sum_{\substack{n>0 \\ n \text{ odd}}} H_{2h}(n\sqrt{\pi y/2})\, e^{\pi i n^2 z/4} \,.$$

A similar calculation applies to the weight 3/2 theta series

$$\theta_{3/2}(z) = \sum_{n=1}^{\infty} \left(\frac{-4}{n}\right) n\, e^{\pi i n^2 z/4} \qquad (z \in \mathcal{H}) \tag{12}$$

and shows that the Fourier expansions of the functions $\theta_{p+1/2}$ $(p \in \mathbf{Z}_{\geq 0})$defined by

$$\theta_{p+1/2}(z) = \begin{cases} 8^h\, \partial_{1/2}^h \theta_{1/2}(z) & \text{if } p = 2h,\ h \geq 0 \\ 8^h\, \partial_{3/2}^h \theta_{3/2}(z) & \text{if } p = 2h+1,\ h \geq 0 \end{cases} \tag{13}$$

can be given by the uniform formula

$$\theta_{p+1/2}(z) = \frac{1}{(2\pi y)^{p/2}} \sum_{\substack{n \geq 1 \\ n \text{ odd}}} \left(\frac{-4}{n}\right)^p H_p(n\sqrt{\pi y/2})\, e^{\pi i n^2 z/4} \,. \tag{14}$$

We remark that the holomorphic theta series defined by (3) and (12) can be expressed in terms of the Dedekind eta-function $\eta(z) = e^{\pi i z/12} \prod (1 - e^{2\pi i n z})$ as

$$\theta_{1/2}(z) = \frac{\eta(2z)^2}{\eta(z)}, \qquad \theta_{3/2}(z) = \eta(z)^3 \,. \tag{15}$$

Finally, from (8) and the binomial theorem we immediately get the identity

$$\partial_k^h \left(\frac{1}{(mz+n)^k} \right) = \frac{\Gamma(h+k)}{\Gamma(k)} \left(\frac{-1}{4\pi y} \frac{m\bar{z}+n}{mz+n} \right)^h \frac{1}{(mz+n)^k} \tag{16}$$

which will be used in the next section to express the values of Hecke L-series at critical points as values of non-holomorphic derivatives of holomorphic Eisenstein series at CM points.

§3. L-series and Eisenstein series

From now on we fix the following notation: K is an imaginary quadratic field of odd discriminant $-d$, $\mathcal{O}_K$ its ring of integers, $\mathfrak{d} = (\sqrt{-d})$ its different, and $\mathrm{Cl}(K)$ its class group. We suppose $d \neq 3$ so that $\mathcal{O}_K^* = \{\pm 1\}$; later we will also suppose that d is prime to avoid complications due to genus characters. We denote by $\varepsilon(n) = \left(\frac{-d}{n}\right) = \left(\frac{n}{d}\right)$ the Dirichlet character associated to K. We can extend it via the isomorphism $\mathbf{Z}/d \xrightarrow{\cong} \mathcal{O}_K/\mathfrak{d}$ to a quadratic character of K of conductor $\mathfrak{d}$. (Explicitly, we can write any $\mu \in \mathcal{O}_K$ as $\mu = \frac{1}{2}(m + n\sqrt{-d})$ with m and n of the same parity, and then $\varepsilon(\mu) = \varepsilon(2m)$.) Finally, we define $\delta = 0$ or 1 by $(d+1)/4 \equiv \delta \pmod 2$ or $\varepsilon(2) = (-1)^\delta$.

We fix a positive integer k and consider Hecke characters ψ of K satisfying

$$\psi((\alpha)) = \varepsilon(\alpha)\,\alpha^{2k-1} \tag{17}$$

for $\alpha \in \mathcal{O}_K$ prime to $\mathfrak{d}$. Clearly the number of such ψ equals $h(-d)$, the class number of K, and each one has conductor $\mathfrak{d}$. Associated to ψ is the Hecke L-series

$$L(\psi, s) = \sum_{\mathfrak{a}} \frac{\psi(\mathfrak{a})}{N(\mathfrak{a})^s}.$$

If we define for any ideal $\mathfrak{a}$ prime to $\mathfrak{d}$ a partial Hecke series by

$$Z(2k-1, \mathfrak{a}, s) = \frac{1}{2} \sum_{\lambda \in \mathfrak{a}}{}' \frac{\varepsilon(\lambda)\,\bar{\lambda}^{2k-1}}{|\lambda|^{2s}} = \sum_{\substack{\lambda \in \mathfrak{a} \\ \varepsilon(\lambda) = +1}} \frac{\bar{\lambda}^{2k-1}}{|\lambda|^{2s}},$$

where the prime indicates that 0 is excluded, then $\psi(\mathfrak{a})N(\mathfrak{a})^{s-2k+1}Z(2k-1, \mathfrak{a}, s)$ depends only on the class $[\mathfrak{a}]$ of $\mathfrak{a}$ in $\mathrm{Cl}(K)$ and a standard one-line calculation gives the decomposition

$$L(\psi, s) = \sum_{[\mathfrak{a}] \in \mathrm{Cl}(K)} \frac{\psi(\mathfrak{a})}{N(\mathfrak{a})^{2k-1-s}}\, Z(2k-1, \mathfrak{a}, s)\,. \tag{18}$$

The series $L(\psi, s)$ and $Z(2k-1, \mathfrak{a}, s)$ converge only for $\Re(s) > k + \frac{1}{2}$, but can be analytically continued to the whole s-plane and satisfy a functional equation under $s \mapsto 2k - s$, with root number $w_k = (-1)^{k+1+\delta}$. Their critical values correspond to $s = k + r$ for integers $0 \le r \le k-1$ and their reflections $s = k - r$. In particular, their center value corresponds to $r = 0$, and $L(\psi, k) = 0$ if $k \equiv \delta \pmod 2$.

Any primitive ideal $\mathfrak{a}$ (i.e., one not divisible by rational integers > 1) can be written as $\mathbf{Z}a + \mathbf{Z}\frac{b+\sqrt{-d}}{2}$ where $a = N(\mathfrak{a})$ and b is an integer, determined mod $2a$, satisfying $b^2 \equiv -d$ mod $4a$. The number $(b + \sqrt{-d})/2a$ in $\mathcal{H}$ is then well-defined modulo $\mathbf{Z}$ and its class in $\mathcal{H}/SL_2(\mathbf{Z})$ depends only the ideal class of $\mathfrak{a}$. However, we will be wanting to evaluate modular forms of level d and for this we have to require additional congruences modulo d. We will choose $\mathfrak{a}$ prime to $\mathfrak{d}$ and then choose b divisible by d (which we can do since $2a$ is prime to d). We set

$$z_{\mathfrak{a}}^{(d)} = \frac{b + \sqrt{-d}}{2ad} \in \mathcal{H}\,, \quad \text{where} \quad \mathfrak{a} = \left[a, \frac{b+\sqrt{-d}}{2}\right], \quad (a,d) = 1\,, \quad b \equiv 0 \pmod d\,. \tag{19}$$

Then $z_{\mathfrak{a}}^{(d)}$ is well-defined modulo $\mathbf{Z}$ and its image in $\mathcal{H}/\Gamma_0(d)$ depends only on the class $[\mathfrak{a}]$.

Each $\lambda \in \mathfrak{a}$ can be written as $a(mdz_{\mathfrak{a}}^{(d)} + n)$ for some integers m, $n \in \mathbf{Z}$, and by virtue of our choice of b we have $\varepsilon(\lambda) = \varepsilon(n)$ (note that $\varepsilon(a) = 1$ automatically). Hence

$$Z(2k-1, \mathfrak{a}, s) = a^{2k-1-2s} \cdot \frac{1}{2} \sum_{m,n}{}' \frac{\varepsilon(n)\,(md\bar{z}_{\mathfrak{a}}^{(d)} + n)^{2k-1}}{|mdz_{\mathfrak{a}}^{(d)} + n|^{2s}} \qquad (\Re(s) > k + \frac{1}{2})\,.$$

For each integer $r \geq 0$ we define an Eisenstein series of weight $2r+1$ and character ε on $\Gamma_0(d)$ by

$$G_{2r+1,\varepsilon}(z) = \frac{1}{2} \sum_{m,n}{}' \frac{\varepsilon(n)}{(mdz+n)^{2r+1}} \qquad (z \in \mathcal{H}) \tag{20}$$

(if $r = 0$ this does not converge absolutely and has to be summed by the usual Hecke trick), with Fourier expansion given by

$$G_{2r+1,\varepsilon}(z) = L(2r+1, \varepsilon) + \frac{(-1)^r (2\pi)^{2r+1}}{(2r)!\, d^{2r+1/2}} \sum_{n \geq 1} \Bigl(\sum_{m|n} \varepsilon(m) m^{2r}\Bigr) q^n \qquad (q = e^{2\pi i z}). \tag{21}$$

As an immediate consequence of (16) we find

$$\partial^{k-r-1} G_{2r+1,\varepsilon}(z) = \frac{1}{2} \frac{(k+r-1)!}{(2r)!} \left(\frac{-1}{4\pi y}\right)^{k-r-1} \sum_{m,n}{}' \frac{\varepsilon(n)\,(md\bar{z}+n)^{2k-1}}{|mdz+n|^{2k+2r}}$$

and hence finally

Proposition. *Let $\mathfrak{a}$ and $z_{\mathfrak{a}}^{(d)}$ be as in (19) and $0 \leq r \leq k-1$. Then*

$$Z(2k-1, \mathfrak{a}, k+r) = \frac{(2r)!}{(k+r-1)!} \frac{(-2\pi/\sqrt{d})^{k-r-1}}{N(\mathfrak{a})^{k+r}}\, \partial^{k-r-1} G_{2r+1,\varepsilon}(z_{\mathfrak{a}}^{(d)})\,. \tag{22}$$

This formula is true also for $r = 0$, the case of primary interest to us, because the summation via Hecke's trick commutes with the differentiation operator ∂.

§4. A factorization identity for theta series

Recall that H_p and L_h^{α} denote the Hermite and Laguerre polynomials. For $\mu, \nu \in \mathbf{Q}$ and $p \in \mathbf{Z}_{\geq 0}$ we define

$$\theta_{(p)}[{}^{\mu}_{\nu}](z) = i^{-p}\,(2\pi y)^{-p/2} \sum_{n \in \mathbf{Z}+\mu} H_p(n\sqrt{2\pi y})\, e^{\pi i n^2 z + 2\pi i \nu n} \qquad (z = x+iy \in \mathcal{H}).$$

For $z = x + iy \in \mathcal{H}$ we set $Q_z(m,n) = |mz-n|^2/2y$, the general positive definite binary quadratic form of discriminant -1 with real coefficients.

THEOREM (FACTORIZATION FORMULA). *For $a \in \mathbf{N}$, $z \in \mathcal{H}$, $\mu, \nu \in \mathbf{Q}$, and $p, \alpha \in \mathbf{Z}_{\geq 0}$,*

$$\begin{aligned}(23)\qquad &\frac{(-1)^p\, p!}{(\pi y)^p} \sum_{m,n\in\mathbf{Z}} e^{2\pi i(m\nu+n\mu)} \Big(\frac{mz-n}{ay}\Big)^{\alpha} L_p^{\alpha}\big(\frac{2\pi}{a} Q_z(m,n)\big)\, e^{\pi(imn-Q_z(m,n))/a} \\ &\qquad = \sqrt{2ay}\,(ay)^{\alpha}\, \theta_{(p)}\big[{a\mu \atop \nu}\big](a^{-1}z)\cdot \theta_{(p+\alpha)}\big[{\mu \atop -a\nu}\big](-a\bar z)\,.\end{aligned}$$

REMARK: For the simplest case $a = 1$, $\alpha = 0$ the right-hand side of the (23) becomes $(-1)^p\sqrt{2y}\,\big|\theta_{(p)}\big[{\mu \atop \nu}\big](z)\big|^2$, so the sum on the left is nonnegative, which is not clear a priori.

PROOF: For u_1, $u_2 \in \mathbf{C}$ we have the identity

$$\begin{aligned}&\Big(e^{\pi u_2^2/2ay} \sum_{n\in\mathbf{Z}+\mu} e^{-\pi i a n^2 \bar z + 2\pi i n(u_2 - a\nu)}\Big)\Big(e^{\pi a u_1^2/2y} \sum_{\ell\in\mathbf{Z}+a\mu} e^{\pi i \ell^2 z/a + 2\pi i \ell(u_1+\nu)}\Big) \\ &= e^{\pi(a^2u_1^2+u_2^2)/2ay} \sum_{m\in\mathbf{Z}} e^{\pi i m^2 z/a + 2\pi i m(u_1+\nu)} \sum_{n\in\mathbf{Z}+\mu} e^{-2\pi a n^2 y + 2\pi i n(mz + au_1 + u_2)} \\ &= \frac{e^{-\pi u_1 u_2/y}}{\sqrt{2ay}} \sum_{m,n\in\mathbf{Z}} e^{2\pi i(m\nu+n\mu)+\pi(imn-Q_z(m,n))/a - \pi[(m\bar z - n)u_1 + (mz-n)u_2/a]/y}\end{aligned}$$

the first equality being obtained by the substitution $\ell = an + m$ and the second by applying the formula

$$\sum_{n\in\mathbf{Z}+\mu} e^{-\pi n^2 A + 2\pi i n B} = \frac{1}{\sqrt{A}} \sum_{n\in\mathbf{Z}} e^{-\pi(n-B)^2/A + 2\pi i n\mu} \qquad (A > 0, \quad B \in \mathbf{C}),$$

which is a standard consequence of the Poisson summation formula, to the inner sum. Identity (23) follows by comparing the Taylor coefficients of $u_1^p\, u_2^{p+\alpha}$ on both sides. ■

REMARKS: 1. This formula in the case $a = 1$ is essentially contained in Kronecker's work ([5], Chapter III). It is also a special case of a general transformation formula for products of Fourier series which is used in the field of radar signal design (cf. Chapter 8 of [7], especially the Corollary to Theorem 8.18).

2. The proof can be expressed in an essentially equivalent but somewhat different way by using the transformation formula of the genus 2 theta series

$$\Theta^{(2)}(u, Z) = \sum_{n\in\mathbf{Z}^2} e^{\pi i\, {}^t n Z n} e^{2\pi i\, {}^t n u} \qquad (z \in \mathcal{H}_2, \quad u \in \mathbf{C}^2)$$

under the action of the matrix

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & -a \\ a & 1 & 0 & 0 \end{pmatrix} \in Sp_4(\mathbf{Z})$$

to relate the values of $\Theta^{(2)}$ at the two points

$$(u, Z) = \left(\begin{pmatrix} u_1 + \mu z + \nu \\ u_2 - a(\mu\bar z + \nu)\end{pmatrix}, \quad \begin{pmatrix} a^{-1}z & 0 \\ 0 & -a\bar z\end{pmatrix}\right)$$

(where it obviously splits into a product of genus one series) and

$$(u, Z) = \left(\frac{i}{2ay} \begin{pmatrix} u_1 a\bar{z} + u_2 z \\ -au_1 - u_2 \end{pmatrix} + \begin{pmatrix} \nu \\ \mu \end{pmatrix}, \quad \frac{i}{2ay} \begin{pmatrix} |z|^2 & -z \\ -z & 1 \end{pmatrix} \right).$$

This is connected with an interesting interpretation of the factorization formula in terms of the geometry of the Siegel modular variety of genus 2 which will be discussed in a later publication.

We now apply the theorem to the case when z is an appropriately chosen CM point in the upper half-plane, in which case the left-hand side of (23) becomes the value at a CM point of a non-holomorphic derivative of a holomorphic theta series of weight $\alpha + 1$. We let $K = \mathbf{Q}(\sqrt{-d})$ and ε have the same meanings as in §3 and define for each ideal $\mathfrak{a}$ and odd integer $h \geq 1$ a theta series

$$\Theta_{\mathfrak{a}}^{(h)}(z) = \frac{1}{2} \sum_{\lambda \in \mathfrak{a}} \lambda^{h-1} \, q^{N(\lambda)/N(\mathfrak{a})} \qquad (z \in \mathcal{H}, \quad q = e^{2\pi i z})$$

which is a modular form in $M_h(\Gamma_0(d), \varepsilon)$ and a cusp form if $h > 1$. Clearly $\Theta_{\lambda\mathfrak{a}}^{(h)}(z) = \lambda^{h-1} \Theta_{\mathfrak{a}}^{(h)}(z)$ (the $\Theta_{\mathfrak{a}}^{(h)}$ are the modular forms corresponding to the partial zeta functions for unramified Hecke L-series of even weight $h-1$). In particular, $\Theta_{\mathfrak{a}}^{(1)}(z)$ depends only on the class $[\mathfrak{a}]$ of $\mathfrak{a}$ in $\mathrm{Cl}(K)$.

On the half-integral weight side we will be evaluating modular forms of level a power of 2 at our CM points, so we have to impose further congruence conditions on the bases of our ideals modulo powers of 2, just as we did modulo d in §3. We set

$$(24) \quad z_{\mathfrak{a}}^{(2)} = \frac{b + \sqrt{-d}}{2a} \in \mathcal{H}, \quad \text{where} \quad \mathfrak{a} = \left[a, \frac{b + \sqrt{-d}}{2}\right], \quad (a, 2) = 1, \quad b \equiv 1 \pmod{16}.$$

which is well defined modulo $8\mathbf{Z}$. (Any other fixed odd choice of b modulo 16 would be just as good.)

THEOREM. *Let $\mathfrak{a}$, $\mathfrak{a}_1$ be coprime ideals of K prime to $2\mathfrak{d}$ and to 2, respectively. Then for k, $h \geq 1$ satisfying $k \equiv 1 + \delta \pmod 2$, $h \equiv 1 \pmod 2$ we have*

(25)

$$\partial^{k-1} \Theta_{\mathfrak{a}\mathfrak{a}_1}^{(h)}(z_{\mathfrak{a}}^{(d)}) = (-1)^{\delta} \Big(\frac{-4}{N(\mathfrak{a})}\Big)^{\delta} \frac{d^{k+h/2-5/4}}{2^{2k+h-4}} \frac{N(\mathfrak{a})^{h-1}}{N(\mathfrak{a}_1)^{k-1/2}} \, \theta_{k-1/2}(z_{\mathfrak{a}^2\mathfrak{a}_1}^{(2)}) \cdot \overline{\theta_{k+h-3/2}(z_{\mathfrak{a}_1}^{(2)})},$$

where $\theta_{*+1/2}$ are the theta series defined by (3), (12) and (13) and $z_{\mathfrak{a}}^{(d)}$ and $z_{\mathfrak{a}}^{(2)}$ have the meanings given in (19) and (24), respectively.

PROOF: We will choose $p = k-1$, $\alpha = h-1$, $\mu = 1/2$, $\nu = \delta/2$ in (23), observing that

$$\theta_{(p)}\Big[{r/2 \atop s/2}\Big](z) = 2(-1)^{(p-s)/2} \, \theta_{p+1/2}(z) \qquad \text{for} \quad r \equiv 1 \pmod 2, \quad s \equiv p \pmod 2$$

by (14) and the definition of $\theta_{(p)}\big[{\mu \atop \nu}\big]$. We further choose $b \equiv 1 \pmod{16}$ such that $(b+\sqrt{-d})/2$ belongs to $\mathfrak{a}^2\mathfrak{a}_1\mathfrak{d}$ (in particular, b is divisible by d and $c := (b^2+d)/4aa_1$ is divisible by a and congruent to δ modulo 2) and set $z = (b + \sqrt{-d})/2aa_1$ in (23). Then

$$\mathfrak{a}\mathfrak{a}_1 = aa_1\,[1, z], \quad z_{\mathfrak{a}}^{(d)} = a_1 z/d, \quad z_{\mathfrak{a}^2\mathfrak{a}_1}^{(2)} = a^{-1} z, \quad z_{\mathfrak{a}_1}^{(2)} = az$$

and

$$\frac{\pi}{a}\left(imn - Q_z(m,n)\right) \equiv 2\pi i\big[(m^2c - bmn + aa_1n^2)z_{\mathfrak{a}}^{(d)} + \frac{m\delta+n}{2}\big] \pmod{2\pi i\mathbf{Z}},$$

and substituting all this into (23) gives the assertion of the theorem. ■

REMARK: Notice that the theorem relates values of modular forms of different levels. This forces ratios of such values to be in smaller fields than one would suspect a priori. Also from the transformation properties of $\partial^p\Theta_{\mathfrak{a}}^{(h)}(z)$ under homotheties of $\mathfrak{a}$ and the action of $\Gamma_0(d)$ on z one immediately gets the following:

COROLLARY. *Let $\mathfrak{a}$ and $\mathfrak{a}_1$ be as in the theorem and ψ a Hecke character of K satisfying (17). Then*

$$(-1)^\delta\Big(\frac{-4}{N(\mathfrak{a})}\Big)^\delta\,\overline{\psi(\mathfrak{a})}^{-1}\,\theta_{k-1/2}(z_{\mathfrak{a}^2\mathfrak{a}_1}^{(2)})$$

depends only on $\mathfrak{a}_1$ and ψ and on the ideal class of $\mathfrak{a}$.

This transformation property is quite non-obvious and was proved in [8] by a long calculation (pp. 559–562) of quadratic symbols. The point is that the theta series $\theta_{k-1/2}$ has level 4 and has nothing to do with d at all, so that the Kronecker symbol $(d/\cdot)$ implicit in the factor ψ has to come from the transformation behavior of the CM point $z_{\mathfrak{a}^2\mathfrak{a}_1}^{(2)}$ under change of ideals and from the automorphy factor of $\theta_{k-1/2}$.

§5. Final formula for the central value of $L(\psi,k)$

The proposition of §3 and the theorem just proved in general involve different modular forms: non-holomorphic derivatives of Eisenstein series (of odd weight and character ε) in one case and non-holomorphic derivatives of theta series (again of odd weight and character ε) in the other. There is one case where these overlap. Namely, for the Eisenstein series of weight one we have

$$\frac{\sqrt{d}}{2\pi}\,G_{1,\varepsilon}(z) = \frac{h(-d)}{2} + \sum_{n=1}^{\infty}\Big(\sum_{m|n}\varepsilon(m)\Big)\,q^n$$

by (21), and since the coefficient of n is the number of integral ideals of K of norm n this is simply $\sum_{[\mathfrak{a}]}\Theta_{\mathfrak{a}}^{(1)}(z)$. Hence we can combine equations (18), (22) (with $r=0$) and (25) (with $h=1$) to get

$$L(\psi,k) = (-1)^\delta\,\frac{(2\pi/\sqrt{d})^k}{(k-1)!}\sum_{[\mathfrak{a}],[\mathfrak{a}_1]\in \mathrm{Cl}(K)}\overline{\psi(\mathfrak{a})}^{-1}\,\partial^{k-1}\Theta_{\mathfrak{a}_1}^{(1)}(z_{\mathfrak{a}}^{(d)})$$

$$(26)\qquad = \frac{\pi^k d^{k/2-3/4}}{2^{k-1}(k-1)!}\sum_{[\mathfrak{a}],[\mathfrak{a}_1]\in \mathrm{Cl}(K)}\overline{\psi(\mathfrak{a})}^{-1}\,N(\mathfrak{a}_1)^{-k+1/2}\,\theta_{k-1/2}(z_{\mathfrak{a}^2\mathfrak{a}_1}^{(2)})\,\overline{\theta_{k-1/2}(z_{\mathfrak{a}_1}^{(2)})}\,.$$

If we assume that d is prime, so that the class number $h(-d)$ is odd, then we can replace first $\mathfrak{a}_1$ by $\mathfrak{a}_1^2$ and then $\mathfrak{a}$ by $\mathfrak{a}\mathfrak{a}_1^{-1}$ to obtain:

MAIN THEOREM. *Let $d > 3$ be a prime $\equiv 3 \mod 4$, k a positive integer satisfying $k \equiv \delta + 1 \pmod 2$ and ψ a Hecke character of $K = \mathbf{Q}(\sqrt{-d})$ satisfying (17). Then*

$$(27) \qquad L(\psi, k) = \frac{\pi^k d^{k/2-3/4}}{2^{k-1}(k-1)!} \left| \sum_{[\mathfrak{a}] \in \mathrm{Cl}(K)} (-1)^\delta \Big(\frac{-4}{N(\mathfrak{a})}\Big)^\delta \overline{\psi(\mathfrak{a})}^{-1} \, \theta_{k-1/2}(z^{(2)}_{\mathfrak{a}^2}) \right|^2 .$$

Notice that the terms of the sum are well-defined by virtue of the corollary in §4.

COROLLARY. *Under the assumptions of the theorem, $L(\psi, k)$ is non-negative.*

REMARK: According to Greenberg ([1], p. 258), the corollary has the application that the values of $L(\psi_1^{2k-1}, k)$ for a fixed Hecke character ψ_1 of weight one have a well-defined average value, equal to $L(1, \varepsilon)$, as $k \to \infty$. He points out that this implies the rather weak estimate $L(\psi_1^{2k-1}, k) = \mathrm{o}(k)$ and asks whether this can be improved. It might be of interest to see whether this can be done using the above theorem.

§6. Recurrences

The results of the preceding sections imply that both the central values of odd weight Hecke L-series and their square roots can be expressed in terms of non-holomorphic derivatives of classical theta series evaluated at CM points. In particular, for the odd powers of the weight one character ψ_1 introduced at the beginning of the paper, for which $d = 7$ with class number $h(-d) = 1$ and $\delta = 0$ (the only such case!), they give

$$(28) \quad L(\psi_1^{2k-1}, k) = \frac{(2\pi/\sqrt{7})^k}{(k-1)!} \, \partial^{k-1} \Theta\Big(\frac{7+\sqrt{-7}}{14}\Big) \; = \; \frac{7^{k/2-3/4} \, \pi^k}{2^{k-3}\,(k-1)!} \left| \theta_{k-1/2}\Big(\frac{1+\sqrt{-7}}{2}\Big) \right|^2 ,$$

for $k \geq 1$ odd, where $\theta_{1/2}$ is the function (3) and

$$(29) \qquad \Theta(z) = \frac{1}{2} \sum_{m,n \in \mathbf{Z}} q^{m^2+mn+2n^2} = \frac{1}{2} + \sum_{n=1}^{\infty} \Big(\frac{-7}{n}\Big) \frac{q^n}{1-q^n} \qquad (q = e^{2\pi i z}).$$

In this section we show how to obtain the values $\{\partial^n f(z_0)\}_{n \geq 0}$ of the non-holomorphic derivatives of a modular form f at a CM point z_0 as (essentially) the constant terms of a sequence of polynomials satisfying a recurrence relation. We will illustrate with the case of the full modular group, treating other groups, and the functions occurring in (28), in the next section.

As well as the differential operator D and $\partial_k = D - k/4\pi y$ of §2, we will use the operator

$$\vartheta_k = D - \frac{k}{12} E_2 = \frac{1}{2\pi i} \frac{d}{dz} - \frac{k}{12} E_2(z),$$

where $E_2(z) = 1 - 24 \sum_{n=1}^{\infty} \sigma_1(n) q^n$ is the Eisenstein series of weight 2 on $SL_2(\mathbf{Z})$. As is well-known, this Eisenstein series is not quite modular, but transforms instead by

$$E_2\Big(\frac{az+b}{cz+d}\Big) = (cz+d)^2 E_2(z) + \frac{6}{\pi i} c(cz+d) \qquad \Big(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbf{Z})\Big).$$

Equivalently, the function $E_2^*(z) = E_2(z) - 3/\pi y$ ($y = \Im(z)$), though not holomorphic, transforms under the action of $SL_2(\mathbf{Z})$ like a holomorphic modular form of weight two. It follows that $\vartheta_k f = \partial_k f - kE_2^* f/12$ transforms like a modular form of weight $k+2$ if $f \in M_k^*(\Gamma)$ for any subgroup Γ of $SL_2(\mathbf{Z})$. On the other hand, $\vartheta_k f$ is clearly holomorphic if f is, so ϑ_k maps $M_k(\Gamma)$ to $M_{k+2}(\Gamma)$. The operator D, which does not preserve the ring $M_*(SL_2(\mathbf{Z})) = \mathbf{C}[E_4, E_6]$, does preserve the larger ring $\mathbf{C}[E_2, E_4, E_6]$. We have

$$(30) \qquad D(E_2) = \frac{1}{12}(E_2^2 - E_4), \quad D(E_4) = \frac{1}{3}(E_2E_4 - E_6), \quad DE_6(z) = \frac{1}{2}(E_2E_6 - E_4^2)$$

and hence—since D clearly acts as a derivation—

$$D = \frac{E_2^2 - E_4}{12}\frac{\partial}{\partial E_2} + \frac{E_2E_4 - E_6}{3}\frac{\partial}{\partial E_4} + \frac{E_2E_6 - E_4^2}{2}\frac{\partial}{\partial E_6} \; : \; \mathbf{C}[E_2, E_4, E_6] \to \mathbf{C}[E_2, E_4, E_6].$$

If f is a holomorphic modular form on $SL_2(\mathbf{Z})$, thought of as a weighted homogeneous polynomial of degree k in E_4 and E_6 (where E_h has weight h), then from $\partial f/\partial E_2 = 0$ and the Euler equation $4E_4\,\partial f/\partial E_4 + 6E_6\,\partial f/\partial E_6 = k\,f$ we get

$$(31) \qquad \vartheta f = -\frac{E_6}{3}\frac{\partial f}{\partial E_4} - \frac{E_4^2}{2}\frac{\partial f}{\partial E_6} \qquad (\,f \in M_*(SL_2(\mathbf{Z})) = \mathbf{C}[E_4, E_6]\,).$$

Each of the three differentiation operators D, ∂_k and ϑ_k has advantages over the others: the first preserves holomorphicity and acts in a simple way on Fourier expansions, but destroys modularity; the second preserves modularity and acts in a simple way on Fourier expansions, but destroys holomorphicity; and the third preserves both the properties of holomorphicity and modularity but has a complicated action on Fourier expansions. The nicest way to understand the action of these operators and their iterates is to put them together into three generating series. The first is the *Kuznetsov-Cohen series*

$$f_D(z, X) = \sum_{n=0}^{\infty} \frac{D^n f(z)}{k(k+1)\cdots(k+n-1)}\frac{X^n}{n!} \qquad (z \in \mathcal{H},\ X \in \mathbf{C},\ f \in M_k(\Gamma))$$

and the other two are

$$(32) \qquad f_\partial(z, X) = e^{-X/4\pi y}\, f_D(z, X) = \sum_{n=0}^{\infty} \frac{\partial^n f(z)}{k(k+1)\cdots(k+n-1)}\frac{X^n}{n!}$$

(the second equality is a restatement of equation (8)) and

$$(33) \qquad f_\vartheta(z, X) = e^{-E_2(z)X/12}\, f_D(z, X) = e^{-E_2^*(z)X/12}\, f_\partial(z, X).$$

The Kuznetsov-Cohen series satisfies the transformation equation

$$f_D\Big(\frac{az+b}{cz+d}, \frac{X}{(cz+d)^2}\Big) = (cz+d)^k\, e^{cX/2\pi i(cz+d)}\, f_D(z, X) \qquad \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma,$$

and the transformation properties of y^{-1} and $E_2(z)$ under Γ imply that f_∂ and f_ϑ satisfy a similar equation but without the exponential factor, which simply says that the nth Taylor coefficient in each of these series transforms like a holomorphic modular form of weight $k+2n$. These Taylor coefficients of f_∂ are the $\partial^n f$, by (32). For f_ϑ they are given by:

PROPOSITION. *Let $f \in M_k(\Gamma)$ for some $\Gamma \subset SL_2(\mathbf{Z})$. Then*

$$f_\vartheta(z,X) = \sum_{n=0}^{\infty} \frac{F_n(z)}{k(k+1)\cdots(k+n-1)} \frac{X^n}{n!} \tag{34}$$

where the modular forms $F_n \in M_{k+2n}(\Gamma)$ are defined recursively by

$$F_0 = f, \qquad F_1 = \vartheta f \qquad F_{n+1} = \vartheta F_n - \frac{n(n+k-1)}{144} E_4 F_{n-1} \quad (n \geq 1) \tag{35}$$

PROOF: From (33) and the definitions we have $F_n = \sum_{\ell=0}^{n} \frac{n!}{\ell!} \binom{n+k-1}{n-\ell} (-\frac{E_2}{12})^{n-\ell} D^\ell f$, and the result follows using (30) and (31). ∎

We now illustrate how to get the recursions for the numbers $\{\partial^n f(z_0)\}$ in the simplest case $\Gamma = SL_2(\mathbf{Z})$, $f = E_4$, $z_0 = i$. By the transformation property of E_2^* under Γ and the fact that i is fixed under $z \mapsto -1/z$, we deduce that $E_2^*(i) = 0$ and hence by (33) that $f_\partial(i,X) = f_\vartheta(i,X)$ for any $f \in M_k(SL_2(\mathbf{Z}))$, so $\partial^n E_4(i) = F_n(i)$ where the polynomials

$$F_0 = E_4, \quad F_1 = -\frac{1}{3} E_6, \quad F_2 = \frac{5}{36} E_4^2, \quad F_3 = -\frac{5}{72} E_4 E_6, \quad F_4 = \frac{5}{288} E_4^2 + \frac{5}{216} E_6^2, \quad \ldots$$

can be computed by (35) and (31). By homogeneity we have $F_n = E_4^{n/2+1} f_n(E_6/E_4^{3/2})$ where the $f_n \in \mathbf{Z}[\frac{1}{6}][t]$ are polynomials in one variable (even if n is even and odd if n is odd, and of degree $\leq (n+2)/3$) given inductively by

$$f_0 = 1, \qquad f_1 = -\frac{1}{3} t, \qquad f_{n+1} = \left(\frac{t^2-1}{2} \frac{d}{dt} - \frac{n+2}{6} t\right) f_n - \frac{n(n+3)}{144} f_{n-1} \,. \tag{36}$$

Since $E_6(i)$ vanishes, we obtain finally

EXAMPLE. *For $n \geq 0$ we have $\partial^n E_4(i) = f_n(0)\omega^{n/2+1}$ ($= 0$ if n is odd), where $\omega = E_4(i)$ ($= 3\Gamma(\frac{1}{4})^8/(2\pi)^6$) and $\{f_n(t)\}$ are the polynomials defined by (36).*

§7. Examples

To calculate the non-holomorphic derivatives for modular forms on other groups Γ than $SL_2(\mathbf{Z})$ and for other CM points z_0, we choose a function $\phi(z)$ satisfying

i) $\phi(z)$ is holomorphic;

ii) $\phi^*(z) = \phi(z) - 1/4\pi y$ transforms like a holomorphic modular form of weight 2 on Γ;

iii) $\phi(z_0) = 0$.

Condition ii) is equivalent to

ii') $\phi(\frac{az+b}{cz+d}) = (cz+d)^2 \phi(z) + \frac{c(cz+d)}{2\pi i}$ for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$,

and a short calculation then shows that

iv) the function $\Phi = D\phi - \phi^2$ belongs to $M_4(\Gamma)$, and

v) if $f \in M_k(\Gamma)$, then $\vartheta_\phi f := Df - k\phi f$ belongs to $M_{k+2}(\Gamma)$.

(In §6, we had $\phi = \frac{1}{12}E_2$, $\Phi = \frac{-1}{144}E_4$, $\vartheta_\phi = \vartheta$.) The analogue of the proposition in §6 is then that the series $e^{-\phi(z)X} f_D(z,X)$ has an expansion as in (35) with $F_0 = f$, $F_1 = \vartheta_\phi f$, and

$F_{n+1} = \vartheta_\phi F_n + n(n+k-1)\,\Phi\, F_{n-1}$ for $n \geq 1$. If we choose an explicit set of generators for the ring $M_*(\Gamma)$ and then express the differential operator ϑ_ϕ in terms of these generators, as was done in (31) for the case of $SL_2(\mathbf{Z})$, we obtain an explicit recursion for the polynomials F_n. The fact that ϕ vanishes at z_0 then shows that $\partial^n f(z_0) = F_n(z_0)$, so we get the numbers $\partial^n f(z_0)$ as special values of a sequence of polynomials satisfying a recursion.

We give the details of this for the functions $\Theta(z)$ and $\theta_{1/2}(z)$ occurring in equation (28), which are modular forms (with multiplier system) on $\Gamma_0(7)$ and $\Gamma_0(2)$, respectively. We give the details $\theta_{1/2}$, since this is the one needed to prove (3) and also because the structure of the corresponding ring of modular forms is simpler.

We abbreviate $\theta = \theta_{1/2}$. By (3), θ^8 is a modular form without character on $\Gamma_0(2)$. (In fact it is the Eisenstein series $\sum_{n\geq 1} n^3 q^n/(1-q^{2n})$.) The ring of modular forms on $\Gamma_0(2)$ is generated by the functions

$$A = A(z) = -E_2(z) + 2E_2(2z) = 1 + 24 \sum_{\substack{n\geq 1\\ n \text{ odd}}} \frac{nq^n}{1-q^n}$$

and θ^8, of weight 2 and 4, respectively. For instance, we have

$$E_4(z) = A^2 + 192\theta^8\,, \qquad E_4(2z) = A^2 - 48\,\theta^8\,.$$

We are interested in the point $z_0 = \dfrac{1+\sqrt{-7}}{2}$. By standard complex multiplication theory,

$$(37)\qquad E_2^*(z_0) = \frac{3}{7}\,\Omega^2\,, \qquad A(z_0) = \frac{3}{2}\,\Omega^2\,, \qquad \theta(z_0)^8 = -\frac{1}{2^8 7}\,\Omega^4\,,$$

with Ω as in equation (1). We therefore choose

$$\phi(z) = \frac{1}{12}\left(E_2(z) - \frac{2}{7}\,A(z)\right) = \frac{3}{28}\,E_2(z) - \frac{1}{21}\,E_2(2z)$$

so that $\phi^*(z) = \phi(z) + 1/4\pi y$ transforms like a form of weight 2 on $\Gamma_0(2)$ and vanishes at z_0. The ϑ_ϕ-derivatives of θ and A are found to be

$$\vartheta_\phi \theta = \frac{2}{21}\,\theta A\,, \qquad \vartheta A = 32\,\theta^8 - \frac{5}{42}\,A^2\,,$$

so (since ϑ is a derivation) the action of ϑ on either $M_*(\Gamma_0(2)) = \mathbf{C}[\theta^8, A]$ or its 8th degree extension $\mathbf{C}[\theta, A]$ is given by

$$(38)\qquad \vartheta_\phi = \frac{2}{21}\,\theta A\,\frac{\partial}{\partial\theta} + (32\,\theta^8 - \frac{5}{42}\,A^2)\,\frac{\partial}{\partial A}\,.$$

Also, $\Phi = D\phi - \phi^2$ equals $-(\frac{5}{84})^2\,A^2 - \frac{44}{21}\,\theta^8$. The above discussion then shows that $\partial^n\theta(z_0) = F_n(z_0)$, where F_n is a weighted homogeneous polynomial in θ and A of weight $2n+\frac{1}{2}$ given inductively by

$$F_0 = \theta\,, \qquad F_1 = \frac{2}{21}A\theta, \qquad F_{n+1} = \vartheta_\phi F_n + n(n-\frac{1}{2})\Phi F_{n-1} \quad (n \geq 1)$$

with ϑ_ϕ as in (38). By homogeneity we have $F_n = \theta^{4n+1} f_n(A/\theta^4)$ where f_n is a polynomial of the same parity as n, and in terms of these the recurrence becomes

$$f_{n+1}(t) = (32 - \frac{1}{2}t^2)\, f_n'(t) + \frac{8n+2}{21}\, t\, f_n(t) - n(n-\frac{1}{2})(\frac{5^2}{84^2}t^2 + \frac{44}{21})\, f_{n-1}(t),$$

the first few values being

$$f_0 = 1, \quad f_1 = \frac{2}{21}t, \quad f_2(t) = -\frac{19}{4704}t^2 + 2, \quad f_3(t) = -\frac{43}{98784}t^3 + \frac{6}{7}t,$$

and (28) and (37) tell us that the numbers

$$B(2n+1) = \frac{(-\sqrt{-7})^n}{2^{3n+1}}\, f_n(24\sqrt{-7}) = \frac{1}{2},\ 1,\ -1,\ -3,\ 7,\ -315,\ -609,\ \ldots$$

satisfy equation (2). A slightly more convenient choice of normalization turns out to be $F_n = 2\,(A/21)^n\,\theta\, b_n(441\,\theta^8/A^2 + 7/64)$ rather than $F_n = \theta^{4n+1} f_n(A/\theta^4)$, so that

$$b_0 = \frac{1}{2}, \quad b_1 = 1, \quad b_2 = x-1, \quad b_3 = 9x-3, \quad b_4 = -2x^2 + 133x + 7, \quad \ldots.$$

With this choice the $B(2n+1)$ are simply the constant terms $b_n(0)$ and the $b_n(x)$ satisfy (5).

For the other assertion of the theorem in §1, giving the numbers $A(k)$ of (1) in terms of the recursion (4), we must use the formula $A(k) = \partial^{k-1}\Theta(z_1)/2\Omega^{2k-1}$, where $z_1 = (7+\sqrt{-7})/14$ (this is just a restatement of (1) and (28)). The calculations here are more complicated because the ring of modular forms with character $M_*(\Gamma_0(7), (\frac{-7}{\cdot})^*)$ is not free, but is generated by the function Θ defined in (29) together with the two weight 3 forms

$$\Theta^{(3)}(z) = \frac{1}{8} \sum_{\substack{r,\, s\in\mathbf{Z} \\ r\equiv s \pmod 2}} (r^2 - 7s^2)\, q^{\frac{r^2+7s^2}{4}}, \qquad E(z) = 1 - \frac{7}{8}\sum_{n=1}^{\infty}\Big(\sum_{d|n}(d^2 + 7\frac{n^2}{d^2})\,(\frac{-7}{n})\Big)\, q^n$$

subject to the relation $E^2 = (\Theta^3 + \Theta^{(3)})(\Theta^3 - 27\Theta^{(3)})$. When we write the non-holomorphic derivatives $\partial^n\Theta(z_1)$ as the values of a sequence of polynomials, then these are polynomials in three algebraically dependent variables. We can use the homogeneity to write them as simple factors times polynomials in $x = \Theta^{(3)}/\Theta^3$ and $\sqrt{(1+x)(1-27x)} = E/\Theta^3$, and after some computation we obtain (4). We leave the details to the reader.

REMARKS: 1. According to the recursion (4), the polynomials a_{2n} have coefficients in $\mathbf{Z}[\frac{1}{3}]$, and this is actually true: the first values are 1, $(2-34x)/9$, and $(8-218x+314x^2+432x^3)/27$. It is not clear on an elementary level why their values at $x=1$ are integers (let alone squares). Similarly, the recursion (5) involves dividing by 21 at each stage, but in fact the polynomials $b_n(x)$ apparently belong to $\mathbf{Z}[x]$.

2. It is now clear that we cannot expect any simple relations between the polynomials $a_{2n}(x-1)$ and $b_n(x)$, even though the constant terms of one are the squares of the constant terms of the other: the variables "x" have completely different meanings in the two polynomials, being a modular function on $X_0(7)$ in the one case and a modular function on $X_0(2)$

in the other. The identity of the constant terms has to do with the way these two modular curves intersect in the Siegel 3-fold $\mathcal{H}_2/Sp_4(\mathbf{Z})$.

Finally, we mention that in the 5 cases $d = 11, 19, 43, 67$ and 163 with $h(-d) = 1$ and $\delta = 1$ the calculation is even easier than the case treated here with $\delta = 0$, since now the function whose non-holomorphic derivatives must be computed at CM points is $\theta_{3/2}$, which is a modular form (with multiplier system) for the full modular group (cf. equation (15)). Here the recursions obtained are essentially the classical recursions for the Taylor coefficients of the Weierstrass σ-function, as given for instance, in [3], Chapter VII, p. 237.

§8. Twists

We finish by mentioning a result on the central values of quadratic twists of Hecke L-series. For simplicity we consider only the case of equation (7), i.e. twists of the L-series for the grossencharacters of conductor $(\sqrt{-7})$ of $\mathbf{Q}(\sqrt{-7})$ by Legendre symbols attached to primes $p \equiv 1 \pmod 4$. Then one of us (F.V.) has found a formula giving integers $B(k,p)$ such that (7) holds. In the simplest case $k = 1$ this formula is

$$B(1,p) = \frac{\sum_n \chi(n^2+7)\,\theta_{1/2}(\frac{n+\sqrt{-7}}{2p}) + \sqrt{p}\,\theta_{1/2}(\frac{p+p\sqrt{-7}}{2})}{S(\chi)\,G(\chi)\,\theta_{1/2}(\frac{1+\sqrt{-7}}{2})},$$

with $\theta_{1/2}(z)$ as in (3); here the sum is over $n \pmod{16p}$) satisfying $n \equiv 1 \pmod{16}$, $\chi(n)$ is one of the two quartic characters modulo p, $G(\chi) = \sum_{n \pmod p} \chi(n)\, e^{2\pi i n/p}$ the associated Gaussian sum, and $S(\chi) = 2$, $1-i$, $2i$ or $1+i$ according as $\chi(7) = 1$, i, -1 or $-i$ (so that $\chi(7)S(\chi)^2 = 3 + (\frac{7}{p})$). For $p < 200$ this gives the values

p	5	13	17	29	37	41	53	61	73	89	97	101	109	113	137	149	157	173	181	193	197
$B(1,p)$	1	−1	−1	−1	1	1	0	1	3	3	−1	−1	−1	2	1	0	−3	1	1	0	0

This will be discussed in a later publication. We have not succeeded in matching these numbers with the coefficients of a modular form weight 3/2.

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