How Often Should You Beat Your Kids?

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A result is proved which shows, roughly speaking, that one should beat one's kids every day except Sunday.

This note is a follow-up to the note "How to Beat Your Kids at Their Own Game," by K. Levasseur [1], in which the author proposes the following game to be played against one's two-year-old children: Starting with a deck consisting of n red cards and n black cards (in typical applications, $n = 26$), the cards are turned up one at a time, each player at each stage predicting the color of the card which is about to appear. The kid is supposed to guess "Red" or "Black" randomly with equal probability (this solves the problem of constructing a perfect random number generator), while you play what is obviously the optimal strategy—guessing randomly (or, if you prefer, always saying "Black") whenever equal numbers of cards of both colors remain in the deck and otherwise predicting the color which is currently in the majority. Levasseur analyzes the game and shows that on the average you will have a score of $n + (\sqrt{\pi n} - 1)/2 + O(n^{-1/2})$, while the kid, of course, will have an average score of exactly n.

We, however, maintain that only the most degenerate parent would play against a two-year-old for money, and that our concern must therefore be, not *by how much* you can expect to win, but with what probability you will win *at all*. Our principal result is that this probability tends asymptotically to 85.4% (more precisely: to $1/2 + 1/\sqrt{8}$) as n tends to infinity. This shows with what unerring instinct Levasseur's mother selected the game—the high 85% loss rate will instill in the young progeny a due respect for the immense superiority of their parents, while the 15% win rate will maintain their interest and prevent them from succumbing to feelings of hopelessness and frustration.

The analysis begins as in Levasseur's article: each of the $\binom{2n}{n}$ possible orderings of the cards into red and black elements corresponds to a path p moving downwards and leftwards from an initial value $(R, B) = (n, n)$ to a final value $(R, B) = (0, 0)$ of the pair (R, B), where R and B denote the numbers of red and black cards remaining, respectively. If this path meets the diagonal $R = B$ a total of $m(p)$ times, where the initial point at (n, n) is counted but the final point at $(0, 0)$ is not, then the expected win of the parent is $m(p)/2$. Indeed, at each meeting point the parent guesses randomly, with an expected score of $1/2$ and hence an expected win over his child of 0; between each pair of meeting points, the parent will consistently guess "Red" or consistently "Black," depending on whether p is now below or above the diagonal, and will be right exactly one more time than he is wrong, gaining exactly half a point over his randomly guessing child. Levasseur shows that the average value of $m(p)$, as p ranges over the set $\mathscr{P}_n$ of paths as described above, is exactly $4^n/\binom{2n}{n} - 1$, leading to the result on the expected win stated above. To solve the problem we have set ourselves, we must answer two questions:

—(i) for a given value of $m(p)$, what is the probability of winning? and

—(ii) with what probability will $m(p)$ take on a given value m, $1 \leqslant m \leqslant n$?

We answer the second question first.

Let $N_m(n)$ denote the number of paths $p \in \mathscr{P}_n$ with $m(p) = m$. For $n = 0$ this equals 1 if $m = 0$ and 0 otherwise, but for positive n we must have $m \geqslant 1$ since the initial point of the path is counted as a meeting with the diagonal. If a path $p \in \mathscr{P}_n$ meets the diagonal more than once, i.e., if $m(p) > 1$, then the first meeting point will be at some value (k, k) of (R, B) with $1 \leqslant k \leqslant n - 1$. Conversely, if we pick such a k, then the number of paths $p \in \mathscr{P}_n$ with $m(p) \doteq m$ and having (k, k) as their first meeting point will be equal to the product of $N_1(n-k)$ (the number of ways of descending from (n, n) to (k, k) without meeting the diagonal on the way) and $N_{m-1}(k)$ (the number of ways of descending from (k, k) to $(0,0)$ with exactly $m-1$ further meetings). Hence

$$N_m(n) = \sum_{k=1}^{n-1} N_1(n-k) N_{m-1}(k) \qquad (m > 1).$$

It follows that the generating function $\mathscr{N}_m(x) = \sum_{n=m}^{\infty} N_m(n) x^n$ is the product of $\mathscr{N}_1(x)$ and $\mathscr{N}_{m-1}(x)$, and hence that $\mathscr{N}_m(x) = \mathscr{N}_1(x)^m$. This formula holds also for $m = 0$ since $N_0(n) = 0$ for all positive n. On the other hand, the sum of all the functions $\mathscr{N}_m(x)$ is the generating function whose nth coefficient is the total number of paths in $\mathscr{P}_n$, i.e., $\binom{2n}{n}$. Hence

$$\frac{1}{1-\mathscr{N}_1(x)} = \sum_{m=0}^{\infty} \mathscr{N}_1(x)^m = \sum_{m=0}^{\infty} \mathscr{N}_m(x) = \sum_{n=0}^{\infty} \binom{2n}{n} x^n = \frac{1}{\sqrt{1-4x}},$$

so

$$\mathscr{N}_1(x) = 1 - \sqrt{1-4x}, \qquad \mathscr{N}_m(x) = \left(1 - \sqrt{1-4x}\right)^m.$$

Using the well-known Taylor expansion of this function, we find:

$$N_m(n) = 2^m \cdot \frac{m}{n} \cdot \binom{2n-m-1}{n-1} \qquad (1 \leqslant m \leqslant n).$$

Therefore, the probability for a random path $p \in \mathscr{P}_n$ to have $m(p) = m$ is given by

$$\operatorname{prob}\{m(p) = m\} = \frac{N_m(n)}{\binom{2n}{n}} = \frac{m}{2n} \cdot \frac{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{m-1}{n}\right)}{\left(1-\frac{1}{2n}\right)\left(1-\frac{2}{2n}\right)\cdots\left(1-\frac{m}{2n}\right)}.$$

For m of the order of $\sqrt{n}$ (the right order according to Levasseur's analysis), this will

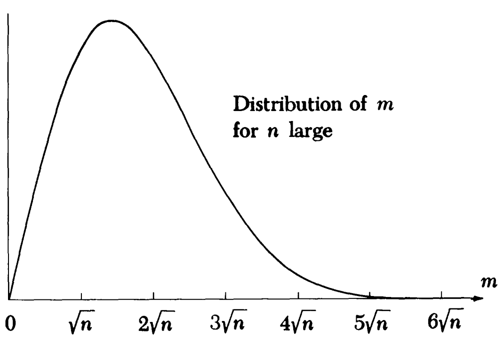

FIGURE 1

be asymptotically equal to $(m/2n)e^{-m^2/4n}$ (cf. FIGURE 1). As a test, when n is large we have for the total probability

$$\sum_{m=0}^{\infty} \text{prob}\{m(p)=m\} \approx \sum_{m=0}^{\infty} \frac{m}{2n} e^{-m^2/4n}$$

$$\approx \int_0^{\infty} \frac{x}{2n} e^{-x^2/4n}\, dx = -e^{-x^2/4n}\Big|_0^{\infty} = 1$$

and for the expected value of m the value

$$\sum_{m=0}^{\infty} \text{prob}\{m(p)=m\}\cdot m \approx \sum_{m=0}^{\infty} m\frac{m}{2n} e^{-m^2/4n}$$

$$\approx \int_0^{\infty} x\frac{x}{2n} e^{-x^2/4n}\, dx = 4\sqrt{n}\int_0^{\infty} t^2 e^{-t^2}\, dt = \sqrt{\pi n},$$

in accordance with Levasseur's result.

We now turn to the first of the two questions above. For the reasons already explained, for an ordering of cards given by a path $p \in \mathcal{P}_n$ with $m(p)=m$, of the $2n-m$ turns corresponding to points on p not on the diagonal one will guess correctly exactly n times and incorrectly exactly $n-m$ times, while the probability of guessing correctly at one of the m turns corresponding to points on the diagonal is 50% each time. Hence one's total number of correct guesses will be described by a bell-shaped curve centered around the expected value $n+\frac{1}{2}m$ and with a width of the order of $\sqrt{m}$, or (for almost all paths p) $\sqrt[4]{n}$ (cf. FIGURE 2). On the other hand, if one guesses correctly $n+k$ times, then one's chance of beating the randomly playing kid is

$$\frac{1}{2^{2n}}\sum_{r=0}^{n+k-1}\binom{2n}{r} \approx \frac{1}{2} + 2^{-2n}\sum_{r=0}^{k}\binom{2n}{n+r},$$

and since $2^{-2n}\binom{2n}{n+r} \approx \sqrt{1/\pi n}\, e^{-r^2/n}$ by Stirling's formula, this is approximately equal to

$$\frac{1}{2} + \frac{1}{\sqrt{\pi n}}\sum_{r=0}^{k} e^{-r^2/n} \approx \frac{1}{2} + \frac{1}{\sqrt{\pi n}}\int_0^k e^{-u^2/n}\, du = \frac{1}{2} + \frac{1}{\sqrt{\pi}}\int_0^{k/\sqrt{n}} e^{-u^2}\, du.$$

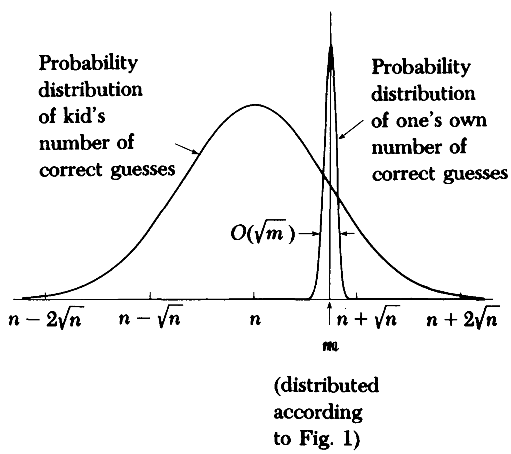

FIGURE 2

Since $k/\sqrt{n}$ is almost always very near to $\frac{1}{2}m/\sqrt{n}$, the probability of winning when $m(p) = m$ is very nearly equal to

$$\frac{1}{2} + \frac{1}{\sqrt{\pi}} \int_0^{m/2\sqrt{n}} e^{-u^2}\, du.$$

Multiplying this by the probability that $m(p) = m$ as computed above, we find finally

$$\begin{aligned}
\text{probability of winning} &\approx \frac{1}{2} + \sum_{m=0}^{\infty} \frac{m}{2n} e^{-m^2/4n} \left(\frac{1}{\sqrt{\pi}} \int_0^{m/2\sqrt{n}} e^{-u^2}\, du \right) \\
&\approx \frac{1}{2} + \int_0^{\infty} \frac{x}{2n} e^{-x^2/4n} \left(\frac{1}{\sqrt{\pi}} \int_0^{x/2\sqrt{n}} e^{-u^2}\, du \right) dx \\
&= \frac{1}{2} + \frac{1}{2\sqrt{\pi}} \int_0^{\infty} x e^{-x^2/4} \left(\int_0^{x/2} e^{-u^2}\, du \right) dx \\
&= \frac{1}{2} + \frac{1}{2\sqrt{\pi}} \int_0^{\infty} e^{-u^2} \left(\int_{2u}^{\infty} x e^{-x^2/4}\, dx \right) du \\
&= \frac{1}{2} + \frac{1}{2\sqrt{\pi}} \int_0^{\infty} e^{-u^2} \left(2e^{-u^2} \right) du \\
&= \frac{1}{2} + \frac{1}{2\sqrt{2}},
\end{aligned}$$

as claimed. This is very nearly 6/7, so the result of our paper can be conveniently implemented by beating one's kids on weekdays and Saturdays, but never on Sunday.

REFERENCE

1. Kenneth M. Levasseur, How to Beat Your Kids at Their Own Game, this MAGAZINE 61 (1988), 301–305.

A Note on the Five-Circle Theorem

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In his paper [1] H. Demir stated and proved

THE FIVE-CIRCLE THEOREM. *Let P and Q be two points on the side BC of a triangle ABC in the order B, P, Q, C. If the triangles ABP, APQ, AQC have congruent incircles, then the triangles ABQ, APC have congruent incircles.*

He also asked for a geometric proof of this theorem.

Here we give such a proof for the following more general