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TWO RESULTS IN NUMBER THEORY WITH ELEMENTARY ASPECTS

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The talk consisted of two parts, entirely unrelated except that each had an "elementary aspect": in the first part, a theorem about heights in algebraic number fields is proved by a completely elementary method (essentially calculus), while in the second part a more difficult theorem about special values of Hecke L-series has as a corollary an elementary identity of which I know no elementary proof. We give only a brief survey since both results will appear shortly.

NO ALGEBRAIC NUMBER CAN BE CLOSE TO BOTH 0 AND 1

If α is an algebraic number in an algebraic number field K, then the *height of α relative to K* is defined by $H_K(\alpha) = \sum_v \log^+ |\alpha|_v$, where the sum runs over all places v of K (with the valuations $|\cdot|_v$ normalized in the usual way, so $\prod_{v|\infty} |\alpha|_v = |N_{K/Q}(\alpha)|$, $\prod_{\text{all } v} |\alpha|_v = 1$) and $\log^+ |x|$ denotes $\log(\max\{1, |x|\})$. The *absolute height* $H(\alpha)$ is defined as $[K : \mathbb{Q}]^{-1} H_K(\alpha)$ and is independent of K.

The height of α with respect to $\mathbb{Q}(\alpha)$ is the same as the logarithm of the "Mahler measure" of the irreducible polynomial of α, and a still-open conjecture of Lehmer of 1933 says that this number has a positive universal lower bound for all α except roots of unity (which by a theorem of Kronecker are the only numbers of height 0). Specifically, the conjecture is that $H_{\mathbb{Q}(\alpha)}(\alpha) \geq H_{\mathbb{Q}(\lambda)}(\lambda) = \log \lambda = 0.1623\ldots$, where λ is the unique root outside the unit circle of the polynomial $x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1$. The absolute height is of course not bounded below in the same way, since $H(\sqrt[n]{2}) = \dfrac{1}{n} \log 2$, but a theorem of Schinzel of 1973 says that $H(\alpha) \geq C_0 = H(\phi) = \frac{1}{2} \log \phi = 0.2460\ldots$ for all totally real α, where $\phi = (1 + \sqrt{5})/2$ is the golden ratio. Recently, Zhang proved a theorem which says that $H(\alpha) + H(1 - \alpha)$ has a positive universal lower bound for all numbers for which it is positive (i.e., all except 0, 1, or 6th roots of unity), as a consequence of some difficult results about hermitian line bundles over arithmetic surfaces. We give a very short and elementary proof of this theorem with a sharp estimate for the lower bound:

Theorem. *For all $\alpha \neq 0, 1, \dfrac{1 \pm \sqrt{-3}}{2}$, we have $H(\alpha) + H(1 - \alpha) \geq C_0 = 0.2460\ldots$, with equality in exactly 8 cases $\alpha = \zeta_{10}, 1 - \zeta_{10}$ (ζ_{10} = primitive 10th root of unity).*

For the proof, one first proves the estimate

$$\log^+ |z| + \log^+ |1 - z| \geq C_1 \log |z^2 - z| + C_2 \log |z^2 - z + 1| + C_0 \qquad (\forall z \in \mathbb{C}),$$

where $C_2 = \dfrac{1}{2\sqrt{5}}$, $C_1 = \dfrac{1}{2} - C_2$. (To do this, one observes that the difference of the right- and left-hand sides is harmonic off the circles $|z| = 1$ and $|z-1| = 1$, so can attain its extreme values only on these circles, and these can be found by writing z or $1-z$ as $x + i\sqrt{1-x^2}$ with $-1 \le x \le 1$ and differentiating with respect to x.) This then gives

$$\log^+ |\alpha|_v + \log^+ |1-\alpha|_v \ge C_1 \log |\alpha^2 - \alpha|_v + C_2 \log |\alpha^2 - \alpha + 1|_v + C_0\, n_v$$

for all places v, where n_v is 1 or 2 for v real or complex and 0 for v non-archimedean. (The proof in the latter case is obtained easily by looking separately at $|\alpha|_v \le 1$ and $|\alpha|_v > 1$.) Summing over all v gives the assertion since $\sum\limits_v \log |\alpha^2 - \alpha|_v = \sum\limits_v \log |\alpha^2 - \alpha + 1|_v = 0$.

References

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Central values of Hecke L-series

There is a general philosophy that certain ("critical") values of certain ("motivic") L-functions arising in number theory, algebraic geometry, or the theory of automorphic forms should be the product of a predictable transcendental number (the "period") and an algebraic number lying in a predictable number field; moreover, if the value in question is at the symmetry point of the functional equation of the L-function, then the algebraic number should be a *square* in the field in question.

Particularly nice examples of motivic L-functions are Hecke L-series, since these belong to all three fields mentioned: they are the L-series of Hecke characters of ideals in a quadratic field, the Hasse-Weil zeta functions of one-dimensional abelian varieties with complex multiplication, and the L-series of theta series associated to binary quadratic forms. In 1980, Gross and I made some numerical computations for higher-weight Hecke characters associated to a simple quadratic field and verified the conjecture on squares mentioned above. Specifically, let $K = \mathbb{Q}(\sqrt{-7})$ and ψ_1 the grossencharacter defined by $\psi_1(\mathfrak{a}) = \varepsilon(\alpha)\alpha$ for $\mathfrak{a} = (\alpha)$, where $\varepsilon(\cdot)$ is the Legendre character $\left(\frac{\cdot}{7}\right)$, extended to the ring of integers $\mathcal{O}$ via the isomorphism $\mathcal{O}/\sqrt{-7} \cong \mathbb{Z}/7$. The Hecke L-series $L(\psi_1^n, s) = \sum \psi_1(\mathfrak{a})^n N(\mathfrak{a})^{-s}$ has a functional equation sending s to $n+1-s$, so there is a central critical point $s = k$ if $n = 2k-1$ is odd. The corresponding value of the L-series vanishes by the functional equation if k is even, but for k odd we found the numerical values

$$L(\psi_1^{2k-1}, k) = 2\,\frac{(2\pi/\sqrt{7})^k\,\Omega^{2k-1}}{(k-1)!}\,A(k) \qquad \left(\Omega = \frac{\Gamma(\frac17)\Gamma(\frac27)\Gamma(\frac47)}{4\pi^2}\right)$$

with $A(1) = \frac{1}{4}$, $A(3) = A(5) = 1$, $A(7) = 9$, $A(9) = 49$, $\ldots$, $A(33) = 44762286327255^2$, and we conjectured that all $A(k)$ with $k > 1$ are integral squares. Recently, by generalizing a beautiful result of Villegas giving the central values of weight one Hecke L-series as the squares of certain sums of values of theta series at CM points, he and I were able to prove this conjecture in the following explicit form:

Theorem. *Define polynomials* $a_{2n}(x)$, $b_n(x) \in \mathbb{Q}[x]$ *by* $a_0(x) = \frac{1}{4}$, $b_0(x) = \frac{1}{2}$, *and*

$$a_{n+1}(x) = \sqrt{(1-x)(1+27x)}\,\big(x\,a_n'(x) - (2n+1)a_n(x)/3\big) - n^2(1-5x)\,a_{n-1}(x)/9,$$
$$21\,b_{n+1}(x) = (32nx - 56n + 42)\,b_n(x) - (x-7)(64x-7)\,b_n'(x) - 2n(2n-1)(11x+7)b_{n-1}(x)$$

for $n \geq 0$. *Then* $A(2n+1) = a_{2n}(-1)$ *and* $A(2n+1) = b_n(0)^2$ *for all* $n \geq 0$.
But no elementary proof that $a_{2n}(-1) = b_n(0)^2$ is known!

The proof of the theorem uses a general factorization formula for derivatives of theta series generalizing the one found by Villegas for the values of theta series. Specifically, the two identities of the theorem follow from the two identities

$$L(\psi^{4n+1}, 2n+1) = \frac{(2\pi/\sqrt{7})^{2n+1}}{(2n)!}\,\Theta^{[2n]}\Big(\frac{7+i\sqrt{7}}{14}\Big)$$

and

$$L(\psi^{4n+1}, 2n+1) = \frac{7^{n-1/4}}{2^{2n-2}(2n)!}\,\Big|\theta^{[n]}\Big(\frac{1+i\sqrt{7}}{2}\Big)\Big|^2,$$

where $\theta(z)$ and $\Theta(z)$ denote the theta-series

$$\Theta(z) = \frac{1}{2}\sum_{m,n\in\mathbb{Z}} q^{m^2+mn+2n^2} = \frac{1}{2} + q + 2q^2 + 3q^4 + \cdots,$$
$$\theta(z) = \frac{1}{2}\sum_{n\in\mathbb{Z}+\frac{1}{2}} q^{n^2/2} = q^{1/8}(1+q+q^3+\cdots) \qquad (q = e^{2\pi i z})$$

and $f^{[n]}(z)$ denotes the nth non-holomorphic derivative of a real-analytic modular form (defined by induction by $f^{[1]}(z) = \frac{1}{2\pi i}\frac{\partial f}{\partial z} - \frac{k\,f}{4\pi\Im(z)}$ if f has weight k; in our case, $\theta(z)$ and $\Theta(z)$ are modular forms of weight $1/2$ and 1 on $\Gamma_0(2)$ and $\Gamma_0(7)$, respectively).

Similar identities are proved for grossencharacters of other imaginary quadratic fields, not necessarily of class number one.

References

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