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L-Series and the Green's Functions of Modular Curves

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The problem we want to discuss in this paper is the following:

Construct on the modular curve $X_0(N)$ explicit divisors of degree 0 defined over $\mathbf{Q}$*, and relate their heights to the derivatives at $s = 1$ of L-series of cusp forms of weight 2 and level N.* (1)

We begin by recalling the definitions of the various terms occurring here and the motivation for the question.

Let X be a curve defined over $\mathbf{Q}$. By a *divisor of degree* 0 on X we mean a finite formal linear combination $\mathfrak{x} = \sum_i n_i(x_i)$ ($x_i \in X(\overline{\mathbf{Q}})$, $n_i \in \mathbf{Z}$) with $\sum_i n_i = 0$. It is *defined over* $\mathbf{Q}$ if $\mathfrak{x}^\sigma = \mathfrak{x}$ for all $\sigma \in \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$. The quotient of the abelian group of such divisors by the subgroup of principal divisors

$$(f) = \sum_{\substack{x \text{ a zero or} \\ \text{pole of } f}} \mathrm{ord}_x(f)\cdot(x) \qquad (f\colon X \to \mathbf{P}^1 \text{ a rational function defined over } \mathbf{Q})$$

is (if X has a $\mathbf{Q}$-rational point) the set $J_X(\mathbf{Q})$ of rational points on a certain abelian variety J_X, the Jacobian of X, and is a finitely generated group by the Mordell-Weil theorem. The Néron-Tate theory associates to each $\mathfrak{x}$ a real number $h(\mathfrak{x}) \geq 0$, called its *height* (or canonical height), which depends only on the class of $\mathfrak{x}$ in $J_X(\mathbf{Q})$ and which defines a quadratic form on $J_X(\mathbf{Q})$, i.e., $h(\mathfrak{x}) = \langle \mathfrak{x}, \mathfrak{x} \rangle$ for a certain symmetric bilinear form $\langle\ ,\ \rangle$ on $J_X(\mathbf{Q})$ (the *height pairing*). Moreover, h is positive definite on the free abelian group $J_X(\mathbf{Q})/\text{torsion}$; i.e., $h(\mathfrak{x})$ vanishes only if $\mathfrak{x}$ is of finite order in $J_X(\mathbf{Q})$. We will explain later how the height is defined.

For $N \in \mathbf{N}$, the *modular curve* $X_0(N)$ is a curve defined over $\mathbf{Q}$ whose $\mathbf{C}$-rational points are given by

$$X_0(N)(\mathbf{C}) = \mathfrak{H}/\Gamma_0(N) \cup (\text{finite set of "cusps"});$$

here $\mathfrak{H} = \{z = x + iy | y > 0\}$ is the upper half-plane and $\Gamma_0(N)$ the group of matrices $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbf{Z})$ with $c \equiv 0 \pmod{N}$, acting in the usual way $z \mapsto \frac{az+b}{cz+d}$ on $\mathfrak{H}$. The points of $X_0(N)$ over a subfield $k \subset \mathbf{C}$ parametrize pairs (E, C) consisting of an elliptic curve E and a cyclic subgroup $C \subset E(\mathbf{C})$ of order

N, both defined over k. A *cusp form of weight* 2 *and level* N is a holomorphic function $f\colon \mathfrak{H} \to \mathbf{C}$ such that the differential form $f(z)\,dz$ on $\mathfrak{H}$ is $\Gamma_0(N)$-invariant and satisfying $f(z) = O(y^{-1})$ in $\mathfrak{H}$. The set of such forms is a finite-dimensional vector space which we will denote by $S_2(N)$. Each $f \in S_2(N)$ has a convergent Fourier development

$$f(z) = \sum_{n=1}^{\infty} a_f(n) e^{2\pi i n z} \qquad (z \in \mathfrak{H}).$$

The *L-series* of f is the associated Dirichlet series

$$L(f,s) = \sum_{n=1}^{\infty} a_f(n) n^{-s} \qquad (s \in \mathbf{C});$$

it converges for $\mathrm{Re}(s) > \frac{3}{2}$ and has a holomorphic continuation to all s. The space $S_2(N)$ has a basis of special modular forms f (*Hecke forms*) whose L-series have Euler products (in particular, $a_f(1) = 1$) and satisfy a functional equation

$$\left(\frac{\sqrt{N}}{2\pi}\right)^{s} \Gamma(s) L(f,s) = w_f \left(\frac{\sqrt{N}}{2\pi}\right)^{2-s} \Gamma(2-s) L(f,2-s)$$

with $w_f = \pm 1$. We will be particularly interested in the value of the derivative $L'(f,1)$ for those Hecke forms with $w_f = -1$ (and hence $L(f,1) = 0$).

The motivation for formulating the problem (1) comes from the conjecture of Birch and Swinnerton-Dyer. Given a Hecke form f whose Fourier coefficients $a_f(n)$ are all rational integers, there exists an elliptic curve E defined over $\mathbf{Q}$ and related to f by

$$a_f(p) \quad + \quad \#E(\mathbf{Z}/p\mathbf{Z}) = p + 1 \quad \text{for all } p \nmid N$$

(Eichler, Shimura); this curve is unique up to isogeny (Faltings). Conversely, it is conjectured that every elliptic curve over $\mathbf{Q}$ arises in this way (Taniyama, Weil). The Birch–Swinnerton-Dyer conjecture predicts that if $L(f,1)$ vanishes (in particular, if $w_f = -1$), then $E(\mathbf{Q})$ has infinite order and

$$L'(f,1) = h(P) \cdot \omega \quad \text{for some } P \in E(\mathbf{Q}) \otimes \mathbf{Q}, \qquad (2)$$

where ω is an explicitly specified positive real number (a certain period of E) and $h\colon E(\mathbf{Q}) \to \mathbf{R}$ is the height function on E, identified with its own Jacobian via $x \mapsto (x) - (0)$. Now the point is that there is a nonconstant map $\phi\colon X_0(N) \to E$ defined over $\mathbf{Q}$. (Over $\mathbf{C}$, ϕ is given as follows; by the $\Gamma_0(N)$-invariance of $f(z)\,dz$, the function $F(z) = \sum_{n=1}^{\infty} n^{-1} a_f(n) e^{2\pi i n z}$ satisfies $F(\gamma z) = F(z) + c_\gamma$ for all $\gamma \in \Gamma_0(N)$; the c_γ all lie in a certain 2-dimensional lattice $\Lambda \subset \mathbf{C}$, with $E(\mathbf{C}) = \mathbf{C}/\Lambda$, so F induces a map $\phi\colon \mathfrak{H}/\Gamma_0(N) \to E(\mathbf{C})$, and this map extends smoothly over the cusps.) Hence to any $\mathbf{Q}$-rational divisor $\mathfrak{x} = \sum n_i(x_i)$ on $X_0(N)$, we can associate a $\mathbf{Q}$-rational divisor $\sum n_i(\phi(x_i))$ and hence—since E is a group—a $\mathbf{Q}$-rational point $P = \sum n_i \phi(x_i)$ on E, the heights of P and $\mathfrak{x}$ being related in a simple way. In this way a solution of (1) can be used to prove (2). We now proceed to describe one such solution.

We must first construct a $\mathbf{Q}$-rational divisor on $X_0(N)$. The construction is based on the theory of complex multiplication and is due to Heegner and Birch (cf. [**1**, **4**]). We need an auxiliary piece of data. This will be an imaginary quadratic field K whose discriminant D is assumed to be prime to $2N$ and congruent to a square modulo $4N$ (equivalently, every prime divisor of N should split in K). Then there are infinitely many $\tau \in \mathfrak{H}$ satisfying a quadratic equation

$$a\tau^2 + b\tau + c = 0, \qquad a, b, c \in \mathbf{Z},\ b^2 - 4ac = D,\ N|a,$$

but only finitely many modulo the action of $\Gamma_0(N)$, say $\tau_{D,1}, \ldots, \tau_{D,h} \in \mathfrak{H}/\Gamma_0(N) \subset X_0(N)(\mathbf{C})$ (h is a certain class number). The theory of complex multiplication tells us that the $\tau_{D,i}$ are algebraic in $X_0(N)(\mathbf{C})$ and are permuted by $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$, so the divisor of degree zero $\mathfrak{x}_D = \sum_{i=1}^{h}(\tau_{D,i}) - h \cdot (\infty)$ is defined over $\mathbf{Q}$ (here "∞" denotes the cusp at infinity on $X_0(N)$, which is defined over $\mathbf{Q}$). If D is -3 or -4, we must divide this by 3 or 2 to get consistent formulas later, because of the presence of extra units in K; the class of $\mathfrak{x}_D$ is then in $J_{X_0(N)}(\mathbf{Q}) \otimes \mathbf{Q}$. We call $\mathfrak{x}_D$ the *Heegner divisor* attached to K. If f is a Hecke form with integral Fourier coefficients associated to an elliptic curve $E/\mathbf{Q}$ and a map $\phi\colon X_0(N) \to E$ as above, then we get a point

$$P_D = \sum_{i=1}^{h} \phi(\tau_{D,i}) \in E(\mathbf{Q}),$$

the *Heegner point* attached to K. A solution to (1) is then provided by the following theorem.

THEOREM 1 [**7**, **9**]. *Let D, N be as above, $h(\cdot)$ the canonical height function for $X_0(N)$ over $\mathbf{Q}$. Assume N is prime. Then*

$$h(\mathfrak{x}_D) = \frac{|D|^{1/2}}{4\pi^2} \sum_f \frac{1}{\|f\|^2} L'(f,1) L(f,D,1), \tag{3}$$

where the sum runs over all Hecke forms $f \in S_2(N)$ with $w_f = -1$, $\|f\|^2$ denotes the Petersson scalar product $\iint_{H/\Gamma_0(N)} |f(z)|^2\, dx\, dy$, and $L(f,D,1)$ is the value at $s = 1$ of the "twisted L-series"

$$L(f,D,s) = \sum_{n=1}^{\infty} a_f(n) \left(\frac{D}{n}\right) n^{-s} \qquad (\mathrm{Re}(s) \gg 0).$$

Actually, the theorem of [**9**] is proved for all N, and we have assumed N prime only for convenience in stating the result. (If N is composite, there is an extra power of 2 in the formula (3) and one has to discuss new and old forms.) It is more general than Theorem 1 in two other respects. First, it gives not only the height pairing of $\mathfrak{x}_D$ with itself, but of the individual $(\tau_{D,i}) - (\infty)$ with one another (over their field of definition, the Hilbert class field of K). Secondly, it includes the action of the Hecke operators T_n ($n \in \mathbf{N}, (n, N) = 1$) on $X_0(N)$; specifically, the height pairing $\langle \mathfrak{x}_D, T_n \mathfrak{x}_D \rangle$ is given by the same expression as in (3) but with an extra factor $a_f(n)$ in the fth summand. This is important since it allows us to use the action of the Hecke algebra on $S_2(N)$ to split up the sum and get a formula in terms of heights for each term $L'(f,1)L(f,D,1)$ separately. In particular, if f is a Hecke form with $w_f = -1$ corresponding to an elliptic curve $E/\mathbf{Q}$ and P_D the Dth Heegner point on E as described above, then we get the formula

$$h(P_D) = \frac{|D|^{1/2}}{\mathrm{vol}(E)} L'(f,1)L(f,D,1), \tag{4}$$

where $h(\cdot)$ is now the height function on $E(\mathbf{Q})$ and $\mathrm{vol}(E)$ the volume of a fundamental parallelogram for a certain lattice Λ with $E(\mathbf{C}) = \mathbf{C}/\Lambda$. The product $L'(f,1)L(f,D,1)$ equals $L'(E/K,1)$, the derivative at $s=1$ of the L-series of E over K. Equation (4) has several consequences, most notably:

A. If $L'(f,1) \neq 0$ (i.e., if the order of $L(f,s)$ at $s=1$ is one), then $E(\mathbf{Q})$ has infinite order.

B. The Birch–Swinnerton-Dyer formula (2) holds up to a nonzero rational number.

C. One gets explicit examples of Hecke forms whose L-series have a zero of order ≥ 3 at $s=1$.

The proof of A uses a theorem of Waldspurger [**16**], which guarantees the existence of a D with $L(f,D,1) \neq 0$. The assertion C is of interest because, in combination with a deep result of Goldfeld [**3**, **12**], it leads to an effective solution of Gauss's problem of showing that there are only finitely many imaginary quadratic fields having a given class number.

Formula (3) has the disadvantage that the values of $L'(f,1)$ in which we are interested do not occur alone, but always multiplied by a twisted L-series value $L(f,D,1)$ for some auxiliary number D. Also, we get only partial information about the positions of the Heegner divisors $\mathfrak{x}_D$ in the Mordell-Weil group $J_{X_0(N)}(\mathbf{Q})$, namely their lengths with respect to the height pairing metric on $J_{X_0(N)}(\mathbf{Q}) \otimes \mathbf{R}$. To understand the dependence on D, we must be able to relate different discriminants, i.e., to compute the height pairing $\langle \mathfrak{x}_D, \mathfrak{x}_{D'} \rangle$ for all D, D', not just for $D = D'$. To state the answer, we recall that Shimura [**13**] defined a correspondence between modular forms of weight 2 and modular forms of weight $\frac{3}{2}$ (or, more generally, weight $2k$ and weight $k + \frac{1}{2}$). If $f \in S_2(N)$ is a Hecke form, we denote its image under this correspondence by g_f and write its Fourier development as $g_f(z) = \sum_{D<0} c_f(D) e^{2\pi i |D| z}$ $(z \in \mathfrak{H})$. We do not recall the exact definition of forms of half-integral weight or of the Shimura correspondence here; roughly, g_f satisfies

$$g_f\left(\frac{az+b}{cz+d}\right)^2 = \pm(cz+d)^3 g_f(z)^2 \qquad \forall \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4N)$$

and the relation between g_f and f is that for all n and D the ratio of $c_f(n^2 D)$ to $c_f(D)$ is given by a simple linear combination of the Fourier coefficients $a_f(d)$, $d|n$. The form g_f is unique up to a scalar multiple and cannot be normalized in any canonical way, but it can be chosen to have all its coefficients algebraic integers in the number field generated by the Fourier coefficients of f. We can now state

THEOREM 2 [**6**]. *Let N be prime, D and D' discriminants of imaginary quadratic fields which are prime to N and squares modulo $4N$. Then*

$$\langle \mathfrak{x}_D, \mathfrak{x}_{D'} \rangle = \frac{3}{4\pi} \sum_f \frac{1}{\|g_f\|^2} L'(f,1) c_f(D) c_f(D'), \tag{5}$$

where the summation is the same as in Theorem 1.

Again the result actually proved is for arbitrary N but is more complicated to state in general, not only because the numerical factor $3/4\pi$ changes and one has to worry about old and new forms, but because one has to replace the theory of modular forms of half-integral weight by the theory of Jacobi forms [**2**] if one wants to get a complete result. The compatibility of Theorems 1 and 2 follows from a theorem of Waldspurger [**14, 15**], which states

$$L(f,D,1) = 3\pi \frac{\|f\|^2}{\|g_f\|^2} \frac{c_f(D)^2}{|D|^{1/2}}$$

(actually, Waldspurger stated only the proportionality of $c_f(D)^2/|D|^{1/2}$ and $L(f,D,1)$ for fixed f; the constant was determined in [**11, 10**]). As before, one has a generalization of (5) involving Hecke operators (replace $\langle \mathfrak{x}_D, \mathfrak{x}_{D'} \rangle$ by $\langle \mathfrak{x}_D, T_n \mathfrak{x}_{D'} \rangle$ and insert a factor $a_f(n)$ before the fth summand) and this can be used to separate the various Hecke forms and get a formula for each $L'(f,1)$ separately. In particular, if f corresponds to an elliptic curve $E/\mathbf{Q}$, we get the following

COROLLARY. *Let $E/\mathbf{Q}$ be an elliptic curve parametrized by a cusp form f of prime level and $L(E/\mathbf{Q},s) = L(f,s)$ its L-series. If $L(E/\mathbf{Q},s)$ has a simple zero at $s = 1$, then the space spanned by all Heegner points in $E(\mathbf{Q}) \otimes \mathbf{Q}$ is one-dimensional; more precisely, there is a nonzero point $P_0 \in E(\mathbf{Q}) \otimes \mathbf{Q}$ such that $P_D = c_f(D)P_0$ for all D, where the $c_f(D)$ are the Fourier coefficients of a form of weight $\frac{3}{2}$ corresponding to f under the Shimura correspondence. There is an analogous result for f of composite level, involving coefficients of Jacobi forms.*

That all Heegner points lie on a line when $\mathrm{ord}_{s=1} L(E/\mathbf{Q},s) = 1$ would follow from—and hence provides additional support for—the Birch–Swinnerton-Dyer conjecture, which predicts that the entire Mordell-Weil group has rank one in this case. If $\mathrm{ord}_{s=1} L(E/\mathbf{Q},s)$ is not equal to one, of course, then all Heegner points vanish (up to torsion) by Theorem 1.

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The Néron-Tate height is a sum of local contributions from all places of $\mathbf{Q}$, and in the proof of Theorems 1 and 2 it is necessary to compute these local heights. In the second part of the talk we discuss some of the arithmetic questions which arise in this context.

We first briefly describe the local height theory. (A good reference is [**5**].) Let $\mathfrak{x} = \sum n_i(x_i)$ and $\mathfrak{y} = \sum m_j(y_j)$ be $\mathbf{Q}$-rational divisors of degree 0 on a curve $X/\mathbf{Q}$, and suppose for simplicity that $\mathfrak{x}$ and $\mathfrak{y}$ have disjoint supports (i.e., $x_i \neq y_j$ for all i, j). Then there is a decomposition

$$\langle \mathfrak{x}, \mathfrak{y} \rangle = \langle \mathfrak{x}, \mathfrak{y} \rangle_\infty + \sum_P \langle \mathfrak{x}, \mathfrak{y} \rangle_p,$$

where the summation runs over all prime numbers p but has only finitely many nonzero terms. Each $\langle \mathfrak{x}, \mathfrak{y} \rangle_v$ (v a place of $\mathbf{Q}$) is a real number depending continuously on the x_i and y_j in the v-adic topology; i.e., we can compute $\langle \mathfrak{x}, \mathfrak{y} \rangle_\infty$ to any degree of accuracy if we know sufficiently accurately the position of the x_i and y_j on the Riemann surface $X(\mathbf{C})$, and $\langle \mathfrak{x}, \mathfrak{y} \rangle_p$ if we know the coordinates of the x_i and y_j modulo a sufficiently large power of p. The local height pairing is bilinear and symmetric and satisfies the identity $\langle \mathfrak{x}, \mathfrak{y} \rangle_v = \sum_i n_i \log |f(x_i)|_v$ if $\mathfrak{y} = (f)$ is a principal divisor (notice that this implies $\langle \mathfrak{x}, (f) \rangle = 0$ by the product formula for valuations, which is why the global height is well defined on $J_X(\mathbf{Q})$). These properties characterize $\langle\ ,\ \rangle_v$ uniquely, since the difference of two such symbols would be a continuous homomorphism of the compact group $J_X(\mathbf{Q}_v)^2$ to $\mathbf{R}$. Moreover, one can find an explicit solution to these axioms by using intersection theory at finite places and potential theory at infinity. Specifically, $\langle \mathfrak{x}, \mathfrak{y} \rangle_p = -n_p(\mathfrak{x}, \mathfrak{y}) \log p$ for p finite, where $n_p(\mathfrak{x}, \mathfrak{y})$ is a rational number which is integral and nonnegative if X has good reduction at p and is then 0 unless some x_i and y_j reduce to the same point modulo p. At infinity we have

$$\langle \mathfrak{x}, \mathfrak{y} \rangle_\infty = \sum_{i,j} n_i m_j G(x_i, y_j),$$

where G is a *Green's function* on X, characterized by the property

$G(\cdot, y)$ is continuous and harmonic on X except for logarithmic singularities of residue $+1$ and -1 at y and y_0, respectively. (6)

Here y_0 is a chosen basepoint on $X(\mathbf{C})$, and by "logarithmic singularity of residue c at a point P" we mean a function which looks like $c \log |z|^2 + O(1)$ near P, where z is a local uniformizing parameter with $z(P) = 0$. (Notice that the function $G(\cdot, y)$ is well defined up to a constant for fixed choice of y_0, since the difference of any two G's would be harmonic and finite on $X(\mathbf{C})$, hence constant; this constant drops out in the sum defining $\langle \mathfrak{x}, \mathfrak{y} \rangle_\infty$ because $\sum_i n_i = 0$, and similarly the choice of y_0 is irrelevant because $\sum_j m_j = 0$.)

Applying this general theory to Heegner divisors on $X_0(N)$, we find

$$\langle \mathfrak{x}_D, \mathfrak{x}_{D'} \rangle = \sum_{i=1}^{h} \sum_{i'=1}^{h'} G(\tau_{D,i}, \tau_{D',i'}) - \sum_p n(D, D', p) \log p \tag{7}$$

where G is an appropriate Green's function as above on $X_0(N)(\mathbf{C})$ (with $y_0 = \infty$) and the $n(D, D', p)$ are rational numbers which can be calculated explicitly using the theory of complex multiplication and our knowledge of a model of $X_0(N)$ over $\mathbf{Z}$. They turn out to be nonnegative and integral if $p \nmid N$ and to vanish unless $p < DD'/4N$ (more precisely, unless $DD' = r^2 + 4Nmp$ for some integers r and $m > 0$).

What about G? We need a function on $\mathfrak{H} \times \mathfrak{H}$ which is $\Gamma_0(N)$-invariant and harmonic in each variable and has logarithmic singularities along the diagonal of $(\mathfrak{H}/\Gamma_0(N))^2$ and at infinity. A natural attempt is to set

$$G(z, z') = \sum_{\gamma \in \Gamma_0(N)/\pm 1} g(z, \gamma z') \tag{8}$$

where g is a function which is invariant under the diagonal action of $\Gamma_0(N)$, harmonic, and logarithmically singular on the diagonal of $\mathfrak{H}^2$, and which drops off rapidly as the hyperbolic distance between z and z' tends to infinity. Such a function is

$$g(z, z') = \log \left| \frac{z - z'}{\bar{z} - z'} \right|^2 \qquad (z, z' \in \mathfrak{H}).$$

Unfortunately, it does not go to zero quite fast enough: the series (8) diverges like $\sum 1/n$. Instead, we replace the harmonicity condition $\Delta g = 0$, where

$$\Delta = \frac{1}{y^2}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$$

is the hyperbolic Laplacian, by $\Delta g = \varepsilon g$ with $\varepsilon > 0$, obtain a function g for which (8) does converge, and then let ε tend to zero. Specifically, set

$$g_s(z, z') = -2Q_{s-1}\left(1 + \frac{|z - z'|^2}{2yy'}\right) \qquad (s \in \mathbf{C}, z, z' \in \mathfrak{H}),$$

where

$$Q_{s-1}(t) = \frac{\Gamma(s)^2}{2\Gamma(2s)}\left(\frac{1+t}{2}\right)^{-s} F\left(s, s; 2s; \frac{2}{1+t}\right) \qquad (t > 1)$$

is the Legendre function of the second kind. It is invariant under the diagonal action of $\Gamma_0(N)$ (or even $\mathrm{SL}_2(\mathbf{R})$) on $\mathfrak{H}^2$, has a logarithmic singularity on the diagonal, and (because of the second order ordinary differential equation satisfied by Q_{s-1}) satisfies $\Delta g_s = s(s-1)g_s$. The following series converges absolutely for $\mathrm{Re}(s) > 1$ and has the analogous properties on $(\mathfrak{H}/\Gamma_0(N))^2$:

$$G_s(z, z') = \sum_{\gamma \in \Gamma_0(N)/\pm 1} g_s(z, \gamma z'). \tag{9}$$

The desired Green's function G is obtained as

$$G(z, z') = \lim_{s \to 1} \left(G_s(z, z') - e_s(z, z')\right)$$

where $e_s(z, z')$ is a certain combination of Eisenstein series and elementary functions which we do not specify here.

Now that we have a formula for the Green's function we obtain its value at Heegner points as a certain explicit infinite sum of Legendre functions. Suprisingly, analytic techniques from the theory of modular forms (specifically: the "Rankin-Selberg method" when $D = D'$, and the theory of modular forms of half-integral weight or the theory of Jacobi forms, combined with the theory of Hilbert modular forms on the real quadratic field $\mathbf{Q}(\sqrt{DD'})$, when $D \neq D'$) produce a different formula containing this same infinite sum. For $D \neq D'$ and N prime, for instance, one gets

$$\sum_{i=1}^{h}\sum_{i'=1}^{h'} G(\tau_{D,i},\tau_{D',i'}) = \sum_p n(D,D',p)\log p + \frac{3}{4\pi}\sum_f \frac{c_f(D)c_f(D')}{\|g_f\|^2}L'(f,1) \tag{10}$$

where, by a further miracle, the coefficients $n(D,D',p)$ are the same numbers as those occurring in (7) from the non-archimedean height computation. Together, of course, formulas (7) and (10) give a (highly nonconceptual) proof of Theorem 2, and similarly for Theorem 1.

It would, of course, be nice to have a more intrinsic proof of Theorems 1 and 2, directly involving the global heights. However, breaking down the identity into local pieces as in (10) does have the advantage of making more structure visible than in the smoother identities (3) and (5). We discuss two aspects of this.

First, consider the case $N = 1$. Here $X_0(N)$ has genus 0, so there are no cusp forms of weight 2 and the global height pairing is trivial (every degree zero divisor on $\mathbf{P}^1$ is principal). Hence both Theorem 1 and Theorem 2 are empty. But for this very reason (10) now becomes interesting. The Green's function for $\mathbf{P}^1$ is clearly $\log|x-y|^2$ (taking $y_0 = \infty$ in the definition (6)), so the Green's function for $X_0(1)$ is $\log|j(z)-j(z')|^2$, where

$$j(z) = e^{-2\pi iz} + 744 + 196884e^{2\pi iz} + \cdots \qquad (z \in \mathfrak{H})$$

is the classical modular function giving the isomorphism from

$$X_0(1) = \mathfrak{H}/\mathrm{SL}_2(\mathbf{Z}) \cup \{\infty\}$$

to $\mathbf{P}^1(\mathbf{C})$. Hence (10) becomes

THEOREM 3 [8]. *Let D, D' be coprime discriminants of imaginary quadratic fields, h and h' their class numbers, and $\tau_{D,i}$ $(1 \le i \le h)$, $\tau_{D',i'}$ $(1 \le i' \le h')$ the $\mathrm{SL}_2(\mathbf{Z})$-inequivalent roots in H of quadratic equations of discriminants D, D'. Then*

$$\prod_{i=1}^{h}\prod_{i'=1}^{h'} |j(\tau_{D,i}) - j(\tau_{D',i'})|^2 = \prod_p p^{n(D,D',p)}$$

with explicitly given nonnegative integral exponents $n(D,D',p)$ which are nonzero only for primes p dividing one of the finitely many integers $(DD'-r^2)/4$, $|r| < \sqrt{DD'}$, $r \equiv DD'$ (mod 2).

The point is that one knows by the classical theory of complex multiplication that the numbers $j(\tau_{D,i})$ and $j(\tau_{D',i'})$ are algebraic integers (in the Hilbert class fields of $\mathbf{Q}(\sqrt{D})$ and $\mathbf{Q}(\sqrt{D'})$, respectively), and Theorem 3 gives an explicit formula for the absolute norm of their difference. As a numerical example, take $D = -163$ and $D' = -4$. Here $h = h' = 1$ and

$$j\left(\frac{1+i\sqrt{163}}{2}\right) - j(i) = -262537412640768000 - 1728$$
$$= -2^6 3^6 7^2 11^2 19^2 127^2 163$$

in accordance with the theorem (the primes 11, 19, 127 and 163 divide $163 - n^2$ for $n = 3, 7, 6$, and 0, respectively).

Secondly, the fact that (10) is proved by purely analytic techniques from the theory of modular forms, rather that by height theory, means that one gets an analogous indentity for forms of higher weight. It takes the form (for $N = 1$)

$$(DD')^{(k-1)/2} \sum_{i=1}^{h} \sum_{i'=1}^{h'} G_k(\tau_{D,i}, \tau_{D',i'})$$
$$= \sum_p n_k(D, D', p) \log p + \frac{3\Gamma(k-1/2)}{2^{2k}\pi^{k+1/2}} \sum_f \frac{c_f(D) c_f(D')}{\|g_f\|^2} L'(f, k), \qquad (k = 3, 5, \dots) \quad (11)$$

where G_k is the resolvent kernel function defined by (9) (with $s = k$), the $n_k(D, D', p)$ are explicitly given integers which are zero unless p divides some positive integer of the form $(DD' - r^2)/4$, and the sum runs over all Hecke forms $f \in S_{2k}(\mathrm{SL}_2(\mathbf{Z}))$, the g_f being the forms of weight $k + \frac{1}{2}$ corresponding to the f and $c_f(D)$ the $|D|$th Fourier coefficient of g_f. This identity suggests two problems:

(i) Find a "higher-weight height theory" which permits us to interpret (11) as a formula for $L'(f, k)$ in terms of heights. Motivated by the identity (11), Deligne and Brylinski have made some progress towards developing such a local height theory, the terms $G_k(\tau, \tau')$ and $n_k \log p$ in (11) appearing as the local contributions from the places ∞ and p to the height pairing of some higher weight Heegner cycles. However, there is as yet no global height theory, so one neither knows that the height pairing of a cycle with itself is nonnegative nor has a criterion for the vanishing of the global height.

(ii) In analogy with Theorem 3, prove that the individual numbers $G_k(\tau, \tau')$ in (11) are logarithms of algebraic numbers when $k = 3, 5$, or 7 (so that the sum over f is empty). Then (11) could be interpreted in these cases as giving the prime decomposition of the absolute norms of these algebraic numbers, just as for the j-values above. The conjectured algebraicity of $\exp(G_k(\tau, \tau'))$ for τ and τ' quadratic imaginary numbers would seem to be an interesting property of the resolvent kernel function, since it shows that G_k is a new transcendental function whose special values can be used to give algebraic extensions, in the spirit of complex multiplication theory and Kronecker's Jugendtraum. It is supported by the analogy with the case $k = 1$, by the fact that (11) provides a proof whenever $h = h' = 1$, and by numerical evidence. (A numerical example for $k = 2$, $D = D' = -23$ is given at the end of [**9**].) The restriction to the handful of cases with $S_{2k}(N) = \{0\}$ can be circumvented by replacing the G_k by appropriate linear combinations of their images under Hecke operators.

BIBLIOGRAPHY

1. B. J. Birch, *Heegner points of elliptic curves*, Symposia Mathematica, Vol. 15, Academic Press, London, 1975, pp. 441–445.

2. M. Eichler and D. Zagier, *The theory of Jacobi forms*, Progr. Math., vol. 55, Birkhäuser, Boston, Mass., 1985.

3. D. Goldfeld, *The class numbers of quadratic fields and the conjectures of Birch and Swinnerton-Dyer*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. **3** (1976), 623–663.

4. B. Gross, *Heegner points on* $X_0(N)$, Modular Forms (R. A. Rankin, ed.), Ellis Horwood, Chichester, 1984, pp. 87–106.

5. ——, *Local heights on curves*, Arithmetic Geometry (G. Cornell, J. H. Silverman, eds.), Springer-Verlag, Berlin-Heidelberg-New York, 1986, pp. 327–339.

6. B. Gross, W. Kohnen, and D. Zagier, *Heegner points and derivatives of L-series. II*, Math. Ann. (to appear).

7. B. Gross and D. Zagier, *Points de Heegner et dérivées de fonctions L*, C. R. Acad. Sci. Paris Ser. I Math. **297** (1983), 85–87.

8. ——, *On singular moduli*, J. Reine Angew. Math. **355** (1985), 191–220.

9. ——, *Heegner points and derivatives of L-series*, Invent. Math. **84** (1986), 225–320.

10. W. Kohnen, *Fourier coefficients of modular forms of half-integral weight*, Math. Ann. **271** (1985), 237–268.

11. W. Kohnen and D. Zagier, *Values of L-series of modular forms at the center of the critical strip*, Invent. Math. **64** (1981), 175–198.

12. J. Oesterlé, *Nombres de classes des corps quadratiques imaginaires*, Séminaire Bourbaki, Vol. 1983/84, Astérisque No. 121–122 (1985), 309–323.

13. G. Shimura, *On modular forms of half-integral weight*, Ann. Math. **97** (1973), 440–481.

14. M.-F. Vignéras, *Valeur au centre de symétrie des fonctions L associées aux formes modulaires*, Séminaire de Théorie des Nombres, Paris 1979–1980, Progr. Math., vol. 12, Birkhäuser, Boston, Mass., 1981, pp. 331–356.

15. J.-L. Waldspurger, *Sur les coefficients de Fourier des formes modulaires de poids demi-entier*, J. Math. Pures Appl. **60** (1981), 375–484.

16. ——, *Correspondances de Shimura et quaternions*, Preprint.

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