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ON THE VALUES AT NEGATIVE INTEGERS OF THE ZETA-FUNCTION OF A REAL QUADRATIC FIELD

by Don ZAGIER [1])

§0. INTRODUCTION

In this paper we will be interested in the numbers $\zeta_K(b)$, where K is a real quadratic field and b a negative odd integer. It has been known for some time [3] that these numbers are rational; indeed, this is true for K any totally real number field [5], [9]. They are interesting on the one hand because they generalize Bernoulli numbers (the special case $K = \mathbf{Q}$) and on the other because they reflect properties of the arithmetic of K. For example, there is a conjecture of Bass, Birch and Tate relating $\zeta_K(-1)$ to the "deviation from the Hasse principle" of K (= order of $\mathrm{Ker}\,(K_2 K \to \prod K_2 K_{\mathfrak{P}})$, with $K_{\mathfrak{P}}$ running over the completions of K). The value of $\zeta_K(b)$, and in particular the problem of estimating its denominator, is related to formulas for the "Euler characteristic" of certain arithmetic groups (see for instance [6]).

Our main object is to give an account of Siegel's formula for $\zeta_K(1-2m)$ for general K, to describe the form it takes when K is quadratic, and prove it in this special case by direct analytic methods. We have tried to keep prerequisites to a minimum by reviewing the main facts about zeta functions of fields (in §1) and the arithmetic of quadratic fields (in §2). We give an exposition of Siegel's theorem and proof in Section 1.

When K is a quadratic field, it is very easy to obtain elementary formulas for $\zeta_K(1-2m)$ directly, using the decomposition $\zeta_K(s) = \zeta(s) L(s, \chi)$. These formulas are discussed in §2. In the simplest case, namely $m = 1$

[1]) This paper was written while the author was at the Forschungsinstitut für Mathematik der Eidgenössischen Technischen Hochschule Zürich and the Sonderforschungsbereich Theoretische Mathematik, Bonn.

and $K = \mathbf{Q}(\sqrt{p})$ with $p \equiv 1 \pmod 4$ a prime number, the formula in question reads

$$\zeta_K(-1) = \frac{1}{24p} \sum_{j=1}^{p-1} \left(\frac{j}{p}\right) j^2 , \tag{1}$$

where $\left(\frac{j}{p}\right)$ is the Legendre-Jacobi symbol.

In §3 we return to the Siegel formula and specialize it to the case of real quadratic fields. Because the arithmetic of quadratic fields is completely known and very simple (the different is a principal ideal; the splitting of a rational prime p depends only on the value $+1$, 0, -1 of $\chi(p)$), we can completely evaluate the terms of this formula, arriving at a formula for $\zeta_K(1-2m)$ not involving any notions of algebraic number theory. For instance, in the case above ($m = 1$, discriminant of K a prime p), the formula is

$$\zeta_K(-1) = \frac{1}{30} \sum a , \tag{2}$$

where the sum is over all ways of writing $p = b^2 + 4ac$ with a, b and c positive integers. We also discuss bounds for the denominator of $\zeta_K(1-2m)$ (the importance of which was mentioned above) and give tables for $m \leqslant 6$, discriminant of $K \leqslant 50$.

The elementary character of the right-hand sides of (1) and (2) suggests the problem of proving their equality directly, by reasoning involving only finite sums. This is probably impossible: it is not even easy to see *a priori* why the sum in (2) must be divisible by 5 if p is a prime different from 5. However, it is possible to study the sum (2) by the methods of analytic number theory. To do this, we observe that the right-hand side of (2) is the coefficient of $e^{2\pi i p z}$ in the Fourier expansion of a function which is (up to a factor) the product of a theta-function and an Eisenstein series. This function transforms in a known way under the action of the modular group, and therefore one can describe its asymptotic behaviour as z tends towards any rational point on the real axis. This is precisely the sort of problem for which the Hardy-Littlewood circle method was designed. When we apply it, we obtain a "singular series" which approximates (2) and which, on the other hand, can be explicitly summed to yield (1). However, we do not obtain a proof of (2): there is a built-in error in the circle method in this situation, and we cannot show that the singular series really sums to the expression in (2), but only that the error is of smaller order than the main term (roughly the square root) as $p \to \infty$. Indeed, in working out the analogous formula for $\zeta_K(1-2m)$, where $m \geqslant 3$, we find that there really is a difference of this order between the Fourier coefficient we are trying to evaluate and the value of the singular series. The calculation of the singular series is carried out in Section 4.

Finally, in §5 we give conjectures concerning the Fourier coefficients of a certain modular form of weight $4m$ related to the value of $\zeta_K(1-2m)$.

§1. Siegel's Formula

In this section, we will state the formula of Siegel for the value of $\zeta_K(b)$ where K is a totally real algebraic number field and b a negative odd integer. We will also give a brief description of the proof.

We begin by reviewing the main properties of the zeta-function of a field. Let K be an algebraic number field of degree n, and $\mathcal{O}$ the ring of integers in K. For any non-zero ideal $\mathfrak{A}$ of $\mathcal{O}$, the *norm* $N(\mathfrak{A})$ is defined as the number of elements in the quotient $\mathcal{O}/\mathfrak{A}$. For $m = 1, 2, \ldots$, let $i(m)$ denote the number of ideals of $\mathcal{O}$ with norm m. This number is finite for each m and has polynomial growth as $m \to \infty$, and so the series $\Sigma_{m=1}^{\infty} i(m) m^{-s}$ makes sense and is convergent if s is a complex number with sufficiently large real part. The function it defines can be extended meromorphically to the whole s-plane, and the function obtained is denoted $\zeta_K(s)$. Thus we have the two representations.

$$\zeta_K(s) = \sum_{\mathfrak{A} \subset \mathcal{O}} \frac{1}{N(\mathfrak{A})^s} \tag{1}$$

$$= \prod_{\mathfrak{P}} (1 - N(\mathfrak{P})^{-1})^{-s}, \tag{2}$$

provided that $Re(s)$ is large enough. The sum in (1) is to be taken over all non-zero ideals of $\mathcal{O}$, and the product in (2) *(Euler product)* over all prime ideals. The function obtained by analytic continuation has a simple pole at $s = 1$ and is holomorphic everywhere else.

Moreover, the function ζ_K satisfies a *functional equation* relating $\zeta_K(s)$ and $\zeta_K(1-s)$. In the case of a totally real field K (i.e. $K = \mathbf{Q}(\alpha)$ where α satisfies a polynomial of degree n with n real roots), this takes the form

$$F(s) = F(1-s), \tag{3}$$

where

$$F(s) = D^{s/2} \pi^{-ns/2} \Gamma\left(\frac{s}{2}\right)^n \zeta_K(s). \tag{4}$$

(Here D is the discriminant of K.) In particular, we have

$$\zeta_K(-2m) = 0, \tag{5}$$

$$\zeta_K(1-2m) = \{(-1)^m (2m-1)!/2^{2m-1}\pi^{2m}\}^n D^{2m-1/2}\zeta_K(2m) \quad (m=1,2,\ldots) \tag{6}$$

It is thus equivalent to give the values of $\zeta_K(s)$ at $s = 2, 4, 6, \ldots$ or at $s = -1, -3, -5, \ldots$; we shall prefer writing our formula for the latter values since, as it turns out, they are always rational numbers. For instance, if $K = \mathbf{Q}$ is the field of rational numbers, then $n = 1$, $D = 1$, $\mathcal{O} = \mathbf{Z}$, and the only ideals are (r) with $r = 1, 2, \ldots$, so

$$\zeta_K(s) = \zeta_{\mathbf{Q}}(s) = \zeta(s) = \sum_{r=1}^{\infty} \frac{1}{r^s} \tag{7}$$

is the ordinary Riemann zeta-function; in this case (6) says

$$\zeta(1-2m) = \frac{(-1)^m (2m-1)!}{2^{2m-1}\pi^{2m}} \zeta(2m) \tag{8}$$

$$= -B_{2m}/2m, \tag{9}$$

where B_i is the i-th Bernoulli number ($B_0 = 1$, $B_1 = -1/2$, $B_2 = 1/6$, $B_3 = 0$, $B_4 = -1/30$, ...) and is always rational.

We now proceed to describe Siegel's formula. We first need some preliminary notation. Recall the definition of the *different* $\mathfrak{d}$ of K: $\mathfrak{d}$ is the inverse of the fractional ideal

$$\mathfrak{d}^{-1} = \{x \in K \mid tr(xy) \in \mathbf{Z} \, (\forall y \in \mathcal{O})\} \tag{10}$$

(here $tr(z) = z^{(1)} + \ldots + z^{(n)}$ denotes the trace of $z \in K$). The ideal $\mathfrak{d}$ is integral, and its norm is related to the discriminant D of K by

$$D = N(\mathfrak{d}). \tag{11}$$

Next, for $r = 0, 1, 2, \ldots$ we define

$$\sigma_r(n) = \sum_{d \mid n} d^r \quad (n=1,2,3,\ldots) \tag{12}$$

to be the sum of the r-th powers of the positive divisors of n. (This is standard notation.) We generalize this definition to number fields by setting

$$\sigma_r(\mathfrak{A}) = \sum_{\mathfrak{B} \mid \mathfrak{A}} N(\mathfrak{B})^r \quad (\mathfrak{A} \subset \mathcal{O} \text{ an ideal}). \tag{13}$$

Here the sum is over all ideals $\mathfrak{B}$ of $\mathcal{O}$ which divide (i.e. contain) $\mathfrak{A}$. If $K = \mathbf{Q}$, $\mathcal{O} = \mathbf{Z}$, $\mathfrak{A} = (n)$, this agrees with (12).

Finally, for $l, m = 1, 2, \ldots$, we define

$$s_l^K(2m) = \sum_{\substack{\nu \in \mathfrak{d}^{-1} \\ \nu \gg 0 \\ tr(\nu) = l}} \sigma_{2m-1}\big((\nu)\,\mathfrak{d}\big). \tag{14}$$

The sum extends over all totally positive (i.e. all conjugates positive) elements of the fractional ideal (10) with given trace l (there are only finitely many such elements). Such a ν need not be integral, but the product of the principal ideal (ν) with the different $\mathfrak{d}$ will be an integral ideal, and therefore $\sigma_{2m-1}\big((\nu)\,\mathfrak{d}\big)$ is defined.

We can now state Siegel's formula.

THEOREM (Siegel [9]). *Let $m = 1, 2, \ldots$ be a natural number, K a totally real algebraic number field, $n = [K:\mathbf{Q}]$, and $h = 2mn$. Then*

$$\zeta_K(1-2m) = 2^n \sum_{l=1}^{r} b_l(h)\, s_l^K(2m). \tag{15}$$

The numbers $r \geqslant 1$ and $b_1(h), \ldots, b_r(h) \in \mathbf{Q}$ depend only on h. In par*ticular,*

$$r = \dim_{\mathbf{C}} \mathfrak{M}_h, \tag{16}$$

where $\mathfrak{M}_h$ is the space of modular forms of weight h; thus by a well-known formula

$$r = \begin{cases} [h/12] & \text{if} \quad h \equiv 2 \pmod{12}, \\ [h/12] + 1 & \text{if} \quad h \not\equiv 2 \pmod{12}, \end{cases} \tag{17}$$

where $[x]$ denotes the greatest integer $\leqslant x$.

(We have given a table of the coefficients $b_l(h)$ on page 60, if for no other reason than to emphasize that they really only depend on the integer h and not on the field. The values for h even, $4 \leqslant h \leqslant 24$, were taken from Siegel [9]; the values for $4 \mid h \leqslant 40$ were calculated on the System 370 computer at Bonn.)

Proof of theorem (sketch): Recall that one can define a modular form of weight $2m$ by the *Eisenstein series*

$$G_{2m}(z) = \sum_{\substack{\lambda, \mu \in \mathbf{Z} \\ (\lambda, \mu) \neq (0,0)}} \frac{1}{(\lambda z + \mu)^{2m}} \tag{18}$$

TABLE 1.

The Siegel coefficients $b_l(h)$

h	$b_1(h)$	$b_2(h)$	$b_3(h)$	$b_4(h)$
4	$\frac{1}{240}$			
6	$\frac{-1}{504}$			
8	$\frac{1}{480}$			
10	$\frac{-1}{264}$			
12	$\frac{-1}{8190}$	$\frac{1}{196560}$		
14	$\frac{-1}{24}$			
16	$\frac{-1}{680}$	$\frac{1}{146880}$		
18	$\frac{-22}{3591}$	$\frac{-1}{86184}$		
20	$\frac{-19}{1650}$	$\frac{1}{39600}$		
22	$\frac{-4}{207}$	$\frac{-1}{14904}$		
24	$\frac{-1087}{291200}$	$\frac{1}{1092000}$	$\frac{1}{52416000}$	
28	$\frac{-2529}{259840}$	$\frac{-1}{81200}$	$\frac{1}{15590400}$	
32	$\frac{837}{43520}$	$\frac{-9}{54400}$	$\frac{1}{2611200}$	
36	$\frac{-274486}{29895075}$	$\frac{-899}{28787850}$	$\frac{1}{86363550}$	$\frac{1}{6218175600}$
40	$\frac{-602849}{39067875}$	$\frac{-1773}{14206500}$	$\frac{-1}{7441500}$	$\frac{1}{1250172000}$

($z \in \mathfrak{H}$ = upper half-plane, i.e. $z \in \mathbf{C}$ and $Im(z) > 0$). Since $G_{2m}(z)$ has period 1, it has a Fourier expansion as a power series in $q = e^{2\pi iz}$,

$$G_{2m}(z) \sim a_0 + a_1 q + a_2 q^2 + \ldots \tag{19}$$

valid as $z \to i\,\infty$ (i.e. $q \to 0$). Then clearly

$$a_0 = \sum_{\substack{\mu \in \mathbf{Z} \\ \mu \neq 0}} \mu^{-2m} = 2\,\zeta(2m)\,, \tag{20}$$

and an easy calculation gives

$$a_n = 2\,\frac{(2\pi i)^{2m}}{(2m-1)\,!}\,\sigma_{2m-1}(n) \quad (n = 1, 2, \ldots)\,. \tag{21}$$

In an entirely analogous way, for the field K one can construct a modular form of weight $2m$ in n variables $z_1, \ldots, z_n \in \mathfrak{H}$ (the *Hecke-Eisenstein series*) and calculate its Fourier coefficients. By setting $z_1 = \ldots = z_n = z$, we obtain a modular form $G_{2m}^K(z)$ in one variable, of weight $2mn = h$, with a known Fourier expansion, namely

$$G_{2m}^K(z) \sim a_0 + a_1 q + a_2 q^2 + \ldots \tag{22}$$

with

$$a_0 = \zeta_K(2m)\,, \tag{23}$$

$$a_l = \{(2\pi i)^{2m}/(2m-1)\,!\}^n\, D^{-2m+1/2}\, s_l^K(2m) \quad (l = 1, 2, \ldots)\,. \tag{24}$$

On the other hand, since the space $\mathfrak{M}_h$ of modular forms of weight h has finite dimension r, there must be a linear relation among the first $r + 1$ coefficients in the Fourier expansion of any such form, i.e. there must exist numbers $c_{h,0}, c_{h,1}, \ldots, c_{h,r}$ depending only on h such that

$$f \in \mathfrak{M}_h\,,\ f \sim a_0 + a_1 q + a_2 q^2 + \ldots$$

$$\Rightarrow c_{h,0}a_0 + c_{h,1}a_1 + \ldots + c_{h,r}a_r = 0\,. \tag{25}$$

Siegel then shows that $c_{h,0}$ is non-zero for all h, so we can set

$$b_l(h) = -\,c_{h,l}/c_{h,0} \quad (l + 1, \ldots, r) \tag{26}$$

to obtain from (25) the relation

$$a_0 = \sum_{l=1}^{r} b_l(h)\, a_l \tag{27}$$

expressing the constant term of a modular form of given weight as a linear combination of finitely many of the other coefficients of its Fourier expansion. Substituting (23) and (24) into (27) gives

$$\zeta_K(2m) = \{(2\pi i)^{2m}/(2m-1)!\}^n D^{-2m+1/2} \sum_{l=1}^{r} b_l(h)\, s_l^K(2m), \tag{28}$$

which in view of the functional equation (6) is equivalent to the assertion of the theorem.

Since the numbers $\sigma_r(\mathfrak{A})$ and hence $s_l^K(2m)$ are clearly (rational) integers, we deduce from (15) not only that $\zeta_K(1-2m)$ is rational, but also that its denominator is bounded by a number depending only on h, i.e. only on the number $1 - 2m$ and the degree of the field K.

We now juggle the terms in the Siegel formula somewhat to rewrite it in a suggestive form. If we substitute the definitions (14) and (13) into equation (15) and invert the order of summation, we obtain

$$\begin{aligned}\zeta_K(1-2m) &= 2^n \sum_{l=1}^{r} b_l(h) \sum_{\substack{\nu\in\mathfrak{d}^{-1}\\ \nu\gg 0\\ tr(\nu)=l}} \sum_{\mathfrak{B}\mid(\nu)\mathfrak{d}} N(\mathfrak{B})^{2m-1} \\ &= \sum_{\mathfrak{B}} w(\mathfrak{B})\, N(\mathfrak{B})^{2m-1}, \end{aligned} \tag{29}$$

where the sum is over all non-zero integral ideals $\mathfrak{B}$ and the "weight" $w(\mathfrak{B})$ is defined by

$$w(\mathfrak{B}) = 2^n \sum_{\substack{\nu\in\mathfrak{B}\mathfrak{d}^{-1}\\ \nu\gg 0}} b_{tr(\nu)}(h). \tag{30}$$

The sum in (30) is always finite and is empty for all but finitely many ideals $\mathfrak{B}$ (because $b_l(h) = 0$ for $l > r$) so the sum (29) is in fact finite. Equation (29) is a rather amusing formulation of Siegel's theorem, for if we had just mechanically substituted $s = 1 - 2m$ into (1) without regard for convergence, we would have obtained

$$\zeta_K(1-2m) = \sum_{\mathfrak{B}} N(\mathfrak{B})^{2m-1}, \tag{31}$$

which is of course nonsense, but then equation (29) tells us that it is all right after all, if we just insert "fudge factors" $w(\mathfrak{B})$ to weight the summands: thus one really *can* evaluate $\zeta_K(1-2m)$ by adding up $(2m-1)$-th powers of norms of ideals.

In this connection, it is perhaps worthwhile to observe that the weights $w(\mathfrak{B})$ are not unique. Indeed, given h, we can choose any $r' \geqslant r$ and find coefficients $b_1'(h), \ldots, b_{r'}'(h)$ expressing the constant term of any form $f \in \mathfrak{M}_h$ in terms of the next r' coefficients (such collections b' will form an affine space of dimension $r' - r$). Then Siegel's theorem is valid with the

b_l' in place of the b_l, and similarly using the b_l' in (30) would give other weights making (29) hold.

Finally, for completeness' sake we should mention that Siegel gave a somewhat more general formula than the one stated. If A denotes any ideal class of the field K, then restricting the ideals A in the sum (1) to ideals in the class A gives rise to another meromorphic function, denoted $\zeta(s, A)$. This function also takes on rational values at negative odd integers, and Siegel's formula for these rational numbers is identical to (15) except that one must modify the definition of $\sigma_r(\mathfrak{A})$ by only allowing those ideal divisors $\mathfrak{B}$ in (13) that lie in the class A. In the formulation of Siegel's result just given, this can be simply stated

$$\zeta(1-2m, A) = \sum_{\mathfrak{B} \in A} w(\mathfrak{B}) N(\mathfrak{B})^{2m-1}, \tag{32}$$

with the same weights $w(\mathfrak{B})$ as before.

§2. Zeta-functions of Quadratic Fields

We now specialize to quadratic fields. A totally real quadratic field can be written uniquely as $\mathbf{Q}(d^{1/2})$ with $d > 1$ a square-free integer. Then it is easy to check that

$$\begin{aligned} D &= d \quad \text{if} \quad d \equiv 1 \pmod 4, \\ D &= 4d \quad \text{if} \quad d \equiv 2 \text{ or } 3 \pmod 4, \end{aligned} \tag{1}$$

and

$$\mathfrak{d} = (\sqrt{D}), \tag{2}$$

i.e. the different is a principal ideal. The decomposition of rational primes in the ring of integers $\mathcal{O}$ is described in terms of the primitive character $\chi \pmod D$ defined by

$$\chi(x) = \left(\frac{D}{x}\right) \tag{3}$$

(here χ is completely multiplicative, and given on primes by: $\chi(p) = 0$ if $p \mid D$; for $p \nmid 2D$, $\chi(p)$ is ± 1 according as D is or is not a quadratic residue $(\text{mod } p)$; for $p = 2$ and $D = d$ odd, $\chi(2) = (-1)^{(d-1)/4}$) as follows: if $p = 2, 3, 5, \ldots$ is a rational prime, then the ideal $(p) \subset \mathcal{O}$ decomposes into prime ideals according to the value of $\chi(p)$ —

$$\chi(p) = 1 \Rightarrow (p) = \mathfrak{P}_1\mathfrak{P}_2\,, \quad \mathfrak{P}_1 = \mathfrak{P}_2'\,,\ N(\mathfrak{P}_i) = p\,, \tag{4a}$$

$$\chi(p) = 0 \Rightarrow (p) = \mathfrak{P}^2\,, \quad N(\mathfrak{P}) = p\,, \tag{4b}$$

$$\chi(p) = -1 \Rightarrow (p) = \mathfrak{P}\,, \quad N(\mathfrak{P}) = p^2\,. \tag{4c}$$

Substituting this into the Euler product 1 (2) gives (for $Re\,(s)$ sufficiently large)

$$\begin{aligned}\zeta_K(s) &= \prod_{\mathfrak{P}} \frac{1}{1 - N(\mathfrak{P})^{-s}} \\ &= \prod_{\chi(p)=1} \frac{1}{1-p^{-s}}\frac{1}{1-p^{-s}} \prod_{\chi(p)=0} \frac{1}{1-p^{-s}} \prod_{\chi(p)=-1} \frac{1}{1-p^{-2s}} \\ &= \prod_{p} \frac{1}{1-p^{-s}}\frac{1}{1-\chi(p)\,p^{-s}} \\ &= \zeta(s)\ L(s,\chi)\,,\end{aligned} \tag{5}$$

where $\zeta\,(s)$ is defined in 1 (7) and

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} \tag{6}$$

is the L-series associated to the character χ. Again, (6) is convergent only for $Re\,(s)$ large enough, but the function $L\,(s, \chi)$ it defines can be extended to the whole s-plane (and (5) is then true everywhere). $L\,(s, \chi)$ is holomorphic everywhere.

Since we know the values of $\zeta\,(2m)$ (equation 1 (9)), we only need calculate $L\,(2m, \chi)$. But $\chi\,(n)$ is periodic with period D and satisfies $\chi\,(n) = \chi\,(-n)$, so we have

$$L(2m,\chi) = \sum_{n=1}^{\infty} \chi(n)\,n^{-2m} = \frac{1}{2}\sum_{a=1}^{D-1} \chi(a)\,\varphi(a,D;2m)\,, \tag{7}$$

where

$$\varphi(a,D;2m) = \sum_{\substack{n=-\infty \\ n\equiv a \ (\mathrm{mod}\ D)}}^{\infty} n^{-2m}\,. \tag{8}$$

This last sum can be evaluated in terms of elementary functions:

$$\varphi(a,D;2m) = \sum_{r\in\mathbf{Z}} (rD+a)^{-2m} = D^{-2m} f_m\!\left(\frac{a}{D}\right), \tag{9}$$

where

$$f_m(x) = \sum_{r\in\mathbf{Z}} \frac{1}{(r+x)^{2m}}$$

$$= \frac{-1}{(2m-1)!} \frac{d^{2m-1}}{dx^{2m-1}} \sum_{-\infty}^{\infty} \frac{1}{r+x}$$

$$= \frac{-\pi}{(2m-1)!} \frac{d^{2m-1}}{dx^{2m-1}} \cot \pi x . \tag{10}$$

Thus

$$f_1(x) = \pi^2 (\csc^2 \pi x) , \quad f_2(x) = \pi^4 (\csc^4 \pi x - \frac{2}{3} \csc^2 \pi x) . \tag{11}$$

This gives a finite and elementary expression for $L(2m, \chi)$. It can be simplified yet further by observing that $f_m\left(\frac{a}{D}\right)$ is periodic in a with period D, and therefore has a finite Fourier expansion as a sum $\Sigma \gamma_n e^{2\pi i n a/D}$. The coefficients γ_n are easy to compute and are rational. If we then put all this into (7), we finally obtain the formula

$$L(2m, \chi) = \frac{(-1)^{m-1} 2^{2m-1} \pi^{2m}}{(2m-1)! \sqrt{D}} \sum_{j=1}^{D} \chi(j) B_{2m}\left(\frac{i}{D}\right), \tag{12}$$

where $B_r(x)$ denotes the r-th *Bernoulli polynomial:*

$$B_r(x) = \sum_{s=0}^{r} \binom{r}{s} B_{r-s} x^s . \tag{13}$$

If we substitute (12) and 1 (9) into equation (5) and apply the functional equation 1 (6), we obtain finally

$$\zeta_K(1-2m) = \frac{B_{2m}}{4m^2} D^{2m-1} \sum_{j=1}^{D} \chi(j) B_{2m}\left(\frac{i}{D}\right). \tag{14}$$

That is, for quadratic fields it is possible to give a completely elementary formula, derived in a completely elementary way, for the value of $\zeta_K(1-2m)$.

As an illustration, we take $m = 1$. Since

$$B_2(x) = x^2 - x + \frac{1}{6}, \tag{15}$$

one gets (after some trivial manipulations)

$$\zeta_K(-1) = \frac{1}{24D} \sum_{j=1}^{D-1} \chi(j) j^2 . \tag{16}$$

For example, with $K = \mathbf{Q}(\sqrt{5})$ we get

$$\zeta_K(-1) = \frac{1}{120} [1^2 - 2^2 - 3^2 + 4^2] = \frac{1}{30}, \tag{17}$$

while for $K = \mathbf{Q}(\sqrt{13})$

$$\zeta_K(-1)$$

$$= \frac{1}{24 \times 13}[1^2 - 2^2 + 3^2 + 4^2 - 5^2 - 6^2 - 7^2 - 8^2 + 9^2 + 10^2 - 11^2 + 12^2]$$

$$= \frac{1}{6}. \tag{18}$$

For a more complete discussion of the formulas treated in this section, see Siegel [8].

§3. The Siegel Formula for Quadratic Fields

In this section we shall exploit the simple arithmetic of quadratic fields to evaluate in elementary form the various terms entering into Siegel's formula, thus arriving at an expression for $\zeta_K(1-2m)$ which is elementary in the sense that it involves only rational integers and not algebraic numbers or ideals.

We have to evaluate $s_l^K(2m)$, and to do so we must first know how to compute $\sigma_r(\mathfrak{A})$ for any ideal $\mathfrak{A}$.

Lemma. *Let* $\mathfrak{A}$ *be any ideal of the ring of integers* $\mathcal{O}$ *of a quadratic field* K. *Let* D *be the discriminant of* K *and* $\chi(j) = \left(\frac{D}{j}\right)$ *the associated character (as in* §2). *Then, for any* $r \geqslant 0$,

$$\sigma_r(\mathfrak{A}) = \sum_{j \mid \mathfrak{A}} \chi(j) j^r \sigma_r(N/j^2), \tag{1}$$

where $N = N(\mathfrak{A})$ *is the norm of* $\mathfrak{A}$, *the function* σ_r *on the right-hand side is the arithmetic function of* 1 (12), *and the sum is over all positive integers* j *dividing* $\mathfrak{A}$ (i.e. $\nu/j \in \mathcal{O}$ for every $\nu \in \mathfrak{A}$; clearly this implies $j^2 \mid N$, so equation (1) makes sense).

Proof: It is very easy to check that both sides of (1) are multiplicative functions, i.e. $\sigma_r(\mathfrak{A}\mathfrak{B}) = \sigma_r(\mathfrak{A})\,\sigma_r(\mathfrak{B})$ for relatively prime ideals $\mathfrak{A}$ and $\mathfrak{B}$, and similarly for the expression on the right-hand side of (1). It therefore suffices to take $\mathfrak{A}$ to be a power $\mathfrak{P}^m$ of a prime ideal $\mathfrak{P}$. Write $N(\mathfrak{P}) = p^i$

where p is a rational prime and $i = 1$ or 2. Then we can evaluate the left-hand side of (1):

$$\sigma_r(\mathfrak{A}) = \sigma_r(\mathfrak{P}^m) = \sum_{\mathfrak{B} \mid \mathfrak{P}^m} N(\mathfrak{B})^r$$

$$= \sum_{n=0}^{m} N(\mathfrak{P}^n)^r = \sum_{n=0}^{m} p^{inr} = \sigma_{ir}(p^m). \tag{2}$$

To evaluate the right-hand side of (1), we must distinguish three cases, according to the value of $\chi(p)$.

Case 1. $\chi(p) = 1$, $(p) = \mathfrak{P}\mathfrak{P}'$ ($\mathfrak{P}'$ = conjugate of $\mathfrak{P}$). Then $N(\mathfrak{A}) = N(\mathfrak{P})^m = p^m$. Clearly $j \mid \mathfrak{A} \Rightarrow j = 1$, for j can only be a power of p (since $j \mid N(\mathfrak{A})$) and cannot be divisible by p (because $\mathfrak{P}' \mid p$, $\mathfrak{P}' \nmid \mathfrak{A}$). Hence the sum in (1) has only one term $\sigma_r(N) = \sigma_r(p^m)$, in agreement with (2).

Case 2. $\chi(p) = 0$, $(p) = \mathfrak{P}^2$. Again j can only be a power of p, and since $\chi(p) = 0$, the only term in (1) that does not vanish is the term $j = 1$, namely $\sigma_r(N)$. Since $N = N(\mathfrak{P})^m = p^m$ and $i = 1$, this again agrees with (2).

Case 3. $\chi(p) = -1$, $(p) = \mathfrak{P}$. Now $\mathfrak{A} = \mathfrak{P}^m = (p^m)$, so j can take on the values $1, p, p^2, \ldots, p^m$, with $\chi(p^n) = (-1)^n$. Here $i = 2$ and $N = N(\mathfrak{P})^m = p^{2m}$, so we must prove

$$\sigma_{2r}(p^m) = \sum_{n=0}^{m} (-1)^n p^{nr} \sigma_r(p^{2m-2n}). \tag{3}$$

This is just an exercise in summing geometric series.

The lemma enables us to calculate the generalized sums-of-powers functions $\sigma_r(\mathfrak{A})$ in terms of the ordinary function $\sigma_r(m)$. It remains to see what ideals $\mathfrak{A}$ occur in Siegel's formula. Recall that

$$s_l^k(2m) = \sum_{\substack{\nu \in \mathfrak{d}^{-1} \\ \nu \gg 0 \\ tr(\nu) = l}} \sigma_{2m-1}((\nu)\,\mathfrak{d}), \tag{4}$$

and that

$$\mathfrak{d} = (\sqrt{D}) \tag{5}$$

for a quadratic field. Furthermore, the ring of integers of K is

$$\mathcal{O} = \left\{ \frac{x + y\sqrt{D}}{2} \;\middle|\; x, y \in \mathbf{Z},\ x^2 \equiv y^2 D \pmod 4 \right\}. \tag{6}$$

We can now describe explicitly the ν occurring in the sum (4). Write such a ν as $\alpha + \beta\sqrt{D}$ with α and β rational. Then

$$\nu \in \mathfrak{d}^{-1} \Leftrightarrow \nu\sqrt{D} \in \mathcal{O}, \tag{7}$$

$$\nu \gg 0 \Leftrightarrow \alpha > |\beta| \sqrt{D}, \tag{8}$$

$$\operatorname{tr}(\nu) = l \Leftrightarrow \alpha = l/2. \tag{9}$$

From (6), (7) and (9) we then get $\beta = b/2D$, where b is a rational integer satisfying

$$b^2 \equiv l^2 D \pmod 4 \tag{10}$$

and (because of (8)) also

$$b^2 < l^2 D. \tag{11}$$

Then $(\nu)\,\delta$ is the principal ideal

$$(\nu)\,\mathfrak{d} = (\nu\sqrt{D}) = \left(\frac{b}{2} + \frac{l}{2}\sqrt{D}\right). \tag{12}$$

An integer j can divide this only if $j \mid b$ and $j \mid l$ and $(b/j)^2 \equiv (l/j)^2 D \pmod 4$, so by the lemma

$$\sigma_r\big((\nu)\,\mathfrak{d}\big) = \sum_{\substack{l = jl' \\ b = jb' \\ b'^2 \equiv l'^2 D \ (\mathrm{mod}\ 4)}} \chi(j)\, j^r\, \sigma_r\left(\frac{l'^2 D - b'^2}{4}\right). \tag{13}$$

We now substitute this into (4), where the summation in (4) is now to be taken over all integers b satisfying (10) and (11), and obtain finally

$$s_l^K(2m) = \sum_{j \mid l} \chi(j)\, j^{2m-1}\, e_{2m-1}\big((l/j)^2 D\big), \tag{14}$$

where the arithmetic function $e_r(n)$ is defined by

$$e_r(n) = \sum_{\substack{x^2 \equiv n \ (\mathrm{mod}\ 4) \\ |x| \leq \sqrt{n}}} \sigma_r\left(\frac{n - x^2}{4}\right) \tag{15}$$

($r = 0, 1, 2, \ldots$; n a positive integer, not a perfect square). Then (15) is a finite sum (empty, if $n \equiv 2$ or $3 \pmod 4$), and so is (14), so that we have completely evaluated $s_l^K(2m)$ in elementary terms. Then Siegel's theorem states

$$\zeta_K(1 - 2m) = 4 \sum_{l=1}^{r} b_l(4m)\, s_l^K(2m), \tag{16}$$

with $r = [m/3]$ and the coefficients $b_l(4m)$ computable rational numbers tabulated on p. 60 for $1 \leqslant l \leqslant 10$.

Using the values of $b_l(4m)$ and equation (14), we can write out the first few cases to illustrate (16): $\underline{m = 1}$. Here $r = 1$, $b_1(4) = 1/240$, and so (16) reduces to

$$\zeta_K(-1) = \frac{1}{60} s_1^K(2) = \frac{1}{60} e_1(D) . \tag{17}$$

Thus for $K = \mathbf{Q}(\sqrt{5})$ we find

$$\begin{aligned}\zeta_K(-1) &= \frac{1}{60} e_1(5) = \frac{1}{60}\left\{\sigma_1\left(\frac{5-1^2}{4}\right) + \sigma_1\left(\frac{5-(-1)^2}{4}\right)\right\} \\ &= 2\sigma_1(1)/60 = 1/30 ,\end{aligned} \tag{18}$$

in agreement with 2 (17), and similarly for $K = \mathbf{Q}(\sqrt{13})$

$$\begin{aligned}\zeta_K(-1) &= \frac{1}{60} e_1(13) = \frac{2}{60}\left\{\sigma_1\left(\frac{13-1^2}{4}\right) + \sigma_1\left(\frac{13-3^2}{4}\right)\right\} \\ &= \frac{2}{60}(3+1+1) = 1/6 ,\end{aligned} \tag{19}$$

in agreement with 2 (18) (but notice how many fewer terms!). $\underline{m = 2}$. Here again $r = 1$, and the formula is just as simple:

$$\zeta_K(-3) = \frac{1}{120} s_1^K(4) = \frac{1}{120} e_3(D) . \tag{20}$$

Thus with $K = \mathbf{Q}(\sqrt{13})$ we find

$$\zeta_K(-3) = \frac{2}{120}(3^3 + 1^3 + 1^3) = \frac{29}{60} . \tag{21}$$

$\underline{m = 3}$. Here $r = 2$ and the formula is more complicated:

$$\begin{aligned}\zeta_K(-5) &= \frac{4}{196560}\left(s_2^K(6) - 24\, s_1^K(6)\right) \\ &= \frac{1}{49140}\{e_5(4D) + 32\,\chi(2)\, e_5(D) - 24\, e_5(D)\} .\end{aligned} \tag{22}$$

Here for $K = \mathbf{Q}(\sqrt{13})$ we get

$$\begin{aligned}\zeta_K(-5) &= (e_5(52) - 56 e_5(13))/49140 \\ &= (\sigma_5(13) + 2\sigma_5(12) + 2\sigma_5(9) + 2\sigma_5(4) \\ &\quad - 112\sigma_5(3) - 112\sigma_5(1))/49140 \\ &= 980370/49140 = 3631/182 .\end{aligned} \tag{23}$$

Table 2.

The Siegel formulas for quadratic fields

$$K = \mathbf{Q}(\sqrt{D}), \quad D = \text{discriminant}, \quad \chi(m) = \left(\frac{D}{m}\right),$$

$$e_r(n) = \sum_{\substack{b^2+4ac=n \\ a,c>0}} a^r .$$

$60\,\zeta_K(-1) = e_1(D)$

$120\,\zeta_K(-3) = e_3(D)$

$49140\,\zeta_K(-5) = e_5(4D) + [32\,\chi(2) + 24]\,e_5(D)$

$36720\,\zeta_K(-7) = e_7(4D) + [128\,\chi(2) - 216]\,e_7(D)$

$9900\,\zeta_K(-9) = e_9(4D) + [512\,\chi(2) - 456]\,e_9(D)$

$13104000\,\zeta_K(-11) = e_{11}(9D) + 48e_{11}(4D) + [177147\,\chi(3)$
$\quad + 98304\,\chi(2) - 195660]\,e_{11}(D)$

$3897600\,\zeta_K(-13) = e_{13}(9D) - 192e_{13}(4D) + [1594323\,\chi(3)$
$\quad - 1572864\,\chi(2) - 151740]\,e_{13}(D)$

$652800\,\zeta_K(-15) = e_{15}(9D) - 432e_{15}(4D) + [14348907\,\chi(3)$
$\quad - 14155776\,\chi(2) - 50220]\,e_{15}(D)$

$1554543900\,\zeta_K(-17) = e_{17}(16D) + 72e_{17}(9D) + [131072\,\chi(2)$
$\quad - 194184]\,e_{17}(4D) + [17179869184\,\chi(4) + 9298091736\,\chi(3)$
$\quad - 25452085248\,\chi(2) - 57093088]\,e_{17}(D)$

$312543000\,\zeta_K(-19) = e_{19}(16D) - 168e_{19}(9D) + [524288\,\chi(2)$
$\quad - 156024]\,e_{19}(4D) + [274877906944\,\chi(4) - 195259926456\,\chi(3)$
$\quad - 81801510912\,\chi(2) - 19291168]\,e_{19}(D)$

$42124500\,\zeta_K(-21) = e_{21}(16D) - 408\,e_{21}(9D) + [2097152\,\chi(2)$
$\quad - 60264]\,e_{21}(4D) + [4398046511104\,\chi(4) - 4267824106824\,\chi(3)$
$\quad - 126382768128\,\chi(2) - 3953248]\,e_{21}(D)$

TABLE 3.

Values of $\zeta_K(1-2m)$ *for quadratic fields*

$Z_1 = 60\zeta_K(-1)$. $Z_3 = 120\zeta_K(-3)$. $Z_5 = 49140\,\zeta_K(-5)$. $Z_7 = 36720\,\zeta_K(-7)$.

$Z_9 = 9900\,\zeta_K(-9)$. $Z_{11} = 13104000\,\zeta_K(-11)$. $(K = \mathbf{Q}(\sqrt{D}),\ D = \text{discriminant})$

D	Z_1	Z_3	Z_5	Z_7	Z_9	Z_{11}
5	2	2	5226	110466	2476506	636229128800
8	5	11	70395	3765483	215478075	141611774080400
12	10	46	655590	78808158	10145592150	1500201727306400
13	10	58	1003890	143106714	21682075650	37653788862335200
17	20	164	4516980	1078232292	277803225300	823821554778449600
21	20	308	14017380	5219942004	2064025431300	9353651984246859200
24	30	522	29672370	14265873306	7346194920450	43450483506376984800
28	40	904	69359160	45338101992	31773438504600	255789968221174153600
29	30	942	82614870	58740797646	44300167762950	382856016709462960800
33	60	1692	173700540	156050858556	151482447747900	1692706573508047636800
37	50	2258	316311450	365256498834	448286221058250	6306377416787885007200
40	70	3154	493274730	658004816322	941093728561050	15461657528842738261600
41	80	3584	572460720	794742744672	1191020559229200	20543995478169063449600
44	70	4306	830983530	1344445147458	2327280476401050	46266888778260351522400

In Table 2 we write out in full the formula for $\zeta_K(1-2m)$ $(1 \leqslant m \leqslant 6)$ in terms of the arithmetical functions $e_r(n)$. In Table 3 we give the values of $\zeta_K(1-2m)$ for $1 \leqslant m \leqslant 6$ and K a quadratic field with discriminant at most 50. Since it is more convenient to tabulate integers, we in fact give the values of

$$Z_{2m-1} = t(m)\,\zeta_K(1-2m)\,, \tag{24}$$

where $t(m)$ is the bound implied by (16) for the denominator of $\zeta_K(1-2m)$, namely

$$t(m) = L.C.M.\,\{\text{denom } 4b_l(4m),\ 1 \leqslant l \leqslant r\}. \tag{25}$$

Because the question of the denominator of $\zeta_K(1-2m)$ is important (namely, a prime p divides this denominator whenever the p-adic analogue of $\zeta_K(s)$ has a pole at $s = 1 - 2m$), it is worthwhile to try to sharpen (25). To do this, we use the result of §2, namely

$$\zeta_K(1-2m) = (B_{2m}/4m^2) \sum_{r=0}^{2m} B_r D^{r-1} \beta_{2m-r}(D)\,, \tag{26}$$

where B_r is the r-th Bernoulli number and

$$\beta_r(D) = \sum_{j=1}^{D} \chi_D(j)\, j^r\,. \tag{27}$$

Set

$$a(m) = \prod_{\substack{3 \leqq p \leqq 2m+1 \\ p \text{ prime}}} p\,. \tag{28}$$

For $0 \leqslant r \leqslant 2m$, $2a(m)B_r$ is an integer, by von Staudt's theorem, and since $\beta_r(D) \equiv 0 \pmod 4$, $\frac{1}{2}a(m)\,B_r\,D^{r-1}\,\beta_{2m-r}(D)$ is an integer for $r \geqslant 1$. There remains the term $r = 0$ of (26). If D is divisible by an odd prime p but $D \neq p$, then (writing $D = pD'$, with $p \nmid D'$)

$$\beta_{2m}(D) \equiv \sum_{k=1}^{p} \chi_p(k)\,k^{2m} \sum_{\substack{j=1 \\ j \equiv k \ (\mathrm{mod}\ p)}}^{D} \chi_{D'}(j) \pmod p\,, \tag{29}$$

and the inner sum is 0 for $D' > 1$. One also checks easily that $\beta_{2m}(D)$ is always even, is divisible by 8 if $D \equiv 0 \pmod 4$ and is divisible by 16 if $D \equiv 0 \pmod 8$. Therefore $\beta_{2m}(D)/D$ is an even integer, unless $D = p$ is a prime ($\equiv 1 \pmod 4$). In that case,

$$\beta_{2m}(p) = \sum_{k=1}^{p-1} \left(\frac{k}{p}\right) k^{2m} \equiv \sum_{k=1}^{p-1} k^{2m+(p-1)/2} \equiv 0 \pmod p \tag{30}$$

if $2m + \frac{p-1}{2}$ is not divisible by $p-1$. Finally, if $2m + \frac{p-1}{2}$ is divisible by $p-1$, then $(p-1) \mid 4m$ and hence $p = 4m+1$ or $p \leqslant 2m+1$. Therefore $a(m)\beta_{2m}(D)/D$ is an even integer here also, except in the one case $D = 4m+1 =$ prime. Thus, if we set

$$s(m) = a(m) \cdot \text{denom } (B_{2m}/2m^2) \cdot \varepsilon_m \,, \tag{31}$$

$$\varepsilon_m = \begin{cases} 4m+1 & \text{if } \quad 4m+1 \quad \text{is prime,} \\ 1 & \text{otherwise,} \end{cases} \tag{32}$$

then $s(m)\zeta_K(1-2m)$ will be an integer for all quadratic fields K, and indeed $(s(m)/\varepsilon_m)\zeta_K(1-2m)$ will be an integer for all fields except $\mathbf{Q}(\sqrt{4m+1})$. We have tabulated the two bounds $t(m)$ and $s(m)$ for $1 \leqslant m \leqslant 17$ in Table 4, putting the factor ε_m of $s(m)$ in brackets because it only occurs in the denominator of $\zeta_K(1-2m)$ for a single exceptional field K. It will be seen that in general neither of $s(m)$, $t(m)$ divides the other, so that

$$u(m) = G.C.D. \{ s(m), t(m) \} \tag{33}$$

gives a better bound than is provided by either the Siegel or the elementary method alone. From the table of values of $u(m)$ one sees that, for instance,

$$3 \mid Z_7 \,, \quad 20 \mid Z_{11} \tag{34}$$

and that

$$5 \mid Z_1 \text{ if } D \neq 5 \,, \quad 13 \mid Z_5 \text{ if } D \neq 13 \,, \quad 17 \mid Z_7 \text{ if } D \neq 17 \,. \tag{35}$$

All of these congruences can be verified in Table 3. Indeed, Table 3 suggests that (34) can be improved to

$$3 \mid Z_5 \,, \quad 9 \mid Z_7 \,, \quad 3 \mid Z_9 \,, \quad 400 \mid Z_{11} \tag{36}$$

and that, as well as the congruences (35), one has

$$5 \mid Z_5 \,, \quad 25 \mid Z_9 \text{ if } D \neq 5 \,. \tag{37}$$

All of these are special cases of the following

CONJECTURE ([6], p. 164). *For any totally real* K,

$$w_m(K)\zeta_K(1-2m) \in \mathbf{Z} \,, \tag{38}$$

where the integer $w_m(K)$ *is defined as*

$$G.C.D. \{ (N\mathfrak{P})^i (N\mathfrak{P}^{2m} - 1), \; i \geqslant m, \; \mathfrak{P} \text{ a prime ideal} \}. \tag{39}$$

TABLE 4.

Bounds for the denominator of $\zeta_K(1-2m)$, *K quadratic*

m	$t(m)$ (Siegel bound)	$s(m)$ (elementary bound)	$u(m) = (t(m), s(m))$
1	$60 = 2^2 3.5$	$2^3 3^2$ (5)	$2^2 3.$ (5)
2	$120 = 2^3 3.5$	$2^5 3^2 5^2$	$2^3 3.5$
3	$49140 = 2^2 3^3 5.7.13$	$2^3 3^4 5.7^2$ (13)	$2^2 3^3 5.7.$ (13)
4	$36720 = 2^4 3^3 5.17$	$2^7 3^2 5^2 7.$ (17)	$2^4 3^2 5.$ (17)
5	$9900 = 2^2 3^2 5^2 11$	$2^3 3^2 5^2 7.11^2$	$2^2 3^2 5^2 11$
6	$13104000 = 2^7 3^2 5^3 7.13$	$2^5 3^4 5^2 7^2 11.13^2$	$2^5 3^2 5^2 7.13$
7	$3897600 = 2^8 3.5^2 7.29$	$2^3 3^2 5.7^2 11.13.$ (29)	$2^3 3.5.7.$ (29)
8	$652800 = 2^9 3.5^2 17$	$2^9 3^2 5^2 7.11.13.17^2$	$2^9 3.5^2 17$
9	$1554543900 = 2^2 3^5 5^2 7.13.19.37$	$2^3 3^6 5.7^2 11.13.17.19^2$ (37)	$2^2 3^5 5.7.13.19.$ (37)
10	$312543000 = 2^3 3^2 5^3 7.11^2 41$	$2^5 3^2 5^4 7.11^2 13.17.19.$ (41)	$2^3 3^2 5^3 7.11^2$ (41)
11	$42124500 = 2^2 3^2 5^3 11.23.37$	$2^3 3^2 5.7.11^2 13.17.19.23^2$	$2^2 3^2 5.23$
12	$141466590720 = 2^9 3^6 5.7^3 13.17$	$2^7 3^4 5^2 7^2 11.13^2 17.19.23$	$2^7 3^4 5.7^2 13.17$
13	$22877225280 = 2^6 3^5 5.7.13.53.61$	$2^3 3^2 5.7.11.13^2 17.19.23.$ (53)	$2^3 3^2 5.7.13.$ (53)
14	$2722083840 = 2^{10} 3^3 5.7.29.97$	$2^5 3^2 5^2 7^2 11.13.17.19.23.29^2$	$2^5 3^2 5.7.29$
15	$11448204768000 = 2^8 3^3 5^3 7^2 11.13.31.61$	$2^3 3^4 5^2 7^2 11^2 13.17.19.23.29.31^2$ (61)	$2^3 3^3 5^2 7^2 11.13.31.$ (61)
16	$1611414604800 = 2^{10} 3^3 5^2 7.11.13.17.137$	$2^{11} 3^2 5^2 7.11.13.17^2 19.23.29.31$	$2^{10} 3^2 5^2 7.11.13.17$
17	$176840092800 = 2^7 3^2 5^2 7.17.51599$	$2^3 3^2 5.7.11.13.17^2 19.23.29.31$	$2^3 3^2 5.7.17$

Define an integer $j(m)$ for $m = 1, 2, \ldots$ by

$$j(m) = G.C.D. \{ n^{m+2}(n^{2m}-1), n \in \mathbf{Z} \}. \tag{40}$$

Thus

$$j(1) = 24, j(2) = 240, j(3) = 504, j(4) = 480, \ldots$$

Then it is easy to check that, for K a quadratic field, $w_m(K) = j(m)$ (independent of K!) unless K is one of the finitely many fields $\mathbf{Q}(\sqrt{p})$ with p a prime such that $(p-1) \mid 4m$, $(p-1) \nmid 2m$, in which case $w_m(K)$ $p^{v+1} j(m)$, where p^v is the largest power of p dividing m. This is interesting because the numbers $j(m)$ occur in topology: it is known (now that the Adams conjecture has been proved) that $j(m)$ is precisely the order of the group $J(S^{4m})$. This may be just a coincidence, of course, but could conceivably reflect some deeper connection between the values of zeta-functions and topological K-theory (the conjectured connection between these values and algebraic K-theory was mentioned in the introduction).

§4. THE CIRCLE METHOD AND THE NUMBERS $e_{2m-1}(n)$

In §3 we defined

$$e_r(n) = \sum_{\substack{k^2 \equiv n \ (\mathrm{mod}\ 4) \\ |k| \leq \sqrt{n}}} \sigma_r\left(\frac{n-k^2}{4}\right), \tag{1}$$

where r and n are positive integers and, for b a positive integer, $\sigma_r(b)$ is defined as the sum of the r-th powers of the positive divisors of b. Since (1) was only needed for n not a perfect square, we are still at liberty to define $\sigma_r(0)$; we set

$$\sigma_r(0) = \frac{1}{2}\zeta(-r) = -\frac{1}{2}\frac{B_{r+1}}{r+1}. \tag{2}$$

This defines $\sigma_r(b)$ for $b = 0, 1, 2, \ldots$; we extend the definition to all real b by setting $\sigma_r(b) = 0$ if $b < 0$ or $b \notin \mathbf{Z}$. Then (1) can be rewritten

$$e_r(n) = \sum_{k=-\infty}^{\infty} \sigma_r\left(\frac{n-k^2}{4}\right). \tag{3}$$

We were led to consider these numbers by Siegel's theorem, which, for real quadratic fields K, expresses the value of $\zeta_K(2m)$ or $\zeta_K(1-2m)$ in terms of the numbers $e_{2m-1}(n)$ with $K = \mathbf{Q}(\sqrt{n})$. In this section we follow a different course, and study the numbers (3) directly by the techniques of analytic number theory—specifically, by means of the Hardy-Littlewood circle method. This will lead to the following formula for $e_{2m-1}(n)$:

THEOREM 1. *Let* m *and* n *be positive integers,* n *not a perfect square. If* $n \equiv 2$ *or* $3 \pmod 4$ *then* $e_{2m-1}(n) = 0$. *If* $n \equiv 0$ *or* $1 \pmod 4$, *write*

$$n = f^2 D \tag{4}$$

with

$$D = \text{discriminant of } K, \quad K = \mathbf{Q}(\sqrt{n}). \tag{5}$$

Then

$$e_{2m-1}(n) = \frac{\zeta_K(1-2m)}{2\zeta(1-4m)} T^{\chi}_{2m}(f) + 0(n^{m+1/4}), \tag{6}$$

where χ *is the character associated to* K *(cf.* §2) *and* $T^{\chi}_{2m}(f)$ *is the multiplicative function given by*

$$T^{\chi}_{2m}(f) = \sum_{t \mid f} t^{4m-1} \sum_{a \mid t} \frac{\mu(a)\chi(a)}{a^{2m}} \tag{7}$$

$$= \sum_{a \mid f} \mu(a)\chi(a)a^{2m-1}\sigma_{4m-1}(f/a) \tag{8}$$

($\mu(a)$ denotes the Möbius function).

Note that the first term in (6) really is of bigger order than the error term, since one easily checks that $T^{\chi}_{2m}(f) > c_1 f^{4m-1}$ and $\zeta_K(1-2m) > c_2 D^{2m-1/2}$ with constants $c_1, c_2 > 0$, and hence the first term is $> c\, n^{2m-1/2}$.

Before turning to the proof of this theorem by means of the Hardy-Littlewood method, we consider its relationship to the results discussed in Sections 1 and 3. We saw in §1 that the Hecke-Eisenstein series $G^K_{2m}(z)$ of K has the Fourier expansion

$$G^K_{2m}(z) \sim a_0 + a_1 q + a_2 q^2 + \ldots \quad (q = e^{2\pi i z}) \tag{9}$$

with

$$a_0 = \zeta_K(2m), \tag{10}$$

$$a_l = k_m s^K_l(2m) = k_m \sum_{j \mid l} \chi(j) j^{2m-1} e_{2m-1}\left(\frac{l^2}{j^2} D\right), \tag{11}$$

where $k_m = (2\pi)^{4m} D^{-2m+1/2} / (2m-1)!^2$. Since $G^K_{2m}(z)$ is a modular form of weight $4m$, the form $G^K_{2m}(z) - a_0\, G_{4m}(z) / 2\zeta(4m)$ is a cusp form of weight $4m$, where $G_{4m}(z)$ is the ordinary Eisenstein series (we have used 1 (20)). But a very well-known theorem of Hecke asserts that the n-th Fourier coefficient of a cusp form of weight $2k$ is $0(n^k)$. Therefore (using 1 (21) for the Fourier coefficients of G_{4m})

$$\begin{aligned} s_l^K(2m) &= \frac{1}{k_m}\frac{a_0}{2\zeta(4m)}\frac{2^{4m+1}\pi^{4m}}{(4m-1)!}\sigma_{4m-1}(l) + 0(l^{2m}) \\ &= D^{2m-1/2}\frac{(2m-1)!^2}{(4m-1)!}\frac{\zeta_K(2m)}{\zeta(4m)}\sigma_{4m-1}(l) + 0(l^{2m}) \\ &= \frac{\zeta_K(1-2m)}{2\zeta(1-4m)}\sigma_{4m-1}(l) + 0(l^{2m}), \end{aligned}$$

where in the last line we have used the functional equations of ζ_K and ζ. Substituting (11) and inverting gives

$$e_{2m-1}(f^2D) = \sum_{a|f}\mu(a)\chi(a)a^{2m-1}s_{f/a}^K(2m) \tag{12}$$

$$= \frac{\zeta_K(1-2m)}{2\zeta(1-4m)}\sum_{a|f}\mu(a)\chi(a)a^{2m-1}\sigma_{4m-1}(a) + 0(f^{2m}), \tag{13}$$

and this is essentially the same as (6)—indeed with a better error term $0(n^m)$ rather than $0(n^{m+1/4})$.

Nevertheless, there is some point to proving Theorem 1 by the circle method. First of all, it provides a direct proof of the relationship between the arithmetic function $e_{2m-1}(n)$ and the value at $s = 2m$ of the zeta-function of $\mathbf{Q}(\sqrt{n})$. Secondly, the evaluation of the "singular series"—which yields the first term of eq. (6)—involves an evaluation of certain Gauss sums and of a Dirichlet series with such Gauss sums as coefficients which are of interest in their own right. Namely, we will prove the following two theorems.

THEOREM 2. *For positive integers a and c, let*

$$\lambda(a,c) = \begin{cases} i^{(1-c)/2}\left(\dfrac{a}{c}\right) & \text{if } c \text{ is odd, } a \text{ even}, \\ i^{a/2}\left(\dfrac{c}{a}\right) & \text{if } a \text{ is odd, } c \text{ even}, \\ 0 & \text{otherwise}, \end{cases} \tag{14}$$

where $\left(\frac{p}{q}\right)$ *(q odd) is the Legendre-Jacobi symbol and* $i^{a/2} = e^{\pi i a/4}$. *Thus* $\lambda(a, c)$ *is* 0 *if* a *and* c *have a common factor or are both odd, and is an* 8th *root of unity otherwise; furthermore,* $\gamma(a, c)$ *is periodic in* a *with period* 2*c*. *We define a Gauss sum* $\gamma_c(n)$ *by*

$$\gamma_c(n) = \frac{1}{\sqrt{c}} \sum_{a=1}^{2c} \lambda(a, c)\, e^{-\pi i n a/c}. \tag{15}$$

Then $\lambda_c(n)$ *is given as follows:*

If c *is odd, write* $c = ld^2$ *with* l *square-free. Then*

$$\gamma_c(n) = \begin{cases} 0 & \text{if } d \nmid n, \\ \displaystyle\sum_{t \mid \left(d, \frac{n}{d}\right)} t\,\mu\left(\frac{d}{t}\right)\left(\frac{n/t^2}{l}\right) & \text{if } d \mid n. \end{cases} \tag{16}$$

If c *is even, write* $c = 2^r c_1$ *with* c_1 *odd,* $r \geqslant 1$. *Then*

$$\gamma_c(n) = Q_r(n)\,\gamma_{c_1}(n), \tag{17}$$

where

$$Q_r(n) = \begin{cases} 2^{r/2}(-1)^{(m-1)/4} & \text{if } r \text{ is even}, \\ & n = 2^{r-2}m, \\ & m \equiv 1 \pmod 4, \\ 2^{\frac{r-1}{2}}(-1)^{m(m-1)/2} & \text{if } r \text{ is odd}, \\ & n = 2^{r-1}m, \\ 0 & \text{otherwise}. \end{cases} \tag{18}$$

THEOREM 3. *Let* n *be a non-zero integer and define a Dirichlet series* $E_n(s)$ *by*

$$E_n(s) = \frac{1}{2} \sum_{\substack{c=1 \\ c \text{ odd}}}^{\infty} \frac{\gamma_c(n)}{c^s} + \frac{1}{2} \sum_{\substack{c=2 \\ c \text{ even}}}^{\infty} \frac{\gamma_c(n)}{(c/2)^s} \tag{19}$$

(i.e. $E_n(s) = \Sigma\, a_m m^{-s}$ with $a_m = \frac{1}{2}(\gamma_m(n) + \gamma_{2m}(n))$ for m odd, $a_m = \frac{1}{2}\gamma_{2m}(n)$ for m even. Clearly $|\gamma_c(n)| \leqslant 2c^{1/2}$, so the series in (19) converge for $Re\ s > \frac{3}{2}$; in fact, $\gamma_c(n) = 0(1)$ as $c \to \infty$ by Theorem 2.

so they even converge for $Re\, s > 1$). *Let* $K = \mathbf{Q}(\sqrt{n})$, $D =$ *discriminant of* K, $\chi =$ *character of* K, $L(s, \chi) =$ *L-series of* χ *(if* n *is a perfect square,* $\chi(m) = 1$ *for all* m *and* $L(s, \chi) = \zeta(s)$). *Then*

$$E_n(s) \equiv 0 \qquad \text{if } n \equiv 2 \text{ or } 3 \pmod 4\,, \tag{20}$$

while, if $n \equiv 0$ *or* $1 \pmod 4$, *then*

$$E_n(s) = \frac{L(s,\chi)}{\zeta(2s)} \sum_{\substack{a,c \geq 1 \\ ac \mid f}} \frac{\mu(a)\,\chi(a)}{c^{2s-1}\, a^s} = \frac{L(s,\chi)}{\zeta(2s)} \frac{T_s^\chi(f)}{f^{2s-1}}\,, \tag{21}$$

where $n = f^2 D$.

As corollaries to Theorem 3, we see that $E_n(s)$ has a meromorphic continuation to the whole s-plane, and that $E_n(s)$ possesses an Euler product whose p-factor is $1 + \chi(p)\, p^{-s}$ if $p \nmid n$ and is a polynomial in p^{-s} in any case.

We will now show how the Dirichlet series (19) arises in connection with the numbers $e_{2m-1}(n)$, deferring to the end of the section the proofs of the two theorems on Gauss sums just enunciated.

Let $G_{2m}(z)$ be the Eisenstein series of weight $2m$, defined in 1 (18), and $\theta(z)$ the theta series

$$\theta(z) = \sum_{k=-\infty}^{\infty} e^{\pi i k^2 z} \qquad (z \in \mathfrak{H})\,, \tag{22}$$

where $\mathfrak{H}$ is the upper half-plane $\{z \in \mathbf{C} \mid Im\, z > 0\}$.

We define

$$F_m(z) = \frac{(-1)^m (2m-1)\,!}{2^{2m+1}\, \pi^{2m}}\, G_{2m}(2z)\, \theta(z) \quad (z \in \mathfrak{H})\,. \tag{23}$$

Clearly $F_m(z+2) = F_m(z)$, so $F_m(z)$ has a Fourier expansion. From (22) and the Fourier expansion of G_{2m} (eqs. 1 (19)-1 (21)), together with eq. (3) and the functional equation of $\zeta(s)$, we obtain

$$\begin{aligned} F_m(z) &= \left(\frac{(-1)^m (2m-1)\,!}{2^{2m}\, \pi^{2m}}\, \zeta(2m) + \sum_{a=1}^{\infty} \sigma_{2m-1}(a)\, e^{4\pi i a z}\right) \\ &\quad \times \sum_{k=-\infty}^{\infty} e^{\pi i k^2 z} \\ &= \sum_{a=0}^{\infty} \sigma_{2m-1}(z)\, e^{4\pi i a z} \sum_k e^{\pi i k^2 z} \\ &= \sum_{n=0}^{\infty} e_{2m-1}(n)\, e^{\pi i n z}\,. \end{aligned} \tag{24}$$

Thus the numbers $e_{2m-1}(n)$ are precisely the Fourier coefficients of $F_m(z)$. By Cauchy's theorem, therefore,

$$e_{2m-1}(n) = \frac{1}{2} \int_{i\varepsilon}^{2+i\varepsilon} e^{-\pi i n z} F_m(z)\, dz \tag{25}$$

for any $\varepsilon > 0$.

The idea of the Hardy-Littlewood method is to replace the integrand in the neighbourhood of each rational point of the interval [0, 2] by an elementary function, integrate this function, and then sum up the contributions obtained in this way from all rational points; this sum, the so-called "singular series," should then be an approximation to the integral. To apply this to (25), we first use the transformation laws of the theta and Eisenstein series under modular transformations to obtain

$$\theta\left(\frac{a}{c}+iy\right) = \lambda(a, c)(cy)^{-1/2} + 0(y^{-1/2}e^{-\pi/4c^2y}), \tag{26}$$

$$G_{2m}\left(\frac{a}{c}+iy\right) = 2(-1)^m \zeta(2m)(cy)^{-2m} + 0(y^{-2m}e^{-\pi/c^2y}) \tag{27}$$

as $y \to 0$ with $Re(y) > 0$, where $\frac{a}{c}$ is a rational number in lowest terms.

Therefore

$$F_m\left(\frac{a}{c}+iy\right) = \frac{(2m-1)!}{2^{4m}\pi^{2m}} \zeta(2m) \frac{(c, 2)^{2m}}{c^{2m+1/2}} \lambda(a, c)\, y^{-2m-1/2}$$
$$+ 0(y^{-2m-1/2}e^{-\pi/4c^2y}) \tag{28}$$

as $y \to 0$, where a and c are relatively prime and $(c, 2)$ is the greatest common divisor of c and 2. To obtain the contribution from the rational point a/c to the singular series, therefore, we replace F_m by the first member of (28) and integrate over y. Since

$$\frac{1}{2} \int_{-i\infty+\varepsilon}^{i\infty+\varepsilon} e^{-\pi i n(iy+a/c)}\, y^{-2m-1/2}\, dy$$
$$= \pi^{2m+1/2} n^{2m-1/2} e^{-\pi i n a/c} / \Gamma(2m+1/2) \tag{29}$$

(this is just the standard integral representation for $1/\Gamma(s)$), we obtain as the contribution from a/c

$$C(m)\, n^{2m-1/2} \frac{(c, 2)^{2m}}{c^{2m+1/2}} \lambda(a, c)\, e^{-\pi i n a/c} \tag{30}$$

with

$$C(m) = \frac{\pi^{1/2}(2m-1)!}{2^{4m}\,\Gamma(2m+1/2)}\,\zeta(2m)\,. \tag{31}$$

Summing this over all rational points $\frac{a}{c} \in [0, 2)$, we obtain the following formula for the singular series:

$$\bar{e}_{2m-1}(n) = C(m)\,n^{2m-1/2} \sum_{c=1}^{\infty} \frac{(c, 2)^{2m}}{c^{2m}}\,\gamma_c(n) \tag{32}$$

$$= 2\,C(m)\,n^{2m-1/2}\,E_n(2m)\,, \tag{33}$$

where $E_n(s)$ is the Dirichlet series of Theorem 3.

We wish to estimate the difference between $e_{2m-1}(n)$ and $\bar{e}_{2m-1}(n)$. To do this, we define a function having the same behaviour in the neighbourhood of each rational point $\frac{a}{c}$ as that described by the leading term of (28):

$$\bar{F}_m(z) = \frac{(2m-1)!}{2^{4m}\,\pi^{2m}}\,\zeta(2m)$$

$$\times \sum_{c=1}^{\infty} \frac{(c, 2)^{2m}}{c^{2m-1/2}} \sum_{a=-\infty}^{\infty} \lambda(a, c) \left(\frac{z - a/c}{i}\right)^{-2m-1/2}. \tag{34}$$

The series is convergent for $z \in \mathfrak{H}$, and

$$F_m(z) - \bar{F}_m(z) = 0\,(y^{-2m-1/2}\,e^{-\pi/4c^2y}) \tag{35}$$

for $z = \frac{a}{c} + iy\,,\ y \to 0$. On the other hand, $\bar{F}_m(z)$ is evidently periodic with period 2, and one easily finds (using the Cauchy integral for the Fourier coefficients and the contour integral (29)) that its Fourier expansion is

$$\bar{F}_m(z) = \sum_{n=1}^{\infty} \bar{e}_{2m-1}(n)\,e^{\pi i n z} \tag{36}$$

with $\bar{e}_{2m-1}(n)$ given by (32). The analysis given by Hardy [2] now permits us to deduce from (35) that

$$e_{2m-1}(n) - \bar{e}_{2m-1}(n) = 0\,(n^{m+1/4}) \tag{37}$$

as $n \to \infty$. We will not reproduce this analysis here, since our main interest is not in a rigorous proof of (6) with error term (in any case, as pointed out above, this error term is not best possible) but in the evaluation of the singular series obtained in the Hardy-Littlewood approach. To see that (37) and (6) are the same, we use equation (33) and Theorem 3 to get

$$\bar{e}_{2m-1}(f^2D) = 2C(m)f^{4m-1}D^{2m-1/2}\frac{L(2m,\chi)}{\zeta(4m)}\sum_{ac\mid f}\frac{\mu(a)\chi(a)}{c^{4m-1}a^{2m}}$$

$$= \frac{\zeta_K(1-2m)}{2\zeta(1-4m)}T^{\chi}_{2m}(f), \tag{38}$$

where in the last line we have used (7) and the functional equations of ζ and ζ_K.

It remains to prove Theorems 2 and 3.

Proof of Theorem 2: We first suppose c is odd. Then the standard Gauss sum

$$\tau_c(n) = \sum_{b=1}^{c}\left(\frac{b}{c}\right)e^{2\pi inb/c} \tag{39}$$

is related to $\gamma_c(n)$ by

$$\gamma_c(n) = \sum_{\substack{a=1\\ a\ \text{even}}}^{2c} c^{-1/2}\, i^{\frac{1-c}{2}}\left(\frac{a}{c}\right)e^{-\pi ina/c}$$

$$= c^{-1/2}\, i^{(1-c)/2}\left(\frac{-2}{c}\right)\tau_c(n), \tag{40}$$

as one sees by setting $a = 2b$. If c is square-free, then the value of (39) is well known to be

$$\tau_c(n) = \begin{cases}\left(\frac{n}{c}\right)\sqrt{c} & \text{if } c \equiv 1 \pmod 4,\\[2ex] \left(\frac{n}{c}\right)i\sqrt{c} & \text{if } c \equiv 3 \pmod 4,\end{cases} \tag{41}$$

or

$$\tau_c(n) = i^{\frac{c-1}{2}}\left(\frac{-2n}{c}\right)c^{1/2}\quad (c \text{ square-free}) \tag{42}$$

Therefore $\gamma_c(n) = \left(\frac{n}{c}\right)$ if c is square-free, in agreement with (16) (since in this case $d = 1$, $l = c$). Now let $c = ld^2$ with l square-free. Then

$$\left(\frac{b}{c}\right) = \left(\frac{b}{l}\right)\left(\frac{b}{d}\right)^2 = \begin{cases} \left(\frac{b}{l}\right) & \text{if } (b, d) = 1\,, \\ 0 & \text{if } (b, d) > 1 \end{cases}$$

$$= \left(\frac{b}{l}\right) \sum_{\substack{j \mid b \\ j \mid d}} \mu(j)\,,$$

where $\mu(j)$ is the Möbius function, so

$$\tau_c(n) = \sum_{j \mid d} \mu(j) \sum_{\substack{b=1 \\ j \mid b}}^{c} \left(\frac{b}{l}\right) e^{2\pi i n b / c}$$

$$= \sum_{j \mid d} \mu(j) \left(\frac{j}{l}\right) \sum_{k=1}^{c/j} \left(\frac{k}{l}\right) e^{2\pi i n k j / c}\,, \tag{43}$$

where we have written $b = jk$. Since $\left(\frac{k}{l}\right)$ only depends on $k \pmod{l}$, the inner sum in (43) equals

$$\sum_{r=1}^{l} \left(\frac{r}{l}\right) \sum_{m=1}^{c/jl} e^{2\pi i n (r + ml) j / c} = \begin{cases} 0 & \text{if } \frac{c}{jl} \nmid n\,, \\ \frac{c}{jl}\, \tau_l\left(\frac{njl}{c}\right) & \text{if } \frac{c}{jl} \mid n\,. \end{cases} \tag{44}$$

Write t for d/j, so $\frac{c}{jl} = dt$. Then, substituting (44) into (43), we find that $\tau_c(n) = 0$ if $d \nmid n$, while if $d \mid n$

$$\tau_c(n) = \sum_{\substack{t \mid d \\ dt \mid n}} \mu\left(\frac{d}{t}\right)\left(\frac{d/t}{l}\right) dt\, \tau_l\left(\frac{n}{dt}\right)$$

Since l is square-free, we can now use (42) to get

$$\tau_c(n) = d l^{1/2} \left(\frac{-2}{l}\right) i^{(l-1)/2} \sum_{t \mid \left(d, \frac{d}{n}\right)} t \mu\left(\frac{d}{t}\right)\left(\frac{d/t}{l}\right)\left(\frac{n/dt}{l}\right). \tag{45}$$

The factor preceding the sum is precisely $c^{1/2}\left(\frac{-2}{c}\right) i^{(c-1)/2}$, since $c = ld^2 \equiv l \pmod{8}$, so combining (45) and (40) yields precisely equation (16).

Now suppose that c is even, $c = 2^r c_1$ ($r \geqslant 1$, c_1 odd). For a odd, we have

$$\lambda(a, c) = i^{a/2}\left(\frac{c}{a}\right) = \left[i^{a/2}\left(\frac{2}{a}\right)^r\left(\frac{-1}{c_1}\right)^{(a-1)/2}\right]\left(\frac{a}{c_1}\right), \tag{46}$$

where we have used the law of quadratic reciprocity. The factor in square brackets has period 8 and the factor $\left(\frac{a}{c_1}\right)$ has period c_1, so

$$\lambda(a + 8c_1, c) = \lambda(a, c). \tag{47}$$

It follows easily that $\gamma_c(n)$ is 0 unless $e^{-8\pi i n c_1/c}$ equals 1, i.e. unless 2^{r-2} divides n (this condition is empty if $r = 1$). Write

$$n = 2^{r-2}\,\nu \tag{48}$$

with ν an integer. Then

$$\gamma_c(n) = \frac{2^{r-2}}{\sqrt{c}} \sum_{\substack{a=1 \\ a \text{ odd}}}^{8c_1} \lambda(a, c)\, e^{-\pi i \nu a/4c_1}. \tag{49}$$

Now write

$$a = k\,{c_1}^2 + 8jy \tag{50}$$

where

$$8y \equiv 1 \pmod{c_1} \tag{51}$$

(e.g. $y = (1 - c_1^2)/8$). Then $a \equiv j \pmod{c_1}$ and $a \equiv k \pmod 8$, so a runs over all odd residue classes (mod $8c_1$) when j runs over the values 1, 2, ..., c_1 and k over the values 1, 3, 5, 7. Therefore (46) and (49) give

$$\gamma_c(n) = \frac{2^{r-2}}{\sqrt{c}} \sum_{j=1}^{c_1} \left(\frac{j}{c_1}\right) e^{-2\pi i \nu y j/c_1}$$
$$\times \sum_{\substack{k=1 \\ k \text{ odd}}}^{8} i^{k/2}\left(\frac{2}{k}\right)^r \left(\frac{-1}{c_1}\right)^{(k-1)/2} e^{-\pi i \nu c_1 k/4}. \tag{52}$$

The first sum is $\tau_{c_1}(-\nu y)$, and by virtue of (51), (48) and (40).

$$\tau_{c_1}(-\nu y) = \left(\frac{-2}{c_1}\right)\left(\frac{2}{c_1}\right)^r \tau_{c_1}(n) = \sqrt{c_1}\left(\frac{2}{c_1}\right)^r i^{\frac{c_1-1}{2}}\, \gamma_{c_1}(n). \tag{53}$$

The second sum in (52) is

$$i^{1/2}\, e^{-\pi i \nu c_1/4} + (-1)^r \left(\frac{-1}{c_1}\right) i^{3/2}\, e^{-3\pi i \nu c_1/4}$$

$$+ (-1)^r i^{5/2} e^{-5\pi i v c_1/4} + \left(\frac{-1}{c_1}\right) i^{7/2} e^{-7\pi i v c_1/4}$$

$$= i^{1/2} e^{-\pi i v c_1/4} \left(1 - (-1)^{r+v}\right) \left(1 + i(-1)^r \left(\frac{-1}{c_1}\right) e^{-\pi i v c_1/2}\right).$$

Putting this all into (52), we obtain

$$\gamma_c(n) = 2^{\frac{r}{2}-2} \left(\frac{2}{c_1}\right)^r i^{\frac{c_1}{2}} e^{-\pi i v c_1/4} \left(1 - (-1)^{r+v}\right)$$

$$\times \left(1 + (-1)^r i^{c_1(1-v)}\right) \gamma_{c_1}(n). \tag{54}$$

Clearly this is 0 if $r \equiv v \pmod 2$, while if $v \equiv r - 1 \pmod 2$ we obtain

$$\gamma_c(n) = 2^{r/2} \left(\frac{2}{c_1}\right)^r \cos \frac{\pi c_1 (v-1)}{4} \gamma_{c_1}(n). \tag{55}$$

If r is even, therefore, v must be odd, and then the cosine in (55) is 0 if $v \equiv 3 \pmod 4$ and $(-1)^{(v-1)/4}$ if $v \equiv 1 \pmod 4$. Thus for r even, $\gamma_c(n)$ is 0 unless $n = 2^{r-2} m$ with $m \equiv 1 \pmod 4$ and is then $2^{r/2}(-1)^{(m-1)/4} \times \gamma_{c_1}(n)$. If r is odd, then v is even, say $v = 2m$, and then the cosine in (55) $= (-1)^{m(m-1)/2} (2/c_1)/\sqrt{2}$. Thus for r odd, $\gamma_c(n)$ is 0 unless $n = 2^{r-1} m$ and is then $2^{(r-1)/2} (-1)^{m(m-1)/2} \gamma_{c_1(n)}$. This proves equation (18).

Proof of Theorem 3. According to eq. (17), we can write

$$E_n(s) = E_n^{\text{odd}}(s) R_n(s), \tag{56}$$

with

$$E_n^{\text{odd}}(s) = \sum_{\substack{c=1 \\ c \text{ odd}}}^{\infty} \frac{\gamma_c(n)}{c^s} \tag{57}$$

and

$$R_n(s) = \frac{1}{2} + \frac{1}{2} \sum_{r=1}^{\infty} \frac{Q_r(n)}{(2^{r-1})^s}. \tag{58}$$

We first evaluate (57). Substituting (16) gives

$$E_n^{\text{odd}}(s) = \sum_{\substack{d|n \\ d \text{ odd}}} \sum_{\substack{l=1 \\ l \text{ odd} \\ l \text{ square-free}}}^{\infty} \frac{\gamma_{ld^2}(n)}{l^s d^{2s}}$$

$$= \sum_{\substack{d|n \\ d \text{ odd}}} d^{-2s} \sum_{t | \left(d, \frac{n}{d}\right)} t\mu\left(\frac{d}{t}\right) \sum_{\substack{l \text{ odd} \\ l \text{ square-free}}} \left(\frac{n/t^2}{l}\right) l^{-s}. \tag{59}$$

Now let r^2 be the largest odd square dividing n, and write $n = Nr^2$. Then $t \mid r$, and N and D differ by an even power of 2, so for any odd l

$$\left(\frac{n/t^2}{l}\right) = \left(\frac{r/t}{l}\right)^2 \left(\frac{D}{l}\right) = \left(\frac{r/t}{l}\right)^2 \chi(l),$$

where χ is the character of $K = \mathbf{Q}(\sqrt{n})$. Also $\left(\frac{r/t}{l}\right)^2$ is 1 or 0 depending whether l is or is not relatively prime to r/t. Therefore (59) can be rewritten

$$E_n^{\mathrm{odd}}(s) = \sum_{\substack{d \mid n \\ d \text{ odd}}} d^{-2s} \sum_{t \mid \left(d, \frac{n}{d}\right)} t\mu\left(\frac{d}{t}\right) \prod_{p \nmid \frac{2r}{t}} (1 + \chi(p) p^{-s}), \tag{60}$$

where the final product extends over primes p not dividing the even integer $2r/t$. Let $u = \frac{r}{t}$, $e = \frac{d}{t}$ then

$$E_n^{\mathrm{odd}}(s) = \sum_{u \mid r} (u/r)^{2s-1} \sum_{\substack{e \mid Nu2 \\ e \text{ odd}}} \frac{\mu(e)}{e^{2s}} \prod_{p \nmid 2u} (1 + \chi(p) p^{-s})$$

$$= \prod_{p \neq 2} \left(1 + \frac{\chi(p)}{p^s}\right) \sum_{u \mid r} \frac{u^{2s-1}}{r^{2s-1}} \prod_{\substack{p \mid Nu2 \\ p \neq 2}} (1 - p^{-2s}) \prod_{p \mid u} (1 + \chi(p) p^{-s})^{-1}$$

$$= \prod_{p \neq 2} \left(1 + \frac{\chi(p)}{p^s}\right) \prod_{\substack{p \mid N \\ p \neq 2}} \left(1 - \frac{1}{p^{2s}}\right) \sum_{u \mid r} \frac{u^{2s-1}}{r^{2s-1}} \prod_{p \mid u} \left(1 - \frac{\chi(p)}{p^s}\right)$$

$$= \prod_{p \neq 2} \frac{1 - p^{-2s}}{1 - \chi(p) p^{-s}} r^{1-2s} T_s^{\chi}(r). \tag{61}$$

We now evaluate the factor (58) of $E_n(s)$ corresponding to the prime 2. Comparing (61) and (20), (21), we see that it remains to prove

$$R_n(s) = \begin{cases} 0 & \text{if } n \equiv 2, 3 \pmod 4, \\ \dfrac{1 - 2^{-2s}}{1 - \chi(2) 2^{-s}} 2^{q(1-2s)} T_s^{\chi}(2^q) & \text{if } n = f^2 D, \end{cases} \tag{62}$$

where in the latter case we have set $f = 2^q r$, r odd.

The first line of (62) follows immediately from (18), since we see that

$$n \equiv 2, 3 \pmod 4 \Rightarrow Q_1(n) = -1,\ Q_r(n) = 0\ (r > 1). \tag{63}$$

We thus suppose $n = f^2 D$, $f = 2^q r$, r odd. We distinguish two cases, according to the parity of D:

Case 1. $D \equiv 0 \pmod 4$, $\chi(2) = 0$. Then either $D = 8d$ with d odd or $D = 4d$ with $d \equiv 3 \pmod 4$. In either case, we deduce easily from (18) that $Q_r(n) = 0$ if r is even or if r is odd and greater than $2q + 3$, that $Q_r(n) = 2^{(r-1)/2}$ if r is odd and less than $2q + 3$, and that $Q_{2q+3}(n) = -2^{q+1}$. Therefore

$$R_n(s) = \frac{1}{2}\left[1 + \sum_{\substack{r=1 \\ r \text{ odd}}}^{2q+1} \frac{2^{(r-1)/2}}{2^{(r-1)s}} - \frac{2^{q+1}}{2^{(q+1)s}}\right]$$

$$= \frac{1}{2}\left[1 + 1 + x^2 + x^4 + \ldots + x^{2q} - x^{2q+2}\right]$$

$$= (1 - x^2/2)(1 + x^2 + \ldots + x^{2q})$$

$$= (1 - 2^{-2s})\, 2^{-q(2s-1)} (1 + 2^{2s-1} + \ldots + 2^{q(2s-1)})$$

$$= (1 - 2^{-2s})\, 2^{-q(2s-1)}\, T_s^{\chi}(2^q)\,,$$

in agreement with (62); in this calculation we have set $x = 2^{-s+\frac{1}{2}}$ for convenience.

Case 2. $D \equiv 1 \pmod 4$, $\chi(2) = (-1)^{(D-1)/4}$. In this case, equation (18) tells us that $Q_r(n) = 2^{(r-1)/2}$ if r is odd and $1 \leqslant r \leqslant 2q + 1$, that $Q_{2q+2}(n) = 2^{q+1}\chi(2)$, and that $Q_r(n) = 0$ for all other values of r. Therefore

$$R_n(s) = \frac{1}{2}\left[1 + \sum_{\substack{r=1 \\ r \text{ odd}}}^{2q+1} \frac{2^{(r-1)/2}}{2^{(r-1)s}} + \frac{2^{q+1}\chi(2)}{2^{(2q+1)s}}\right]$$

$$= \frac{1}{2}\left[1 + 1 + x^2 + x^4 + \ldots + x^{2q} + \chi(2)\sqrt{2}\,x^{2q+1}\right]$$

$$= \left[1 + \frac{\chi(2)}{\sqrt{2}}x\right]\left[1 + x^2 + \ldots + x^{2q} - \frac{\chi(2)}{\sqrt{2}}(x + x^3 + \ldots + x^{2q-1})\right]$$

$$= 2^{-q(2s-1)}\left[1 + \frac{\chi(2)}{2^s}\right]\left[1 + 2^{2s-1} + \ldots + 2^{q(2s-1)} - \frac{\chi(2)}{2^s}(2^{2s-1} + 2^{2(2s-1)} + \ldots + 2^{q(2s-1)})\right]$$

$$= 2^{-q(2s-1)}\,\frac{1 - 2^{-2s}}{1 - \chi(2)\,2^{-s}}\, T_s^{\chi}(2^q)\,.$$

This proves (62) in this case also, and completes the evaluation of $E_n(s)$.

§5. Congruences for the Hecke-Eisenstein series

For K a totally real number field and $m \geqslant 1$, define

$$\bar{G}_{2m}^{K}(z) = \{(2\pi i)^{2m}/(2m-1)!\}^{-n} D^{2m-1/2} G_{2m}^{K}(z), \tag{1}$$

where $n = [K:\mathbf{Q}]$ and $G_{2m}^{K}(z)$ (as in §1) is the restriction to the diagonal of the Hecke-Eisenstein series of weight $2m$. Then $\bar{G}_{2m}^{K}$ is a modular form of weight $h = 2mn$ whose Fourier expansion (cf. eqs. (22), (23), (24) and (6) of §1) is

$$\bar{G}_{2m}^{K}(z) = 2^{-n}\zeta_K(1-2m) + \sum_{l=1}^{\infty} s_l^K(2m)\, e^{2\pi i l z} \tag{2}$$

with $s_l^K(2m) \in \mathbf{Z}$.

In the space $\mathfrak{M}_h$ of all modular forms of weight h, let

$$\mathfrak{M}_h^{\mathbf{Z}} = \{f \in \mathfrak{M}_h \mid f(z) = \sum_{n=0}^{\infty} a_n e^{2\pi i n z}, \ a_n \in \mathbf{Z} \text{ for } n \geqslant 1\}$$

be the set of modular forms whose Fourier coefficients, apart from the constant term, are all integral. Then $\mathfrak{M}_h^{\mathbf{Z}}$ is a free $\mathbf{Z}$-module of rank $r = \dim_{\mathbf{C}} \mathfrak{M}_h$ and $\mathfrak{M}_h = \mathfrak{M}_h^{\mathbf{Z}} \otimes_{\mathbf{Z}} \mathbf{C}$. Write

$$c: \mathfrak{M}_h^{\mathbf{Z}} \to \mathbf{C}$$

for the map sending a modular form $f(z) = \Sigma\, a_n e^{2\pi i n z}$ to its constant term a_0. Then

$$c(\mathfrak{M}_h^{\mathbf{Z}}) = \frac{M_h}{N_h}\mathbf{Z}$$

for some coprime integers M_h and N_h, and N_h is then a universal bound for the denominators of the constant terms of forms in $\mathfrak{M}_h^{\mathbf{Z}}$ and in particular of $\bar{G}_{2m}^{K}$, i.e.

$$N_h 2^{-n}\zeta_K(1-2m) \in \mathbf{Z}. \tag{3}$$

This is the essence of Siegel's theorem as discussed in §1.

But we know that (3) is not the best possible bound for the denominator of $\zeta_K(1-2m)$ (cf. the remarks at the end of §3), and this means that the modular forms $\bar{G}_{2m}^{K}$ must be contained in some *smaller* lattice than

$\mathfrak{M}_h^{\mathbf{Z}}$. For example, if K is a real quadratic field, then Serre's bound for the denominator of $\frac{1}{4}\zeta_K(1-2m)$, at least for K not in the set

$$\{\mathbf{Q}(\sqrt{2})\} \cup \{\mathbf{Q}(\sqrt{p}) \mid p \text{ prime}, \quad (p-1) \mid 4m, \quad (p-1) \nmid 2m\}, \qquad (4)$$

is the number $j(m)$ defined in §3, eq. (40), and this is always smaller than $N_h = N_{4m}$ (for $m = 1, 2, 3, 4, 5$ the values of N_{4m} are $2^4.3.5$, $2^5.3.5$, $2^4.3^2.5.7.13$, $2^6.3.5.17$ and $2^4.3.5^2.11$, whereas those of $j(m)$ are $2^3.3$, $2^4.3.5$, $2^3.3^2.7$, $2^5.3.5$ and $2^3.3.11$). Therefore, if K is not one of the finitely many exceptional fields (4), the modular form $\bar{G}_{2m}^K$ lies in the proper sublattice

$$\mathfrak{M}_{4m}^{Se} = \mathfrak{M}_{4m}^{\mathbf{Z}} \cap c^{-1}\left(\frac{1}{j(m)}\mathbf{Z}\right) \qquad (5)$$

of $\mathfrak{M}_{4m}^{\mathbf{Z}}$. We want to describe some numerical evidence that, although $j(m)$ is the best possible bound for the denominator of $\frac{1}{4}\zeta_K(1-2m)$, the modular forms $\bar{G}_{2m}^K$ are contained in a much smaller sublattice than (5). This means that the coefficients $s_l^K(2m)$ satisfy congruences (modulo certain powers of certain primes) above and beyond those required to obtain the correct bound for the denominator of ζ.

For $m = 1$ or $m = 2$, $\mathfrak{M}_{4m}$ is one-dimensional, so a modular form is completely determined by its constant term and (5) is best possible. Consider $m = 3$. A basis for $\mathfrak{M}_{12}$ is given by Q and R^2, where

$$Q = E_4(z) = 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n)\, e^{2\pi i n z}$$

$$R = E_6(z) = 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n)\, e^{2\pi i n z}$$

(Ramanujan's notation). The lattice $\mathfrak{M}_{12}^{\mathbf{Z}}$ has the basis $\frac{1}{720}Q^3$, $\frac{1}{156}\left(\frac{1}{720}Q^3 + \frac{1}{1008}R^2\right)$. We conjecture, however, that for all real quadratic fields K different from $\mathbf{Q}(\sqrt{2})$, $\mathbf{Q}(\sqrt{5})$ and $\mathbf{Q}(\sqrt{13})$, the modular form $\bar{G}_6^K$ lies in the sublattice generated by $\frac{1}{24}Q^3$ and $\frac{5}{504}R^2$ i.e. that if we write

$$\bar{G}_6^K = \frac{x}{24}Q^3 + \frac{5y}{504}R^2 \quad (x, y \in \mathbf{Q}),$$

TABLE 5

The modular form $G_6^K(z)$

$K = \mathbf{Q}(\sqrt{D})$, D = discriminant of K

$$\bar{G}_6^K(z) = \frac{225}{64\pi^{12}} D^{11/2} G_6^K(z)$$

$$= \frac{1}{4}\zeta_K(-5) + \sum_{l=1}^{\infty} s_l^K(6)\, q^l \quad (q = e^{2\pi i z})$$

$$E_4(z) = \frac{45}{\pi^4} G_4(z) = 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n)\, q^n$$

$$E_6(z) = \frac{945}{2\pi^6} G_6(z) = 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n)\, q^n$$

$$\bar{G}_6^K(z) = \frac{x}{24} E_4(z)^3 + \frac{5y}{504} E_6(z)^2$$

D	x	y
5	2/5	1
8	11/2	13
12	51	122
13	1018/13	2417/13
17	352	838
21	1092	2602
24	2313	5502
28	5404	12872
29	6438	15327
33	13536	32226
37	24650	58681
40	38437	91526
41	44608	106216
44	64757	154166

then the coefficients x and y will be integral for all quadratic fields K except the three mentioned. Some numerical evidence for this is presented in Table 5 (x and y were calculated for much larger discriminants and were always integers). Similar data for $m=4$ and $m=5$ leads to the conjectures

$$\bar{G}_8^K \in \frac{7Q^4}{480}\mathbf{Z} + \frac{5QR^2}{12}\mathbf{Z} \quad (K \neq \mathbf{Q}(\sqrt{2}),\ \mathbf{Q}(\sqrt{17})\ ,\ ..$$

$$\bar{G}_{10}^K \in \frac{147Q^5}{8}\mathbf{Z} + \frac{5Q^2R^2}{264}\mathbf{Z} \quad (K \neq \mathbf{Q}(\sqrt{2}),\ \mathbf{Q}(\sqrt{5})).$$

These assertions imply highly non-trivial congruences for the coefficients $s_l^K(2m)$ of the Hecke-Eisenstein series, since (for example) the lattice generated by $\frac{147}{8}Q^5$ and $\frac{5}{264}Q^2R^2$ has index 7,938,000 in $\mathfrak{M}_{20}^{\mathbf{Z}}$ (whereas $[\mathfrak{M}_{20}^{\mathbf{Z}} : \mathfrak{M}_{20}^{Se}]$ is only 50). This leads to the following

CONJECTURE. *For each* $m \geqslant 1$, *define the "Hecke-Eisenstein lattice"* $\mathfrak{M}_{4m}^{HE}$ *as the sublattice of* $\mathfrak{M}_{4m}^{\mathbf{Z}}$ *generated by the modular forms* $\bar{G}_{2m}^K$, *where* K *runs over all real quadratic fields not in the finite set* (4). *Then*

(i) $\mathfrak{M}_{4m}^{HE}$ *has finite index in* $\mathfrak{M}_{4m}^{\mathbf{Z}}$.

(ii) *If we replace* (4) *by any larger finite set in the definition of* $\mathfrak{M}_{4m}^{HE}$, *we obtain the same lattice* (in other words, the only fields which are exceptional with respect to the congruence properties of their Hecke-Eisenstein series are those for which the denominator of $\zeta_K(1-2m)$ is exceptionally large).

(iii) *For* $m \leqslant 5$, $\mathfrak{M}_{4m}^{HE}$ *is as given in* Table 6.

(iv) $\mathfrak{M}_{4m}^{HE}$ *has a basis consisting of monomials in* Q *and* R.

(v) *For* $m > 2$, *the primes dividing* $[\mathfrak{M}_{4m} : \mathfrak{M}_{4m}^{HE}]$ *are: all primes* $\leqslant 2m$ *and* $4m+1$ *(if the latter is prime).*

It would be of interest to have numerical data on $\bar{G}_{2m}^K$ for $m > 5$ and for $[K:\mathbf{Q}] > 2$, especially to test the somewhat rash conjecture (iv). Particularly interesting would be to fix a prime p and study the behaviour at p of the sublattice $\mathfrak{M}_{4m}^{HE}$ for varying m, since this could give information about the p-adic analogue of the zeta-function of K.

Table 6

The "Hecke-Eisenstein lattice" for $m \leqslant 5$

(In the table, $Q = E_4(z)$, $R = E_6(z)$. The data for $m = 3, 4, 5$ is conjectural only.)

m	Basis for $\mathfrak{M}_{4m}^{\mathbf{Z}}$	Basis for $\mathfrak{M}_{4m}^{HE}$	$[\mathfrak{M}_{4m}^{\mathbf{Z}} : \mathfrak{M}_{4m}^{HE}]$	Exceptional discriminants
1	$\frac{1}{240} Q$	$\frac{1}{24} Q$	$2.5 = 10$	5, 8
2	$\frac{1}{480} Q^2$	$\frac{1}{240} Q^2$	2	8
3	$\frac{1}{720} Q^3$, $\frac{1}{156}\left(\frac{Q^3}{720} + \frac{R^2}{1008}\right)$	$\frac{1}{24} Q^3$, $\frac{5}{504} R^2$	$2^4.3^2.5^2.13 = 46800$	5, 8, 13
4	$\frac{1}{960} Q^4$, $\frac{1}{153}\left(\frac{Q^4}{240} + \frac{QR^2}{192}\right)$	$\frac{7}{480} Q^4$, $\frac{5}{12} QR^2$	$2^5.3^2.5.7.17 = 171360$	8, 17
5	$\frac{1}{1200} Q^5$, $\frac{1}{36}\left(\frac{Q^5}{1200} + \frac{Q^2R^2}{528}\right)$	$\frac{147}{8} Q^5$, $\frac{5}{264} Q^2R^2$	$2^4.3^4.5^3.7^2$ 7938000	5, 8

Afterword

The original version of this paper was written three years ago. To bring it up to date, we must comment on two developments which have occurred in the intervening time.

1. The conjecture of Serre quoted at the end of Section 3 is now (almost) a theorem. In the original paper [6], Serre proved the partial result that,

for any totally real field K and positive integer n, $\prod_{m=1}^{n} w_m(K)\,\zeta_K(1-2m)$ is an integer (the product occurs when one calculates the "Euler characteristic" of the discrete group $Sp_{2n}\mathcal{O}$, $\mathcal{O}$ = ring of integers of K). For the case of *abelian* totally real fields (and thus in particular the case of quadratic fields), the conjecture is much easier, since it can be reduced to the evaluation of L-series, and it was proved independently by several people (e.g. J. Fresnel, "Valeurs des fonctions zêta aux entiers negatifs", *Séminaire de Théorie de Nombres*, 1970-1971, Bordeaux). In [7], Serre obtained better bounds than 3 (25), still by using Siegel's idea, but studying in more detail the p-adic behaviour of the coefficients $s_l^K(2m)$ of the Hecke-Eisenstein series. Finally, Deligne, using p-adic modular forms in several variables and a strengthened version of Mumford's results on compactifications of modular schemes (of which the details have apparently not yet been checked completely), proved Serre's conjecture for arbitrary totally real fields modulo the question of the irreducibility of a certain p-adic representation, and this question was resolved affirmatively by K. Ribet.

Related to the question of the denominator of $\zeta_K(1-2m)$ is the question of its exact fractional part (resolved for $K = \mathbf{Q}$ by the theorem of von Staudt). In connection with his work on the Hilbert modular group (*L'Enseignement Mathématique* (3-4) *19* (1973) 183-283). Hirzebruch found formulas for the fractional part of $\zeta_K(-1)$, K a real quadratic field, in terms of the class numbers of certain imaginary quadratic fields. This formula has been generalized to arbitrary totally real fields by Brown ("Euler characteristics of discrete groups and G-spaces", *Inv. Math. 27* (1974), 229-264), using the methods of [6], and by Vignéras-Guého ("Partie fractionnaire de $\zeta_K(-1)$", *C. R. Acad. Sciences, Paris* (10) *279* (1974), 359-361, "Nombres de classes d'un ordre d'Eichler et valeur au point -1 de la fonction zêta d'un corps quadratique réel", *l'Ens. Math.*, *21* (1975) 69-105) using a formula of Eichler for class numbers of orders in totally definite quaternion fields.

2. The aim of Section 4, namely to explain without the use of modular forms in two variables Siegel's formula for $\zeta_K(1-2m)$, can now be achieved in another way, both simpler and more enlightening than the application of the circle method outlined in §4. In that section, we observed that the number

$$e_{2m-1}(n) = \sum_{0 \leq n-k^2 \equiv 0 \pmod 4} \sigma_{2m-1}\left(\frac{n-k^2}{4}\right)$$

is the coefficient of $e^{\pi i n z}$ in the Fourier expansion of a function $F_m(z)$ (eq. 4 (23)) which is up to a factor the product of the ordinary theta series $\theta(z)$ and the Eisenstein series $G_{2m}(2z)$. The function $F_m(2z)$ (at least if $m > 1$) is a modular form of weight $2m + \frac{1}{2}$ for $\Gamma_0(4)$ in the sense of Shimura's paper "Modular functions of half integral weight", (*Modular Functions of One Variable* I, Lecture Notes 320, Springer Verlag, Berlin/Heidelberg/New York 1973, pp. 57-74). In this paper, Shimura discusses how to set up for such forms a theory of Hecke operators with many of the usual properties but with the essential difference that there are now Hecke operators T_n only for n a perfect square. He also shows that the two Eisenstein series of weight $2m + \frac{1}{2}$ for $\Gamma_0(4)$ have n-th Fourier coefficients related to $\zeta_{\mathbf{Q}(\sqrt{n})}(1-2m)$. In fact, one can check that there is a linear combination of these two Eisenstein series whose n-th Fourier coefficient is precisely the number

$$\bar{e}_{2m-1}(n) = \begin{cases} 0 & \text{if } n \equiv 2 \text{ or } 3 \pmod 4, \\ \dfrac{\zeta_K(1-2m)}{2\zeta(1-4m)} T_{2m}^{\chi}(f) & \text{if } n = f^2 D,\ D = \text{discriminant} \\ & \text{of } K = \mathbf{Q}(\sqrt{n}),\ \chi = (\tfrac{D}{\cdot}) \end{cases}$$

which arose in our §4 as the sum of the singular series for $e_{2m-1}(n)$. The identities of Siegel expressing $\bar{e}_{2m-1}(n)$ as a linear combination of

$$e_{2m-1}(n),\ e_{2m-1}(4n),\ e_{2m-1}(9n),\ \ldots,\ e_{2m-1}(r^2 n)\ \left(r = \left[\frac{m}{3}\right] + 1\right)$$

can now be interpreted as saying that the modular form $\sum_{n=0}^{\infty} \bar{e}_{2m-1}(n)\, e^{2\pi i n z}$ of weight $2m + \frac{1}{2}$ can be expressed as a linear combination of the function $F_m(2z)$ and its images under the Hecke operators $T_4, T_9, \ldots, T_{r^2}$. These ideas have been worked out by Cohen in three papers,

Cohen, H. Sommes de carrés, fonctions L et formes modulaires. *C. R. Acad. Sci. Paris* (A) *277* (1973), 827-830.

—— Variations sur un thème de Siegel et Hecke. *To appear in Acta Arithm. 30* (1975).

—— Sums involving the values at negative integers of L-functions of quadratic characters. *Math. Annalen 217* (1975), 271-285,

especially the last, in which he studies an arithmetic function $H(r, N)$ which is related to our function by

$$H(2m, n) = \frac{2\zeta(1-4m)}{\zeta(1-2m)} \bar{e}_{2m-1}(n).$$

However, despite these new approaches to Siegel's formula, I have retained Section 4 because the calculations of the Gauss sums $\gamma_c(n)$ and of the Dirichlet series $\sum \gamma_c(n) c^{-s}$ (Theorems 2 and 3 of §4) are often useful to have (for example, the calculation of the Fourier coefficients of the Eisenstein series of weight $2m + \frac{1}{2}$, of which is not carried out in detail in Shimura's paper, depends on them) and also because the application of the circle method in the context of forms of half-integral weight seemed novel.

REFERENCES

[1] Hardy, G. H. On the representation of a number as the sum of any number of squares, and in particular of five or seven. *Proc. Nat. Acad. Sci. 4* (1918), pp. 189-193, (*Collected Papers of G. H. Hardy*, Vol. I, p. 340, Clarendon Press, Oxford 1966).

[2] —— On the representation of a number as the sum of any number of squares, and in particular of five. *Trans. AMS 21* (1920), pp. 255-284 (*Collected Papers*, Vol. I, p. 345).

[3] Hecke, E. Analytische Funktionen und algebraische Zahlen, Zweiter Teil. *Abh. Math. Sem. Hamb. Univ. 3* (1924), pp. 213-236 (No. 20 of *Mathematische Werke*, Vandenhoeck und Ruprecht, Göttingen 1959).

[4] Iwasawa, K. *Lectures on p-adic L-functions* Annals of Math. Studies No. 74, Princeton University Press, Princeton 1972.

[5] Klingen, H. Über die Werte der Dedekindschen Zetafunktion. *Math. Annalen 145* (1962), pp. 265-272.

[6] Serre, J. P. Cohomologie des groupes discrets. *Prospects in Mathematics*, Annals of Math. Studies No. 70, Princeton University Press 1971, pp. 77-170.

[7] —— Congruences et formes modulaires. *Séminaire Bourbaki, vol. 1971/1972*, Lecture Notes in Mathematics No. 317, Springer, Berlin, Heidelberg, New York 1973, exposé 416, p. 319.

[8] Siegel, C. L. Bernoullische Polynome und quadratische Zahlkörper. *Nachr. Akad. Wiss. Göttingen, Math.-Phys. Klasse 2* (1968), pp. 7-38.

[9] —— Berechnung von Zetafunktionen an ganzzahligen Stellen. *Nachr. Akad. Wiss. Göttingen, Math.-Phys. Klasse 10* (1969), pp. 87-102.

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