A Converse to Cauchy's Inequality

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Denote by $\mathcal{M}$ the set of monotone decreasing functions $f: [0, \infty) \to [0, 1]$ for which $I(f) = \int_0^\infty f(x) \, dx$ converges. For $f, g \in \mathcal{M}$ the scalar product $(f, g) = I(fg)$ converges, and the Cauchy-Schwarz inequality and the inequalities $0 \leq f(x) \leq 1$ imply the estimate

$$(f, g) \leq \min(I(f), I(g), (f, f)^{1/2}(g, g)^{1/2}) \quad (f, g \in \mathcal{M}). \quad (1)$$

An inequality for $(f, g)$ in terms of the same data but in the other direction, namely

$$(f, g) \geq \frac{(f, f)(g, g)}{\max(I(f), I(g))} \quad (f, g \in \mathcal{M}), \quad (2)$$

was proved in an earlier article with the same title (up to translation) by a trick involving a quadruple integral [2]. We give here a more general result with a much simpler proof.

**Theorem.** Let $f$ and $g$ be monotone decreasing nonnegative functions on $[0, \infty)$. Then

$$(f, g) \geq \frac{(f, F)(g, G)}{\max(I(F), I(G))} \quad (3)$$

for any integrable (but not necessarily monotone) functions $F, G: [0, \infty) \to [0, 1]$.

**Proof:** For all $x \geq 0$ we have

$$(f, F) = I(F)f(x) + \int_0^\infty [f(t) - f(x)]F(t) \, dt$$

$$\leq I(F)f(x) + \int_0^x [f(t) - f(x)] \, dt,$$

and hence, since $\int_0^x G(t) \, dt$ is bounded from above by both $x$ and $I(G)$,

$$(f, F)\int_0^x G(t) \, dt \leq I(F)xf(x) + I(G)\int_0^x [f(t) - f(x)] \, dt$$

$$\leq \max(I(F), I(G))\int_0^x f(t) \, dt.$$

Now multiply by $-dg(x)$ and integrate by parts from 0 to $\infty$. The left-hand side gives $(f, F)(g, G)$, the right-hand side gives $\max(I(F), I(G))(f, g)$, and the inequality remains true because the measure $-dg(x)$ is nonnegative.

**Remarks.** 1. Another proof of (3) can be obtained as follows. It is geometrically clear (and easily proved) that for $f$ monotone decreasing the largest value of $(f, F)$ as $F$ ranges over integrable functions $[0, \infty) \to [0, 1]$ with a given value of $I(F)$ is attained by taking $F$ to be "as far left as possible," i.e., equal to 1 for $0 \leq x \leq I(F)$ and to 0 otherwise. Therefore the maximum of $(f, F)(g, G)/A$ as $F$ and $G$ range
over functions \([0, \infty) \to [0, 1]\) with \(\max(I(F), I(G)) \leq A\) is equal to \(A^{-1} \int_0^A f(x) \, dx \cdot \int_0^A g(x) \, dx\), and this is \(\leq (f, g)\) because the average of the product of two decreasing functions on an interval is at least equal to the product of their averages.

2. A further generalization of (2) is the inequality

\[
W(fg) \geq \frac{W(fF)W(gG)}{\max(W(F), W(G))}
\]

valid for any positive linear functional \(W(f) = \int_0^\infty f(x)w(x) \, dx\) \((w(x) > 0)\), monotone decreasing functions \(f\) and \(g\), and functions \(F, G: [0, \infty) \to [0, 1]\) with \(W(F)\) and \(W(G)\) finite. To prove it, apply (3) to the functions \(f \circ \nu, g \circ \nu, F \circ \nu\) and \(G \circ \nu\), where \(\nu(x) = \int_0^x w(x') \, dx'\). The case \(F = f, G = g\) is the weighted generalization of (2) proved in [1].

3. As pointed out in [1], both bounds (1) and (2) are best possible in terms of the four parameters \(I(f), (f, f), I(g),\) and \((g, g)\). The bound (2) cannot be attained for generic values of these parameters but can be approached arbitrarily closely by taking \(f\) and \(g\) to be step functions with only two non-zero values (i.e. equal to 1 for \(x \leq x_0\), to \(C\) for \(x_0 < x \leq x_1\), and to 0 for \(x > x_1\), where \(0 < x_0 < x_1\) and \(0 < C < 1\)). Such functions with given values of \(I(f)\) and \((f, f)\) form a one-parameter family (the numbers \(x_0, x_1\) and \(C\) determine each other). If \(I(g) < I(f)\) and we let \(f\) move to the left \((x_0 \to 0)\) and \(g\) to the right \((C \to 0)\) in their respective families, then \(\int_0^\infty f(x)g(x) \, dx\) tends to \((f, f)(g, g)/I(f)\).

4. Monotone decreasing functions \(f: [0, \infty) \to [0, 1]\) can be interpreted as the integrals of probability measures \((f(x) = \int_x^\infty d\mu\) where \(d\mu\) is a nonnegative measure with integral 1). Hence (2) can be interpreted as a statement about correlations of statistical distributions. One such result, which was the original motivation for the inequality, is an estimate of the possible values of the “Gini coefficient” for a population consisting of two sub-populations, when the size, average income, and Gini coefficients of each of these is given [3]. (The Gini coefficient is a measure of the inequity of distributions of income in a large population which is used widely in mathematical economics.) Since the inequalities (2) and (3) are very general, they should have other applications, perhaps also in pure mathematics.

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REFERENCES


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