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# A Converse to Cauchy's Inequality

D. Zagier

Denote by $\mathfrak{M}$ the set of monotone decreasing functions $f\colon [0,\infty) \to [0,1]$ for which $I(f) = \int_0^\infty f(x)\,dx$ converges. For $f, g \in \mathfrak{M}$ the scalar product $(f,g) = I(fg)$ converges, and the Cauchy-Schwarz inequality and the inequalities $0 \le f(x) \le 1$ imply the estimate

$$(f,g) \le \min\bigl(I(f), I(g), (f,f)^{1/2}(g,g)^{1/2}\bigr) \qquad (f,g \in \mathfrak{M}). \tag{1}$$

An inequality for $(f,g)$ in terms of the same data but in the other direction, namely

$$(f,g) \ge \frac{(f,f)(g,g)}{\max(I(f), I(g))} \qquad (f,g \in \mathfrak{M}), \tag{2}$$

was proved in an earlier article with the same title (up to translation) by a trick involving a quadruple integral [2]. We give here a more general result with a much simpler proof.

**Theorem.** *Let $f$ and $g$ be monotone decreasing nonnegative functions on $[0,\infty)$. Then*

$$(f,g) \ge \frac{(f,F)(g,G)}{\max(I(F), I(G))} \tag{3}$$

*for any integrable (but not necessarily monotone) functions $F, G\colon [0,\infty) \to [0,1]$.*

*Proof:* For all $x \ge 0$ we have

$$\begin{aligned}(f,F) &= I(F)f(x) + \int_0^\infty [f(t) - f(x)]F(t)\,dt \\ &\le I(F)f(x) + \int_0^x [f(t) - f(x)]\,dt,\end{aligned}$$

and hence, since $\int_0^x G(t)\,dt$ is bounded from above by both $x$ and $I(G)$,

$$\begin{aligned}(f,F)\int_0^x G(t)\,dt &\le I(F)xf(x) + I(G)\int_0^x [f(t) - f(x)]\,dt \\ &\le \max(I(F), I(G))\int_0^x f(t)\,dt.\end{aligned}$$

Now multiply by $-dg(x)$ and integrate by parts from 0 to $\infty$. The left-hand side gives $(f,F)(g,G)$, the right-hand side gives $\max(I(F), I(G))(f,g)$, and the inequality remains true because the measure $-dg(x)$ is nonnegative. ■

*Remarks.* **1.** Another proof of (3) can be obtained as follows. It is geometrically clear (and easily proved) that for $f$ monotone decreasing the largest value of $(f,F)$ as $F$ ranges over integrable functions $[0,\infty) \to [0,1]$ with a given value of $I(F)$ is attained by taking $F$ to be "as far left as possible," i.e., equal to 1 for $0 \le x \le I(F)$ and to 0 otherwise. Therefore the maximum of $(f,F)(g,G)/A$ as $F$ and $G$ range over functions $[0, \infty) \to [0,1]$ with $\max(I(F), I(G)) \le A$ is equal to $A^{(-1)}\int_0^A f(x)\,dx \cdot \int_0^A g(x)\,dx$, and this is $\le (f,g)$ because the average of the product of two decreasing functions on an interval is at least equal to the product of their averages.

**2.** A further generalization of (2) is the inequality

$$W(fg) \ge \frac{W(fF)W(gG)}{\max(W(F), W(G))}$$

valid for any positive linear functional $W(f) = \int_0^\infty f(x)w(x)\,dx$ $(w(x) > 0)$, monotone decreasing functions $f$ and $g$, and functions $F, G$: $[0, \infty) \to [0,1]$ with $W(F)$ and $W(G)$ finite. To prove it, apply (3) to the functions $f \circ \nu$, $g \circ \nu$, $F \circ \nu$ and $G \circ \nu$, where $\nu(x) = \int_0^x w(x')\,dx'$. The case $F = f$, $G = g$ is the weighted generalization of (2) proved in **[1]**.

**3.** As pointed out in **[1]**, both bounds (1) and (2) are best possible in terms of the four parameters $I(f)$, $(f,f)$, $I(g)$, and $(g,g)$. The bound (2) cannot be attained for generic values of these parameters but can be approached arbitrarily closely by taking $f$ and $g$ to be step functions with only two non-zero values (i.e. equal to 1 for $x \le x_0$, to $C$ for $x_0 < x \le x_1$, and to 0 for $x > x_1$, where $0 < x_0 < x_1$ and $0 < C < 1$). Such functions with given values of $I(f)$ and $(f,f)$ form a one-parameter family (the numbers $x_0, x_1$ and $C$ determine each other). If $I(g) \le I(f)$ and we let $f$ move to the left $(x_0 \to 0)$ and $g$ to the right $(C \to 0)$ in their respective families, then $\int_0^\infty f(x)g(x)\,dx$ tends to $(f,f)(g,g)/I(f)$.

**4.** Monotone decreasing functions $f$: $[0, \infty) \to [0,1]$ can be interpreted as the integrals of probability measures ($f(x) = \int_x^\infty d\mu$ where $d\mu$ is a nonnegative measure with integral 1). Hence (2) can be interpreted as a statement about correlations of statistical distributions. One such result, which was the original motivation for the inequality, is an estimate of the possible values of the "Gini coefficient" for a population consisting of two sub-populations, when the size, average income, and Gini coefficients of each of these is given **[3]**. (The Gini coefficient is a measure of the inequity of distributions of income in a large population which is used widely in mathematical economics.) Since the inequalities (2) and (3) are very general, they should have other applications, perhaps also in pure mathematics.

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