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Asymptotics of classical spin networks

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A spin network is a cubic ribbon graph labeled by representations of SU(2). Spin networks are important in various areas of Mathematics (3–dimensional Quantum Topology), Physics (Angular Momentum, Classical and Quantum Gravity) and Chemistry (Atomic Spectroscopy). The evaluation of a spin network is an integer number. The main results of our paper are: (a) an existence theorem for the asymptotics of evaluations of arbitrary spin networks (using the theory of G–functions), (b) a rationality property of the generating series of all evaluations with a fixed underlying graph (using the combinatorics of the chromatic evaluation of a spin network), (c) rigorous effective computations of our results for some $6j$–symbols using the Wilf–Zeilberger theory and (d) a complete analysis of the regular Cube $12j$ spin network (including a nonrigorous guess of its Stokes constants), in the appendix.

57N10; 57M25

1 Introduction

1.1 Spin networks in mathematics, physics and chemistry

A (classical) *spin network* (Γ, γ) consists of a *cubic ribbon graph* Γ (ie, an abstract trivalent graph with a cyclic ordering of the edges at each vertex) and a coloring γ of its set of edges by natural numbers. According to Penrose, spin networks correspond to a diagrammatic description of tensors of representations of SU(2). Here a color k on an edge indicates the $k+1$ dimensional irreducible representation of SU(2), and their evaluation is a contraction of the above tensors. Spin networks originated in work by Racah and Wigner in atomic spectroscopy in the late forties [38; 39; 40; 41; 52]. Exact or asymptotic evaluations of spin networks is a useful and interesting topic studied by Ponzano and Regge [37], Biedenharn and Louck [6; 7] and Varshalovich, Moskalev and Khersonskiĭ [48]. In the past three decades, spin networks have been used in relation to classical and quantum gravity and angular momentum in 3 dimensions; see Engle, Pereira and Rovelli [14], Penrose [35; 36] and Rovelli and Smolin [44]. In mathematics, q–deformations of spin networks (so called quantum spin networks) appeared in the

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eighties in the work of Kirillov and Reshetikhin [28]. *Quantum spin networks* are *knotted framed trivalent graphs* embedded in 3–space with a cyclic ordering of the edges near every vertex, and their evaluations are rational functions of a variable q. The quantum theta and $6j$–symbols are the building blocks for topological invariants of closed 3–manifolds in the work of Turaev and Viro [47; 46]. Quantum spin networks are closely related to a famous invariant of knotted 3–dimensional objects, the celebrated *Jones polynomial*; see [23]. A thorough discussion of quantum spin networks and their relation to the Jones polynomial and the *Kauffman bracket* is given by Kauffman and Lins [27] and Carter, Flath and Saito [10]. Recent papers on asymptotics of spin networks in physics and mathematics include Aquilanti, Haggard and Hedeman [4], Littlejohn and Yu [29] and Costantino and Marché [12]. Aside from the appearances of spin networks in the above mentioned areas, their evaluations and their asymptotics lead to challenging questions even for simple networks such as the cube, discussed in detail in the appendix. Some examples of spin networks that will be discussed in the paper are shown in Figure 1.

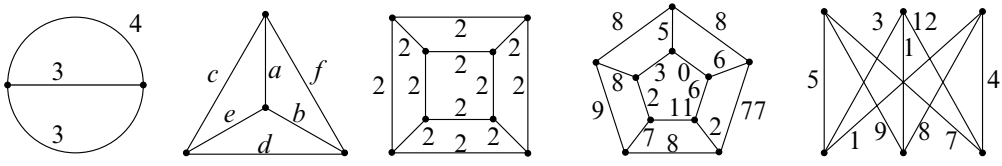


Figure 1: From left to right: The theta, the tetrahedron or $6j$–symbol, the Cube, the 5–sided prism and the complete bipartite graph $K_{3,3}$ or $9j$–symbol. The cyclic order of the edges around each vertex is counterclockwise. The left three spin networks are admissible, and the right two are not.

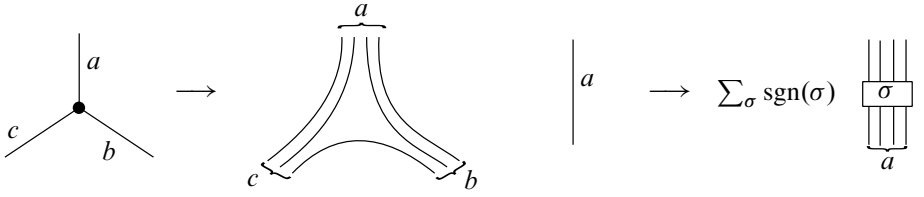
1.2 The evaluation of a spin network

Definition 1.1 (1) We say a spin network is *admissible* when the sum of the three colors a, b, c around every vertex is even and a, b, c satisfy the triangle inequalities $|a-b| \le c \le a+b$.

(2) The Penrose evaluation $\langle \Gamma, \gamma \rangle^P$ of a spin network (Γ, γ) is defined to be zero if it is not admissible. If it is admissible, its evaluation is given by the following algorithm:

- Use the cyclic ordering to thicken the vertices into disks and the edges into untwisted bands.

- Replace the vertices and edges by the linear combinations of arcs as follows:

(1) 

- Finally the resulting linear combination of closed loops is evaluated by assigning the value $(-2)^n$ to a term containing n loops.

In the above definition the summation is over all permutations σ of the a arcs at an edge colored a. The Penrose evaluation $\langle \Gamma, \gamma \rangle^P$ of a spin network is always an integer. Note that the admissibility condition is equivalent to saying that the strands can be connected at each vertex as in (1). Note also that cubic ribbon graphs Γ are allowed to have multiple edges, loops and several connected components including components that contain no vertices. In addition, Γ is allowed to be nonplanar (contrary to the requirement of many authors such as Westbury [50], Moussouris [31] and Kauffman and Lins [27]), as long as one fixes a cyclic ordering of the edges at each vertex. The latter condition is implicit in [35]. It turns out that changing the cyclic ordering at a vertex of a spin network changes its evaluation by a single sign; see Lemma 2.1 below.

1.3 Three fundamental problems

It is easy to see that if (Γ, γ) is an admissible spin network and n is a natural number, then $(\Gamma, n\gamma)$ is also admissible. A fundamental problem is to study the asymptotic behavior of the sequence of evaluations $\langle \Gamma, n\gamma \rangle^P$ when n is large. This problem actually consists of separate parts. Fix an admissible spin network (Γ, γ).

Problem 1.2 Prove the existence of an asymptotic expansion of the sequence $\langle \Gamma, n\gamma \rangle^P$ when n is large.

Problem 1.3 Compute the asymptotic expansion of the sequence $\langle \Gamma, n\gamma \rangle^P$ to all orders in n effectively.

Problem 1.4 Identify the terms in the asymptotic expansion of $\langle \Gamma, n\gamma \rangle^P$ with geometric invariants of the spin network.

These problems are motivated by the belief that the quantum mechanics of particles with large spin will approximate the classical theory. To the best of our knowledge,

the literature for Problem 1.2 is relatively new and short and concerns only thetas and $6j$–symbols with certain labelings. For Problem 1.3, it should be noted that even for the $6j$–symbols not much is known about the subleading terms in the asymptotic expansion. Some terms are found by Dupuis and Livine in [13] but no general algorithm is given. As for the geometric interpretation in Problem 1.4 there is a well known conjecture in the case of the $6j$–symbol [37]. Roberts used geometric quantization techniques to prove this conjecture on the leading asymptotic behavior of $6j$–symbols in the so-called Euclidean case; see [42; 43]. Some results on the $9j$–symbol have been found by Haggard and Littlejohn [22]. Finally a more general interpretation for the leading order asymptotics appears in [12], however this assumes a hypothesis that has not been shown to hold in cases other than the $6j$–symbol. Problems 1.2–1.4 can also be viewed as the classical analogue of the problem of understanding the asymptotics of quantum spin networks and quantum invariants. Even less is known in the quantum case, but see the authors' earlier work [20]. A well known conjecture in this context is the volume conjecture; see Kashaev [24] and Murakami and Murakami [32].

1.4 A solution to Problem 1.2

In this paper we give a complete solution to Problem 1.2 in full generality. A convenient role is played by the following normalization of the spin network evaluation. This normalization was introduced independently by Costantino [11] in the q–deformed case.

Definition 1.5 We define the *standard normalization* of a spin network evaluation by

$$\langle \Gamma, \gamma \rangle = \frac{1}{\mathcal{I}!} \langle \Gamma, \gamma \rangle^{P}, \tag{2}$$

where $\mathcal{I}!$ is defined to be the product

$$\mathcal{I}! = \prod_{v \in V(\Gamma)} \Big(\frac{-a_v + b_v + c_v}{2}\Big)! \Big(\frac{a_v - b_v + c_v}{2}\Big)! \Big(\frac{a_v + b_v - c_v}{2}\Big)!, \tag{3}$$

where a_v, b_v, c_v are the colors of the edges adjacent to vertex v, and $V(\Gamma)$ is the set of vertices of Γ.

The standard normalization has a number of useful properties (see Theorem 1.7 below) that can be stated conveniently in terms of a generating function that we now define. If we fix a cubic ribbon graph Γ one can consider many spin network evaluations, one for each admissible labeling γ of Γ. We organize these in a generating function by taking a formal variable for every edge and encoding γ in the exponents of monomials in these variables.

Definition 1.6 Given a cubic ribbon graph Γ define a formal power series in the variables $z = (z_e)_{e\in E(\Gamma)}$ by

$$F_\Gamma(z) = \sum_{\gamma\geq 0} \langle \Gamma, \gamma\rangle z^\gamma,$$

where $z^\gamma = \prod_{e\in E(\Gamma)} z_e^{\gamma(e)}$, and $E(\Gamma)$ denotes the set of edges of Γ.

By virtue of our use of the standard normalization we can prove the following theorem about our generating function F_Γ.

Theorem 1.7 (1) *For all spin networks (Γ,γ), the standard evaluation $\langle\Gamma,\gamma\rangle$ is an integer.*

(2) *The sequence $\langle\Gamma, n\gamma\rangle$ is exponentially bounded.*

(3) *For any cubic ribbon graph Γ the generating series F_Γ is a rational function explicitly defined in terms of Γ.*

To illustrate the last part of the theorem let us mention the special case in which Γ is planar with the counterclockwise orientation. In this case a result from [50] states

$$F_\Gamma(z) = \frac{1}{P_\Gamma(z)^2},$$

where $P_\Gamma(z) = \sum_{c\in C_\Gamma} z^c$ and C_Γ is the set of 2–regular subgraphs of Γ. Our theorem generalizes this result to arbitrary Γ; the precise statement can be found in Theorem 2.9. The next result gives a complete answer to Problem 1.2. To state it, we need to recall a useful type of sequence; see the first author's works [16; 17].

Definition 1.8 We say that a sequence (a_n) is of *Nilsson type* if it has an *asymptotic expansion* of the form

$$a_n \sim \sum_{\lambda,\alpha,\beta} \lambda^n n^\alpha (\log n)^\beta S_{\lambda,\alpha,\beta} h_{\lambda,\alpha,\beta}(1/n), \tag{4}$$

where

- the summation is over a finite set of triples (λ,α,β);
- the *growth rates* λ are algebraic numbers of equal magnitude;
- the *exponents* α are rational and the *nilpotency exponents* β are natural numbers;
- the *Stokes constants* $S_{\lambda,\alpha,\beta}$ are complex numbers;
- the $h_{\lambda,\alpha,\beta}(x)$ are formal power series with coefficients in a number field K such that the coefficient of x^n is bounded by $C^n n!$ for some $C>0$ and the constant coefficient is 1.

Note that a sequence of Nilsson type uniquely determines its asymptotic expansion (see (4)) as was explained in detail in [17]. Using the theory of G–functions, (discussed in Section 3.2), we prove the following theorem.

Theorem 1.9 *For any spin network (Γ,γ) the sequence $\langle\Gamma,n\gamma\rangle$ is of Nilsson type.*

1.5 A partial solution to Problem 1.3

Regarding Problem 1.3, we introduce a new method (the Wilf–Zeilberger theory) which

- computes a linear recursion for the sequence $\langle\Gamma,n\gamma\rangle$;
- given a linear recursion for the sequence $\langle\Gamma,n\gamma\rangle$, effectively computes the corresponding triples (λ,α,β), number field K and any number of terms of the power series $h_{\lambda,\alpha,b}(x)\in 1+xK[\![x]\!]$ in Definition 1.8;
- numerically computes the Stokes constants $S_{\lambda,\alpha,\beta}$.

Given this information, one may guess exact values of the Stokes constants. In some cases, we obtain an alternative exact computation of the Stokes constants, too. As an illustration of the theorem we will present computations of the asymptotic expansions of three representative $6j$–symbols up to high order using the Wilf–Zeilberger method in Section 4.1. In the Appendix we will present additional numerical results on the asymptotic expansion of the case of the cube spin network. About 20 more examples of spin network evaluations (including the s–sided prisms for $s=2,\dots,7$ and the twisted s–sided prisms for $s=2,\dots,5$) have been computed, and the data is available from the first author upon request.

1.6 A conjecture regarding Problem 1.4

The example of the cube spin network also provides evidence for the following conjecture on the growth rates λ in the Nilsson type expansion. The conjecture connects the growth rates of suitable spin networks to the total mean curvature of a related Euclidean polyhedron. Let P be a convex polyhedron in three dimensional Euclidean space. Denote by $M(P)$ the total mean curvature of P. Recall that $M(P)=\frac{1}{2}\sum_e \ell_e\phi_e$, where ϕ_e is the exterior dihedral angle at edge e and ℓ_e is the length of the edge.

Conjecture 1.10 *Let (Γ,γ) be a planar spin network such that the dual of Γ is realized as the 1–skeleton of a convex Euclidean polyhedron P with edge lengths given by γ. The numbers $e^{\pm iM(P)}$ are growth rates in the asymptotic expansion of the unitary evaluations of $\langle\Gamma,n\gamma\rangle^U$.*

In the conjecture we are using the so called unitary evaluation of a spin network defined in Section 4. This evaluation is still of Nilsson type since it differs from the standard one by an explicit factor.

After this work was completed, an approach to Problems 1.2–1.4 was proposed by Costantino and Marche in [12] using generating functions. Their approach requires certain nondegeneracy conditions, and in particular does not give a solution to Problem 1.2 or 1.3 for the regular cube spin network; see the Appendix.

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2 Evaluation of spin networks

In this section we treat two ways of calculating the evaluation of a spin network. The first is by recoupling theory and leads to practical but noncanonical formulas for the evaluations as multisums. The second way is the method of chromatic evaluation. This leads to the proof of the generating function result, Theorem 1.7 announced above.

We start by recording some elementary facts about spin network evaluations. First of all our definition of the standard evaluation assumes that there are no edges without vertices. By definition we will add an $(a,a,0)$ colored vertex to any a–colored component that has none. This makes sense because of part (a) of the following.

Lemma 2.1 *Let (Γ,γ) be a spin network and consider the standard evaluation.*

(a) *Inserting a vertex colored $(0,a,a)$ in the interior of an edge of Γ colored a does not change the standard evaluation of the spin network.*

(b) *Changing the cyclic ordering at a vertex whose edges are colored a, b, c changes the evaluation by a sign $(-1)^{(a(a-1)+b(b-1)+c(c-1))/2}$.*

Proof (a) The chosen normalization introduces an extra factor $1/a!$ for the new vertex labeled $(0,a,a)$, while it follows from the definition that one also inserts an extraneous summation over permutations in the pre-existing edge labeled a. Since

$$\sum_{\sigma\in S_a}\sum_{\tau\in S_a}\operatorname{sgn}(\sigma)\sigma\ \operatorname{sgn}(\tau)\tau = a!\sum_{\sigma\in S_a}\operatorname{sgn}(\sigma)\sigma,$$

the evaluation is unchanged.

(b) Changing the cyclic order at a vertex with edge labels a, b, c has the following effect. The alternating sum at each of the adjacent edges is multiplied by the permutation that turns the arcs in the edge 180 degrees. This element has sign $a(a-1)/2$ in S_a. $\square$

As a consequence of part (a) of the above lemma, an edge labeled 0 in a spin network can be removed without affecting the evaluation. There is an alternative *bracket normalization* $\langle \Gamma, \gamma \rangle^B$ of the evaluation of a spin network (Γ, γ) which agrees with a specialization of the *Jones polynomial* or *Kauffman bracket*.

Definition 2.2 The *bracket normalization* of a spin network (Γ, γ) is defined by

$$\langle \Gamma, \gamma \rangle^B = \frac{1}{\mathcal{E}!} \langle \Gamma, \gamma \rangle^P, \tag{5}$$

where

$$\mathcal{E}! = \prod_{e \in E(\Gamma)} \gamma(e)!. \tag{6}$$

This normalization has the property that it coincides with the Kauffman bracket (Jones polynomial) of a quantum spin network evaluated at $A = -1$; see [27]. However, $\langle \Gamma, \gamma \rangle^B$ is not necessarily an integer number, and the analogous generating series does not satisfy the rationality property of Theorem 1.7.

2.1 Evaluation of spin networks by recoupling

In this subsection we describe a way of evaluating spin networks by recoupling. We will reduce the evaluation of spin networks to multidimensional sums of $6j-$ and theta-symbols. The value of the $6j-$ and theta-symbols is given by the following lemma from [27; 50], using our normalization. The choice of letters in coloring the $6j$–symbol is traditional following for example [27].

Lemma 2.3 (1) *Let $(\triangle, \gamma)$ denote a tetrahedron colored and oriented as in Figure 1 with $\gamma = (a, b, c, d, e, f)$. Its standard evaluation is given by*

$$\langle \triangle, \gamma \rangle = \sum_{k=\max T_i}^{\min S_j} (-1)^k (k+1) \binom{k}{S_1-k, S_2-k, S_3-k, k-T_1, k-T_2, k-T_3, k-T_4}, \tag{7}$$

where, as usual

$$\binom{a}{a_1, a_2, \dots, a_r} = \frac{a!}{a_1! \cdots a_r!}$$

denotes the multinomial coefficient when $a_1+\cdots+a_r=a$, and S_i are the half sums of the colors in the three quadrangular curves in the tetrahedron and T_j are the half sums of the colors of the edges adjacent to a given vertex. In other words, the S_i and T_j are given by

$$S_1=\tfrac{1}{2}(a+d+b+c),\quad S_2=\tfrac{1}{2}(a+d+e+f),\quad S_3=\tfrac{1}{2}(b+c+e+f), \tag{8}$$

$$\begin{aligned} T_1&=\tfrac{1}{2}(a+b+e),\quad T_2=\tfrac{1}{2}(a+c+f),\\ T_3&=\tfrac{1}{2}(c+d+e),\quad T_4=\tfrac{1}{2}(b+d+f). \end{aligned} \tag{9}$$

(2) *Let (Θ,γ) denote the Θ spin network of Figure 1 admissibly colored by $\gamma=(a,b,c)$. Then we have*

$$\langle\Theta,\gamma\rangle=(-1)^{(a+b+c)/2}\left(\frac{a+b+c}{2}+1\right)\left(\begin{matrix}\frac{a+b+c}{2}\\ \frac{-a+b+c}{2},\ \frac{a-b+c}{2},\ \frac{a+b-c}{2}\end{matrix}\right). \tag{10}$$

Finally note that the evaluation of an n–labeled unknot is equal to $(-1)^n(n+1)$.

Recoupling is a way to modify a spin network locally, while preserving its evaluation. This is done as in Figure 2. The topmost formula is called the recoupling formula and follows from the recoupling formula in [27], using our conventions. The other two pictures in the figure show the bubble formula and the triangle formula.

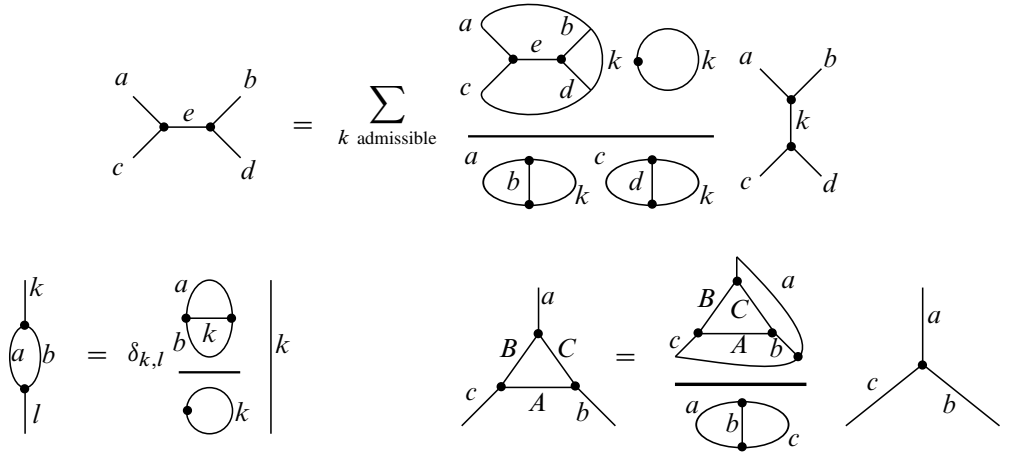


Figure 2: The recoupling formula (top), the bubble formula (left) and the triangle formula (right). The sum is over all k for which the network is admissible, and $\delta_{k,l}$ is the *Kronecker delta function*.

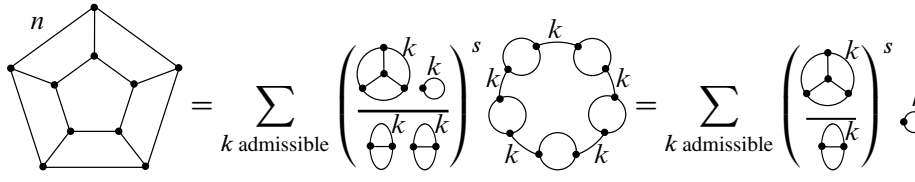
The bubble formula shown on the left of Figure 2 serves to remove all bigon faces. Likewise the triangle formula can be used to remove triangles. The recoupling and

bubble formulas suffice to write any spin network as a multisum of products of $6j$–symbols divided by thetas. To see why, we argue by induction on the number of edges. Applying the recoupling formula to a cycle in the graph reduces its length by one and preserves the number of edges. Keep going until you get a multiple edge which can then be removed by the bubble formula.

Although the triangle formula follows quickly from the bubble formula and the recoupling formula it is important enough to state on its own. For example the triangle formula shows that the evaluation of the class of *triangular networks* is especially simple. The triangular networks are the planar graphs that can be obtained from the tetrahedron by repeatedly replacing a vertex by a triangle. By the triangle formula the evaluation of any triangular network is simply a product of $6j$–symbols divided by thetas. No extraneous summation will be introduced.

To illustrate how recoupling theory works, let us evaluate the regular s–sided prism and $K_{3,3}$. Consider the s–sided prism network as shown in (11) (for $s=5$) where every edge is colored by the integer n. In the figure we have left out most of the labels n for clarity. By convention unlabeled edges are colored by n. Performing the recoupling move on every inward pointing edge we transform the prism into a string of bubbles that is readily evaluated.

(11)



Observing that if n is odd the network is not admissible (and thus evaluates to zero), and denoting the tetrahedron and the theta with one edge colored by k and the others by n by $S(n,k)$ and $\theta(n,k)$ we conclude the following formula for the n–colored s–sided prism.

Proposition 2.4 *If $n=2N$ is even we have*

$$\langle \mathrm{Prism}_s, 2N\rangle = \sum_{j=0}^{2N} (2j+1)\left(\frac{S(2N,2j)}{\theta(2N,2j)}\right)^s$$

and if n is odd we have $\langle \mathrm{Prism}_s, n\rangle = 0$.

For small values of s the prism can be evaluated in a more straightforward way, thus providing some well known identities of $6j$–symbols. Namely when $s=1$ we get zero, when $s=2$ we find some thetas and when $s=3$ we have by the triangle formula

a product of two $6j$–symbols thus giving a special case of the Biedenharn–Elliott identity; see [27]. For $s=4$ we find a formula for the regular cube, that will be used in the appendix. We know of no easier expression for the evaluation in this case. A similar computation for $K_{3,3}$ cyclically ordered as a plane hexagon with its three diagonals gives the following.

Proposition 2.5 *If $n=2N$ is even we have*

$$\langle K_{3,3}, 2N\rangle = \sum_{j=0}^{2N} (-1)^j (2j+1) \left(\frac{S(2N,2j)}{\theta(2N,2j)}\right)^3$$

and if n is odd we have $\langle K_{3,3}, n\rangle = 0$.

Note the similarity between Prism_3 and $K_{3,3}$. The only difference is the sign that comes up in the calculation when one needs to change the cyclic order. The extra sign makes $\langle K_{3,3}, 2N\rangle = 0$ for all odd N. This is because changing the cyclic ordering at a vertex takes the graph into itself, while it produces a sign $(-1)^N$ when all edges are colored $2N$.

2.2 Generating series and chromatic evaluation

Recall the generating function $F_\Gamma(z)$ for all spin network evaluations with the same underlying graph Γ from Definition 1.6. We are using variables $z=(z_e)_{e\in E(\Gamma)}$, one for each edge, and abbreviate monomials $\prod_{e\in E(\Gamma)} z_e^{\gamma(e)}$ as z^γ. Our goal is to express F_Γ explicitly in terms of Γ. To do so we need a couple of definitions.

Definition 2.6 Given a cubic ribbon graph Γ define a *cycle* to be a (possibly disconnected) 2–regular subgraph of Γ. The set of all cycles is denoted by C_Γ.

In other words, a cycle is a subgraph such that at any vertex an even number of edges meet. In terms of the cycles we define a polynomial and a quadratic form.

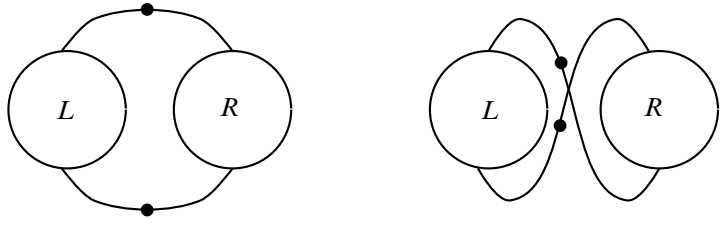
Definition 2.7 Given a cubic ribbon graph Γ and $X\subset C_\Gamma$ we define

$$P_{\Gamma,X}(z) = \sum_{c\in C_\Gamma} \epsilon_X(c) \prod_{e\in c} z_e, \tag{12}$$

where $\epsilon_X(c)=-1$ (resp. 1) when $c\in X$ (resp. $c\notin X$). Also define the function Q_Γ on the subsets of C_Γ as follows. Let $Q_\Gamma(X)$ be the number of unordered pairs $\{c,c'\}\subset X$ with the property that c and c' intersect in an odd number of places when drawn on the thickening of Γ.

Note that the cyclic orientation of Γ defines a unique thickening. We call $P_\Gamma = P_{\Gamma,\varnothing}$ the *cycle polynomial* of Γ. It is independent of the cyclic orientation of Γ. Notice how the other $P_{\Gamma,X}$ only differ from the cycle polynomial in the signs of the individual monomials. In particular, the polynomials $P_{\Gamma,X}$ all have constant coefficient 1.

It is interesting to remark that the cycle polynomial determines the cubic graph, up to a well-determined ambiguity. First of all we can restrict to connected graphs since the cycle polynomial is multiplicative under disjoint union. For connected graphs we will make use of a classic theorem of Whitney, which we quote for the benefit of the reader. Recall that a connected graph Γ is 2–connected (resp. 3–connected) if it remains connected after removing any one (resp. any two) vertices of Γ. A *Whitney flip* is the following move on a graph (where R and L contain at least two edges):



A Whitney flip is the graph-theoretic analogue of *a knot mutation* (see Adams [2]) and can only be applied to graphs which are not 3–connected.

Proposition 2.8 (Whitney [51]) (a) Let Γ_1, Γ_2 be two 2–connected cubic graphs with same cycle polynomial. Then Γ_2 is obtained from Γ_1 by a sequence of Whitney flips.

(b) Let Γ be a 3–connected cubic graph. The cycle polynomial P_Γ uniquely determines Γ.

Proof Since Γ_1 and Γ_2 have the same cycle polynomial we have a bijection on the sets of edges preserving the set of cycles. We can extend this bijection to a bijection on the set of vertices so that the result follows from a more general theorem of Whitney [51] that states the following. Two finite 2–connected graphs with a bijection on the set of vertices that preserves the set of cycles are related by a sequence of Whitney flips. This also works for noncubic graphs if we define a *cycle* C of a finite graph Γ to be a subgraph of Γ with the same vertex set as Γ such that every vertex of C has even valency. In case Γ is a cubic graph, a cycle of Γ in the above sense exactly coincides with Definition 2.6. Thus, part (a) follows.

Part (b) follows from (a) and the fact that 3–connected graphs cannot be Whitney flipped. □

With the definitions in place we can finally state the precise version of the last part of Theorem 1.7.

Theorem 2.9 *For every cubic ribbon graph Γ we have*

$$F_\Gamma(z) = \sum_{X \subset C_\Gamma} \frac{a_X}{P_{\Gamma,X}^2} \in \mathbb{Z}[\![z]\!] \cap \mathbb{Q}(z), \tag{13}$$

where the coefficients are given by

$$a_X = \frac{1}{2^{|C_\Gamma|}} \sum_{Y \subset C_\Gamma} (-1)^{Q_\Gamma(Y)+|X\cap Y|}.$$

Corollary 2.10 *For every spin network (Γ, γ), the evaluation $\langle \Gamma, \gamma\rangle$ is an integer number and $\langle \Gamma, n\gamma\rangle$ is exponentially bounded.*

In particular Theorem 1.7 reduces to Theorem 2.9 above. To see how the particular case of planar spin networks comes about we note the following.

Corollary 2.11 *When Γ is planar with the counterclockwise orientation, then all cycles intersect an even number of times so $(-1)^{Q(X)} = 1$ and hence only $a_\varnothing$ is nonzero. It follows that*

$$F_\Gamma = \frac{1}{P_{\Gamma,\varnothing}^2},$$

recovering an earlier theorem by Westbury [50].

The proof of this theorem uses the *chromatic evaluation method* which goes back to [36]. Our proof builds on earlier work in [50; 27] on planar spin networks and will be given in the next subsection.

2.3 Chromatic evaluation

Definition 2.12 For $N \in \mathbb{Z}$ define the evaluation $\langle \Gamma, \gamma\rangle_N^P$ just as in Definition 1.1 except that the value of a loop is now N instead of -2.

Note that by definition $\langle \Gamma, \gamma\rangle_{-2}^P = \langle \Gamma, \gamma\rangle^P$. However for positive N the evaluations are easier to work with combinatorially. Also since the evaluations depend polynomially on N, the values of the evaluations at positive N will together determine the original evaluation at $N = -2$.

Definition 2.13 Define a *cycle configuration* to be a function $L\colon C_\Gamma \to \mathbb{N}$ such that $L(\varnothing) = 0$. A cycle configuration L defines a coloring $\gamma(L)$ as

$$\gamma(L)(e) = \sum_{c \in C_\Gamma : e \in c} L(c), \quad |L| = \sum_{c \in C_\Gamma} L(c), \quad L! = \prod_{c \in C_\Gamma} L(c)!.$$

Define the quadratic form Q_Γ on cycle configurations as $Q_\Gamma(L) = \sum_{\{c,d\}} L(c)L(d)$, where $\{c, d\} \subset C$ runs over the unordered pairs of cycles that intersect in an odd number of places.

Viewing a (nonempty) subset $X \subset C_\Gamma$ as the cycle configuration that is 1 on X and 0 elsewhere, this definition of Q_Γ coincides with the one given in Definition 2.7. We can now state and prove the main lemma that expresses the evaluation for positive N in terms of cycle configurations.

Lemma 2.14 *For positive integers N we have*

$$\langle \Gamma, \gamma \rangle_N^P = \sum_{L : \gamma(L) = \gamma} (-1)^{Q_\Gamma(L)} \binom{N}{L} \mathcal{I}!.$$

Here $\binom{N}{L}$ is defined as $\frac{N(N-1)\ldots(N-|L|+1)}{L!}$ and recall $\mathcal{I}!$ is the normalization factor from Definition 1.5.

Proof For convenience the proof is summarised in the following string of equalities. We will comment on each step in turn, introducing new notation as it comes up.

$$\langle \Gamma, \gamma \rangle_N^P = \sum_{\sigma \in S_\gamma} \operatorname{sgn}(\sigma) N^{U/\sigma} \tag{14}$$

$$= \sum_{\sigma \in S_\gamma} \operatorname{sgn}(\sigma) \sum_{f : U \to B} \langle \sigma, f \rangle \tag{15}$$

$$= \sum_{\text{good } f} \sum_{\sigma} \operatorname{sgn}(\sigma) \langle \sigma, f \rangle \tag{16}$$

$$= \sum_{L : \gamma(L) = \gamma} \sum_{f : L_f = L} (-1)^{Q(L)} \tag{17}$$

$$= \sum_{L : \gamma(L) = \gamma} (-1)^{Q(L)} \binom{N}{L} \mathcal{I}!. \tag{18}$$

In (14) we have made precise the process of the evaluation of a spin network. Recall that we replace each vertex by a system of arcs; see (1). Let U be the set of all such arcs. Next these arcs are connected at the edges of the graph by permutations in the

product of symmetric groups $S_\gamma = \prod_{e\in E(\Gamma)} S_{\gamma(e)}$. An element $\sigma \in S_\gamma$ gives rise to an equivalence relation on the set of arcs U indicating which arcs are connected. We used the notation U/σ to mean the number of equivalence classes under this relation (ie the number of closed loops).

The next equality, (15), is a reformulation of $N^{U/\sigma}$ in terms of maps $f\colon U \to B$ where B is an abstract N–element set. Define $\langle \sigma, f\rangle$ to be 1 if σ only connects arcs of U with the same value of f and define $\langle \sigma, f\rangle = 0$ otherwise. By definition we then have $N^{U/\sigma} = \sum_{f\colon U\to B}\langle \sigma, f\rangle$.

The next step, (16), is merely an interchange of the two summations. This is important because in the innermost sum all terms cancel out except for the *good* f that we will define now. Let us call a function $f\colon U \to B$ *good* if there exists a σ such that $\langle \sigma, f\rangle = 1$ and for every vertex of Γ it assigns distinct elements of B to all arcs at that vertex. To see why only the good f contribute, suppose f is not good so there will be an edge at which two arcs u, u' satisfy $f(u) = f(u')$. If $\langle \sigma, f\rangle = 1$ then composing σ with a transposition exchanging u and u' produces a term $\langle \sigma', f\rangle = 1$ such that the signs of σ and σ' are opposite.

Notice that a good f determines a unique σ such that $\langle \sigma, f\rangle = 1$, since the values of f must all be distinct at each edge. f also determines uniquely a cycle configuration L_f defined for $c \in C_\Gamma$ by

$$L_f(c) = \#\{b \in B \,|\, f^{-1}(b) \in c\}.$$

Here $f^{-1}(b) \in c$ means the arcs in the inverse image of b trace out the cycle c of Γ.

In (17) we have arranged all the good f according to what cycle configuration they represent. Note that the sign of the permutation σ corresponding to the good f only depends on L_f and actually equals $(-1)^{Q(L_f)}$. This follows directly from the interpretation of the sign as $\operatorname{sgn}(\sigma) = (-1)^{\#\text{crossings}}$.

The final step, (18), consists of counting the number of good f corresponding to a given cycle configuration L. To obtain such an f from L we first assign disjoint unordered $L(c)$–tuples of distinct elements of B to all cycles $c \in C_\Gamma$. This can be done in $\binom{N}{L}$ ways. Finally we have to fix an ordering of the chosen elements of B at the arcs at each side of every vertex. By the arcs at a side of a vertex we mean the arcs that run between a fixed pair of half edges at the vertex. This ordering can be fixed in $\mathcal{I}!$ ways so the proof is complete. □

Since the evaluation $\langle \Gamma, \gamma\rangle_N^P$ is a polynomial in N, the conclusion of Lemma 2.14 actually holds for all N, in particular $N = -2$. We record this for future use as the following corollary, where we have switched back to the standard normalization.

Corollary 2.15

$$\langle \Gamma, \gamma \rangle = \sum_{L:\gamma(L)=\gamma} (-1)^{Q_\Gamma(L)} \binom{-2}{L}.$$

2.4 Proof of Theorem 2.9

To find a generating function for these evaluations, we first need to expand the sign $(-1)^{Q_\Gamma(L)}$ in terms of characters. That is we use Fourier analysis on the group $(\mathbb{Z}/2\mathbb{Z})^{|C_\Gamma|}$. We often write elements of this group as subsets of C_Γ. For every fixed $X \subset C_\Gamma$ we have a character $(-1)^{X(L)} = (-1)^{\sum_{x\in X} L(x)}$. In case L is a cycle configuration we extend the character to a Dirichlet character. So for some coefficients a_X we have

$$(-1)^{Q_\Gamma(L)} = \sum_{X\subset C_\Gamma} a_X (-1)^{X(L)}.$$

Taking inner products, the coefficients are given by

$$a_X = \frac{1}{2^{|C_\Gamma|}} \sum_{Y\subset C_\Gamma} (-1)^{Q_\Gamma(Y)+|X\cap Y|}.$$

To rewrite the generating function let us introduce a variable $w = (w_c)_{c\in C_\Gamma}$ for each cycle and set $w_c = \prod_{e\in c} z_e$. If $\gamma(L) = \gamma$ then the color of an edge is the sum of the number of cycles (with multiplicity) containing that edge, hence

$$(-1)^{X(L)} z^\gamma = (-1)^{X(L)} w^L = \prod_{c\in C_\Gamma} (\epsilon_X(c) w_c)^{L(c)},$$

where $\epsilon_X(c)$ is the function that is 1 if $c \notin X$ and -1 if $c \in X$. Recall the cycle polynomial $P_{\Gamma,X}(z) = \sum_{c\in C_\Gamma} \epsilon_X(c) w_c$ so we can compute

$$\begin{aligned}\sum_\gamma \sum_{L:\gamma(L)=\gamma} \binom{N}{L} (-1)^{X(L)} z^\gamma &= \sum_L \binom{N}{L} \prod_{c\in C_\Gamma} (\epsilon_X(c) w_c)^{L(c)} \\ &= \Big(1 + \sum_{\varnothing\neq c\in C_\Gamma} \epsilon_X(c) w_c\Big)^N = P_{\Gamma,X}^N.\end{aligned}$$

Now setting $N = -2$ everywhere and applying Corollary 2.15 we have

$$F_\Gamma(z) = \sum_\gamma \langle \Gamma, \gamma \rangle z^\gamma = \sum_{X\subset C_\Gamma} a_X P_{\Gamma,X}^{-2},$$

which completes the proof. □

3 Asymptotic expansions

The goal of this section is to prove Theorem 1.9 that provides a Nilsson type asymptotic expansion for spin network evaluations. The general idea is to define the following (single variable) generating function.

Definition 3.1 Let (Γ, γ) be a spin network. The single variable generating function $F_{\Gamma,\gamma}$ is the formal power series

$$F_{\Gamma,\gamma}(z) = \sum_z \langle \Gamma, n\gamma \rangle z^n. \tag{19}$$

Our goal is to show this function is a G–function (defined in Section 3.2 below). It then follows from the theory of G–functions the sequence $\langle \Gamma, n\gamma \rangle$ is of Nilsson type. Before doing so we first make some comments on Nilsson type sequences in general.

3.1 Sequences of Nilsson type

Recall from Definition 1.8 that a Nilsson type sequence (a_n) has the following asymptotic expansion as $n \to \infty$:

$$a_n \sim \sum_{\lambda,\alpha,\beta} \lambda^n n^\alpha (\log n)^\beta S_{\lambda,\alpha,\beta} h_{\lambda,\alpha,\beta}(1/n).$$

The meaning of this expansion is entirely analogous to the more familiar special case where there is only one growth rate:

$$a_n \sim \lambda^n n^\alpha (\log n)^\beta \sum_{k=0}^{\infty} \frac{\mu_k}{n^k},$$

which goes back to Poincaré; see Olver [33]. In this case the meaning is that for every $r \in \mathbb{N}$ we have

$$\lim_{n\to\infty} n^r \left(a_n \lambda^{-n} n^{-\alpha} (\log n)^{-\beta} - \sum_{k=0}^{r-1} \frac{\mu_k}{n^k} \right) = \mu_r.$$

The general case is similar but to express it we would need more notation; see [17]. It can be shown that a Nilsson type sequence has a unique asymptotic expansion [17].

An important source of Nilsson type sequences are G–functions that we will introduce next.

3.2 G–functions

In this section we recall the notion of a G–function, introduced by Siegel [45] in relation to transcendence problems in number theory. Many of their arithmetic and algebraic properties were established by André in [3]. G–functions appear naturally in Geometry (as Variations of Mixed Hodge Structures), in Arithmetic and most recently in Enumerative Combinatorics. For a detailed discussion, see [3; 16] and references therein.

Definition 3.2 We say that a series $G(z) = \sum_{n=0}^{\infty} a_n z^n$ is a *G–function* if

- the coefficients a_n are algebraic numbers;
- there exists a constant $C > 0$ so that for every $n \geq 1$ the absolute value of every conjugate of a_n is less than or equal to C^n;
- the common denominator of the algebraic numbers $a_0, \dots, a_n$ is less than or equal to C^n;
- $G(z)$ is holonomic, ie, it satisfies a linear differential equation with coefficients polynomials in z.

For the purposes of this paper the most important property of G–functions is expressed in the following lemma.

Lemma 3.3 [16, Proposition 2.5; 17, Theorem 4.1] *The sequence of Taylor coefficients of a G–function at $z = 0$ is a sequence of Nilsson type.*

With the help of Lemma 3.3 we can now reduce the proof of Theorem 1.9 to the following lemma.

Lemma 3.4 *For any admissible spin network (Γ, γ) the generating function $F_{\Gamma,\gamma}(z)$ is a G–function.*

In the next subsection we will give a proof of Lemma 3.4.

3.3 Hypergeometric terms

In this subsection we prove Lemma 3.4 (and hence Theorem 1.9) by showing that the standard evaluations of spin networks are a certain type of hypergeometric multisums that we will first describe in general.

Definition 3.5 An r–dimensional *balanced hypergeometric datum* $\mathfrak{t}$ (in short, *balanced datum*) in variables (n,k), where $n\in\mathbb{N}$ and $k=(k_1,\dots,k_r)\in\mathbb{N}^r$, is

- a finite list $\{(\epsilon_j, A_j(n,k)) \mid j\in J\}$ where $A_j\colon \mathbb{N}^{r+1}\longrightarrow\mathbb{Z}$ is a linear form in (n,k) and $\epsilon_j\in\{-1,1\}$ for all $j\in J$,
- a vector $(C_0,\dots,C_r)$ of algebraic numbers,
- a polynomial $p(n,k)\in\mathbb{Q}[n,k]$,

that satisfies the *balancing condition*

$$\sum_{j=1}^{J}\epsilon_j A_j = 0 \tag{20}$$

and moreover, the set

$$P_{\mathfrak{t}} = \{x\in\mathbb{R}^r_{\geq 0} \mid A_j(1,x)\geq 0 \text{ for all } j\in J\} \tag{21}$$

is a *compact* rational convex polytope.

A balanced datum $\mathfrak{t}$ gives rise to a balanced term $\mathfrak{t}(n,k)$ (defined for $n\in\mathbb{N}$ and $k\in\mathbb{Z}^r\cap nP_{\mathfrak{t}}$), to a sequence $(a_{\mathfrak{t},n})$ and to a generating series $G_{\mathfrak{t}}(z)$ defined by:

$$\mathfrak{t}(n,k) = C_0^n\prod_{i=1}^{r}C_i^{k_i}\prod_{j=1}^{J}A_j(n,k)!^{\epsilon_j}\,p(n,k), \tag{22}$$

$$a_{\mathfrak{t},n} = \sum_{k\in\mathbb{Z}^r\cap nP_{\mathfrak{t}}}\mathfrak{t}(n,k), \tag{23}$$

$$G_{\mathfrak{t}}(z) = \sum_{n=0}^{\infty}a_{\mathfrak{t},n}z^n. \tag{24}$$

We will call the sequences $(a_{\mathfrak{t},n})$ balanced multisums. The connection between balanced multisum sequences and their asymptotics was given using the theory of G–functions [16]. More precisely, we have the following lemma.

Lemma 3.6 [16, Theorem 2] *If $\mathfrak{t}$ is a balanced datum, then the corresponding series $G_{\mathfrak{t}}(z)$ is a G–function.*

Using this, we can now easily prove Lemma 3.4.

Proof of Lemma 3.4 Using the recoupling formulae from Section 2.1 we can write $\langle \Gamma, \gamma \rangle$ as a multidimensional sum of products of $6j$–symbols, theta-symbols and unknots (ie, $1j$–symbols) with a denominator consisting of theta-symbols. It follows from Equations (7) and (10) that the $6j$–symbols (resp. theta-symbols) are balanced 1–dimensional (resp. 0–dimensional) sums, thus the ratio of the product of the theta-symbols by the product of the theta-symbols is a balanced multidimensional sum. The unknots can be written as $(-1)^k(k+1)!/k!$ and are therefore balanced as well. It is easy to check admissibility guarantees the multidimensional sum has finite range. □

Beware that the term $\mathrm{t}(n,k)$ constructed in the above proof is neither unique nor canonical in any sense.

3.4 Integral representation of spin network evaluations

In this final subsection we comment on the connection between Lemma 3.4 and Theorem 1.7 on the rationality of the multivariate generating function. The idea is that the single variable generating function $F_{\Gamma,\gamma}$ is a diagonal of the multivariate generating function F_Γ, where the diagonal is defined as follows.

Definition 3.7 Given a power series $f(x_1,\dots,x_r) \in \mathbb{Q}[\![x_1,\dots,x_r]\!]$ and an exponent $J=(j_1,\dots,j_r)\in\mathbb{N}_+^r$, we define the J–diagonal of f by

$$(\Delta_J f)(z) = \sum_{n=0}^{\infty} [x^{nJ}](f) z^n \in \mathbb{Q}[\![z]\!], \tag{25}$$

where $[x^{nJ}](f)$ denotes the coefficient of $x_1^{nj_1}\cdots x_r^{nj_r}$ in f.

For every spin network (Γ,γ) we have

$$F_{\Gamma,\gamma} = \Delta_\gamma F_\Gamma. \tag{26}$$

Consequently, the G–function $F_{\Gamma,\gamma}(z)$ is the diagonal of a rational function, and thus it comes from geometry in the following sense. Fix a power series $f(x_1,\dots,x_r)\in\mathbb{Q}[\![x_1,\dots,x_r]\!]$ convergent at the origin and an exponent $J=(j_1,\dots,j_r)\in\mathbb{N}_+^r$ and consider the diagonal $(\Delta_J f)(z)\in\mathbb{Q}[\![z]\!]$ as in Definition 3.7. Let $\mathcal{C}$ denote a small real r–dimensional torus around the origin. Then we have the following.

Lemma 3.8 *With the above assumptions, we have*

$$(\Delta_J f)(z) = \frac{1}{(2\pi i)^r}\int_{\mathcal{C}} \frac{f(x_1,\dots,x_r)}{x_1^{j_1}\cdots x_r^{j_r}-z} dx_1\wedge\cdots\wedge dx_r. \tag{27}$$

Proof With the notation of Equation (25), an application of Cauchy's theorem gives for every natural number n,

$$[x^{nJ}](f) = \frac{1}{(2\pi i)^r} \int_{\mathcal{C}} \frac{f(x_1, \dots, x_r)}{(x_1^{j_1} \cdots x_r^{j_r})^{n+1}} dx_1 \wedge \cdots \wedge dx_r.$$

Multiplying by z^n and summing up for n and interchanging summation and integration concludes the proof. □

If in addition $f(x_1, \dots, x_r)$ is a rational function, then the singularities of the analytic continuation of the right hand side of (27) can be analyzed by deforming the integration cycle $\mathcal{C}$ and studying the corresponding variation of Mixed Hodge Structure; see Bloch and Kreimer [8]. Such G–functions come from geometry; see [3; 8].

4 Examples and a conjecture on growth rates

In this section we illustrate the result of Theorem 1.9 on the asymptotic expansions in the case of the $6j$–symbol. We also review the well known geometric interpretation of the leading asymptotics in this case. Finally we formulate a conjecture on the geometric meaning of the growth rates in the asymptotic expansion of more general spin networks. To discuss the geometric aspects of the asymptotics of spin networks it is convenient to introduce one more normalization of spin network evaluations.

Definition 4.1 We define the *unitary normalization* $\langle \Gamma, \gamma \rangle^U$ of a spin network evaluation (Γ, γ) to be

$$\langle \Gamma, \gamma \rangle^U = \frac{1}{\Theta(\gamma)} \langle \Gamma, \gamma \rangle,$$

where

$$\Theta(\gamma) = \prod_{v \in V(\Gamma)} \sqrt{|\langle \Theta, a_v, b_v, c_v \rangle|}$$

and a_v, b_v, c_v are the colors of the edges at vertex v.

Since the asymptotics of the normalization factor $\Theta(\gamma)$ is of Nilsson type by Stirling's formula [33], we see that $\langle \Gamma, n\gamma \rangle^U$ is still of Nilsson type.

4.1 The $6j$–symbol and the tetrahedron

The special case of the tetrahedral spin network or $6j$–symbol motivates much of the questions we asked in the introduction. There is a well known interpretation of the leading asymptotics in terms of a metric tetrahedron T dual to Γ such that the length of a (dual) edge e is given by $\gamma(e)$ [37]. Provided the $6j$ symbol is admissible, such a tetrahedron T can always be found uniquely in either $\mathbb{R}^3, \mathbb{R}^2$ or Minkowski space $\mathbb{R}^{2,1}$; see Blumenthal [9, Chapter 8]. We say the $6j$–symbol is Euclidean, Plane or Minkowskian depending on the type of T. The type is determined by the sign of the Cayley–Menger determinant of T. Let us be more specific in the Euclidean case. Denote by ϕ_e is the exterior dihedral angle of T at edge e.

Theorem 4.2 *Let (Γ, γ) be a Euclidean $6j$ symbol. The sequence $\langle \Gamma, n\gamma \rangle^U$ is of Nilsson type where the growth rates, Stokes constants and powers of n and $\log n$ are*

$$\lambda_\pm = e^{i \sum_j \frac{\gamma(e_j)\phi_j}{2}}, \qquad S_\pm = \frac{e^{i \sum_j \frac{\phi_j}{2} + \frac{i\pi}{4}}}{\sqrt{6\pi \operatorname{Vol}(T)}},$$

$$\alpha = \frac{3}{2}, \qquad \beta = 0.$$

These formulae have been proven in [42]. By analytically continuing the Euclidean formula for dihedral angles in terms of edge lengths, the results can be extended to the Minkowskian case. This will be postponed to a future publication. The Plane case must be different since the volume vanishes in this case. Also any interpretation of the terms in the asymptotic expansion of the $6j$–symbol beyond the ones just given is very much an open problem; see [13]. This warrants a detailed and exact investigation of the asymptotics of three representative $6j$–symbols using the Wilf–Zeilberger method. In this method we compute a recursion for the sequence from which all terms in the asymptotic expansion except for the Stokes constants may be computed.

We have chosen the simplest examples of a Euclidean $6j$–symbol, a Plane one and a Minkowskian $6j$–symbol. Their colorings are given by

$$\begin{aligned} \gamma_{\text{Euclidean}} &= (2,2,2,2,2,2), \\ \gamma_{\text{Plane}} &= (3,4,4,3,5,5), \\ \gamma_{\text{Minkowskian}} &= (4,4,4,4,6,6). \end{aligned}$$

Using the unitary evaluation (Definition 4.1) we thus consider the sequences (a_n), (b_n) and (c_n):

$$a_n := \langle \triangle, n\gamma_{\text{Euclidean}} \rangle^U = \frac{n!^6}{(3n+1)!^2} \sum_{k=3n}^{4n} \frac{(-1)^k (k+1)!}{(k-3n)!^4 (4n-k)!^3},$$

$$b_n := \langle \triangle, n\gamma_{\text{Plane}} \rangle^U = \frac{n!^2 (2n)!^2 (3n)!^2}{(6n+1)!^2} \sum_{k=6n}^{7n} \frac{(-1)^k (k+1)!}{(k-6n)!^4 (7n-k)!(8n-k)!(9n-k)!},$$

$$c_n := \langle \triangle, n\gamma_{\text{Minkowskian}} \rangle^U = \frac{n!^2 (3n)!^4}{(7n+1)!^2} \sum_{k=7n}^{8n} \frac{(-1)^k (k+1)!}{(k-7n)!^4 (8n-k)!(10n-k)!^2}.$$

In what follows we denote by $\det(C)$ the Cayley–Menger determinant and by K the field generated by the coefficients of the power series $h_{\lambda,\alpha,\beta}$ in the asymptotic expansion. The command

```
<< zb.m
```

loads the package of Paule and Riese [34] into `Mathematica`. The command

```
teucl[n_, k_] := n!^6 / (3 n + 1)!^2 (-1)^k (k + 1)! 1 / ((4 n - k)!^3 (k - 3 n)!^4)
```

defines the summand of the sequence (a_n), and the command

```
Zb[teucl[n, k], {k, 3 n, 4 n}, n, 2]
```

computes the following second order linear recursion relation for the sequence (a_n):

```
-9 (1 + n) (2 + 3 n)^2 (4 + 3 n)^2 (451 + 460 n + 115 n^2) a[n] +
  (3 + 2 n) (319212 + 1427658 n + 2578232 n^2 + 2423109 n^3 + 1255139 n^4 + 340515 n^5 + 37835 n^6) a[1 + n] -
  9 (2 + n) (5 + 3 n)^2 (7 + 3 n)^2 (106 + 230 n + 115 n^2) a[2 + n] == 0
```

This linear recursion has two formal power series solutions of the form

$$a_{\pm,n} = \frac{1}{n^{3/2}} \Lambda_\pm^n \Big(1 + \frac{-432 \pm 31i\sqrt{2}}{576n} + \frac{109847 \mp 22320i\sqrt{2}}{331776n^2} + \frac{-18649008 \pm 4914305i\sqrt{2}}{573308928n^3} + \frac{14721750481 \pm 45578388960i\sqrt{2}}{660451885056n^4} + \frac{-83614134803760 \pm 7532932167923i\sqrt{2}}{380420285792256n^5} + \frac{-31784729861796581 \mp 212040612888146640i\sqrt{2}}{657366253849018368n^6} + O\Big(\frac{1}{n^7}\Big)\Big),$$

where

$$\Lambda_\pm = \frac{329 \mp 460i\sqrt{2}}{729} = e^{\mp i\, 6 \arccos(1/3)}$$

are two complex numbers of absolute value 1. Notice the growth rates indeed match the interpretation in terms of dihedral angles of the regular tetrahedron predicted in Theorem 4.2.

The coefficients of the formal power series $a_{\pm,n}$ are in the number field $K = \mathbb{Q}(\sqrt{-2})$ and the Cayley–Menger determinant is $\det(C) = 2^5$. More is actually true. Namely, the sequence (a_n) generates two new sequences $(\mu_{+,n})$ and $(\mu_{-,n})$ defined by

$$a_{\pm,n} = \frac{1}{n^{3/2}} \Lambda_\pm^n \sum_{l=0}^{\infty} \frac{\mu_{\pm,l}}{n^l},$$

where $\mu_{\pm,0} = 1$. Each of the sequences $(\mu_{\pm,n})$ are factorially divergent. However, the generating series $\sum_{n=0}^{\infty} z^n \mu_{\pm,n+1}/n!$ are G–functions (as follows from [3]), and the sequences $(\mu_{\pm,n+1}/n!)$ are of Nilsson type, with exponential growth rates $\Lambda_\pm - \Lambda_\mp$. The asymptotics of each sequence $(\mu_{\pm,n+1}/n!)$ gives rise to finitely many new sequences, and so on. All those sequences span a finite dimensional vector space, canonically attached to the sequence (a_n). This is an instance of resurgence, and is explained in detail by the first author and Mariño in [19, Section 4]. The second order recursion relation for the Plane and the Minkowskian examples has lengthy coefficients, and leads to the following sequences for $(b_{\pm,n})$ and $(c_{\pm,n})$:

$$b_{+,n} = \frac{1}{n^{4/3}} \Lambda_+^n \Big(1 - \frac{1}{3n} + \frac{3713}{46656n^2} - \frac{25427}{2239488n^3} + \frac{9063361}{17414258688n^4} - \frac{109895165}{104485552128n^5} + \frac{1927530983327}{2437438960041984n^6} + \cdots\Big),$$

$$b_{-,n} = \frac{1}{n^{5/3}} \Lambda_-^n \Big(1 - \frac{37}{96n} + \frac{3883}{46656n^2} - \frac{13129}{4478976n^3} - \frac{5700973}{8707129344n^4} - \frac{14855978561}{3343537668096n^5} + \frac{2862335448661}{2437438960041984n^6} + \cdots\Big),$$

$$c_{\pm,n} = \frac{1}{n^{3/2}} \Lambda_\pm^n \Big(1 + \frac{336 \mp 1369\sqrt{2}}{4032n} + \frac{1769489 \mp 831792\sqrt{2}}{1806336n^2} + \frac{67925105712 \mp 66827896993\sqrt{2}}{21849440256n^3} + \frac{5075437500833257 \mp 2589265090380768\sqrt{2}}{176193886224384n^4}$$

$$+\frac{1009784057599974429 92 \mp 98904713360431641651\sqrt{2}}{552544027199668224n^5}$$
$$+\frac{6851035127390585267584 57 \mp 34978263160288715171777 6\sqrt{2}}{2475397241854513643 52n^6}+\cdots\Big),$$

where in the Plane case we have

$$\Lambda_- = \Lambda_+ = -1, \quad K = \mathbb{Q}, \quad \det(C) = 0,$$

and in the Minkowskian case we have

$$\Lambda_+ = \frac{696321931873 - 111529584108\sqrt{2}}{678223072849} = 0.794127\ldots,$$
$$\Lambda_- = \frac{696321931873 + 111529584108\sqrt{2}}{678223072849} = 1.25924\ldots,$$
$$K = \mathbb{Q}(\sqrt{2}), \quad \det(C) = -2^5 3^4.$$

Again as in the Euclidean case the growth rates may be interpreted in terms of dihedral angles. Finally note that the growth rates $\Lambda_\pm$ have norm 1 in $\mathbb{Q}(\sqrt{2})$.

4.2 A conjecture on growth rates

In the special case where (Γ, γ) is an admissible tetrahedron, we have seen a geometric interpretation for the growth rates of $\langle \Gamma, n\gamma\rangle^U$. We can reformulate this more concisely using mean curvature. Recall that for a convex Euclidean polyhedron P in $\mathbb{R}^3$ the mean curvature is defined by $M(P) = \frac{1}{2}\sum_e \ell_e\phi_e$, where ϕ_e is the exterior dihedral angle at edge e and ℓ_e is the length of the edge. So in the case of the tetrahedron Theorem 4.2 says that if there exists a Euclidean tetrahedron T whose 1–skeleton is dual to Γ and whose edge lengths are given by γ, then the growth rates are given by $\{e^{\pm iM(T)}\}$. We would like to conjecture that the growth rates of a spin network always include a growth rate corresponding to the mean curvature of the dual polyhedron. For simplicity we formulate the conjecture for planar spin networks only.

Conjecture 4.3 *Let (Γ, γ) be a planar spin network with the counterclockwise orientation. Suppose that Γ is 3–connected and that its dual can be viewed as the 1–skeleton of a convex Euclidean polyhedron P whose edge lengths are given by γ. The set of growth rates of the Nilsson type sequence $\langle \Gamma, n\gamma\rangle^U$ contains $e^{\pm iM(P)}$.*

By Cauchy's theorem the dual polyhedron P is determined up to isometry by its 1–skeleton and its edge lengths, ie by (Γ, γ). This follows from the fact that P has only triangular faces and is convex.

As a first test of the conjecture we show that it behaves well under the triangle formula on spin networks defined in Section 2.1. In particular this will verify the conjecture for all triangular networks as defined in Section 2.1. Let (Γ,γ) and (Γ',γ') be two spin networks that both satisfy the hypotheses of Conjecture 4.3 and denote their dual polyhedra by P and P'. Furthermore, suppose that (Γ',γ') is obtained from (Γ,γ) by replacing a vertex $v\in\Gamma$ by a triangle. Dually this implies that P' can be produced by attaching a tetrahedron to a (triangular) face of P.

Lemma 4.4 *If Conjecture 4.3 is true for (Γ,γ) then it is also true for (Γ',γ').*

Proof Let the labels around the vertex v be a,b,c and call the labels of the edges of new triangle A,B,C as in Figure 2 (lower right) and denote by $(\triangle,\psi)$ the tetrahedron spin network with labels a,b,c,A,B,C that shows up in the triangle formula. This formula shows that

$$\langle\Gamma',n\gamma'\rangle^U=(-1)^{(n(a+b+c))/2}\langle\triangle,n\psi\rangle^U\langle\Gamma,n\gamma\rangle^U,$$

since the theta only contributes a sign in the unitary evaluation. We already know Conjecture 4.3 holds for tetrahedra with Euclidean duals, including $(\triangle,\psi)$. Let us call the dual Euclidean tetrahedron T. Multiplying the asymptotic expansions on the right hand side we see that the growth rates will include

$$(-1)^{(a+b+c)/2}e^{\pm i(M(P)+M(T))}=e^{\pm iM(P')}.$$

To see why the equality holds note that we can dissect P' into P and T along the triangle with labels a,b,c that is dual to the vertex v. The minus sign coming from the theta accounts for the fact that we are working with exterior dihedral angles and these add an additional factor of π when comparing the angles in a dissection. □

The Euclidean volume also appears in the asymptotic expansion of the tetrahedral spin network, as part of the Stokes constants; see Section 4.1. However this does not generalize well to larger networks since the volumes do not add under the triangle formula. In the appendix we will see a less trivial confirmation of the above conjecture for the cube spin network.

5 Challenges and future directions

In this section we list some challenges and future directions. Our first problem concerns a bound on the unitary evaluations.

Problem 5.1 Show that the unitary evaluation of a spin network (Γ, γ) satisfies

$$|\langle \Gamma, \gamma \rangle^U| \leq 1.$$

This problem may be solved using unitarity and locality in a way similar to the proof that the Reshetikhin–Turaev invariants of a closed 3–manifold grow at most polynomially with respect to the level; see the first author [15, Theorem 2.2]. Our next problem is a version of the Volume Conjecture for classical spin networks with all edges colored by 2. Problem 5.1 also suggests that the growth rates must be less than or equal to 1. In the case $\gamma = 2n$ more seems to be true.

Problem 5.2 The growth rates of the sequence $\langle \Gamma, 2n \rangle^U$ are on the unit circle.

A positive solution to this problem is known for the following ribbon graphs: the Θ, the tetrahedron, the 3–faced prism and more generally for the infinite family of drums; see Abdesselam [1]. More generally one may pose the following.

Problem 5.3 Give a geometric meaning to the set of growth rates of a spin network.

We have formulated Conjecture 4.3 as a partial answer to this question but that concerns only a single special growth rate among many. Along the same lines one may ask for an interpretation of the rest of the asymptotic expansion. Looking at the case of the $6j$–symbol it seems reasonable to consider the number field $K_{\Gamma,\gamma}$ generated by the coefficients of the power series $h_{\lambda,\alpha,\beta}$ in the Nilsson type asymptotic expansion of $\langle \Gamma, n\gamma \rangle^U$.

Problem 5.4 Give a geometric interpretation of the number field $K_{\Gamma,\gamma}$ of a spin network (Γ, γ).

Also the Stokes constant may have a geometric meaning as in the case of the tetrahedron.

Problem 5.5 Give a geometric meaning to the Stokes constants of the sequence $\langle \Gamma, n\gamma \rangle^U$.

The next problem is a computational challenge to all the known asymptotic methods, and shows their practical limitations.

Problem 5.6 Compute the asymptotics of the evaluation $\langle K_{3,3}, 2n \rangle$ (given explicitly in Proposition 2.5) and $\langle \text{Cube}, 2n \rangle$.

The next problem is formulated by looking at the examples from Section 4.1.

Problem 5.7 Prove that for every coloring γ of the tetrahedron spin network $(\triangle, \gamma)$, the sequence $\langle \triangle, n\gamma \rangle$ satisfies a second order recursion relation with coefficients polynomials in n. Can you compute the coefficients of this recursion from γ alone?

Let us end this section with a remark. The main results of our paper can be extended to evaluations of spin networks corresponding to higher rank Lie groups. This will be discussed in a later publication.

Appendix A: Asymptotics of the regular cube

We give the asymptotic expansion of the standard evaluation a_n of the 1–skeleton of the 3–dimensional cube, with all edges colored by $2n$. Proposition 2.4 implies (a_n) is given by

$$a_n = \sum_{k=0}^{2n} (2k+1) a_{n,k}^4, \tag{28}$$

$$a_{n,k} = \sum_j (-1)^j \binom{k}{j-3n}^2 \binom{2n-k}{4n-j} \binom{j+1}{2n+k+1},$$

making it clear that the numbers (a_n) are integral and positive. The first few values of a_n are given by

$$\begin{aligned}
a_0 &= 1,\\
a_1 &= 6144,\\
a_2 &= 505197000,\\
a_3 &= 77414400000000,\\
a_4 &= 13937620296600000000,\\
a_5 &= 3685480142898164744060928,\\
a_6 &= 1038107879077276408534853271552,\\
a_7 &= 297223547548257752224492840550400000,\\
a_8 &= 104193297934159421485149830847575156250000,\\
a_9 &= 35577316035253000096415678610598379040000000000,\\
a_{10} &= 12357485751601160513255660198337121351402277161410240.
\end{aligned}$$

We first look for a recursion of the form

$$\sum_{j=0}^{J} c_j(n)\, a_{n+j} = 0, \tag{29}$$

with J not too large and $c_j(n)$ polynomials of n of some not too large degree d. Using the first few hundred values of (a_n), we find experimentally a recursion of this form with $J=4$, $d=61$ and with $c_j(n)$ given by

$$\begin{aligned}
c_0(n) &= 3^{16}(2n+7)(3n+2)^8(3n+4)^8(3n+5)^7(3n+7)^7(3n+8)(3n+10)\\
&\qquad\cdot(4n+3)(4n+5)P_0(n+3),\\
c_1(n) &= -2\cdot 3^8(n+1)^5(2n+3)(2n+7)(3n+5)^7(3n+7)^7(3n+8)(3n+10)P_1(n),\\
c_2(n) &= -2\cdot 3^4(n+1)^5(n+2)^7(2n+5)(3n+8)(3n+10)P_2(n),\\
c_3(n) &= -2\cdot(n+1)^5(n+2)^7(n+3)^9(2n+3)(2n+7)P_1(-n-5),\\
c_4(n) &= (n+1)^5(n+2)^7(n+3)^9(n+4)^{11}(2n+3)(4n+15)(4n+17)P_0(n+2),
\end{aligned}$$

where P_0, P_1 and P_2 are irreducible polynomials (normalized to have integral coefficients with no common factor) with leading terms

$$\begin{aligned}
P_0(n) &= 2^{11}3^75^57\cdot 23^547(n^{26}+\mathrm{O}(n^{24})),\\
P_1(n) &= 2^{15}3^75^57^323^547^3(n^{38}+94n^{37}+\mathrm{O}(n^{36})),\\
P_2(n) &= 2^{15}3^{13}5^57\cdot 19\cdot 23^547\cdot 71\cdot 73(n^{46}+115n^{45}+\mathrm{O}(n^{44})),
\end{aligned}$$

as $n\to\infty$, and with the polynomial P_0 being even. The full values are given at the end of the appendix.

To analyze the asymptotics of the solutions of the above recursion, we will use the standard Frobenius theory; see for example Miller [30], Olver [33], Wasow [49] and Wimp and Zeilberger [53]. If C_j denotes the top coefficient of the polynomial $c_j(n)$, then we find $\sum_{j=0}^4 C_j\lambda^j$ factors as $(\lambda-3^{12})^2\,(\lambda-(1+\sqrt{-2})^{24})\,(\lambda-(1-\sqrt{-2})^{24})$ and that the indicial equation of the root 3^{12} has a double root at $-9/2$, while the indicial equations of the roots $(1\pm\sqrt{-2})^{24}$ both have root -4. This implies that (a_n) has an asymptotic expansion

$$a_n \sim S_0\frac{3^{12n}}{n^4}\Big((\log n+c)M_1\Big(\frac{1}{n}\Big)+M_2\Big(\frac{1}{n}\Big)\Big)+\Re\Big(S_1\frac{(1+\sqrt{-2})^{24n}}{n^{9/2}}M_3\Big(\frac{1}{n}\Big)\Big) \tag{30}$$

for some constants S_0, $c\in\mathbb{R}$, $S_1\in\mathbb{C}$ and power series $M_1(x)$, $M_2(x)\in\mathbb{Q}[\![x]\!]$, $M_3(x)\in\mathbb{Q}[\sqrt{-2}][\![x]\!]$, normalized by requiring that M_1 and M_3 have constant term 1. Notice that the three roots 3^{12}, $(1\pm\sqrt{-2})^{24}$ have the same absolute value, so that the different terms of this expansion all have the same order of magnitude up to powers of n. Using the acceleration method[1] described by Zagier [54, page 954] and Grünberg and Moree [21, Section 4], applied to the values of a_n for $n=1000,\dots,1050$, we

[1] This method is equivalent to the Richardson transform, explained in detail by the first author, Its, Kapaev and Mariño in [18, Section 5.2], and also by Bender and Orszag in [5].

find the numerical values of the constants S_j and c and the first few coefficients of the power series M_i. The former are then recognized as

$$S_0 = \frac{3^5}{2^4\pi^6}, \quad c = \frac{7}{4}\log 2 + \log 3 + \gamma, \quad S_1 = \frac{(1+i)(1+\sqrt{-2})^{12}}{2^{31/4}\pi^{11/2}}, \tag{31}$$

and the latter as

$$\begin{aligned}
M_1(x) &= 1 - \frac{14}{9}x + \frac{419}{324}x^2 - \frac{5659}{8748}x^3 + \frac{84769}{629856}x^4 + \cdots,\\
M_2(x) &= \frac{1}{2}x - \frac{689}{864}x^2 + \frac{4771}{7776}x^3 - \frac{3799441}{22394880}x^4 + \cdots,\\
M_3(x) &= 1 - \frac{2080 - 43\sqrt{-2}}{1152}x + \frac{1985023 - 114208\sqrt{-2}}{1327104}x^2 + \cdots.
\end{aligned}$$

The acceleration method can give many more terms, but it is easier to simply substitute the Ansatz (30) into the recursion for the a_n, thus obtaining as many terms as desired. The approximation works very well in practice, eg, the maximal relative error between a_n and the right hand side of (30) with 50 terms of the power series $M_j(1/n)$ is about one part in 10^{105} for n between 900 and 1000. To first order, the above formulas say that the asymptotics of a_n are given by

$$a_n \sim 3^{12n+5}\frac{\log(2^{7/4}3n) + \gamma + \mathrm{O}(1/\sqrt{n})}{\pi^6\,(2n)^4}.$$

We end by giving the complete values of the polynomials $P_j(n)$ that appear in the recursion relation:

$$\begin{aligned}
P_0(n) = {} & 296390196760896000000n^{26} - 150687090646682256000n^{24}\\
& + 30665002281010487154 0n^{22} - 331831776907297971277n^{20}\\
& + 219414205267920364521n^{18} - 96826696589802950226n^{16}\\
& + 29683042452642732342n^{14} - 6233837158945489065n^{12}\\
& + 868763697226715493n^{10} - 77173811768742984n^8\\
& + 4094153904684504n^6 - 111886799053248n^4 + 797085625600n^2\\
& + 17508556800,\\
P_1(n) = {} & 513305140601538302976000000n^{38}\\
& + 48250683216544600479744000000n^{37}\\
& + 21995755293187341482403686400 0n^{36}\\
& + 64786940771956771719460400640000n^{35}
\end{aligned}$$

$$+ 1385960180588775176675737464662 40n^{34}$$
$$+ 2295022658488679405177615124025920n^{33}$$
$$+ 3061492998404649816251959572272850 8n^{32}$$
$$+ 3380720878366674197377642334399225 30n^{31}$$
$$+ 3151590998989517431768295237323718623n^{30}$$
$$+ 2516902360588581958593215891274441490 6n^{29}$$
$$+ 1741467163088784869225465657227912254 48n^{28}$$
$$+ 1053195250756920493731804102357697945572n^{27}$$
$$+ 5606361750518381240594997946959656401095n^{26}$$
$$+ 2641473679486192520967396205375485000212 4n^{25}$$
$$+ 1106426993668985265429757756458862576678 32n^{24}$$
$$+ 4134473456000502285211369919704494042609 66n^{23}$$
$$+ 1381980145537336658418260176761712507602933n^{22}$$
$$+ 4140268295003002172648827386155584658850114n^{21}$$
$$+ 1113242373371885259047253773582227287743659 2n^{20}$$
$$+ 2688584959402454142161306026880958006866901 6n^{19}$$
$$+ 5833497019961435249918671596660130510129977 3n^{18}$$
$$+ 1136756570068660494961205432511991608235389 84n^{17}$$
$$+ 1987746779911709028251825093422223627590669 32n^{16}$$
$$+ 3114436106568701932426294905767809444391118 36n^{15}$$
$$+ 4363344195442832645037679647165308688563806 48n^{14}$$
$$+ 5450973363815798648778905914411218303112428 64n^{13}$$
$$+ 6050444584814311117350142503526507509799965 44n^{12}$$
$$+ 5940073135745796837741456890723681826591973 76n^{11}$$
$$+ 5128861292228050602761630966560472778214137 60n^{10}$$
$$+ 3867093687435146900184465014437640218807303 68n^{9}$$
$$+ 2523423742269371317664774723793927152466493 44n^{8}$$
$$+ 1408884626475717853658110309707067480987607 04n^{7}$$

$$
\begin{aligned}
&+ 6631285204220480832585734640556244103359692 8n^6\\
&+ 2579673325468703653708889053984809723113472 0n^5\\
&+ 8069974595385074631605661061376909102284800n^4\\
&+ 1950251347843211319569463651786279088128000n^3\\
&+ 3415551508448266833096304004279894016000 00n^2\\
&+ 3855416349711228534647288752436674560000 0n\\
&+ 2104728968892765569954334578933760000000,\\
P_2(n) = {}& 3404443694285150188922880000 0n^{46}\\
&+ 3915110248427922717261312000000n^{45}\\
&+ 2196627067915657828489263928320 00n^{44}\\
&+ 8013067972307054904678991211520000n^{43}\\
&+ 2136932144184316194192980872154851 20n^{42}\\
&+ 4441366173640824819720536004281937600n^{41}\\
&+ 7489438471892887139721860626216552484 4n^{40}\\
&+ 1053306557874097263334396534371665684400n^{39}\\
&+ 1260393334121832469977502315956706676696 7n^{38}\\
&+ 1302705261999293710527933241635231412768 65n^{37}\\
&+ 1176678761157600477488177183321806479261195n^{36}\\
&+ 9375125430920826953262133708501549064882175n^{35}\\
&+ 6638353787356179957040710517708064336897073 8n^{34}\\
&+ 4203073099505456277902712780253740523129314 80n^{33}\\
&+ 2391581100396727961084015883784259869439400176n^{32}\\
&+ 1228071469777833171547275894751306165762094558 0n^{31}\\
&+ 5710556141067718110971414789913029140929246105 0n^{30}\\
&+ 2411471518807279455690925907185656239646774987 50n^{29}\\
&+ 9269191510847393021489245285511966020483294140 90n^{28}\\
&+ 3249130587428846232389342739110375794038903230050n^{27}\\
&+ 1040139558452243313722372904584727425120493883128 0n^{26}
\end{aligned}
$$

$$
\begin{aligned}
&+ 3044307818378504217323593639254580111073727860070 0n^{25}\\
&+ 8152347129504110112170214873982216698252738124968 0n^{24}\\
&+ 1998285679761681351587311963346051339711595369183 00n^{23}\\
&+ 4483946475788168084735555760997961463817670025570 15n^{22}\\
&+ 9208803286213498581975460618384229396838758302508 25n^{21}\\
&+ 1730060383317082970163176773077295571528776228823 811n^{20}\\
&+ 2970768265449411986606322605605748766920711781281 175n^{19}\\
&+ 4657049644272293081174713334539070013552116895070 038n^{18}\\
&+ 6654448413726919814684796853369712557012807014275 460n^{17}\\
&+ 8650083204510826813891208265741376798744454496660 220n^{16}\\
&+ 1020435093608162322715142312160908352932997428941 4800n^{15}\\
&+ 1089257892991156360883467377512790002524248874789 2352n^{14}\\
&+ 1048372358480988942962366727231831952705820129673 2320n^{13}\\
&+ 9059104715506400545606048153659029802430897952414 784n^{12}\\
&+ 6992047378850409545281115023066707881078577900939 520n^{11}\\
&+ 4790327226399433431900479655636468590589446522365 440n^{10}\\
&+ 2891085435841974142824330005531725765542027596096 000n^{9}\\
&+ 1522663617068026467557018527073299284644425562419 200n^{8}\\
&+ 6916040381466444171537278418706251077388810531840 00n^{7}\\
&+ 2668225165203749016196216325546681927715633408000 00n^{6}\\
&+ 8569733024671315291886433875334466830204503040000 0n^{5}\\
&+ 2228766307487877964839702817567682350111795200000 0n^{4}\\
&+ 4507926265942350227509023190826979902899200000000 n^{3}\\
&+ 6650458956249292938305985123626848419840000000000 n^{2}\\
&+ 6363528251174783559928067350587667660800000000 0n\\
&+ 2962949736504660768760707593463767040000000000.
\end{aligned}
$$

Appendix B: Further comments on the asymptotics of the regular cube

The guessed recursion relation of the sequence (a_n) from the previous section agrees with the result of the independent guessing program `Guess` of Kauers [26; 25]. The recursion for (a_n) was verified for $n = 0, \dots, 2996$, where the height (ie, the number of digits) of a_{3000} is 17162. On the other hand, the coefficients of the polynomials $c_k(n)$ are integers with a much smaller height 73. In addition, the root λ_1 of the characteristic polynomial can be written in the form

$$\lambda_1 = (1 + i\sqrt{2})^{24} = 3^{12} e^{12i \arccos(-1/3)},$$

where $e^{12i \arccos(-1/3)}$ is the exponentiated total mean curvature of the regular Euclidean octahedron (dual to the regular Euclidean cube) with unit sides. This confirms Conjecture 4.3 on the asymptotics of evaluations of classical spin networks. The factor 3^{12} comes from the fact that we are considering the standard normalization and not the unitary one as is done in Conjecture 4.3.

The asymptotic expansion (30) is clearly of Nilsson type, with the presence of logarithms, and Stokes constants which are no longer algebraic, up to rational powers of π. This makes it unlikely that stationary phase type methods will be able to obtain the asymptotic expansion for the regular cube.

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