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Logarithmic Series and Hodge Integrals in the Tautological Ring

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(with an Appendix by D. ZAGIER)

Dedicated to William Fulton on the occasion of his 60th birthday

0. Introduction

0.1. OVERVIEW. Let $\bar{M}_g$ be the moduli space of Deligne–Mumford stable curves of genus $g \geq 2$. The study of the Chow ring of the moduli space of curves was initiated by Mumford in [Mu]. In the past two decades, many remarkable properties of these intersection rings have been discovered. Our first goal in this paper is to describe a new perspective on the intersection theory of the moduli space of curves that encompasses advances from both classical degeneracy studies and topological gravity. This approach is developed in Sections 0.2–0.7.

The main new results of the paper are computations of basic Hodge integral series in $A^*(M_g)$ encoding the canonical evaluations of $\kappa_{g-2-i}\lambda_i$. The motivation for the study of these tautological elements and the series results are given in Section 0.8. The body of the paper contains the Hodge integral derivations.

0.2. MODULI FILTRATION. We will consider the moduli filtration

$$\bar{M}_g \supset M_g^c \supset M_g \supset \{[X_g]\}. \tag{1}$$

Here, X_g is a fixed nonsingular curve, M_g is the moduli space of nonsingular genus g curves, and M_g^c is the moduli space of stable curves of compact type (curves with tree dual graphs or, equivalently, with compact Jacobians).

Let $A^*(\bar{M}_g)$ denote the Chow ring with $\mathbb{Q}$-coefficients. Intersection theory on $\bar{M}_g$ may be naturally viewed in four stages corresponding to the filtration (1). There is an associated sequence of successive quotients:

$$A^*(\bar{M}_g) \to A^*(M_g^c) \to A^*(M_g) \to A^*([X_g]) \cong \mathbb{Q}. \tag{2}$$

We develop here a uniform approach to the study of these quotient rings.

0.3. TAUTOLOGICAL RINGS. The study of the structure of the entire Chow ring of the moduli space of curves appears quite difficult at present. While presentations are known in a few genera [F1; F2; I; Mu], no general results have yet been conjectured. Since the principal motive is to understand cycle classes obtained from algebro-geometric constructions, it is natural to restrict inquiry to the *tautological ring* $R^*(\bar{M}_g) \subset A^*(\bar{M}_g)$.

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It is most convenient to define the full system of tautological rings of all the moduli spaces of pointed curves simultaneously:

$$\{R^*(\bar{M}_{g,n}) \subset A^*(\bar{M}_{g,n})\}. \tag{3}$$

The first step is to define the cotangent line classes ψ_i. The class

$$\psi_i \in A^1(\bar{M}_{g,n})$$

is the first Chern class of the line bundle with fiber $T^*_{p_i}(C)$ over the moduli point $[C, p_1, \ldots, p_n] \in \bar{M}_{g,n}$. The tautological system (3) is defined to be the set of smallest $\mathbb{Q}$-subalgebras satisfying the following three properties.

(i) $R^*(\bar{M}_{g,n})$ contains the cotangent line classes $\psi_1, \ldots, \psi_n$.

(ii) The system is closed under push-forward via all maps forgetting markings:

$$\pi_*\colon R^*(\bar{M}_{g,n}) \to R^*(\bar{M}_{g,n-1}).$$

(iii) The system is closed under push-forward via all gluing maps:

$$\pi_*\colon R^*(\bar{M}_{g_1,n_1\cup\{*\}}) \otimes_{\mathbb{Q}} R^*(\bar{M}_{g_2,n_2\cup\{\cdot\}}) \to R^*(\bar{M}_{g_1+g_2,n_1+n_2}),$$
$$\pi_*\colon R^*(\bar{M}_{g_1,n_1\cup\{*,\cdot\}}) \to R^*(\bar{M}_{g_1+1,n_1}).$$

Natural algebraic constructions typically yield Chow classes lying in the tautological ring.

We point out four additional properties of the tautological system that are consequences of the definition.

(iv) The system is closed under pull-back via the forgetting and gluing maps.

(v) $R^*(\bar{M}_{g,n})$ is an $\mathbb{S}_n$-module via the permutation action on the markings.

(vi) The κ classes lie in the tautological rings.

(vii) The λ classes lie in the tautological rings.

Property (iv) follows from the well-known boundary geometry of the moduli space of curves. Since properties (i)–(iii) are symmetric under the marking permutation action, property (v) is obtained. Property (vi) is true by definition because

$$\pi_*(\psi_{n+1}^{l+1}) = \kappa_l \in R^*(\bar{M}_{g,n}),$$

where π is the map forgetting the marking $n+1$ (see [AC]). Recall that the λ classes are the Chern classes of the Hodge bundle $\mathbb{E}$ on the moduli space of curves. Property (vii) is a consequence of Mumford's Grothendieck–Riemann–Roch computation [Mu].

The tautological rings for the other elements of the filtration (1) are defined by the images of $R^*(\bar{M}_g)$ in the quotient sequence (2):

$$R^*(\bar{M}_g) \to R^*(M_g^c) \to R^*(M_g) \to R^*([X_g]) \cong \mathbb{Q}. \tag{4}$$

0.4. EVALUATIONS. The quotient rings (4) exhibit several parallel structures that serve to guide their study. Each admits a canonical *nontrivial* linear evaluation ε to $\mathbb{Q}$ obtained by integration. For $\bar{M}_g$, ε is defined by

$$\xi \in R^*(\bar{M}_g), \qquad \varepsilon(\xi) = \int_{\bar{M}_g} \xi.$$

The other three evaluations involve the λ classes.

Recall that the fiber of $\mathbb{E}$ over a moduli point $[C] \in \bar{M}_g$ is the rank-g vector space $H^0(C, \omega_C)$. Let $\Delta_0 = \bar{M}_g \setminus M_g^c$. A basic vanishing holds:

$$\lambda_g\big|_{\Delta_0} = 0. \tag{5}$$

To prove (5), consider the standard ramified double cover $\bar{M}_{g-1,2} \to \Delta_0$,

$$[\tilde{C}, p_1, p_2] \mapsto [C],$$

obtained by identifying the markings p_1, p_2 of $\tilde{C}$ to form a nodal curve C. The pull-back of $\mathbb{E}$ to $\bar{M}_{g-1,2}$ admits a surjection to the trivial bundle $\mathbb{C}$ over $\bar{M}_{g-1,2}$ obtained from the residue of $\sigma \in H^0(C, \omega_C)$ at the distinguished node of C. Hence, the pull-back of λ_g vanishes on $\bar{M}_{g-1,2}$. As we consider Chow groups with $\mathbb{Q}$-coefficients, the vanishing (5) follows.

For M_g^c, evaluation is defined by

$$\xi \in R^*(M_g^c), \qquad \varepsilon(\xi) = \int_{\bar{M}_g} \xi \cdot \lambda_g,$$

well-defined by the vanishing property of λ_g. Similarly, the vanishing of the restriction of $\lambda_g\lambda_{g-1}$ to $\bar{M}_g \setminus M_g$ is proven in [F3]. Define evaluation for M_g by

$$\xi \in R^*(M_g), \qquad \varepsilon(\xi) = \int_{\bar{M}_g} \xi \cdot \lambda_g\lambda_{g-1}.$$

Finally, define evaluation for $[X_g]$ by

$$\xi \in R^*([X_g]), \qquad \varepsilon(\xi) = \int_{\bar{M}_g} \xi \cdot \lambda_g\lambda_{g-1}\lambda_{g-2}.$$

These four evaluations do *not* commute with the quotient structure.

The nontriviality of the ε evaluations is proven by explicit integral computations. The integral computation

$$\int_{\bar{M}_g} \kappa_{3g-3} = \frac{1}{24^g g!} \tag{6}$$

explicitly shows that ε is nontrivial on $R^*(\bar{M}_g)$. Equation (6) follows from Witten's conjectures and Kontsevich's theorem (or, alternatively, via an algebraic computation in [FP2]). The integral

$$\int_{\bar{M}_g} \kappa_{2g-3}\lambda_g = \frac{2^{2g-1}-1}{2^{2g-1}} \frac{|B_{2g}|}{(2g)!} \tag{7}$$

shows nontriviality on $R^*(M_g^c)$ [FP2]. The integral

$$\int_{\bar{M}_g} \kappa_{g-2}\lambda_g\lambda_{g-1} = \frac{1}{2^{2g-1}(2g-1)!!} \frac{|B_{2g}|}{2g} \tag{8}$$

shows nontriviality on $R^*(M_g)$. Equation (8) is proven in Section 1. Finally, the computation

$$\int_{\bar{M}_g} \lambda_g \lambda_{g-1} \lambda_{g-2} = \frac{1}{2(2g-2)!} \frac{|B_{2g-2}|}{2g-2} \frac{|B_{2g}|}{2g} \tag{9}$$

establishes the last nontriviality [FP2]. We note the Bernoulli number convention used in these formulas is

$$\frac{t}{e^t - 1} = \sum_{m=0}^{\infty} B_m \frac{t^m}{m!}.$$

It is known B_{2g} never vanishes.

The ε evaluation maps are well-defined on the quotient sequence (2) of full Chow rings. To see the difference in perspective, the nontriviality of ε for $A^*(\bar{M}_g)$ is established by considering any point class whereas the nontriviality for $R^*(\bar{M}_g)$ requires a tautological point class—such as a maximally degenerate stratum or, alternatively, (6).

0.5. GORENSTEIN ALGEBRAS. Computations of $R^*(M_g)$ for genera $g \leq 15$ have led to the following conjecture for the ring structure for all genera [F3].

CONJECTURE 1. *$R^*(M_g)$ is a Gorenstein algebra with socle in codimension $g-2$.*

The evaluation ε is then a canonically normalized function on the socle. It is natural to hope analogous Gorenstein properties hold for $R^*(\bar{M}_g)$ and $R^*(M_g^c)$, but the data in these cases is very limited. The following conjectures are therefore really speculations.

SPECULATION 2. *$R^*(M_g^c)$ is a Gorenstein algebra with socle in codimension $2g-3$.*

SPECULATION 3. *$R^*(\bar{M}_g)$ is a Gorenstein algebra with socle in codimension $3g-3$.*

Conjecture 1 was verified for $g \leq 15$ via relations found by classical degeneracy loci techniques [F3] and the nonvanishing result (8)—see Section 1. In fact, a complete presentation of $R^*(M_g)$ has been conjectured in [F3] from these low-genus studies. Such calculations become much more difficult in $R^*(M_g^c)$ and $R^*(\bar{M}_g)$ because of the inclusion of nodal curves. It is known that $R^*(M_g^c)$ and $R^*(\bar{M}_g)$ are Gorenstein algebras for $g \leq 3$. It would be very interesting to find further evidence for or against Speculations 2 and 3.

A stronger version of Conjecture 1 was made in [HL]. Also, Speculation 3 was raised as a question in [HL].

The tautological ring $R^*([X_g]) \cong \mathbb{Q}$ is obviously Gorenstein. While this case of fixed moduli appears trivial in the present context, interesting geometry emerges when marked points are considered. An extension of the perspective on the tautological ring presented here to $\bar{M}_{g,n}$ and fiber products of the universal curve will be discussed in [FP3].

The moduli space of stable curves $\bar{M}_{g,n}$ may be viewed as a special case of the moduli space of stable maps $\bar{M}_{g,n}(X, \beta)$, so it is natural to investigate tautological

rings in the more general setting of stable maps. The first obstacle is finding the appropriate definitions in the context of the virtual class. However, in the case of genus-0 maps to homogeneous varieties, it is straightforward to define the tautological ring because the moduli space is a nonsingular Deligne–Mumford stack. In [P1], the tautological ring $R^*(M_{0,0}(\mathbf{P}^r, d))$ is proven to be a Gorenstein algebra.

0.6. Socle Rank and Higher Vanishing Predictions. The Gorenstein conjectures and speculations of Section 0.5 imply the ranks of the tautological rings are 1 in the expected socle codimension. Moreover, vanishing above the socle codimension is implied in each case. The socle and vanishing results

$$R^{g-2}(M_g) \cong \mathbb{Q} \quad \text{and} \quad R^{>g-2}(M_g) = 0$$

are a direct consequence of Looijenga's theorem [L] and the nonvanishing (8) proven in Section 1. Looijenga's theorem states the tautological ring of the n-fold fiber product C_g^n of $C_g = M_{g,1}$ over M_g is *at most* rank 1 in codimension $g-2+n$ and vanishes in all codimensions greater than $g-2+n$.

It is natural to ask whether the tautological rings satisfy the usual exact sequences via restriction:

$$R^*(\partial \bar{M}_g) \to R^*(\bar{M}_g) \to R^*(M_g) \to 0. \tag{10}$$

Here, $R^*(\partial \bar{M}_g) \subset A^*(\partial \bar{M}_g)$ is generated by tautological classes pushed forward to the boundary $\partial \bar{M}_g$ of the moduli space of curves. Pointed generalizations of the restriction sequences (10) together with Looijenga's theorem and the nonvanishings (6) and (7) imply the socle and vanishing results for $R^*(M_g^c)$ and $R^*(\bar{M}_g)$. However, at present, the exactness of sequence (10) is not proven.

We note the socle dimension proof for $R^*(\bar{M}_g)$ in [HL, Sec. 5.1] is incomplete as it stands since (10) is assumed there (the error is repeated in [FL]).

0.7. Virasoro Constraints. The tautological rings (4) each have an associated Virasoro conjecture. For $\bar{M}_g$, the original Virasoro constraints (conjectured by Witten and proven by Kontsevich [K1]) compute all the integrals

$$\int_{\bar{M}_{g,n}} \psi_1^{\alpha_1} \cdots \psi_n^{\alpha_n}. \tag{11}$$

These integrals determine the ε evaluations in the ring $R^*(\bar{M}_g)$. The methods for calculating ε evaluations from the integrals (11) are effective but quite complicated (see [F3; HL; W]).

Eguchi, Hori, and Xiong (and S. Katz) have conjectured Virasoro constraints in Gromov–Witten theory for general target varieties V that specialize to Witten's conjectures in case V is a point [EHX]. In [GP], these general constraints are applied to *collapsed* maps to target curves, surfaces, and threefolds in order to study integrals of the Chern classes of the Hodge bundle. The Virasoro constraints for curves then imply:

$$\int_{\bar{M}_{g,n}} \psi_1^{\alpha_1} \cdots \psi_n^{\alpha_n} \lambda_g = \binom{2g+n-3}{\alpha_1, \ldots, \alpha_n} \int_{\bar{M}_{g,1}} \psi_1^{2g-2} \lambda_g, \tag{12}$$

where $\alpha_i \geq 0$. Equation (12) determines (up to scalars) the ε evaluations in the ring $R^*(M_g^c)$. This Virasoro conjecture for M_g^c has been proven in [FP1].

The Virasoro constraints for surfaces imply a formula previously conjectured in [F3] determining evaluations in $R^*(M_g)$:

$$\int_{\bar{M}_{g,n}} \psi_1^{\alpha_1} \cdots \psi_n^{\alpha_n} \lambda_g \lambda_{g-1} = \frac{(2g+n-3)!\,(2g-1)!!}{(2g-1)! \prod_{i=1}^n (2\alpha_i - 1)!!} \int_{\bar{M}_{g,1}} \psi_1^{g-1} \lambda_g \lambda_{g-1}, \quad (13)$$

where $\alpha_i > 0$ (see [GP]). Formula (13) is currently still conjectural.

Finally, the Virasoro constraints for threefolds yield relations among the integrals

$$\int_{\bar{M}_{g,n}} \psi_1^{\alpha_1} \cdots \psi_n^{\alpha_n} \lambda_g \lambda_{g-1} \lambda_{g-2}. \quad (14)$$

In fact, all integrals (14) are determined in terms of $\int_{\bar{M}_g} \lambda_g \lambda_{g-1} \lambda_{g-2}$ by the string and dilaton equations (which leads to a proof of the Virasoro constraints in this case [G]).

We note that the ring structure of a finite-dimensional Gorenstein algebra is *determined* by the socle evaluation of polynomials in the generators. Hence, if the Gorenstein properties of Section 0.5 hold for any of the tautological rings, the Virasoro constraints then determine the ring structure. This concludes our general discussion of the tautological rings of the moduli space of curves.

0.8. RESULTS. A basic generating series for 1-pointed Hodge integrals was computed in [FP2]:

$$1 + \sum_{g \geq 1} \sum_{i=0}^{g} t^{2g} k^i \int_{\bar{M}_{g,1}} \psi_1^{2g-2+i} \lambda_{g-i} = \left(\frac{t/2}{\sin(t/2)} \right)^{k+1}. \quad (15)$$

Equation (15) may be interpreted as determining ε evaluations of the monomials

$$\kappa_{3g-3-i} \lambda_i \in R^{3g-3}(\bar{M}_g).$$

The main result of this paper is a determination of related evaluations in $R^{g-2}(M_g)$.

First, the basic series for the nontriviality of ε on $R^*(M_g)$ is calculated.

THEOREM 1. *For genus $g \geq 2$,*

$$\int_{\bar{M}_g} \kappa_{g-2} \lambda_g \lambda_{g-1} = \frac{1}{2^{2g-1}(2g-1)!!} \frac{|B_{2g}|}{2g}. \quad (16)$$

Two proofs of Theorem 1 are given in the paper. The first uses Mumford's Grothendieck–Riemann–Roch formulas for the Chern character of $\mathbb{E}$ and the Witten–Kontsevich theorem in KdV form. The derivation appears in Section 1, following a discussion of the context of this calculation. The second proof appears in Section 5 as a combinatorial consequence of Theorem 3 below. The required combinatorics is explained in the Appendix by D. Zagier.

Next, integrals encoding the values of all the monomials

$$\kappa_{g-2-i} \lambda_i \in R^{g-2}(M_g)$$

are studied. For positive integers g and k, let

$$I(g,k) = \int_{\bar{M}_{g,1}} \frac{1-\lambda_1+\lambda_2-\cdots+(-1)^g\lambda_g}{\prod_{i=1}^{k}(1-i\psi_1)}\lambda_g\lambda_{g-1}.$$

The integrals $I(g,k)$ arise geometrically in the following manner. Let

$$\pi\colon M_{g,1} \to M_g$$

be the universal curve. Let J_k denote the rank-k vector bundle with fiber $H^0(C,\omega_C/\omega_C(-kp))$ at the moduli point $[C,p]$; J_k is a bundle of π-vertical $(k-1)$-jets of ω_π. There is a canonical (dualized) evaluation map

$$J_k^* \to \mathbb{E}^* \tag{17}$$

on $M_{g,1}$. For $g \geq 2$,

$$I(g,k) = \varepsilon(\pi_* c_{g-1}(\mathbb{E}^*/J_k^*)),$$

where the ε evaluation is taken in $R^*(M_g)$.

For $k=1$ we have that $J_1 = \omega_\pi$ and the map (17) is a bundle injection. Then $I(g,1)$ is the evaluation of the π-push-forward of the Euler class of the quotient:

$$I(g,1) = \varepsilon(\pi_* c_{g-1}((\mathbb{E}^*/\omega_\pi^*)).$$

The integrals $I(g,2)$ are easily related to the (stack) classes of the hyperelliptic loci $[H_g] \in R^{g-2}(M_g)$ by the equation

$$I(g,2) = (2g+2)\cdot\varepsilon([H_g]) \tag{18}$$

(see [Mu]). For $k>2$, $I(g,k)$ does not admit such simple interpretations. However, the generating series of these integrals appear to be the best-behaved analogs of (15) in $R^*(M_g)$. The search for such an analog was motivated by the parallel structure view of these tautological rings.

For each positive integer k, define

$$G_k(t) = \sum_{g\geq 1} t^{2g+k-1} I(g,k).$$

These generating series are uniquely determined as follows.

THEOREM 2. *For all integers $k \geq 1$, the series $G_k(t)$ satisfies*

$$\frac{d^{k-1}G_k}{dt^{k-1}} = \sum_{j=1}^{k}(-1)^{k-j}\frac{j^{k-1}}{k}\binom{k}{j}\log\left(\frac{jt/2}{\sin(jt/2)}\right). \tag{19}$$

In case $k=1$ we obtain the following corollary, first encountered in the study of degenerate threefold contributions in Gromov–Witten theory [P2].

COROLLARY 1.

$$\sum_{g\geq 1} t^{2g}\int_{\bar{M}_g}\left(\sum_{i=0}^{g-2}(-1)^i\kappa_{g-2-i}\lambda_i\right)\lambda_g\lambda_{g-1} = \log\left(\frac{t/2}{\sin(t/2)}\right).$$

In case $k = 2$, we find

$$(G_2)' = \log\left(\frac{2\sin(t/2)}{\sin(t)}\right) = -\log\left(\cos\left(\frac{t}{2}\right)\right).$$

The generating series for the evaluations of the hyperelliptic loci in $R^*(M_g)$ (with an appropriate genus-1 term) is

$$H(t) = \frac{t^2}{96} + \sum_{g\geq 2} t^{2g}\varepsilon([H_g]).$$

By Mumford's calculation (18),

$$(t^2 H)' = G_2.$$

Theorem 2 then yields the following result.

COROLLARY 2. *The hyperelliptic evaluations are determined by*

$$(t^2 H)'' = -\log(\cos(t/2)). \tag{20}$$

Equation (20) was conjectured previously in an equivalent Bernoulli number form in [F3]: for $g \geq 2$,

$$\varepsilon([H_g]) = \frac{(2^{2g}-1)|B_{2g}|}{(2g+2)!\,2g}.$$

Theorem 2 is derived here from relations obtained by virtual localization in Gromov–Witten theory (see [FP1; FP2; GrP]). In addition to the cohomology classes on the moduli space of stable maps $\bar{M}_{g,n}(\mathbf{P}^1, d)$ considered in [FP1], new classes obtained from the ramification map of [FanP] play an essential role. The Hodge integral series (15) and Virasoro constraints (12) for M_g^c are also used. This derivation appears in Sections 2 and 3.

In case $k = 2$, the integrals $I(g, 2)$ may be computed by reduction to the moduli space of hyperelliptic curves. This classical derivation provides a contrast to the more formal Gromov–Witten arguments. Section 4 contains these hyperelliptic computations.

In Section 5, the standard 1-point Hodge integral series for $R^*(M_g)$ is studied, and the following consequence of Theorem 2 is found.

THEOREM 3. *For positive integers g and k,*

$$\sum_{i=0}^{g-1}(-1)^i k^{g-1-i}\int_{\bar{M}_{g,1}} \psi_1^{g-1-i}\lambda_i\lambda_g\lambda_{g-1} = \frac{|B_{2g}|}{2g}\sum_{l=1}^{k}\frac{(k-1)!}{(k-l)!}\frac{l!}{k^l}\frac{\mathfrak{S}^{(l)}_{2g-1+l}}{(2g-1+l)!}.$$

Here, $\mathfrak{S}^{(l)}_{n+l}$ is the Stirling number of the second kind: $\mathfrak{S}^{(l)}_{n+l}$ equals the number of partitions of a set of $n + l$ elements into l nonempty subsets.

Theorem 3 and the Appendix together provide proofs of all previously conjectured formulas for 1-point integrals in the tautological ring. In particular, closed forms for the evaluations in $R^*(M_g)$ of

$$\kappa_{g-2},\ \kappa_{g-3}\lambda_1,\ \kappa_1\lambda_{g-3},\ \lambda_{g-2} \tag{21}$$

are found, providing an alternate derivation of Theorem 1 and settling conjectures of [F3; F4]. A list of these formulas is provided in Section 5.2. In fact, the combinatorial results of the Appendix lead to proofs of natural extensions of the formulas for (21).

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1. Theorem 1

1.1. CONTEXT. Looijenga proved in [L] that the tautological ring $R^*(M_g)$ vanishes in degrees greater than $g-2$ and is at most 1-dimensional in degree $g-2$, generated by the class of the hyperelliptic locus. Theorem 1 shows that

$$\varepsilon(\kappa_{g-2}) = \int_{\bar{M}_g} \kappa_{g-2}\lambda_g\lambda_{g-1}$$

is nonzero, where ε is the evaluation on $R^*(M_g)$ (see Section 0.4). Hence, κ_{g-2} is nonzero in $R^{g-2}(M_g)$. In Section 1.2 we present the first proof of Theorem 1, relying upon an explicit calculation using the Witten–Kontsevich theorem in KdV form. The resulting nonvanishing of the tautological ring $R^*(M_g)$ in degree $g-2$ completed the verification for $5 \le g \le 15$ of the conjectural description of $R^*(M_g)$ given in [F3]. A second, more geometric proof of this nonvanishing appears in Section 4 using the defining property of hyperelliptic curves. Later proofs may be found in [FP2] and [P2], showing the nonvanishing in $R^{g-2}(M_g)$ of λ_{g-2} and $\sum_{i=0}^{g-2}(-1)^i\kappa_i\lambda_{g-2-i}$, respectively. Theorem 1 is re-derived in Section 5 from Theorem 3 (together with the Appendix), providing an alternative to the KdV derivation here.

1.2. FIRST PROOF OF THEOREM 1. Using Mumford's expression [Mu] for the Chern character of the Hodge bundle and the resulting identity

$$\lambda_g\lambda_{g-1} = (-1)^{g-1}(2g-1)!\,\mathrm{ch}_{2g-1}(\mathbb{E})$$

[FP2, (4.3)], Theorem 1 is reduced to the identity

$$\begin{aligned}\frac{1}{2^{2g-1}(2g-1)!!} &= \langle\tau_{2g}\tau_{g-1}\rangle - \langle\tau_{3g-2}\rangle + \frac{1}{2}\sum_{j=0}^{2g-2}(-1)^j\langle\tau_{2g-2-j}\tau_j\tau_{g-1}\rangle \\ &\quad + \frac{1}{2}\sum_{h=1}^{g-1}\Big((-1)^{g-h}\langle\tau_{3h-g}\tau_{g-1}\rangle\langle\tau_{3g-3h-2}\rangle \\ &\qquad + (-1)^h\langle\tau_{3h-2}\rangle\langle\tau_{2g-3h}\tau_{g-1}\rangle\Big) \qquad (22)\end{aligned}$$

(see [FP2, (16)]). Here, the second sum equals

$$\sum_{h=1}^{g-1} \frac{(-1)^{g-h}}{24^{g-h}(g-h)!} \langle \tau_{3h-g}\tau_{g-1}\rangle,$$

since $\langle \tau_{3k-2}\rangle = 1/(24^k k!)$ by Equation (0.7). Hence, it suffices to prove the two identities

$$\sum_{h=1}^{g} \frac{(-1)^{g-h}}{24^{g-h}(g-h)!} \langle \tau_{3h-g}\tau_{g-1}\rangle = \frac{1}{24^g g!} \tag{23}$$

and

$$\sum_{j=0}^{2g-2} (-1)^j \langle \tau_{2g-2-j}\tau_j\tau_{g-1}\rangle = \frac{g!}{2^{g-2}(2g)!}. \tag{24}$$

Both are consequences of the following equation for coefficients resulting from Witten's KdV equation for power series [W, (2.33) and (2.19)]. For any monomial

$$T = \prod_{j=0}^{k} \tau_j^{d_j},$$

the coefficient equation holds:

$$\begin{aligned}(2n+1)\langle \tau_n\tau_0^2 T\rangle &= \tfrac{1}{4}\langle \tau_{n-1}\tau_0^4 T\rangle \\ &\quad + \sum_{0\le a_j\le d_j} \left(\prod_{j=0}^{k} \binom{d_j}{a_j}\right)\big(\langle \tau_{n-1}\tau_0 T_1\rangle\langle \tau_0^3 T_2\rangle \\ &\qquad + 2\langle \tau_{n-1}\tau_0^2 T_1\rangle\langle \tau_0^2 T_2\rangle\big), \end{aligned} \tag{25}$$

where the sum is over factorizations $T = T_1T_2$ with $T_1 = \prod_{j=0}^{k} \tau_j^{a_j}$.

For $T = \tau_b$ and $n = a$, this gives

$$\begin{aligned}&(2a+1)\langle \tau_0^2\tau_a\tau_b\rangle \\ &\quad = \tfrac{1}{4}\langle \tau_{a-1}\tau_0^4\tau_b\rangle + \langle \tau_{a-1}\tau_0\tau_b\rangle\langle \tau_0^3\rangle \\ &\qquad + \langle \tau_{a-1}\tau_0\rangle\langle \tau_0^3\tau_b\rangle + 2\langle \tau_{a-1}\tau_0^2\tau_b\rangle\langle \tau_0^2\rangle + 2\langle \tau_{a-1}\tau_0^2\rangle\langle \tau_0^2\tau_b\rangle. \end{aligned} \tag{26}$$

Consider now the two-point function $D(w,z) = \sum_{a,b\ge 0}\langle \tau_0\tau_a\tau_b\rangle w^a z^b$. Equation (26) is equivalent to the differential equation

$$\begin{aligned}&\left(2w\frac{\partial}{\partial w} + 1\right)((w+z)D(w,z)) \\ &\quad = \tfrac{1}{4}(w+z)^3 wD(w,z) + wD(w,z) \\ &\qquad + D(w,0)zD(0,z) + 2wD(w,0)D(0,z). \end{aligned} \tag{27}$$

It is easy to verify that the unique solution of this equation satisfying $D(w,0) = \exp(w^3/24)$ and $D(0,z) = \exp(z^3/24)$ is given by

$$D(w,z)=\exp\left(\frac{(w^3+z^3)}{24}\right)\sum_{n\geq 0}\frac{n!}{(2n+1)!}\left[\tfrac{1}{2}wz(w+z)\right]^n.$$

We learned this formula from Dijkgraaf [Dij]. Consequently, for all $k\geq 1$,

$$\sum_{h=0}^{g}\frac{(-1)^{g-h}}{24^{g-h}(g-h)!}\langle\tau_0\tau_{3h-g+k}\tau_{g-k}\rangle=0, \tag{28}$$

since this is the coefficient of $w^{2g+k}z^{g-k}$ in

$$\langle\tau_0\tau(w)\tau(z)\rangle\cdot\langle\tau_0\tau(-w)\tau_0\rangle=\exp\left(\frac{z^3}{24}\right)\sum_{n\geq 0}\frac{n!}{(2n+1)!}\left[\tfrac{1}{2}wz(w+z)\right]^n,$$

in which all terms of total degree $3g$ have degree at least g in z. Therefore, by applications of the string equation to (28), we find:

$$\begin{aligned}
\sum_{h=0}^{g}\frac{(-1)^{g-h}}{24^{g-h}(g-h)!}\langle\tau_{3h-g}\tau_{g-1}\rangle &= -\sum_{h=0}^{g}\frac{(-1)^{g-h}}{24^{g-h}(g-h)!}\langle\tau_{3h-g+1}\tau_{g-2}\rangle\\
&= +\sum_{h=0}^{g}\frac{(-1)^{g-h}}{24^{g-h}(g-h)!}\langle\tau_{3h-g+2}\tau_{g-3}\rangle\\
&\ \ \vdots\\
&= (-1)^{g-1}\sum_{h=0}^{g}\frac{(-1)^{g-h}}{24^{g-h}(g-h)!}\langle\tau_{3h-1}\tau_0\rangle\\
&= \sum_{h=1}^{g}\frac{1}{24^h h!}\frac{(-1)^{h+1}}{24^{g-h}(g-h)!}\\
&= \frac{1}{24^g g!}\left(\sum_{h=1}^{g}(-1)^{h+1}\binom{g}{h}\right)=\frac{1}{24^g g!},
\end{aligned}$$

which proves (23) for $g\geq 1$.

To prove (24), we use (25) for $T=\tau_b\tau_c$ and $n=a$. This is equivalent to a differential equation for the general three-point function $E(x,y,z)=\sum_{a,b,c\geq 0}\langle\tau_a\tau_b\tau_c\rangle x^a y^b z^c$ that specializes to the following differential equation for the special three-point function $F(w,z)=E(w,z,-z)$:

$$\begin{aligned}
&4w^2F(w,z)+2w^3\frac{\partial F}{\partial w}(w,z)-\frac{1}{4}w^5F(w,z)\\
&\qquad=w(2w+z)D(w,z)D(0,-z)+w(2w-z)D(w,-z)D(0,z).
\end{aligned}$$

It is clear that it has a unique solution. One verifies easily that the solution is

$$F(w,z)=\exp\left(\frac{w^3}{24}\right)\sum_{a,b\geq 0}(w^3)^a(wz^2)^b\frac{(a+b)!}{2^{a+b-1}(2a+2b+2)!}\binom{a+b+1}{2a+1}.$$

The coefficient of $w^g z^{2g}$ equals $(g+1)!/2^{g-1}(2g+2)!$, which gives (24). This finishes the (first) proof of Theorem 1. □

2. Localization Relations

2.1. RESULTS. In this section we use the localization method to find relations among Hodge integrals [FP1; FP2]. Define the Hodge integral Q_g^e for $g, e \geq 1$ by

$$Q_g^e = \int_{\bar{M}_{g,1}} \frac{1 - \lambda_1 + \lambda_2 - \cdots + (-1)^g \lambda_g}{1 - e\psi_1} \lambda_g \lambda_{g-1}. \tag{29}$$

The first step in the proof of Theorem 2 is the computation of Q_g^e.

To state the relations determining Q_g^e, we will need the following combinatorial coefficients. For any formal series $t(x) = \sum t_i x^i$, define

$$\mathcal{C}(x^i, t(x)) = t_i.$$

Let $\tau(x)$ be the series inverse of xe^{-x}:

$$\tau(x) = \sum_{r \geq 1} \frac{r^{r-1}}{r!} x^r.$$

For $d \geq e$, define f_{gde} by

$$f_{gde} = \frac{e^{e+1}}{e!} \sum_{l=0}^{2g} \frac{(2g + d - l - 1)!}{(2g - l)!} \frac{(-d)^l}{l!} \mathcal{C}(x^{d-e}, \tau^l(x)). \tag{30}$$

PROPOSITION 1. *For $d \geq 1$,*

$$\sum_{e=1}^{d} \sum_{g=1}^{\infty} Q_g^e f_{gde} t^{2g} = d^{d-1} \log\left(\frac{dt/2}{\sin(dt/2)} \right).$$

The proof of Proposition 1 depends upon almost all of the main results of [FP1; FP2; FanP; GrP]. Theorem 2 will be derived as a consequence of Proposition 1 in Section 3.

2.2. THE TORUS ACTION. Let $\mathbf{P}^1 = \mathbf{P}(V)$, where $V = \mathbb{C} \oplus \mathbb{C}$. Let $\mathbb{C}^*$ act diagonally on V:

$$\xi \cdot (v_1, v_2) = (v_1, \xi \cdot v_2). \tag{31}$$

Let p_1, p_2 be the fixed points $[1, 0]$, $[0, 1]$ of the corresponding action on $\mathbf{P}(V)$. An equivariant lifting of $\mathbb{C}^*$ to a line bundle L over $\mathbf{P}(V)$ is uniquely determined by the weights $[l_1, l_2]$ of the fiber representations at the fixed points

$$L_1 = L\big|_{p_1}, \qquad L_2 = L\big|_{p_2}.$$

The canonical lifting of $\mathbb{C}^*$ to the tangent bundle $T_{\mathbf{P}}$ has weights $[1, -1]$. We will utilize the equivariant liftings of $\mathbb{C}^*$ to $\mathcal{O}_{\mathbf{P}(V)}(1)$ and $\mathcal{O}_{\mathbf{P}(V)}(-1)$ with weights $[0, -1]$ and $[0, 1]$, respectively.

Let $\bar{M}_{g,n}(\mathbf{P}(V), d)$ be the moduli stack of stable, genus-g, degree-d maps to $\mathbf{P}^1$ (see [FuP; K2]). There are canonical maps

$$\pi : U \to \bar{M}_{g,n}(\mathbf{P}(V), d), \qquad \mu : U \to \mathbf{P}(V),$$

where U is the universal curve over the moduli stack. The representation (31) canonically induces $\mathbb{C}^*$-actions on U and $\bar{M}_{g,n}(\mathbf{P}(V), d)$ compatible with the maps π and μ (see [GrP]).

2.3. The Branch Morphism. In [FanP], a canonical branch divisor morphism γ is constructed using derived category techniques:

$$\gamma\colon \bar{M}_{g,n}(\mathbf{P}(V), d) \to \mathrm{Sym}^r(\mathbf{P}(V)) = \mathbf{P}(\mathrm{Sym}^r(V^*)), \tag{32}$$

where $r = 2d + 2g - 2$. We review the point-theoretic description of γ. Let

$$[f\colon C \to \mathbf{P}(V)]$$

be a moduli point, where C is a possibly singular curve. Let $N \subset C$ be the cycle of nodes of C, and let $\nu\colon \tilde{C} \to C$ be the normalization of C. Let $A_1, \ldots, A_a$ be the components of $\tilde{C}$ that dominate D, and let

$$\{a_i\colon A_i \to D\}$$

denote the natural maps. Since a_i is a surjective map between nonsingular curves, the classical branch divisor $\mathrm{br}(a_i)$ is well-defined. Let $B_1, \ldots, B_b$ be the components of $\tilde{C}$ contracted over D, and let $f(B_j) = p_j \in D$; then the following formula holds:

$$\gamma([f]) = \mathrm{br}(f) = \sum_i \mathrm{br}(a_i) + \sum_j (2g(B_j) - 2)[p_j] + 2f_*(N). \tag{33}$$

We note that γ commutes with the forgetful maps

$$\bar{M}_{g,n}(\mathbf{P}(V), d) \to \bar{M}_g(\mathbf{P}(V), d)$$

and that γ is equivariant with respect to the canonical action of $\mathbb{C}^*$ defined by the representation (31).

2.4. Equivariant Cycle Classes. We now describe the equivariant Chow classes that arise in the proof of Proposition 1.

First consider the $\mathbb{C}^*$-action on $\mathbf{P}(\mathrm{Sym}^r(V^*))$. There are exactly $r+1$ distinct $\mathbb{C}^*$-fixed points. For $0 \le a \le r$, let q_a denote the fixed point $v_1^{*(r-a)} v_2^{*a}$. The canonical $\mathbb{C}^*$-linearization on $S = \mathcal{O}(1)$ has weight w_a at q_a equal to a. Let S_i denote the unique $\mathbb{C}^*$-linearization of S for which the weight w_i at q_i equals zero. We note the weight at q_a of S_i is $a - i$. The first equivariant Chow classes considered are

$$s_i = \gamma^*(c_1(S_i))$$

for all $0 \le i \le r$.

Second, there is a natural rank $d + g - 1$ bundle on $\bar{M}_{g,n}(\mathbf{P}(V), d)$:

$$\mathbb{R} = R^1\pi_*(\mu^*\mathcal{O}_{\mathbf{P}(V)}(-1)). \tag{34}$$

The linearization $[0, 1]$ on $\mathcal{O}_{\mathbf{P}(V)}(-1)$ defines an equivariant $\mathbb{C}^*$-action on $\mathbb{R}$. We will require the equivariant top Chern class $c_{\mathrm{top}}(\mathbb{R})$.

Third, there is a canonical lifting of the $\mathbb{C}^*$-action on $\bar{M}_{g,n}(\mathbf{P}(V), d)$ to the Hodge bundle $\mathbb{E}$ over $\bar{M}_{g,n}(\mathbf{P}(V), d)$. Hence, the Chern classes λ_i yield equivariant cycle classes.

Finally, let

$$\mathrm{ev}_i : \bar{M}_{g,n}(\mathbf{P}(V), d) \to \mathbf{P}(V)$$

denote the ith evaluation morphism, and let

$$\rho_i = c_1(\mathrm{ev}_i^* \mathcal{O}_{\mathbf{P}(V)}(1)),$$

where we fix the $\mathbb{C}^*$-linearization $[0, -1]$ on $\mathcal{O}_{\mathbf{P}(V)}(1)$.

2.5. VANISHING INTEGRALS. We will obtain relations among Q_g^e from a sequence of vanishing integrals. Let $g, d \geq 1$, and let $P(g, d)$ denote the integral

$$P(g, d) = \int_{\bar{M}_{g,1}(\mathbf{P}^1, d)} \lambda_{g-1} c_{\mathrm{top}}(\mathbb{R}) \rho_1^2 \prod_{i=0}^{d-2} s_i = 0.$$

Since the virtual dimension of $\bar{M}_{g,1}(\mathbf{P}^1, d)$ equals $2d + 2g - 1$ and since the total dimension of the integrand is

$$(g-1) + (d+g-1) + 2 + (d-1) = 2d + 2g - 1,$$

it follows that the integral $P(g, d)$ is well-defined. Since $\rho_1^2 = 0$, we have $P(g, d) = 0$.

2.6. LOCALIZATION TERMS. Because all the integrand terms in $P(g, d)$ have been defined with $\mathbb{C}^*$-equivariant lifts, the virtual localization formula of [GrP] yields a computation of these integrals in terms of Hodge integrals over moduli spaces of stable curves.

The integrals $P(g, d)$ are expressed via localization as a sum over connected decorated graphs Γ (see [GrP; K2]) indexing the $\mathbb{C}^*$-fixed loci of $\bar{M}_{g,n}(\mathbf{P}(V), d)$. The vertices of these graphs lie over the fixed points $p_1, p_2 \in \mathbf{P}(V)$ and are labeled with genera (which sum over the graph to $g - h^1(\Gamma)$). The edges of the graphs lie over $\mathbf{P}^1$ and are labeled with degrees (which sum over the graph to d). Finally, the graphs carry a single marking on one of the vertices. The edge valence of a vertex is the number of incident edges (markings excluded).

The equivariant integrand of $P(g, d)$ has been chosen to force vanishing contributions for most graphs (see [FP1; FP2]). By the linearization choice on the bundle $\mathbb{R}$, we find: If a graph Γ contains a vertex lying over p_1 of edge valence greater than 1, then the contribution of Γ to $P(g, d)$ vanishes. This basic vanishing was first used in $g = 0$ by Manin in [M]. Additional applications have been pursued in [FP1; FP2; GrP].

By this vanishing, only *comb* graphs Γ contribute to $P(g, d)$. Comb graphs contain $k \leq d$ vertices lying over p_1, each connected by a distinct edge to a unique vertex lying over p_2. These graphs carry the usual vertex genus and marking data.

If the (unique) marking of Γ lies over p_1, then the contribution of Γ to $P(g, d)$ vanishes by the linearization choice for ρ_1. We may thus assume the marking of Γ lies over p_2.

A comb graph Γ is defined to have complexity $n \geq 0$ if exactly n vertices lying over p_1 have positive genus. A vertex v of positive genus $g(v)$ over p_1 yields the moduli space $\bar{M}_{g(v),1}$ occurring as a factor in the fixed point locus corresponding to Γ. Let $v_1, \ldots, v_{k'}$ denote the positive genus vertices over p_1. The fixed point locus corresponding to Γ is a quotient of

$$\prod_{i=1}^{k'} \bar{M}_{g(v_i),1} \times \bar{M}_{g',k+1}. \tag{35}$$

Here, the unique vertex over p_2 is of genus g', the comb consists of k total vertices over p_1, and the marking lies over p_2. The restriction of the integrand term $c_{\text{top}}(\mathbb{R})$ to the fixed locus yields the class

$$\prod_{i=1}^{k'} \lambda_{g(v_i)}$$

as a factor. The integrand term λ_{g-1} contributes the sum

$$\prod_{i=1}^{k'} \lambda_{g(v_i)}\lambda_{g'-1} + \sum_{i=1}^{k'} \lambda_{g(v_i)-1} \prod_{j\neq i} \lambda_{g(v_j)}\lambda_{g'}. \tag{36}$$

By (36) and the basic vanishing $\lambda_h^2 = 0 \in A^*(\bar{M}_{h,1})$ for $h > 0$, we easily see that graphs Γ of complexity greater than 1 contribute 0 to $P(g,d)$. We have now shown that only graphs of complexity 0 or 1 may contribute to $P(g,d)$.

Consider first a graph Γ of complexity 0. As before, let k be the total number of vertices over p_1. The image under γ of the fixed point locus corresponding to Γ is the point q_{d-k}. By the term $\prod_{i=0}^{d-2} s_i$ in the integrand, all such graphs contribute 0 unless $k = 1$. Therefore there is a unique complexity-0 graph Γ that contributes to $P(g,d)$. The contribution of this graph is

$$-(-1)^{d-g} d^{d-2} d^{2g} \int_{\bar{M}_{g,1}} \psi_1^{2g-1} \lambda_{g-1}. \tag{37}$$

The contribution is computed via a direct application of the virtual localization formula [GrP]. The string equation and the identity $c(\mathbb{E})c(\mathbb{E}^*) = 1$ are used as well. Only one Hodge integral (occurring at the vertex lying over p_2) appears.

Next, consider a graph Γ of complexity 1. Let v_1 denote the unique positive genus vertex. Let $h = g(v_1)$, and let e be the degree of the unique edge incident to v_1. Let $m = \{m_1, \ldots, m_l\}$ be the degrees of remaining edges of Γ. The triple (h, e, m) satisfies $h \leq g$ and $e \leq d$, and m is a partition of $d - e$. The set of such triples is in bijective correspondence to the set of complexity-1 graphs

$$(h, e, m) \leftrightarrow \Gamma(h, e, m).$$

The contribution of $\Gamma(h, e, m)$ to $P(g,d)$ contains two Hodge integrals: at the vertex v_1 and at the vertex v lying over p_2. The Hodge integral at v_1 is Q_h^e (up to signs). The Hodge integral at v is a λ_g integral (see [FP1]) and may be integrated by the Virasoro constraints (12). A direct computation then yields that the contribution of Γ is

$$\frac{(-1)^{d-g}}{d} Q_h^e \frac{e^{e+1}}{e!} \frac{(2h+d-l-1)!}{(2h-l)!} \frac{(-d)^l}{|\mathrm{Aut}(m)|} \prod_{i=1}^{l} \frac{m_i^{m_i-1}}{m_i!}$$
$$\cdot\, d^{2g-2h} \int_{\bar{M}_{g-h,1}} \psi_1^{2g-2h-2} \lambda_{g-h}. \quad (38)$$

Here, $\mathrm{Aut}(m)$ is the group that permutes equal parts of m. The contribution vanishes unless $2h \geq l$. Finally, the integral $\int_{\bar{M}_{0,1}} \psi_1^{-2}\lambda_0$ occurring in (38) in case $g = h$ is defined to be 1.

The integral $P(g,d)$ equals the sum of all graph contributions from (37) and (38). Since $P(g,d) = 0$, we have found a relation among the Hodge integrals, including the Q integrals.

2.7. PROOF OF PROPOSITION 1. The Hodge relation found in Section 2.6 can be rewritten using the following observations.

The Hodge integrals other than the Q integrals appearing in (37) and (38) are determined in [FP2] as follows:

$$\sum_{g\geq 0} d^{2g} t^{2g} \int_{\bar{M}_{g,1}} \psi_1^{2g-2}\lambda_g = \left(\frac{dt/2}{\sin(dt/2)}\right), \quad (39)$$

$$\sum_{g\geq 1} d^{2g} t^{2g} \int_{\bar{M}_{g,1}} \psi_1^{2g-1}\lambda_{g-1} = \left(\frac{dt/2}{\sin(dt/2)}\right) \cdot \log\left(\frac{dt/2}{\sin(dt/2)}\right). \quad (40)$$

Let $\mathrm{Part}(a,b)$ denote the set of partitions of a of length b. The equality

$$f_{hde} = \frac{e^{e+1}}{e!} \sum_{l=0}^{2h} \frac{(2h+d-l-1)!}{(2h-l)!} \sum_{m\in \mathrm{Part}(d-e,l)} \frac{(-d)^l}{|\mathrm{Aut}(m)|} \prod_{i=1}^{l} \frac{m_i^{m_i-1}}{m_i!}$$

follows directly from the definition (30).

Let $d \geq 1$ be fixed. The Hodge integral relations obtained from the vanishing of $P(g,d)$ for all $g \geq 1$ may then be expressed as a series equality:

$$\left(\sum_{e=1}^{d} \sum_{g=1}^{\infty} Q_g^e f_{gde} t^{2g}\right) \cdot \left(\frac{dt/2}{\sin(dt/2)}\right) = d^{d-1}\left(\frac{dt/2}{\sin(dt/2)}\right) \cdot \log\left(\frac{dt/2}{\sin(dt/2)}\right).$$

Proposition 1 follows from cancelling the invertible series (39). □

3. Theorem 2

3.1. REDUCTION. The derivation of Theorem 2 from Proposition 1 requires some knowledge of $\tau(x)$ and a significant amount of binomial combinatorics.

Let k be a fixed positive integer. We start by summing the right side of (19) using Proposition 1:

$$\sum_{j=1}^{k}(-1)^{k-j}\frac{j^{k-1}}{k}\binom{k}{j}\log\left(\frac{jt/2}{\sin(jt/2)}\right)$$
$$=\sum_{g=1}^{\infty}t^{2g}\sum_{e=1}^{k}Q_g^e\sum_{j=e}^{k}(-1)^{k-j}\frac{j^{k-j}}{k}\binom{k}{j}f_{gje}. \quad (41)$$

A direct partial fraction expansion shows the equality

$$I(g,k)=\sum_{e=1}^{k}Q_g^e(-1)^{k-e}\frac{e^k}{k!}\binom{k}{e}.$$

Hence, Theorem 2 is a direct consequence of (41) and the following proposition.

PROPOSITION 2. *Let $k \geq e$. Then*

$$\sum_{j=e}^{k}(-1)^{k-j}\frac{j^{k-j}}{k}\binom{k}{j}f_{gje}=\frac{(2g+k-1)!}{(2g)!}\cdot(-1)^{k-e}\frac{e^k}{k!}\binom{k}{e}.$$

3.2. POWERS OF τ. In order to prove Proposition 2, we will need a formula for the coefficients of $\tau^l(x)$ appearing in the definition (30) of f_{gje}.

LEMMA 1. *Let $r, l \geq 0$. Then*

$$\frac{1}{l!}\mathcal{C}(x^r,\tau^l(x))=\binom{r-1}{l-1}\frac{r^{r-l}}{r!}.$$

Proof. This is a direct application of the Lagrange inversion formula (see [dB, (2.2.4)]). Solving $x=z/f(z)$ with $f(z)=e^z$ gives

$$z=\tau(x)=\sum_{r=1}^{\infty}c_r x^r,$$
$$c_r=\frac{1}{r!}\left[\left(\frac{d}{dz}\right)^{r-1}(f(z))^r\right]_{z=0}=\frac{r^{r-1}}{(r!)}.$$

This is simply the well-known formula stated in Section 2.1. More generally,

$$g(z)=g(0)+\sum_{r=1}^{\infty}d_r x^r,$$
$$d_r=\frac{1}{r!}\left[\left(\frac{d}{dz}\right)^{r-1}\{g'(z)(f(z))^r\}\right]_{z=0}.$$

Applying this with $g(z)=z^l$ yields the result. □

3.3. PROOF OF PROPOSITION 2. Using definition (30), Lemma 1, and simple manipulations, we find that Proposition 2 is equivalent to the equation

$$\sum_{j=e+1}^{k}\sum_{l=1}^{j-e}\binom{2g+j-l-1}{j-1}\binom{k}{j}\binom{j-1}{e-1}\binom{j-e-1}{l-1}j^{k-j+l}(e-j)^{j-e-l}$$
$$= e^{k-e}\binom{k}{e}\left(\binom{2g+k-1}{k-1}-\binom{2g+e-1}{e-1}\right). \quad (42)$$

To proceed, we may write the left and right sides of the above equation canonically in terms of the binomials

$$\binom{2g+e-1}{t+e-1}$$

for $0 \le t \le k-e$, using the relations

$$\binom{2g+j-l-1}{j-1} = \sum_{t=l}^{j-e}\binom{j-e-l}{t-l}\binom{2g+e-1}{t+e-1},$$
$$\binom{2g+k-1}{k-1} = \sum_{t=0}^{k-e}\binom{k-e}{t}\binom{2g+e-1}{t+e-1}.$$

Then it suffices to match the coefficients

$$\sum_{j=e+1}^{k}\sum_{l=1}^{j-e}\binom{j-e-l}{t-l}\binom{k}{j}\binom{j-1}{e-1}\binom{j-e-1}{l-1}j^{k-j+l}(e-j)^{j-e-l}$$
$$= e^{k-e}\binom{k}{e}\binom{k-e}{t} \quad (43)$$

for $1 \le t \le k-e$ (the matching at $t=0$ is trivial). Equation (43) simplifies to

$$\sum_{j=e+1}^{k}\sum_{l=1}^{j-e}\binom{k-e-t}{j-e-t}\binom{t-1}{l-1}j^{k-j-1+l}(e-j)^{j-e-1-l} = -\frac{e^{k-e-1}}{t},$$

and summing over l yields

$$\sum_{j=e+t}^{k}\binom{k-e-t}{j-e-t}j^{k-j}(e-j)^{j-e-t-1} = -\frac{e^{k-e-t}}{t}.$$

Substitute $z = k-e$ and $s = j-e-t$. Then we must prove that

$$\sum_{s=0}^{z-t}\binom{z-t}{s}(e+s+t)^{z-t-s}(-s-t)^{s-1} = -\frac{e^{z-t}}{t} \quad (44)$$

for all $1 \le t \le z$. If the left side of (44) is viewed as a polynomial in e, then the coefficient of e^{z-t} clearly matches the right side. Hence, it suffices to show that the coefficient of e^q vanishes for $0 \le q < z-t$:

$$\sum_{s=0}^{z-t-q}\binom{z-t}{s}\binom{z-t-s}{q}(s+t)^{z-t-s-q}(-s-t)^{s-1} = 0,$$

which is equivalent to

$$\sum_{s=0}^{z-t-q}\binom{z-t-q}{s}(s+t)^{z-t-s-q}(-s-t)^{s-1}=0.$$

Substituting $n=z-t-q$ and simplifying, we must prove that

$$\sum_{s=0}^{n}(-1)^s\binom{n}{s}(s+t)^{n-1}=0 \tag{45}$$

for all $n>0$. Finally, the proof of Proposition 2 (and hence of Theorem 2) is completed by observing that (45) follows from the well-known relation

$$\sum_{s=0}^{n}(-1)^s\binom{n}{s}s^{\gamma}=0$$

for all $0\le\gamma\le n-1$. □

4. Hyperelliptic Hodge Integrals

In this section we compute, for all g, the M_g-evaluation of the class of the hyperelliptic locus H_g. As explained in Section 0, this provides an alternative proof of Theorem 1 in the case $k=2$ and its Corollary 2.

As in Section 1, the starting point is the identity

$$\lambda_g\lambda_{g-1}=(-1)^{g-1}(2g-1)!\,\mathrm{ch}_{2g-1}(\mathbb{E}). \tag{46}$$

Mumford's calculation [Mu] of the Chern character of the Hodge bundle then gives an expression for $\lambda_g\lambda_{g-1}$ in terms of κ and ψ classes. This expression lends itself very well to a direct evaluation on the hyperelliptic locus: in the usual model of hyperelliptic curves as double covers of rational curves, all relevant classes are pull-backs from the moduli of rational curves, where evaluation is straightforward. In the process one finds simple expressions (in the rational model) for all components of the restriction of $\mathrm{ch}(\mathbb{E})$ to the hyperelliptic locus. This generalizes the formula of Cornalba and Harris [CH] for λ_1 on $\bar{H}_g$. It seems plausible that these expressions will allow the evaluation of other hyperelliptic Hodge integrals.

We may view $\bar{M}_{0,2g+2}$ as the coarse moduli space of stable hyperelliptic curves of genus g with an ordering of the Weierstrass points (see [HM, 6C] or [FP2, Sec. 3.2]). The universal hyperelliptic curve is then the (stack) double cover of $\bar{M}_{0,2g+3}$ branched over B, the disjoint union of the $2g+2$ sections:

$$\begin{array}{ccc}\mathcal{C} & \xrightarrow{f} & \bar{M}_{0,2g+3}\\ {\scriptstyle\varpi}\downarrow & & \downarrow{\scriptstyle\pi}\\ \bar{H}_g^{\mathrm{ord}} & = & \bar{M}_{0,2g+2}.\end{array}$$

We have $\psi_1=f^*(\psi_{2g+3}-B/2)$. Writing h_i for the genus-g class κ_i viewed on $\bar{M}_{0,2g+2}$, we obtain

$$h_i = \varpi_*\psi_1^{i+1} = \varpi_*(f^*(\psi_{2g+3} - B/2))^{i+1} = \pi_* f_* f^*((\psi_{2g+3} - B/2)^{i+1})$$
$$= 2\pi_*((\psi_{2g+3} - B/2)^{i+1}) = 2\pi_*(\psi_{2g+3}^{i+1} + (-B/2)^{i+1})$$
$$= 2\kappa_i + 2\sum_{j=1}^{2g+2}\left(-\tfrac{1}{2}\right)^{i+1}(-\psi_j)^i = 2\kappa_i - 2^{-i}\sum_{j=1}^{2g+2}\psi_j^i.$$

(Here the genus-0 class κ_i in the last line is the generalization to $\bar{M}_{g,n}$ by Arbarello–Cornalba [AC] of Mumford's class for $\bar{M}_g$.) Writing $\chi_i = \mathrm{ch}_i(\mathbb{E})$, we have computed the first term in Mumford's formula,

$$\frac{(2k)!}{B_{2k}}\chi_{2k-1} = \kappa_{2k-1} + \frac{1}{2}\sum_{h=0}^{g-1} i_{h,*}\frac{\psi_1^{2k-1} + \psi_2^{2k-1}}{\psi_1 + \psi_2},$$

in the rational model; it remains to evaluate the boundary terms. (Recall that $\chi_{2k} = 0$ for positive k.)

Boundary divisors of $\bar{M}_{0,2g+2}$ come in two types: odd boundary divisors, with an underlying partition of $2g+2$ in two odd numbers (≥ 3); and even boundary divisors. As described in [CH] and [HM], the hyperelliptic curves corresponding to an odd boundary divisor generically have one disconnecting node and four automorphisms, whereas those corresponding to an even boundary divisor generically have two nondisconnecting nodes and two automorphisms.

As a result, Mumford's formula in codimension 1 reads on the rational model as follows:

$$12\chi_1 = 2\kappa_1 - \tfrac{1}{2}\psi + \tfrac{1}{2}\delta_{\text{odd}} + 2\delta_{\text{even}},$$

with evident notations. Since $\kappa_1 = \psi - \delta$ in genus 0, this simplifies to

$$8\chi_1 = \psi - \delta_{\text{odd}} = \kappa_1 + \delta_{\text{even}}.$$

The higher-codimension case is very similar. The terms with $1 \leq h \leq g-1$ in Mumford's formula correspond to the odd boundary divisors; in the rational model, they appear with an extra factor $\frac{1}{2}$. Now $\psi_1 = f_h^*(\psi_{2h+3} - B/2)$; since this is here a cotangent line at a Weierstrass point, we must evaluate $\psi_{2h+3} - B/2$ on a Weierstrass point divisor in $\bar{M}_{0,2h+3}$. It is easy to check that the result, as a class on a boundary divisor of $\bar{M}_{0,2g+2}$ with underlying partition $[2h+1,\, 2(g-h)+1]$, is $\frac{1}{2}\psi_*$, where ψ_* is the cotangent line in the node to the branch with $2h+1$ marked points. Analogously, for ψ_2 and genus $g-h$, we find $\frac{1}{2}\psi_\cdot$, where $\psi_\cdot$ is the cotangent line in the node to the other branch. Therefore the odd boundary contribution to $(2k)!/B_{2k})\chi_{2k-1}$ equals

$$\frac{1}{2}\sum_{\text{odd } D}\frac{\left(\frac{1}{2}\psi_*\right)^{2k-1} + \left(\frac{1}{2}\psi_\cdot\right)^{2k-1}}{\frac{1}{2}\psi_* + \frac{1}{2}\psi_\cdot}\Bigg|_D = \frac{1}{2^{2k-1}}\sum_{\text{odd } D}\frac{\psi_*^{2k-1} + \psi_\cdot^{2k-1}}{\psi_* + \psi_\cdot}\Bigg|_D.$$

The $h = 0$ term in Mumford's formula breaks up into terms corresponding to the even boundary divisors; each of these appears with an extra factor 2. To identify the classes ψ_1 and ψ_2, we need to construct the family of hyperelliptic curves corresponding to an even boundary divisor with underlying partition $[2h+2,\, 2k+2]$

(hence $h+k=g-1$). The base of the family is $\mathcal{C}_h \times \mathcal{C}_k$. The idea is to glue $\mathcal{C}_h \times_{\bar{H}_h} \mathcal{C}_h$ and $\mathcal{C}_k \times_{\bar{H}_k} \mathcal{C}_k$ along two sections on either side, the diagonal Δ and its image $\Delta' = \{(p, p')\}$ under the hyperelliptic involution on the second factor. However, Δ and Δ' intersect along $\Delta(W)$, where W is the Weierstrass divisor in $\mathcal{C}$. Hence $\mathcal{C} \times_{\bar{H}} \mathcal{C}$ must be blown up along $\Delta(W)$, on either side. The relative canonical divisor induced on the second factor after the blow-up can be identified with the class $\psi_1 + W$ on the second factor before blowing up. Therefore, the classes ψ_1 and ψ_2 in Mumford's formula correspond on the rational model to $f_h^*(\psi_{2h+3})$ and $f_k^*(\psi_{2k+3})$, respectively, and the even boundary contribution to $((2k)!/B_{2k})\chi_{2k-1}$ equals simply

$$2 \sum_{\text{even } D} \left. \frac{\psi_*^{2k-1} + \psi_\cdot^{2k-1}}{\psi_* + \psi_\cdot} \right|_D .$$

We have proven the following.

PROPOSITION 3. *In the coarse rational model $\bar{M}_{0,2g+2} = \bar{H}_g^{\mathrm{ord}}$, the Chern character of the genus-g Hodge bundle equals*

$$\begin{aligned} \mathrm{ch}(\mathbb{E}) = g + \sum_{k=1}^{g} \frac{B_{2k}}{(2k)!} \Bigg[& 2\kappa_{2k-1} - \frac{1}{2^{2k-1}} \sum_{j=1}^{2g+2} \psi_j^{2k-1} \\ & + \frac{1}{2^{2k-1}} \sum_{\text{odd } D} \left. \frac{\psi_*^{2k-1} + \psi_\cdot^{2k-1}}{\psi_* + \psi_\cdot} \right|_D \\ & + 2 \sum_{\text{even } D} \left. \frac{\psi_*^{2k-1} + \psi_\cdot^{2k-1}}{\psi_* + \psi_\cdot} \right|_D \Bigg]. \end{aligned}$$

(The vanishing of $\mathrm{ch}(\mathbb{E})$ in degrees $\geq 2g$—here, trivial—holds on $\bar{M}_g$ as well; see e.g. [FP2, (4.3)].)

In fact, these formulas can be simplified, just as in codimension 1:

$$\begin{aligned} \frac{(2k)!}{B_{2k}} \chi_{2k-1} &= \frac{2^{2k}-1}{2^{2k-1}} \left(\sum_{j=1}^{2g+2} \psi_j^{2k-1} - \sum_{\text{odd } D} \left. \frac{\psi_*^{2k-1} + \psi_\cdot^{2k-1}}{\psi_* + \psi_\cdot} \right|_D \right) \\ &= \frac{2^{2k}-1}{2^{2k-1}} \left(\kappa_{2k-1} + \sum_{\text{even } D} \left. \frac{\psi_*^{2k-1} + \psi_\cdot^{2k-1}}{\psi_* + \psi_\cdot} \right|_D \right). \end{aligned}$$

This follows from the identity

$$\kappa_{2k-1} = \sum_{j=1}^{n} \psi_j^{2k-1} - \left. \frac{\psi_*^{2k-1} + \psi_\cdot^{2k-1}}{\psi_* + \psi_\cdot} \right|_\delta$$

on $\bar{M}_{0,n}$, a consequence of Proposition 1 in [FP2].

COROLLARY. *On $\bar{H}_g^{\mathrm{ord}}$,*

$$\mathrm{ch}_{2g-1}(\mathbb{E}) = \frac{B_{2g}}{(2g)!}(2^{2g+1} - 2).$$

Hence, on the stack $\bar{H}_g$,

$$\lambda_g\lambda_{g-1} = \frac{(2^{2g}-1)|B_{2g}|}{(2g+2)!\,2g}.$$

Proof. By the foregoing we have

$$\frac{(2g)!}{B_{2g}}\chi_{2g-1} = \frac{2^{2g}-1}{2^{2g-1}}\left(1+\frac{1}{2}\sum_{h=1}^{g}\binom{2g+2}{2h}\right) = \frac{2^{2g}-1}{2^{2g-1}}2^{2g} = 2^{2g+1}-2,$$

whence the first formula. The second formula follows by using (46) and dividing by $2\cdot(2g+2)!$. The factor of 2 is required to account for the hyperelliptic automorphism groups in the stack $\bar{H}_g$. □

5. Theorem 2 Revisited

5.1. REFORMULATION. In this section we present a reformulation of Theorem 2 that reduces all known (and several conjectured) nonvanishing results to combinatorial identities. For $g \geq 1$, consider the polynomial $P_g(k)$ in k of degree $g-1$ (with zero constant term for $g \geq 2$) defined by

$$\frac{|B_{2g}|}{2g}P_g(k) = \sum_{i=0}^{g-1}(-1)^i k^{g-1-i}\int_{\bar{M}_{g,1}}\psi_1^{g-1-i}\lambda_i\lambda_g\lambda_{g-1}.$$

Note that the right-hand side equals Q_g^k as in (29) for positive integers k.

THEOREM 3. *For positive integers* g *and* k,

$$P_g(k) = \sum_{l=1}^{k}\frac{(k-1)!}{(k-l)!}\frac{1}{k^l}\sum_{m=1}^{l}(-1)^{l-m}\binom{l}{m}\frac{m^{2g+l-1}}{(2g+l-1)!}.$$

Proof. This follows directly from Theorem 2. By expanding the logarithmic series as in [FP2, Lemma 3], one obtains

$$I(g,k) = \frac{(k-1)!}{(2g+k-1)!}\frac{|B_{2g}|}{2g}\sum_{j=1}^{k}\frac{(-1)^{k-j}}{(k-j)!}\frac{j^{k-1}}{j!}j^{2g}.$$

Since

$$\frac{1}{\prod_{i=1}^{k}(1-i\psi_1)} = \sum_{n=0}^{\infty}\psi_1^n\frac{(-1)^k}{k!}\sum_{j=1}^{k}(-1)^j j^{k+n}\binom{k}{j},$$

we also have

$$I(g,k) = \int_{\bar{M}_{g,1}}\lambda_g\lambda_{g-1}c(\mathbb{E}^*)\sum_{n=0}^{\infty}\psi_1^n\frac{1}{k!}\sum_{j=1}^{k}(-1)^{k-j}j^{k-1}\binom{k}{j}j^{n+1}.$$

Now observe that the resulting identity can be written as $BA = DBV$, where A is the infinite vector with entries

$$A(j) = \int_{\bar{M}_{g,1}} \lambda_g \lambda_{g-1} c(\mathbb{E}^*) \sum_{n=0}^{g-1} j^{n+1} \psi_1^n$$

(for a fixed g), B is the infinite lower-triangular matrix with entries

$$B(i, j) = (-1)^{i+j} j^{i-1} \binom{i}{j}$$

for $1 \leq j \leq i$, D is the infinite diagonal matrix with entries

$$D(k, k) = \frac{(k-1)!}{(2g+k-1)!} \frac{|B_{2g}|}{2g},$$

and V is the infinite vector with entries $V(j) = j^{2g}$.

One easily shows that the inverse of B has entries $B^{-1}(i, j) = \binom{i-1}{j-1} i^{1-j}$ for $1 \leq j \leq i$. The theorem follows by writing out $A = B^{-1} D B V$ and using $(|B_{2g}|/2g) P_g(k) = A(k)/k$. □

The connection to the Stirling number formula in Section 0.8 is obtained from the equation

$$\mathfrak{S}^{(l)}_{2g-1+l} = \frac{1}{l!} \sum_{m=1}^{l} (-1)^{l-m} \binom{l}{m} m^{2g+l-1}.$$

5.2. Nonvanishing Results. We present here the reformulations of four nonvanishing results. All four are proved (from Theorem 3) by D. Zagier in the Appendix. Equivalently, these are identities in the socle of the tautological ring $R^*(M_g)$. First, the leading coefficient in $P_g(k)$ is

$$\mathcal{C}(k^{g-1}, P_g(k)) = \frac{1}{2^{2g-1}(2g-1)!!}. \tag{47}$$

Equation (47) is equivalent to Theorem 1 (providing an alternate proof that avoids the KdV equations). The next highest coefficient is

$$\mathcal{C}(k^{g-2}, P_g(k)) = \frac{-g(g-2)}{3^2 2^{2g-1}(2g-1)!!}, \tag{48}$$

in agreement with the prediction for $\kappa_{g-3}\lambda_1$ in [F3]. Zagier has found generalizations of these combinatorial formulas for the coefficient of k^{g-1-i} in $P_g(k)$ (for fixed codegree i).

Similarly, Bernoulli number formulas are found in the Appendix for the coefficient of k^i in $P_g(k)$ for fixed degree i. The coefficient of the linear term in $P_g(k)$ is

$$\mathcal{C}(k^1, P_g(k)) = \frac{B_{2g-2}}{2 \cdot (2g-2)!}, \tag{49}$$

in agreement with (9) and as previously calculated in [FP2]. The quadratic coefficient in $P_g(k)$ is

$$\mathcal{C}(k^2, P_g(k)) = \frac{-g\, B_{2g-2}}{2 \cdot (2g-2)!}. \tag{50}$$

Equation (50) determines the evaluation of $\kappa_1 \lambda_{g-3}$ for $g \geq 3$, and so it implies Conjecture 2 in [F4].

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Appendix: Polynomials Arising from the Tautological Ring

DON ZAGIER

1. Statement of Results

For positive integers g and k, define

$$P_g(k) = \sum_{l=1}^{k} \frac{(k-1)!}{(k-l)!}\frac{1}{k^l}\sum_{m=1}^{l}(-1)^{l-m}\binom{l}{m}\frac{m^{2g+l-1}}{(2g+l-1)!} \tag{1}$$

(the inner sum here is a Stirling number). For example, for $k \le 3$ we have

$$P_g(1) = \frac{1}{(2g)!}, \qquad P_g(2) = \frac{2^{2g-1}+g}{(2g+1)!},$$
$$P_g(3) = \frac{2(3^{2g+1}+2^{2g+2}g+6g^2+5g)}{9(2g+2)!}.$$

A property of the function P_g which is far from obvious—and which is false if the number $2g-1$ on the right-hand side of (1) is replaced by an even number—is that it is a polynomial in k for each fixed g; the first values are

$$P_1(k) = \frac{1}{2}, \quad P_2(k) = \frac{k}{24}, \quad P_3(k) = \frac{3k^2-k}{1440}, \quad P_4(k) = \frac{9k^3-8k^2+2k}{120960}.$$

This fact was discovered and proved by Faber and Pandharipande [FP] using an indirect argument in which the coefficients of the polynomials $P_g(k)$ were interpreted as intersection numbers of certain cycles in the moduli space of curves of genus g. Here we will give a more direct combinatorial proof and will also obtain alternative expressions for the polynomial $P_g(k)$ and explicit formulas for its highest and lowest coefficients. The formulas for the coefficients of k^{g-1}, k^{g-2}, k^2 and k^1 were quoted in Section 5.2 of [FP].

THEOREM 1. (i) *For each integer $g \ge 1$, the function $P_g(k)$ defined by* (1) *is a polynomial of degree $g-1$ in k.*

(ii) *Write $P_g(k) = \sum_{i=0}^{g-1} c_{g,i}k^i$. Then, for fixed $j \ge 0$ and $g > j$, we have*

$$c_{g,g-j-1} = \frac{(g-1)!}{2^g(2g-1)!}C_j(g), \tag{2}$$

where

$$C_0(g) = 1, \quad C_1(g) = -\frac{g(g-2)}{9}, \quad C_2(g) = \frac{g(g-3)(5g^2-9g+1)}{810},$$

and in general $C_j(g)$ is a polynomial of degree $2j$ with leading coefficient $(-1/9)^j/j!$.

(iii) *For fixed $i \geq 0$ and $g > i+1$ we have $c_{g,i} = \sum_{j=0}^{i} \gamma_{i,j}(g)\beta_{2g-j-1}$, where $\beta_n = B_n/n!$ ($B_n = n$th Bernoulli number) and $\gamma_{i,j}(g)$ is a polynomial of degree $i-j$. In particular (for $g > 2$),*

$$c_{g,0} = 0, \quad c_{g,1} = \frac{1}{2}\beta_{2g-2}, \quad c_{g,2} = -\frac{g}{2}\beta_{2g-2},$$
$$c_{g,3} = \frac{g(g+2)}{6}\beta_{2g-2} + \frac{1}{24}\beta_{2g-4}.$$

Parts (i) and (ii) of Theorem 1 are equivalent to the following amusing result. Let us define numbers $A(g,n)$ ($g \geq 1$, $n \geq 0$) by

$$\sum_{n=0}^{\infty} A(g,n)x^n = e^{-x}\sum_{k=0}^{\infty} P_g(k)\frac{x^k}{k!}$$

or, equivalently, by

$$A(g,n) = \sum_{k=0}^{n} \frac{(-1)^{n-k}}{k!\,(n-k)!}P_g(k), \qquad P_g(k) = \sum_{n=0}^{k} \frac{k!}{(k-n)!}A(g,n). \tag{3}$$

THEOREM 2. *The numbers $A(g,n)$ vanish for $n \geq g$. For $n \leq g-1$ we have*

$$A(g,n) = \frac{(g-1)!}{2^g(2g-1)!}C^*_{g-n-1}(g-n-1), \tag{4}$$

where

$$C^*_0(h) = 1, \quad C^*_1(h) = \frac{7h^2+5h}{18}, \quad C^*_2(h) = \frac{245h^4+594h^3+283h^2-42h}{3240},$$

and in general $C^*_r(h)$ is a polynomial of degree $2r$ in h with leading coefficient $(7/18)^r/r!$.

This theorem, as well as more general results concerning the numbers

$$A_\nu(g,n) = \sum_{k=1}^{n} \frac{(-1)^{n-k}}{k!\,(n-k)!}k^{-\nu}P_g(k) \quad (\nu \geq 0),$$

that are related to part (iii) of Theorem 1, will be proved in Section 3. For instance, we have

$$A_1(g,n) = \frac{(-1)^{n-1}}{2n!}\beta_{2g-2} \quad \text{and} \quad A_2(g,n) = A_1(g,n)\left(g - \sum_{k=1}^{n}\frac{1}{k}\right) \tag{5}$$

for $n+2 \geq g > 2$.

To state the remaining results, and for the proofs, we will need some more notation. As in [FP], we write $\mathcal{C}(x^n, f(x))$ to denote the coefficient of x^n in a power series $f(x)$ and $h_n(\alpha_1,\ldots,\alpha_l) = \mathcal{C}(x^n, \prod_{i=1}^{l}(1-\alpha_i x)^{-1})$ for the full symmetric function of degree n in variables $\alpha_1,\ldots,\alpha_l$. For any integer $n \geq 0$, we define

$$S_n(l) = \mathcal{C}\left(x^n, \left(\frac{e^x-1}{x}\right)^l\right). \tag{6}$$

For $l \in \mathbb{N}$ we have the formulas

$$\frac{(n+l)!}{l!}S_n(l) = \frac{1}{l!}\sum_{m=0}^{l}(-1)^{l-m}\binom{l}{m}m^{n+l} = h_n(1,2,\dots,l) = \mathfrak{S}_{n+l}^{(l)},$$

where $\mathfrak{S}_{n+l}^{(l)}$ denotes the Stirling number of the second kind (= number of partitions of a set of $n+l$ elements into l nonempty subsets). In particular, equation (1) can be written as

$$P_g(k) = \sum_{l=1}^{k}\frac{(k-1)!}{(k-l)!}\frac{1}{k^l}S_{2g-1}(l). \tag{7}$$

However, $S_n(l)$ is a polynomial (of degree n) in l whose first values are

$$S_0(l) = 1, \quad S_1(l) = \frac{l}{2}, \quad S_2(l) = \frac{3l^2+l}{24}, \quad S_3(l) = \frac{l^3+l^2}{48},$$

so it makes sense for any complex value of l. For $l = 0$ we clearly have $S_n(l) = 0$ for all $n > 0$. For $l = -1$ we have $S_n(l) = \beta_n$ by definition, where $\beta_n = B_n/n!$ as in Theorem 1; more generally, $S_n(l)$ for fixed negative l is a finite combination of Bernoulli numbers (Lemma 3), where the first three cases for n odd are

$$S_{2g-1}(-1) = 0, \quad S_{2g-1}(-2) = -\beta_{2g-2}, \quad S_{2g-1}(-3) = \tfrac{3}{2}(2g-3)\beta_{2g-2}$$

for $g \geq 3$. Using these numbers, we can now state a formula for $P_g(t)$ as a power series in t.

THEOREM 3. *Define the function $S_n(l)$ by equation* (6). *Then, for each integer $g \geq 1$, we have*

$$P_g(t) = -\sum_{r=1}^{\infty}\frac{S_{2g-1}(-r)t^{r-1}}{(1+t)\cdots(r+t)} \in \mathbb{Q}[[t]]. \tag{8}$$

In particular, the power series on the right-hand side of (8) *is in fact a polynomial in t.*

This theorem gives an alternative definition of the polynomials $P_g(t)$. As with (1), however, the polynomial property is not clear from this definition, and it is not true if the index $2g-1$ on the right-hand side of (8) is replaced by an even number.

The next result gives a closed-form expression for the generating function of the $P_g(t)$ as an integral. This looks less elementary than the preceding results but has the advantage of making it obvious that P_g is a polynomial.

THEOREM 4. *Define a power series $F(x)$ by*

$$\begin{aligned} F(x) &= \frac{\sinh x/2}{x/2}\exp\left(\frac{x/2}{\tanh x/2} - 1\right) = \exp\left(\sum_{n=2}^{\infty}\frac{n+1}{n}\beta_n x^n\right) \\ &= 1 + \frac{1}{8}x^2 + \frac{7}{1152}x^4 + \frac{61}{414720}x^6 + \cdots. \end{aligned} \tag{9}$$

Then the $P_g(t)$ are given by the generating function identity

$$\sum_{g=1}^{\infty} P_g(t)x^{2g-1} = \frac{1}{2}F(x)^t \int_0^x F(y)^{-t}\,dy. \tag{10}$$

The polynomiality of the functions $P_g(t)$ follows immediately because we can rewrite the generating series identity (10) in the form

$$P_g(t) = \sum_{n=0}^{g-1} \frac{p_{g-1-n}(t)\,p_n(-t)}{2(2n+1)},$$

where $p_n(t)$ denotes the coefficient of x^{2n} in $F(x)^t$, which is clearly a polynomial in t of degree n. Equation (10) is also equivalent to the following recursion for the polynomials P_g.

THEOREM 5. *The polynomials $P_g(t)$ can be given recursively by the formulas*

$$P_1(t) = \frac{1}{2}, \qquad P_g(t) = \frac{t}{2g-1}\sum_{n=1}^{g-1}(2n+1)\beta_{2n}P_{g-n}(t) \quad (g \geq 2). \tag{11}$$

The final result describes the coefficients $c_{g,i}$ (which are actually the numbers of interest, since it is they—and not the values of the polynomial $P_g(k)$—that occur in [FP] as intersection numbers) via a generating series with respect to the variable g rather than i. We begin with the well-known fact that the inverse power series of $x = ye^{-y}$ is given by $y = \sum_{k\geq 1} k^{k-1}x^k/k!$. A simple generalization of this states that the power series

$$Q_i(y) = (-1)^i \sum_{k=1}^{\infty} \frac{k^{k-1-i}}{k!}(ye^{-y})^k \tag{12}$$

is in fact a polynomial in y for every integer $i \geq 0$; the first few values are

$$Q_0(y) = y, \qquad Q_1(y) = \frac{1}{2}y^2 - y,$$
$$Q_2(y) = \frac{1}{6}y^3 - \frac{3}{4}y^2 + y, \qquad Q_3(y) = \frac{1}{24}y^4 - \frac{11}{36}y^3 + \frac{7}{8}y^2 - y.$$

The polynomials $Q_i(y)$ can also be defined and computed using the recursion

$$Q_0(y) = y, \qquad Q_{i+1}(y) = \int_0^y \frac{x-1}{x}Q_i(x)\,dx \quad (i \geq 0) \tag{13}$$

or the generating function identity

$$\sum_{i=0}^{\infty} Q_i(y)t^i = \sum_{r=1}^{\infty} \frac{t^{r-1}y^r}{(1+t)\cdots(r+t)}. \tag{14}$$

The following theorem provides yet another characterization of these polynomials and a new generating function for the rational numbers $c_{g,i}$.

THEOREM 6. (i) *The polynomial Q_i is, up to a constant, the unique polynomial with constant term* 0 *and degree* $\le i+1$ *satisfying*

$$Q_i\left(\frac{x}{1-e^{-x}}\right) - Q_i\left(\frac{x}{e^x-1}\right) = O(x^{2i+1}) \quad (x \to 0). \tag{15}$$

(ii) *For all integers* $g \ge 1$ *and* $i \ge 0$, *we have*

$$c_{g,i} = \mathcal{C}\left(x^{2g-1}, Q_i\left(\frac{x}{1-e^{-x}}\right)\right). \tag{16}$$

The proof of this theorem will be given in Section 5.

2. Polynomials Defined by Functional Equations

We begin by giving two simple (and well-known) lemmas that will be used several times in the sequel.

LEMMA 1. *Let r be a nonnegative integer and let z be a variable. Then*

$$\frac{1}{z(z-1)\cdots(z-r)} = \sum_{m=0}^{r} \frac{(-1)^{r-m}}{m!\,(r-m)!}\,\frac{1}{z-m}.$$

Proof. Compare residues on the two sides. □

LEMMA 2. *Let z and y be two free variables. Then*

$$\sum_{r=0}^{\infty} \frac{y^r}{z(z-1)\cdots(z-r)} = e^{-y} \sum_{m=0}^{\infty} \frac{y^m}{m!}\,\frac{1}{z-m}.$$

Proof. The equality of the coefficients of y^r is Lemma 1. Alternatively, we can prove the identity directly by observing that it holds for $y = 0$ and that

$$\begin{aligned}\frac{\partial}{\partial y}(y^{-z}e^y \cdot \mathrm{LHS}) &= \sum_{r=0}^{\infty}\left(\frac{e^y y^{r-z}}{z\cdots(z-r)} - \frac{e^y y^{r-z-1}}{z\cdots(z-r+1)}\right)\\ &= e^y y^{-z-1} = \frac{\partial}{\partial y}(y^{-z}e^y \cdot \mathrm{RHS}).\end{aligned}$$

□

We now prove several results showing that certain generating functions that are a priori power series are in fact polynomials. We denote by $(x)_n$ the ascending Pochhammer symbol $x(x+1)\cdots(x+n-1)$.

PROPOSITION 1. *For each $n \ge 0$, there is a unique polynomial $B_n(z, y, t)$ in three variables of degree $n-1$ satisfying the identity*

$$(z-t)B_n(z,y,t) - yB_n(z-1,y,t) = (z)_n - \sum_{m=0}^{n}\binom{n}{m} y^m (t)_{n-m}. \tag{17}$$

EXAMPLES. For $0 \le n \le 3$, the polynomials B_n are given by

$$B_0 = 0, \quad B_1 = 1, \quad B_2 = z+y+t+1,$$
$$B_3 = (z+1)(z+2) + (y+t)(z+y+t) + y + 3t.$$

Proof of Proposition 1. The recursion is equivalent to the functional equation

$$(z-t)\mathcal{B}(z,y,t,u) - y\mathcal{B}(z-1,y,t,u) = (1-u)^{-z} - e^{yu}(1-u)^{-t} \tag{18}$$

for the generating function $\mathcal{B}(z,y,t,u) = \sum_{n=0}^{\infty} B_n(z,y,t)(u^n/n!)$. The solution of this is

$$\mathcal{B}(z,y,t,u) = (1-u)^{-t}\mathcal{B}_0(z-t,y,u), \tag{19}$$

where $\mathcal{B}_0(z,y,u)$ $(= \mathcal{B}(z,y,0,u))$ satisfies the simpler functional equation

$$z\mathcal{B}_0(z,y,u) - y\mathcal{B}_0(z-1,y,u) = (1-u)^{-z} - e^{yu}. \tag{20}$$

Write $\mathcal{B}_0(z,y,u)$ as $\sum_{r\geq 0} \beta_r(z,u)y^r$. Then (20) is equivalent to

$$z\beta_r(z) = \begin{cases} (1-u)^{-z} - 1 & \text{if } r = 0, \\ \beta_{r-1}(z-1,u) - u^r/r! & \text{if } r > 0, \end{cases}$$

which can be solved by induction on r to give the closed formula

$$\beta_r(z,u) = \frac{(1-u)^{-z+r}}{z(z-1)\cdots(z-r)} - \sum_{s=0}^{r} \frac{1}{z(z-1)\cdots(z-s)} \frac{u^{r-s}}{(r-s)!}. \tag{21}$$

Using Lemma 1, we can rewrite (21) as

$$\beta_r(z,u) = \sum_{m=0}^{r} \frac{(-1)^{r-m}}{m!\,(r-m)!} \frac{(1-u)^{-z+r} - (1-u)^{r-m}}{z-m}$$

or, going back to the generating function $\mathcal{B}_0$, as

$$\mathcal{B}_0(z,y,u) = e^{y(u-1)} \sum_{m=0}^{\infty} \frac{y^m}{m!} \frac{(1-u)^{-z+m} - 1}{z-m}. \tag{22}$$

Substituting this into (19) gives the generating series $\mathcal{B}(z,y,t,u)$ in the form

$$\mathcal{B}(z,y,t,u) = e^{y(u-1)} \sum_{m=0}^{\infty} \frac{y^m}{m!} \frac{(1-u)^{-z+m} - (1-u)^{-t}}{z-m-t}. \tag{23}$$

To see that the coefficients of this with respect to u are polynomials, we rewrite (22) as

$$\begin{aligned} \mathcal{B}_0(z,y,u) &= e^{y(u-1)} \sum_{m=0}^{\infty} \frac{y^m}{m!} \int_0^u (1-v)^{-z+m-1}\,dv \\ &= \int_0^u (1-v)^{-z-1} e^{y(u-v)}\,dv \\ &= \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \frac{(z+1)_p y^q}{p!\,q!} \int_0^u v^p (u-v)^q\,dq \\ &= \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \frac{(z+1)_p y^q u^{p+q+1}}{(p+q+1)!} \end{aligned} \tag{24}$$

(the last equality by Euler's beta integral). Now substituting this into (19) and using the binomial expansion of $(1-u)^{-t}$ yields the explicit polynomial expression

$$B_n(z,y,t) = \sum_{\substack{p,q,l\ge 0\\ p+q+l+1=n}} \binom{n}{l}(z-t+1)_p(t)_l y^q \in \mathbb{Z}[z,y,t]. \tag{25}$$

□

Of course, we could have simply written down (25) and checked that it satisfies the identity (17); we gave the full derivation for clarity and because some of the formulas found along the way will be needed in what follows. In particular, from (24) and (19) we obtain the integral representation

$$\mathcal{B}(z,y,t,u) = (1-u)^{-t}\int_0^u (1-v)^{-z+t-1}e^{y(u-v)}\,dv, \tag{26}$$

and from (21) and (19) (or (23) and Lemma 2) we have the generating function identity

$$\begin{aligned}\mathcal{B}(z,y,t,u) = &\sum_{r=0}^{\infty}\frac{(1-u)^{-z+r}y^r}{(z-t)\cdots(z-t-r)}\\ &-(1-u)^{-t}e^{uy}\sum_{r=0}^{\infty}\frac{y^r}{(z-t)\cdots(z-t-r)}.\end{aligned} \tag{27}$$

This can also be obtained from (26) by writing $\int_0^u = -\int_u^1 + \int_0^1$ (for $\Re(z-t) < 0$).

We now consider the specialization of these functions to the case $y=-t$.

PROPOSITION 2. *For each $n \ge 0$ there is a unique polynomial $\hat{B}_n(z,t)$, in z and t, of degree $[(n-1)/2]$ in t, satisfying the identity*

$$(z-t)\hat{B}_n(z,t) + t\hat{B}_n(z-1,t) = (z)_n - \sum_{m=0}^{n}\binom{n}{m}(-t)^m(t)_{n-m}. \tag{28}$$

EXAMPLES. For $0 \le n \le 4$ we have

$$\hat{B}_0 = 0, \quad \hat{B}_1 = 1, \quad \hat{B}_2 = z+1, \quad \hat{B}_3 = 2t+(z+1)_2, \quad \hat{B}_4 = 3(z+3)t+(z+1)_3.$$

Proof. Since (28) is just the specialization of (17) to $y=-t$, its solution is of course given simply by $\hat{B}_n(z,t) = B_n(z,-t,t)$; what we have to show is that the degree with respect to t drops by a factor of 2 under this specialization. To do this we expand $(1-v)^{-z-1}$ in the integral representation (26) by the binomial theorem and change v to uv to obtain

$$\mathcal{B}(z,-t,t,u) = \sum_{r=0}^{\infty}\binom{z+r}{r}u^{r+1}\int_0^1 v^r\left[\frac{1-uv}{1-u}e^{uv-u}\right]^t dv.$$

The expression in square brackets has a power series expansion in u beginning $1+O(u^2)$, so the integrand is a power series in tu^2 and u. It follows that $\mathcal{B}(z,-t,t,u)$ is u times a power series in tu^2 and u and hence that the coefficient

$\hat{B}_n(z,t)$ of u^n has degree $\le (n-1)/2$ in t for every n, as claimed. Specifically, from the expansion

$$\frac{1-uv}{1-u}e^{uv-u} = \exp\left(\sum_{m=2}^{\infty}\frac{u^m}{m}(1-v^m)\right)$$

we find the closed form

$$\hat{B}_n(z,t) = \sum_{\substack{r,k_2,k_3,\ldots\ge 0 \\ r+2k_2+3k_3+\cdots=n-1}} \binom{z+r}{r}\frac{t^{k_2+k_3+\cdots}}{2^{k_2}k_2!\,3^{k_3}k_3!\cdots}\int_0^1 v^r(1-v^2)^{k_2}(1-v^3)^{k_3}\cdots dv,$$

from which the coefficients of $\hat{B}_n$ can be computed explicitly. In particular, we see that $l+2m \le n-1$ for all monomials $z^l t^m$ occurring in $\hat{B}_n$ and that, in the case of equality, the coefficient of this monomial comes only from the term $r = l$, $k_2 = m$, $k_3 = k_4 = \cdots = 0$ in the above sum and equals the beta integral $\int_0^1 v^l(1-v^2)^m\,dv/2^m l!\,m!$. □

Now comes the second point. The specialization $y = -t$ had the effect in the above proof of making the linear term in the power series expansion of $\left(\frac{1-uv}{1-u}\right)^t e^{-uy(1-v)}$ vanish, but it also has a second, less obvious effect: if we denote by $U(x)$ the power series

$$U(x) := 1 - \frac{x}{e^x-1} = \frac{x}{2} - \frac{x^2}{12} + \frac{x^4}{720} - \cdots,$$

then we have

$$u = U(x) \implies \frac{e^{-u}}{1-u} = \frac{e^x-1}{xe^{x/2}}\exp\left(\frac{x}{e^x-1}+\frac{x}{2}-1\right) = F(x), \tag{29}$$

where $F(x)$ is the power series defined in Theorem 4 and is an *even* function of x. This leads immediately to the following definition and proposition.

PROPOSITION 3. *For each positive integer g, the function*

$$P_g(z,t) := \mathcal{C}(x^{2g-1}, \mathcal{B}(z,-t,t,U(x))) \tag{30}$$

is a polynomial of degree $2g-2$ in z and $g-1$ in t, and it satisfies the identities

$$(z-t)P_g(z,t) + tP_g(z-1,t) = S_{2g-1}(z) \tag{31}$$

and

$$P_g(z,t) = \sum_{r=0}^{\infty}\frac{S_{2g-1}(z-r)(-t)^r}{(z-t)\cdots(z-t-r)} \in \mathbb{Q}(z)[[t]]. \tag{32}$$

Proof. Equation (31) follows by substituting $y = -t$ and $u = U(x)$ into the generating series identity (18), since the second term $e^{-tu}(1-u)^{-t}$ on the right is an even power series in x by virtue of equation (29), while the coefficient of x^{2g-1} in the first term $(1-u)^{-z}$ is $S_{2g-1}(z)$ by definition. Similarly, equation (32) is obtained by substituting $y = -t$ and $u = U(x)$ into (27) and noting that the second term is an even power series in x. □

3. Proof of Theorems 2–5

We begin with Theorem 2. From (3) and (7) we have

$$A(g,n) = \sum_{1\le l\le k\le n} \frac{(-1)^{n-k}k^{-l-1}}{(n-k)!\,(k-l)!} S_{2g-1}(l).$$

For fixed l, the coefficient of $S_{2g-1}(l)$ can be rewritten as

$$\begin{aligned}\sum_{k=l}^{n} \frac{(-1)^{n-k}}{(n-k)!\,(k-l)!}k^{-l-1} &= \mathcal{C}\left(t^l, \sum_{k=l}^{n} \frac{(-1)^{n-k}}{(n-k)!\,(k-l)!}\frac{1}{k-t}\right)\\ &= \mathcal{C}\left(t^l, \frac{(-1)^{n-l}}{(n-t)(n-t-1)\cdots(l-t)}\right)\end{aligned}$$

(the latter by Lemma 1 with $r = n-l$ and $z = n-t$); so, replacing l by $r = n-l$, we have

$$A(g,n) = \mathcal{C}\left(t^n, \sum_{r=0}^{n-1} \frac{S_{2g-1}(n-r)(-t)^r}{(n-t)(n-t-1)\cdots(n-r-t)}\right). \tag{33}$$

The key observation is now that, if we replace the summation on the right by one from $r = 0$ to ∞, then its value does not change: the terms $r = n$ and $r = n+1$ contribute nothing because $S_{2g-1}(0) = S_{2g-1}(-1) = 0$, and the terms with $r \ge n+2$ contribute nothing because the rational function $1/(n-t)(n-t-1)\cdots(n-t-r)$ has only a simple pole at $t = 0$ and hence its product with t^r has no coefficient of t^n. Hence equation (32) gives

$$A(g,n) = \mathcal{C}(t^n, P_g(n,t)).$$

This proves the vanishing of $A(g,n)$ for $n \ge g$ (since $P_g(z,t)$ is a polynomial of degree $\le g-1$ in t for all z) and hence also that $P_g(k)$ is a polynomial in k of degree $g-1$. The statement (4) about the values of the numbers $A(g,n)$ for $g-n$ fixed can be proved by using the integral representation of the generating function $\mathcal{B}(z,-t,t,u)$, but since the argument is similar to the one we shall give for equation (2) (to which (4) is in fact equivalent), and since the statement about the form of the $A(g,n)$ was included only for amusement, we omit the derivation.

We now turn to $A_\nu(g,n)$. The same argument as used to derive (33) gives

$$A_\nu(g,n) = \mathcal{C}\left(t^{n+\nu}, \sum_{r=0}^{n-1} \frac{S_{2g-1}(n-r)(-t)^r}{(n-t)(n-t-1)\cdots(n-r-t)}\right)$$

for any $\nu > 0$, but now changing the sum to one over all $r \ge 0$ does change the right-hand side, since the terms $r = n+\mu+1$ of the sum have nonzero coefficients of t^n for $1 \le \mu \le \nu$. Equation (32) therefore now gives

$$\begin{aligned}A_\nu(g,n) &= \mathcal{C}(t^{n+\nu}, P_g(n,t))\\ &\quad - \mathcal{C}\left(t^\nu, \frac{(-1)^n}{(n-t)\cdots(1-t)} \sum_{\mu=1}^{\nu} \frac{S_{2g-1}(-\mu-1)t^\mu}{(1+t)\cdots(\mu+1+t)}\right).\end{aligned}$$

Again the first term vanishes for n sufficiently large ($n \geq g - \nu$), so for small ν we obtain explicit formulas for ν; two examples were given by equation (5). By analyzing these formulas we could deduce the statement in part (iii) of Theorem 1 about the lowest coefficients of $P_g(k)$. But it will be easier to work directly with $P_g(k)$, using the following result.

PROPOSITION 4. *For each positive integer k, the polynomials $P_g(z,t)$ defined by* (30) *satisfy the identity*

$$P_g(t-k,t) = \sum_{l=1}^{k} \frac{(k-1)!}{(k-l)!} t^{-l} S_{2g-1}(l+t-k). \tag{34}$$

In particular, the function $P_g(k)$ defined by(1) *is equal to the polynomial $P_g(0,k)$.*

Proof. We prove this by induction on k: setting $z = t$ in (31) gives the case $k = 1$ of (34), and setting $z = t - k$ in (31) gives the induction step from k to $k+1$. □

The remaining results stated in Section 1 follow easily from the last statement of Proposition 4. Theorem 3 is obtained immediately by taking $z = 0$ in equation (32). For Theorem 4, we first use the integral representation (26) to write

$$\mathcal{B}(0,-t,t,u) = \left(\frac{e^{-u}}{1-u}\right)^t \int_0^u \left(\frac{e^{-v}}{1-v}\right)^{-t} \frac{dv}{1-v}.$$

Now making the substitutions $u = U(x)$ and $v = U(y)$ and using equation (29), we have

$$\mathcal{B}(0,-t,t,U(x)) = F(x)^t \int_0^x F(y)^{-t} \frac{U'(y)}{1-U(y)}\, dy.$$

But

$$\frac{U'(y)}{1-U(y)} = \frac{e^y}{e^y-1} - \frac{1}{y} = \frac{1}{2} + (\text{odd power series in } y),$$

so

$$\mathcal{B}(0,-t,t,U(x)) = \frac{1}{2} F(x)^t \int_0^x F(y)^{-t}\, dy + (\text{even power series in } x).$$

Equation (10) now follows from the equality $P_g(t) = P_g(0,t)$ and the definition of $P_g(z,t)$. Finally, the recursion (11) is, as already stated in Section 1, equivalent to equation (10): If we denote by $\mathfrak{P}(x,t)$ the generating function occurring on the left-hand side of (10), then

$$(10) \iff \frac{1}{2} = F(x)^t \frac{\partial}{\partial x}(F(x)^{-t}\mathfrak{P}(t,x)) = \frac{\partial \mathfrak{P}(x,t)}{\partial x} - t\frac{F'(x)}{F(x)}\mathfrak{P}(t,x), \tag{35}$$

and this is seen to be equivalent to (11) by substituting

$$\frac{F'(x)}{F(x)} = \sum_{n\geq 1} (2n+1)\beta_{2n} x^{2n-1}$$

from (9) and comparing the coefficients of x^{2g-2} on both sides.

4. Proof of Theorem 1

We now know, from Proposition 4 or from Theorem 4 or 5, that $P_g(k)$ is a polynomial. It remains to prove the statements made in Theorem 1 about the coefficients $c_{g,g-j-1}$ (j fixed) and $c_{g,i}$ (i fixed). We start with the "top" coefficients $c_{g,g-j-1}$. Writing $y = vx$ in (10) we find

$$\sum_{g=1}^{\infty} P_g(t)x^{2g-2} = \frac{1}{2}\int_0^1 \exp\biggl(\sum_{r=1}^{\infty} \lambda_r t x^r (1-v^r)\biggr)\,dv,$$

where $\lambda_r = \mathcal{C}(x^{2r}, \log F(x)) = (1+1/2r)\beta_{2r}$. Expanding the integral as in the proof of Proposition 2 and comparing the coefficients of $x^{2g-2}t^{g-j-1}$ on both sides, we find that

$$\begin{aligned} c_{g,g-j-1} &= \frac{1}{2} \sum_{\substack{\alpha,\beta,\gamma\cdots\ge 0 \\ \alpha+2\beta+3\gamma+\cdots=g-1 \\ \beta+2\gamma+\cdots=j}} \frac{\lambda_1^\alpha}{\alpha!}\frac{\lambda_2^\beta}{\beta!}\frac{\lambda_3^\gamma}{\gamma!}\cdots \int_0^1 (1-v^2)^\alpha (1-v^4)^\beta (1-v^6)^\gamma \cdots dv \\ &= \frac{1}{2}\sum_{j\le d\le 2j} \frac{\lambda_1^{g-d-1}}{(g-d-1)!}\int_0^1 (1-v^2)^{g-j-1} H_{j,d}(v^2)\,dv \end{aligned}$$

with

$$H_{j,d}(x) = \sum_{\substack{\alpha,\beta,\gamma\cdots\ge 0 \\ \beta+2\gamma+\cdots=j \\ 2\beta+3\gamma+\cdots=d}} \frac{\lambda_2^\beta}{\beta!}\frac{\lambda_3^\gamma}{\gamma!}\cdots (1+x)^\beta (1+x+x^2)^\gamma \cdots .$$

This can now be computed by expanding $H_{j,d}$ as a polynomial and computing each term $\int_0^1 (1-v^2)^{g-j-1}v^{2n}\,dv$ as a beta integral, and can easily be seen to have the form (2) for some polynomial $C_j(g)$. The highest power of g occurs for the maximal value $d = 2j$, corresponding to taking $\beta = j$ and $\gamma = \cdots = 0$. Also, to compute the coefficient of the highest power of g we may replace $H_{j,d}(x)$ by its constant term $H_{j,d}(0)$, since the main contribution to the integral for g large comes from v near 0 and since the asymptotic value of $\int_0^1 (1-v^2)^{g-j-1}dv$ is $C(g) = 2^{2g-2}(g-1)!^2/(2g-1)!$ (independent of j) by the beta integral formula. It follows that the asymptotic formula for $c_{g,g-j-1}$ is

$$c_{g,g-j-1} \sim \frac{C(g)}{2}\frac{\lambda_1^{g-2j-1}}{(g-2j-1)!}\frac{(2\lambda_2)^j}{j!} \sim \frac{C(g)\lambda_1^{g-1}}{2(2g-1)!}g^{2j}\frac{(2\lambda_2/\lambda_1^2)^j}{j!},$$

and this agrees with the result stated in Theorem 1 because $\lambda_1 = 1/8$ and $2\lambda_2/\lambda_1^2 = -2/9$. One can also prove equation (2), and obtain explicit recursion relations for the polynomials $C_j(g)$, from the recursion relation given in Theorem 5. The details are left to the reader.

For the "bottom" coefficients $c_{g,i}$ (i fixed) we use the expansion (8) together with the following lemma, which expresses the "negative Stirling numbers" $S_n(-r)$ for r fixed as finite linear combinations of Bernoulli numbers.

LEMMA 3. *For $n \geq r \geq 1$ we have the identity*

$$S_n(-r) = \sum_{j=0}^{r-1} (-1)^{r-1-j} \binom{n-j-1}{r-j-1} S_j(-r)\beta_{n-j}.$$

Proof. One sees by induction that the powers of the function $1/(e^x - 1)$ are linear combinations of its derivatives. From the formulas

$$\left(\frac{1}{e^x - 1}\right)^r = \sum_{s=1}^{r} S_{r-s}(-r)\frac{1}{x^s} + O(1) \quad (x \to 0)$$

and

$$\frac{(-1)^{s-1}}{(s-1)!}\frac{d^{s-1}}{dx^{s-1}}\left(\frac{1}{e^x - 1}\right) = \frac{1}{x^s} + (-1)^{s-1}\sum_{l=s}^{\infty}\binom{l-1}{s-1}\beta_l x^{l-s},$$

we deduce that

$$\left(\frac{1}{e^x - 1}\right)^r = \sum_{s=1}^{r} S_{r-s}(-r)\left(\frac{1}{x^s} + (-1)^{s-1}\sum_{l=s}^{\infty}\binom{l-1}{s-1}\beta_l x^{l-s}\right);$$

the desired result follows by comparing coefficients of x^{n-r} on both sides. □

Part (iii) of Theorem 1 follows immediately from (8) and Lemma 3. Explicitly, we have

$$c_{g,i} = \sum_{j=0}^{i}\left(\sum_{r=j+1}^{i+1}(-1)^{r-j}\binom{2g-j-2}{r-j-1}S_j(-r)\alpha_{i-r+1}(r)\right)\beta_{2g-j-1},$$

where

$$\alpha_n(r) := \mathcal{C}\left(t^n, \frac{1}{(1+t)\cdots(r+t)}\right) = \frac{(-1)^n}{r!}h_n\left(1, \frac{1}{2}, \ldots, \frac{1}{r}\right) \tag{36}$$

and the coefficient of β_{2g-j-1} is a polynomial of degree $i - j$ in g. □

5. The Polynomials $Q_i(y)$ and the Second Generating Function for the $c_{g,i}$

In this section we will discuss the polynomials defined by equations (12)–(14) and prove Theorem 6. We must first check that the power series in (12) is indeed a polynomial of degree $i+1$ and that the three definitions are indeed equivalent. For the first statement, note that if $n \geq i+2$ then

$$\mathcal{C}(y^n, Q_i(y)) = \sum_{k=1}^{n}\frac{k^{k-1-i}}{k!}\cdot\frac{(-k)^{n-k}}{(n-k)!} = \frac{1}{n!}\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}k^{n-1-i} = 0$$

(the nth difference of a polynomial of degree $< n$ vanishes). For the second, note that the system of integral recursions (13) is equivalent to the system of differential recursions

$$Q_0(y) = y, \qquad yQ'_{i+1}(y) = (y-1)Q_i(y) \quad (i \geq 0) \tag{37}$$

(no initial values are needed here because the $(i+1)$th equation in this system implies that $Q_{i+1}(0) = 0$, which is the needed initial condition to solve the ith equation). It is easy to check that the functions satisfied by (12) or by (14) both satisfy the system (37), so they are all equal. We can write out (14) more explicitly as

$$Q_i(y) = \sum_{r=1}^{i+1} \alpha_{i+1-r}(r) y^r, \tag{38}$$

with $\alpha_n(r)$ defined by (36). It is obvious that these numbers satisfy $\alpha_n(r-1) = r\alpha_n(r) + \alpha_{n-1}(r)$, and this is equivalent to the statement that the polynomials given in (38) satisfy (37).

Now set $Y(x) = x/(1-e^{-x})$ and $\tilde{Q}_i(x) = Q_i(Y(x))$. Then (37) gives

$$\tilde{Q}'_{i+1}(x) = Y'(x)\frac{Y(x)-1}{Y(x)}\tilde{Q}_i(x) =: \gamma(x)Q_i(x). \tag{39}$$

Yet an easy calculation shows that the function $\gamma(x)$ is nothing other than the logarithmic derivative $F'(x)/F(x)$ of the function defined in (9). In particular it is an *odd* function of x, so that from (39) we deduce that also

$$\frac{d}{dx}(\tilde{Q}_{i+1}(x) - \tilde{Q}_{i+1}(-x)) = \gamma(x)(\tilde{Q}_i(x) - \tilde{Q}_i(-x)). \tag{40}$$

This equation and the fact that $\tilde{Q}_i(x) - \tilde{Q}_i(-x)$ vanishes at $x = 0$ imply by induction on i that $\tilde{Q}_i(x) - \tilde{Q}_i(-x)$ vanishes to order $2i+1$ at the origin for all $i \geq 0$, which is the first assertion of Theorem 6. (The uniqueness statement follows immediately from the existence, since the polynomials $Q_0, Q_1, \ldots, Q_i$ form a basis for the space of polynomials of degree $\leq i+1$ with no constant term.) Equation (16), which can be written as the generating function identity

$$2\sum_{g=1}^{\infty} P_g(t) x^{2g-1} = \sum_{i=0}^{\infty} (\tilde{Q}_i(x) - \tilde{Q}_i(-x)) t^i, \tag{41}$$

follows at the same time, since the differential equation (40) is equivalent to the differential equation in (35) for the generating series $\sum P_g(t) x^{2g-1}$ or to the recursion (11) for its coefficients. □

Reference

[1] C. Faber and R. Pandharipande, *Logarithmic series and Hodge integrals in the tautological ring*, Michigan Math. J. 48 (2000), 215–239.

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