A COMBINATORIAL REFINEMENT OF THE KRONECKER-HURWITZ CLASS NUMBER RELATION

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(Communicated by Kathrin Bringmann)

Abstract. We give a refinement of the Kronecker-Hurwitz class number relation, based on a tesselation of the Euclidean plane into semi-infinite triangles labeled by $\text{PSL}_2(\mathbb{Z})$ that may be of independent interest.

1. A refinement of a classical class number relation

We give a refinement, and a new proof, of the following classical result [1–3].

Theorem 1 (Kronecker, Gierster, Hurwitz). For any $n \geq 1$ we have

$$\sum_{t^2 \leq 4n} H(4n - t^2) = \sum_{n = ad \atop a, d > 0} \max(a, d).$$

Here $H(D) (D \geq 0, D \equiv 0, 3 \text{ mod } 4)$ is the Kronecker-Hurwitz class number, which has initial values

\begin{tabular}{|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|}
\hline
$D$ & 0 & 3 & 4 & 7 & 8 & 11 & 12 & 15 & 16 & 19 & 20 & 23 & 24 \\
\hline
$H(D)$ & $-\frac{1}{12}$ & $\frac{1}{3}$ & $\frac{1}{2}$ & 1 & 1 & 1 & $\frac{4}{3}$ & 2 & $\frac{3}{2}$ & 1 & 2 & 3 & 2 \\
\hline
\end{tabular}

and for $D > 0$ equals the number of $\text{PSL}_2(\mathbb{Z})$-equivalence classes of positive definite integral binary quadratic forms of discriminant $-D$, with those classes that contain a multiple of $x^2 + y^2$ or of $x^2 - xy + y^2$ counted with multiplicity $1/2$ or $1/3$, respectively.

Let $\Gamma = \text{PSL}_2(\mathbb{Z})$. By the $\Gamma$-equivariant bijection $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \leftrightarrow cx^2 + (d - a)xy - by^2$ between integral matrices of determinant $n$ and trace $t$ and quadratic forms of discriminant $t^2 - 4n$, the class number relation can be written as

$$\sum_{M \in \mathcal{M}_n^\text{elliptic}} \chi(z_M) = \sum_{n = ad \atop a, d > 0} \max(a, d) + \begin{cases} 1/6 & \text{if } n \text{ is a square}, \\
0 & \text{otherwise}, \end{cases}$$

where $\mathcal{M}_n$ is the set of integral matrices of determinant $n$ modulo $\pm 1$, $z_M$ is the fixed point of an elliptic $M$ in the upper half-plane $\mathfrak{H}$, and $\chi : \mathfrak{H} \rightarrow \mathbb{Q}$ is the modified

Received by the editors April 11, 2016 and, in revised form, May 7, 2016.

2010 Mathematics Subject Classification. Primary 11E41.

The first author was partly supported by CNCSIS grant TE-2014-4-2077. He would like to thank the MPIM in Bonn and the IHES in Bures-sur-Yvette for providing support and a stimulating research environment while he was working on this paper.
characteristic function of the standard fundamental domain
\[ \mathcal{F} = \{ z \in \mathcal{H} : -1/2 \leq \Re(z) \leq 1/2, \ |z| \geq 1 \} \]
of \( \Gamma \) acting on \( \mathcal{H} \) such that \( \chi(z) \) is 1/2\( \pi \) times the angle subtended by \( \mathcal{F} \) at \( z \) (so \( \chi \) is 1 in the interior of \( \mathcal{F} \), 0 outside of \( \mathcal{F} \), 1/2 on the boundary points different from the corners \( \rho = e^{\pi i/3} \) and \( \rho^2 \), and 1/6 at the corners).

We will prove a refinement of (1) saying that the subsum of the expression on the left over all \( M \) in a given orbit of the right action of \( \Gamma \) on \( \mathcal{M}_n \) always takes on one of the values 0, 1, 2 (or 7/6 for the orbit \( \sqrt{n} \Gamma \) if \( n \) is a square). Specifically, let us define for any right coset \( K \) in \( \mathcal{M}_n / \Gamma \) (more precisely, \( K \) is a right coset in \( \text{PGL}_2(\mathbb{Q}) / \Gamma \), since \( \mathcal{M}_n \) is not a group) two positive integers \( \delta_K \) and \( \delta'_K \) by
\[ \delta_K = \gcd(c, d), \delta'_K = n/\delta_K, \]
where \( \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \) is any representative of \( K \). Then we have:

**Theorem 3.** For each right coset \( K \in \mathcal{M}_n / \Gamma \) we have
\[ \sum_{M \in K} \chi(z_M) = 1 + \text{sgn}(\delta'_K - \delta_K) + \begin{cases} 1/6 & \text{if } K = \sqrt{n} \Gamma, \\ 0 & \text{otherwise}. \end{cases} \]

Equation (1) follows immediately by summing the relations in Theorem 2 over all cosets in the disjoint decomposition \( \mathcal{M}_n = \bigsqcup (\delta' / \delta) \Gamma \) with \( n = \delta' \delta \) and \( 0 \leq \beta < \delta' \).

Theorem 2 provides a correspondence between right cosets and \( \Gamma \)-conjugacy classes in \( \mathcal{M}_n \), which generically assigns two conjugacy classes to each coset with \( \delta' > \delta \). We will deduce it from a similar statement, Theorem 3 which is sharper in two respects (it counts the number of matrices with a fixed point in a smaller domain, and it allows real coefficients), and which gives a generically one-to-one correspondence between cosets and conjugacy classes. To state it, we consider a half-fundamental domain
\[ \mathcal{F}^- = \{ z \in \mathcal{H} : -1/2 \leq \Re(z) \leq 0, \ |z| \geq 1 \}, \]
and define a function \( \alpha : \text{GL}_2^+ (\mathbb{R}) \to \mathbb{Q} \) by
\[ \alpha(M) = \begin{cases} \chi^-(z_M) & \text{if } M \text{ is elliptic with fixed point } z_M \in \mathcal{H}, \\ -1/12 & \text{if } M \text{ is scalar}, \\ 0 & \text{if } M \text{ is parabolic or hyperbolic}, \end{cases} \]
where \( \chi^- \) is defined in the same way as \( \chi \) (and hence equals 1 in the interior of \( \mathcal{F}^- \), 0 outside \( \mathcal{F}^- \), 1/2 on the internal boundary points of \( \mathcal{F}^- \), and 1/4 and 1/6 at the corners \( i \) and \( \rho^2 \), respectively). Note that \( \alpha(-M) = \alpha(M) \), so \( \alpha \) is well defined on \( M \Gamma \).

**Theorem 3.** For \( M = \left( \begin{smallmatrix} y & x \\ 0 & 1 \end{smallmatrix} \right) \in \text{GL}_2(\mathbb{R}) \) with \( y > 0 \), we have
\[ \sum_{\gamma \in \Gamma} \alpha(M \gamma) = \frac{1 + \text{sgn}(y - 1)}{2}. \]
Since each coset \( K \in \mathcal{M}_n / \Gamma \) contains a representative \( M \) with \( M \infty = \infty \), Theorem 2 immediately follows from (2), and the fact that the map \( \pm \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \mapsto \pm \left( \begin{smallmatrix} -a & b \\ -c & -d \end{smallmatrix} \right) \) is a bijection between the sets of elements in \( \mathcal{M}_n \) having a fixed point in the left half and in the right half of the standard fundamental domain for \( \Gamma \).

Theorem 3 is proved in Section 3 as an easy consequence of a triangulation of a Euclidean half-plane by triangles associated to elements of \( \Gamma \) (Theorem 4).
This triangulation may be of independent interest, and we give a self-contained treatment in the next section.

2. A triangulation of a Euclidean half-plane

Let \( \Gamma_\infty = \{ \gamma \in \Gamma \mid \gamma \infty = \infty \} \). We identify \( \Gamma \setminus \Gamma_\infty \) with a subset of \( \text{SL}_2(\mathbb{Z}) \) by choosing representatives \( \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) with \( c > 0 \), and for such \( \gamma \) we define a semi-infinite triangle

\[
(3) \quad \Delta(\gamma) = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq d - cx - ay \leq c \leq -dx - by\}.
\]

(The motivation for this definition is that \((x, y) \in \Delta(\gamma)\) if and only if \( \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} \gamma \) has a fixed point in \( \mathcal{F}^- \).) Note that \( \Delta(\gamma) \) is contained in the half-plane

\[
\mathcal{H} = \{(x, y) \in \mathbb{R}^2 \mid y \geq 1\},
\]

since \( y = c(-dx - by) + d^2 - d(d - cx - ay) \geq c^2 + d^2 - c|d| \geq 1 \).

**Theorem 4.** We have a tessellation

\[
\mathcal{H} = \bigcup_{\gamma \in \Gamma \setminus \Gamma_\infty} \Delta(\gamma)
\]

of the half-plane \( \mathcal{H} \) into semi-infinite triangles with disjoint interiors.

**Remark.** We can extend the triangulation of Theorem 4 to a triangulation of all of \( \mathbb{R}^2 \) by triangles labeled by all of \( \Gamma \) if we define \( \Delta(\gamma) \) also for \( \gamma \in \Gamma_\infty \) by

\[
\Delta\left( \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \right) = [-n - 1, -n] \times (-\infty, 1],
\]

and can then interpret the extended triangulation as giving a piecewise-linear action of \( \Gamma \) on \( \mathbb{R}^2 \), with each triangle being a fundamental domain. However we will not use this in the sequel.

**Proof.** The group \( \Gamma \) is a free product of its two subgroups generated by the elements \( S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \) and \( U = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} \) of orders 2 and 3, respectively, which fix the two corners of \( \mathcal{F}^- \). Therefore we can view elements of \( \Gamma \) as words in \( S, U, U^2 \) or as vertices of the tree shown in Figure 1. The proof of both Theorems 3 and 4 will

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**Figure 1.** A tree associated to \( \Gamma = \text{PSL}_2(\mathbb{Z}) \): the vertices are labeled by the elements of \( \Gamma \) and the edges by the generators \( S, U \) and \( U^2 \) as shown.
follow from the following decomposition into triangles with disjoint interiors:

$$\mathcal{R} := \{(x,y) \in \mathbb{R}^2 \mid 0 \leq x \leq y - 1\} = \bigcup_{\gamma \in T} \Delta(\gamma),$$

where $T \subset \Gamma$ is the set of words starting in $U$. The regions $\mathcal{H}$ and $\mathcal{R}$ and a few triangles corresponding to words of small length are pictured in Figure 2.

To prove (4), let $T = T^+ \cup T^-$, where $T^+$ consists of the elements of $T$ ending in $U$ or $U^2$, while $T^- := T^+ S$ consists of those elements ending in $S$. The set $T^+$ can be enumerated recursively by starting at $U$ and replacing $\gamma = (ab\,cd)$ at each step by $\gammaSU = (a\,a+b\,d), \quad \gammaSU^2 = (a+b\,d\,c+d\,a)$. From this description we easily obtain the following equivalent characterizations:

$$\gamma \in T^+ \iff 0 \leq -\frac{a}{c} < -\frac{b}{d} \leq 1, \quad \gamma \in T^- \iff 0 \leq -\frac{b}{d} < -\frac{a}{c} \leq 1.$$

Alternatively, $T^+$ consists of those $\gamma \in T$ having $d > 0$.

For $\gamma \in \Gamma \setminus \Gamma_{\infty}$, the triangle $\Delta(\gamma)$ has two vertices given by

$$P_3(-ac - bd + bc, c^2 + d^2 - cd), \quad P_2(-ac - bd, c^2 + d^2),$$

connected by a line segment of slope $-d/b$, and it has two infinite parallel sides of slope $-c/a$. For $\gamma \in T$ we denote by $C(\gamma) \subset \mathcal{H}$ the half-cone containing $\Delta(\gamma)$, bounded by half-lines of slopes $-c/a$ and $-d/b$, and having as vertex $P_3$ or $P_2$, depending on whether $\gamma \in T^+$ or $\gamma \in T^-$ respectively (see Figure 3).
Using this information, it is easy to check that for \( \gamma \in \mathcal{T}^+ \) and \( \gamma' = \gamma S \in \mathcal{T}^- \) we have the following decompositions into sets with disjoint interiors (see the right picture in Figure 3):

\[
\mathcal{C}(\gamma) = \Delta(\gamma) \cup \mathcal{C}(\gamma') , \quad \mathcal{C}(\gamma') = \Delta(\gamma') \cup \mathcal{C}(\gamma' U) \cup \mathcal{C}(\gamma' U^2) .
\]

By induction we obtain that \( \mathcal{R} = \mathcal{C}(U) \) is the union of the triangles indexed by \( \mathcal{T} \), proving (4).

Finally we show that the decomposition in (4) implies the decomposition of \( \mathcal{H} \) given in Theorem 4. From the parenthetical remark following (3) it is clear that \( \Delta(T \gamma) = T \Delta(\gamma) \), where \( T = SU = (1 1 0 1) \) and \( \Gamma_\infty \) acts on \( \mathcal{H} \) by \( T^n(x, y) = (x - ny, y) \). The region

\[
\mathcal{R'} = \mathcal{R} \cup \Delta(U^2) = \{(x, y) \in \mathcal{H} : 0 \leq x < y\}
\]

(see Figure 2) is a fundamental domain for this action of \( \Gamma_\infty \) on \( \mathcal{H} \), and we obtain the following decompositions into triangles with disjoint interiors:

\[
\{(x, y) \in \mathcal{H} : y - 1 \leq x\} = \bigcup_{\gamma \in \mathcal{T}'} \Delta(\gamma) , \quad \{(x, y) \in \mathcal{H} : x \leq 0\} = \bigcup_{\gamma \in \mathcal{T}''} \Delta(\gamma) ,
\]

where \( \mathcal{T}' \) consists of words starting with \( U^2 \), but different from \( (U^2 S)^n = T^{-n} \) with \( n > 0 \), while \( \mathcal{T}'' \) consists of words starting with \( S \), but different from \( (SU)^n = T^n \) with \( n > 0 \). Theorem 4 follows since \( \Gamma \setminus \Gamma_\infty = \mathcal{T} \cup \mathcal{T}' \cup \mathcal{T}'' \). \( \square \)

3. Proof of Theorem 3

Since (2) is invariant under multiplying \( M = (y x \ y) \) on the right by elements in \( \Gamma_\infty \), we assume without loss of generality that \( 0 \leq x < y \). If \( M\gamma \) is scalar for \( \gamma \in \Gamma \), the only possibility is easily seen to be \( M = 1 \). In this case, \( \alpha(\gamma) \neq 0 \) for \( \gamma \in \{1, S, U, U^2\} \), and (2) holds since \(-1 \frac{1}{2} \frac{1}{2} \frac{1}{6} \frac{1}{6} \frac{1}{6} = \frac{1}{2} \).
Assuming that $M \neq 1$, it follows that $\alpha(M\gamma) \neq 0$ if and only if $M\gamma$ has a fixed point in $\mathcal{F}^-$, that is, $(x, y) \in \Delta(\gamma)$. We conclude from Section 2 that $y \geq 1$, so the point $(x, y)$ belongs to the region $\mathcal{R}'$ in (5), and $\gamma = U^2$ or $\gamma \in \mathcal{T}$ by (4). Therefore the elements $\gamma$ such that $\alpha(M\gamma) \neq 0$ depend on the position of the point $(x, y)$ with respect to the triangulation of $\mathcal{R}'$ as follows (see Figure 3):

- $y = 1$ and $0 < x < 1$: $\alpha(MU^2) = 1/2$;
- $(x, y)$ is in the interior of a triangle $\Delta(\gamma)$: $\alpha(M\gamma) = 1$;
- $(x, y)$ is on a common side between $\Delta(\gamma)$ and $\Delta(\gamma')$, but it is not a vertex:
  $$\alpha(M\gamma) + \alpha(M\gamma') = \frac{1}{2} + \frac{1}{2} = 1;$$
- $(x, y) \in \mathcal{R}$ is the $P_2$ vertex of the triangle $\Delta(\gamma)$ for $\gamma \in \mathcal{T}^+$:
  $$\alpha(M\gamma) + \alpha(M\gamma S) + \alpha(M\gamma U) = \frac{1}{4} + \frac{1}{4} + \frac{1}{2} = 1;$$
- $(x, y) \in \mathcal{R}$ is the $P_3$ vertex of $\Delta(\gamma')$ with $\gamma' \in \mathcal{T}^-$:
  $$\alpha(M\gamma') + \alpha(M\gamma' U) + \alpha(M\gamma' U^2) + \alpha(M\gamma' S) = \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{2} = 1.$$

**References**


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