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A COMBINATORIAL REFINEMENT OF THE KRONECKER-HURWITZ CLASS NUMBER RELATION

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Abstract. We give a refinement of the Kronecker-Hurwitz class number relation, based on a tesselation of the Euclidean plane into semi-infinite triangles labeled by $\mathrm{PSL}_2(\mathbb{Z})$ that may be of independent interest.

1. A refinement of a classical class number relation

We give a refinement, and a new proof, of the following classical result [1–3].

Theorem 1 (Kronecker, Gierster, Hurwitz). *For any $n \geqslant 1$ we have*

$$\sum_{t^2 \leqslant 4n} H(4n - t^2) \;=\; \sum_{\substack{n=ad \\ a,\, d>0}} \max(a, d)\,.$$

Here $H(D)$ ($D \geqslant 0, D \equiv 0, 3 \bmod 4$) is the Kronecker-Hurwitz class number, which has initial values

D	0	3	4	7	8	11	12	15	16	19	20	23	24
$H(D)$	$-\frac{1}{12}$	$\frac{1}{3}$	$\frac{1}{2}$	1	1	1	$\frac{4}{3}$	2	$\frac{3}{2}$	1	2	3	2

and for $D > 0$ equals the number of $\mathrm{PSL}_2(\mathbb{Z})$-equivalence classes of positive definite integral binary quadratic forms of discriminant $-D$, with those classes that contain a multiple of $x^2 + y^2$ or of $x^2 - xy + y^2$ counted with multiplicity $1/2$ or $1/3$, respectively.

Let $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$. By the Γ-equivariant bijection $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \leftrightarrow cx^2 + (d-a)xy - by^2$ between integral matrices of determinant n and trace t and quadratic forms of discriminant $t^2 - 4n$, the class number relation can be written as

$$\sum_{\substack{M \in \mathcal{M}_n \\ M \text{ elliptic}}} \chi(z_M) \;=\; \sum_{\substack{n=ad \\ a,\, d>0}} \max(a, d) + \begin{cases} 1/6 & \text{if } n \text{ is a square,} \\ 0 & \text{otherwise,} \end{cases} \tag{1}$$

where $\mathcal{M}_n$ is the set of integral matrices of determinant n modulo ± 1, z_M is the fixed point of an elliptic M in the upper half-plane $\mathfrak{H}$, and $\chi : \mathfrak{H} \to \mathbb{Q}$ is the modified

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characteristic function of the standard fundamental domain

$$\mathcal{F} \;=\; \{z \in \mathfrak{H} : -1/2 \leqslant \mathrm{Re}(z) \leqslant 1/2,\ |z| \geqslant 1\}$$

of Γ acting on $\mathfrak{H}$ such that $\chi(z)$ is $1/2\pi$ times the angle subtended by $\mathcal{F}$ at z (so χ is 1 in the interior of $\mathcal{F}$, 0 outside of $\mathcal{F}$, 1/2 on the boundary points different from the corners $\rho = e^{\pi i/3}$ and ρ^2, and 1/6 at the corners).

We will prove a refinement of (1) saying that the subsum of the expression on the left over all M in a given orbit of the right action of Γ on $\mathcal{M}_n$ always takes on one of the values 0, 1, 2 (or 7/6 for the orbit $\sqrt{n}\,\Gamma$ if n is a square). Specifically, let us define for any right coset K in $\mathcal{M}_n/\Gamma$ (more precisely, K is a right coset in $\mathrm{PGL}_2(\mathbb{Q})/\Gamma$, since $\mathcal{M}_n$ is not a group) two positive integers δ_K and δ'_K by $\delta_K = \gcd(c,d)$, $\delta'_K = n/\delta_K$, where $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is any representative of K. Then we have:

Theorem 2. *For each right coset $K \in \mathcal{M}_n/\Gamma$ we have*

$$\sum_{\substack{M\in K \\ M \text{ elliptic}}} \chi(z_M) \;=\; 1 + \mathrm{sgn}(\delta'_K - \delta_K) + \begin{cases} 1/6 & \text{if } K = \sqrt{n}\,\Gamma, \\ 0 & \text{otherwise.} \end{cases}$$

Equation (1) follows immediately by summing the relations in Theorem 2 over all cosets in the disjoint decomposition $\mathcal{M}_n = \bigsqcup \left(\begin{smallmatrix} \delta' & \beta \\ 0 & \delta \end{smallmatrix}\right) \Gamma$ with $n = \delta'\delta$ and $0 \leqslant \beta < \delta'$.

Theorem 2 provides a correspondence between right cosets and Γ-conjugacy classes in $\mathcal{M}_n$, which generically assigns two conjugacy classes to each coset with $\delta' > \delta$. We will deduce it from a similar statement, Theorem 3, which is sharper in two respects (it counts the number of matrices with a fixed point in a smaller domain, and it allows real coefficients), and which gives a generically one-to-one correspondence between cosets and conjugacy classes. To state it, we consider a half-fundamental domain

$$\mathcal{F}^- \;=\; \{z \in \mathfrak{H} : -1/2 \leqslant \mathrm{Re}(z) \leqslant 0,\ |z| \geqslant 1\}\,,$$

and define a function $\alpha : \mathrm{GL}_2^+(\mathbb{R}) \to \mathbb{Q}$ by

$$\alpha(M) \;=\; \begin{cases} \chi^-(z_M) & \text{if } M \text{ is elliptic with fixed point } z_M \in \mathfrak{H}, \\ -\dfrac{1}{12} & \text{if } M \text{ is scalar}, \\ 0 & \text{if } M \text{ is parabolic or hyperbolic}, \end{cases}$$

where χ^- is defined in the same way as χ (and hence equals 1 in the interior of $\mathcal{F}^-$, 0 outside $\mathcal{F}^-$, 1/2 on the internal boundary points of $\mathcal{F}^-$, and 1/4 and 1/6 at the corners i and ρ^2, respectively). Note that $\alpha(-M) = \alpha(M)$, so α is well defined on $M\Gamma$.

Theorem 3. *For $M = \left(\begin{smallmatrix} y & x \\ 0 & 1 \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{R})$ with $y > 0$, we have*

$$\sum_{\gamma\in\Gamma} \alpha\big(M\gamma\big) \;=\; \frac{1 + \mathrm{sgn}(y-1)}{2}\,. \tag{2}$$

Since each coset $K \in \mathcal{M}_n/\Gamma$ contains a representative M with $M\infty = \infty$, Theorem 2 immediately follows from (2), and the fact that the map $\pm\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mapsto \pm\left(\begin{smallmatrix} -a & b \\ c & -d \end{smallmatrix}\right)$ is a bijection between the sets of elements in $\mathcal{M}_n$ having a fixed point in the left half and in the right half of the standard fundamental domain for Γ.

Theorem 3 is proved in Section 3, as an easy consequence of a triangulation of a Euclidean half-plane by triangles associated to elements of Γ (Theorem 4).

This triangulation may be of independent interest, and we give a self-contained treatment in the next section.

2. A triangulation of a Euclidean half-plane

Let $\Gamma_\infty = \{\gamma \in \Gamma \mid \gamma\infty = \infty\}$. We identify $\Gamma \smallsetminus \Gamma_\infty$ with a subset of $\mathrm{SL}_2(\mathbb{Z})$ by choosing representatives $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with $c > 0$, and for such γ we define a semi-infinite triangle

$$\Delta(\gamma) \;=\; \{(x,y) \in \mathbb{R}^2 \mid 0 \leqslant d - cx - ay \leqslant c \leqslant -dx - by\} \,. \tag{3}$$

(The motivation for this definition is that $(x,y) \in \Delta(\gamma)$ if and only if $\left(\begin{smallmatrix} y & x \\ 0 & 1 \end{smallmatrix}\right)\gamma$ has a fixed point in $\mathcal{F}^-$.) Note that $\Delta(\gamma)$ is contained in the half-plane

$$\mathcal{H} \;=\; \{(x,y) \in \mathbb{R}^2 \mid y \geqslant 1\} \,,$$

since $y = c(-dx - by) + d^2 - d(d - cx - ay) \geqslant c^2 + d^2 - c|d| \geqslant 1$.

Theorem 4. *We have a tesselation*

$$\mathcal{H} \;=\; \bigcup_{\gamma \in \Gamma \smallsetminus \Gamma_\infty} \Delta(\gamma)$$

of the half-plane $\mathcal{H}$ into semi-infinite triangles with disjoint interiors.

Remark. We can extend the triangulation of Theorem 4 to a triangulation of all of $\mathbb{R}^2$ by triangles labeled by all of Γ if we define $\Delta(\gamma)$ also for $\gamma \in \Gamma_\infty$ by

$$\Delta\big(\left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right)\big) \;=\; [-n-1, -n] \times (-\infty, 1] \,,$$

and can then interpret the extended triangulation as giving a piecewise-linear action of Γ on $\mathbb{R}^2$, with each triangle being a fundamental domain. However we will not use this in the sequel.

Proof. The group Γ is a free product of its two subgroups generated by the elements $S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ and $U = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 1 \end{smallmatrix}\right)$ of orders 2 and 3, respectively, which fix the two corners of $\mathcal{F}^-$. Therefore we can view elements of Γ as words in S, U, U^2 or as vertices of the tree shown in Figure 1. The proof of both Theorems 3 and 4 will

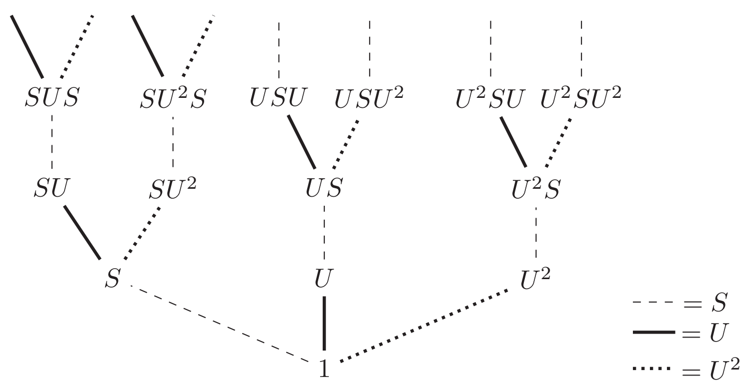

FIGURE 1. A tree associated to $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$: the vertices are labeled by the elements of Γ and the edges by the generators S, U and U^2 as shown.

follow from the following decomposition into triangles with disjoint interiors:

$$\mathcal{R} := \{(x,y) \in \mathbb{R}^2 \mid 0 \leqslant x \leqslant y-1\} \;=\; \bigcup_{\gamma\in\mathcal{T}} \Delta(\gamma)\ , \tag{4}$$

where $\mathcal{T} \subset \Gamma$ is the set of words starting in U. The regions $\mathcal{H}$ and $\mathcal{R}$ and a few triangles corresponding to words of small length are pictured in Figure 2.

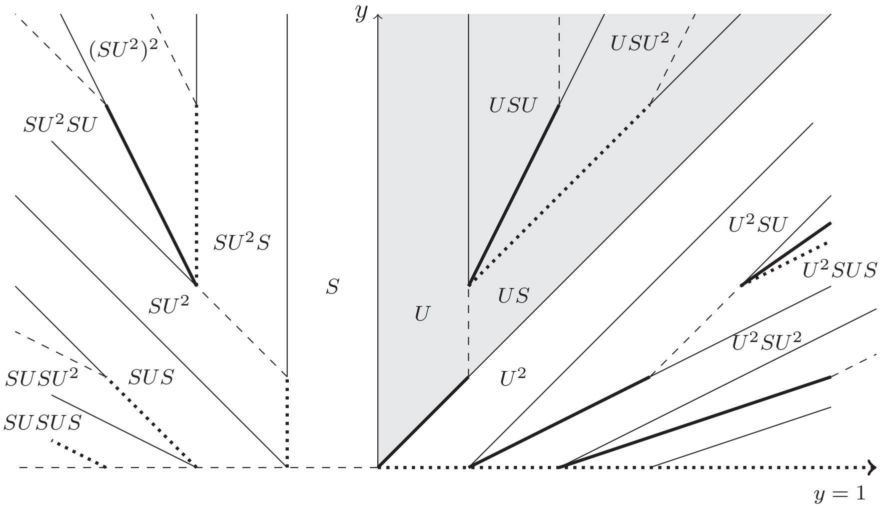

FIGURE 2. The region $\mathcal{R}$ (shaded) and a few triangles $\Delta(\gamma)$. The finite side of a triangle $\Delta(\gamma)$ has been labeled by the final letter of γ as a word in S, U, U^2, with the same convention as in Figure 1.

To prove (4), let $\mathcal{T} = \mathcal{T}^+ \cup \mathcal{T}^-$, where $\mathcal{T}^+$ consists of the elements of $\mathcal{T}$ ending in U or U^2, while $\mathcal{T}^- := \mathcal{T}^+S$ consists of those elements ending in S. The set $\mathcal{T}^+$ can be enumerated recursively by starting at U and replacing $\gamma = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)$ at each step by

$$\gamma SU = \left(\begin{smallmatrix} a & a+b \\ c & c+d\end{smallmatrix}\right)\ , \quad \gamma SU^2 = \left(\begin{smallmatrix} a+b & b \\ c+d & d\end{smallmatrix}\right)\ .$$

From this description we easily obtain the following equivalent characterizations:[1]

$$\gamma \in \mathcal{T}^+ \iff 0 \leqslant \frac{-a}{c} < \frac{-b}{d} \leqslant 1, \quad \gamma \in \mathcal{T}^- \iff 0 \leqslant \frac{-b}{d} < \frac{-a}{c} \leqslant 1\ .$$

Alternatively, $\mathcal{T}^+$ consists of those $\gamma \in \mathcal{T}$ having $d > 0$.

For $\gamma \in \Gamma \smallsetminus \Gamma_\infty$, the triangle $\Delta(\gamma)$ has two vertices given by

$$P_3(-ac-bd+bc, c^2+d^2-cd), \quad P_2(-ac-bd, c^2+d^2)\ ,$$

connected by a line segment of slope $-d/b$, and it has two infinite parallel sides of slope $-c/a$. For $\gamma \in \mathcal{T}$ we denote by $\mathcal{C}(\gamma) \subset \mathcal{H}$ the half-cone containing $\Delta(\gamma)$, bounded by half-lines of slopes $-c/a$ and $-d/b$, and having as vertex P_3 or P_2, depending on whether $\gamma \in \mathcal{T}^+$ or $\gamma \in \mathcal{T}^-$ respectively (see Figure 3).

[1]Recall our convention that $c > 0$.

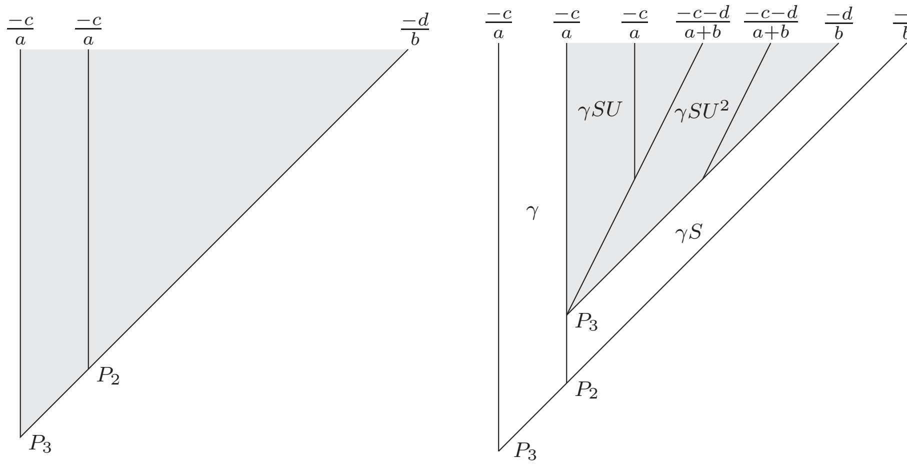

FIGURE 3. Left: The cone $\mathcal{C}(\gamma)$ and the triangle $\Delta(\gamma) \subset \mathcal{C}(\gamma)$ in the case $\gamma \in \mathcal{T}^+$. Right: The cone $\mathcal{C}(\gamma)$ decomposes into two triangles and two smaller, higher-up cones. On top of each line we have marked its slope.

Using this information, it is easy to check that for $\gamma \in \mathcal{T}^+$ and $\gamma' = \gamma S \in \mathcal{T}^-$ we have the following decompositions into sets with disjoint interiors (see the right picture in Figure 3):

$$\mathcal{C}(\gamma) = \Delta(\gamma) \cup \mathcal{C}(\gamma')\,, \quad \mathcal{C}(\gamma') = \Delta(\gamma') \cup \mathcal{C}(\gamma' U) \cup \mathcal{C}(\gamma' U^2)\,.$$

By induction we obtain that $\mathcal{R} = \mathcal{C}(U)$ is the union of the triangles indexed by $\mathcal{T}$, proving (4).

Finally we show that the decomposition in (4) implies the decomposition of $\mathcal{H}$ given in Theorem 4. From the parenthetical remark following (3) it is clear that

$$\Delta(T\gamma) = T\Delta(\gamma)\,,$$

where $T = SU = \left(\begin{smallmatrix}1 & 1\\ 0 & 1\end{smallmatrix}\right)$ and Γ_∞ acts on $\mathcal{H}$ by $T^n(x,y) = (x - ny, y)$. The region

$$\mathcal{R}' = \mathcal{R} \cup \Delta(U^2) \;=\; \{(x,y) \in \mathcal{H} : 0 \leqslant x < y\} \tag{5}$$

(see Figure 2) is a fundamental domain for this action of Γ_∞ on $\mathcal{H}$, and we obtain the following decompositions into triangles with disjoint interiors:

$$\{(x,y) \in \mathcal{H} \mid y - 1 \leqslant x\} \;=\; \bigcup_{\gamma \in \mathcal{T}'} \Delta(\gamma), \quad \{(x,y) \in \mathcal{H} \mid x \leqslant 0\} \;=\; \bigcup_{\gamma \in \mathcal{T}''} \Delta(\gamma)\,,$$

where $\mathcal{T}'$ consists of words starting with U^2, but different from $(U^2S)^n = T^{-n}$ with $n > 0$, while $\mathcal{T}''$ consists of words starting with S, but different from $(SU)^n = T^n$ with $n > 0$. Theorem 4 follows since $\Gamma \smallsetminus \Gamma_\infty = \mathcal{T} \sqcup \mathcal{T}' \sqcup \mathcal{T}''$. $\square$

3. Proof of Theorem 3

Since (2) is invariant under multiplying $M = \left(\begin{smallmatrix}y & x\\ 0 & 1\end{smallmatrix}\right)$ on the right by elements in Γ_∞, we assume without loss of generality that $0 \leqslant x < y$. If $M\gamma$ is scalar for $\gamma \in \Gamma$, the only possibility is easily seen to be $M = 1$. In this case, $\alpha(\gamma) \neq 0$ for $\gamma \in \{1, S, U, U^2\}$, and (2) holds since $-\frac{1}{12} + \frac{1}{4} + \frac{1}{6} + \frac{1}{6} = \frac{1}{2}$.

Assuming that $M \neq 1$, it follows that $\alpha(M\gamma) \neq 0$ if and only if $M\gamma$ has a fixed point in $\mathcal{F}^-$, that is, $(x, y) \in \Delta(\gamma)$. We conclude from Section 2 that $y \geqslant 1$, so the point (x, y) belongs to the region $\mathcal{R}'$ in (5), and $\gamma = U^2$ or $\gamma \in \mathcal{T}$ by (4). Therefore the elements γ such that $\alpha(M\gamma) \neq 0$ depend on the position of the point (x, y) with respect to the triangulation of $\mathcal{R}'$ as follows (see Figure 3):

- $y = 1$ and $0 < x < 1:\ \alpha(MU^2) = 1/2$;
- (x, y) is in the interior of a triangle $\Delta(\gamma):\ \alpha(M\gamma) = 1$;
- (x, y) is on a common side between $\Delta(\gamma)$ and $\Delta(\gamma')$, but it is not a vertex:
$$\alpha(M\gamma) + \alpha(M\gamma') \ = \ \frac{1}{2} + \frac{1}{2} \ = \ 1\,;$$
- $(x, y) \in \mathcal{R}$ is the P_2 vertex of the triangle $\Delta(\gamma)$ for $\gamma \in \mathcal{T}^+$:
$$\alpha(M\gamma) + \alpha(M\gamma S) + \alpha(M\gamma U) \ = \ \frac{1}{4} + \frac{1}{4} + \frac{1}{2} \ = \ 1\,;$$
- $(x, y) \in \mathcal{R}$ is the P_3 vertex of $\Delta(\gamma')$ with $\gamma' \in \mathcal{T}^-$:
$$\alpha(M\gamma') + \alpha(M\gamma' U) + \alpha(M\gamma' U^2) + \alpha(M\gamma' S) \ = \ \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{2} \ = \ 1\,.$$

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