JOURNAL OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 23, Number 4, October 2010, Pages 1013–1040
S 0894-0347(2010)00672-8
Article electronically published on June 9, 2010

NOETHER-LEFSCHETZ THEORY AND THE YAU-ZASLOW CONJECTURE

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Contents

Received by the editors December 28, 2008 and, in revised form, April 13, 2010.
2010 *Mathematics Subject Classification.* Primary 14N35.
The first author was partially supported by DOE grant DE-FG02-95ER40896.
The second author was partially supported by a Clay research fellowship.
The third author was partially support by NSF grant DMS-0500187.

0. Introduction

0.1. **Yau-Zaslow conjecture.** Let S be a nonsingular projective $K3$ surface, and let

$$\beta \in \mathrm{Pic}(S) = H^2(S,\mathbb{Z}) \cap H^{1,1}(S,\mathbb{C})$$

be a nonzero effective curve class. The moduli space $\overline{M}_0(S,\beta)$ of genus 0 stable maps (with no marked points) has the expected dimension

$$\dim^{vir}_{\mathbb{C}} \left(\overline{M}_0(S,\beta)\right) = \int_\beta c_1(S) + \dim_{\mathbb{C}}(S) - 3 = -1.$$

Hence, the virtual class $[\overline{M}_0(S,\beta)]^{vir}$ vanishes, and the standard Gromov-Witten theory is trivial.

Curve counting on $K3$ surfaces is captured instead by the *reduced* Gromov-Witten theory constructed first via the twistor family in [6]. An algebraic construction following [1, 2] is given in [31]. Since the reduced class

$$[\overline{M}_0(S,\beta)]^{red} \in H_0(\overline{M}_0(S,\beta),\mathbb{Q})$$

has dimension 0, the reduced Gromov-Witten integrals of S,

$$R_{0,\beta}(S) = \int_{[\overline{M}_0(S,\beta)]^{red}} 1 \ \in \mathbb{Q}, \tag{1}$$

are well-defined. For deformations of S for which β remains a $(1,1)$-class, the integrals (1) are invariant.

The second cohomology of S is a rank 22 lattice with intersection form

$$H^2(S,\mathbb{Z}) \cong U \oplus U \oplus U \oplus E_8(-1) \oplus E_8(-1), \tag{2}$$

where

$$U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and

$$E_8(-1) = \begin{pmatrix} -2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & -2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 \end{pmatrix}$$

is the (negative) Cartan matrix. The intersection form (2) is even.

The *divisibility* $m(\beta)$ is the maximal positive integer dividing the lattice element $\beta \in H^2(S,\mathbb{Z})$. If the divisibility is 1, β is *primitive*. Elements with equal divisibility and norm are equivalent up to orthogonal transformations of $H^2(S,\mathbb{Z})$. By straightforward deformation arguments using the Torelli theorem for $K3$ surfaces, $R_{0,\beta}(S)$ depends, for effective classes, *only* on the divisibility $m(\beta)$ and the norm $\langle \beta, \beta \rangle$. We will omit the argument S in the notation.

The genus 0 BPS counts associated to $K3$ surfaces have the following definition. Let $\alpha \in \mathrm{Pic}(S)$ be a nonzero class which is both effective and primitive. The

Gromov-Witten potential $F_\alpha(v)$ for classes proportional to α is

$$F_\alpha = \sum_{m>0} R_{0,m\alpha}\ v^{m\alpha}.$$

The BPS counts $r_{0,m\alpha}$ are uniquely defined via the Aspinwall-Morrison formula,

$$F_\alpha = \sum_{m>0} r_{0,m\alpha} \sum_{d>0} \frac{v^{dm\alpha}}{d^3}, \tag{3}$$

for both primitive and divisible classes.

The Yau-Zaslow conjecture [36] predicts the values of the genus 0 BPS counts for the reduced Gromov-Witten theory of $K3$ surfaces. We interpret the conjecture in two parts.

Conjecture 1. *The BPS count $r_{0,\beta}$ depends upon β only through the norm $\langle\beta,\beta\rangle$.*

Conjecture 1 is rather surprising from the point of view of Gromov-Witten theory since $R_{0,\beta}$ certainly depends upon the divisibility of β. Let $r_{0,m,h}$ denote the genus 0 BPS count associated to a class β of divisibility m satisfying

$$\langle\beta,\beta\rangle = 2h-2.$$

Assuming Conjecture 1 holds, we define

$$r_{0,h} = r_{0,m,h},$$

independent[1] of m.

Conjecture 2. *The BPS counts $r_{0,h}$ are uniquely determined by*

$$\sum_{h\geq 0} r_{0,h}\ q^h = \prod_{n=1}^{\infty}(1-q^n)^{-24}. \tag{4}$$

Conjecture 2 can be written in terms of the Dedekind η function

$$\sum_{h\geq 0} r_{0,h}\ q^{h-1} = \eta(\tau)^{-24},$$

where $q = e^{2\pi i\tau}$.

The conjectures have been previously proven in very few cases. A proof of the Yau-Zaslow formula (4) for primitive classes β via Euler characteristics of compactified Jacobians following [36] can be found in [3, 7, 11]. The Yau-Zaslow formula (4) was proven via Gromov-Witten theory for primitive classes β by Bryan and Leung [6]. An early calculation by Gathmann [13] for a class β of divisibility 2 was important for the correct formulation of the conjectures. Conjectures 1 and 2 have been proven in the divisibility 2 case by Lee and Leung [26] and Wu [35]. The main result of the paper is a proof of Conjectures 1 and 2 in all cases.

Theorem 1. *The Yau-Zaslow conjecture holds for all nonzero effective classes $\beta \in \mathrm{Pic}(S)$ on a $K3$ surface S.*

[1] Independence of m holds when $2m^2$ divides $2h-2$. Otherwise, no such class β exists and $r_{0,m,h}$ is defined to vanish.

0.2. **Noether-Lefschetz theory.**

0.2.1. *Lattice polarization.* Let S be a $K3$ surface. A primitive class $L \in \text{Pic}(S)$ is a *quasi-polarization* if

$$\langle L, L\rangle > 0 \quad \text{and} \quad \langle L, [C]\rangle \geq 0$$

for every curve $C \subset S$. A sufficiently high tensor power L^n of a quasi-polarization is base point free and determines a birational morphism

$$S \to \tilde{S}$$

contracting A-D-E configurations of (-2)-curves on S. Hence, every quasi-polarized $K3$ surface is algebraic.

Let Λ be a fixed rank r primitive[2] embedding

$$\Lambda \subset U \oplus U \oplus U \oplus E_8(-1) \oplus E_8(-1)$$

with signature $(1, r-1)$, and let $v_1, \ldots, v_r \in \Lambda$ be an integral basis. The discriminant is

$$\Delta(\Lambda) = (-1)^{r-1} \det \begin{pmatrix} \langle v_1, v_1\rangle & \cdots & \langle v_1, v_r\rangle \\ \vdots & \ddots & \vdots \\ \langle v_r, v_1\rangle & \cdots & \langle v_r, v_r\rangle \end{pmatrix} .$$

The sign is chosen so that $\Delta(\Lambda) > 0$.

A Λ-*polarization* of a $K3$ surface S is a primitive embedding

$$j : \Lambda \to \text{Pic}(S)$$

satisfying two properties:

(i) the lattice pairs $\Lambda \subset U^3 \oplus E_8(-1)^2$ and $\Lambda \subset H^2(S, \mathbb{Z})$ are isomorphic via an isometry which restricts to the identity on Λ,
(ii) $\text{Im}(j)$ contains a quasi-polarization.

By (ii), every Λ-polarized $K3$ surface is algebraic.

The period domain M of Hodge structures of type $(1, 20, 1)$ on the lattice $U^3 \oplus E_8(-1)^2$ is an analytic open set of the 20-dimensional nonsingular isotropic quadric Q,

$$M \subset Q \subset \mathbb{P}\big((U^3 \oplus E_8(-1)^2) \otimes_{\mathbb{Z}} \mathbb{C}\big).$$

Let $M_\Lambda \subset M$ be the locus of vectors orthogonal to the entire sublattice $\Lambda \subset U^3 \oplus E_8(-1)^2$.

Let Γ be the isometry group of the lattice $U^3 \oplus E_8(-1)^2$, and let

$$\Gamma_\Lambda \subset \Gamma$$

be the subgroup restricting to the identity on Λ. By global Torelli, the moduli space $\mathcal{M}_\Lambda$ of Λ-polarized $K3$ surfaces is the quotient

$$\mathcal{M}_\Lambda = M_\Lambda / \Gamma_\Lambda.$$

We refer the reader to [10] for a detailed discussion.

[2]An embedding of lattices is primitive if the quotient is torsion free.

0.2.2. *Families.* Let X be a compact 3-dimensional complex manifold equipped with holomorphic line bundles

$$L_1, \dots, L_r \ \rightarrow X$$

and a holomorphic map

$$\pi : X \rightarrow C$$

to a nonsingular complete curve.

The tuple $(X, L_1, \dots, L_r, \pi)$ is a 1-*parameter family of nonsingular* Λ-*polarized* $K3$ *surfaces* if

(i) the fibers $(X_\xi, L_{1,\xi}, \dots, L_{r,\xi})$ are Λ-polarized $K3$ surfaces via

$$v_i \mapsto L_{i,\xi}$$

for every $\xi \in C$,

(ii) there exists a $\lambda^\pi \in \Lambda$ which is a quasi-polarization of all fibers of π simultaneously.

The family π yields a morphism,

$$\iota_\pi : C \rightarrow \mathcal{M}_\Lambda,$$

to the moduli space of Λ-polarized $K3$ surfaces.

Let $\lambda^\pi = \lambda_1^\pi v_1 + \cdots + \lambda_r^\pi v_r$. A vector $(d_1, \dots, d_r)$ of integers is *positive* if

$$\sum_{i=1}^{r} \lambda_i^\pi d_i > 0.$$

If $\beta \in \mathrm{Pic}(X_\xi)$ has intersection numbers

$$d_i = \langle L_{i,\xi}, \beta \rangle,$$

then β has positive degree with respect to the quasi-polarization if and only if $(d_1, \dots, d_r)$ is positive.

0.2.3. *Noether-Lefschetz divisors.* Noether-Lefschetz numbers are defined in [31] by the intersection of $\iota_\pi(C)$ with Noether-Lefschetz divisors in $\mathcal{M}_\Lambda$. We briefly review the definition of the Noether-Lefschetz divisors.

Let $(\mathbb{L}, \iota)$ be a rank $r+1$ lattice $\mathbb{L}$ with an even symmetric bilinear form $\langle , \rangle$ and a primitive embedding

$$\iota : \Lambda \rightarrow \mathbb{L}.$$

Two data sets $(\mathbb{L}, \iota)$ and $(\mathbb{L}', \iota')$ are isomorphic if there is an isometry which restricts to the identity on Λ. The first invariant of the data $(\mathbb{L}, \iota)$ is the discriminant $\Delta \in \mathbb{Z}$ of $\mathbb{L}$.

An additional invariant of $(\mathbb{L}, \iota)$ can be obtained by considering any vector $v \in \mathbb{L}$ for which[3]

$$\mathbb{L} = \iota(\Lambda) \oplus \mathbb{Z}v. \tag{5}$$

The pairing

$$\langle v, \cdot \rangle : \Lambda \rightarrow \mathbb{Z}$$

determines an element of $\delta_v \in \Lambda^*$. Let $G = \Lambda^*/\Lambda$ be the quotient defined via the injection $\Lambda \rightarrow \Lambda^*$ obtained from the pairing $\langle , \rangle$ on Λ. The group G is abelian of order equal to the discriminant $\Delta(\Lambda)$. The image

$$\delta \in G/\pm$$

[3]Here, $\oplus$ is used just for the additive structure (not the orthogonal direct sum).

of δ_v is easily seen to be independent of v satisfying (5). The invariant δ is the *coset* of $(\mathbb{L}, \iota)$.

By elementary arguments, two data sets $(\mathbb{L}, \iota)$ and $(\mathbb{L}', \iota')$ of rank $r+1$ are isomorphic if and only if the discriminants and cosets are equal.

Let $v_1, \ldots, v_r$ be an integral basis of Λ as before. The pairing of $\mathbb{L}$ with respect to an extended basis $v_1, \ldots, v_r, v$ is encoded in the matrix

$$\mathbb{L}_{h,d_1,\ldots,d_r} = \begin{pmatrix} \langle v_1, v_1 \rangle & \cdots & \langle v_1, v_r \rangle & d_1 \\ \vdots & \ddots & \vdots & \vdots \\ \langle v_r, v_1 \rangle & \cdots & \langle v_r, v_r \rangle & d_r \\ d_1 & \cdots & d_r & 2h-2 \end{pmatrix}.$$

The discriminant is

$$\Delta(h, d_1, \ldots, d_r) = (-1)^r \det(\mathbb{L}_{h,d_1,\ldots,d_r}).$$

The coset $\delta(h, d_1, \ldots, d_r)$ is represented by the functional

$$v_i \mapsto d_i.$$

The Noether-Lefschetz divisor $P_{\Delta,\delta} \subset \mathcal{M}_\Lambda$ is the closure of the locus of Λ-polarized $K3$ surfaces S for which $(\mathrm{Pic}(S), j)$ has rank $r+1$, discriminant Δ, and coset δ. By the Hodge index theorem, $P_{\Delta,\delta}$ is empty unless $\Delta > 0$.

Let $h, d_1, \ldots, d_r$ determine a positive discriminant

$$\Delta(h, d_1, \ldots, d_r) > 0.$$

The Noether-Lefschetz divisor $D_{h,(d_1,\ldots,d_r)} \subset \mathcal{M}_\Lambda$ is defined by the weighted sum

$$D_{h,(d_1,\ldots,d_r)} = \sum_{\Delta,\delta} m(h, d_1, \ldots, d_r | \Delta, \delta) \cdot [P_{\Delta,\delta}],$$

where the multiplicity $m(h, d_1, \ldots, d_r | \Delta, \delta)$ is the number of elements β of the lattice $(\mathbb{L}, \iota)$ of type (Δ, δ) satisfying

$$\langle \beta, \beta \rangle = 2h-2, \quad \langle \beta, v_i \rangle = d_i. \tag{6}$$

If the multiplicity is nonzero, then $\Delta | \Delta(h, d_1, \ldots, d_r)$, so only finitely many divisors appear in the above sum.

If $\Delta(h, d_1, \ldots, d_r) = 0$, the divisor $D_{h,(d_1,\ldots,d_r)}$ has an alternate definition. The tautological line bundle $\mathcal{O}(-1)$ is Γ-equivariant on the period domain M_Λ and descends to the *Hodge line bundle*

$$\mathcal{K} \to \mathcal{M}_\Lambda.$$

We define $D_{h,(d_1,\ldots,d_r)} = \mathcal{K}^*$. See [31] for an alternate view of degenerate intersection.

If $\Delta(h, d_1, \ldots, d_r) < 0$, the divisor $D_{h,(d_1,\ldots,d_r)}$ on $\mathcal{M}_\Lambda$ is defined to vanish by the Hodge index theorem.

0.2.4. *Noether-Lefschetz numbers.* Let Λ be a lattice of discriminant $l = \Delta(\Lambda)$, and let $(X, L_1, \ldots, L_r, \pi)$ be a 1-parameter family of Λ-polarized $K3$ surfaces. The Noether-Lefschetz number $NL^\pi_{h,d_1,\ldots,d_r}$ is the classical intersection product

$$NL^\pi_{h,(d_1,\ldots,d_r)} = \int_C \iota_\pi^* [D_{h,(d_1,\ldots,d_r)}]. \tag{7}$$

Let $\mathrm{Mp}_2(\mathbb{Z})$ be the metaplectic double cover of $SL_2(\mathbb{Z})$. There is a canonical representation [4] associated to Λ,

$$\rho^*_\Lambda : \mathrm{Mp}_2(\mathbb{Z}) \to \mathrm{End}(\mathbb{C}[G]).$$

The full set of Noether-Lefschetz numbers $NL^\pi_{h,d_1,\ldots,d_r}$ defines a vector-valued modular form

$$\Phi^\pi(q) = \sum_{\gamma \in G} \Phi^\pi_\gamma(q) v_\gamma \in \mathbb{C}[[q^{\frac{1}{2l}}]] \otimes \mathbb{C}[G],$$

of weight $\frac{22-r}{2}$ and type ρ^*_Λ by results[4] of Borcherds and Kudla-Millson [4, 25]. The Noether-Lefschetz numbers are the coefficients[5] of the components of Φ^π,

$$NL^\pi_{h,(d_1,\ldots,d_r)} = \Phi^\pi_\gamma \left[\frac{\Delta(h, d_1, \ldots, d_r)}{2l} \right],$$

where $\delta(h, d_1, \ldots, d_r) = \pm\gamma$. The modular form results significantly constrain the Noether-Lefschetz numbers.

0.2.5. *Refinements.* If $d_1, \ldots, d_r$ do not simultaneously vanish, refined Noether-Lefschetz divisors are defined. If $\Delta(h, d_1, \ldots, d_r) > 0$,

$$D_{m,h,(d_1,\ldots,d_r)} \subset D_{h,(d_1,\ldots,d_r)}$$

is defined by requiring the class $\beta \in \mathrm{Pic}(S)$ to satisfy (6) and have divisibility $m > 0$. If $\Delta(h, d_1, \ldots, d_r) = 0$, then

$$D_{m,h,(d_1,\ldots,d_r)} = D_{h,(d_1,\ldots,d_r)}$$

if $m > 0$ is the greatest common divisor of $d_1, \ldots, d_r$ and 0 otherwise.

Refined Noether-Lefschetz numbers are defined by

$$NL^\pi_{m,h,(d_1,\ldots,d_r)} = \int_C \iota^*_\pi [D_{m,h,(d_1,\ldots,d_r)}]. \tag{8}$$

In Section 2.5, the full set of Noether-Lefschetz numbers $NL^\pi_{h,(d_1,\ldots,d_r)}$ is easily shown to determine the refined numbers $NL^\pi_{m,h,(d_1,\ldots,d_r)}$.

0.3. **Three theories.** The main geometric idea in the proof is the relationship of three theories associated to a 1-parameter family

$$\pi : X \to C$$

of Λ-polarized $K3$ surfaces:

(i) the Noether-Lefschetz numbers of π,
(ii) the genus 0 Gromov-Witten invariants of X,
(iii) the genus 0 reduced Gromov-Witten invariants of the $K3$ fibers.

The Noether-Lefschetz numbers (i) are classical intersection products while the Gromov-Witten invariants (ii)-(iii) are quantum in origin. For (ii), we view the theory in terms of the Gopakumar-Vafa invariants[6] [16, 17].

Let $n^X_{0,(d_1,\ldots,d_r)}$ denote the Gopakumar-Vafa invariant of X in genus 0 for π-vertical curve classes of degrees $d_1, \ldots, d_r$ with respect to the line bundles $L_1, \ldots, L_r$. Let $r_{0,m,h}$ denote the reduced $K3$ invariant defined in Section 0.1. The following

[4]While the results of the papers [4, 25] have considerable overlap, we will follow the point of view of Borcherds.
[5]If f is a series in q, $f[k]$ denotes the coefficient of q^k.
[6]A review of the definitions can be found in Section 2.5.

result is proven[7] in [31] by a comparison of the reduced and usual deformation theories of maps of curves to the $K3$ fibers of π.

Theorem 2. *For degrees $(d_1, \ldots, d_r)$ positive with respect to the quasi-polarization λ^π,*

$$n^X_{0,(d_1,\ldots,d_r)} = \sum_{h=0}^{\infty}\sum_{m=1}^{\infty} r_{0,m,h} \cdot NL^\pi_{m,h,(d_1,\ldots,d_r)}.$$

0.4. **Proof of Theorem 1.** The STU model described in Section 1 is a special family of rank 2 lattice polarized $K3$ surfaces

$$\pi^{STU} : X^{STU} \to \mathbb{P}^1.$$

The fibered $K3$ surfaces of the STU model are themselves elliptically fibered. The proof of Theorem 1 proceeds in four basic steps:

(i) The modular form [4, 25] determining the intersections of the base $\mathbb{P}^1$ with the Noether-Lefschetz divisors is calculated. For the STU model, the modular form has vector dimension 1 and is proportional to the product E_4E_6 of Eisenstein series.
(ii) Theorem 2 is used to show the 3-fold BPS counts $n^{X^{STU}}_{0,(d_1,d_2)}$, then *determine* all the reduced $K3$ invariants $r_{0,m,h}$. Strong use is made of the rank 2 lattice of the STU model.
(iii) The BPS counts $n^{X^{STU}}_{0,(d_1,d_2)}$ are calculated via mirror symmetry. Since the STU model is realized as a Calabi-Yau complete intersection in a nonsingular toric variety, the genus 0 Gromov-Witten invariants are obtained after proven mirror transformations from hypergeometric series. The Klemm-Lerche-Mayr identity, proven in Section 3, shows that the invariants $n^{X^{STU}}_{0,(d_1,d_2)}$ are themselves related to modular forms.
(iv) Theorem 1 then follows from the Harvey-Moore identity which simultaneously relates the modular structures of

$$n^{X^{STU}}_{0,(d_1,d_2)}, \quad r_{0,m,h}, \quad \text{and} \quad NL^{\pi^{STU}}_{m,h,(d_1,d_2)}$$

in the form specified by Theorem 2. D. Zagier's proof of the Harvey-Moore identity is presented in Section 4.

The strategy of proof is special to genus 0. Much less is known in higher genus. The Katz-Klemm-Vafa conjecture [21, 31] for the integral[8]

$$\int_{[\overline{M}_g(S,\beta)]^{red}} (-1)^g \lambda_g$$

is a particular generalization of the Yau-Zaslow formula to higher genera. The KKV formula does not yet appear easily approachable in Gromov-Witten theory.[9] However, a proof of the KKV formula for primitive $K3$ classes in the conjecturally equivalent theory of stable pairs in the derived category is given in [22, 34].

[7] The result of [31] is stated in the rank $r = 1$ case, but the argument is identical for arbitrary r.

[8] The integrand λ_g is the top Chern class of the Hodge bundle on $\overline{M}_g(X, \beta)$.

[9] For $g = 1$, the KKV formula follows for all classes on $K3$ surfaces from the Yau-Zaslow formula via the boundary relation for λ_1.

1. The STU model

1.1. **Overview.** The STU model[10] is a particular nonsingular projective Calabi-Yau 3-fold X equipped with a fibration

$$\pi : X \to \mathbb{P}^1. \tag{9}$$

Except for 528 points $\xi \in \mathbb{P}^1$, the fibers

$$X_\xi = \pi^{-1}(\xi)$$

are nonsingular elliptically fibered $K3$ surfaces. The 528 singular fibers X_ξ have exactly 1 ordinary double point singularity each.

The 3-fold X is constructed as an anticanonical section of a nonsingular projective toric 4-fold Y. The Picard rank of Y is 6. The fibration (9) is obtained from a nonsingular toric fibration

$$\pi^Y : Y \to \mathbb{P}^1.$$

The image of

$$\mathrm{Pic}(Y) \to \mathrm{Pic}(X_\xi)$$

determines a rank 2 sublattice of each fiber $\mathrm{Pic}(X_\xi)$ with intersection form

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The toric data describing the construction of $X \subset Y$ and the fibration structure are explained here.

1.2. **Toric varieties.** Let N be a lattice of rank d,

$$N \cong \mathbb{Z}^d.$$

A fan Σ in N is a collection of strongly convex rational polyhedral cones containing all faces and intersections. A toric variety V_Σ is canonically associated to Σ. The variety V_Σ is complete of dimension d if the support of Σ covers $N \otimes_{\mathbb{Z}} \mathbb{R}$. If all cones are simplicial and if all maximal cones are generated by a lattice basis, then V_Σ is nonsingular. See [8, 12, 32] for the basic properties of toric varieties.

Let Σ be a fan corresponding to a nonsingular complete toric variety. A 1-dimensional cone of Σ is a ray with a unique primitive vector. Let $\Sigma^{(1)}$ denote the set of 1-dimensional cones of Σ indexed by their primitive vectors

$$\{\rho_1, \ldots, \rho_n\}. \tag{10}$$

Let $r^1, \ldots, r^\ell$ be a basis over the integers of the module of relations among the vectors (10). We write the j^{th} relation as

$$r_1^j \rho_1 + \ldots + r_n^j \rho_n = 0.$$

Define a torus

$$(\mathbb{C}^*)^\ell \cong \prod_{j=1}^{\ell} \mathbb{C}_j^*$$

with factors indexed by the relations.

[10]The model has been studied in physics since the 1980s. The letter S stands for the dilaton and T and U label the torus moduli in the heterotic string. The STU model was an important example for the duality between type IIA and heterotic strings formulated in [20]. The ideas developed in [18, 19, 23, 24, 30] about the STU model play an important role in our paper.

A simple description of V_Σ is obtained via a quotient construction. Let $\{z_i\}_{1\le i\le n}$ be coordinates on $\mathbb{C}^n$ corresponding to the primitives ρ_i of the rays in $\Sigma^{(1)}$. An action of $\mathbb{C}^*_j$ on $\mathbb{C}^n$ is defined by

$$\lambda_j \cdot \big(z_1, \dots, z_n\big) = \big(\lambda_j^{r^j_1} z_1, \dots, \lambda_j^{r^j_n} z_n\big), \quad \lambda_j \in \mathbb{C}^*_j. \tag{11}$$

In order to obtain a well-behaved quotient for the induced $(\mathbb{C}^*)^\ell$-action on $\mathbb{C}^n$, an exceptional set $Z(\Sigma) \subset \mathbb{C}^n$ consisting of a finite union of linear subspaces is excluded. The linear space defined by $\{z_i = 0 \,|\, i \in I\}$ is contained in $Z(\Sigma)$ if there is no single cone in Σ containing all of the primitives $\{\rho_i\}_{i\in I}$. After removing $Z(\Sigma)$, the quotient

$$V_\Sigma = \Big(\mathbb{C}^n \setminus Z(\Sigma)\Big)\Big/\big(\mathbb{C}^*\big)^\ell \tag{12}$$

yields the toric variety associated to Σ.

Since $\ell = n - d$, the complex dimension of the quotient V_Σ equals the rank d of the lattice N. The variety V_Σ is equipped with the action of the quotient torus

$$T = (\mathbb{C}^*)^n/(\mathbb{C}^*)^\ell.$$

The rank of $\mathrm{Pic}(V_\Sigma)$ is ℓ. The primitives ρ_i are in 1–to–1 correspondence with the T-invariant divisors D_i on V_Σ defined by

$$D_i = \big\{z_i = 0\big\} \ \subset V_\Sigma. \tag{13}$$

Conversely, the homogeneous coordinate z_i is a section of the line bundle $\mathcal{O}(D_i)$. The anticanonical divisor class of V_Σ is determined by

$$-K_{V_\Sigma} = \sum_{i=1}^n D_i. \tag{14}$$

1.3. **The toric 4-fold Y.** The fan Σ in $\mathbb{Z}^4$ defining the toric 4-fold Y has 10 rays with primitive elements:

$$\begin{array}{lll}
\rho_1 = (1,0,2,3), & \rho_2 = (-1,0,2,3), & \\
\rho_3 = (0,1,2,3), & \rho_4 = (0,-1,2,3), & \\
\rho_5 = (0,0,2,3), & \rho_6 = (0,0,-1,0), & \rho_7 = (0,0,0,-1), \\
\rho_8 = (0,0,1,2), & \rho_9 = (0,0,0,1), & \rho_{10} = (0,0,1,1).
\end{array}$$

The full fan Σ is obtained from the convex hull of the 10 primitives. By explicitly checking each of the 24 dimension-4 cones, Y is seen to be a complete nonsingular toric 4-fold.

Generators $r^1, \dots, r^6$ of the rank 6 module of relations among the primitives can be taken to be

$$\begin{array}{cccccccccccc}
\rho_1 & +\rho_2 & & & & +4\rho_6 & +6\rho_7 & & & & = 0, \\
& & \rho_3 & +\rho_4 & & +4\rho_6 & +6\rho_7 & & & & = 0, \\
& & & & \rho_5 & +2\rho_6 & +3\rho_7 & & & & = 0, \\
& & & & & \rho_6 & +2\rho_7 & +\rho_8 & & & = 0, \\
& & & & & & +\ \rho_7 & & +\rho_9 & & = 0, \\
& & & & & \rho_6 & +\ \rho_7 & & & +\rho_{10} & = 0.
\end{array}$$

By the identification (14) of $-K_Y$, the product $\prod_{i=1}^{10} z_i$ defines an anticanonical section. Hence, every product

$$\prod_{i=1}^{10} z_i^{m_i}, \quad m_i \geq 0,$$

which is homogeneous of degree $\sum_{i=1}^{10} r_i^j$ with respect to the action (11) of $\mathbb{C}_j^*$ also defines an anticanonical section. Hence,

$$(15) \qquad \begin{array}{rr} {z_1}^{12}{z_4}^{12}{z_5}^6{z_8}^4{z_9}^2{z_{10}}^3, & {z_1}^{12}{z_3}^{12}{z_5}^6{z_8}^4{z_9}^2{z_{10}}^3, \\ {z_2}^{12}{z_4}^{12}{z_5}^6{z_8}^4{z_9}^2{z_{10}}^3, & {z_2}^{12}{z_3}^{12}{z_5}^6{z_8}^4{z_9}^2{z_{10}}^3, \\ {z_6}^3 z_8 {z_9}^2, & {z_7}^2 z_{10} \end{array}$$

are all sections of $-K_Y$.

From the definitions, we find that $Z(\Sigma)$ consists of the union of the following 11 linear spaces of dimension 2 in $\mathbb{C}^4$:

$$(16) \qquad \begin{array}{llll} I_1 = \{1,2\}, & I_2 = \{3,4\}, & I_3 = \{5,6\}, & I_4 = \{5,7\}, \\ I_5 = \{5,9\}, & I_6 = \{6,8\}, & I_7 = \{6,10\}, & I_8 = \{7,8\}, \\ I_9 = \{7,9\}, & I_{10} = \{8,10\}, & I_{11} = \{9,10\}\ . & \end{array}$$

Recall, I_k indexes the coordinates which vanish.

A simple verification shows that the 6 sections (15) of $-K_Y$ do not have a common zero on the prequotient $\mathbb{C}^n \setminus Z(\Sigma)$. Hence, $-K_Y$ is generated by global sections on Y. A hypersurface

$$X \subset Y$$

defined by a generic section of $-K_Y$ is nonsingular by Bertini's Theorem. By adjunction, X is Calabi-Yau.

1.4. **Fibrations.** The toric variety Y admits two obvious fibrations

$$\pi^Y : Y \to \mathbb{P}^1, \quad \mu^Y : \to \mathbb{P}^1$$

given in homogeneous coordinates by

$$\pi^Y(z_1, \ldots, z_{10}) = [z_1, z_2], \quad \mu^Y(z_1, \ldots, z_{10}) = [z_3, z_4].$$

Since $Z(\Sigma)$ contains the linear spaces

$$I_1 = \{1,2\}, \quad I_2 = \{3,4\},$$

both π^Y and μ^Y are well-defined.

Consider first π^Y. The fibers of π^Y are nonsingular complete toric 3-folds defined by the fan in

$$\mathbb{Z}^3 \subset \mathbb{Z}^4, \quad (c_1, c_2, c_3) \mapsto (0, c_1, c_2, c_3)$$

determined by the primitives $\rho_3, \ldots, \rho_{10}$.

Let X be obtained from a generic section of $-K_Y$. Let

$$\pi : X \to \mathbb{P}^1$$

be the restriction $\pi^Y|_X$.

Proposition 1. *Except for* 528 *points* $\xi \in \mathbb{P}^1$, *the fibers*

$$X_\xi = \pi^{-1}(\xi)$$

are nonsingular elliptically fibered $K3$ *surfaces. The* 528 *singular fibers* X_ξ *each have exactly* 1 *ordinary double point singularity.*

Proof. Let $P_{k,k}(z_1, z_2|z_3, z_4)$ denote a bihomogeneous polynomial of degree k in (z_1, z_2) and degree k in (z_3, z_4). Let

$$F = P_{12,12}(z_1, z_2|z_3, z_4), \quad G = P_{8,8}(z_1, z_2|z_3, z_4), \quad H = P_{4,4}(z_1, z_2|z_3, z_4)$$

be bihomogeneous polynomials. Then

$$Fz_5^6 z_8^4 z_9^2 z_{10}^3, \quad Gz_5^4 z_6 z_8^3 z_9^2 z_{10}^2, \quad Hz_5^2 z_6^2 z_8^2 z_9^2 z_{10}, \quad z_6^3 z_8 z_9^2, \quad z_7^2 z_{10} \tag{17}$$

all determine sections of $-K_Y$.

Let X be defined by a generic linear combination of the sections (17). Since the base point free system (15) is contained in (17), X is nonsingular. We will prove that all the fibers X_ξ are nonsingular, except for finitely many with exactly 1 ordinary double point each, by an explicit study of the equations.

Since $I_7 = \{6, 10\}$, $I_{10} = \{8, 10\}$, and $I_{11} = \{9, 10\}$ are in $Z(\Sigma)$, we easily see that $X \cap D_{10} = \emptyset$ if the coefficient of $z_6^3 z_8 z_9^2$ is nonzero. Similarly,

$$X \cap D_8 = \emptyset, \quad X \cap D_9 = \emptyset.$$

Hence, using the last 3 factors of the torus $(\mathbb{C}^*)^\ell$, the coordinates z_8, z_9, and z_{10} can all be set to 1. The equation for X simplifies to

$$Fz_5^6 + Gz_5^4 z_6 + Hz_5^2 z_6^2 + \alpha z_6^3 + \beta z_7^2.$$

The coordinates z_1 and z_2 do not simultaneously vanish on Y. There are two charts to consider. By symmetry, the analysis on each is identical, so we assume $z_1 \neq 0$. Using the first factor of $(\mathbb{C}^*)^\ell$, we set $z_1 = 1$. By the same reasoning, we set $z_3 = 1$ using the second factor of $(\mathbb{C}^*)^\ell$. Since $I_3 = \{5, 6\}$ and $I_4 = \{5, 7\}$ are in $Z(\Sigma)$, either $z_5 \neq 0$ or *both* z_6 and z_7 do not vanish.

Case $z_5 \neq 0$. Using the third factor of $(\mathbb{C}^*)^\ell$ to set $z_5 = 1$, we obtain the equation

$$F(1, z_2|1, z_4) + H(1, z_2|1, z_4)z_6 + G(1, z_2|1, z_4)z_6^2 + \alpha z_6^3 + \beta z_7^2 \tag{18}$$

in $\mathbb{C}^4$ with coordinates z_2, z_4, z_6, z_7. The map π is given by the z_2 coordinate. The partial derivative of (18) with respect to z_7 is $2\beta z_7$. Hence, if $\beta \neq 0$, all singularities of π occur when $z_7 = 0$.

We need only analyze the reduced dimension case

$$F(1, z_2|1, z_4) + H(1, z_2|1, z_4)z_6 + G(1, z_2|1, z_4)z_6^2 + z_6^3 \tag{19}$$

with coordinates z_2, z_4, z_6. Here, α has been set to 1 by scaling the equation. We must show that all the fibers of π are nonsingular curves except for finitely many with simple nodes. We view equation (19) as defining a 1-parameter family of paths $\gamma_{z_2}(z_4)$ in the space

$$\mathcal{C} = \{\gamma_0 + \gamma_1 z_6 + \gamma_2 z_6^2 + z_6^3 \mid \gamma_0, \gamma_1, \gamma_2 \in \mathbb{C}\}$$

of cubic polynomials in the variable z_6. The coordinate of the path is z_4. The variable z_2 indexes the family of paths.

Let $\Delta \subset \mathcal{C}$ be the codimension 1 discriminant locus of cubics with double roots. The discriminant is irreducible with cuspidal singularities in codimension 2 in $\mathcal{C}$. The possible singularities of the fiber $\pi^{-1}(\lambda)$ occur only when the path $\gamma_\lambda(z_4)$ intersects Δ. The fiber $\pi^{-1}(\lambda)$ is nonsingular over such an intersection point if either

(i) γ_λ is transverse to Δ at a nonsingular point of Δ,

(ii) γ_λ is transverse to the codimension 1 tangent cone of a singular point of Δ.

The fiber $\pi^{-1}(\lambda)$ has a simple node over an intersection point of the path $\gamma_\lambda(z_4)$ with Δ if

(iii) γ_λ is tangent to Δ at a nonsingular point of Δ.

The above are all the possibilities which can occur in a generic 1-parameter family of paths in the space of cubic equations.[11] Possibility (iii) can happen only for finitely many λ and just once for each such λ.

Case $z_6 \neq 0$ *and* $z_7 \neq 0$. Using the third factor of $(\mathbb{C}^*)^\ell$ to set $z_6 = 1$, we obtain the equation

$$F(1, z_2|1, z_4) + H(1, z_2|1, z_4) + G(1, z_2|1, z_4) + \alpha + \beta z_7^2 \tag{20}$$

in $\mathbb{C}^4$ with coordinates z_2, z_4, z_5, z_7. The partial derivative of (20) with respect to z_7 is not 0 for $z_7 \neq 0$. Hence, there are no singular fibers of π on the chart.

We have proven that all the fibers X_ξ of π are nonsingular except for finitely many with exactly 1 ordinary double point each. Let X_ξ be a nonsingular fiber. Let

$$\mu : X \to \mathbb{P}^1$$

be the restriction $\mu^Y|_X$. The fibers of the product

$$(\pi, \mu) : X \to \mathbb{P}^1 \times \mathbb{P}^1$$

are easily seen to be anticanonical sections of the nonsingular toric surface[12] W with fan in $\mathbb{Z}^2$ determined by the primitives $\rho_5, \ldots, \rho_{10}$. These anticanonical sections are elliptic curves. Since X_ξ has trivial canonical bundle by adjunction and the map

$$\mu : X_\xi \to \mathbb{P}^1$$

is dominant with elliptic fibers, we conclude that X_ξ is an elliptically fibered $K3$ surface.

The Euler characteristic of X can be calculated by toric intersection in Y,

$$\chi_{top}(X) = -480.$$

The Euler characteristic of a nonsingular $K3$ fibration over $\mathbb{P}^1$ is 48. Since each fiber singularity reduces the Euler characteristic by 1, we conclude that π has exactly 528 singular fibers. □

For emphasis, we will sometimes denote the STU model by

$$\pi^{STU} : X^{STU} \to \mathbb{P}^1.$$

1.5. **Divisor restrictions.** The divisors D_1, D_2, D_8, D_9, and D_{10} have already been shown to restrict to the trivial class in $\mathrm{Pic}(X_\xi)$. The divisors D_3 and D_4 restrict to the fiber class $F \in \mathrm{Pic}(X_\xi)$ of the elliptic fibration

$$\mu : X_\xi \to \mathbb{P}^1. \tag{21}$$

Certainly $F^2 = 0$. Let $S \in \mathrm{Pic}(X_\xi)$ denote the restriction of D_5. Toric calculations yield the products

$$F \cdot S = 1, \quad S \cdot S = -2.$$

[11] A cusp of $\pi^{-1}(\lambda)$ occurs, for example, when the path has contact order 3 at a nonsingular point of the discriminant.

[12] Since the product $(\pi^Y, \mu^Y) : Y \to \mathbb{P}^1 \times \mathbb{P}^1$ has fibers isomorphic to the nonsingular complete (hence projective) toric surface W, the 4-fold Y is projective.

Hence, S may be viewed as the section class of the elliptic fibration (21). The divisors D_6 and D_7 restrict to classes in the rank 2 lattice generated by F and S.

The restriction of $\mathrm{Pic}(Y)$ to each fiber X_ξ is a rank 2 lattice generated by F and S with intersection form

$$\begin{pmatrix} 0 & 1 \\ 1 & -2 \end{pmatrix}.$$

We may also choose generators $L_1 = F$ and $L_2 = F + S$ with intersection form

$$\Lambda = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

1.6. **1-parameter families.** Let X be a compact 3-dimensional complex manifold equipped with two holomorphic line bundles

$$L_1,\ L_2 \to X$$

and a holomorphic map

$$\pi : X \to C$$

to a nonsingular complete curve.

The data (X, L_1, L_2, π) determine a *family of* Λ*-polarized* $K3$ *surfaces* if the fibers $(X_\xi, L_{1,\xi}, L_{2,\xi})$ are $K3$ surfaces with intersection form

$$\begin{pmatrix} L_{1,\xi} \cdot L_{1,\xi} & L_{2,\xi} \cdot L_{1,\xi} \\ L_{1,\xi} \cdot L_{2,\xi} & L_{2,\xi} \cdot L_{2,\xi} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and there exists a simultaneous quasi-polarization. The 1-parameter family (X, L_1, L_2, π) yields a morphism,

$$\iota_\pi : C \to \mathcal{M}_\Lambda,$$

to the moduli space of Λ-polarized $K3$ surfaces.

The construction $(X^{STU}, L_1, L_2, \pi^{STU})$ of the STU model in Sections 1.3-1.5 is almost a 1-parameter family of Λ-polarized $K3$ surfaces. The only failing is the 528 singular fibers of π^{STU}. Let

$$\epsilon : C \xrightarrow{2-1} \mathbb{P}^1$$

be a hyperelliptic curve branched over the 528 points of $\mathbb{P}^1$ corresponding to the singular fibers of π. The family

$$\epsilon^*(X^{STU}) \to C$$

has 3-fold double point singularities over the 528 nodes of the fibers of the original family. Let

$$\widetilde{\pi}^{STU} : \widetilde{X}^{STU} \to C$$

be obtained from a small resolution

$$\widetilde{X}^{STU} \to \epsilon^*(X^{STU}).$$

Let $\widetilde{L}_i \to \widetilde{X}^{STU}$ be the pull-back of L_i by ϵ. The data

$$(\widetilde{X}^{STU}, \widetilde{L}_1, \widetilde{L}_2, \widetilde{\pi}^{STU})$$

determine a 1-parameter family of Λ-polarized $K3$ surfaces; see Section 5.3 of [31]. The simultaneous quasi-polarization is obtained from the projectivity of X^{STU}.

1.7. **Gromov-Witten invariants.** Since X^{STU} is defined by an anticanonical section in a semi-positive nonsingular toric variety Y, the genus 0 Gromov-Witten invariants have been proven by Givental [14, 15, 29, 33] to be related by mirror transformation to hypergeometric solutions of the Picard-Fuchs equations of the Batyrev-Borisov mirror. By Section 5.3 of [31], the Gromov-Witten invariants of $\widetilde{X}^{STU}$ are exactly twice the Gromov-Witten invariants of X^{STU} for curve classes in the fibers.

2. Noether-Lefschetz numbers and reduced $K3$ invariants

2.1. **Refined Noether-Lefschetz numbers.** Following the notation of Section 0.2, let

$$\Lambda \subset U \oplus U \oplus U \oplus E_8(-1) \oplus E_8(-1)$$

be primitively embedded with signature $(1, r-1)$ and integral basis $v_1, \ldots, v_r$. Let $(X, L_1, \ldots, L_r, \pi)$ be a 1-parameter family of Λ-polarized $K3$ surfaces. Let $d_1, \ldots, d_r$ be integers which do not all vanish.

Lemma 1. *The Noether-Lefschetz numbers $NL^\pi_{h,(d_1,\ldots,d_r)}$ completely determine the refinements $NL^\pi_{m,h,(d_1,\ldots,d_r)}$.*

Proof. By definition, the refined Noether-Lefschetz numbers satisfy two elementary identities. The first is

$$NL^\pi_{h,(d_1,\ldots,d_r)} = \sum_{m=1}^{\infty} NL^\pi_{m,h,(d_1,\ldots,d_r)}.$$

If m does not divide all d_i, then $NL^\pi_{m,h,(d_1,\ldots,d_r)}$ vanishes. If m divides all d_i, then a second identity holds:

$$NL^\pi_{m,h,(d_1,\ldots,d_r)} = NL^\pi_{1,h',(d_1/m,\ldots,d_r/m)},$$

where $2h-2 = m^2(2h'-2)$.

If $\Delta(h, d_1, \ldots, d_r) = 0$, the refined number $NL^\pi_{m,h,(d_1,\ldots,d_r)}$ vanishes by definition unless m is the GCD of $(d_1, \ldots, d_r)$. In the latter case,

$$NL^\pi_{h,(d_1,\ldots,d_r)} = NL^\pi_{m,h,(d_1,\ldots,d_r)}.$$

Hence the lemma is trivial in the $\Delta(h, d_1, \ldots, d_r) = 0$ case.

If $\Delta(h, d_1, \ldots, d_r) > 0$, we prove the lemma by induction on Δ. The second identity reduces us to the case where $m = 1$. The first identity determines the $m = 1$ case in terms of the Noether-Lefschetz number $NL_{h,(d_1,\ldots,d_r)}$ and refined numbers with

$$\Delta(h', d'_1, \ldots, d'_r) < \Delta(h, d_1, \ldots, d_r).$$

□

2.2. **STU model.** The resolved version of the STU model

$$\widetilde{\pi}^{STU} : \widetilde{X}^{STU} \to C$$

is lattice polarized with respect to

$$\Lambda = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The application of the results of [4, 25] to the STU model is extremely simple. Since the lattice Λ is unimodular, the corresponding representation ρ^*_Λ is 1-dimensional and, in fact, is the trivial representation of $\mathrm{Mp}_2(\mathbb{Z})$. The Noether-Lefschetz degrees are thus encoded by a scalar modular form of weight $\frac{22-r}{2} = 10$. The space of such forms is well known to be of dimension 1 and spanned by the product of the Eisenstein series[13]

$$E_{10}(q) = E_4(q)E_6(q) = 1 - 264 \sum_{n \geq 1} \sigma_9(n) q^n.$$

Hence, a single Noether-Lefschetz calculation determines the full series.

Lemma 2. $NL^{\widetilde{\pi}}_{0,(0,0)} = 1056$.

Proof. By Proposition 1, the STU model

$$\pi^{STU} : X^{STU} \to \mathbb{P}^1$$

has 528 nodal fibers. Let S be a fiber of the resolved family $\widetilde{\pi}^{STU}$ lying over a singular fiber of π. The Picard lattice of S certainly contains

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix} \tag{22}$$

spanned by L_1, L_2, and the (-2)-curve E of the small resolution. Let

$$\widetilde{\iota} : C \to \mathcal{M}_\Lambda$$

be the map to moduli. Since a class β satisfying

$$\langle \beta, \beta \rangle = -2$$

on a $K3$ surface is either effective or anti-effective, the set-theoretic intersections of $\widetilde{\iota}$ with $D_{0,(0,0)}$ correspond to fibers of $\widetilde{\pi}$, where L_1 and L_2 do not generate an ample class, precisely, the 528 fibers of $\widetilde{\pi}$ lying over the singular fibers of π.

The divisor $D_{0,(0,0)}$ has multiplicity exactly 2 at the 528 intersections with $\widetilde{\iota}$ since E and $-E$ are the only -2 classes orthogonal to L_1 and L_2. Finally, since E has normal bundle $(-1,-1)$ in $\widetilde{X}^{STU}$, the curve $\widetilde{\iota}$ is transverse to the reduced divisor $\frac{1}{2}D_{0,(0,0)}$ at the 528 intersections. We conclude that $NL^{\widetilde{\pi}}_{0,(0,0)} = 528 \cdot 2 = 1056$. □

Proposition 2. *The Noether-Lefschetz degrees of the resolved STU model are given by the equation*

$$NL^{\widetilde{\pi}}_{h,(d_1,d_2)} = -4E_4(q)E_6(q) \left[\frac{\Delta(h, d_1, d_2)}{2} \right].$$

[13] The Eisenstein series E_{2k} is the modular form defined by the equation

$$-\frac{B_{2k}}{4k} E_{2k}(q) = -\frac{B_{2k}}{4k} + \sum_{n \geq 1} \sigma_{2k-1}(n) q^n,$$

where B_{2n} is the $2n^{\text{th}}$ Bernoulli number and $\sigma_n(k)$ is the sum of the k^{th} powers of the divisors of n,

$$\sigma_k(n) = \sum_{i|n} i^k.$$

2.3. **BPS states.** Let $(\widetilde{X}^{STU}, \widetilde{L}_1, \widetilde{L}_2, \widetilde{\pi}^{STU})$ be the Λ-polarized STU model. The vertical classes are the kernel of the push-forward map by $\widetilde{\pi}$,

$$0 \to H_2(\widetilde{X}, \mathbb{Z})^{\widetilde{\pi}} \to H_2(\widetilde{X}, \mathbb{Z}) \to H_2(C, \mathbb{Z}) \to 0.$$

While $\widetilde{X}$ need not be a projective variety, $\widetilde{X}$ carries a $(1,1)$-form ω_K which is Kähler on the $K3$ fibers of $\widetilde{\pi}$. The existence of a fiberwise Kähler form is sufficient to define the Gromov-Witten theory for vertical classes,

$$0 \neq \gamma \in H_2(\widetilde{X}, \mathbb{Z})^{\widetilde{\pi}}.$$

The fiberwise Kähler form ω_K is obtained by a small perturbation of the quasi-Kähler form obtained from the quasi-polarization. The associated Gromov-Witten theory is independent of the perturbation used.

Let $\overline{M}_0(\widetilde{X}, \gamma)$ be the moduli space of stable maps from connected genus 0 curves to $\widetilde{X}$. Gromov-Witten theory is defined by integration against the virtual class,

$$N_{0,\gamma}^{\widetilde{X}} = \int_{[\overline{M}_0(\widetilde{X},\gamma)]^{vir}} 1. \tag{23}$$

The expected dimension of the moduli space is 0.

The genus 0 Gromov-Witten potential $F^{\widetilde{X}}(v)$ for nonzero vertical classes is the series

$$F^{\widetilde{X}} = \sum_{0 \neq \gamma \in H_2(\widetilde{X},\mathbb{Z})^{\widetilde{\pi}}} N_{0,\gamma}^{\widetilde{X}} \ v^{\gamma},$$

where v is the curve class variable. The BPS counts $n_{0,\gamma}^{\widetilde{X}}$ of Gopakumar and Vafa are uniquely defined by the following equation:

$$F^{\widetilde{X}} = \sum_{0 \neq \gamma \in H_2(\widetilde{X},\mathbb{Z})^{\widetilde{\pi}}} n_{0,\gamma}^{\widetilde{X}} \sum_{d>0} \frac{v^{d\gamma}}{d^3}.$$

Conjecturally, the invariants $n_{0,\gamma}^{\widetilde{X}}$ are integral and obtained from the cohomology of an as yet unspecified moduli space of sheaves on $\widetilde{X}$. We do not assume that the conjectural properties hold.

Using the Λ-polarization, we define the BPS counts

$$n_{0,(d_1,d_2)}^{\widetilde{X}} = \sum_{\gamma \in H_2(\widetilde{X},\mathbb{Z})^{\widetilde{\pi}}, \ \int_\gamma \widetilde{L}_i = d_i} n_{0,\gamma}^{\widetilde{X}} \tag{24}$$

when d_1 and d_2 are not both 0.

The original STU model,

$$\pi^{STU} : X^{STU} \to \mathbb{P}^1,$$

with 528 singular fibers is a nonsingular, projective, Calabi-Yau 3-fold. Hence the Gromov-Witten invariants are well-defined. Let $n_{0,(d_1,d_2)}^{X}$ denote the fiberwise Gopakumar-Vafa invariant with degrees d_i measured by L_i. By the argument of Section 1.7,

$$n_{0,(d_1,d_2)}^{\widetilde{X}} = 2 n_{0,(d_1,d_2)}^{X}$$

when d_1 and d_2 are not both 0.

2.4. **Invertibility of constraints.** Let $\mathcal{P} \subset \mathbb{Z}^2$ be the set of pairs

$$\mathcal{P} = \{ \ (d_1, d_2) \neq (0,0) \mid d_1 \geq 0, \ d_1 \geq -d_2 \ \} \ .$$

Pairs $(d_2, d_2) \in \mathcal{P}$ are certainly positive with respect to any quasi-polarization for $\widetilde{\pi}^{STU}$ since such (d_1, d_2) can be realized by linear combinations of the effective classes F and S.

Theorem 2 applied to the resolved STU model yields the equation

$$n^{\widetilde{X}}_{0,(d_1,d_2)} = \sum_{h=0}^{\infty} \sum_{m=1}^{\infty} r_{0,m,h} \cdot NL^{\widetilde{\pi}}_{m,h,(d_1,d_2)} \tag{25}$$

for $(d_1, d_2) \in \mathcal{P}$. The BPS states on the left side will be computed by mirror symmetry in Section 3. The refined Noether-Lefschetz degrees are determined by Lemma 1 and Proposition 2. Consequently, equation (25) provides constraints on the reduced $K3$ invariants $r_{0,m,h}$.

The integrals $r_{0,m,h}$ are very simple in case $h \leq 0$. By Lemma 2 of [31], $r_{0,m,h} = 0$ for $h < 0$,

$$r_{0,1,0} = 1,$$

and $r_{0,m,0} = 0$ otherwise.

Proposition 3. *The set of integrals* $\{r_{0,m,h}\}_{m \geq 1, h > 0}$ *is uniquely determined by the set of constraints* (25) *for* $(d_1 \geq 0, \ d_2 > 0)$ *and the integrals* $r_{0,m,h \leq 0}$.

Proof. A certain subset of the linear equations with $d_2 > 0$ will be shown to be upper triangular in the variables $r_{0,m,h}$. Picard rank 2 is crucial for the argument.

Let us fix in advance the values of $m \geq 1$ and $h > 0$. We proceed by induction on m assuming the reduced invariants $r_{0,m',h}$ have already been determined for all $m' < m$. The assumption is vacuous when $m = 1$. We can also assume that $r_{0,m,h'}$ has been determined inductively for $h' < h$. If $2h - 2$ is not divisible by $2m^2$, then we have $r_{0,m,h} = 0$, so we can further assume that

$$2h - 2 = m^2(2s - 2)$$

for an integer $s > 0$.

Consider equation (25) for $(d_1, d_2) = (m(s-1), m)$. Certainly

$$NL^{\widetilde{\pi}}_{m',h',(m(s-1),m)} = 0$$

unless m' divides m. By the Hodge index theorem, we must have

$$\Delta(h', m(s-1), m) = 2 - 2h' + m^2(2s - 2) \geq 0 \tag{26}$$

if $NL^{\widetilde{\pi}}_{m,h',(m(s-1),m)} \neq 0$. Inequality (26) implies that $h' \leq h$.

Therefore, the constraint (25) takes the form

$$n^{\widetilde{X}}_{0,(m(s-1),m)} = r_{0,m,h} NL^{\widetilde{\pi}}_{m,h,(m(s-1),m)} + \dots ,$$

where the dots represent terms involving $r_{0,m',h'}$ with either

$$m' < m \quad \text{ or } \quad m' = m, \ h' < h.$$

The leading coefficient is given by

$$NL^{\widetilde{\pi}}_{m,h,(m(s-1),m)} = NL^{\widetilde{\pi}}_{h,(m(s-1),m)} = -4.$$

As the system is upper-triangular, we can invert to solve for $r_{0,m,h}$. □

2.5. **Proof of the Yau-Zaslow conjecture.** By Proposition 3, we need only show that the answer for $r_{0,m,h}$ predicted by the Yau-Zaslow conjecture satisfies the constraints (25) for all pairs $(d_1 \geq 0,\ d_2 > 0)$.

Let X^{STU} be the original Calabi-Yau 3-fold of the STU model. Let

$$D_2^3 F^X = \sum_{(d_1,d_2)\in\mathcal{P}} d_2^3 \ N^X_{0,(d_1,d_2)} \ q_1^{d_1} q_2^{d_2} \tag{27}$$

be the third derivative[14] of the genus 0 Gromov-Witten series for π-vertical classes in $\mathcal{P}$.

We can calculate $D_2^3 F^X$ by the constraint (25) *assuming the validity of the Yau-Zaslow conjecture*,

$$D_2^3 F^X = \sum_{(d_1,d_2)\in\mathcal{P}} d_2^3 \ c(d_1,d_2) \ \frac{q_1^{d_1} q_2^{d_2}}{1 - q_1^{d_1} q_2^{d_2}}, \tag{28}$$

where $c(k,l)$ is the coefficient of q^{kl} in

$$-2\frac{E_4(q)E_6(q)}{\eta^{24}(q)}.$$

Proposition 4. *The Yau-Zaslow conjecture is implied by the identity*

$$\sum_{(d_1,d_2)\in\mathcal{P}} d_2^3 N^X_{0,(d_1,d_2)} q_1^{d_1} q_2^{d_2} = \sum_{(d_1,d_2)\in\mathcal{P}} d_2^3 \ c(d_1,d_2) \ \frac{q_1^{d_1} q_2^{d_2}}{1 - q_1^{d_1} q_2^{d_2}}.$$

Proof. The $q_1^{d_1} q_2^{d_2}$ coefficient of the above identity is simply d_2^3 times the constraint (25). Since we only require the constraints in case

$$(d_1 \geq 0,\ d_2 > 0) \in \mathcal{P},$$

the identity implies all the constraints we need. □

The remainder of the paper is devoted to the proof of Proposition 4. The genus 0 Gromov-Witten invariants of X are related, after a mirror transformation, to hypergeometric solutions of the associated Picard-Fuchs system of differential equations. Hence, Proposition 4 amounts to a subtle identity among special functions.

3. Mirror transform

3.1. **Picard-Fuchs.** Let $\pi : X \to \mathbb{P}^1$ be the STU model. Let

$$\delta_0 \in H^*(X, \mathbb{C})$$

denote the identity class. A basis of $H^2(X,\mathbb{C})$ is obtained from the restriction of the toric divisors of Y discussed in Section 1.5,

$$\delta_1 = 2D_1 + 2D_3 + D_5, \quad \delta_2 = D_3, \quad \delta_3 = D_1.$$

Recall, δ_3 vanishes on the fibers of π. Let $\{\delta_j\}$ be a full basis of $H^*(X,\mathbb{C})$ extending the above selections.

Let u_1, u_2, u_3 be the canonical coordinates for the mirror family with respect to the divisor basis $\delta_1, \delta_2, \delta_3$. Let

$$\theta_i = u_i \frac{\partial}{\partial u_i}.$$

[14] $D_2 = q_2 \frac{d}{dq_2}$.

The Picard-Fuchs system associated to the mirror of X^{STU} is:

$$\begin{aligned} \mathcal{L}_1 &= \theta_1\,(\theta_1 - 2\,\theta_2 - 2\,\theta_3) - 12\,(6\,\theta_1 - 5)\,(6\,\theta_1 - 1)\,u_1, \\ \mathcal{L}_2 &= {\theta_2}^2 - (2\,\theta_2 + 2\,\theta_3 - \theta_1 - 2)\,(2\,\theta_2 + 2\,\theta_3 - \theta_1 - 1)\,u_2, \\ \mathcal{L}_3 &= {\theta_3}^2 - (2\,\theta_2 + 2\,\theta_3 - \theta_1 - 2)\,(2\,\theta_2 + 2\,\theta_3 - \theta_1 - 1)\,u_3\,. \end{aligned} \tag{29}$$

The system is obtained canonically from the Batyrev-Borisov construction; see [9] for the formalism.

3.2. **Solutions.** A fundamental solution to the Picard-Fuchs system can be written in terms of GKZ hypergeometric series,

$$\varpi \in H^*(X,\mathbb{C}) \otimes_{\mathbb{C}} \mathbb{C}[\log(u_1), \log(u_2), \log(u_3)][[u_1,u_2,u_3]]. \tag{30}$$

Let $\varpi(u,\delta_j)$ be the corresponding coefficient of (30). Then

$$\mathcal{L}_i\ \varpi(u,\delta_j) = 0.$$

The standard normalization of ϖ satisfies two important properties:

(i) The δ_0 coefficient is the unique solution
$$\varpi(u,\delta_0) = 1 + O(u)$$
holomorphic at $u = 0$.

(ii) For $1 \le i \le 3$,
$$\varpi(u,\delta_i) = \frac{\varpi(u,\delta_0)}{2\pi i}\log(u_i) + O(u)$$
are the logarithmic solutions.

Let T_1, T_2, T_3 be coordinates on $H^2(X,\mathbb{C})$ with respect to the basis δ. The mirror transformation is defined by

$$T_i = \frac{\varpi(u,\delta_i)}{\varpi(u,\delta_0)} = \frac{1}{2\pi i}\log(u_i) + O(u)$$

for $1 \le i \le 3$.

The mirror transformation relates the genus 0 Gromov-Witten theory of X to the Picard-Fuchs system for the mirror family. For anticanonical hypersurfaces in toric varieties, a proof is given in [15].

3.3. **Mirror transform for $q_3 = 0$.** We introduce two modular parameters

$$\tau_1 = T_1, \quad \tau_2 = T_1 + T_2\,. \tag{31}$$

For $i = 1$ and 2, let

$$\widehat{q}_i = \exp(2\pi i \tau_i),$$

and let $q_3 = \exp(2\pi i T_3)$.

Our first step is to find a modular expression for the mirror map and the period $\varpi(u,\delta_0)$ to leading order in q_3. We prove two formulas discovered by Klemm, Lerche, and Mayr in [24].

Lemma 3. *We have*

$$u_1 = \frac{2(j(\widehat{q}_1) + j(\widehat{q}_2) - \mu)}{j(\widehat{q}_1)j(\widehat{q}_2) + \sqrt{j(\widehat{q}_1)(j(\widehat{q}_1) - \mu)}\sqrt{j(\widehat{q}_2)(j(\widehat{q}_2) - \mu)}} + O(q_3),$$

$$u_2 = \frac{(j(\widehat{q}_1)j(\widehat{q}_2) + \sqrt{j(\widehat{q}_1)(j(\widehat{q}_2) - \mu)}\sqrt{j(\widehat{q}_2)(j(\widehat{q}_2) - \mu)})^2}{4j(\widehat{q}_1)j(\widehat{q}_2)(j(\widehat{q}_1) + j(\widehat{q}_2) - \mu)^2} + O(q_3),$$

where $\mu = 1728$ *and*

$$j(q) = \frac{E_4^3}{\eta^{24}} = \frac{1}{q} + 744 + 196884q + O(q^2) \tag{32}$$

is the normalized j *function.*

Lemma 4. $\mathrm{Lim}_{q_3 \to 0}\ \varpi(u, \delta_0) = E_4(\widehat{q}_1)^{\frac{1}{4}} E_4(\widehat{q}_2)^{\frac{1}{4}}$.

Proof. We prove Lemmas 3 and 4 together. The first step is to perform the following change of variables:

$$u_1 = z_1, \qquad u_2 = \frac{z_2}{2}\left(1 + \sqrt{1 - 4z_3}\right), \qquad u_3 = \frac{z_2}{2}\left(1 - \sqrt{1 - 4z_3}\right),$$

with the inverse change

$$z_1 = u_1, \quad z_2 = u_2 + u_3, \quad z_3 = \frac{u_2 u_3}{(u_2 + u_3)^2}.$$

In the new variables, the limit $u_3 \to 0$ becomes the limit $z_3 \to 0$.

The statement of Lemma 3 in the variables z_i remains unchanged to first order in q_3. We will prove

$$z_1 = \frac{2\left(j(\widehat{q}_1) + j(\widehat{q}_2) - \mu\right)}{j(\widehat{q}_1) j(\widehat{q}_2) + \sqrt{j(\widehat{q}_1)(j(\widehat{q}_1) - \mu)}\sqrt{j(\widehat{q}_2)(j(\widehat{q}_2) - \mu)}} + O(q_3),$$
$$z_2 = \frac{(j(\widehat{q}_1) j(\widehat{q}_2) + \sqrt{j(\widehat{q}_1)(j(\widehat{q}_2) - \mu)}\sqrt{j(\widehat{q}_2)(j(\widehat{q}_2) - \mu)})^2}{4 j(\widehat{q}_1) j(\widehat{q}_2)(j(\widehat{q}_1) + j(\widehat{q}_2) - \mu)^2} + O(q_3)\,.$$

The Picard-Fuchs differential operators (29) can be rewritten as

$$\begin{aligned}
\mathcal{L}_1'(z) &= \mathcal{L}_1(u), \\
z_2\sqrt{1 - 4z_3}\ \mathcal{L}_2'(z) &= \mathcal{L}_2(u) - \mathcal{L}_3(u), \\
z_2\sqrt{1 - 4z_3}\ \mathcal{L}_3'(z) &= u_3 \mathcal{L}_2(u) - u_2 \mathcal{L}_3(u),
\end{aligned}$$

with

$$\begin{aligned}
\mathcal{L}_1' &= \theta_1\left(\theta_1 - 2\,\theta_2\right) - 12\ \left(6\,\theta_1 - 5\right)\left(6\,\theta_1 - 1\right) z_1, \\
\mathcal{L}_2' &= \theta_2\left(\theta_2 - 2\,\theta_3\right) - \left(2\,\theta_2 - \theta_1 - 2\right)\left(2\,\theta_2 - \theta_1 - 1\right) z_2, \\
\mathcal{L}_3' &= {\theta_3}^2 - \left(2\,\theta_3 - \theta_2 - 2\right)\left(2\,\theta_3 - \theta_2 - 1\right) z_3,
\end{aligned}$$

where now $\theta_i = z_i \frac{d}{dz_i}$. Since $\mathcal{L}_3'(z) \to 0$ in the limit $z_3 \to 0$, we need only focus on $\mathcal{L}_1'(z)$ and $\mathcal{L}_2'(z)$.

Next, we transform $\mathcal{L}_1'(z)$ and $\mathcal{L}_2'(z)$ to new variables y_1, y_2, y_3 via the change

$$\begin{aligned}
z_1 &= \frac{2\left(y_1 + y_2 - \mu\right)}{y_1 y_2 + \sqrt{y_1(y_1 - \mu)}\sqrt{y_2(y_2 - \mu)}}, \\
z_2 &= \frac{(y_1 y_2 + \sqrt{y_1(y_1 - \mu)}\sqrt{y_2(y_2 - \mu)})^2}{4 y_1 y_2 (y_1 + y_2 - \mu)^2}, \\
z_3 &= y_3.
\end{aligned}$$

We obtain

$$\begin{aligned}\mathcal{L}_1'' &= y_1^2 y_2 (y_1-\mu)\partial_{y_1}^2 + y_1 y_2 (y_1 - \frac{\mu}{2})\partial_{y_1} - y_1 y_2^2 (y_2-\mu)\partial_{y_2}^2 \\ &\quad - y_1 y_2 (y_2 - \frac{\mu}{2})\partial_{y_2} + 60(y_1 - y_2), \\ \mathcal{L}_2'' &= -y_1^2 (y_1-\mu)\,\partial_{y_1}^2 + y_1(\frac{\mu}{2} - y_1)\partial_{y_1} + y_2^2(y_2-\mu)\,\partial_{y_2}^2 + y_2(y_2 - \frac{\mu}{2})\partial_{y_2} \\ &\quad - 2y_1 y_3 (y_1-\mu)\partial_{y_1}\partial_{y_3} + 2y_2 y_3 (y_2-\mu)\partial_{y_2}\partial_{y_3}.\end{aligned}$$

In the limit $y_3 \to 0$, the second line on the right for $\mathcal{L}_2''$ vanishes. We can combine $\mathcal{L}_1''$ and $\mathcal{L}_2''$ to obtain the following simple forms:

$$\begin{aligned}\mathcal{L}_1'' + y_1 \lim_{y_3\to 0} \mathcal{L}_2'' &= (y_1 - y_2)\left(60 - \left(y_1 - \frac{\mu}{2}\right) y_1\, \partial_{y_1} - (y_1-\mu)\; y_1^2\, \partial_{y_1}^2\right), \\ \mathcal{L}_1'' + y_2 \lim_{y_3\to 0} \mathcal{L}_2'' &= (y_1 - y_2)\left(60 - \left(y_2 - \frac{\mu}{2}\right) y_2\, \partial_{y_2} - (y_2-\mu)\; y_2^2\, \partial_{y_2}^2\right).\end{aligned}$$

The solution $\varpi(y,\delta_0)_{y_3=0}$ therefore satisfies the differential equation

$$\mathcal{L} = (y-\mu)\, y^2 \partial_y^2 + \left(y - \frac{\mu}{2}\right) y\, \partial_y - 60 \tag{33}$$

in both y_1 and y_2.

Changing (33) to the variable $t = \frac{1728}{y}$ yields

$$\mathcal{L} = t(1-t)\partial_t^2 + (1 - \tfrac{3}{2}t)\partial_t - \tfrac{5}{144}\ ,$$

which by comparing with the general hypergeometric differential operator

$$\mathcal{L} = t(1-t)\partial_t^2 + (c - (1+a+b)t)\partial_t - ab$$

is identified with the system

$${}_2F_1(a,b;c;t) = {}_2F_1(\frac{1}{12}, \frac{5}{12}; 1; t(\tau)).$$

According to the results of Klein and Fricke as reviewed in [37], we have a unique (up to scaling) solution g_0 to (33) locally analytic at $y = \infty$. The solution can be written as

$$g_0(j(\tau)) = (E_4)^{\frac{1}{4}}(\tau), \qquad\qquad y(\tau) = j(\tau)\ .$$

Moreover, the inverse is

$$\tau(y) = \frac{g_1(y)}{2\pi i g_0(y)},$$

where g_1 is a logarithmic solution at $y = \infty$ of $\mathcal{L}$, unique up to normalization and addition of g_0.

Transformation of the solution $\varpi(u,\delta_0)$ is seen to be analytic in a neighborhood of $t_1 = t_2 = 0$. We conclude that

$$\varpi(u,\delta_0)_{u_3=0} = E_4^{\frac{1}{4}}(\tau_1) E_4^{\frac{1}{4}}(\tau_2).$$

By comparing the first few coefficients of the actual solutions $\varpi(u,\delta_i)$ in the $u_3 \to 0$ limit, we can uniquely identify

$$\tau_1(u) = T_1(u), \qquad \tau_2(u) = T_1(u) + T_2(u)\ .$$

Hence, Lemma 4 is established. Lemma 3 is proven by transforming back to the u_1 and u_2 variables. □

Restricted to a $K3$ fiber of $\pi : X \to \mathbb{P}^1$, we have

$$\delta_1 = 2F + S, \quad \delta_2 = F.$$

The coordinates $2\pi i\tau_1$ and $2\pi i\tau_2$ correspond to the divisor basis

$$L_2 = F + S, \quad L_1 = F$$

of the $K3$ fiber. Since the variables q_1 and q_2 of Section 2 measure degrees against L_1 and L_2, we see that

$$\widehat{q}_1 = q_2 \quad \text{and} \quad \widehat{q}_2 = q_1$$

for the fiber geometry.

3.4. **B-model.** The mirror transformation results of Section 3.3 together with a B-model calculation of the periods will be used to prove the following result discovered by Klemm, Mayr, and Lerche [24].

Proposition 5. *We have*

$$2 + \sum_{(d_1,d_2)\in\mathcal{P}} d_2^3 N^X_{0,(d_1,d_2)} q_1^{d_1} q_2^{d_2} = 2\frac{E_4(q_1)E_6(q_1)}{\eta^{24}(q_1)} \frac{E_4(q_2)}{j(q_1) - j(q_2)}.$$

The left side of Proposition 5 is the left side of Proposition 4 with an added degree 0 constant 2.

Proof. We will use following universal expression for the Gromov-Witten invariants of X in terms of the periods of the mirror:

$$2 + \sum_{(d_1,d_2)\in\mathcal{P}} d_2^3 N^X_{0,(d_1,d_2)} q_1^{d_1} q_2^{d_2}$$
$$= \lim_{q_3\to 0} \frac{1}{\varpi(u(T),\delta_0)^2} \sum_{i,j,k=1}^{3} \frac{\partial u_i}{\partial \tau_1}\frac{\partial u_j}{\partial \tau_1}\frac{\partial u_k}{\partial \tau_1} Y_{i,j,k}(u(T)),$$

where the $Y_{i,j,k}$ are the Yukawa couplings of the mirror family; see [9, 24].

The periods $Y_{i,j,k}$ can be explicitly computed via Griffith transversality [24] and greatly simplify in the $q_3 \to 0$ limit. We tabulate the results below:

$$Y_{111} = \frac{8(1-\tilde{u}_1)}{\tilde{u}_1^3\Delta_1}, \qquad Y_{133} = \frac{2\tilde{u}_1(1-\tilde{u}_1)}{\tilde{u}_3\Delta_1},$$
$$Y_{112} = \frac{2(1-\tilde{u}_1)^2 + \tilde{u}_1^2(\tilde{u}_2 - \tilde{u}_3)}{\tilde{u}_1^2\tilde{u}_2\Delta_1}, \qquad Y_{222} = \frac{(1-2\tilde{u}_1)\,A_2}{2\tilde{u}_2^2\Delta_1\Delta_2},$$
$$Y_{113} = \frac{2(1-\tilde{u}_1)^2 + \tilde{u}_1^2(\tilde{u}_3 - \tilde{u}_2)}{\tilde{u}_1^2\tilde{u}_3\Delta_1}, \qquad Y_{223} = \frac{(1-2\tilde{u}_1)\,A_3}{2\tilde{u}_3\tilde{u}_2\Delta_1\Delta_2},$$
$$Y_{122} = \frac{2\tilde{u}_1(1-\tilde{u}_1)}{\tilde{u}_2\Delta_1}, \qquad Y_{233} = \frac{(1-2\tilde{u}_1)\,A_2}{2\tilde{u}_3\tilde{u}_2\Delta_1\Delta_2},$$
$$Y_{123} = \frac{(1-\tilde{u}_1)\left((1-\tilde{u}_1)^2 - (\tilde{u}_2+\tilde{u}_3)\tilde{u}_1^2\right)}{\tilde{u}_1\tilde{u}_2\tilde{u}_3\Delta_1}, \qquad Y_{333} = \frac{(1-2\tilde{u}_1)\,A_3}{2\tilde{u}_3^2\Delta_1\Delta_2}.$$

Here, we have introduced the variables

$$\tilde{u}_1 = 432u_1, \quad \tilde{u}_2 = 4u_2, \quad \tilde{u}_3 = 4u_3$$

and the discriminant loci

$$\begin{aligned}\Delta_1 &= (1-\tilde{u}_1)^4 - 2(\tilde{u}_2+\tilde{u}_3)\tilde{u}_1^2(1-\tilde{u}_1)^2 + (\tilde{u}_2-\tilde{u}_3)^2\tilde{u}_1^4,\\ \Delta_2 &= (1-\tilde{u}_2-\tilde{u}_3)^2 - 4\tilde{u}_2\tilde{u}_3.\end{aligned} \tag{34}$$

The quantities A_2 and A_3 are defined by

$$\begin{aligned}A_2 &= (1+\tilde{u}_2-\tilde{u}_3)\,(1-\tilde{u}_1)^2 + \tilde{u}_1^2\,(1-\tilde{u}_3-3\,\tilde{u}_2)\,(\tilde{u}_2-\tilde{u}_3)\,,\\ A_3 &= (1+\tilde{u}_3-\tilde{u}_2)\,(1-\tilde{u}_1)^2 + \tilde{u}_1^2\,(1-\tilde{u}_2-3\,\tilde{u}_3)\,(\tilde{u}_3-\tilde{u}_2)\,.\end{aligned} \tag{35}$$

The normalizations of the Yukawa couplings $Y_{i,j,k}$ are fixed by the classical intersections.

The leading behavior of the mirror map for u_1, u_2 is obtained by rewriting Lemma 3 in terms of $E_4(\tau_i)$ and $E_6(\tau_i)$ as

$$\begin{aligned}u_1 &= \frac{1}{864}\left(1 - \frac{E_6(\tau_1)\,E_6(\tau_2)}{E_4(\tau_1)^{\frac{3}{2}}\,E_4(\tau_2)^{\frac{3}{2}}}\right) + \mathcal{O}(q_3)\ ,\\ u_2 &= \frac{\left(E_4(\tau_1)^3 - E_6(\tau_1)^2\right)\,\left(E_4(\tau_2)^3 - E_6(\tau_2)^2\right)}{4\left(E_4(\tau_1)^{\frac{3}{2}}\,E_4(\tau_2)^{\frac{3}{2}} - E_6(\tau_1)\,E_6(\tau_2)\right)^2} + \mathcal{O}(q_3).\end{aligned} \tag{36}$$

Denote the leading behavior of the last mirror map by

$$u_3 = q_3 f_3(\widehat{q}_1, \widehat{q}_2) + \mathcal{O}(q_3^2)\ . \tag{37}$$

The derivatives of the mirror maps with respect to T_2 are easily evaluated using the standard identities

$$\begin{aligned}q\frac{d}{dq}E_2 &= \tfrac{1}{12}(E_2^2 - E_4),\\ q\frac{d}{dq}E_4 &= \tfrac{1}{3}(E_2E_4 - E_6),\\ q\frac{d}{dq}E_6 &= \tfrac{1}{2}(E_2E_6 - E_4^2),\\ q\frac{d}{dq}j &= -j\frac{E_6}{E_4}.\end{aligned}$$

We find, to leading order in q_3,

$$\begin{aligned}\frac{\partial u_1}{\partial \tau_1} &= \frac{E_6(\tau_2)\left(E_4(\tau_1)^3 - E_6(\tau_1)^2\right)}{1728\,E_4(\tau_2)^{\frac{3}{2}}\,E_4(\tau_1)^{\frac{5}{2}}},\\ \frac{\partial u_2}{\partial \tau_1} &= \frac{\sqrt{E_4(\tau_1)}\left(E_4(\tau_2)^3 - E_6(\tau_2)^2\right)\left(-\left(E_4(\tau_1)^{\frac{3}{2}}\,E_6(\tau_2)\right) + E_4(\tau_2)^{\frac{3}{2}}\,E_6(\tau_1)\right)\left(E_4(\tau_1)^3 - E_6(\tau_1)^2\right)}{4\left(E_4(\tau_2)^{\frac{3}{2}}\,E_4(\tau_1)^{\frac{3}{2}} - E_6(\tau_2)\,E_6(\tau_1)\right)^3}.\end{aligned}$$

The derivative $\frac{\partial u_3}{\partial \tau_1}$ can be written to this order as

$$\frac{\partial u_3}{\partial \tau_1} = \frac{u_3}{f_3(\widehat{q}_1, \widehat{q}_2)}\frac{\partial}{\partial \tau_1} f_3(\widehat{q}_1, \widehat{q}_2) + O(u_3^2)\ . \tag{38}$$

There are many simplifications in the limit $u_3 \to 0$. First the triple couplings

$$Y_{133},\quad Y_{233},\quad Y_{333}$$

do not have enough inverse powers of u_3 and therefore do not contribute by the vanishing (38). Second, the surviving $Y_{i,j,k}$ simplify in the limit. We evaluate

$$\lim_{q_3\to 0} \frac{1}{\varpi(u(T),\delta_0)^2} \sum_{i,j,k=1}^{3} \frac{\partial u_i}{\partial \tau_1}\frac{\partial u_j}{\partial \tau_1}\frac{\partial u_k}{\partial \tau_1} Y_{i,j,k}(u(T)) \tag{39}$$
$$= -2\, \frac{E_4(\tau_2)\, E_4(\tau_1)\, E_6(\tau_2)\, \left(E_4(\tau_1)^3 - E_6(\tau_1)^2\right)}{E_4(\tau_2)^3\, E_6(\tau_1)^2 - E_4(\tau_1)^3\, E_6(\tau_2)^2}\ .$$

The possible linear dependence on $f_3(\widehat{q}_1, \widehat{q}_2)$ drops out as claimed in [24]! Using the standard identities

$$j = \frac{E_4^3}{\eta^{24}}, \quad \eta^{24} = E_4^3 - E_6^2,$$

we obtain the right side of Proposition 5. □

4. The Harvey-Moore identity

4.1. **Proof of Proposition 4.** After evaluating the left side via Proposition 5 and dividing by 2, Proposition 4 amounts to a modular form identity. Let

$$f(\tau) = \frac{E_4(\tau)E_6(\tau)}{\eta(\tau)^{24}} = \sum_{n=-1}^{\infty} c(n) q^n,$$

where $q = \exp(2\pi i \tau)$. Then, we must prove

$$\frac{f(\tau_1)E_4(\tau_2)}{j(\tau_1) - j(\tau_2)} = \frac{q_1}{q_1 - q_2} + E_4(\tau_2) - \sum_{d,k,\ell>0} \ell^3 c(k\ell)\ q_1^{kd} q_2^{\ell d}. \tag{40}$$

Equation (40) is the Harvey-Moore identity conjectured in [18].

4.2. **Zagier's proof of the Harvey-Moore identity.** The Harvey-Moore identity implies Proposition 4 and concludes the proof of the Yau-Zaslow conjecture. We present here Zagier's argument from [38].

Let $S_k \subset M_k \subset M_k^!$ denote the spaces of cusp forms, modular forms, and weakly holomorphic[15] modular forms for $\Gamma = \mathrm{SL}(2,\mathbb{Z})$. Certainly

$$f(\tau) \in M_{-2}^!.$$

For each $n \geq 0$, there is a unique function $F_n \in M_4^!$ satisfying

$$F_n(\tau) = q^{-n} + \mathcal{O}(q)$$

as $\Im(\tau) \to \infty$. Uniqueness follows from the vanishing of S_4. Existence follows by writing $F_n(\tau)$ as $E_4(\tau)$ times a polynomial in $j(\tau)$,

$$F_0 = E_4, \quad F_1 = E_4(j - 984), \quad F_2 = E_4(j^2 - 1728j + 393768)\ \ldots\ .$$

We draw several consequences:

(i) $F_1|T_n = n^3 F_n$ for all $n \geq 1$, where T_n is the n^{th} Hecke operator in weight 4. Indeed, T_n sends $M_4^!$ to itself and, by standard formulas for the action of T_n on Fourier expansions, T_n sends $q^{-1} + \mathcal{O}(q)$ to $n^3 q^{-n} + \mathcal{O}(q)$.

[15] Holomorphic except for a possible pole at infinity.

(ii) $F_1 = -f'''$, where the prime denotes differentiation by

$$\frac{1}{2\pi i}\frac{d}{d\tau} = q\frac{d}{dq}.$$

We see that f''' lies in $M_4^!$ by the $k = 4$ case of Bol's identity,

$$\frac{d^{k-1}}{d\tau^{k-1}}(f|_{2-k}\gamma) = \left(\frac{d^{k-1}f}{d\tau^{k-1}}\right)|_k\gamma \quad \forall\gamma \in \Gamma.$$

Since the Fourier expansion of f''' begins as $-q^{-1} + \mathcal{O}(q)$, the claim is proven.

(iii) For $\Im(\tau_1) > \max_{\gamma\in\Gamma} \Im(\gamma\tau_2)$,

$$\frac{f(\tau_1)E_4(\tau_2)}{j(\tau_1) - j(\tau_2)} = \sum_{n=0}^{\infty} F_n(\tau_2)q_1^n.$$

Let $L(\tau_1, \tau_2)$ denote the left side of (4.2). We see that $L(\tau_1, \tau_2)$ is a meromorphic modular form in τ_2 with a simple pole of residue $-\frac{1}{2\pi i}$ at $\tau_2 = \tau_1$ (since $j' = -E_4^2E_6/\eta^{24}$) and no poles outside $\Gamma\tau_1$. Moreover, $L(\tau_1, \tau_2)$ tends to 0 as $\Im(\tau_2) \to \infty$. These properties characterize $L(\tau_1, \tau_2)$ uniquely and show that the n^{th} Fourier coefficient with respect to τ_1 for $\Im(\tau_1) \to \infty$ has the properties characterizing $F_n(\tau_2)$.

Combining (i) and (ii) with the formula for the action of T_n on Fourier expansions, we obtain

$$(41) \qquad \begin{aligned} F_n(\tau) &= (-n^{-3}f''')|T_n = n^{-3}\left(q^{-1} - \sum_{m=1}^{\infty} m^3c(m)\ q^m\right)|T_n \\ &= q^{-n} - \sum_{\substack{k,\ell,d>0 \\ kd=n}} \ell^3c(k\ell)\ q^{\ell d} \end{aligned}$$

for $n > 0$. The Harvey-Moore identity follows from (41) and (iii) together with the equality $F_0 = E_4$. □

Acknowledgments

We thank R. Borcherds, J. Bruinier, J. Bryan, B. Conrad, I. Dolgachev, J. Lee, E. Looijenga, G. Moore, Y. Ruan, R. Thomas, G. Tian, W. Zhang, and A. Zinger for conversations about Noether-Lefschetz divisors, reduced invariants of $K3$ surfaces, and modular forms. We are especially grateful to D. Zagier for providing us with a proof of the Harvey-Moore identity.

The Clay Institute workshop on $K3$ surfaces in March 2008, which all of us attended, played a crucial role in the completion of the paper. We thank J. Carlson, D. Ellwood, and the Clay Institute staff for providing a wonderful research environment. Parts of the paper were written during visits to the Scuola Normale Superiore in Pisa and the Hausdorff Institute for Mathematics in Bonn in the summer of 2008.

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