LETTER TO THE EDITOR

Ground state of the quantum symmetric finite-size XXZ spin chain with anisotropy parameter $\Delta = \frac{1}{2}$

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Abstract. We find an analytic solution of the Bethe Ansatz equations for the special case of a finite XXZ spin chain with free boundary conditions and with a complex surface field which provides for $U_q(sl(2))$ symmetry of the Hamiltonian. More precisely, we find one nontrivial solution, corresponding to the ground state of the system with anisotropy parameter $\Delta = \frac{1}{2}$ corresponding to $q^2 = -1$.

It is widely accepted that the Bethe Ansatz equations (BAE) for an integrable quantum spin chain can be solved analytically only in the thermodynamic limit or for a small number of spin waves or short chains. In this letter, however, we have managed to find a special solution of the BAE for a spin chain of arbitrary length $N$ with $\frac{N}{2}$ spin waves.

It is well known (see, for example [1] and references therein) that there is a correspondence between the $Q$-state Potts models and the ice-type models with anisotropy parameter $\Delta = \sqrt{Q}$. The coincidence in the spectrum of an $N$-site self-dual $Q$-state quantum Potts chain with free ends with a part of the spectrum of the $U_q(sl(2))$ symmetrical $2N$-site XXZ Hamiltonian (1) is to some extent a manifestation of this correspondence:

$$H_{\text{XXZ}} = \sum_{n=1}^{N-1} \left\{ \sigma_n^+ \sigma_{n+1}^- + \sigma_n^- \sigma_{n+1}^+ + \frac{q+q^{-1}}{4} \sigma_n^z \sigma_{n+1}^z + \frac{q-q^{-1}}{4} (\sigma_n^- - \sigma_{n+1}^+) \right\}$$

(1)

where $\Delta = (q + q^{-1})/2$. This Hamiltonian was considered by Alcaraz et al [1] and its $U_q(sl(2))$ symmetry was described by Pasquier and Saleur [2]. The family of commuting transfer matrices that commute with $H_{\text{XXZ}}$ was constructed by Sklyanin [3] incorporating a method of Cherednik [4].

Baxter’s $T$–$Q$ equation for the case under consideration can be written as [5]

$$t(u) Q(u) = \phi \left( u + \frac{N}{2} \right) Q(u - \eta) + \phi \left( u - \frac{\eta}{2} \right) Q(u + \eta)$$

(2)

where $q = \exp i \eta, \phi(u) = \sin 2u \sin^2 N u$ and $t(u) = \sin 2u T(u)$. The $Q(u)$ are eigenvalues of Baxter’s auxiliary matrix $Q(u)$, where $Q(u)$ commutes with the transfer matrix $T(u)$. The

* Dedicated to Rodney Baxter on the occasion of his 60th birthday.
eigenvalue $Q(u)$ corresponding to an eigenvector with $M = \frac{N}{2} - S$; reversed spins has the form

$$Q(u) = \prod_{m=1}^{M} \sin(u - u_m) \sin(u + u_m).$$

Equation (2) is equivalent to the BAE [6]

$$\left[ \frac{\sin(u_k + \eta/2)}{\sin(u_k - \eta/2)} \right]^{2N} = \prod_{m \neq k}^{M} \frac{\sin(u_k - u_m + \eta) \sin(u_k + u_m + \eta)}{\sin(u_k - u_m - \eta) \sin(u_k + u_m - \eta)}. \tag{3}$$

In a recent article [7] Belavin and Stroganov argued that the criteria for the above-mentioned correspondence is the existence of a second trigonometric solution for Baxter’s $T$-$Q$ equation and it was shown that in the case $\eta = \frac{\pi}{2}$ the spectrum of $H_{x\sigma}$ contains the spectrum of the Ising model. In this letter we limit ourselves to the case $\eta = \frac{\pi}{2}$. This case is in some sense trivial since for this value of $\eta$, $H_{x\sigma}$ corresponds to the one-state Potts model.

We find only one eigenvalue $T_0(u)$ of the transfer matrices $T(u)$ when Baxter’s equation (2) has two independent trigonometric solutions. Solving for $T(u) = T_0(u)$ analytically we find a trigonometric polynomial $Q_0(u)$, the zeros of which satisfy the BAE (3). The number of spin waves is equal to $M = \frac{N}{2}$. The corresponding eigenstate is the ground state of $H_{x\sigma}$ with eigenvalue $E_0 = \frac{1}{2}(1 - N)$, as numerically discovered by Alcaraz et al [1].

When does a second independent periodic solution exist? This question was considered in [7]. Here we use a variation more convenient for our goal (see also [8]).

Let us consider the $T$-$Q$ equation (2) for $\eta = \frac{\pi}{2}$, where $L \geq 3$ is an integer. Let us fix a sequence of spectral parameter values $v_k = v_0 + \eta k$, where $k$ are integers and write $\phi_k = \phi(v_k - \eta/2), Q_k = Q(v_k)$ and $t_k = t(v_k)$. The functions $\phi(u), Q(u)$ and $t(u)$ are periodic with period $\pi$. Consequently, the sequences we have introduced are also periodic with period $L$, i.e., $\phi_{k+L} = \phi_k$, etc.

Setting $u = v_k$ in (2) gives the linear system

$$t_k Q_k = \phi_{k+1} Q_{k-1} + \phi_k Q_{k+1}. \tag{4}$$

The matrix of coefficients for this system has a tridiagonal form. Taking $v_0 \neq \frac{\pi m}{2}$, where $m$ is an integer, we have $\phi_k \neq 0$ for all $k$.

It is straightforward to calculate the determinant of the $L - 2 \times L - 2$ minor obtained by deleting the two left-most columns and two lower-most rows. It is equal to the product $-\phi_1^2 \phi_2 \phi_3 \ldots \phi_{L-1}$, which is nonzero, hence the rank of $M$ cannot be less than $L - 2$. Here we are interested in the case when the rank of $M$ is precisely $L - 2$ and we have two linearly independent solutions for equation (4). Let us consider the three simplest cases $L = 3, 4$ and $5$. The parameter $\eta$ is equal to $\frac{\pi}{2}, \frac{\pi}{2}$ and $\frac{\pi}{2}$ respectively.

For $L = 3$ the rank of $M$ is unity and we immediately get $t_0 = -\phi_2, t_1 = -\phi_0$ and $t_2 = -\phi_1$. Returning to the functional form, we can write

$$T_0(u) = \frac{t_0(u)}{\sin 2u} = \frac{-\phi(u + \frac{\pi}{2})}{\sin 2u} = \cos 2^N u. \tag{5}$$

This is the unique eigenvalue of the transfer matrix for which the $T$-$Q$ equation has two independent periodic solutions. It is well known (see, for example, [6]) that the eigenvalues of $H_{x\sigma}$ are related to the eigenvalues $t(u)$ by

$$E = -\cos \eta(N + 2 - \tan^2 \eta) + \sin \eta \frac{t'(\eta/2)}{t(\eta/2)}.$$

For the eigenstate corresponding to eigenvalue (5) we obtain $E_0 = \frac{1}{2}(1 - N)$. This is the ground state energy which was discovered by Alcaraz et al [1] numerically.
Below we find all solutions of Baxter’s $T$–$Q$ equation corresponding to $T(u) = T_0(u)$. Zeros of these solutions satisfy the BAE (3). In particular, we find $Q_0(x)$ corresponding to physical Bethe state.

For $L = 4$, deleting the second row and the forth column of $M$ we obtain a minor with determinant $-\phi_0\phi_1(t_0 + t_2)$. It is zero when $t_2 = -t_0$, i.e., $t(u + \frac{\pi}{2}) = -t(u)$. Considering the other minors we obtain the functional equation

$$t\left(u + \frac{\pi}{8}\right)t\left(u - \frac{\pi}{8}\right) = \phi\left(u + \frac{\pi}{4}\right)\phi\left(u - \frac{\pi}{4}\right) - \phi(u)\phi\left(u + \frac{\pi}{2}\right).$$

This functional equation was used in [7] to find $t(u)$ and show that this part of the spectrum of $H_{xxz}$ coincides with the Ising model. It would be interesting to find a corresponding $Q(u)$. Lastly, for $L = 5$, minor $M_{35}$ (the third row and the fifth column are deleted) has determinant $\phi_0\phi_2(t_0t_1 + \phi_1t_3 - \phi_2\phi_3)$. Setting this to zero we have

$$t(u)t\left(u + \frac{\pi}{5}\right) + \phi\left(u + \frac{\pi}{10}\right)t\left(u + \frac{3\pi}{5}\right) - \phi\left(u - \frac{\pi}{10}\right)\phi\left(u + \frac{3\pi}{10}\right) = 0. \quad (6)$$

It is not difficult to check that in this case all $4 \times 4$ minors have zero determinant and that the rank of $M$ is 3. Thus we have two independent periodic solutions of Baxter’s $T$–$Q$ equation.

Note that this functional relation coincides with the Baxter–Pearce relation for the hard hexagon model [9]. The connection between (6) and a special value of the rank of the matrix of coefficients for system (4) was remarked upon in [10] by Andrews et al (see also [8]).

For general $L$ we obtain the same truncated functional relations that have been obtained in [7] with the same assumptions. Note that for the ABF models [10], which are a generalization of the hard hexagon model, the truncated functional relations have been proved by Behrend et al [11].

We now consider the solution of Baxter’s equation for $\eta = \frac{\pi}{3}$ and $T = T_0$. For $\eta = \frac{\pi}{3}$ and transfer-matrix eigenvalue $T_0(u) = \cos^{2N} u$, the $T$–$Q$ equation (2) reduces to

$$\phi\left(u + \frac{3u}{2}\right)Q(u) + \phi\left(u - \frac{\pi}{2}\right)Q(u + \pi) + \phi\left(u + \frac{\eta}{2}\right)Q(u - \eta) = 0.$$

This equation can be rewritten as

$$f(v) + f\left(v + \frac{2\pi}{3}\right) + f\left(v + \frac{4\pi}{3}\right) = 0 \quad (7)$$

where $f(v) = \sin v \cos^{2N}(v/2)Q(v/2)$ has period $2\pi$. The trigonometric polynomial $f(v)$ is an odd function, so it can be written

$$f(v) = \sum_{k=1}^{K} c_k \sin kv \quad (8)$$

where $K$ is the degree of $f(v)$. Then equation (7) is equivalent to $c_{3m} = 0, m \in \mathbb{Z}$.

The condition that $f(v)$ be divisible by $\sin v \cos^{2N}(v/2)$ is equivalent to

$$\left(\frac{d}{dv}\right)^i f(v)|_{v=\pi} = 0 \quad i = 0, 1, \ldots, 2N.$$ 

For even $i$ this condition is immediate, whereas for $i = 2j - 1$ we use (8) to obtain

$$\sum_{k=1, k \neq 3m}^{K} (-1)^k c_k k^{2j-1} = 0 \quad j = 1, 2, \ldots, N. \quad (9)$$

Our problem is thus to find $[c_k]$ satisfying the last equation. This problem is a special case of a more general problem which can be formulated as follows. Given a set of different
complex numbers $X = \{x_1, x_2, \ldots, x_I\}$ we seek another complex set $B = \{\beta_1, \beta_2, \ldots, \beta_I\}$ where $\beta_i \neq 0$ for some $i$, so that

$$\sum_{i=1}^I \beta_i P(x_i) = 0 \quad (10)$$

for any polynomial $P(x)$ of degree not more than $N - 1$. It is clear that for $I \leq N$ the system $B$ does not exist. If $\beta_i \neq 0$, for example, the product $(x - x_2)(x - x_3)\ldots(x - x_I)$ provides a counter-example.

Let $I = N + 1$. We try the polynomials

$$P_r = \prod_{i=1, i \neq r}^N (x - x_i) \quad r = 1, 2, \ldots, N. \quad (11)$$

Condition (10) gives $\beta_r P_r(x_r) + \beta_I P_I(x_I) = 0$ and we immediately obtain

$$\beta_r = \text{const} \prod_{i=1, i \neq r}^{N+1} (x_r - x_i)^{-1} \quad (12)$$

which is a solution because the system (11) forms a basis of the linear space of $N - 1$ degree polynomials. So for $I \leq N + 1$ we have a unique solution (up to an arbitrary nonzero constant) given by (12). It is easy to show that for $I = N + \nu$ we obtain a $\nu$-dimensional linear space of solutions.

Returning to (9) we consider $N = 2n$, $n$ a positive integer. Fix $I = N + 1 = 2n + 1$. The degree $K$ becomes $3n + 1$. It is convenient to use a new index $\kappa$, where $|\kappa| \leq n$ and $k = |3\kappa + 1|$. Equation (9) can be rewritten as

$$\sum_{\kappa = -n}^n \beta_\kappa (3\kappa + 1)^{2j-1} = 0 \quad j = 1, 2, \ldots, N$$

where we use new unknowns $\beta_\kappa = (-1)^\kappa c_{3\kappa + 1} |3\kappa + 1|$ instead of $c_\kappa$. Using (12) and (8) we obtain the function $f(\nu)$

$$f(\nu) = \sum_{\kappa = -n}^n (-1)^\kappa \left( \frac{2n + \frac{2}{3}}{n - \kappa} \right) \left( \frac{2n - \frac{2}{3}}{n + \kappa} \right) \sin(3\kappa + 1)\nu. \quad (13)$$

We recall that the solution of Baxter’s T–Q equation for $T(u) = T_0(u)$ is given by

$$Q_0(u) = \frac{f(2n)}{(\sin 2\mu \cos 2N \mu)} \quad (14)$$

and its zeros $\{\nu_k\}$ satisfy the BAE (3). In a similar manner we have obtained the second independent solution which we have used to find the first $\eta$-derivative of the transfer-matrix eigenvalue [12].

Another way to derive the above solution is to observe that the function $f(\nu)$ satisfies a simple second-order linear differential equation. Indeed, it is easily seen that the functions $F^+(x)$ and $F^-(x)$, where

$$F^+(x) = \sum_{\kappa = -n}^n (-1)^\kappa \left( \frac{2n + \frac{2}{3}}{n - \kappa} \right) \left( \frac{2n - \frac{2}{3}}{n + \kappa} \right) x^{\kappa+\frac{1}{3}}$$

and

$$F^-(x) = F^+ \left( \frac{1}{x} \right)$$

are the two linearly independent solutions of the differential equation

$$\left[(\theta + n)^2 - \frac{1}{9} \right] x^{-1} + (\theta - n)^2 - \frac{1}{9} F^+ = 0 \quad (15)$$

where $\theta = x^{\frac{1}{3}}$ (up to a change of variables this is just the standard hypergeometric differential equation, and in fact $F^+(x) = \text{const} F(-2n, \frac{2}{3} - 2n, \frac{2}{3}, -x)x^{1/3-n}$). Now the fact that there is
a combination $f(v)$ of $F^+(e^{3i\nu})$ and $F^-(e^{3i\nu})$ which vanishes to order $2N+1$ at $v = \pi$ follows immediately from the fact that $x = -1$ is a singular point of the differential equation (15) and that the indicial equation at this point has roots 0 and $2n + 1$. In terms of the variable $v$, equation (15) becomes

$$\frac{d^2 f}{dv^2} + 6n \tan \left( \frac{3v}{2} \right) \frac{df}{dv} + (1 - 9n^2) f = 0.$$  

The zeros of $f(v)$, the density of which is important in the thermodynamic limit, are located on the imaginary axis in the complex $v$-plane. So it is convenient to make the change of variable $v = is$. It is also useful to introduce another function $g(s) = f(is)/\cosh^{2n}(\frac{3s}{2})$. The differential equation for $g(s)$ is then

$$g'' + \left( \frac{9n(2n+1)}{2 \cosh^{2n}(\frac{3s}{2})} - 1 \right) g = 0.$$  

Let $g(s_0) = 0$. For large $n$ we have in a small vicinity of $s_0$ an approximate equation $g'' + \omega_0^2 g = 0$. This equation describes a harmonic oscillator with frequency $\omega_0 = 3n/\cosh(\frac{3s_0}{2})$. The distance between nearest zeros is approximately $\Delta s = \frac{\pi}{\omega_0}$ and we obtain the following density function which describes the number of zeros per unit length:

$$\rho(s) = \frac{1}{\Delta s} = \frac{\omega}{\pi} = \frac{3n}{(\pi \cosh(\frac{3s_0}{2}))}.$$  

We note that equation (16) has a history as rich as the BAE. Eckart [13] used the Schroedinger equation with bell-shaped potential $V(r) = -G/\cosh^2 r$ for phenomenological studies in atomic and molecular physics. Later it was used in chemistry, biophysics and astrophysics, to name just a few. For more recent references see, for example, [14].

After the completion of this letter, we were informed that in Baxter’s review [8] he noticed the possibility of a simple eigenvalue of the transfer matrix for the XYZ model for the special value $\mu = \frac{\pi}{3}$ of the crossing parameter.

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References