

Multiple zeta values of fixed weight, depth, and height

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ABSTRACT

We give a generating function for the sums of multiple zeta values of fixed weight, depth and height in terms of Riemann zeta values.

For any multi-index $\mathbf{k} = (k_1, k_2, \dots, k_n)$ ($k_i \in \mathbf{Z}_{>0}$), the *weight*, *depth*, and *height* of \mathbf{k} are by definition the integers $k = k_1 + k_2 + \dots + k_n$, n , and $s = \#\{i \mid k_i > 1\}$, respectively. We denote by $I(k, n, s)$ the set of multi-indices \mathbf{k} of weight k , depth n , and height s , and by $I_0(k, n, s)$ the subset of *admissible* indices, i.e., indices with the extra requirement that $k_n \geq 2$. For any admissible index $\mathbf{k} = (k_1, k_2, \dots, k_n) \in I_0(k, n, s)$, the *multiple zeta value* $\zeta(\mathbf{k})$ is defined by

$$\zeta(\mathbf{k}) = \zeta(k_1, k_2, \dots, k_n) = \sum_{0 < m_1 < m_2 < \dots < m_n} \frac{1}{m_1^{k_1} m_2^{k_2} \dots m_n^{k_n}}.$$

We denote by $G_0(k, n, s)$ the value of the sum

$$(1) \quad G_0(k, n, s) = \sum_{\mathbf{k} \in I_0(k, n, s)} \zeta(\mathbf{k}).$$

Since the set $I_0(k, n, s)$ is non-empty only if the indices k , n and s satisfy the in-

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equalities $s \geq 1$, $n \geq s$, and $k \geq n + s$, we can collect all the numbers $G_0(k, n, s)$ into a single generating function

$$(2) \quad \Phi_0(x, y, z) = \sum_{k, n, s} G_0(k, n, s) x^{k-n-s} y^{n-s} z^{s-1} \in \mathbf{R}[[x, y, z]].$$

Our main result will then be

Theorem 1. *The power series (2) is given by*

$$(3) \quad \Phi_0(x, y, z) = \frac{1}{xy - z} \left(1 - \exp \left(\sum_{n=2}^{\infty} \frac{\zeta(n)}{n} S_n(x, y, z) \right) \right),$$

where the polynomials $S_n(x, y, z) \in \mathbf{Z}[x, y, z]$ are defined by the formula

$$(4) \quad S_n(x, y, z) = x^n + y^n - \alpha^n - \beta^n. \quad \alpha, \beta = \frac{x + y \pm \sqrt{(x + y)^2 - 4z}}{2}$$

or alternatively by the identity

$$(5) \quad \log \left(1 - \frac{xy - z}{(1 - x)(1 - y)} \right) = \sum_{n=2}^{\infty} \frac{S_n(x, y, z)}{n}$$

together with the requirement that $S_n(x, y, z^2)$ is a homogeneous polynomial of degree n . In particular, all of the coefficients $G_0(k, n, s)$ of $\Phi_0(x, y, z)$ can be expressed as polynomials in $\zeta(2), \zeta(3), \dots$ with rational coefficients.

In view of (5), we can also restate (3) in the alternative form

$$(6) \quad 1 - (xy - z) \Phi_0(x, y, z) = \prod_{m=1}^{\infty} \left(1 - \frac{xy - z}{(m - x)(m - y)} \right),$$

which is simpler looking but does not directly give the coefficients of the power series as finite expressions in terms of Riemann zeta values.

Proof. A convenient approach to the multiple zeta value $\zeta(\mathbf{k})$ is to consider it as the limiting value at $t = 1$ of the function

$$L_{\mathbf{k}}(t) = L_{k_1, k_2, \dots, k_n}(t) = \sum_{0 < m_1 < m_2 < \dots < m_n} \frac{t^{m_n}}{m_1^{k_1} m_2^{k_2} \dots m_n^{k_n}} \quad (|t| < 1).$$

(Note that we consider $L_{\mathbf{k}}(t)$ not just for $\mathbf{k} \in I_0$ but for all $\mathbf{k} \in I$.) For \mathbf{k} empty we define $L_{\mathbf{k}}(t)$ to be 1. For non-negative integers k, n and s set

$$G(k, n, s; t) = \sum_{\mathbf{k} \in I(k, n, s)} L_{\mathbf{k}}(t)$$

(so $G(0, 0, 0; t) = 1$ and $G(k, n, s; t) = 0$ unless $k \geq n + s$ and $n \geq s \geq 0$), and let $G_0(k, n, s; t)$ be the function defined by the same formula but with the summation restricted to $\mathbf{k} \in I_0(k, n, s)$. We denote by $\Phi = \Phi(x, y, z; t)$ and $\Phi_0 = \Phi_0(x, y, z; t)$ the corresponding generating functions

$$\Phi = \sum_{k,n,s \geq 0} G(k,n,s;t) x^{k-n-s} y^{n-s} z^s = 1 + L_1(t)y + L_{1,1}(t)y^2 + \dots$$

and

$$\Phi_0 = \sum_{k,n,s \geq 0} G_0(k,n,s;t) x^{k-n-s} y^{n-s} z^{s-1} = L_2(t) + L_{1,2}(t)y + L_3(t)x + \dots$$

Our object is to express the generating function $\Phi_0(x, y, z) = \Phi_0(x, y, z; 1)$ in terms of Riemann zeta values. Using the obvious formula

$$\frac{d}{dt} L_{k_1, \dots, k_n}(t) = \begin{cases} t^{-1} L_{k_1, \dots, k_{n-1}, k_n-1}(t) & \text{if } k_n \geq 2, \\ (1-t)^{-1} L_{k_1, \dots, k_{n-1}}(t) & \text{if } k_n = 1 \end{cases}$$

for the derivative of $L_k(t)$, we obtain

$$\begin{aligned} \frac{d}{dt} G_0(k,n,s;t) &= \frac{1}{t} \left(G(k-1,n,s-1;t) - G_0(k-1,n,s-1;t) + G_0(k-1,n,s;t) \right), \\ \frac{d}{dt} \left(G(k,n,s;t) - G_0(k,n,s;t) \right) &= \frac{1}{1-t} G(k-1,n-1,s;t) \end{aligned}$$

or, in terms of generating functions,

$$\frac{d\Phi_0}{dt} = \frac{1}{yt} (\Phi - 1 - z\Phi_0) + \frac{x}{t} \Phi_0. \quad \frac{d}{dt} (\Phi - z\Phi_0) = \frac{y}{1-t} \Phi.$$

Eliminating Φ , we obtain the differential equation

$$t(1-t) \frac{d^2\Phi_0}{dt^2} + \left((1-x)(1-t) - yt \right) \frac{d\Phi_0}{dt} + (xy-z)\Phi_0 = 1$$

for the power series Φ_0 . The unique solution of this vanishing at $t = 0$ is given by

$$\Phi_0(x, y, z; t) = \frac{1}{xy-z} \left(1 - F(\alpha-x, \beta-x; 1-x; t) \right),$$

where $\alpha + \beta = x + y$, $\alpha\beta = z$ and $F(a, b; c; x)$ denotes the Gauss hypergeometric function. Specializing to $t = 1$ and using Gauss's formula for $F(a, b; c; 1)$ gives

$$1 - (xy-z)\Phi_0(x, y, z; 1) = F(\alpha-x, \beta-x; 1-x; 1) = \frac{\Gamma(1-x)\Gamma(1-y)}{\Gamma(1-\alpha)\Gamma(1-\beta)},$$

and now using the expansion $\Gamma(1-x) = \exp(\gamma x + \sum_{n \geq 2} \zeta(n) x^n/n)$ yields equation (3).

We end by mentioning several special cases of the theorem which were previously known or are of special interest.

- (1) Specializing (3) to $z = xy$ corresponds to dropping all information

about s , the number of indices k_i greater than 1, so the function $\Phi_0(x, y, xy)$ equals $\sum_{k>n>0} G_0(k, n)x^{k-n-1}y^{n-1}$ where $G_0(k, n) = \sum_s G_0(k, n, s)$ is the sum of all multiple zeta values of weight k and depth n . On the other hand, taking the limit as $z \rightarrow xy$ in (6), we find

$$\Phi_0(x, y, xy) = \sum_{m=2}^{\infty} \frac{1}{(m-x)(m-y)} = \sum_{k>n>0} \zeta(k) x^{k-n-1} y^{n-1},$$

so we obtain the sum formula $G_0(k, n) = \zeta(k)$ already proved in [1] and [6].

(2) If $s = 1$, then the only admissible multi-index of weight k and depth n is $(1, \dots, 1, k-n)$ (with $n-1$ 1's), so $G(k, n, 1) = \zeta(1, \dots, 1, k-n)$. On the other hand, we have $S_n(x, y, 0) = x^n + y^n - (x+y)^n$, so (3) for $z = 0$ reduces to

$$\begin{aligned} \sum_{a, b \geq 1} \zeta(\underbrace{1, \dots, 1}_{a-1}, b+1) x^a y^b &= \Phi_0(x, y, 0) \\ &= \frac{1}{xy} \left(1 - \exp\left(\sum_{n=2}^{\infty} \zeta(n) \frac{x^n + y^n - (x+y)^n}{n}\right) \right), \end{aligned}$$

a formula given also in [6].

(3) The well-known duality relation for multiple zeta values says that there is a bijection $\mathbf{k} \rightarrow \mathbf{k}'$ from $I_0(k, n, s)$ to $I_0(k, k-n, s)$ such that $\zeta(\mathbf{k}) = \zeta(\mathbf{k}')$ for all \mathbf{k} . In particular, $G_0(k, n, s) = G_0(k, k-n, s)$, so the generating function $\Phi_0(x, y, z)$ must be symmetric in x and y , a symmetry which is of course evident in the formula (3).

(4) Specializing (3) to $x = 0$ and $y = 0$ gives formulas for the sums of all multiple zeta values having all $k_i \geq 2$ or all $k_i \leq 2$, respectively. The simultaneous specialization to $x = y = 0$ corresponds to the unique zeta value $\zeta(2, \dots, 2)$ (with $k = 2n = 2s$), so from (3) we get

$$\begin{aligned} \sum_{s=1}^{\infty} \zeta(\underbrace{2, \dots, 2}_s) z^{s-1} &= \Phi_0(0, 0, z) = -\frac{1}{z} \left(1 - \exp\left(-\sum_{n=1}^{\infty} \frac{\zeta(2n)}{n} (-z)^n\right) \right), \\ &= \frac{1}{z} \left(\frac{\sinh \pi\sqrt{z}}{\pi\sqrt{z}} - 1 \right) = \sum_{s=1}^{\infty} \frac{\pi^{2s}}{(2s+1)!} z^{s-1} \end{aligned}$$

and hence

$$\zeta(\underbrace{2, \dots, 2}_s) = \frac{\pi^{2s}}{(2s+1)!},$$

a formula also already given in [6].

(5) Finally, by specializing to $y = -x$ in Theorem 1, we obtain the formula

$$\sum_{s \leq n \leq 2k-s} (-1)^n G_0(2k, n, s) = \frac{(-1)^k \pi^{2k}}{(2k+1)!} \sum_{r=0}^{k-s} \binom{2k+1}{2r} (2-2^{2r}) B_{2r} \quad (k \geq s \geq 1)$$

proved by Le and Murakami in [3]. Indeed, from equation (4) or equation (5) we have

$$S_n(x, -x, z) = \begin{cases} 2(x^{2m} - (-z)^m) & \text{if } n = 2m, \\ 0 & \text{if } n \equiv 1 \pmod{2}, \end{cases}$$

so (3) and the standard Taylor expansion of $\log((x/2)/\sinh x/2)$ give

$$\begin{aligned} \Phi_0(x, -x, z) &= \frac{1}{x^2 + z} \left(\exp \left(\sum_{m=1}^{\infty} \frac{x^{2m} - (-z)^m}{m} \zeta(2m) \right) - 1 \right) \\ &= \frac{1}{x^2 + z} \left(\frac{\pi x}{\sin \pi x} \frac{\sinh \pi \sqrt{z}}{\pi \sqrt{z}} - 1 \right) \\ &= \frac{\pi x}{\sin \pi x} \frac{(\sinh \pi \sqrt{z})/\pi \sqrt{z} - (\sin \pi x)/\pi x}{z + x^2} \\ &= \left(\sum_{r=0}^{\infty} (-1)^r \frac{B_{2r}}{(2r)!} (2 - 2^{2r}) \pi^{2r} x^{2r} \right) \times \\ &\quad \times \left(\sum_{p \geq 0, s \geq 1} (-1)^p \frac{\pi^{2p+2s} x^{2p} z^{s-1}}{(2p+2s+1)!} \right). \end{aligned}$$

The required identity now follows by comparing the coefficients of $x^{2k-2s} z^{s-1}$ on both sides.

REFERENCES

- [1] Granville, A. – A decomposition of Riemann’s Zeta-Function. In *Analytic Number Theory*. London Mathematical Society Lecture Note Series, **247**, Y. Motohashi (ed.), Cambridge University Press, 95-101 (1997).
- [2] Hoffman, M. and Y. Ohno – Relations of multiple zeta values and their algebraic expression. Preprint.
- [3] Le, T.Q.T. and J. Murakami – Kontsevich’s integral for the Homfly polynomial and relations between values of multiple zeta functions. *Topology and its Applications*, **62**, 193–206 (1995).
- [4] Ohno, Y. – A generalization of the duality and sum formulas on the multiple zeta values. *J. Number Theory*, **74**, 39–43 (1999).
- [5] Zagier, D. – Values of zeta functions and their applications. In *Proceedings of ECM 1992*. *Progress in Math.*, **120**, 497–512 (1994).
- [6] Zagier, D. – Multiple zeta values. Unpublished manuscript, Bonn (1995).

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