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# THE PONTRJAGIN CLASS OF AN ORBIT SPACE

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THE OBJECT of this paper is to show that, if a finite group $G$ acts by orientation-preserving diffeomorphisms on an oriented closed manifold $X$, then it is possible to determine the (total) Pontrjagin class of the quotient space $X/G$. This exists according to the work of Thom [9] and Milnor [7], since $X/G$ is a "rational homology manifold" (a triangulated space in which the boundary of the star of each vertex has the rational homology of a sphere). As an illustration, we shall apply the formula obtained to the quotient of complex projective space by a product of finite cyclic groups acting in the obvious manner. In this case the result had previously been obtained by Bott and Sullivan using somewhat different methods (unpublished; see, however, [5]). We end the paper with a sketch of another but unfortunately very much more complicated application to symmetric powers of a manifold (here $X$ is the $n$th Cartesian product with itself of an even-dimensional manifold and $G$ is the symmetric group acting by permutation of the factors); the details of the results described will be given in [10].

The formula actually obtained will give the Hirzebruch $L$-class $L(X/G)$ rather than the Pontrjagin class, though it is of course possible to pass from one to the other via a multiplicative sequence. In fact the definition of Milnor and Thom also yields the $L$-class, and it seems to be the case that this is the more natural class in the situation of rational homology manifolds, where signatures are defined, while the Pontrjagin class belongs properly to the theory of differentiable manifolds where the structure is given by a bundle.

The motivation for the formula is the well-known relation

$$\operatorname{Sign}(X/G) = \frac{1}{|G|} \sum_{g \in G} \operatorname{Sign}(g, X), \tag{1}$$

where Sign $(g, X)$ is the equivariant signature defined (for $X$ of even dimension; for $X$ odd-dimensional it is zero) and studied in [1]. The left-hand side of this is the top-dimensional component of $L(X/G)$, so we expect that the whole class $L(X/G)$ (lifted up to $X$ by the projection) will be given by the average over $g$ in $G$ of some *equivariant* $L$-class

$$L(g, X) \in \mathrm{H}^*(X; \mathbb{C}). \tag{2}$$

This is in fact the form that the theorem will take. The formulation of the theorem will be given in Section 1 and its proof in Section 2, while Section 3 contains the application to the homology manifold studied by Bott and Section 4 the discussion of symmetric products.

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## §1. STATEMENT OF THE MAIN THEOREM

It is clear from the remarks above that the class $L(g, X)$ should have Sign $(g, X)$ as its top-dimensional component, and thus we naturally look at the $G$-signature theorem of Atiyah and Singer [1], which states

$$\operatorname{Sign}(g, X) = L'(g, X)[X^g] \tag{3}$$

where $L'(g, X)$ is a certain cohomology class of the fixed point set $X^g$ of the action of $g$ on $X$ (with twisted coefficients if $X^g$ is non-orientable). To obtain a class in $H^*(X)$ we apply the "Umkehr" (or Gysin) homomorphism $j_*$ from the cohomology of $X^g$ to that of $X$ (where $j$ is the embedding of $X^g$ in $X$):

$$L(g, X) = j_* L'(g, X). \tag{4}$$

We will give the precise definitions of $L'(g, X)$ and $j_*$ after stating the theorem; our definition of $L'(g, X)$ will not quite be the class introduced by Atiyah and Singer but will differ from it only by powers of two (and of course agree in the top dimension).

THEOREM 1. *Let $X$ be a closed, oriented, smooth manifold, and $G$ a finite group acting on $X$ by orientation-preserving diffeomorphisms. Let*

$$\pi: X \to X/G \tag{5}$$

*be the projection of $X$ onto the orbit space $X/G$. Then*

$$\frac{1}{\deg \pi}\pi^* L(X/G) = \frac{1}{|G|}\sum_{g \in G} L(g, X), \tag{6}$$

*where* $\deg \pi$ *denotes the degree of the map* $\pi$. Note that if $G$ acts effectively (which can always be assumed to be the case), then $\deg \pi = |G|$, so the numerical factors in (6) can be omitted.

To define $L'(g, X)$ we first define certain (non-stable) characteristic classes of complex and real bundles, i.e. multiplicative sequences in their Chern and Pontrjagin classes, respectively. For $\theta$ real and $\xi$ a $U(q)$-bundle over a space $Y$, we write

$$c(\xi) = \prod_{j=1}^{q} (1 + x_j), \quad x_j \in H^2(Y; \mathbb{Z}), \tag{7}$$

$$L_\theta(\xi) = \prod_{j=1}^{q} \coth\left(x_j + \frac{i\theta}{2}\right); \tag{8}$$

as usual, this means that the symmetric expression (8) is to be expanded as a power series in the $x_j$'s in which the elementary symmetric polynomials in the $x_j$'s are then replaced by the corresponding Chern classes $c_k(\xi)$. This only makes sense if $\theta$ is not a multiple of $2\pi$. We have

$$L_\theta(\xi) \in H^*(Y; \mathbb{Q})[e^{i\theta}] \to H^*(Y; \mathbb{C}). \tag{9}$$

Similarly, if $\xi$ is an $O(2q)$-bundle over $Y$, we write

$$p(\xi) = \prod_{j=1}^{q} (1 + x_j^2), \quad x_j \in H^2(Y, \mathbb{Z}), \tag{10}$$

so that the Hirzebruch $L$-class of $\xi$ is defined by

$$L(\xi) = \prod_{j=1}^{q} \frac{x_j}{\tanh x_j} \in H^*(Y; \mathbb{Q}), \tag{11}$$

and then define

$$L_\pi(\xi) = e(\xi)L(\xi)^{-1} \in H^*(Y; or_\xi \otimes \mathbb{Q}). \tag{12}$$

Here $or_\xi$ is the local system of coefficients defined by $\xi$, namely

$$or_\xi(y) = H_{2q}(E_y, E_y - \{0\}), \qquad (y \in Y) \tag{13}$$

where $E_y$ is the fibre of $\xi$ over $y$, and $e(\xi) \in H^{2q}(Y; or_\xi)$ is the Euler class, defined as the restriction to the zero-section of the Thom class

$$U_\xi \in H^{2q}(E, E_0; \pi^* or_\xi). \tag{14}$$

The Thom class is defined (without assuming that the bundle $\xi$ is oriented) by the requirement that its restriction to any fibre is the identity under the isomorphism

$$H^{2q}(E_y, E_{oy}; \pi^* or_\xi) \approx \mathrm{Hom}(H_{2q}(E_y, E_{oy}), H_{2q}(E_y, E_{oy})) \tag{15}$$

induced by (13). (A good discussion of the local coefficient system and Thom class of a non-oriented bundle, as well as of the orientation class and Poincaré isomorphism for a non-oriented manifold, needed below, can be found in the 1970 Bonn Diplomarbeit of J. Heithecker: *Homologietheorie mit lokalen Koeffizienten und einige Anwendungen.*)

We have thus defined classes $L_\theta(\xi) \in H^*(Y, \mathbb{C})$ for $\xi$ a complex bundle over $Y$ and $\theta$ not divisible by $2\pi$ and a class $L_\pi(\xi) \in H^*(Y; or_\xi \otimes \mathbb{Q})$ for a real bundle. Notice that these classes (tensored with $\mathbb{C}$) agree if $\theta = \pi$ and $\xi$ is complex, for in that case $or_\xi = \mathbb{Z}$ since $\xi$ is oriented, and the Euler class of $\xi$ is $e(\xi) = c_q(\xi) = x_1 \ldots x_q$.

We next recall that $g$ acts on $N_x{}^g$, the fibre at $x$ of the normal bundle of $X^g$ in $X$, orthogonally and effectively, so that by standard representation theory $N_x{}^g$ decomposes into a sum of subspaces on which $g$ acts by $-1$ or by

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}. \tag{16}$$

This splitting clearly extends to the whole bundle, and since the equivalence class of the representation (16) is unchanged by $\theta \to -\theta$, we can write

$$N^g = N_\pi{}^g \oplus \sum_{0<\theta<\pi} N_\theta{}^g \tag{17}$$

where $N_\pi{}^g$ is a bundle on which $g$ acts by multiplication with $-1$ and where $N_\theta{}^g$ acquires from (16) a complex structure with respect to which $g$ acts by multiplication with $e^{i\theta}$. (In particular, $N_\theta{}^g$ is naturally oriented.) We can therefore define

$$L'(g, X) = L(X^g)L_\pi(N_\pi{}^g) \prod_{0<\theta<\pi} L_\theta(N_\theta{}^g) \in H^*(X^g; or_{N_\pi{}^g} \otimes \mathbb{C}). \tag{18}$$

We want to use (4) to define a class $L(g, X)$ in $H^*(X)$. The Umkehr homomorphism $j_*$ is defined by using Poincaré duality isomorphisms to identify cohomology with homology and then applying the ordinary induced map $j_*$ in homology. Recall that for any closed $n$-manifold $M$ there is a canonical generator $[M]$ of $H_n(M, or_{T(M)})$, and that the Poincaré map is defined by cupping with $[M]$:

$$D_M : H^i(M; \Gamma) \xrightarrow{\cong} H_{n-i}(M; \Gamma \otimes or_{T(M)}). \tag{19}$$

Therefore $D_{X^g}L'(g, X)$ is in $H_*(X^g; or_{T(X^g)} \otimes or_{N_\pi{}^g} \otimes \mathbb{C})$. But from (17) we conclude that $or_{N_\pi{}^g} = or_{N^g}$, since the bundles $N_\theta{}^g$ are oriented. Therefore:

$$or_{T(X^g)} \otimes or_{N_\pi{}^g} \otimes \mathbb{C} \approx or_{T(X^g)\oplus N^g} \otimes \mathbb{C} \approx or_{j^*T(X)} \otimes \mathbb{C} \approx j^* or_{T(X)} \otimes \mathbb{C}, \tag{20}$$

$$j_* D_{X^g} L'(g, X) \in H_*(X; or_{T(X)} \otimes \mathbb{C}), \tag{21}$$

$$L(g, X) = j_* L'(g, X) = (D_X{}^{-1} j_* D_{X^g}) L'(g, X) \in H^*(X; \mathbb{C}). \tag{22}$$

More properly, $L(g, X)$ is the sum over the connected components of $X^g$ of the corresponding expression on the right of (22), a point that will be tacitly assumed in the future. Note that $L(g, X)$ is an untwisted class.

This completes the list of definitions and notations that were necessary to the statement of Theorem 1; before turning to the proof we make one or two comments about the expression on the right-hand side of (6). First of all, it is easy to show from the definitions that (for $h \in G$) the map $h\colon X \to X$ takes $X^g$ to $X^{hgh^{-1}}$ and pulls back the normal bundle and the splitting (17), so by the functoriality of the characteristic classes $L_\pi$ and $L_\theta$ we find

$$L'(g, X) = h^* L'(hgh^{-1}, X). \tag{23}$$

Moreover, since $h$ is an isomorphism we have $h^* = h_*{}^{-1}$, so that we easily find that

$$L(g, X) = h^* L(hgh^{-1}, X). \tag{24}$$

It follows that the right-hand side of (6) is invariant under the action of $G$ on $H^*(X)$. But it is known ([3] p. 202, [2] III.2.3, p. 38) that, with (possibly twisted) coefficients which are (locally) isomorphic to a field of characteristic 0, the map $\pi^*$ is an isomorphism onto the $G$-invariant part of $X$:

$$\pi^*\colon H^*(X/G) \xrightarrow{\cong} H^*(X)^G \xrightarrow{\subset} H^*(X). \tag{25}$$

It follows that (assuming the action is effective and omitting the numbers in the denominator) the class on the right-hand side of (6) can be written as $\pi^* L$ for a unique $L \in H^*(X/G)$. The assertion of the theorem is then that $L = L(X/G)$. But even without proving the whole theorem we can check that $L$ has some of the right properties, namely:

(i) The 0-dimensional component of $L$ is 1
(ii) The top-dimensional component of $L$ is Sign $(X/G)$
(iii) The $k$-dimensional component of $L$ is 0 unless $k \equiv 0 \pmod 4$.

For (i), we note that any Umkehr homomorphism between manifolds raises dimensions by the codimension of the smaller manifold. Therefore we end up in positive dimension unless the codimension is 0; if $X$ is connected and the action effective this implies $g = 1$ and the corresponding 0-dimensional component of $L(g, X) = L(1, X) = L(X)$ is indeed one. Statement (ii) follows immediately from equations (1) and (3) since clearly for any inclusion $j\colon Y \to X$ and class $y \in H^*(Y)$ we have

$$y[Y] = j_* y[X]. \tag{26}$$

The third assertion follows immediately from

$$L(g, X) + L(g^{-1}, X) \in H^{4*}(X). \tag{27}$$

To see that this holds, note that $g$ and $g^{-1}$ have the same fixed-point sets and decomposition of normal bundles, except that $g^{-1}$ acts on $N_\theta{}^g$ as $e^{-i\theta}$. Since $\coth(x_j - i\theta) = -\coth(-x_j + i\theta)$, we deduce that $\prod_{\theta \neq \pi} L_\theta(N_\theta^{g^{-1}}) + \prod_{\theta \neq \pi} L_\theta(N_\theta^g)$ is only non-zero in dimensions equal (mod 4) to $2 \sum_{\theta \neq \pi} \dim_{\mathbb{C}} N_\theta^g$. Multiplication with $e(N_\pi^g)$ increases dimension by $\dim_{\mathbb{R}} N_\pi^g$, and the effect of $j_*$ is a further increase by $\dim_{\mathbb{R}} N_\pi^g + 2\sum_{\theta \neq \pi} \dim_{\mathbb{C}} N_\theta^g$. Since $\dim_{\mathbb{R}} N_\pi^g$ is even, this completes the proof of (27).

## §2. PROOF

The proof of Theorem 1 will use Milnor's formulation of the definition of the $L$-class of a rational homology manifold, which we now recall.

From the work of Serre [8] we know that, if $Y$ is a $CW$-complex of dimension $n \leq 2i - 2$, then $H^i(Y; \mathbb{Q})$ is generated as a vector space over $\mathbb{Q}$ by classes of the form $f^*\sigma$, where

$$f\colon Y \to S^i \tag{28}$$

is a continuous map and

$$\sigma \in H^i(S^i; \mathbb{Q}) \tag{29}$$

is the generator. More precisely, in these dimensions the homotopy classes of maps (28) form an abelian group $\pi^i(Y)$ and there is a map $\pi^i(Y) \to H^i(Y; \mathbb{Q})$ defined by $[f] \to f^*\sigma$ which becomes an isomorphism after tensoring with $\mathbb{Q}$.

Now let $Y$ be an (oriented) rational homology manifold of (formal) dimension $n$; that is, $Y$ is a topological space with a triangulation in which the star of each vertex has a boundary with the same rational cohomology groups as $S^{n-1}$. Then $Y$ satisfies Poincaré duality, so, by the above, a class

$$l \in H^{n-i}(Y; \mathbb{Q}) \qquad (2i - 2 \geq n) \tag{30}$$

is completely determined by the values of

$$(l \cup f^*\sigma)[Y] \in \mathbb{Q} \tag{31}$$

for each map $f$ as in (28).

What Milnor showed is that, for $f$ a simplicial map (for some fixed triangulation of $S^i$), the inverse image of a point,

$$A = f^{-1}(p) \tag{32}$$

is a rational homology manifold (of formal dimension $n - i$) for almost all $p \in S^i$, and that, again only for almost all $p$, the cobordism class and hence signature of $A$ are independent of $p$. Moreover, this integer only depends on the homotopy class of the map $f$. This defines a homomorphism from $\pi^i(Y)$ to $\mathbb{Z}$, and hence, by Serre's results, a class

$$l_{n-i} \in H^{n-i}(Y; \mathbb{Q}) \tag{33}$$

such that

$$(l_{n-i} \cup f^*\sigma)[Y] = \mathrm{Sign}(A) \tag{34}$$

for all maps (28). All of this holds only for

$$n \leq 2i - 2, \qquad n - i \leqslant \frac{n-2}{2}. \tag{35}$$

However, for $j > \frac{n-2}{2}$ we can define $l_j' \in H^j(Y \times S^N)$ $\left(\text{where } \frac{N+n-2}{2} > j\right)$ and then let $l_j$ be the restriction to $H^j(Y)$ of $l_j'$. Then the classes $l_j$ are defined for each $j$, and we define the $L$-class of $Y$ as their sum:

$$L(Y) = \sum_{j=0}^{\infty} l_j \in H^*(Y; \mathbb{Q}). \tag{36}$$

It is easy to show from (34) that this gives the usual definition if $Y$ is a differentiable manifold. The proof uses the fact that $f^*\sigma$ is the Poincaré dual of $j_*[A] \in H_{n-i}(Y)$, where $j: A \subset Y$ is the inclusion. Furthermore $A$ has trivial normal bundle in $Y$, so $L(A) = j^*L(Y)$. Therefore

$$\operatorname{Sign}(A) = \langle L(A), [A]\rangle = \langle j^*L(Y), [A]\rangle = \langle L(Y), j_*[A]\rangle = \langle L(Y) \cup f^*\sigma, [Y]\rangle. \tag{37}$$

Thus the $i$-dimensional component of $L(Y)$ is $l_i$. If $i > \frac{n-2}{2}$ we use the fact $L(Y \times S^N) = L(Y) \times L(S^N) = L(Y) \times 1$.

Now let $X$ and $G$ be as in Section 1 and $Y = X/G$. Then maps

$$\bar{f}: X/G \to S^i \tag{38}$$

correspond under $\bar{f} \to \bar{f} \circ \pi = f$ (where $\pi: X \to Y$ is the projection) to $G$-equivariant maps

$$f: X \to S^i \tag{39}$$

where $G$ acts trivially on $S^i$. Then

$$\bar{A} = \bar{f}^{-1}(p) \tag{40}$$

is the quotient $A/G$, where

$$A = f^{-1}(p) \tag{41}$$

is a $G$-invariant subset of $X$. We can change $f$ in its equivariant homotopy class to make it differentiable (replace $f: X \to S^i \subset \mathbb{R}^{i+1}$ by a differentiable map $f': X \to \mathbb{R}^{i+1}$ with max $|f(x) - f'(x)| < \varepsilon$, and then set $f''(x) = \frac{1}{|G|}\sum_{g \in G} f'(g \circ x)$. Then $f''$ is $G$-equivariant, differentiable and close to $f$; in particular $f''(X) \subset \mathbb{R}^{i+1} - \{0\}$ so we can project back onto $S^i$ to get a $G$-equivariant differentiable map $f''': X \to S^i$. Applying the process to a homotopy from $f$ $f'$ yields an equivariant homotopy from $f$ to $f'''$, proving the assertion.) Excepting sets of measure 0 for $p$, we can assume that $p$ is a regular value of $f$ and of each $f | X^g (g \in G)$ by Sard's theorem; then $A$ is a differentiable submanifold of $X$ and $A$ meets each submanifold $X^g$ transversally with intersection $A^g$. Again we can ignore difficulties with dimensional restrictions by multiplying everything with a sphere of large dimension on which $G$ acts trivially; then Milnor's definition tells us that the $L$-class $L$ of $X/G$ is defined by

$$(L \cup \bar{f}^*\sigma)[X/G] = \operatorname{Sign}(\bar{A}). \tag{42}$$

We now want to evalaute the right-hand side of (42) in terms of the action of $G$ on $X$. By equation (1), we have

$$\begin{aligned}\operatorname{Sign}(\bar{A}) &= \operatorname{Sign}(A/G) \\ &= \frac{1}{|G|}\sum_{g \in G} \operatorname{Sign}(g, A)\end{aligned} \tag{43}$$

and by the Atiyah–Singer theorem (which is applicable since $A$ is a differentiable manifold) we have

$$\mathrm{Sign}(g, A) = L'(g, A)[A^g]. \tag{44}$$

Now $A$ in $X$ and $A^g$ in $X^g$ are submanifolds with trivial normal bundle, since both are the inverse images of a point under a differentiable map. Thus in the diagram of inclusions

$$\begin{array}{ccc} A^g = A \cap X^g & \xrightarrow{j'} & X^g \\ \downarrow{\scriptstyle i'} & & \downarrow{\scriptstyle i} \\ A & \xrightarrow{j} & X \end{array}$$

the inclusions $j$ and $j'$ have trivial normal bundles, so ($\varepsilon$ = trivial bundle)

$$N(A^g) \oplus \varepsilon = j'^*N(X^g) \oplus \varepsilon \tag{45}$$

where $N(A^g)$ is the normal bundle of $A^g$ in $A$ and $N(X^g)$ that of $X^g$ in $X$. This isomorphism is $G$-equivariant since $A = f^{-1}(p)$ with $f$ equivariant and $S^i$ a trivial $G$-space; therefore the various eigenvalues of the action of $g$ and the corresponding subbundles correspond, and we deduce

$$L'(g, A) = j'^*L'(g, X). \tag{46}$$

We now put this into (44) and calculate just as we did in the non-equivariant case (equation (37)):

$$\begin{aligned} \mathrm{Sign}(g, A) &= \langle j'^*L'(g, X), [A^g]\rangle \\ &= \langle L'(g, X), j_*'[A^g]\rangle \\ &= \langle L'(g, X), D_{X^g}(f|X^g)^*\sigma\rangle \\ &= \langle L'(g, X), (f\circ i)^*\sigma \cap [X^g]\rangle \\ &= \langle L'(g, X) \cup i^*f^*\sigma, [X^g]\rangle \\ &= \langle i^*f^*\sigma \cup L'(g, X), [X^g]\rangle, \end{aligned} \tag{47}$$

where the last line is justified since $L'$ is an even-dimensional class, and where we have used the usual relationship between cup and cap products. If $X^g$ is non-orientable, everything in the calculation is to be taken in the sense of twisted coefficients. Proceeding in the same way, we get

$$\begin{aligned} \mathrm{Sign}(g, A) &= \langle i^*f^*\sigma, L'(g, X) \cap [X^g]\rangle \\ &= \langle f^*\sigma, i_*(D_{X^g}L'(g, X))\rangle \\ &= \langle f^*\sigma, D_X L(g, X)\rangle \\ &= \langle f^*\sigma, L(g, X) \cap [X]\rangle \\ &= \langle L(g, X) \cup f^*\sigma, [X]\rangle. \end{aligned} \tag{48}$$

Combining equations (42), (43) and (48), we find that $L = L(X/G)$ is determined by the equation

$$(L \cup \bar{f}^*\sigma)[X/G] = \frac{1}{|G|} \sum_{g\in G} (L(g, X) \cup f^*\sigma)[X]. \tag{49}$$

But $[X/G] = \frac{1}{\deg \pi} \pi_*[X]$, so the left-hand side of (49) is

$$\frac{1}{\deg \pi}(L \cup \bar{f}^*\sigma)(\pi_*[X]) = \frac{1}{\deg \pi}(\pi^*L \cup f^*\sigma)[X] \tag{50}$$

since $\bar{f} \circ \pi = f$. Comparing equations (49 and (50) yields Theorem 1.

## §3. THE QUOTIENT OF $P_n(\mathbb{C})$ BY A PRODUCT OF CYCLIC GROUPS

In this section we compute $L(g, X)$ for $X = P_n(\mathbb{C})$ and

$$g = (\zeta_0, \dots, \zeta_n) \in T^{n+1} = S^1 \times \cdots \times S^1 \tag{51}$$

acting on $X$ by

$$g \circ (z_0 : \dots : z_n) = (\zeta_0 z_0 : \dots : \zeta_n z_n). \tag{52}$$

Then if we take a finite subgroup

$$G = \mu_{a_0} \times \cdots \times \mu_{a_n} \subset S^1 \times \cdots \times S^1 \tag{53}$$

(where $\mu_a$ is the subgroup of $a^{\text{th}}$ roots of unity), we know from Theorem 1 that the $L$-class of $X/G$ is (up to a factor) the average over $G$ of $L(g, X)$.

Clearly

$$X^g = \{(z_0 : \dots : z_n) \mid \zeta_i z_i = \zeta z_i,\ i = 0, \dots, n \text{ for some } \zeta\}. \tag{54}$$

Since at least one of the $z_i$'s is non-zero, the number $\zeta$ is determined by $z$ and must belong to $\{\zeta_0, \dots, \zeta_n\}$. Therefore, rewriting (54) as

$$X^g = \bigcup_{\zeta \in S^1} X(\zeta) \tag{55}$$

with

$$X(\zeta) = \{(z_0 : \dots : z_n) \mid z_i = 0 \text{ for } \zeta_i \neq \zeta\}, \tag{56}$$

we see that (55) is a finite disjoint union (at most $n + 1$ of the sets $X(\zeta)$ are non-empty). Since $X(\zeta)$ is isomorphic to $P_s(\mathbb{C})$ (with $s + 1$ the number of $\zeta_i$'s equal to $\zeta$; if $s = -1$ then $X(\zeta)$ is empty) and therefore empty or connected, (55) gives exactly the decomposition of $X^g$ into connected components.

We wish to calculate the contribution to $L(g, X)$ of a component $X(\zeta)$ of $X^g$. For convenience we renumber the coordinates so that $\zeta_0, \dots, \zeta_s$ are equal and $\zeta_{s+1}, \dots, \zeta_n$ unequal to $\zeta$. Thus

$$X(\zeta) = \{(z_0 : \dots : z_s : 0 : \dots : 0) \in P_n(\mathbb{C})\} \cong P_s(\mathbb{C}) \tag{57}$$

and the normal bundle of $X(\zeta)$ in $X$ is a complex bundle isomorphic to $n - s$ copies of the Hopf bundle $H$. Let

$$x \in H^2(X) = H^2(P_n(\mathbb{C})), \tag{58}$$

$$y \in H^2(X(\zeta)) = H^2(P_s(\mathbb{C})) \tag{59}$$

be the generators in cohomology; then $c_1(H) = y$, so the Chern class of the normal bundle $N$ of $X(\zeta)$ in $X$ is $(1 + y)^{n-s}$. Moreover, $g$ acts on $N \cong H \oplus \cdots \oplus H$ as multiplication with

$\zeta_i\zeta^{-1}$ on the $(i-s)^{\text{th}}$ factor $(i = s+1, \ldots, n)$. To see this, we identify $N$ with a tubular neighbourhood of $X(\zeta)$. Then the fibre $N_z$ at $z \in X(\zeta)$ is identified with $\mathbb{C}^{n-s}$ by

$$(y_{s+1}, \ldots, y_n) \leftrightarrow (z_0 : \ldots : z_s : y_{s+1} : \ldots : y_n) \tag{60}$$

so the action of $g$ on the fibre is

$$\begin{aligned} g\circ(y_{s+1}, \ldots, y_n) &\leftrightarrow g\circ(z_0 : \ldots : z_s : y_{s+1} : \ldots : y_n) \\ &= (\zeta z_0 : \ldots : \zeta z_s : \zeta_{s+1} y_{s+1} : \ldots : \zeta_n y_n) \\ &= (z_0 : \ldots : z_s : \zeta_{s+1}\zeta^{-1} y_{s+1} : \ldots : \zeta_n \zeta^{-1} y_n) \\ &\leftrightarrow (\zeta_{s+1}\zeta^{-1} y_{s+1}, \ldots, \zeta_n\zeta^{-1} y_n). \end{aligned} \tag{61}$$

Clearly in §1, for a complex bundle $\xi$ the definition of $L_\theta$ includes that of $L_\pi$, so that for any $\theta \neq 0 \pmod{2\pi}$ we have

$$L_\theta(\xi) = \prod_{j=1}^{q} \frac{e^{i\theta}e^{2x_j} + 1}{e^{i\theta}e^{2x_j} - 1} \tag{62}$$

where $\prod(1 + x_j)$ is the Chern class of $\xi$. Therefore with the above description of $N$ we get

$$\begin{aligned} L'(g, X)_\zeta &= L(X(\zeta)) \prod_\theta L_\theta(N^g_{\theta,\zeta}) \\ &= \left(\frac{y}{\tanh y}\right)^{s+1} \prod_{j=s+1}^{n} \frac{\zeta_j\zeta^{-1}e^{2y} + 1}{\zeta_j\zeta^{-1}e^{2y} - 1}. \end{aligned} \tag{63}$$

Now the Umkehr homomorphism of the inclusion $i : X(\zeta) \to X$ is clearly

$$i_*(y^r) = x^{n-s+r} \tag{64}$$

since the Poincaré dual of $y^r$ in $P_s(\mathbb{C})$ or of $x^{n-s+r}$ in $P_n(\mathbb{C})$ is $[P_{s-r}(\mathbb{C})]$. Therefore from (63) we get

$$\begin{aligned} L(g, X)_\zeta &= i_* L'(g, X)_\zeta \\ &= x^{n-s}\left(\frac{x}{\tanh x}\right)^{s+1} \prod_{j=s+1}^{n} \frac{\zeta_j\zeta^{-1}e^{2x} + 1}{\zeta_j\zeta^{-1}e^{2x} - 1} \\ &= \prod_{j=0}^{n} \left(x\,\frac{\zeta_j\zeta^{-1}e^{2x} + 1}{\zeta_j\zeta^{-1}e^{2x} - 1}\right). \end{aligned} \tag{65}$$

The last line is symmetric and therefore does not depend on the numbering of the coordinates chosen. Summing this over the components (55) yields:

THEOREM 2. *Let $g \in T^{n+1}$ act on $X = P_n(\mathbb{C})$ as in (52). Then*

$$L(g, X) = \sum_{\zeta \in S^1} \prod_{j=0}^{n} \left(x\,\frac{\zeta_j\zeta^{-1}e^{2x} + 1}{\zeta_j\zeta^{-1}e^{2x} - 1}\right) \tag{66}$$

*where $x \in H^2(X)$ is the generator.* The sum is finite since for $\zeta \notin \{\zeta_0, \ldots, \zeta_n\}$ each term in the product is a power series beginning with a multiple of $x$, and $x^{n+1} = 0$, so that the product vanishes for such $\zeta$.

COROLLARY (Bott). *Let G be as in* (53), *other notation as in the theorem. Let $\pi$ be the projection $X \to X/G$. Then the L-class of $X/G$ is given by*

$$\pi^* L(X/G) = \frac{1}{d} \sum_{0 \leqslant \xi < \pi} \prod_{j=0}^{n} \frac{a_j x}{\tanh a_j(x + i\xi)} \tag{67}$$

*where d is the greatest common divisor of* $a_0, \ldots, a_n$.

*Proof.* The degree of $\pi$ is $|G/H|$, where $H$ is the subgroup of $G$ acting trivially on $X$, i.e.

$$H = \{(\zeta_0, \ldots, \zeta_n) \in G : \zeta_0 = \ldots = \zeta_n\} = \{\zeta : \zeta^{a_0} = \ldots = \zeta^{a_n} = 1\} \cong \mu_d. \tag{68}$$

Therefore Theorems 1 and 2 combine to give

$$\begin{aligned} \pi^* L(X/G) &= \frac{1}{d} \sum_{g \in G} L(g, X) \\ &= \frac{1}{d} \sum_{\zeta_0^{a_0} = 1} \cdots \sum_{\zeta_n^{a_n} = 1} \sum_{\zeta \in S^1} \prod_{j=0}^{n} \left( x \frac{\zeta_j \zeta^{-1} e^{2x} + 1}{\zeta_j \zeta^{-1} e^{2x} - 1} \right) \\ &= \frac{1}{d} \sum_{\zeta \in S^1} \prod_{j=0}^{n} \left( \sum_{\zeta_j^{a_j} = 1} x \frac{\zeta_j \zeta^{-1} e^{2x} + 1}{\zeta_j \zeta^{-1} e^{2x} - 1} \right) \\ &= \frac{1}{d} \sum_{\zeta \in S^1} \prod_{j=0}^{n} \left( a_j x \frac{\zeta^{-a_j} e^{2a_j x} + 1}{\zeta^{-a_j} e^{2a_j x} - 1} \right) \end{aligned} \tag{69}$$

where in the last line we have used an elementary trigonometric identity. Writing $\zeta = e^{-2i\xi}$ changes equation (69) into (67). Again we observe that the sum in (67) is a finite one, since if $\xi$ is such that no $a_j \xi$ is divisible by $\pi$, then the product in (67) is a power series whose leading term is a multiple of $x^{n+1}$, and therefore equals zero in $H^*(X)$.

## §4. SYMMETRIC PRODUCTS

If $X$ is a compact oriented manifold of even dimension $2s$, then a permutation of two factors in the Cartesian product $X^n$ does not reverse the orientation, so the symmetric group $G = S_n$ acts orientably on the manifold $X^n$. We can therefore apply Theorem 1 to find the $L$-class of the quotient $X(n) = X^n/S_n$, the $n$th symmetric power of $X$. The complete calculation is rather long and will be given in [10]. In this section we just describe the main results. It will be necessary to introduce some preliminary results and notation.

We work with rational cohomology so that there is no torsion; then the cohomology of $X^n$ is the tensor product $H^*(X) \otimes \cdots \otimes H^*(X)$ and that of $X(n)$ is the $S_n$-invariant part of this, i.e. the $n$th symmetric tensor product of $H^*(X; \mathbb{Q})$ with itself. There is an inclusion map

$$j : X(n) \to X(n+1) \tag{70}$$

sending an unordered set $\{x_1, \ldots, x_n\}$ of points of $X$ to its union with $\{x_0\}$, where $x_0$ is the basepoint of $X$. These inclusions serve to define a limit $X(\infty)$ with inclusion maps

$$j : X(n) \to X(\infty). \tag{71}$$

This limit has a well-defined cohomology group $H^i(X(\infty))$ in each degree. Indeed, a basis of $H^i(X(n))$ is given by the elements $\langle e_0^{n_0}, \cdots, e_b^{n_b} \rangle$ (the symmetrization of $e_0 \times \cdots \times e_0 \times \cdots \times e_b \times \cdots \times e_b \in H^i(X^n)$), where $n_0 + \cdots + n_b = n$ and $\sum n_j \deg e_j = i$ (here $\{e_0 = 1,$

$e_1, \cdots, e_b\}$ is an additive basis of $H^*(X)$). Since deg $e_i > 0$ for $i > 0$, there are only finitely many possible values for $n_1, \ldots, n_b$, so for $n$ sufficiently large a basis of $H^i(X(n))$ is given by elements $e(n_1, \ldots, n_b) = \langle e_0{}^{n-n_1-\cdots-n_b}, e_1{}^{n_1}, \ldots, e_b{}^{n_b}\rangle$ with $i = n_1 \deg e_1 + \cdots + n_b \deg e_b$. Thus $H^i(X(\infty))$ is finitely generated with the $e(n_1, \ldots, n_b)$ as a basis, and the map

$$j^*: H^i(X(\infty)) \to H^i(X(n)) \tag{72}$$

is surjective, with kernel the space spanned by the $e(n_1, \ldots, n_b)$ with $n_1 + \cdots + n_b > n$. In particular there is an element

$$\eta = e(0, \ldots, 0, 1) \in H^{2s}(X(\infty)), \tag{73}$$

where $e_b \in H^{2s}(X)$ is the fundamental class in cohomology; thus

$$j^*\eta = e_b \times 1 \times \cdots \times 1 + 1 \times e_b \times 1 \times \cdots \times 1 + \cdots + 1 \times \cdots \times 1 \times e_b$$

in $H^*(X(n))$. We can now state the main result of [10]:

THEOREM 3. *Let $X(n)$ be the* $n$th *symmetric product of an oriented differentiable manifold of dimension* 2s *and all notation as above. Then*

$$L(X(n)) = j^*(Q^{n+1}G) \tag{74}$$

*where $Q$ and $G$ in $H^*(X(\infty))$ are independent of n. The factor $Q$ is a power series in $\eta$, only depending on s, more precisely,*

$$Q = Q_s(\eta) = \frac{\eta}{f_s(\eta)} \tag{75}$$

*where $f_s(t) = t - t^3/3^s + \cdots$ is the inverse power series of*

$$g_s(x) = x + \frac{x^3}{3^s} + \frac{x^5}{5^s} + \cdots. \tag{76}$$

The content of the theorem as stated, i.e. without giving a formula for the second factor $G$, is that the various $L$-classes of symmetric products are related by

$$j^*L(X(n+1)) = L(X(n)) \cdot j^*Q_s(\eta) \tag{77}$$

where the $j$ on the left-hand side is that of (70) and that on the right-hand side is the map (71). To see that this is equivalent to (74), note that equation (74) for all $n$ certainly implies (77), while if the equations (77) all hold, the equations $j^*G = L(X(n))(j^*Q)^{-n-1}$ are consistent and together define a class $G$ in $H^*(X(\infty))$ satisfying (74). It is, in fact, easier to prove (77) directly than to prove the full theorem with an evaluation of $G$, since the expression for $G$ turns out to be much more complicated than that for $Q$, involving not only the dimension of $X$ but its whole structure (a knowledge of the cup products $e_i e_j$ and of the $L$-class $L(X)$ in terms of the basis elements $e_i$). However, the formula for $G$ is needed in applying Theorem 3. Two situations which are simple enough to be evaluated explicitly are those of a sphere ($X = S^{2s}$) and of a Riemann surface ($s = 1$). In the former case $H^*(X(n))$ is generated by $\eta$ and we have

$$L(X(n)) = \frac{f_s'(\eta)}{1 - f_s(\eta)^2} \left(\frac{\eta}{f_s(\eta)}\right)^{n+1}; \tag{78}$$

in the latter case $X(n)$ is a complex manifold whose Chern class is known (Macdonald [6]), so that the $L$-class found by Theorem 3 can be checked.

Equation (77) is interesting since it suggests that $X(n)$ in $X(n+1)$ has a normal bundle $\nu_n$ with $L(\nu_n) = j^*Q_s(\eta)$ (so that $\nu_n$ would be the restriction of some bundle $\nu_\infty$ over $X(\infty)$ with $L$-class $Q_s(\eta)$). This would fit in with Thom's theory of $L$-classes of rational homology manifolds, in which a rational homology manifold $A$ of dimension $a$ in another one $B$ of dimension $b$ can under certain circumstances have a "normal bundle" $\nu$ which is a $(b-a)$-dimensional vector bundle with $L(\nu)L(A) = i^*L(B)$. However, this is certainly not the situation here, since if one applies to $Q_s(\eta)$ the inverse multiplicative sequence of that defining $L$-polynomials (Hirzebruch [4]), the resulting power series in $\eta^2$ does not break off after the first term and therefore cannot be the Pontrjagin class of any real bundle of dimension $2s$. However, (77) still looks formally like the usual relationship involving the $L$-class of a normal bundle. This suggests that there is some sort of generalized or "rational homology" bundle which is modelled on the inclusion $X(n) \subset X(n+k)$ in the same way as complex vector bundles are locally modelled on the inclusion $P_n(\mathbb{C}) \subset P_{n+k}(\mathbb{C})$. Thus generalized line bundles would correspond to real codimension $2s$ instead of 2 and be classified by a first Chern class in $H^{2s}(X)$ rather than in $H^2(X)$ (i.e. a classifying space would be the Eilenberg–Maclane space $K(\mathbb{Z}, 2s)$ rather than the classifying space $K(\mathbb{Z}, 2) = P_\infty(\mathbb{C})$ of complex bundles; this is to be expected in this context since $K(\mathbb{Z}, 2s)$ is the infinite symmetric product $S^{2s}(\infty)$). The multiplicative sequence giving the $L$-class from the Chern class would be $Q_s(t)$ rather than $Q_1(t) = \dfrac{t}{\tanh t}$. However, it is hard to see exactly what form such a theory would take since the local structure of a symmetric power of $X$ embedded in a higher one is rather complex.

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