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A Property of L-Functions on the Real Line

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Let $L(s) = L_d(s)$ be the Dirichlet L-series associated to the quadratic field of discriminant d and set

$$\Lambda_d(s) = \left(\frac{|d|}{\pi}\right)^{s/2} \Gamma\left(\frac{s+1}{2}\right) L_d(s) \qquad (d < 0),$$

$$= \left(\frac{|d|}{\pi}\right)^{s/2} \Gamma\left(\frac{s}{2}\right) L_d(s) \qquad (d > 0).$$

Numerical evidence [1] indicates that $\Lambda_d(s)$ is convex upwards for real s, $0 < s < 1$. One cannot hope to prove this easily since one consequence would be that $L_d(s)$ has at most one real zero $\geqslant 1/2$. However, we will show here that $\Lambda_d(s)$ is convex upward on the real line if $L_d(s)$ satisfies the Riemann hypothesis or even if $L_d(s) \neq 0$ in the triangle $|t| \leqslant \sigma - \frac{1}{2} \leqslant \frac{1}{2}$ where $s = \sigma + it$ (see Fig. 1). Indeed we will show that if for some $\sigma_0 \geqslant \frac{1}{2}$, the function $\Lambda_d(s)$ has no zeros in the hyperbolic region

$$(\sigma - \tfrac{1}{2})^2 - t^2 > (\sigma_0 - \tfrac{1}{2})^2$$

(see Fig. 2), then $\Lambda_d(\sigma)$ is monotone increasing and convex for $\sigma > \sigma_0$.

THEOREM. *Let $\Lambda(s)$ be an entire function of order 1 which is real and positive for large real s and which satisfies the functional equation*

$$\Lambda(s) = \pm\Lambda(k - s) \tag{1}$$

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for some $k > 0$. Suppose that for some $\sigma_0 \geqslant k/2$ the function $\Lambda(s)$ has no zeros in the hyperbolic region

$$(\sigma - k/2)^2 - t^2 > (\sigma_0 - k/2)^2. \tag{2}$$

Then $\Lambda^{(n)}(\sigma) > 0$ for all $\sigma > \sigma_0$ and all $n \geqslant 0$. In particular, $\Lambda(\sigma)$ is monotone increasing and convex for $\sigma > \sigma_0$.

Proof. It is convenient to translate s by $k/2$. We let

$$z = s - k/2$$

and

$$F(z) = \Lambda(z + k/2),$$

so that the functional equation (1) becomes

$$F(z) = \pm F(-z). \tag{3}$$

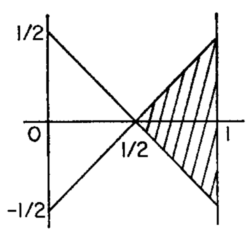

FIGURE 1

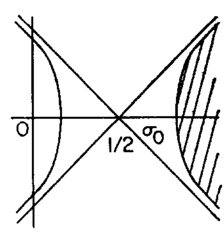

FIGURE 2

By the Hadamard product theorem,

$$F(z) = Az^m e^{Bz} \prod_{\alpha} (1 - z/\alpha)\, e^{z/\alpha},$$

where $m \geqslant 0$ and the product runs over all zeros $\alpha \neq 0$ of $F(z)$ counted according to multiplicity and converges absolutely and uniformly on compact sets. If $\alpha = \beta + i\gamma$ then the condition that $\alpha + k/2$ not be in the region (2) is

$$\beta^2 - \gamma^2 \leqslant x_0^2 \tag{4}$$

where $x_0 = \sigma_0 - k/2$.

If α is a zero of $F(z)$, then so are $\bar{\alpha}$, $-\alpha - \bar{\alpha}$. We define the equivalence class $[\alpha]$ of α to be the set

$$\alpha] = \{\alpha, \bar{\alpha}, -\alpha, -\bar{\alpha}\}$$

of four elements (if $\beta\gamma \neq 0$) or two elements (if one of β and γ equal zero) or one element (if $\alpha = 0$). Set

$$\begin{aligned} f_\alpha(z) &= \prod_{\delta \in [\alpha]} (1 - z/\delta) && \text{if } \gamma \neq 0 \\ &= - \prod_{\delta \in [\alpha]} (1 - z/\delta) && \text{if } \gamma = 0, \quad \beta \neq 0 \\ &= z && \text{if } \alpha = 0. \end{aligned}$$

Then

$$F(z) = Ce^{Bz} \prod_{[\alpha]} f_\alpha(z),$$

where $C = \pm A$. Since $f_\alpha(z) = f_\alpha(-z)$ for $\alpha \neq 0$, Eq. (3) shows that $B = 0$. Further, for $x > x_0$, $f_\alpha(x) > 0$ and hence $C > 0$. We have thus arrived at

$$F(z) = C \prod_{[\alpha]} f_\alpha(z), \tag{5}$$

where each α is taken according to multiplicity and where the product converges absolutely and uniformly on compact sets.

Thus if we can show that each of the infinitely many $f_\alpha(z)$ has positive first derivatives and nonnegative derivatives of all other orders for $z = x > x_0$, the theorem will follow. But

$$\begin{aligned} f_\alpha(z) &= |\alpha|^{-4}[z^4 + 2(\gamma^2 - \beta^2) z^2 + (\gamma^2 + \beta^2)^2] && \text{if } \beta \neq 0, \ \gamma \neq 0 \\ &= \gamma^{-2}(z^2 + \gamma^2) && \text{if } \beta = 0, \ \gamma \neq 0 \\ &= \beta^{-2}(z^2 - \beta^2) && \text{if } \beta \neq 0, \ \gamma = 0 \\ &= z && \text{if } \alpha = 0 \end{aligned}$$

from which it is easily seen that $f_\alpha^{(n)}(x) \geqslant 0$ for $x > x_0$ with inequality for $n = 1$.

It is possible to prove the monotonicity and convexity results using the logarithmic derivative of $F(x)$ but the proof is essentially the same. For instance, from (5) we obtain

$$\frac{F'(z)}{F(z)} = \sum_{[\alpha]} \frac{f'_\alpha(z)}{f_\alpha(z)}$$

and this returns us to the question of showing that each $f'_\alpha(x) > 0$ for $x > x_0$. Likewise, we have

$$\frac{F''(z)}{F(z)} = \left(\frac{F'(z)}{F(z)}\right)^2 + \sum_{[\alpha]} \left[\frac{f''_\alpha(z)}{f_\alpha(z)} - \left(\frac{f'_\alpha(z)}{f_\alpha(z)}\right)^2\right]$$

$$= \sum_{[\alpha]} \frac{f'_\alpha(z)}{f_\alpha(z)} \sum_{[\alpha']}' \frac{f'_{\alpha'}(z)}{f_{\alpha'}(z)} + \sum_{[\alpha]} \frac{f''_\alpha(z)}{f_\alpha(z)}$$

where $\sum'_{[\alpha']}$ means that if $[\alpha]$ has multiplicity m then $[\alpha'] = [\alpha]$ should be taken with multiplicity $m - 1$. Again we come to the question of showing that each $f'_\alpha(x) > 0$ and $f''_\alpha(x) \geqslant 0$ for $x > x_0$.

Reference

1. G. Purdy, R. Terras, A. Terras, and H. Williams, Graphing L-functions of Kronecker symbols in the real part of the critical strip, *Math. Student*, in press.