GROUPS AND GROUP FUNCTORS ATTACHED TO KAC-MOODY DATA

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1. The finite-dimensional complex semi-simple Lie algebras.

To start with, let us recall the classification, due to W. Killing and E. Cartan, of all complex semi-simple Lie algebras. (The presentation we adopt, for later purpose, is of course not that of those authors.) The isomorphism classes of such algebras are in one-to-one correspondence with the systems

$$\mathfrak{H}\ ,\ (\alpha_i)_{1\leq i\leq \ell}\ ,\ (h_i)_{1\leq i\leq \ell}\ , \tag{1.1}$$

where $\mathfrak{H}$ is a finite-dimensional complex vector space (a Cartan subalgebra of a representative $\mathfrak{G}$ of the isomorphism class in question), $(\alpha_i)_{1\leq i\leq \ell}$ is a basis of the dual $\mathfrak{H}^*$ of $\mathfrak{H}$ (a basis of the root system of $\mathfrak{G}$ relative to $\mathfrak{H}$) and $(h_i)_{1\leq i\leq \ell}$ is a basis of $\mathfrak{H}$ indexed by the same set $\{1,\ldots,\ell\}$ (h_i is the coroot associated with α_i), such that the matrix $\underline{\underline{A}} = (A_{ij}) = (\alpha_j(h_i))$ is a Cartan matrix, which means that the following conditions are satisfied:

- (C1) the A_{ij} are integers ;
- (C2) $A_{ij} = 2$ or ≤ 0 according as $i =$ or $\neq j$;
- (C3) $A_{ij} = 0$ if and only if $A_{ji} = 0$;
- (C4) $\underline{\underline{A}}$ is the product of a positive definite symmetric matrix and a diagonal matrix (by abuse of language, we shall simply say that $\underline{\underline{A}}$ is positive definite).

More correctly: two such data correspond to the same isomorphism class of algebras if and only if they differ only by a permutation of the indices $1,\ldots,\ell$. Following C. Chevalley, Harish-Chandra and J.-P. Serre, one can give a simple presentation of the algebra corresponding to the system (1.1): it is generated by $\mathfrak{H}$ and a set of 2ℓ elements $e_1,\ldots,e_\ell$, $f_1,\ldots,f_\ell$ subject to the following relations (besides the vector space structure of $\mathfrak{H}$):

$$\begin{cases} [\mathfrak{H},\mathfrak{H}] = \{0\} \; ; \\ [h,e_i] = \alpha_i(h).e_i \qquad (h \in \mathfrak{H}) \; ; \\ [h,f_i] = -\alpha_i(h).f_i \qquad (h \in \mathfrak{H}) \; ; \\ [e_i,f_i] = -h_i \; ; \\ [e_i,f_j] = 0 \quad \text{if} \quad i \neq j \; ; \\ (\mathrm{ad}\, e_i)^{-A_{ij}+1}(e_j) = 0 \quad \text{if} \quad i \neq j \; ; \\ (\mathrm{ad}\, f_j)^{-A_{ij}+1}(f_j) = 0 \quad \text{if} \quad i \neq j \, . \end{cases}$$

If one does no longer assume that the α_i and the h_i generate $\mathfrak{H}^*$ and $\mathfrak{H}$ respectively, one obtains in that same way all complex <u>reductive</u> Lie algebras.At this point, the generalization is rather harmless (reductive = semi-simple × commutative), but it becomes more significant at the group level and will turn out to be quite essential in the Kac-Moody situation.

2. <u>Reductive algebraic groups and Chevalley schemes</u>.

It is well known that a complex Lie algebra determines a Lie group only up to local isomorphism. Thus, in order to characterize a reductive algebraic group, over $\underline{\mathbb{C}}$, say, an extra-information, besides the data (1.1), is needed. It is provided by a lattice Λ in $\mathfrak{H}$ (i.e. a $\underline{\mathbb{Z}}$ -submodule of $\mathfrak{H}$ generated by a basis of $\mathfrak{H}$) such that $h_i \in \Lambda$ and $\alpha_i \in \Lambda^*$ (the $\underline{\mathbb{Z}}$ -dual of Λ), namely the lattice of rational co-characters of a maximal torus of the group one considers. To summarize: the isomorphism classes of complex reductive groups are in one-to-one correspondence (again up to permutation of the indices) with the systems

(2.1) $\qquad S = (\Lambda, (\alpha_i)_{1\leq i\leq \ell}, (h_i)_{1\leq i\leq \ell})$,

where Λ is a finitely generated free $\underline{\underline{Z}}$ -module, $\alpha_i \in \Lambda^*$, $h_i \in \Lambda$ and $\underline{\underline{A}} = (\alpha_j(h_i))$ is a Cartan matrix.

A remarkable result of C. Chevalley [Ch2] is that the same classification holds when one replaces $\underline{\underline{C}}$ by any algebraically closed field. Furthermore, to any system (2.1), Chevalley [Ch3] and Demazure [De2] associate a group-scheme over $\underline{\underline{Z}}$, hence, in particular, a group functor G_S on the category of rings. Thus, the main result of [Ch2] asserts that the reductive algebraic groups over an algebraically closed field K are precisely the groups $G_S(K)$, where S runs over the systems (2.1) described above.

Question: what happens if, in the above considerations, one drops Condition (C4) (in which case, the matrix $\underline{\underline{A}}$ is called a generalized Cartan matrix, or GCM)? This is what the Kac-Moody theory is about.

3. Kac-Moody Lie algebras.

From now on, when talking about a system (1.1), we only assume that $\alpha_i \in \mathcal{H}^*$, $h_i \in \mathcal{H}$ (the α_i and h_i need not generate $\mathcal{H}^*$ and $\mathcal{H}$) and that $\underline{\underline{A}} = (\alpha_j(h_i))$ is a GCM. To such a system, the presentation (1.2) associates a Lie algebra which is infinite-dimensional whenever $\underline{\underline{A}}$ is not a Cartan matrix. The Lie algebras one obtains that way are called Kac-Moody algebras. A large part of the classical theory - root systems, linear representations etc. - extends to them, with a bonus: the study of root multiplicities (roots do have multiplicities in the general case) and of character formulas for linear representations with highest weights have a number-theoretic flavour which is not apparent in the finite-dimensional situation. For those questions, which are outside the subject of the present survey, see [Ka3] and its bibliography.

In general, Kac-Moody algebras are entirely new objects, but there is a case, besides the positive definite one, where they are still closely related to finite-dimensional simple Lie algebras, namely the "semi-definite" case: by the same abuse of language as above, we say that the matrix $\underline{\underline{A}}$ is semi-definite if it is the product of a semi-definite symmetric matrix and a diagonal matrix.

The simplest example of Kac-Moody algebras of semi-definite type is provided by the so-called <u>loop algebras</u>. Let $\mathfrak{G}$ be a complex semi-simple Lie algebra, $\mathfrak{H}$ a Cartan subalgebra of $\mathfrak{G}$, $(\alpha_i)_{1\leq i\leq \ell}$ a basis of the root system of $\mathfrak{G}$ relative to $\mathfrak{H}$, α_0 the opposite of the dominant root and h_j, for $0\leq j\leq \ell$, the coroot corresponding to α_j. Then, the system

$$\mathfrak{H}\ ,\ (\alpha_j)_{0\leq j\leq \ell}\ ,\ (h_j)_{0\leq j\leq \ell}$$

satisfies our conditions and the corresponding Kac-Moody algebra turns out to be the "loop algebra" $\mathfrak{G}\otimes\underline{\underline{C}}[z,z^{-1}]$. In this case, the GCM $\tilde{\underline{\underline{A}}} = (\alpha_k(h_j))_{1\leq j,k\leq \ell}$ is described by the well-known extended Dynkin diagram ("graphe de Dynkin complété" in [Bo]) of $\mathfrak{G}$; we shall call it the <u>extended Cartan matrix</u> of $\mathfrak{G}$.

Let us modify the previous example slightly: instead of $\mathfrak{H}$, we take a direct sum $\tilde{\mathfrak{H}} = \coprod_{0\leq j\leq \ell} \underline{\underline{C}}.\tilde{h}_j$, where the $\tilde{h}_j$'s are "copies" of the h_j's, and we choose the elements $\tilde{\alpha}_j$ of $\tilde{\mathfrak{H}}^*$ in such a way that the matrix $(\tilde{\alpha}_k(\tilde{h}_j))$ be the same $\tilde{\underline{\underline{A}}}$ as before. Then, $\tilde{\mathfrak{H}}$ is the extension of $\mathfrak{H}$ by a one-dimensional subspace $\mathfrak{r} = \underline{\underline{C}}.(\Sigma d_j\tilde{h}_j)$ (where the d_j's are nonzero coefficients such that $\Sigma d_j h_j = 0$), and it is readily seen that the Kac-Moody algebra defined by the system $(\tilde{\mathfrak{H}},(\tilde{\alpha}_j),(\tilde{h}_j))$ is a perfect central extension of $\mathfrak{G}\otimes\underline{\underline{C}}[z,z^{-1}]$ by the one-dimensional algebra $\mathfrak{r}$. In fact, it is the <u>universal</u> central extension of $\mathfrak{G}\otimes\underline{\underline{C}}[z,z^{-1}]$: this is a special case of the following, rather easy proposition, proved independently by Kac ([Ka3], exercise 3.14), Moody (unpublished) and the author ([Ti4]):

PROPOSITION 1. - <u>If the h_i's form a basis of $\mathfrak{H}$, the Kac-Moody algebra defined by</u> (1.2) (<u>for</u> $(\alpha_j(h_i))$ <u>an arbitrary GCM) has no nontrivial central extension</u>.

The existence of a nontrivial central extension of $\mathfrak{G}\otimes\underline{\underline{C}}[z,z^{-1}]$ by $\underline{\underline{C}}$ plays an important role in the applications of the Kac-Moody theory for instance to physics and to the theory of differential equations (cf. e. g. [Ve1], [SW] and the literature cited in those papers). It is worth noting that the Kac-Moody presentation provides a natural approach to that extension and a very simple proof of its universal property, which is much less evident when one uses direct (e. g. cohomological) methods (cf. [Ga], [Wi]). (NB. In the literature, the expression "Kac-Moody algebras" is frequently used to designate merely the loop algebras and/or their universal central extension; this unduly restrictive usage explains itself by the importance of those special cases for the applications.)

Here, a GCM will be called "<u>of affine type</u>" if it is semi-definite, nondefinite and indecomposable; we say that it is of <u>standard</u> (resp. <u>twisted</u>) affine type if it is (resp. is not) the extended Cartan matrix of a finite-dimensional simple Lie algebra. (In the literature, one often finds the words "affine" and "euclidean" to mean "standard affine" and "twisted affine" in our terminology.) In rank 2, there are two GCM of affine type, one standard $\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$ and one twisted $\begin{pmatrix} 2 & -1 \\ -4 & 2 \end{pmatrix}$ (up to permutation of the indices). When the rank is ≥ 3, the coefficients of a GCM (A_{ij}) of affine type always satisfy the relation $A_{ij}A_{ji} \leq 3$ (for $i \neq j$), so that the matrix can be represented by a Dynkin diagram in the usual way (cf. e. g. [BT3], 1.4.4 or [Bo], p. 195); then, it turns out that the diagrams representing the twisted types are obtained by reversing arrows in the diagrams representing standard types (i.e. in extended Dynkin diagrams of finite-dimensional simple Lie algebras). For instance,

$(\tilde{F}_4)$ 

is standard, whereas

$({}^2\tilde{E}_6)$ 

is twisted.

The most general Kac-Moody algebra of standard affine type is a semi-direct product of an abelian algebra by a central extension of a loop algebra. There is a similar description for the algebras of twisted affine type, in which the loop algebras must be replaced by suitable twisted forms. For instance, if $\mathfrak{G}$ is a complex Lie algebra of type E_6 and if σ denotes an involutory automorphism of the loop algebra $\mathfrak{G} \otimes \underline{\underline{C}}[z,z^{-1}]$ operating on the first factor by an outer automorphism and on the second by $z \longmapsto -z$, then the fixed point algebra $(\mathfrak{G} \otimes \underline{\underline{C}}[z,z^{-1}])^{\sigma}$ is a Kac-Moody algebra of type ${}^2\tilde{E}_6$ above (hence the notation !). The connection between Kac-Moody algebras of affine type and the loop algebras and their twisted analogues was first made explicit in [Ka2], but the corresponding relation at the group level had been known for some time: cf. [IM] and [BT2] (where, however, a local field - such as $\underline{\underline{C}}((z))$ - replaces $\underline{\underline{C}}[z,z^{-1}]$).

4. Associated groups: introductory remarks.

In the classical, finite-dimensional theory, the Lie algebras often appear as intermediate step in the study of Lie groups. It is therefore natural to try similarly to "integrate" Kac-Moody Lie algebras and to define "Kac-Moody groups". More precisely, let S be as in (2.1) except that, now, the matrix $(\alpha_j(h_i))$ is only assumed to be a GCM. To such a system S, one wishes to associate an "infinite-dimensional group over $\underline{\underline{C}}$", let us call it $G_S(\underline{\underline{C}})$, or, more ambitiously, a group functor G_S on the category of rings.

Before passing in quick review the methods that have been used to define such groups, let us make a preliminary comment. As may be expected, since the groups in question are "infinite-dimensional", one is led, for a given S, to define not one but several groups which are various completions of a smallest one (those completions corresponding usually to various completions of the Kac-Moody Lie algebra). Thus, the group theory can be developed at different levels (or, if one prefers, in different categories); roughly speaking, one may distinguish a <u>minimal</u> (or purely algebraic) level, a <u>formal</u> level and an <u>analytic</u> level, with many subdivisons.

Instead of trying to define those terms formally, I shall just illustrate them with one example. Let $\mathfrak{G}$ be a complex, quasi-simple simply connected algebraic group (Lie algebras will now play a minor role, and we are free to use gothic letters for other purposes !), Λ^* the lattice of rational characters of a maximal torus of $\mathfrak{G}$, Λ its $\underline{\underline{Z}}$-dual, $(\alpha_i)_{1\leq i\leq \ell}$ a basis of the root system of $\mathfrak{G}$ with respect to the torus in question, α_0 the opposite of the dominant root, h_j (for $0 \leq j \leq \ell$) the coroot corresponding to α_j and $S = (\Lambda, (\alpha_j)_{0 \leq j \leq \ell}, (h_j)_{0 \leq j \leq \ell})$. In § 3, we have seen that the Lie algebra associated with S (in which Λ is replaced by $\underline{\underline{C}} \otimes \Lambda$) is Lie $\mathfrak{G} \otimes \underline{\underline{C}}[z,z^{-1}]$. Clearly, the group most naturally associated with S over $\underline{\underline{C}}$ must – and will – be the group $\mathfrak{G}(\underline{\underline{C}}[z,z^{-1}])$ of all "polynomial maps" $\underline{\underline{C}}^{\times} \longrightarrow \mathfrak{G}$. In that special case, this is the answer to our question at the minimal level. At the formal level, we find $\mathfrak{G}(\underline{\underline{C}}((z)))$. Now, the points of $\mathfrak{G}(\underline{\underline{C}}[z,z^{-1}])$ can also be viewed as certain special loops $S^1 \longrightarrow \mathfrak{G}$ (by restricting $\underline{\underline{C}}^{\times} \longrightarrow \mathfrak{G}$ to the complex numbers of absolute value one) and this opens the way to a great variety of completions of $\mathfrak{G}(\underline{\underline{C}}(z,z^{-1}))$, leading to groups of loops $S^1 \longrightarrow \mathfrak{G}$ in various categories (L^2, continuous, C^{∞}, etc.): this is the analytic level.

In the case of the above system S, there is no difficulty in guessing what should be the group functor G_S : at the minimal level, we shall have $G_S(R) = \mathfrak{G}(R[z,z^{-1}])$, where $\mathfrak{G}$ now denotes the <u>Chevalley scheme</u> corresponding to the system $(\Lambda, (\alpha_i)_{1 \leq i \leq \ell}, (h_i)_{1 \leq i \leq \ell})$, and the corresponding formal group will be $\mathfrak{G}(R((z)))$. (In this generality, I do not know what "analytic" should mean.) As one sees, all those groups can be described with elementary means, without reference to Kac-Moody algebras. But things change as soon as one slightly modifies the system S as in § 3 by taking for instance $\Lambda = \coprod_{j=0}^{\ell} \underline{\underline{Z}}\tilde{h}_j$ (and keeping the GCM unchanged, as before). The corresponding group is then a central extension of the loop group (whichever category one is in) by $\underline{\underline{C}}^{\times}$ or, in the ring situation, by $R^{\times}$. As in the Lie algebra case, the existence of that extension comes out of the general theory quite formally, but in the loop group case, it reflects rather deep properties of those groups (cf. e. g. [SW]), and direct existence proofs are not easy. Note that if R is a finite field k, one gets a central extension of $\mathfrak{G}(k((z)))$ by $k^{\times}$ which appears in the work of C. Moore [Mo2] and H. Matsumoto [Ma3].

Here, we shall most of the time adopt either the minimal or the formal viewpoint (the analytic ones are usually deeper and more important for the applications, but unfortunately less familiar to the speaker). Let us briefly mention some contrasting features of those. The formal groups are usually simpler to handle (as are local fields compared to global ones !). This is due in particular to the fact that they contain "large" proalgebraic subgroups (cf. e. g. [BT2], § 5, and [Sℓ], Kap. 5). Also, they seem to be the right category for simplicity theorems (cf. [Mo1]; observe that if $\mathfrak{G}$ denotes a complex simple Lie group, then $\mathfrak{G}(\underline{\underline{C}}((z)))$ is a simple group, which is far from true for $\mathfrak{G}(\underline{\underline{C}}[z,z^{-1}])$. On the other hand, the minimal theory presents a certain symmetry (the symmetry between the e_i's and f_i's or, in the example of $\mathfrak{G}(\underline{\underline{C}}[z,z^{-1}])$, the symmetry between z and z^{-1}), which gets lost in the formal completion.

Let us mention an important aspect of that symmetry. All the groups $G = G_S(\underline{\underline{C}})$ we are talking about (and, in fact, the groups $G_S(K)$, for K a field), whether minimal or formal, are equipped with a BN-pair (B,N) (or Tits system: cf. [Bo]) whose Weyl group $W = N/B \cap N$ is the Coxeter group $W(\underline{\underline{A}})$ defined as follows:

$$W(A) = \langle r_i \mid 1 \leq i \leq \ell \;;\; r_i^2 = 1 \;;\; (r_i r_j)^{c_{ij}} = 1 \text{ if } i \neq j, \; A_{ij}A_{ji} \leq 3, \text{ and } c_{ij} = 2,3,4 \text{ or } 6 \text{ according as } A_{ij}A_{ji} = 0,\,1,\,2 \text{ or } 3 \rangle$$

(cf. [MT], [Ma1], [Ti3] and also, for the affine case, [IM], [BT2] and [Ga]). In particular, G has a Bruhat decomposition $G = \bigcup_{w \in W} BwB$, leading to a "cell decomposition" of G/B : the quotients BwB/B have natural structures of finite-dimensional affine spaces. Now, in the minimal situation, the same N is the group N of another BN-pair (B^-,N), not conjugate to the previous one except in the finite-dimensional case (i.e. when $\underline{\underline{A}}$ is positive-definite). Furthermore, one also has a partition $G = \bigcup_{w \in W} B^-wB$, called the Birkhoff decomposition of G (because of the special case considered in [Bi]; for the general result, cf. [Ti4]). While the cells BwB/B are finite-dimensional, the "cells" B^-wB are finite-codimensional in G, in a suitable sense, and, unlike the Bruhat decomposition, the Birkhoff decomposition always has a big cell, namely B^-B if one chooses B^- in its conjugacy class by N so that the intersection $B^- \cap B$ is minimum with respect to the inclusion (we then say that B and B^- are opposite). In the formal situation, a Birkhoff decomposition (and hence a big cell) still exists, but here, the groups B^- and B play completely asymmetric roles: B_- is <u>much smaller</u> than B in that, for instance, $B^-\backslash G/B^-$ is now highly uncountable (always excepting the case where $\underline{\underline{A}}$ is positive-definite). We can be more explicit: if $G = \cup B^- wB$ is the Birkhoff decomposition of the <u>minimal</u> group G, and if $\hat{G}$ denotes the formal completion of G, then the Birkhoff decomposition of $\hat{G}$ is $\cup B^- w\bar{B}$, where $\bar{B}$ is the closure of B in $\hat{G}$; the group B^- is closed (and even discrete) in $\hat{G}$.

Different methods have been used to attach groups to Kac-Moody data. Roughly, one can classify them into four types, according to which techniques they are based upon, namely:

linear representations (cf. § 5 below);
generators and relations (cf. § 6);
Hilbert manifolds and line bundles;
axiomatic (cf. [Ti4]).

About the third approach, which is handled in Graeme Segal's lecture at this Arbeitstagung, let us just say that it gives a deeper geometric insight in the situation than the other methods, but that, at present, it concerns only the affine case. Also the axiomatic approach has been used only in the affine case so far: we shall briefly indicate below (§ 6 and Appendix 2) to which purpose.

5. <u>Construction of the groups via representation theory</u>.

One of the simplest way to prove the existence of a Lie group with a given (finite-dimensional) Lie algebra L consists in embedding L in the endomorphism algebra $\mathrm{End}\, V$ of a vector space V (by Ado's theorem) and then considering the group generated by $\exp L$.

If L is a Kac-Moody algebra, linear representations are infinite-dimensional and $\exp L$ is no longer defined in general. However, suppose that the linear representation $L \hookrightarrow \mathrm{End}\, V$ is such that the elements e_i, f_i, considered as endomorphisms of V, are <u>locally nilpotent</u> (an endomorphism φ of V is said to be locally nilpotent if, for any $v \in V$, $\varphi^n(v) = 0$ for almost all $n \in \underline{\underline{N}}$). Then, if the ground field K has characteristic zero, say, $\exp Ke_i$ and $\exp Kf_i$ are well-defined "one-parameter" automorphism groups of V which generate the group $G_S(K)$ one is looking for, at least if the h_i's generate Λ. Otherwise, one must also require that, as a Λ-module (remember that $\Lambda \subset L$), V is a direct sum $\coprod V_s$ of one-dimensional modules on which Λ operates through "integral characters" $\chi_s \in \Lambda^*$; then, one adds to the above generators the "one-parameter groups" $\lambda(K^\times)$, with $\lambda \in \Lambda$, where, by definition, $\lambda(k)$ operates on V_s via the multiplication by $k^{\langle\lambda,\chi_s\rangle}$. An L-module V is said to be integrable if it satisfies the above conditions (local nilpotency of e_i, f_i, plus the extra-requirement on Λ, which however follows from the first condition when the h_i's generate Λ).

That method for integrating L, inspired by C. Chevalley's Tohôku paper [Ch1], was first devised by R. Moody and K. Teo [MT], who used the adjoint representation of L. In that way, of course, they only get the minimal adjoint group. (More precisely, the group they construct is the analogue of Chevalley's simple group, namely the subgroup of the adjoint group generated by the $\exp Ke_i$ and $\exp Kf_i$; here, we say that the system S defines an adjoint group if the α_i's generate Λ^* and if $\mathbb{Q} \otimes \Lambda$ is generated as a $\mathbb{Q}$-vector space by the h_i's.) On the other hand, a suitable variation of the method described above enables them to include the case of a ground field with sufficiently large characteristics. Later on, Moody [Mo1] has applied the same ideas at the formal level, starting from a suitable completion of the Kac-Moody algebra.

In [Ma1], R. Marcuson works with highest weight modules, at the formal level. His method requires the characteristic to be zero.

In [Ga], H. Garland also uses highest weight representations. He restricts himself to the standard affine case - and makes heavy use of the relation between L and the loop algebra -, but in that special case, his results go much beyond those of Marcuson in that he essentially works over $\mathbb{Z}$ (with $\mathbb{Z}$-forms of the universal enveloping algebra of L and of the representation space), which enables him to define groups over arbitrary fields.

One drawback of the approach by means of linear representations is that it is not clear, *a priori*, how the group one associates to a given Kac-Moody algebra (over $\underline{\underline{C}}$, say) varies with the chosen representation. In [Ma1], this question is left open. Garland answers it by using the fact that the groups he constructs are central extensions of loop groups, and computing a cocycle which describes the extension.

V. Kac and D. Peterson [KP] obviate that inconvenient of the method by considering all integrable modules simultaneously. They start from the free product G^* of the additive groups Ke_i, Kf_i for all i . For any integrable module V , the maps $te_i \longrightarrow \exp te_i$, $tf_i \longrightarrow \exp tf_i$ extend to a representation $\exp_V : G^* \longrightarrow GL(V)$, and the group they consider is $G^*/\bigcap_V (\mathrm{Ker}\ \exp_V)$, where V runs through all integrable representations. This is the minimal group, in the sense of § 4, and corresponds to the case where the h_i's form a basis of Λ . (An other, earlier approach of that same group, but without this last restriction on the h_i's , can be found in [Ti3] : cf. § 6).

R. Goodman and N. Wallach [GW] are concerned with the standard affine case over $\underline{\underline{C}}$. Working within the theory of Banach Lie algebras and groups, they consider a large variety of Banach completions of the Kac-Moody algebras, and integrate them by using suitable topologizations of certain highest weight (so-called standard) modules. One of their purposes is to define the central extension of loop groups by $\underline{\underline{C}}^\times$ at various analytic levels. An alternative, more elementary approach to that problem (not touching, however, the main body of results of [GW]) may possibly be suggested by the remark of Appendix 1 below.

6. Generators and relations.

In a course of lectures summarized in [Ti3] (cf. also [Sℓ] and [Ma2]), I gave another construction for groups associated with Kac-Moody data. In order to sketch the main idea, let us return to the case of a finite-dimensional complex semi-simple Lie group G . Such a group is known to be the amalgamated product of the normalizer N of a maximal torus T and the parabolic subgroups $P_1,\ldots,P_\ell$ containing properly a given Borel subgroup B containing T and minimal with that property, with amalgamation of the intersections $P_i \cap P_j = B$ and $P_i \cap N$. (cf.[Ti2],13.3). Furthermore, P_i is the semi-direct product of its Levi subgroup L_i containing T by a unipotent group U_i . Thus, we have a presentation of G whose ingredients are the subgroups

N, L_i, U_i . The groups N and L_i can be reconstructed from the system S of (2.1) in a uniform way, without reference to the positivity of the matrix $\underline{\underline{A}}$: the group N is generated by $T = \mathrm{Hom}(\Lambda^*, C^\times)$ and ℓ elements m_i $(1 \leq i \leq \ell)$ submitted to the relations

(6.1) m_i normalizes T , and the automorphism of T it induces is the adjoint of the <u>reflection</u>

$$\lambda \longmapsto \lambda - \langle\lambda, h_i\rangle \cdot \alpha_i \quad \text{of} \quad \Lambda^* ,$$

(6.2) $m_i^2 = \eta_i \in T = \mathrm{Hom}(\Lambda^*, \underline{\underline{C}}^\times)$, with $\eta_i(\lambda) = (-1)^{\langle\lambda, h_i\rangle}$ for $\lambda \in \Lambda^*$

and

(6.3) if $A_{ij}A_{ji} = 0$ (resp. 1;2;3), then $m_i m_j = m_j m_i$ (resp. $m_i m_j m_i = m_j m_i m_j$; $(m_i m_j)^2 = (m_j m_i)^2$; $(m_i m_j)^3 = (m_j m_i)^3$),

whereas L_i is nothing else but the reductive group of semi-simple rank one corresponding to the system (Λ, h_i, α_i) . As for the U_i's , being unipotent, they are easily described in terms of their Lie algebras $\mathrm{Lie}\, U_i$, either by means of the Campbell-Hausdorff formula or, more conceptually, by exponentiating $\mathrm{Lie}\, U_i$ in the completion (for the natural filtration) of its universal enveloping algebra $\mathcal{U}(\mathrm{Lie}\, U_i)$.

All this can be carried over to an arbitrary system S , with the only difference that, now, $\mathrm{Lie}\, U_i$ is infinite-dimensional and no longer nilpotent but only pro-nilpotent (more precisely, the $\mathrm{Lie}\, U_i$'s are certain subalgebras of codimension 1 of the Lie algebra generated by the e_j's , and the latter has a <u>pro-nilpotent</u> <u>completion</u>). Moreover, by using a suitable $\underline{\underline{Z}}$-form of the universal enveloping algebra of the Kac-Moody algebra (generalizing the $\underline{\underline{Z}}$-form used by H. Garland in the affine case: cf. § 5), one is able to do everything over $\underline{\underline{Z}}$ and, by reduction, over an arbitrary ring R . Thus, one is led to attach to S a group functor on the category of rings, call it E_S . But this group functor E_S is <u>not</u> the "good" functor G_S one is looking for: indeed, if $\underline{\underline{A}}$ is a Cartan matrix, that is, in the positive definite case, G_S should of course be the Chevalley group-scheme corresponding to the Chevalley-Demazure data S , and one finds that the functor E_S

coincides with that scheme only over the principal ideal domain. This suggests that, in general, E_S may be the good functor when restricted to those rings. This is undoubtedly so in the affine case. Indeed, in that case, one can characterize the functor E_S restricted to principal ideal domains - call it $E^{(pid)}$ - by a system of very natural axioms which, it seems, should be satisfied by the "good" functor G_S (cf. [Ti4], 7.6 b)). Another application of those axioms is that they enable one to determine explicitly the functor $E_S^{(pid)}$ (whereas the more abstract definition by generators and relations is much less manageable), and that the result one obtains suggests (always in the affine case) what must be the functor G_S for arbitrary rings. We shall come back to that question in the next section (and in Appendix 2), but let us first conclude the present one by two remarks.

The above considerations can be developed both at the minimal and the formal level. In fact, the construction of [Ti3] depends on the choice of a certain subgroup X (subject to some simple conditions) of the multiplicative group of the completed universal enveloping algebra of the Lie algebra generated by the e_j's . Among the possible X , there is a minimal one, leading to the minimal group $G_S(\underline{\underline{C}})$ (and functor G_S), and a maximal one (which has been determined by O. Mathieu [Ma2]), leading to the formal group (and functor) associated with S .

The groups we have been considering are the generalizations, in the Kac-Moody framework, of the split reductive groups but, as in the finite-dimensional case, one can define non-split forms of those groups. In particular, over $\underline{\underline{R}}$, there is a "compact" form (which is by no means compact in the topological sense !): in the minimal set-up, it can be defined as the fixed-point group of the "anti-analytic" involution of $G_S(\underline{\underline{C}})$ induced by the semi-linear involution of the Kac-Moody algebra which permutes e_i and f_i and inverts the elements of Λ . (Another definition, involving hermitian forms in representation spaces, also works at the formal level: cf. [Ga], [KP]). Generalizing a result which was known in the finite-dimensional case [Ka1], V. Kac [KDP] has observed that that compact form can be defined as the amalgamated product of its rank 2 subgroups corresponding to the pairs of indices $i,j, \in \{1,\dots,\ell\}$, with amalgamation of the rank 1 subgroups (of type SU_2) corresponding to the indices (here, one must assume that the h_i's form a basis of Λ, or add a compact torus to the amalgam). As for the rank 2 groups, which are the ingredients of that definition, they are known in case they are finite-dimensional (i.e. when $\alpha_i(h_j).\alpha_j(h_i) \leqq 3$) ; otherwise, they are shown to be amalgamated products of two groups of type U_2 with suitable amalgamation of a two-dimensional torus. (N.B. A result similar to the above is known to hold for <u>finite-dimensional</u> <u>split</u> groups or, more generally, for groups having a BN-pair with finite Weyl group: cf. [Ti2], 13.32, and, for earlier versions and special cases, [Cu] and [Ti1], 2.12.)

The fact that the definition by generators and relations does not provide the "good" functor G_S for rings that are not principal ideal domains probably lies in the nature of things (as K-theory suggests). A more likely way to get at the "right" G_S would consist in exhibiting a suitable $\underline{\underline{Z}}$-form of the affine algebra of $G_S(\underline{\underline{C}})$ (cf. § 8 below).

7. <u>An example: groups of type</u> ${}^2\tilde{E}_6$.

In this section, we adopt the <u>formal</u> viewpoint; to emphasize the fact, we shall use the notation $\hat{G}_S$, instead of G_S as above.

Let S be the system $(\Lambda,(\alpha_j,h_j)_{0\leqq j\leqq 4})$, where the matrix $(\alpha_j(h_i))$ is of type ${}^2\tilde{E}_6$ (cf. § 3), and where the α_i generate Λ^* whereas the h_i generate Λ : these properties characterize S . Our purpose is to describe the groups $\hat{G}_S(K)$ when K is a <u>field</u>. We discuss only this special example for the sake of concreteness, but similar results hold for any other twisted type (the type

is briefly examined in [Ti4], 7.4 , and general statements, concerning all affine types and arbitrary rings, will be given in Appendix 2).

From the explicit description of the Kac-Moody algebras of type ${}^2\tilde{E}_6$ given in § 3, one readily guesses what must be the group $\hat{G}_S(K)$ when K is a field <u>of</u> <u>characteristic</u> <u>not</u> 2, namely

$$\hat{G}_S(K) = \mathfrak{G}(K((z))) ,$$

where $\mathfrak{G}$ is a quasi-split algebraic group of type ${}^2E_6$, defined over $K((z))$ and whose splitting field is $K((\sqrt{z}))$. If $K = \underline{\underline{C}}$, one proves this by straightforward "integration" (for arbitrary affine types, this part of the work is done in [Mo3]), and the general case ensues via the axiomatic method mentioned above (cf. § 6 and [Ti4], 7.6 b)).

Now, suppose that car $K = 2$. The above description cannot hold in that case since the extension $K((\sqrt{z})/K((z))$ is not separable, hence is improper for the definition of a quasi-split group. But there is a circumstance which enables one again to guess the result, at least when K is perfect. Indeed, one knows that, in the finite-dimensional theory, the arrows carried by the double bonds of Dynkin diagrams "disappear" over perfect fields of characteristic 2: more precisely, reversing such an arrow corresponds to an inseparable isogeny which is bijective on rational points. Here, the diagram becomes "the same as" F_4 =  , hence the (correct) guess

$$G_S(K) = F_4(K((z))) .$$

But how can it be that a 78-dimensional (quasi-split) group of type E_6 suddenly degenerates into a 52-dimensional (split) group of type F_4 ?! The answer is simple: $F_4(K((z)))$ must be viewed as the group of rational points of a suitable 78-dimensional group defined over $K((z))$. The existince of such a group is not so surprising when one considers the isomorphism $(a,b) \longmapsto a^2+zb^2$ of $K((z))\times K((z))$ onto $K((z))$, hence of a 2-dimensional group onto a 1-dimensional group (K perfect).

To be more specific, set $L = \underline{\underline{F}}_2((z))$, $L' = \underline{\underline{F}}((\sqrt{z}))$ and denote by $\mathcal{F}$ a split group of type F_4 over L' , by $\sigma : \mathcal{F} \longrightarrow \mathcal{F}$ a special isogeny of $\mathcal{F}$ into itself whose square is the Frobenius endomorphism (cf. [BT1], 3.3), by $R_{L'/L}$ the restriction of scalars and by $\mathfrak{G}$ the image of $R_{L'/L}\sigma : R_{L'/L}\mathcal{F} \longrightarrow R_{L'/L}\mathcal{F}$. Then

the algebraic group $\mathfrak{G}$ _is_ 78-_dimensional_. _For any_ (_non necessarily perfect_) _field_ K _of characteristic_ 2, _one has_ $G_S(K) = \mathfrak{G}(K((z)))$;

<u>if</u> K <u>is perfect, the map</u> $R_{L'/L}\sigma : R_{L'/L}\mathcal{F} \longrightarrow \mathfrak{G}$ <u>is bijective on rational points, therefore</u> $G_S(K) \cong (R_{L'/L}\mathcal{F})(K((z))) = \mathcal{F}(K((\sqrt{z}))) \cong \mathcal{F}(K((z)))$.

Let us explain briefly where the 78 dimensions of $\mathfrak{G}$ come from. The group F_4 has an open set Ω which is the product, in a suitable order, of 48 additive groups $\mathfrak{U}_a$ corresponding to the 48 roots a and a 4-dimensional torus $\mathfrak{T}$. The isogeny σ induces a bijection $a \longmapsto \sigma(a)$ of the root system into itself which maps short roots onto long roots and vice versa. The groups $R_{L'/L}\mathfrak{U}_a$ are 2-dimensional and $\dim R_{L'/L}\mathfrak{T} = 8$. Now, it is readily checked that:

if a is short, $R_{L'/L}\sigma$ maps $R_{L'/L}\mathfrak{U}_a$ isomorphically onto $R_{L'/L}\mathfrak{U}_{\sigma(a)}$;

if a is long, $R_{L'/L}\sigma$ maps $R_{L'/L}\mathfrak{U}_a$ onto a one-dimensional subgroup of $R_{L'/L}\mathfrak{U}_{\sigma(a)}$;

$R_{L'/L}\sigma$ maps $R_{L'/L}\mathfrak{T}$ onto a six-dimensional subtorus of itself.

Thus, $\dim (R_{L'/L}\sigma)(\Omega) = 2.24 + 24 + 6 = 78$.

We propose the following exercise to the interested reader: for K perfect of characteristic 2, write $SL_2(K((z)))$ as the group of rational points on $K((z))$ of an 8-dimensional algebraic group. This arises when one studies the case of the GCM $\begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix}$; the 8-dimensional group in question appears as a characteristic 2 "degeneracy" of $SU_3(K((\sqrt{z})))$. Cf. [Ti4], 7.4, for more details.

8. <u>The algebro-geometric nature of the groups</u> $G_S(\underline{\underline{C}})$.

What kind of algebro-geometric objects are the functors G_S and, in particular, the groups $G_S(\underline{\underline{C}})$? Little is known for G_S is general, but something can be said about $G_S(\underline{\underline{C}})$ (here, $\underline{\underline{C}}$ could be replaced by any field of characteristic zero).

Set $G = G_S(\underline{\underline{C}})$. We have already mentioned the Bruhat decomposition $G = \bigcup BwB$, where B is a certain subgroup of G , which we may call "Borel subgroup", and w runs over a Coxeter group W . Coxeter groups, endowed as usual with a distinguished generating set $S = \{r_1,\ldots,r_\ell\}$ (cf. § 4), have a natural ordering: for $w,w' \in W$, one sets $w \geq w'$ if there exists a reduced expression $w = s_1 \ldots s_n$ $(s_i \in S)$ and a subsequence $(i_1,\ldots,i_m)$ of $(1,\ldots,n)$ such that $w' = s_{i_1} \ldots s_{i_m}$. Then:

<u>for any</u> $w \in W$, <u>the subset</u> Schub $w = \bigcup_{w' \leq w} (Bw'B)/B$ of G/B , <u>called the</u> Schubert variety <u>corresponding to</u> w , <u>has a natural structure of projective manifold</u> (cf. [Ti4]); <u>thus</u>, G/B <u>is a direct limit of projective manifolds</u>.

(In [Ti4], the projective structure of Schub w is defined by means of a highest weight representation of G , and is then shown not to depend on the choice of that representation. It would be desirable to have a more intrinsic definition, using for instance the big cell of the Birkhoff decomposition, as was suggested to the speaker by G. Lusztig.) The set G/B , and its description as a limit of projective varieties, does not depend on whether one adopts the minimal or the formal viewpoint (more precisely, the formal group is the completion of the minimal one for a topology for which B is an open subgroup). Also, since B contains $H = B \cap N$, the quotient G/B and the varieties Schub w depend only on the GCM $\underline{\underline{A}}$, and not on Λ ; when the choice of $\underline{\underline{A}}$ needs to be specified, we shall write $\mathrm{Schub}_{\underline{\underline{A}}} w$ instead of Schub w .

If we now take the formal viewpoint, the Borel subgroup, or rather, to remain consistent with the notation of the end of § 4, the closure $\overline{B}$ of B in $\hat{G} = \hat{G}_S(\underline{\underline{C}})$, is a proalgebraic group, semi-direct extension of a torus by a prounipotent group (cf. [BT2], [Sℓ]).

Having thus described both $G/B = \hat{G}/\overline{B}$ and $\overline{B}$, we have gained some understanding of the algebro-geometric nature of $\hat{G}$ itself. But a more direct and promising picture is given by V. Kac and D. Peterson [KP 1] who attach to G (the minimal group) a "coordinate ring", or rather two rings, the ring $\underline{\underline{C}}[G]$ of "strongly regular" functions, and the ring $\underline{\underline{C}}[G]_r$ of "regular" functions. The first one is generated by the coefficients of all highest weight representations (in [KP1], this is not chosen as definition of $\underline{\underline{C}}[G]$ but is proved to be a property of the ring, defined in a different way) and provides a Peter-Weyl type theorem. That ring is not invariant by the map $i : g \longmapsto g^{-1}$ (under that map, highest weight representations become lowest weight representations !); for a suitable topology, $\underline{\underline{C}}[G]_r$ is topologically generated by $\underline{\underline{C}}[G]$ and $i^*(\underline{\underline{C}}[G])$. It is shown in [KP] that G is an affine (infinite-dimensional) algebraic group with coordinate ring $\underline{\underline{C}}[G]_r$, in the sense of Shafarevitch [Sh] : this implies, in particular, that G can be identified with a subset of $\underline{\underline{C}}^{\underline{\underline{N}}}$ in such a way that $\underline{\underline{C}}[G]_r$ is the restriction to G of the ring $\underline{\underline{C}}[\underline{\underline{C}}^{\underline{\underline{N}}}]_r$ of regular functions on $\underline{\underline{C}}^{\underline{\underline{N}}}$ (i.e. the ring of functions whose restriction to $\underline{\underline{C}}^{[0,n]}$ is polynomial for all n), and that G is the vanishing set of an ideal of $\underline{\underline{C}}[\underline{\underline{C}}^{\underline{\underline{N}}}]_r$.

9. Applications.

"Kac-Moody groups" have been used in a variety of domains such as topology, differential and partial differential equations, singularity theory, etc. Those applications, a fast growing subject, are beyond both the scope of this survey and the competence of the speaker. Let me just unsystematically list a few basic references, which will give access to at least part of the literature on that topic: [SW] (cf. also the reference[5] of [SW]), [Ve1], [Ve2] (these concern applications of Kac-Moody Lie algebras, rather than groups), [RS], [Sℓ].

Most applications so far use only groups of affine type, and there may still be doubts about the usefulness of the general theory. To finish with, I would like to give an argument in favour of it. We have seen that to every GCM $\underline{\underline{A}} = (\underline{\underline{A}}_{ij})$ and every element of the corresponding Coxeter group $W(\underline{\underline{A}})$, the theory associates a certain complex projective variety $\mathrm{Schub}_{\underline{\underline{A}}}w$. If w is one of the canonical generators r_i of W , $\mathrm{Schub}_{\underline{\underline{A}}}w$ is just a projective line. The next simple case is $w = r_ir_j$; then, $\mathrm{Schub}_{\underline{\underline{A}}}w$ is a rational ruled surface, i.e. a surface fibered over $\underline{\underline{P}}_1(\underline{\underline{C}})$ with projective lines as fibers. It is well known that such a surface X is characterized up to isomorphism by a single invariant $\nu(X)$ which is a negative integer (if $\nu(X) \neq 0$, X is obtained by blowing up the vertex of a cone of degree $\nu(X)$ in a $(\nu(X)+1)$-dimensional projective space). Now, one shows that $\nu(\mathrm{Schub}_{\underline{\underline{A}}}(r_ir_j)) = A_{ij}$. This gives a geometric interpretation of the matrix $\underline{\underline{A}}$. Moreover, observe that, if one accepts only to consider GCM of affine type, only the surfaces X with $\nu(X) \in [-4,0]$, among the rational ruled surfaces, have the right to be called "Schubert varieties", which seems rather unnatural ! I should think that the class of all $\mathrm{Schub}_{\underline{\underline{A}}} w$, for all $\underline{\underline{A}}$ and w , will turn out to be a very natural and interesting class of projective varieties to consider.

Appendix 1. Central extension.

For arbitrary S, the "minimal group" $G_S(\underline{\underline{C}})$ can be constructed by the methods described in §§ 5 and 6. In particular, those methods provide very simple, purely formal existence proofs for a nontrivial central extension of the "polynomial" loop groups by $\underline{\underline{C}}^{\times}$. The situation is quite different when one starts from loop groups defined by analytic conditions. However, the following rather trivial considerations may conceivably enable one to exploit the result known for polynomial loops in the analytic case. Here, all topological spaces are assumed to be Hausdorff.

Let $\pi : G' \longrightarrow G$ be a central group extension and let U'_-, H', U'_+ be three subgroups of G' such that $\operatorname{Ker}\pi \subset H'$, that H' normalizes $U'_\pm$ and that the product mapping $U'_- \times H' \times U'_+ \longrightarrow G'$ is injective. Thus, $\pi_+ = \pi|_{U'_+}$ and $\pi_- = \pi|_{U'_-}$ are isomorphisms of U'_+ and U'_- onto two subgroups U_+ and U_- of G. We set $H = \pi(H')$.

Now, let us embed G in a complete topological group $\hat{G}$ and suppose that, if $\overline{U}_-$ and $\overline{U}_+$ denote the closures of U_- and U_+ in $\hat{G}$, the product mapping in $\hat{G}$ defines a homeomorphism of $\overline{U}_- \times H \times \overline{U}_+$ onto a dense open subset Ω of $\hat{G}$. Let us also endow H' with a topology making it into a complete topological group, such that $\operatorname{Ker}\pi$ is closed in H' and that the canonical algebraic isomorphism $H'/\operatorname{Ker}\pi \longrightarrow H$ is an isomorphism of topological groups as well (observe that, by hypthesis, H is locally closed, hence closed in $\hat{G}$, and is therefore a complete topological group).

Set $X = \{(u,u') \in U_+ \times U_- \mid uu' \in \Omega\}$. This is a dense open subset of $U_+ \times U_-$ (endowed with the topology induced by that of $\overline{U}_+ \times \overline{U}_-$). For $(u,u') \in X$, there is a unique element $\varphi(u,u') \in H'$ such that

$$\pi_+^{-1}(u) \cdot \pi_-^{-1}(u') \in U'_- \cdot \varphi(u,u') \cdot U'_+ \quad .$$

The topology of $\hat{G}$ induces a topology on $U_\pm$ which we lift to $U'_\pm$ by means of $\pi_\pm^{-1}$, and we endow $\Omega' = U'_- H' U'_+$ with the product topology. The following proposition is easy.

PROPOSITION 2. *If the function $\varphi : X \longrightarrow H'$ is continuous, there is a unique topology on G' making G' into a topological group and Ω', topologized as above, into a dense open subset of G'. Suppose further that there is a neigborhood $\overline{X}_1$ of (1.1) in $\overline{U}_+ \times \overline{U}_-$ such that the restriction of φ to $X \cap \overline{X}_1$ extends to a continuous map $\overline{X}_1 \longrightarrow H'$. Then, the topological group G' admits a completion $\hat{G}'$, $\operatorname{Ker} \pi$ is a closed subgroup of $\hat{G}'$ and the homomorphism $\hat{G}' \longrightarrow \hat{G}$ extending π factors through an isomorphism of topological groups $\hat{G}'/\operatorname{Ker}\pi \longrightarrow \hat{G}$.*

Note that the left (or right) translates of all open subsets of Ω obviously form a basis of the topology of G' (hence the uniqueness assertion).

In the application I have in mind, G would be a "polynomial" loop group, $\hat{G}$ some other loop group, $\pi : G' \longrightarrow G$ the "natural" central extension of G by $\operatorname{Ker}\pi \cong \underline{\underline{C}}^\times$ (whose existence is easily proved by any of the methods described in §§ 5 and 6), U'_- and U'_+ the (non complete) "prounipotent radicals" of two opposite Borel subgroups of G' (cf. § 4) and H' the intersection of those Borel subgroups, a direct product of copies of $\underline{\underline{C}}^\times$ which one endows with its natural topology. The main problem, which I have not investigated, is of course to prove (in the interesting cases) that φ is continuous and extends to a neighborhood of (1,1) in $\overline{U}_+ \times \overline{U}_-$.

Appendix 2. The group functor $\hat{G}_S$ in the affine case.

In this appendix, we shall use the techniques and terminology of [BT4] to describe the formal functors $\hat{G}_S$ for all systems

$$S = (\Lambda, (\alpha_i)_{0 \leq i \leq \ell}, (h_i)_{0 \leq i \leq \ell})$$

satisfying the following conditions:

(A1) the matrix $\underline{\underline{A}} = (\alpha_j(h_i))$ is of irreducible, affine type;

(A2) the set $\{h_i \mid 0 \leq i \leq \ell\}$ generates Λ;

(A3) the set $\{\alpha_i \mid 0 \leq i \leq \ell\}$ contains a $\underline{\underline{Q}}$-basis of $\underline{\underline{Q}} \otimes \Lambda^*$.

More precisely, for any such S , we shall describe a topological group functor $\hat{G}_S$ having the following properties.

(P0) There is a Lie algebra functor $\mathrm{Lie}\,\hat{G}_S$ defined as follows (compare [DG], pp. 209-210). For any ring R , set $R(\varepsilon,\varepsilon') = R[t,t']/(t^2,t'^2)$, where ε,ε' are the canonical images of t,t' in the quotient; in other words, $R(\varepsilon,\varepsilon')$ is the tensor product of two algebras $R(\varepsilon)$, $R(\varepsilon')$ of dual numbers. For $r \in R$, let $\pi : R(\varepsilon) \longrightarrow R$, $\iota : R(\varepsilon) \longrightarrow R(\varepsilon')$, $\sigma : R(\varepsilon) \longrightarrow R(\varepsilon,\varepsilon')$ and $\mu_r : R(\varepsilon) \longrightarrow R(\varepsilon)$ be the R-homomorphisms sending ε onto $0,\varepsilon',\varepsilon\varepsilon'$ and $r\varepsilon$ respectively. Then, the additive group $(\mathrm{Lie}\,\hat{G}_S)(R)$ is the kernel of the homomorphism

$$\hat{G}_S(\pi) \; : \; \hat{G}_S(R(\varepsilon)) \longrightarrow \hat{G}_S(R) \;,$$

the scalar multiplication by r is induced by the automorphism $\hat{G}_S(\mu_r)$ of $\hat{G}_S(R(\varepsilon))$ and the commutator of two elements $x,y \in (\mathrm{Lie}\,\hat{G}_S)(R) \subset \hat{G}_S(R(\varepsilon))$ is the only element $[x,y]$ such that

$$\hat{G}_S(\sigma)([x,y]) = (x,\hat{G}_S(\iota)(y))$$

where $(\ ,\ )$ stands for the usual commutator in the group $\hat{G}_S(R(\varepsilon,\varepsilon'))$.

(P1) $(\mathrm{Lie}\,\hat{G}_S)(\underline{\underline{C}})$ is the Kac-Moody algebra associated to the system $(\underline{\underline{C}} \otimes \Lambda,\ (\alpha_i)_{0 \leqq i \leqq \ell}, (h_i)_{0 \leqq i \leqq \ell})$ completed with respect to the natural gradation $(\deg e_i = 1,\ \deg f_i = -1,\ \deg h_i = 0)$.

(P2) The group $\hat{G}_S(\underline{\underline{C}})$ coincides with the formal group over $\underline{\underline{C}}$ attached to S by any one of the construction processes described in §§ 5 and 6; in particular, it contains (a canonical image of) $\mathrm{Hom}(\Lambda^*,\underline{\underline{C}}^\times)$ and its center consists of all $\xi \in \mathrm{Hom}(\Lambda^*,\underline{\underline{C}}^\times)$ such that $\xi(\alpha_i) = 0$ for all i .

(P3) Modulo its center, $G_S(\underline{\underline{C}})$ is the subgroup of $\mathrm{Aut}((\mathrm{Lie}\,\hat{G}_S)(\underline{\underline{C}}))$ generated by all converging exp ad g , with $g \in (\mathrm{Lie}\,\hat{G}_S)(\underline{\underline{C}})$ (this turns out to be identical with the adjoint group considered by R.V. Moody [Mo1] and J.I. Morita [Mo3]; about this group, cf. also the last sentence of this appendix).

(P4) The functor $\hat{G}_S$ restricted to principal ideal domains, together with suitably defined functorial homomorphisms $\sigma_i : SL_2 \longrightarrow \hat{G}_S$, $\eta : \mathrm{Hom}(\Lambda^*, ?^\times) \longrightarrow \hat{G}_S$ (which we leave as an exercise to determine explicitly), satisfies the axioms (i') to (iv') of [Ti4], 7.5, and is characterized by them, once $\hat{G}_S(\underline{\underline{C}})$, $\sigma_i(\underline{\underline{C}})$, $\eta(\underline{\underline{C}})$ are given.

Those properties clearly indicate that the functor $\hat{G}_S$ which we are going to define is the "right one", at least when restricted to principal ideal domains but maybe also for general rings, considering its fairly simple and natural definition (though it is conceivable that some algebro-geometric invariants of the ring, such as Pic R , should be brought into play).

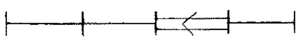
Let e be an integer and let G be a quasi-split, simply connected absolutely almost simple group defined over the field $K = \underline{\underline{Q}}(Z)$, whose splitting field over K is generated by the e-th roots of Z ; thus, $e = 1, 2$ or 3 and, in the latter case, G is of type D_4 . Let S be a maximal split torus of G , Φ the system of roots of G with respect to S , $\underline{\underline{\Phi}}$ the system of root rays ("rayons radiciels": cf. [BT4], 1.1.2), i.e. of half-lines $\underline{\underline{R}}_+ . a$ with $a \in \Phi$, T the centralizer of S in G and U_a (for $a \in \underline{\underline{\Phi}}$) the root subgroup corresponding to a . We also denote by $G, S, T, \ldots,$ the groups of K-rational points of $G, S, T, \ldots$.

Let now $\mathcal{S} = (\Lambda, (\alpha_i)_{0 \leq i \leq \ell}, (h_i)_{0 \leq i \leq \ell})$ be defined as follows:

$\Lambda = X_*(S) = \mathrm{Hom}(\mathrm{Mult}, S)$ is the group of cocharacters of S ;

$(\alpha_1, \ldots, \alpha_\ell)$ is a basis of Φ and $-\alpha_0$ is the maximal root if $e = 1$ or if $e = 2$ and G is of type A_{2n} , and it is the maximal "short" root in the remaining cases; h_i is the coroot associated α_i .

Varying e and the type of G , one gets all systems $\mathcal{S}$ satisfying the conditions (A1) to (A3) above in this way. If G has type X , we say that $\mathcal{S}$ has type ${}^{e}\tilde{X}$. The Dynkin diagram representing the GCM $(\alpha_j(h_i))$ is given by the following table:

type of $\mathcal{S}$	diagram	
${}^{1}\tilde{X}$	extended Dynkin diagram of X	
${}^{2}\tilde{A}_{2n}$		(n+1 vertices)
${}^{2}\tilde{A}_{2n-1}$		(n+1 vertices)
${}^{2}\tilde{D}_{n}$		(n vertices)
${}^{2}\tilde{E}_{6}$		
${}^{3}\tilde{D}_{4}$		

We shall now choose a system of "épinglages" of the U_a's (cf. [BT4], 4.1). This is a system $(x_a)_{a\in\Phi}$ where, for all a , x_a is one of three things:

(i) an isomorphism $K \longrightarrow U_a$;

(ii) an isomorphism $K(Z^{1/e}) \longrightarrow U_a$ (here $Z^{1/e}$ denotes any e-th root of Z and, when $e = 3$, all cubic roots involved may be chosen equal);

(iii) an isomorphism $H \longrightarrow U_a$, where H is the product $K(Z^{1/2}) \times Z^{1/2}\cdot K$ endowed with the group structure

$$(*) \qquad (u,v)\cdot(u',v') = (u+u',\ v+v'+(u^{\sigma}u' - u'^{\sigma}u))$$

in which σ represents the nontrivial K-automorphism of $K(Z^{1/2})$ (observe that H is nothing else but the group H^{λ} of [BT4], 4.1.15, for $\lambda = 1/2$,) transformed by the automorphism $(x,y) \longmapsto (x,2y)$ of the underlying variety).

In all cases except ${}^{2}\tilde{A}_{2n}$ (i.e. when G is of type A_{2n} and

$e = 2$), we take for (x_a) a coherent system of "épinglages" deduced from a Chevalley-Steinberg system (cf. [BT4], 4.1.16). In order to describe the system (x_a) in the case ${}^2\tilde{A}_{2n}$, let us choose an orthogonal basis $(a_i)_{1 \leq i \leq n}$ of the (relative) root lattice: thus, $\Phi = \{\pm a_i, \pm 2a_i, \pm a_i \pm a_j \text{ with } i \neq j\}$. For $a \in \underline{\underline{\Phi}}$, let θ_a denote the automorphism of the source of x_a defined as follows: if a contains $a_i + a_j$ (resp. $a_i - a_j$; resp. $-a_i - a_j$), $\theta_a(k) = 2k$ (resp. $k/2$) and if a contains a_i (resp. $-a_i$), $\theta_a(u,v) = (2u,4v)$ (resp. (u,v)). Finally, we set $x_a = x'_a \circ \theta_a$, where $(x'_a)_{a \in \underline{\underline{\Phi}}}$ is a coherent system of "épinglagles", as in loc. cit.

Let us now describe a certain schematic root datum $(\mathfrak{T}, (\mathfrak{U}_a)_{a \in \underline{\underline{\Phi}}})$ in G over the ring $K = \underline{\underline{Z}}[Z, Z^{-1}]$ (cf. BT4 , 3.1.1). The scheme is the "canonical group-scheme associated with the torus T" , defined as in [BT4], 4.4.5 (as in [BT4], it can be shown that $\mathfrak{T}$ does not depend on the way T is expressed as a product of tori of the form $\prod_{L/K} \mathfrak{Mult}_L$) and the scheme $\mathfrak{U}_a$ is the "image by x_a "of:

the additive group-scheme canonically associated with the module $\mathfrak{K}$ (resp. $\mathfrak{K}[Z^{1/e}]$) in case (i) (resp. (ii)) (cf. [BT4], 1.4.1);

the group-scheme whose underlying scheme is canonically associated with the module $\mathfrak{H} = \mathfrak{K}[Z^{1/2}] \times Z^{1/2}\cdot\mathfrak{K}$ and whose product operation is given by (*) in case (iii).

It is readily verified, using the appendix of [BT4], that the system $(\mathfrak{T}, (\mathfrak{U}_a))$ is indeed a schematic root datum. By Section 3.8.4 of [BT4], there exists a unique smooth connected group-scheme $\mathfrak{G}$ with generic fibre G containing the direct product

$$\prod_{a \in \underline{\underline{\Phi}}_-} \mathfrak{U}_a \times \mathfrak{T} \times \prod_{a \in \underline{\underline{\Phi}}_+} \mathfrak{U}_a$$

as an open subscheme ("big cell") (here, $\underline{\underline{\Phi}}_+ \subset \underline{\underline{\Phi}}$ denotes a system of positive root rays and $\underline{\underline{\Phi}}_- = -\underline{\underline{\Phi}}_+$). Finally, S being as above, the announced functor $\hat{G}_S$ is defined by

$$\hat{G}_S(R) = \mathfrak{G}(R((Z))) ,$$

this group being given the natural topology, induced by that of $R((Z))$.

Suppose now that R is a perfect field of characteristic e (which implies that $e = 2$ or 3). There is a "natural" isomorphism of each $\mathfrak{U}_a(R((Z)))$ onto $R((Z))$, namely

x_a^{-1} in case (i),

$x_a(r) \longmapsto r^e$ in case (ii),

$x_a(r,r') \longmapsto r'^2 + r^4$ in case (iii),

and $\mathfrak{T}(R((Z)))$, which is a product of groups of the form $R((Z))^\times$ and $R((Z^{1/e}))^\times$, is clearly isomorphic to the group $\mathfrak{T}'(R((Z)))$ of rational points of a split torus $\mathfrak{T}'$. It is then readily verified (using [BT3], § 10, and the appendix of [BT4]), that, via those isomorphisms, the system $(\mathfrak{T}(R((Z))), (\mathfrak{U}_a(R((Z))))_{a \in \underline{\underline{\Phi}}})$ "is" the standard root datum of the group of rational points of an $R((Z))$-split simple group of type

C_n if $\underline{\underline{A}} = (\alpha_j(h_i))$ has type ${}^2\tilde{A}_{2n}$,

B_n if $\underline{\underline{A}}$ has type ${}^2\tilde{A}_{2n-1}$,

C_{n-1} if $\underline{\underline{A}}$ has type ${}^2\tilde{D}_n$,

F_4 if $\underline{\underline{A}}$ has type ${}^2\tilde{E}_6$,

G_2 if $\underline{\underline{A}}$ has type ${}^3\tilde{D}_4$.

This is the phenomenon already mentioned in § 7 for the special case of type ${}^2\tilde{E}_6$.

Let us return to the group-scheme $\mathfrak{G}$. In the classical cases ${}^2\tilde{A}_m$ and ${}^2\tilde{D}_n$, it can be given a more direct and more elementary description. Here, we shall only briefly treat the types ${}^2\tilde{A}_m$ (the case of ${}^2\tilde{D}_n$ is slightly more complicated because one must work with the spin group). According as $m = 2n-1$ or $2n$, set $I = \{\pm 1, \pm 2, \ldots, \pm n\}$ or $I = \{0, \pm 1, \ldots, \pm n\}$. Let V be the $K[Z^{1/2}]$-module $(K[Z^{1/2}])^I$ endowed with a coordinate system $\underline{\underline{z}} = (z_i)_{i \in I}$, let τ denote the K-automorphism of $K[Z^{1/2}]$ defined by $\tau(Z^{1/2}) = -Z^{1/2}$ and consider the hermitian form

$$h(\underline{\underline{z}};\underline{\underline{z}}') = \Sigma(z'^{\tau}_{-i} z_i + z'^{\tau}_i z_{-i}) ,$$

where i runs from 1 to n or from 0 to n according as $m = 2n-1$ or $2n$. We represent by V_K the module V considered as a K-module; in it, we use the coordinate system $(\underline{\underline{x}},\underline{\underline{y}}) = (x_i, y_i)_{i\in I}$, where $x_i, y_i \in K$ and $z_i = x_i + y_i \cdot Z^{1/2}$. Separating the "real and imaginary parts" of h, we get $h = s + Z^{1/2} \cdot a$, where s and a are a symmetric and an alternating bilinear form in V_K respectively. Similarly, the determinant in $\mathrm{End}\, V$ can be written $\det_0 + Z^{1/2} \cdot \det_1$, where $\det_0$ and $\det_1$ are K-polynomials in $\mathrm{End}\, V$ considered as a K-module. Let q be the quadratic form $q(\underline{\underline{x}},\underline{\underline{y}}) = \frac{1}{2}s(\underline{\underline{x}},\underline{\underline{y}};\underline{\underline{x}},\underline{\underline{y}})$ in V_K. The multiplication by $Z^{1/2}$ is an automorphism J of the K-module V_K. Finally, the group-scheme $\mathbb{G}$ (corresponding to the type ${}^2\tilde{A}_m$) can be described as the subgroup-scheme of $\mathbb{GL}(V_K)$ defined by the equations $g \cdot a = a$, $g \cdot q = q$ (hence $g \cdot s = s$), ${}^gJ = J$, $\det_0 g = 1$, $\det_1 g = 0$. In other words, if $\mathbb{R}$ is a K-algebra, $\mathbb{G}(\mathbb{R})$ is the subgroup of all elements of $SL(V \otimes \mathbb{R}[Z^{1/2}])$ preserving the ($\mathbb{R}$-valued) "forms" a and q. (For the case $m = 2n$, see [Ti4], 7.4.)

Now, consider again the case $\mathbb{R} = R((Z))$, where R is a perfect field of characteristic 2 (in fact, any ring $\mathbb{R}$ such that the map $x \longmapsto x^2$ is a bijection of $\mathbb{R}[Z^{1/2}]$ onto $\mathbb{R}$ would do). Let V' (resp. V'') denote the $\mathbb{R}[Z^{1/2}]$-module, product of $2n+1$ (resp. $2n$) factors $\mathbb{R}[Z^{1/2}]$ indexed by $\{0, \pm 1, \ldots, \pm n\}$ (resp. $\{\pm 1, \ldots, \pm n\}$). In those modules, we use again coordinates z_i where i runs through the same index sets. In V', consider the quadratic form $q'(\underline{\underline{z}}) = \sum_{i=1}^{n} z_{-i}z_i + Z^{1/2} \cdot z_0^2$, and in V'', the alternating bilinear form $a'(\underline{\underline{z}};\underline{\underline{z}}') = \sum_{i=1}^{n} (z'_{-i}z_i - z'_i z_{-i})$. If $m = 2n-1$, $V \otimes \mathbb{R}[Z^{1/2}]$ can be identified with V'', hence with a quotient of V', the "bilinearization" and the "real part" (K-part) of q' are the inverse images in V' of the "forms" $a_{\mathbb{R}}$ and $q_{\mathbb{R}}$ (with obvious notational conventions), and it is easy to verify that the projection $V' \longrightarrow V''$ induces an isomorphism $SO(q') \xrightarrow{\sim} \mathbb{G}(\mathbb{R})$. If $m = 2n$, $V \otimes \mathbb{R}[Z^{1/2}]$ can be identified with V', the bilinear form $h_{\mathbb{R}[Z^{1/2}]}$ is the inverse image of a' by the projection $V' \longrightarrow V''$ and, this time, the latter induces an isomorphism $\mathbb{G}(\mathbb{R}) \xrightarrow{\sim} Sp(a')$. Thus we have found again the two isomorphisms obtained earlier in a different way.

The description of the functor $\hat{G}_S$ associated to an <u>arbitrary</u> system S of affine type, i.e. a system satisfying (A1) but not necessarily (A2) and (A3) now amounts to a combination of extension problems. In particular, when $\Lambda = \coprod \underline{\underline{Z}}.h_i$, one must define a central extension of the above functor $\hat{G}$ by the multiplicative group-scheme $\mathfrak{Mult}$ [1]; this is related to work of C. Moore [Mo2], H. Matsumoto [Ma3] and P. Deligne [De1]. Note that if, with the notation used throughout this appendix, we assume $e = 1$, we denote by S_{ad} the system obtained in the same way as S but replacing Λ by the dual of the lattice of roots and by $\mathfrak{G}_{ad}$ the split <u>adjoint</u> group-scheme of the same type as G , then the functor $\hat{G}_{S_{ad}}$ is <u>not</u> equal to $R \longmapsto \mathfrak{G}_{ad}(R((Z)))$ in general; for instance, $\hat{G}_{S_{ad}}(\underline{\underline{C}})$ is the image of the canonical map

$$\mathfrak{G}(\underline{\underline{C}}((Z))) \longrightarrow \mathfrak{G}_{ad}(\underline{\underline{C}}((Z))) ,$$

whose cokernel is isomorphic to the center of G .

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[1] As P. Deligne pointed out to me, the word "extension" must be understood here in a "schematic sense"; one should not expect the extension map to be surjective for rational points over an arbitrary ring R .

REFERENCES

[Bi] G.D. BIRKHOFF, *A theorem on matrices of analytic function,* Math, Ann. 74 (1913), 122-133.

[Bo] N. BOURBAKI, *Groupes et Algébras de Lie,* Chap. IV, V, VI, Hermann, Paris, 1968.

[BT1] A. BOREL et J. TITS, *Homomorphismes "abstracts" de groupes algébriques simples*, Ann. of Math., 97 (1973), 499-571.

[BT2] F. BRUHAT et J. TITS, *Groups algébriques simples sur un corps local,* Proc. Conf. on local fields (Driebergen, 1966), Springer, 1967, 23-36.

[BT3] F. BRUHAT et J. TITS, *Groupes réductifs sur un corps local, I: Données radicielles valuées,* Publ. Math. I.H.E.S. 41 (1972), 5-251.

[BT4] F. BRUHAT et J. TITS, *Groupes réductifs sur un corps local, II: Schémas en groupes, Existence d'une donnée radicielle valuée,* Publ. Math. I.H.E.S. 60 (1984), 5-184.

[Ch1] C. CHEVALLEY, *Sur certains groupes simples,* Tohoku Math. J. (2)7 (1955), 14-66.

[Ch2] C. CHEVALLEY, *Séminaise sur la classification des groupes de Lie algébriques,* 2 vol., Inst. H. Poincaré, Paris, 1958, mimeographed.

[Ch3] C. CHEVALLEY, *Certains schémas de groupes semi-simples,* Sém. N. Bourbaki, exposé 219 (1980-1981), Benjamin, New York, 1966.

[Cu] C.W. CURTIS, *Groups with a Bruhat decomposition,* Bull. Amer. Math. Soc. 70 (1964), 357-360.

[D21] P. DELIGNE, *Extention de* G(K) *par* K_2(K), Séminaire I.H.E.S., 1979, non publié.

[De2] M. DEMAZURE, *Schémas en groupes réductifs,* Bull. Soc. Math. Fr. 93 (1965), 369-413; cf. aussi le *Séminaire de Géométrie algébrique, de* M. DEMAZURE er A. GROTHENDIECK, vol. 3, Springer Lecture Notes in Math. n° 153, 1970.

[DG] M. DEMAZURE et P. GABRIEL, *Groupes algébriques,* I, Masson, Paris, 1970.

[Ga] H. GARLAND, *The arithmetic theory of loop groups,* Publ. Math. I.H.E.S. 52 (1980), 5-136.

[GW] R. GOODMAN and N.R. WALLACH, *Structure and unitary cocyle representations of loop groups and the group of diffeomorphisms of the circle,* Jour. r. angew. Math. 347 (1984), 69-133.

[IM] N. IWAHORI and H. MATSUMOTO, *On some Bruhat decomposition and the structure of the Hecke ring of* p-*adic Chevalley groups*, Publ. Math. I.H.E.S. 25 (1965), 5-48.

[Ka1] V. KAC, *An algebraic definition of compact Lie groups*, Trudy M.I.E.M. 5 (1969), 36-47.

[Ka2] V. KAC, *Simple irreducible Lie algebras of finite growth*, Izvestija Akad. Nauk S.S.S.R. (ser. mat.) 32 (1968), 1923-1967; English transl., Math. USSR Izvestija 2 (1968), 1271-1311.

[Ka3] V. KAC, *Infinite dimensional Lie algebras*, Birkhäuser, Boston, 1983.

[KP1] V. KAC and D. PETERSON, *Regular functions on certain infinite dimensional groups*, *in* Arithmetic and Geometry (ed. M. ARTIN and J. TATE), Birkhäuser, Boston, 1983, 141-166.

[KP2] V. KAC and D. PETERSON, *Cohomology of infinite-dimensional groups and their flag varieties: algebraic part*, Colloque Elie Cartan, Lyon June 1984, to appear.

[Ma1] R. MARCUSON, *Tits's system in generalized non-adjoint Chevalley groups*, J. of Algebra 34 (1975), 84-96.

[Ma2] O. MATHIEU, *Sur la construction de groupes associés aux algêbres de Kac-Moody*, C.R. Acad. Sci. série I, 299 (1984), 161-164.

[Ma3] H. MATSUMOTO, *Sur les sous-groupes arithmétiques des groupes semi-simples déployés*, Ann. Sci. Ec. Normale Sup. 2 (4 - série) (1969), 1-62.

[Mo1] R.V. MOODY, *A simplicity theorem for Chevalley groups defined by generalized Cartan matrices*, preprint, 1982.

[Mo2] C. MOORE, *Group extensions of* p-*adic and adelic linear groups*, Publ. Math. I.H.E.S. 35 (1968), 5-70.

[Mo3] J. MORITA, *On adjoint Chevalley groups associated with completed Euclidean Lie algebras*, Comm. in Algebra 12 (1984), 673-690.

[MT] R. MOODY and K. TEO, *Tits systems with crystallographic Weyl groups*, J. Algebra 21 (1972), 178-190.

[RS] A. REIMAN and M. SEMENOV-TJAN-SHANSKII, *Reduction of Hamiltonian systems, affine Lie algebras and Lax equations*, I, Invent. Math.54 (1979), 81-100; II, Invent. Math.63 (1981), 423-432.

[SW] G. SEGAL and G. WILSON, *Loop groups and equations of KdV type*, Publ. Math. I.H.E.S. 61 (1985).

[Sh] I. SHAFAREVICH, *On some infinite dimensional groups*, II, Izvestija Akad. Nauk S.S.S.R. (ser. mat.) 45 (1981); English transl., Math. USSR, Izvestija 18 (1982), 214-226.

[Sl] P. SLODOWY, *Singularitäten, Kac-Moody Liealgebren, assoziierte Gruppen und Verallgemeinerungen*, Habilitationsschrift, Bonn, 1984.

[Ti1] J. TITS, *Groupes semi-simples isotropes*, Coll. théor. gr. alg., C.B.R.M. (Bruxelles, 1962), Librairie Universitaire, Louvain, et Gauthier-Villars, Paris, 1962, 137-147.

[Ti2] J. TITS, *Buildings of spherical type and finite BN-pairs*, Springer Lecture Notes in Math. n° 386 (1974).

[Ti3] J. TITS, Résumé de cours, Annaire du Collge de France, 81 e année (1980)-1981), 75-86.

[Ti4] J. TITS, Résumé de cours, Annuaire du Collège de France, 82 e an 82 e année (1981-1982), 91-105.

[Ve1] J.-L. VERDIER, *Algèbres de Lie, systèmes hamiltoniens, courbes algébriques* (*d'après* M. ADLER *et* P. van MOERBEKE), Sém. N. Bourbaki, exposé n° 566 (1980-1981), Springer Lecture Notes in Math. n° 901 (1982), 85-94.

[Ve2] J.-L. VERDIER, *Les représentations des algèbres de Lie affines: Applications à quelques problèmes de physique* (*d'après* E. DATE, M. JIMPB, M. KASHIWARA, T. MIWA), Sém. N. Bourbaki, exposé n° 596 (1981-1982), Astérisque 92-93 (1983), 365-377.

[Wi] R. WILSON, *Euclidean Lie algebras are universal central extensions*, *in* Lie algebras and related topics, Springer Lecture Notes in Math. n° 933, 210-213.