NEW DIMENSIONS IN GEOMETRY

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Introduction

Twenty-five years ago André Weil published a short paper entitled "De la métaphysique aux mathématiques" [37]. The mathematicians of the XVIII century, he says, used to speak of the "methaphysics of the calculus" or the "metaphysics of the theory of equations". By this they meant certain dim analogies which were difficult to grasp and to make precise but which nevertheless were essential for research and discovery.

The inimitable Weil style requires a quotation.

"Rien n'est plus fécond, tous les mathématiciens le savent, que ces obscures analogies, ces troubles reflets d'une théorie à une autre, ces furtives caresses, ces brouilleries inexplicables; rien aussi ne donne plus de plaisir au chercheur. Un jour vient où l'illusion se dissipe, le pressentiment se change en certitude; les théories jumelles révèlent leur source commune avant de disparaître; comme l'enseigne la Gitá on atteint à la connaissance et à l'indifférence en même temps. La métaphysique est devenu mathématique, prête à former la matière d'un traité dont la beauté froide ne saurait plus nous émouvoir".

I think it is timely to submit to the 25th Arbeitstagung certain variations on this theme. The analogies I want to speak of are of the following nature.

The archetypal m-dimensional geometric object is the space $\mathbb{R}^m$ which is, after Descartes, represented by the polynomial ring $\mathbb{R}[x_1,...,x_m]$.

Consider instead the ring $\mathbb{Z}[x_1,...,x_m;\xi_1,...,\xi_n]$, where $\mathbb{Z}$ denotes the integers and ξ_i are "odd" variables anticommuting among themselves and commuting with the "even" variables x_K . It is convenient to associate with this ring a certain geometric object of dimension $1+m+n$, or better still $(1;m|n)$, where 1 refers to the "arithmetic dimension" $\mathbb{Z}$, m to the ordinary geometric dimensions $(x_1,\ldots,x_m)$ and n to the new "odd dimensions" represented by the coordinates ξ_i.

Before the advent of ringed spaces in the fifties it would have been difficult to say precisely what we mean when we speak about this geometric object. Nowadays we simply define it as an "affine superscheme" Spec $\mathbb{Z}\,[x_i,\xi_K]$, an object of the category of topological spaces locally ringed by a sheaf of $\mathbb{Z}_2$-graded supercommutative rings (cf. n°4 below) I have tried to draw the "three-space-2000", whose plain x-axis is supplemented by the set of primes and by the "black arrow", corresponding to the odd dimension.

Three-space-2000

The message of the picture is intended to be the following methaphysics underlying certain recent developments in geometry: "<u>all three types of geometric dimensions are on an equal footing</u>".

Actually the similarity of Spec $\mathbb{Z}$ to Spec k[x], or in general of algebraic number fields to algebraic function fields, is a well known heuristic principle which led to the most remarkable discoveries in the diophantine geometry of this century. This similarity was in fact the subject matter of the Weil paper I just quoted. Weil likens the three theories, those of Riemann surfaces, algebraic numbers and algebraic curves over finite fields, to a trilingual inscription with parallel texts. The texts have a common theme but not identical. Also they have been partly destroyed, each in different places, and we are to decipher the enigmatic parts and to reconstruct the missing fragments.

In this talk I shall be concerned with only one aspect of this similarity, reflected in the idea that one may compactify a projective scheme over $\mathbb{Z}$ by adding to it a fancy infinite closed fibre. In the remarkable papers [1], [2] S. Arakelov has shown convincingly that in this way the arithmetic dimension acquires truly geometric global properties, not just by itself, but in its close interaction with the "functional" coordinates. G. Faltings [10], [11] has pushed through Arakelov's idea much further and beyond doubt (for me) the existence of a general arithmetic geometry, or A-geometry. This A-geometry is expected to contain the analogues of all main results of conventional algebraic geometry. The leading idea for the construction of the arithmetic compactifications seems to be as follows:

<u>Kähler-Einstein geometry</u> = ∞-<u>adic arithmetic</u>

I have tried in sections 1-3 of this talk to bring together our scattered knowledge on this subject.

Starting with section 4 the odd dimensions enter the game. The algebraic geometers are well accustomed to envisage the spectrum of the dual numbers Spec $\mathbb{R}[\varepsilon]$, $\varepsilon^2 = 0$, as the infinitesimal arrow and will hardly object to a similar visualization of Spec $\mathbb{R}[\xi]$. Still, there is an essential difference between these two cases. The even arrow Spec $\mathbb{R}[\varepsilon]$ is not a manifold but only an infinitesimal part of a manifold. This can be seen e.g. in the fact that $\Omega^1[\varepsilon]/\mathbb{R}$ is not $\mathbb{R}[\varepsilon]$-free, since from $\varepsilon^2 = 0$ it follows that $\varepsilon d\varepsilon = 0$. By contrast, the odd arrow Spec $\mathbb{R}[\xi]$ is an honest manifold from this point of view, since the $\mathbb{Z}_2$-graded Leibniz formula for, say, the even differential $d\xi$, is valid automatically, $\xi \cdot d\xi + d\xi \cdot \xi = 0$ and one easily sees that $\Omega^1 \mathbb{R}[\xi]/\mathbb{R}$ is $\mathbb{R}[\xi]$-free.

In spite of the elementary nature of this example it shows why the odd nilpotents in the structure sheaf may deserve the name of coordinates. But of course this is only a beginning.

The most remarkable result of supergeometry up to now is probably the extension of the Killing-Cartan classification to the finite dimensional simple Lie supergroups made in [15], [16]. The Lie supergroups acting on supermanifolds mix the even and the odd coordinates, which is one reason more to consider them on an equal footing.

In sections 5 and 6 we state some recent results of A. Vaintrob, J. Skornyakov, A. Voronov, I. Penkov and the author on the geometry of supermanifolds. They refer to the Kodaira deformation theory and the construction of the Schubert supercells and show that in this respect also supergeometry is a natural extension of the pure even geometry.

The following radical idea seems more fascinating:

> <u>the even geometry is a collective effect in the ∞-dimensional odd geometry</u>

There is a very simple algebraic model showing how this might happen. The homomorphism of the formal series in ∞ variables

(1) $$\mathbb{R}[[x_1,x_2,\ldots]] \to \mathbb{R}[[\ldots\xi_{-1},\xi_0,\xi_1,\ldots]] ,$$
$$x_i \to \sum_{n=-\infty}^{\infty} \xi_n\xi_{n+i}$$

is injective.

A considerably more refined version of this construction has recently emerged in the work on representations of Kac-Moody algebras [17],[7]. This result establishes that the two realizations of $\mathfrak{gl}(\infty)$ in the differential operator algebras $\mathcal{D}_{ev} = \mathrm{Diff}^{\infty}(\mathbb{R}[x_1,x_2,\ldots])$ and $\mathcal{D}_{odd} = \mathrm{Diff}(\mathbb{R}[[\ldots\xi_{-1},\xi_0,\xi_1,\ldots]])$ are explicitly isomorphic:

(2) $$\xi_i\frac{\partial}{\partial\xi_j} - \delta_{ij} \leftrightarrow Z_{ij}(x,\frac{\partial}{\partial x}) .$$

Here Z_{ij} are defined from the formal series

$$\Sigma Z_{ij}p^iq^{-j} = qp^{-1}(1-qp^{-1})[\exp(\sum_1^{\infty} x_i(p^i-q^i))\exp(\sum_1^{\infty}\frac{\partial}{\partial x_i}\frac{q^{-i}-p^{-i}}{i})-1].$$

The isomorphism (2) is established by comparison of two natural representations, that of $\mathcal{D}_{ev}$ on $\mathbb{R}[x_1,x_2,\ldots]$ and that of $\mathcal{D}_{odd}$ on $F = \mathbb{R}[\xi_i,\frac{\partial}{\partial\xi_j}]/I$, where I is the left ideal generated by ξ_i, $i<0$, and $\frac{\partial}{\partial\xi_j}$, $j\geq 0$. The generator 1 mod I of the cyclic $\mathbb{R}[\xi_i,\frac{\partial}{\partial\xi_j}]$-module F can be conveniently represented as the infinite wedge-product $\bigwedge_{n=-\infty}^{-1}\xi_i$ and the total module F as the span of half-infinite monomials $\bigwedge_{i\in J}\xi_i$, $J\subset\mathbb{Z}$, card $\mathbb{Z}\setminus J<\infty$. The isomorphism

(3) $$F \cong \mathbb{R}[x_1,x_2,\ldots]$$

may then be considered as the development of the simplistic idea (1).

The investigation of geometry with odd coordinates was started by physicists and is continued mainly in the physically motivated work [12],[13],[34]. In particular, the mathematical foundations of supergeometry were laid by F.A. Berezin [5] who early understood the role and the necessity of this extension of our geometric intuition. Of course the general philosophy of algebraic geometry is of great help.

Odd functions serve for modelling the internal degrees of freedom of the fundamental matter fields, leptons and quarks. Their quanta have spin $\frac{1}{2}$ and obey the Fermi-Dirac statistics. On the other hand the quanta of gauge fields (photons, gluons, $W^{\pm}$, Z,...) have spin 1 and are bosons. The map (1) is a toy model of the bosonic collective excitations in the condensate of pairs of fermions. The formulas (2) and (3) also were essentially known to specialists in dual strings theory.

The idea that fermionic coordinates are primary with respect to the bosonic ones has been repeatedly advertised in various disguises. It is still awaiting the precise mathematical theory. It may well prove true that our four space-time coordinates $(x_0 = ct, x_1, x_2, x_3)$ are only the phenomenologically effective entities convenient for the description of the low energy world in which our biological life can exist only, but not really fundamental ones.

Meanwhile physicists are discussing grand unification schemes and supergravity theories which account for all fundamental interactions (or some of them) united in a Lagrangian invariant with respect to a Lie supergroup or covariant with respect to the general coordinate transform in a superspace.

Section 6 of this talk describes the geometry of simple supergravity from a new viewpoint which presents superspace as a "curved flag space" keeping a part of its Schubert cells.

The geometry of supergravity being essentially different from the simple-minded super-riemannian geometry, one is led to believe that the substitute of the Kählerian structure in supergeometry must be rather sophisticated. Therefore I do not venture here to make any guesses about the $\mathbb{Z}$-geometry with odd coordinates.

Comparing our present understanding of the arithmetic dimension with that of the odd ones we discover that the destroyed texts are reconstructed in different parts of the parallel texts. Trying to guess more, we can ask two questions.

a) Is it possible to compactify a supermanifold with respect to the odd dimensions ?

We seemingly need a construction of such a compactification if we want to have a cohomology theory in which the Schubert supercells would have nontrivial (i.e. depending essentially on the odd part) cohomology classes.

D. Leites has conjectured that in an appropriate category an "odd projective space" might exist, that is the quotient of $\mathrm{Spec}\ k[\xi_1,\ldots,\xi_n] \setminus \mathrm{Spec}\ k$ modulo the multiplicative group action $(t,(\xi_i)) \mapsto (t\xi_i)$. Of course, in the ordinary sense it is empty.

b) Does there exist a group, mixing the arithmetic dimension with the (even) geometric ones ?

There is no such group naively, but a "category of representations of this group" may well exist. There may exist also certain correspondence rings (or their representations) between $\mathrm{Spec}\ \mathbb{Z}$ and x . A recent work by Mazur and Wiles [27] shows that the p-adic Kubota-Leopoldt ζ-function divides a certain modular p-adic ζ-function defined in characteristic p. Such things usually happen if a correspondence exists.

Finally, I would like to acknowledge my gratitude to many friends whose ideas helped to consolidate certain beliefs expressed here. I am particularly grateful to I.R. Shafarevitch who taught the arithmetic-geometry analogies to his students for three decades, to A.A. Beilinson who has generously shared his geometric insight with the author.

1. A-manifolds and A-divisors

1. A-manifolds. Let K be a finite algebraic number field, R its ring of integers, $S = S_f \cup S_\infty$ the set of finite and infinite places of K. If $v \in S$, K_v denotes the completion of K with respect to the valuation $|\,|_v : K \to \mathbb{R}^*$. We put $|a|_v = |a|$ if $K_v = \mathbb{R}$, $|a|_v = |a|^2$, if $K_v = \mathbb{C}$. Then $\prod_v |a|_v = 1$ for all $a \in K$. Moreover, $R = \{a \in K \mid |a|_v \leq 1$ for all $v \in S_f\}$.

We shall call the following data an A-manifold:

(1) $$X = (X_f ; \omega_v, v \in S_\infty)$$

Here X_f is a scheme of finite type, proper, surjective and flat over Spec R, with smooth irreducible generic fiber. Furthermore, ω_v is a Kählerian form on the complex variety $X_v = (X_f \otimes_{\mathbb{R}} K_v)(\mathbb{C})$, and $\overline{\omega}_v = \omega_v$ if $K_v = \mathbb{R}$; if $K_v = \mathbb{C}$, then the forms corresponding to the two embeddings $K_v \to \mathbb{C}$ should be conjugate.

We shall denote by $\mathrm{vol}_v = \omega_v^{\dim X_v}$ the corresponding volume forms.

The simplest example of an A-manifold is the A-curve $X_f = \mathrm{Spec}\, R$ endowed with the volumes of all points $v \in S_\infty$; ω_v do not exist in this case.

I want to stress the preliminary nature of the definition (1). First of all, one should not restrict oneself to the relatively proper schemes. If X_v is not proper, the ω_v presumably may have logarithmic growth at infinity, cf. [8]. Furthermore, a very special role is played by the Kähler-Einstein forms ω_v, see n°5 below.

2. Invertible A-sheaves. An invertible A-sheaf on the A-manifold (1) is the data

(2) $$L = (L_f ; h_v , v \in S_\infty) .$$

Here L_f is an invertible sheaf on X_f, h_v - a Hermitian metric on $L_v = L_f \otimes_R K_v$ with the evident reality conditions and the following property:

(3) <u>the curvature form</u> F_v <u>of the hermitian connection corresponding to</u> h_v <u>is</u> ω_v<u>-harmonic.</u>

We recall that if s is a local holomorphic section of L_v , then $F_v = \bar{\partial}\partial \log h_v(s,s)$ in the domain of s. For an A-curve the condition (3) is empty.

It is evident how to define the tensor product $L \otimes L'$ of two invertible A-sheaves. The group of isomorphism classes of invertible A-sheaves is denoted by $\mathrm{Pic}_A X$. The identity is the class of the structure sheaf

$$\mathcal{O}_X = (\mathcal{O}_{X_f};\ h_v \mid h_v(1,1) = 1 \quad \text{for all} \quad v \in S_\infty)$$

Later on we shall use the following fact: for a fixed ω_v , h_v is defined by the condition (3) up to a multiplicative constant.

3. <u>Sections.</u> Let L be the invertible sheaf (2). Set $H\ (X,L) = H^i(X_f,L_f)$. These cohomology groups are the R-moduli of finite type, and the ordinary Riemann-Roch-Grothendieck theorem for schemes [35] tells much about their structure. An essentially new object in A-geometry is the Euler A-characteristic $\chi_A(L)$. If a canonical map $\rho: X \to$"A-point" were to exist, one would define $\chi_A(L)$ as $R_{\rho_*}(L)$. This being otherwise, only certain ad hoc definitions of $\chi_A(L)$ in a few particular cases are known, which are reviewed in n°2. The general idea is that in case $H^i(X,L) = 0, 1 > 0$, one must define $\chi_A(L)$ as the covolume of the image

$$H^\circ(X,L) \to \bigoplus_{v \in S_\infty} H^\circ(X_v, L_v) = H^\circ_\infty(X,L)$$

relative to a certain volume form on H°_∞ . The general definition of this volume form is still lacking. Following Faltings [10] , one may conjecture that to construct it one can use a canonical metric on the bundle on $\mathrm{Pic}_0 X_v$ with fiber $\bigotimes_i \det H^i(X_v,L_v)^{(-1)^i}$ and to supplement this by inductive reasoning on the Néron-Severi group.

A correctly defined $\chi_A(L)$ should be calculable via an A-Riemann-Roch theorem so that we shall need divisors of sections of L and, more generally, A-characteristic classes.

4. <u>A-divisors.</u> We shall mean by an A-divisor on X the following data:

$$D = (D_f; r_v \mid r_v \in \mathbb{R}\ ,\ v \in S_\infty).$$

where D_f is a Cartier divisor on X_f. The following symbolic notation is more convenient

$$D = D_f + \sum_{v \in S_\infty} r_v X(v) \quad .$$

The A-divisors $X(v)$ (not to be confused with X_v) are called the "closed fibers" of X at infinity. The A-divisors form a group $\mathrm{Div}_A X$.

By the A-divisor of a section $s \in H^\circ(X,L)$ we shall mean the following element of $\mathrm{Div}_A X$:

(4) $$\mathrm{div}\ s = \mathrm{div}_f s - \sum_{v \in S_\infty} \left(\int_{X_v} \log |s|_v \cdot \mathrm{vol}_v\right) X(v),$$

$$|s|_v^{[\mathbb{C}:K_v]} = h_v(s,s).$$

Here $\mathrm{div}_f s$ is the Cartier divisor of s . If a rational function g on X_f is a quotient of two sections of L , it is natural to define its principal A-divisor by the formula

(5) $$\mathrm{div}\ g = \mathrm{div}_f g - \sum_{v \in S_\infty} \left(\int_{X_v} \log |g|_v\ \mathrm{vol}_v\right) X(v),$$

which does not depend on L . Finally, the same formula (4) may be used to define that A-divisor of a meromorphic section of L.

Now we can easily introduce the A-sheaves $\mathcal{O}(D)$ where D is an arbitrary A-divisor, together with the canonical section whose A-divisor is D . First, for $D = D_f$ we set:

(6) $$\mathcal{O}(D_f) = (\mathcal{O}_{X_f}(D_f);\ h_v)$$

where h_v is the unique metric on $\mathcal{O}_{X_f}(D_f) \otimes K_v$ satisfying equation (3) and normalized by

(7) $$\int_{X_v} \log |1_{D_f}|_v \cdot \mathrm{vol}_v = 0,\ v \in S_\infty\ ,$$

where 1_{D_f} is the meromorphic section of $\mathcal{O}_{X_f}(D_f)$ whose Cartier divisor is D_f. Using (4) and (7) we get $\operatorname{div} 1_D = \operatorname{div}_f 1_D = D_f$ which justifies (6).

Furthermore, we set

(8) $$\mathcal{O}\big(\sum_{v\in S_\infty} r_v X(v)\big) = (\mathcal{O}_{X_f}; h_v \quad \|1\|_v = \exp(-r_v(\int_{X_v} \mathrm{vol}_v)^{-1})).$$

Again using (4) we obtain

$$\operatorname{div} 1 = \sum_{v\in S} r_v X(v).$$

as is to be expected.

As in the geometric case we can construct the exact sequence

$$\begin{array}{ccccccc} 0 \to \operatorname{Div}^\circ_A X \to & \operatorname{Div}_A X & \to & \operatorname{Pic}_A X & \to 0, \\ & \cup & & \cup & \\ & D & \to & \text{class of } \mathcal{O}(D) & \end{array}$$

where $\operatorname{Div}^\circ_A X$ is the group of principal A-divisors.

5. Green's functions. It is clear from the previous definitions that the essential information about the Archimedean part of the A-divisors is encoded in the functions

(9) $$G_v(D_v,x) = |1_{D_v}|_v(x) \quad , \quad x \in X_v$$

On the compact Kählerian manifold (X_v,ω_v) they are uniquely defined by the following conditions:

a) $G_v(D_v,x)$ is real analytic for $x \notin \operatorname{supp.} D_v$. The function $G_v(D_v,x)/|g(x)|_v$, where g_v is a local equation of D_v, is extendable to $\operatorname{supp} D_v$.

b) The (1,1)-form $\bar\partial\partial \log G_v(D_v,x)$ is ω_v-harmonic outside of D_v. The corresponding current is a linear combination of a harmonic form and the δ_{D_v}.

c) $\int_{X_v} \log G_v(D_v,x)\cdot \mathrm{vol}_v = 0$

Furthermore,

d) $G_v(D'_v + D''_v,x) = G_v(D'_v,x)\cdot G_v(D''_v,x)$.

e) $G_v(\mathrm{div}\, g_v,x) = c_g\cdot |g_v(x)|_v$ for each meromorphic function g_v where c_v is defined by c).

It is explained in the last chapter of Lang's book [22] how to calculate Green's functions on abelian varieties and algebraic curves using theta-functions and differentials of the third kind respectively. The Kählerian metric involved is flat in the first case and induced by the flat metric of the Jacobian in the second one. The same metrics are used in the Arakelov-Faltings-Riemann-Roch theorem on A-surfaces which we shall state in n°2.

Since the function (9) is not constant except in trivial cases, the closed fibers $X(v)$, $v \in S_\infty$, should be imagined as "infinitely degenerate". To make it more credible note that if for $v \in S_f$ the closed fiber is degenerate then a meromorphic function or a section of an invertible sheaf can have different orders at different components of $X(v)$. In other words, instead of $|1_{D_f}|_v$ one should consider in this case $|1_{D_f}|_{v_i}$ where the valuations v_i correspond to the components $X(v)_i$ of $X(v)$. Finally, one can unify these numbers into a function $|1_{D_f}|(x)$, $x \in X(R_v) = X_v(K_v)$ setting $|1_{D_f}|(x) = |1_{D_f}|_{v_i}$ if the section x intersects $X(v)_i$.

This analogy suggests refining the definition of the divisor supported by $X(v)$, $v \in S_\infty$. Conjecturally, instead of a constant r_v one should consider a volume form ρ_v and delete the integrals from (4) and (5). A comparison with the Mumford-Schottky curves may serve to clarify the situation. Meanwhile we shall use the coarse definitions.

6. The intersection index. Let $\sigma: \mathrm{Spec}\, R \to X_f$ be a section of the structural morphism $\pi: X_f \to \mathrm{Spec}\, R$. We shall consider the image of σ as the closed A-curve Y lying in X. We define the intersection index of Y with an A-divisor D such that $\mathrm{supp}\, Y \cap \mathrm{supp}\, D$ is disjoint from the generic fiber of π:

$$(10)\qquad \langle Y,D\rangle = \sum_{v\in S_f} (Y,D_f)_v \log q_v + \sum_{v\in S_\infty} \langle Y,D\rangle_v .$$

Here (Y,D_f) for $v \in S_f$ denotes the sum of the local intersection indices of Y and D in the closed points of X_f over v, q_v is the order of the residue field. Furthermore, for $v \in S_\infty$ we set

$$\langle Y,D_f\rangle_v = -\log G_v(D_{f,v},y_v) ,$$

$$\langle Y,\sum_{v\in S_\infty} r_v X(v)\rangle = \sum_{v\in S_\infty} r_v(\int_{X_v} \mathrm{vol}_v)^{-1} .$$

An equivalent definition is obtained if one puts (10) and (4) together. Denote by $\sigma^*(1_D)$ the section of the A-sheaf $\sigma^*(\mathcal{O}(D))$ on Y, induced by $\mathcal{O}(D)$. Then

$$(11)\qquad \langle Y,D\rangle = -\log \prod_v |\sigma^*(1_D)|_v$$

From the product formula one sees that the right hand side of (11) remains unchanged if one takes a different non zero section of $\sigma^*(\mathcal{O}(D))$ instead of $\sigma^*(1_D)$. This justifies the following general definition of degree of an invertible A-sheaf L on the A-curve $Y = \operatorname{Spec} R$:

$$(12)\qquad \deg L = -\log \prod_v |s|_v$$

One can take any non zero meromorphic section s of L in (12).

2. The Riemann-Roch theorems.

1. The geometric Riemann-Roch theorems. We shall recall first the simplest Riemann-Roch-Hirzebruch theorem for projective manifolds over a field. Let X be a d-dimensional manifold, L an invertible sheaf on it. Set $\chi(L) = \sum_i (-1)^i \dim H^i(L)$ and denote by $c_1(L)$, $td_i(X)$ the Chern and Todd classes respectively. Then

$$(1)\qquad \chi(L) = \langle \sum_{i=0}^{d} \frac{1}{i!} c_1(L)^i \, td_{d-i}(X)\rangle$$

where $\langle\ \rangle$ in the right hand side of (1) means the intersection index calculated in the Chow ring or in a cohomology ring with characteristic zero coefficients. In particular

(2) $$\chi(L) = \sum_{i=0}^{d} \frac{n^i}{i!} \langle c_1(L)^i \, td_{d-i}(X) \rangle$$

In this section we shall describe three particular cases of a would be Riemann-Roch theorem for A-manifolds, for the projective space, A-curve and A-surface respectively, the last case being by far the deepest one. As we have said already, the first problem is to define $\chi_A(L)$.

2. Projective A-space. Let us consider the A-manifold, for simplicity over $\mathbb{Z}$, $\mathbb{P}^d = (\mathbb{P}^d_{\mathbb{Z}}, \omega)$, where ω is a Kählerian form on $\mathbb{P}^d(\mathbb{C})$. We shall realize $\mathbb{P}^d_{\mathbb{Z}}$ as Proj $S(T_{\mathbb{Z}})$, where $T_{\mathbb{Z}}$ is a $\mathbb{Z}$-free module of the rank $d+1$, and we set $T = \mathbb{R} \otimes T_{\mathbb{Z}}$. There is a canonical hermitian metric on $\mathcal{O}_f(n)$ whose curvature from is a multiple of ω . We shall denote by $\mathcal{O}(n)$ the corresponding A-sheaf. Since

$$H^i(\mathbb{P}^d, \mathcal{O}(n)) = S^n(T_{\mathbb{Z}}) \quad , \quad H^i(\mathbb{P}^d, \mathcal{O}(n)) = 0,$$

for $i > 0$, $n \geq 0$, we must choose a volume form $w_n \in \Lambda_{\mathbb{R}}^{\binom{n+d}{d}}(S^n(T))$ and then define

$$\chi_A(\mathcal{O}(n)) = \log \left|\frac{w_n}{v_n}\right| \quad ,$$

where $v_n \in \Lambda_{\mathbb{Z}}^{\binom{n+d}{d}}(S^n(T_{\mathbb{Z}}))$ is one of the generators of this cyclic group.

The simplest imaginable choice of w_n is the following one. Consider the isomorphism

$$\varphi_n : \Lambda_{\mathbb{Z}}^{\binom{n+d}{d}} (S^n(T_{\mathbb{Z}})) \to [\Lambda_{\mathbb{Z}}^{d+1}(T_{\mathbb{Z}})]^{\otimes \frac{n}{d+1}\binom{n+d}{d}}$$

which maps v_n onto $v_1^{\otimes \frac{n}{d+1}\binom{n+d}{d}}$ (if $\frac{n}{d+1}\binom{n+d}{d} \notin \mathbb{Z}$ one can still correctly define $\varphi_n^{\otimes(d+1)}$, which suffices for our needs). Now choose somehow w_1 and set

$$w_n = (\varphi_n \otimes id_{\mathbb{R}})^{-1}\left(w_1^{\otimes \frac{n}{d+1}\binom{n+d}{d}}\right) .$$

Then

$$\chi_A(\mathcal{O}(n)) = \log \left|\frac{w_1}{v_1}\right|^{\otimes \frac{n}{d+1}\binom{n+d}{d}} = \frac{n}{d+1}\binom{n+d}{d} \chi_A(\mathcal{O}(1)) .$$

In view of (8), n°1, the tensor multiplication of an A-sheaf L by $\mathcal{O}(\frac{r}{n}\mathbb{P}(\infty))$ multiplies the metric on L_∞ by $\exp(-\frac{r}{n \operatorname{vol} \mathbb{P}_\infty})$. Assuming the corresponding change of w_1, we get

$$\chi_A(\mathcal{O}(1) \otimes \mathcal{O}(\frac{r}{n}\mathbb{P}(\infty))) = \chi_A(\mathcal{O}(1)) + \frac{r(d+1)}{n \operatorname{vol} \mathbb{P}_\infty}$$

and finally

$$(3) \qquad \chi_A(\mathcal{O}(n) \otimes \mathcal{O}(r\mathbb{P}(\infty))) = \frac{n}{d+1}\binom{n+d}{d}\chi_A(\mathcal{O}(1)) + \binom{n+d}{d}\frac{r}{\operatorname{vol} \mathbb{P}_\infty}$$

Comparing (3) with (1) and (2) we see that $\mathbb{P}^d$ looks like a $(d+1)$-dimensional geometric manifold. We can also guess the Todd A-classes $td^A_{d+1-i}(\mathbb{P}^d)$.

3. <u>A-curve</u>. Let $X = \operatorname{Spec} R$, $n = [R:\mathbb{Z}] = r_1 + 2r_2$, $r_1 = \operatorname{card}\{v \in S_\infty | K_v = \mathbb{R}\}$. Denote by $L = (L_f; h_v)$ an invertible A-sheaf on X, $L = H^\circ(X, L_f)$. According to n°.1.3, we have

$$\chi_A(L) = \operatorname{vol}((\bigoplus_{v \in S_\infty} L \otimes K_v)/L).$$

We choose the volume form on $\bigoplus_{v \in S_\infty} L \otimes K_v$ implicit in this definition following A. Weil [36] and Szpiro [33] in the following way:

$$w = \frac{2^n}{2^{r_1}\pi^{r_2}} \prod_{v \in S_\infty} w_v ,$$

where w_v is the volume form corresponding to the Euclidean metric on L_v defined by h_v. With this choice, the following statements, closely parallel to the case of curves over finite fields, are valid.

The Riemann-Roch theorem:

$$\chi_A(L) = \deg L + \chi_A(\mathcal{O}_X)$$

where $\deg L$ is defined by (12), n°1.

The Euler number of the structure sheaf:

$$\chi_A(\mathcal{O}_X) = r_2 \log \frac{\pi}{2} - \frac{1}{2} \log |\Delta_K| ,$$

where Δ_K is the discriminant of K.

Furthermore, set $H_A^\circ(L) = \{s \in L \mid |s|_v \leq 1$ for all $v\}$. Then $H_A^\circ(\mathcal{O}) = \{0\} \cup$ {roots of unity in R}, which is the analog of the constant field. From the Minkowski theorem one easily deduces that $\chi_A(L) \geq 0$ implies $H_A^\circ(L) \neq 0$.

4. <u>A-surface.</u> An A-surface $X = (X_f, \omega_v)$, according to Arakelov and Faltings, is a semistable family of curves $X_f \overset{\pi}{\to} \operatorname{Spec} R$ with smooth irreducible generic fiber of genus $g > 0$ and the following metrics at infinity:

$$\omega_v = \mathrm{vol}_v = -\frac{1}{2\pi i} \sum_{k=1}^{g} \nu_{k,v} \wedge \bar{\nu}_{k,v}$$

Here $(\nu_{1,v}, \dots, \nu_{g,v})$ is a base of the differentials of the first kind on X_v orthonormal with respect to the scalar product

$$\langle \nu, \nu' \rangle = -\frac{1}{2\pi i} \int_{X_v(\mathbb{C})} \nu \wedge \bar{\nu}'.$$

We denote by Ω_f the relative dualizing sheaf of π. Then $\Omega_v = \Omega^1 X_v$. The canonical A-sheaf $\Omega = (\Omega_f, h_v)$ is <u>unambigously</u> defined by the following prescription which normalizes h_v. For an arbitrary point $x \in X_v$ the residue map $\mathrm{res}_x : \Omega^1 X_v \otimes \mathcal{O}_{X_v}(x) \to \mathbb{C}$ is an isometry of the geometric fiber of the former sheaf and of $\mathbb{C}$.

5. <u>The Euler characteristic.</u> Faltings [10] defines $\chi_A(L)$ for an invertible A-sheaf L on X in the following way.

The decisive step is the definition of canonical metrics on the spaces $\det_{K_v} H^\circ(L_v) \otimes \det_{K_v}^{-1} H^1(L_v) \otimes \bar{K}_v$, $v \in S_\infty$, in the case $H^\circ(L_v) = H^1(L_v) = 0$. This being done, Faltings uses this case as the induction base with respect to the ordinary degree of L on the generic fiber. To this end he represents L in the form $L_0(D)$, where $H^\circ(L_{0,v}) = H^1(L_{0,v}) = 0$ and D is a horizontal A-divisor which can be taken as a sum of sections after a base extension. One can then simultaneously define $\chi_A(L_0(D))$ and prove the Riemann-Roch formula if only one establishes the independence of this construction on the choice of the isomorphism $L = L_0(D)$ which is highly non-unique.

This independence is valid for the following particular choice of metrics on all L_0's simultaneously. Set $\mathrm{Pic}_{g-1} X_v = M_v$ and denote by E_v the universal sheaf over $X_v \times M_v$. Let $\pi_2 : X_v \times M_v \to M_v$ by the projection map. We can construct the invertible sheaf $\det R\pi_{2*} E_v$ on M_v. Its geometric fiber at a point $y \in M_v$ corresponding to the sheaf $L_v(y) = E_v \mid (X_v \times \{y\})$ can be canonically identified with

$$\det H^\circ(L_v(y)) \otimes \det^{-1} H^1(L_v(y)) \otimes \overline{K}_v \ .$$

Over the set $U = \{y \in M_v \mid H^\circ(L_v(y)) = H^1(L_v(y)) = 0\}$ the sheaf $\det R\pi_{2*} E$ has a canonical unit section. On the other hand, $M_v \setminus U$ is the theta-divisor, and an easy consideration shows that under the suitable identification $\det R\pi_{2*} E = \mathcal{O}_{M_v}(-\theta)$ the unit section goes into 1. Therefore, a choice of an A-structure on $\mathcal{O}_M(-\theta)$ normalizes all $\chi_A(L_0)$. Faltings proves that the θ-polarization induces precisely the A-structure suitable for the inductive argument.

We can now state the Riemann-Roch.

6. <u>Theorem.</u> a) $\chi_A(\mathcal{O}(D)) = \frac{1}{2} \langle D, D-K \rangle + \chi_A(\Omega)$, where $\Omega = \mathcal{O}(K)$, $\langle\ \rangle$ is the intersection index defined in n°1.6.

b) $$\chi_A(\Omega) = \frac{1}{12}(\langle K,K \rangle + \delta) \ , \delta = \sum_{v \in S_\infty} \delta_v(X_v) \ ,$$

where $\delta_v(X_v) = \log$ card (singular points of $X(v)$) for $v \in S_f$; for $v \in S_\infty$, δ_v is a real analytic function on the moduli space of Riemann surfaces which measures the distance of X_v to the boundary. ∎

Of course, in the geometric case, the Noether formula b) follows from the Riemann-Roch-Grothendieck theorem applied to the morphism π.

The structure of $\delta_v(X_v)$ for $v \in S_\infty$ vaguely agrees with our philosophy about $X(v)$ as a degenerate fiber.

Faltings proves the Noether formula by an argument using the moduli space of X instead of Pic_{g-1} of the first part.

The governing idea always is to use some canonical A-structures on the moduli spaces and their tautological sheaves, to apply the ordinary

Riemann-Roch-Grothendieck to the finite part and then to "compactify" this information by the Kählerian geometry.

Hence we need the A-geometry of arbitrary dimension anyway, even to deal with A-surfaces only. In the next section we shall discuss what is to be done to put this program on a firm foundation.

3. Prospects and problems of A-geometry

1. The problem of the definition of the fundamental categories.

In the definition of A-manifolds given in 1.1. no conditions on the Kählerian forms ω_v were imposed. However, the Arakelov and Faltings theorems are proved for distinguished Kählerian structures. We shall give the tentative definitions in a more general context.

Let E_v be a locally free sheaf on a compact Kählerian manifold (X_v,ω_v) and h_v a Hermitian metric on E_v. We choose holomorphic local coordinates (z^α) on X_v and a base of local holomorphic sections (s_i) of E_v and set $h_{ij} = h_v(s_i,s_j)$. The curvature tensor of the canonical connection associated with h_v is

$$F_{ij\alpha\beta} = -\frac{\partial^2 h_{ij}}{\partial z^\alpha \partial \bar{z}^\beta} + h^{ab}\frac{\partial h_{ib}}{\partial z^\alpha}\frac{\partial h_{ij}}{\partial \bar{z}^\beta},$$

where $(h^{ab}) = (h_{ij})^{-1}$. Set $\omega_v = \sqrt{-1}\, g_{\alpha\beta} dz^\alpha \wedge d\bar{z}^\beta$ and $(g^{\gamma\delta}) = (g_{\alpha\beta})^{-1}$. Then (E_v,h_v) is called a Hermite-Einstein sheaf if

$$g^{\alpha\beta}F_{ij\alpha\beta} = \lambda h_{ij},$$

where λ is a constant. (It can be explicity calculated: setting $n = \dim X_v$ we get $\lambda = (2\pi n \int_{X_v} c_1(E_v)\omega_v^{n-1}) / (\mathrm{rk}\, E \int_{X_v} \omega_v^n))$.

On the holomorphic tangent sheaf TX_v there is the hermitian metric $g_v = 2g_{\alpha\beta}\, dz^\alpha d\bar{z}^\beta$. The manifold (X_v,ω_v) is called the Hermite-Einstein manifold if (TX_v,g_v) is a Hermite Einstein sheaf.

The existence and uniqueness problems for Hermite-Einstein structures on a sheaf E_v were considered by Kobayashi [19] and Donaldson [9]. Kobayashi has shown that the existence of such structure implies the semistability of E_v and that any semistable Hermite-Einstein sheaf (E_v, h_v) is a direct sum $\oplus (E_v^{(i)}, h_v^{(i)})$ with stable $E_v^{(i)}$. (Stability here means that the function $\mu(F) = \deg_{\omega_r} F/\mathrm{rk}\, F$, where $\deg_{\omega_v} F = \int_{X_v} c_1(F)\omega_v^{n-1}$, is monotonous on subsheaves $F \subset E$).

On the other hand, Donaldson proved that on projective algebraic surfaces (X_v, ω_v) any stable sheaf has a unique Hermite-Einstein metric (up to a multiplicative constant). The same is true for algebraic curves and, conjecturally, for all projective manifolds.

Deep existence and uniqueness properties of Kähler-Einstein metrics on manifolds were obtained by Yau [38] and Aubin [3]. From Yau's results it follows in particular that for $c_1(X_v) = 0$ each cohomology class of Kählerian metrics contains a unique Kähler-Einstein metric. Aubin has established the existence on X_v of a unique Kähler-Einstein metric with the constant $\lambda = -1$ under condition that $c_1(X_v)$ contains a form with negative definite metric.

Our limited understanding of A-geometry suggests the special role of those A-manifolds for which (X_v, ω_v) are Kähler-Einstein. This condition appears to be a reasonable analog of the minimality of X_f over Spec R. Furthermore, on a given A-manifold, the following definition of a locally free A-sheaf seems plausible enough: it is the data $E = (E_f; h_v, v \in S_\infty)$ for which E_f is a locally free sheaf on X_f and (E_v, h_v) are Hermite-Einstein sheaves on (X_v, ω_v). For $\mathrm{rk} E_f = 1$ this is our initial definition.

The category-theoretic aspects of these definitions need clarification.

Since the Hermite-Einstein property may possibly be relevant only for locally free and semistable sheaves, to define a substitute for coherent A-sheaves one probably is bound to consider something like "perfect complexes of locally free A-sheaves", as in [35]. Unfortunately, the differential geometry of complexes of sheaves in a derived category is not sufficiently developed.

The complexes of A-sheaves must have a torsion invariant χ_A. For

$v \in S_\infty$ the corresponding component of χ_A under certain conditions should be given by the Quillen construction [30].

I do not know how to define morphisms of A-manifolds. The problem seems to be related to the hyperbolicity theory by Kobayashi [18]. In fact, it is based on the study of morphisms $D \to X_v$ and $X_v \to D$, where $D = \{z \in \mathbb{C},\ |z| \leq 1\}$ is the analog of Spec $\mathbb{Z}_p$. Therefore it can be considered as the counterpart of the theory of $\mathbb{Z}_p$-models of $\mathbb{Q}_p$-manifolds.

2. The problem of canonical A-structures on moduli spaces and of A-moduli spaces. The Arakelov and Faltings work shows the existence of distinguished A-structures on the moduli spaces of curves and of invertible sheaves on a curve, by which one arrives at good statements of the principal results. (For the moduli space of curves having a boundary this statement should be considered as heuristic, granting the existence of good definitions).

One should study from this view point the moduli spaces of stable sheaves on a curve, with rank and degree relatively prime. The first unsolved problem is to generalize the Riemann-Roch-Arakelov-Faltings theorem to the A-sheaves of arbitrary rank, where the second Chern A-class $c_2^A(E)$ should emerge,an intersecting new invariant.

When a category of A-spaces is properly defined one would naturally hope for existence of moduli-objects in this category. Of course, the first problems here are again connected with the situation "at infinity", i.e. the Kählerian geometry. In this respect a recent work of Koiso [20] deserves to be mentioned. Koiso shows in particular that the base space of normal and stable family of Kähler-Einstein structures carries a canonical Kähler structure. Unfortunately in most cases it is unknown whether it in addition satisfies the Einstein equation.

3. The problem of intersection theory of A-manifolds. In the important paper [4] A. Beilinson defined regulators for K-theory and introduced the general technique for construction of intersection theory on A-manifolds. We shall briefly describe here a part of his results, stressing the role of K-theory as a cohomology theory.

Let X_f be a regular projective scheme flat over $\mathbb{Z}$, $\dim X_f = d + 1$.

We fix two cycles z_i of pure codimension ℓ_i in X_f. Assume that $\ell_0 + \ell_1 = d+1$ and $\operatorname{supp}(z_0 \wedge z_1) \cap X_{f,\mathbb{Q}} = \emptyset$. We shall describe Beilinson's construction of the A-intersection index $\langle z_0, z_1 \rangle$. For simplicity, we shall assume that both cycles have zero cohomology classes on the generic fiber. In this case we can forget ω_∞ since the intersection index will not depend on this metric. We get

$$\langle z_0, z_1 \rangle = \langle z_0, z_1 \rangle_\infty + \sum_{p \in S_f} \log p \cdot \langle z_0, z_1 \rangle_p$$

and define the fiberwise indices $\langle\ \rangle_v$ with the help of several cohomology theories.

To calculate $\langle\ \rangle_\infty$ we shall use the Deligne-Beilinson cohomology $H^j_{\mathcal{D}}$. For a smooth compact complex manifold X_∞ and a coefficient ring $B \subset \mathbb{R}$, $B(j) = (2\pi\sqrt{-1})^j B$, this cohomology is defined as follows:

$$H^K_{\mathcal{D}}(X_\infty, B(j)) = R^K\Gamma(X_\infty, B(j)_{\mathcal{D}}),$$

$$B(j)_{\mathcal{D}} = \operatorname{Cone}(F^j \oplus B(j) \xrightarrow{\alpha} \Omega^\cdot)[-1],$$

where $F^j = \Omega^{\geq j}$ (the truncated complex of holomorphic forms), $\alpha = \alpha_1 - \alpha_2$, α_i the natural injections. In the noncompact case, the forms with logarithmic singularities are used. The now standard homological methods permit us to define the $\mathcal{D}$-cohomology of simplicial schemes, the relative $\mathcal{D}$-cohomology, to define the classes of algebraic cycles and to prove the Poincaré duality theorem.

Now we return to the situation described earlier and set $U_i = X_\infty - \operatorname{supp} z_{i,\infty}$. If the classes of $z_{i,\infty}$ in $H^{2\ell_i}_{\mathcal{D}}(X_\infty, \mathbb{R}(\ell_i))$ vanish, the Mayer-Vietoris sequence shows that the classes $c\ell_{\mathcal{D}} z_i \in H^{2\ell_i}_{\mathcal{D}}(X, U_i, \mathbb{R}(\ell_i))$ are of the form $\partial\zeta_i$ where $\zeta_i \in H_{\mathcal{D}}^{2\ell_i - 1}(U_i, \mathbb{R}(\ell_i))$. We can construct the class

$$\zeta_0 \cup \zeta_1 \in H^{2d}_{\mathcal{D}}(U_0 \cap U_1)\mathbb{R}(d+1))$$

and its image

$$(z_0 \cap z_1)_\infty = \partial(\zeta_0 \cup \zeta_1) \in H_{\mathcal{D}}^{2d+1}(X, \mathbb{R}(d+1))$$

Let $\pi : X_\infty \to \operatorname{Spec} \mathbb{R}$ be the structure morphism. The final formula for the A-intersection index at the arithmetical infinity is

$$\langle z_0, z_1\rangle_\infty = \pi_*(z_0 \cap z_1)_\infty \in H^1_{\mathcal{D}}(\mathrm{Spec}\ \mathbb{R}, \mathbb{R}(1)) = \mathbb{C}/\mathbb{R}(1) = \mathbb{R}\ .$$

To define $\langle z_1, z_2\rangle_p$ in a similar way, Beilinson introduces the K-cohomology:

$$H^j_K(X_f, \mathbb{Q}(i)) = K^{(i)}_{2i-j}(X_f) \otimes \mathbb{Q} = \{a \in K_{2i-j}(X_f) \otimes \mathbb{Q} \mid \Psi^p(a) = p^i a\},$$

where Ψ^p are the Adams operations. In this case also the relative version and the formalism of the cyclic classes can be defined. Setting $U_i = X_f \setminus \mathrm{supp}\ z_i$ as earlier we can now construct the intersection class

$$(z_0 \cap z_1)_f \in H^{2d+2}_K(X_f, U_0 \cup U_1; \mathbb{Q}(d+1))\ .$$

Set $S' = \mathrm{supp}\ \pi_f(X_f \setminus (U_0 \cup U_1))$, where $\pi_f : X_f \to \mathrm{Spec}\ \mathbb{Z}$ is the structure morphism. This is a finite set of primes. We have $\langle z_0, z_1\rangle_p = 0$ for for $p \notin S'$, and for $p \in S'$ the index $\langle z_0, z_1\rangle_p$ is a sort of direct image $\pi_f(z_0 \cap z_1)_f$ localized at p.

4. The problem of the Euler A-characteristic and of the Riemann-Roch A-theorem. I cannot add much to what has been said earlier. Two remarks may be in order.

First, granting that the definition of χ_A in a general situation can be done in terms of the analytic torsion of the Dalbeault complexes, we shall need the relative analytic torsion to treat the general Riemann-Roch-Grothendieck case.

Second, independently of the conjectural general theory, very interesting and directly accessible problems of A-geometry may be found, e.g., in the theory of flag manifolds $G/P_{\mathbb{Z}}$. A recent work by Bombieri-Vaaler [6] is an example. It suggests in particular that the classical Minkowski "geometry of numbers" should be interpreted in A-geometry as a theory of characteristic classes at the arithmetic infinity.

4. Superspace

1. Examples of superspaces. A smooth or analytic manifold can be described

by a family of local coordinate systems and transition functions. Before introducing a formal definition of superspace, we shall give several examples of supermanifolds with the help of local coordinates.

a) The $m|n$ -dimensional affine superspace. It has global coordinates $(x_1,\dots,x_m;\xi_1,\dots,\xi_n)$, where x_i commute among themselves and with ξ_j and ξ_j anticommute. In the category of superschemes over a commutative ring A the ring of functions on the relative affine $m|n$-space is the Grassmann algebra with generators ξ_j over polynomial ring $A[x_1,\dots,x_m]$. In the category of C^∞-supermanifolds the ring of functions is $C^\infty(x_1,\dots,x_m)\,[\xi_1,\dots,\xi_n]$.

b) The $m|n$-dimensional projective superspace. It is defined by the atlas U_i, $i=0,\dots,m$, each U_i being a $m|n$-dimensional affine space. It is convenient to introduce a homogeneous coordinate system $(X_0,X_1,\dots,X_m;Z_1,\dots,Z_n)$ and to relate the coordinates $(x^i_j,\ j\neq i\ |\ \xi^i_1,\dots,\xi^i_r)$ by setting $x^i_j = X_j/X_i$, $\xi^i_j = Z_j/X_i$.

c) The supergrassmannian of the $d_0|d_1$-dimensional linear superspaces in the $(d_0+c_0|d_1+c_1)$-dimensional linear superspace. We shall describe it by the following standard atlas. Consider matrices of the form $(d_0+d_1)\times(d_0+c_0+d_1+c_1)$ divided into four blocks such that the format of the upper left block is $d_0\times(d_0+c_0)$. For each subset I of columns containing d_0 columns of the left part and d_0 columns of the right part consider the matrix

$$(1)\qquad Z_I = \begin{array}{c} \begin{array}{cccc} c_0 & d_0 & d_1 & c_1 \end{array} \\ \left(\begin{array}{c|cc|cc|c} x_I & 1 & 0 & \multicolumn{2}{c|}{0} & \xi_I \\ & 0 & 1 & & & \\ \hline \xi_I & \multicolumn{2}{c|}{0} & 1 & 0 & x_I \\ & & & 0 & 1 & \end{array}\right) \begin{array}{c} d_0 \\ \\ d_1 \end{array} \\ \underbrace{\qquad\qquad\qquad}_{I} \end{array}$$

The columns I in Z_I form the identity matrix. All the remaining places are filled by the independent even and odd variables x^{ab}_I ,ξ^{cd}_I, even places being in the upper left and lower right blocks. These variables (x_I, ξ_I) are the coordinates of the local chart U_I. Denote by B_{IJ} the submatrix of Z_I formed by the columns with indices in J. Then the transition rules are $Z_J = B_{IJ}^{-1} Z_I$.

Setting in this prescription $d_0|d_1 = 1|0$, $d_0 + c_0|d_1 + c_1 = m+1|n$, we get the projective superspace. On the other hand, setting $d_1 = c_1 = 0$, we get an ordinary grassmannian.

Proceeding in a more systematic way, we shall start with several basic notions of superalgebra and then define superspaces by means of a structure sheaf.

2. Superalgebra. The algebraic composition laws relevant in geometry are naturally divided into additive and multiplicative ones. All additive groups in superalgebra are endowed with a $\mathbb{Z}_2$-gradation and all multiplications are compatible with it. We use notation $A = A_0 \oplus A_1$ and $\tilde{a} = \varepsilon$ in case $a \in A_\varepsilon$, then $\widetilde{ab} = \tilde{a} + \tilde{b}$. The elements of A_0 are called even ones and those of A_1 odd ones. The characteristic feature of the superalgebra is the appearance of certain signs ± 1 in all definitions, axioms and polynomial identities of fundamental structures.

We shall give a representative list of examples.

Let $A = A_0 \oplus A_1$ be an associative ring. The supercommutator of homogeneous elements $a,b \in A$ is defined by the formula $[a,b] = ab - (-1)^{\tilde{a}\tilde{b}} ba$. The ring A is called supercommutative iff $[a,b] = 0$ for all a,b. If 2 is invertible (which we shall always assume), $a^2 = \frac{1}{2}[a,a] = 0$ for all $a \in A_1$. The supercommutators in general satisfy two identities

$$[a,b] = -(-1)^{\tilde{a}\tilde{b}} [b,a] ,$$

$$[a,[b,c]] + (-1)^{\tilde{a}(\tilde{b}+\tilde{c})} [b,[c,a]] + (-1)^{\tilde{c}(\tilde{a}+\tilde{b})} [c,[a,b]] = 0 .$$

These identities (together with superbilinearity) are taken as the definition of Lie superalgebras. The ring morphisms, by definition, respect the gradation.

Let A be a supercommutative ring. The notions of ($\mathbb{Z}_2$-graded, of course) left, right and bimodule S over A coincide, just as in the commutative case, left and right multiplications being connected by the formula $a s = (-1)^{\tilde{a}\tilde{s}} s a$. A new feature is the parity-change functor: $(\Pi S)_0 = S_1, (\Pi S)_1 = S_0$, right multiplication by A coincide on S and ΠS. An A-module S is called free of rank $p|q$ iff it is isomorphic to $A^{p|q} = A^p \oplus (\Pi A)^q$. The tensor algebra of A-modules differs from the ordinary one by the introduction of Z sign into certain canonical isomorphisms, e.g. $\varphi : S \otimes T \tilde{\to} T \otimes S$ is defined by $\varphi(s \otimes t) = (-1)^{\tilde{s}\tilde{t}} t \otimes s$. There are internal Hom's in the category of A-modules consisting of ordinary morphisms and also of odd ones, with the linearity rule $f(a s) = (-1)^{\tilde{a}} a f(s)$.

The morphisms between the free A-modules can be given by matrices.

One must not forget that the passage from the left (row) coordinates to the right (column) coordinates of an even element implies sign change in odd coordinates etc. The matrices are often written in the standard format, like (1) where the even-even places are kept in the upper left block. The group GL represents the functor of the invertible matrices corresponding to even morphisms.

F. Berezin has invented the superdeterminant, or Berezinian, $\mathrm{Ber} : GL(p|q,A) \to A_1^*$. It is a rational function of the elements of the matrix which in the standard format is given by the formula

$$\mathrm{Ber}\begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix} = \det\,(B_1 - B_2 B_4^{-1} B_3)\ \det B_4^{-1}.$$

The kernel of Ber is denoted SL. Since Ber is rational, supergrassmannians fail to have Plücker coordinates, as we shall see later.

The Berezinian of a free module $\mathrm{Ber}\,S$ is defined as a free module of rank $1|0$ or $0|1$ (depending on the parity of q in $\mathrm{rk}S = p(q)$ freely generated by any element of the form $D(s_1, \ldots, s_{p+q})$ where (s_i) is a free base of S, with the relations $D(f(s_1), \ldots, f(s_{p+q})) = \mathrm{Ber} f \cdot D(s_1, \ldots, s_{p+q})$. This notion is a specific substitute for the maximal exterior power in commutative algebra.

The bilinear forms on a free A-module T with the symmetry conditions are divided into four main types: OSp (even symmetric), SpO (even alternate), ΠO (odd symmetric), ΠSp (odd alternate). The form b^{π} on ΠT is defined by the formula $b^{\pi}(\Pi t_1, \Pi t_2) = (-1)^{\tilde{t}} b(t_1, t_2)$. This construction preserves parity but changes the symmetry of the form so that usually it suffices to consider only types OSp and ΠSp.

The automorphism functor of a non-degenerate form defines the supergroups of the corresponding type (we shall consider below only the split ones). Besides SL, OSp and ΠSp, there is in superalgebra one more series Q of supergroups of classical type: the centralizer group of an odd involution $p: T \to T$, $\tilde{p} = 1$, $p^2 = \mathrm{id}$. The Lie superalgebras of these groups, slightly diminished if necessary (to kill a center etc.) constitute the classical part of the Kac classification [15] of simple finite dimensional Lie superalgebras. There are also two exceptional types (one having parameters) and Catzdan-type superalgebras of formal vector fields which happen to be finite-dimensional when defined on Grassmann algebras.

A superderivation $X: A \to A$ verifies the Leibniz formula $X(ab) = (Xa)b + (-1)^{\tilde{X}\tilde{a}} aXb$. There are two natural modules of relative differentials of a commutative A-algebra B: $\Omega^1_{ev} B/A$ and $\Omega^1_{odd} B/A$, classifying even and odd differentials respectively. Later on we use mainly $\Omega^1_{odd} B/A$ since the corresponding de Rham complex is supercommutative while for $\Omega^1_{ev} B/A$ it is super anticommutative.

3. Supergeometry. The most general known notion of a "space with even and odd coordinates" is that of superspace. Superspace is a pair $(M, \mathcal{O}_M)$, where M is a toplogical space, $\mathcal{O}_M$ a sheaf of local supercommutative rings on it. Morphisms of superspaces are morphisms of locally ringed spaces compatible with the gradations of structure sheaves.

All objects of the main geometric categories,-differentiable and analytic manifolds, analytic spaces, schemes, - are trivially superspaces, with $\mathcal{O}_M = \mathcal{O}_{M,0}$. Such superspaces we shall call purely even ones.

In the general case we set $J_M = \mathcal{O}_M \cdot \mathcal{O}_{M,1}$, $Gr_i\, \mathcal{O}_M = J^i_M / J^{i+1}_M$. Furthermore, $M_{rd} = (M, Gr_0 \mathcal{O}_M)$, $GrM = (M, \bigoplus_{i=0}^{\infty} Gr_i \mathcal{O}_M)$.

The structure sheaf GrM has a natural $\mathbb{Z}$-gradation. To consider GrM as a superspace we reduce it modulo 2.

With the help of these constructions we can define the most simple and important class of superspaces. We shall call a superspace $(M,\mathcal{O}_M)$ a supermanifold, analytic or algebraic, iff a) M_{rd} is a pure even manifold of the respective class; b) the sheaf $\mathcal{O}_M$ is locally isomorphic to the sheaf $Gr\mathcal{O}_M$, which is in turn isomorphic to the Grassmann algebra of the locally free (over Gr_0M) sheaf J_M/J_M^2 of finite rank. (Note that this Grassmann algebra should be called symmetric in the superalgebra since J_M/J_M^2 is of pure odd rank).

One proves then that C^∞ and analytic supermanifolds can be described by local charts $(x_1,\ldots,x_m;\ \xi_1,\ldots,\xi_n)$. The sheaf $\mathcal{O}_M$ locally consists of the expressions $\sum_\alpha f_\alpha(x)\xi^\alpha$ where f_α are even functions of the corresponding type. An essential feature of supergeometric constructions is their invariance with respect to the coordinate changes mixing even and odd functions. Set, e.g., $y = x + \xi_1\xi_2, \eta_1 = (1+x^2)\xi_1,\ \eta_2 = \xi_2$. Then the local function $f(x)$ goes into $f(y - \frac{1}{1+x^2}\ \eta_1\eta_2) = f(y) - \frac{f'(y)}{1+x^2}\ \eta_1\eta_2$. The appearance of derivatives in the coordinate change formulas plays an essential role in the Langrangian formalism of supersymmetric field theoretic models. It also shows that in continuous supergeometry a natural structure sheaf ought to contain certain distributions. The peculiarities of continuous supergeometry were not studied for this reason.

The most important superspaces which are not necessarily supermanifolds can be easily defined in the analytic category. They are the superspaces $(M,\mathcal{O}_M)$ such that $(M,\mathcal{O}_{M,0})$ is an analytic space and $\mathcal{O}_{M,1}$ is $\mathcal{O}_{M,0}$-coherent. In the same way one defines superschemes.

The notion of a locally free sheaf of $\mathcal{O}_M$-modules is a natural substitute for vector bundles. For supermanifolds over a field the tangent sheaf $\mathcal{T}M$ and the cotangent sheaf $\Omega^1M = \Omega^1_{odd}M$ are defined in the usual way, using superderivations over the ground field. The generalization to the relative case is selfevident. The rank of $\mathcal{T}M$ is called the dimension of the supermanifold.

Now the reader will easily transcribe the descriptions given in n°4.1 into the definitions of the superspaces in the algebraic, analytic or C^∞ categories. Notice that the grassmannian is endowed with the tautological sheaf, which is generated by the rows of Z_I over U_I. Over the projective superspace it is denoted $\mathcal{O}(-1)$.

4. Methods of construction of superspaces. a) Let $(M,\mathcal{O}_M)$ be a pure even locally ringed space, E a locally free sheaf of modules of rank $0|q$ over $\mathcal{O}_M$. Then $(M,\mathcal{O}_{M,s} = S(E))$ is a superspace, which is called split. In the C^∞ category every supermanifold is split, i.e. can be obtained in this way from a manifold and a vector bundle on it. In the analytic and algebraic categories this is not true anymore, e.g. the Grassmannians are not analytically split unless the tautological bundle is of pure even or pure odd rank. (cf. below).

b) As in pure even analytic and algebraic geometry, very important superspaces are defined by their functor of points. We have already mentioned the algebraic supergroups GL. SL , OSp, ΠSp and in the next section we shall work with the flag superspaces $F(d_1,\ldots,d_K;T)$, where $d_1 < d_2 < \ldots < d_K$ are the dimensions of the components of a flag in the linear superspace T. We expect that the main theorems on the representability of various functors and moduli problems admit their counterparts in supergeometry although the systematic work has barely begun. We shall state two results proved by A. Vaintrob which show the existence of a local deformation theory of Kodaira-Spencer type. The basic definitions are readily stated in the context of analytic superspaces. The infinitesimal deformations are represented by the ring $\mathbb{C}[x,\xi] \mid (x^2,x\xi)$.

Let M be a compact supermanifold, TM its tangent sheaf.

5. Theorem. a) Let $\dim H^1(M,TM) = a|b$. If $H^2(M,TM) = 0$, then in the category of analytic supermanifolds there exists a local deformation of M over $B = \mathbb{C}^{a|b}$ such that the Kodaira-Spencer map $\rho: T_0B \to H^1(M,TM)$ is an isomorphism.

b) Any deformation of M over a supermanifold with surjective Kodaira-Spencer map is complete; in particular, it is versal, if ρ is isomorphic. ∎

Let us give an example. Let M be a compact analytic supermanifold of dimension $1|1$. It is completely defined by the Riemann surface $M_0 = M_{rd}$ and the invertible sheaf $\Pi J_M = L$ on it. Assume that genus $M = g > 1$, $\deg L = 0$ and L is not isomorphic to $\mathcal{O}_{M_0}$. In this case $\mathrm{rk}\, H^1(M, TM) = 4g-3|4g-4$. The even part $4g-3$ of this dimension corresponds to the classical manifold Z of deformations of the pair (M_0, L) which is fibered by Jacobians over the coarse moduli space of curves. Theorem 5 shows that outside of the zero section this manifold Z is naturally extended to the supermanifold of odd dimension $4g-4$. This structure deserves further study.

6. <u>Theorem</u>. Let M be a closed compact subsupermanifold of the complex supermanifold M'. Let N be a normal sheaf to M.

a) If $\dim H^0(M,N) = a|b$ and $H^1(M,N) = 0$, there exists a versal local deformation of M in M' over $B = \mathbb{C}^{a|b}$.

b) A deformation of M in M' over B is complete iff the Kodaira-Spencer map $\rho: T_0 B \to H^0(M,N)$ is surjective.

7. <u>Example.</u> Let us return to the definition of a supergrassmannian and illustrate certain of our constructions. The supergrassmannian $G = G(1|1;\, \mathbb{C}^{2|2})$ is covered by four $2|2$-dimensional affine superspaces. The corresponding Z_I-matrices are

$$\begin{pmatrix} x_1 & 1 & \xi_1 & 0 \\ \eta_1 & 0 & y_1 & 1 \end{pmatrix}, \qquad \begin{pmatrix} x_2 & 1 & 0 & \xi_2 \\ \eta_2 & 0 & 1 & y_2 \end{pmatrix},$$

$$\begin{pmatrix} 1 & x_3 & \xi_3 & 0 \\ 0 & \eta_3 & y_3 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 1 & x_4 & 0 & \xi_4 \\ 0 & \eta_4 & 1 & y_4 \end{pmatrix}$$

Using the prescription in the beginning of this section, one calculates the transition functions, e.g.

$$\begin{pmatrix} x_1 & \xi_1 \\ \eta_1 & y_1 \end{pmatrix} = \begin{pmatrix} x_4 & \xi_4 \\ \eta_4 & y_4 \end{pmatrix}^{-1} =$$

$$= \begin{pmatrix} x_4^{-1} + x_4^{-2} y_4^{-1} \xi_4\eta_4 \, , & -x_4^{-1}y_4^{-1}\xi_4 \\ -x_4^{-1}y_4^{-1}\eta_4 & , \; y_4^{-1}-x_4^{-1}y_4^{-2}\xi_4\eta_4 \end{pmatrix}$$

It follows immediately, that $G_{rd} = P^1 \times P^1$. A calculation shows also that

$$J_G^2 \simeq \Omega^2(P^1 \times P^1)$$

so that the obstruction ω to the splitting of G lies in the group

$$H^1(T(P^1{\times}P^1) \otimes \Omega^2(P^1{\times}P^1)) = H^1(P^1,\Omega^1P^1)^2 = \mathbb{C} \oplus \mathbb{C} \; .$$

One can check directly, using the Čech cocycle in the standard atlas, that this obstruction is (1,1). Hence G is not split. (Notice that the projective superspace is split: $P^{m|n} = (P^m, S(\prod \mathcal{O}^n_{pm}(-1)))$.

Moreover, G is not a projective supermanifold. In fact, the image of the Picard group $H^1(G,\mathcal{O}^*_{G,0}) \to \mathrm{Pic}\,(P^1{\times}P^1)$ consists of the classes of sheaves $\mathcal{O}(a,-a)$, $a \in \mathbb{Z}$, since the obstruction to extending $\mathcal{O}(a,b)$ from G_{rd} to G is essentially $(a+b)\omega$. Therefore, any supergrassmannian $G(a|b;\mathbb{C}^{m|n})$ with $0 < a < m$, $0 < b < n$ is non-projective since G admits a closed embedding in such a Grassmannian.

This example shows that the use of projective technique in the algebraic supergeometry is restricted, and one is obliged to generalize those methods of algebraic and analytic geometry which do not rely upon the existence of ample invertible sheaves.

For example, P. Deligne conjectured that the dualizing sheaf on a smooth complex supermanifold X in $\mathrm{Ber}\,\Omega^1_{ev}X$. This was proved by I. Penkov [29] who has demonstrated that in this case working with $\mathcal{D}$-modules on a supermanifold permits one to effectively reduce the situation to the pure even one.

5. Schubert supercells.

1. Basic notions. Let G be a semisimple algebraic group, $B \subset G$ its Borel subgroup. The G-orbits of $G/B \times G/B$ form a finite stratification on this manifold whose strata Y_w are numbered by elements w of the Weyl group $W = W(G)$ (Bruhat decomposition). We shall call locally closed submanifolds $X_w(b) = (\{b\} \times G/B) \cap Y_w \subset G/B$ the Schubert cells. In the same way, using G-orbits of $G/B \times G/P$, one defines the Schubert cells for a parabolic subgroup $P \subset G$. The geometry of the Schubert cells plays an important role in many developments of characteristic classes theory and representation theory.

In this section we shall define Schubert supercells for complete flag superspaces of classical type and explain how some classical results generalize in this context.

Let T be a linear superspace of dimension $m|n$ over a field. We shall consider the following algebraic supergroups G given together with their fundamental representation T: a) $G = SL(T)$; b) $G = OSp(T)$, the automorphism group of a nondegenerate even symmetric form $b: T \to T^*$; c) $G = \Pi Sp(T)$, the automorphism group of a nondegenerate odd alternate form $b: T \to T^*$; d) $G = Q(T)$, the automorphism group of an odd involution $p: T \to T$, $p^2 = \mathrm{id}$. In the cases ΠSp and Q we have $m = n$.

The counterpart of the classical manifold G/B is the supermanifold F of complete flags in T, which are in addition invariant with respect to b or p in the cases $G = OSp$, ΠSp, Q (the exact definitions are given below). Several differences between this situation and the classical one are worth mentioning.

First, the stabilizers of complete flags B in general are not maximal solvable subsupergroups. However, they play the same role as the classical Borel subgroups both in the theory of highest weight [16] and in the theory of the Schubert supercells. Second, not all subgroups B are pairwise conjugate, and the flag manifolds F consist of several components. Third, the stratification we want to construct is not purely set-theoretic. In fact it will be a decomposition of $F \times F$ into a union of locally closed subsuperschemes. The flag realization is suitable for this construction.

The first unsolved problem is to define a cohomology theory in which the classes of the Schubert cells would be free generators. As we conjectured in the introduction, this may require a sort of compactification along the odd dimensions.

We shall now give some details.

2. <u>The connected components of flag supermanifolds</u>. These connected components are naturally numbered by the sets ${}^{G}I$ which are defined as follows. Set

$$^{SL}I = \{(\delta_1,\ldots,\delta_{m+n}) \mid \delta_i = 1|0 \quad \text{or} \quad 0|1, \ \sum_{i=1}^{m+n} \delta_i = m|n\}.$$

Furthermore,

$$^{OSp}I = \{(\delta_1,\ldots,\delta_{m+n}) \in {}^{SL}I \mid \delta_1 = \delta_{m+n+1-i}\ , \ i = 1,\ldots,m+n\} \ ,$$

$$^{\Pi Sp}I = \{(\delta_1,\ldots,\delta_{2m}) \in {}^{SL}I \mid \delta_i = \delta^c_{2m+1-i}\ , \ i = 1,\ldots,2m\} \ ,$$

where $(p|q)^c = q|p$. Finally, ${}^{Q}I = \{(\overbrace{1|1,\ldots,1|1}^{m})\}$ (the one element set). We say that the flag $f\colon 0 = S_0 \subset S_1 \subset \ldots \subset S_{m+n} = T$ is of type $I \in {}^{G}I$, if $\delta_i(I) = \operatorname{rk} S_i/S_{i+1}$. For groups $G = OSp, \Pi Sp, Q$ the flag f is called G-stable if the following conditions are fulfilled:

$$b(S_i) = S^{\perp}_{m+n-i} \quad \text{for} \quad OSp,\ \Pi Sp\ ;\ p(S_i) = S_i \quad \text{for} \quad Q\ .$$

The functor ${}^{G}F_I$ on the category of superschemes over ground field associates with a superscheme S the set of flags of type $I \in {}^{G}I$ in the sheaf $\mathcal{O}_S \otimes T$; for $G \neq SL$ the flags should be G-stable. (A flag is a filtration of $\mathcal{O}_S \otimes T$ by subsheaves S_i such that all injections $S_i \subset S_j$ locally split; G-stability is defined with respect to $id_{\mathcal{O}_S} \otimes b$ or $id_{\mathcal{O}_S} \otimes p$).

3. Theorem. Functor ${}^{G}F_{I}$ is representable by a supermanifold which is irreducible except for $G = \mathcal{O}Sp(2r|2s)$, $r \geq 1$, in which case ${}^{G}F_{I}$ consists of two ismorphic components. Furthermore,

$$\dim {}^{SL}F_{I} = \left(\frac{m(m-1)}{2} + \frac{n(n-1)}{2} \mid mn\right);$$

$$\dim {}^{OSp}F_{I} = \begin{cases} (r^2 + s^2 \mid (2r+1)s) & \text{for } m = 2r+1,\ n = 2s > 0; \\ (r^2 - r + s^2 \mid 2rs) & \text{for } m = 2r,\ n = 2s > 0 \end{cases}$$

$$\dim {}^{Q}F_{I} = \left(\frac{m(m-1)}{2} \mid \frac{m(m-1)}{2}\right) \quad ;$$

$$\dim {}^{\Pi Sp}F_{I} = \left(rs + \frac{r(r-1)}{2} + \frac{s(s+1)}{2} \quad rs + \frac{r(r+1)}{2} + \frac{s(s-1)}{2}\right),$$

where $(r|s) = \sum_{i=1}^{m} \delta_i(I)$. ■

Of course, the functors of noncomplete flags also are representable and the morphisms of projection onto a subflag are representable by morphisms of supermanifolds. It is convenient to prove theorem 3 by induction on the length of a flag, starting with relative Grassmannians as in section 4. The reader shall find most of details in [24].

Now we shall set ${}^{G}F = \coprod_{I \in G_I} {}^{G}F_{I}$ and denote by $S_i \subset \mathcal{O}_F \otimes T$ the components of the tautological flag on ${}^{F}G$. There are two natural flags on ${}^{G}F \times {}^{G}F$, $\{p_1^*(S_i)\}$ and $\{p_2^*(S_j)\}$, where $p_{1,2}$ are the projections. In the classical theory every G-orbit consists of those points of ${}^{G}F \times {}^{G}F$, over which the type of relative position of flags $\{p_1^*(S_i)\}$ and $\{p_2^*(S_j)\}$ is fixed. We can imitate this definition in supergeometry taking functor of points instead of geometric points.

The type of relative position of complete flags (S_i) and (S_j') in T is, by definition, the matrix $d_{ij} = \mathrm{rk}(S_i + S_j')$.

Let us introduce the Weyl groups G_W , acting on ${}^{G}I$:

$${}^{SL}W = S_{m+n} \ ; \ {}^{Q}W = S_m \ ; \ {}^{G}W = \{g \in S_{m+n} | g({}^{G}I) \subset {}^{G}I\} \quad \text{for } G = OSp, \Pi Sp.$$

The reader will notice that our ${}^{G}W$ is in general different from the Weyl group of G_{rd}, e.g. the latter is $S_m \times S_n$ for SL and not S_{m+n}. As we shall see in a moment, in the theory of Schubert supercells it is this big group, which contains the odd reflections, which is the right one.

4. Lemma. The types of relative positions of complete G-stable flags are in (1,1)-correspondence with the triples (I,J,w), where $I,J \in {}^{G}I$, $w \in {}^{G}W$, $J = w(I)$ ∎.

The proof is purely combinatorial.

5. Bruhat subsets. We set now for $w \in {}^{G}W$, $I,J = {}^{G}I$:

$$|Y_{w,IJ}| = \{x \in {}^{G}F_I \times {}^{G}F_J \mid rk(p_1{*}S_i(x) + p_2{*}S_j(x)) = d_{ij,w}\}$$

where $(d_{ij,w})$ is the type of relative position corresponding to the triple (I,J,w) in view of Lemma 4 (if $J \neq w(I)$ we set $|Y_{w,IJ}| = \emptyset$). Furthermore, put

$$|Y_w| = \coprod_{I,J} |Y_{w,IJ}| \subset {}^{G}F \times {}^{G}F \quad .$$

6. Theorem. Each set $|Y_w|$ carries a canonical structure of the locally closed subsuperscheme $Y_w \subset {}^{G}F \times {}^{G}F$, such that the decomposition $\coprod_w Y_w$ is the flattening stratification for the family of sheaves $S_{ij} = p_1{*}\, S_i + p_2{*}S_j$ on ${}^{G}F \times {}^{G}F$ ∎.

We recall that by definition of a flattening stratification this condition means that each morphism $q: X \to {}^{G}F \times {}^{G}F$ for which all the sheaves $q^*(S_{ij})$ are flat uniquely decomposes as $X \to \coprod_w Y_w \to {}^{G}F \times {}^{G}F$. The proof of the existence of the flattening stratification is the same as in the pure even case.

7. Superlength. In the classical theory the dimension of a Schubert cell associated with $w \in W$ equals the minimal length of a decomposition of w into a product of basic reflections. To state the counterpart of this fact in supergeometry we need several definitions.

We shall call the following elements of ${}^{G}W$ the basic reflections:

$\sigma_i = (i,i+1)$ for $G = SL, Q$;

$\sigma_i = (i,i+1)\ (m+n+1-i,\ m+n-i)$, $i+1 \leq [\frac{m+n}{2}]$;

$\tau_\ell = (\ell, m+n+1-\ell)$, $\ell = [\frac{m+n}{2}]$ for $G = OSp, \prod Sp$.

The superlength ${}^G\ell(w)$, $w \in {}^GW$ will be defined inductively . This is a vector of superdimensions $({}^G\ell_{IJ}(w) \mid I,J \in {}^GI)$ such that

a) For a basic reflection $\sigma \in {}^GW$ we have ${}^G\ell_{IJ}(\sigma) = 0$ if $J \neq \sigma(I)$; the other possibilities are contained in the table:

σ \ G	SL	OSp	$\prod$Sp	Q
σ_i	$1\|0$ $(I=J)$; $0\|1$ $(I\neq J)$;	$1\|0$ $(I=J)$; $0\|1$ $(I\neq J)$;	$1\|0$ $(I=J)$; $0\|1$ $(I\neq J)$,	$1\|1$
τ_ℓ	-	$1\|0$ $(\delta_\ell(I) = 1\|0,$ $m\|n = 2r+1\|2s)$; $1\|1$ $(\delta_\ell(I) = 0\|1,$ $m\|n = 2r+1\|2s)$; $0\|0$ $(\delta_\ell(I) = 1\|0,$ $m\|n = 2r\|2s)$; $1\|0$ $(\delta_\ell(I) = 0\|1,$ $m\|n = 2r\|2s)$.	$0\|1$ $(\delta_\ell(I)=1\|0)$ $0\|0$ $(\delta_\ell(r)=0\|1)$	-

b) Let $w = \sigma^K \ldots \sigma^1$ be an irreducible decomposition of w as a product of basic reflections. Set $I_i = \sigma^i \ldots \sigma^1(I)$ and

$${}^G\ell_{IJ}(w) = \sum_{i=0}^{K-1} {}^G\ell_{I_i, I_{i+1}}(\sigma^{i+1}) \text{ , if } J = w(I);$$

$${}^G\ell_{IJ}(w) = 0 \text{ , if } J \neq w(I).$$

8. Theorem. The projection map $Y_{w,IJ} \to {}^GF_I$ is surjective, and locally over ${}^GF_I$ the Bruhat manifold $Y_{W,IJ}$ is a relative affine superspace

of dimension ${}^{G}\ell_{IJ}(w)$. ■

In other words, the dimension of a Schubert cell $Y_w(b)$ coincides with the superlength of w.

From the geometric proof of the theorem some purely combinatorial facts follow. For example, ${}^{G}\ell_{IJ}(w)$ does not depend on the choice of an irreducible decomposition of w; furthermore, ${}^{G}\ell_{IJ}(w) = {}^{G}\ell_{IJ}(w^{-1})$ for $G \neq \Pi Sp$, finally for $G = \Pi Sp$

$$ {}^{G}\ell_{IJ}(w) + \dim {}^{G}F_I = {}^{G}\ell_{JI}(w^{-1}) + \dim {}^{G}F_J, $$

if $J = w(I)$.

6. Geometry of supergravity

1. Minkowski space and Schubert cells. The objective of this section is to describe a model of simple supergravity from the view point which was introduced in [26] where the kinematic constraints of supergravity were interpreted as the integrability conditions for a curved version of a flag superspace.

To explain the essence of our approach let us recall the usual exposition of general relativity. The space-time without gravitational field is the Minkowski space of special relativity $\mathbb{R}^4$ with a metric which in an inertial frame takes the form $dx^2 - \sum_{i=1}^{3} dx_i^2$. The gravitation field reflects itself in the curvature of space-time which becomes a smooth four-manifold M^4 with the pseudoriemannian metric $g_{ab}dx_a dx_b$. The dynamics is governed by the Lagrangian (action density) $R \operatorname{vol}_g$ where R is the scalar curvature, vol_g the volume form of g.

The models of supergravity in superspace studied in many recent works [12],[13],[34] also start with certain geometric structures on a differentiable supermanifold $M^{m|n}$ which are then used to define a (super) Lagrangian which is a section of the sheaf $\operatorname{Ber} M = \operatorname{Ber} \Omega^1_{ev} M$. There are physically meaningful cases with $m \neq 4$, e.g. the case $m|n = 11|32$ is now considered as the most fundamental one.

What is still very much unclear, is the question what exactly is the geometry to start with, i.e. the kinematics of supergravity. The naive suggestion to use a supermetric was quickly seen inadequate. The most universal known method is the Cartan approach. One starts with an affine connection and then painstakingly guesses the so called constraints and the action density. The constraints are the differential equations which must imply no equations of motion. The physical interpretation and quantization of constrained fields is a difficult task and one faces the problem of solving constraints and expressing everything in terms of free fields. This approach was successful more than once but the poor command of underlying geometry hinders the work considerably.

Our approach essentially interprets the constraints as integrability conditions ensuring the existence of certain families of submanifolds in $M^{m|n}$, the geometry of these families being a curved geometry of Schubert supercells.

Let us first describe from this viewpoint the simplest example, the Plücker-Klein-Penrose model of Minkowski space.

Let T be a four dimensional complex space (Penrose's twistor space). Let $G = G(2;T)$ be the Grassmannian of planes in T , $S = S_\ell$ the tautological sheaf on G , $S_r = (T \otimes \mathcal{O}_G/S_\ell)^*$. There is a canonical isomorphism $\Omega^1 G = S_\ell \otimes S_r$, and the subsheaf $\Lambda^2 S_\ell \otimes \Lambda^2 S_r \subset S^2(\Omega^1 G)$ can be interpreted as the holomorphic conformal metric on G. Choose a big cell $U \subset G$. The complement $G \setminus U$ is a singular divisor, the light-cone at infinity, and there are sections $\varepsilon_\ell, \varepsilon_r$ of the sheaves $\Lambda^2 S_\ell, \Lambda^2 S_r$ on U having a pole of first order at this divisor. The complex metric $\varepsilon_\ell \otimes \varepsilon_r \in \Gamma(U, S^2\Omega^1 G)$ is well defined up to multiplicative constant. Now introduce a real structure ρ on $T \oplus T^*$ interchanging T and T^*. The involution ρ acts on $\mathbb{C}$-points of $G(2,T)$ since $G(2,T)$ canonically identifies with $G(2;T^*)$. Let ρ be compatible with $(U, \varepsilon_\ell, \varepsilon_r)$ in the sense that $U^\rho = U$, $\varepsilon_\ell^{\ \rho} = \varepsilon_r$. The following statements can be directly verified.

a) The real (i.e. ρ-invariant) points of the big cell U form the space $\mathbb{R}^4$. The restriction of $\varepsilon_\ell \otimes \varepsilon_r$ to it is a Minkowski metric.

b) The real three-dimensional Schubert manifolds in the Grassmannian

G intersected with $U(\mathbb{R})$ form the system of light cones of this metric.

c) There are no real two-dimensional Schubert cells in G. In $U(\mathbb{C})$ they define two connected families of complex planes. These families play an essential role in the theory of Yang-Mills fields. In fact, the integrability of a connection along one family means that this connection is an (anti) self dual solution of the Yang-Mills equation.

In a curved space-time of general relativity null geodesics and light cones still exist and, moreover, define the corresponding metric up to a conformal factor. To break the conformal invariance one may choose metrics $\varepsilon_\ell, \varepsilon_r$ for two-component Weyl spinors, left and right.

We describe supergravity along these lines. In nn° 2,3 a flag model of Minkowski superspace is introduced, In n°4 we explain that in a curved superspace two families of $0|2N$ - dimensional Schubert supercells should be preserved. Finally, in nn°5,6 we define the dynamics by means of an action density, expressed through the Ogievetsky-Sokachev prepotential [28].

2. Minkowski superspace. Fix an integer $N \geq 1$ and a linear complex superspace T of dimension $4|4N$. Set $M = F(2|0, 2|N;T)$, i.e. a S-point of M is a flag $S^{2|0} \subset S^{2|N}$ in $\mathcal{O}_S \otimes T$. Moreover, define the left and right superspaces as Grassmannians

$$M_\ell = G(2|0;T), \; M_r = G(2|N;T) = G(2|0;T^*).$$

Denote by $S^{2|0} \subset S^{2|N}$ the tautological flag in $\mathcal{O}_M \otimes T$, by $\tilde{S}^{2|0} \subset \tilde{S}^{2|N}$ the orthogonal flag in $\mathcal{O}_M \otimes T^*$. Set $F_\ell = S^{2|N}/S^{2|0}$, $F_r = \tilde{S}^{2|N}/\tilde{S}^{2|0}$. Let $\pi_{\ell,r}: M \to M_{\ell,r}$ be the canonical maps. Let $T_\ell M = TM/M_r$, $T_r M = TM/M_\ell$ (recall that we work in the category of complex superspaces). Since M over $M_{\ell,r}$ is a relative Grassmannian, a standard argument gives canonical isomorphisms

$$T_\ell M = (S^{2|0})^* \otimes F_\ell \quad , \; T_r M = F_\ell^* \otimes (\tilde{S}^{2|0})^*.$$

Combining this with the map $F_\ell \otimes F_\ell \to \mathcal{O}_M$ we get a natural map

(1) $a: T_\ell M \otimes T_r M \to (S^{2|0})^* \otimes (\tilde{S}^{2|0})^*.$

On the other hand, the relative tangent sheaves $T_{\ell,r}M$ are integrable distributions, i.e. locally free subsheaves of Lie superalgebras in TM, of rank $0|2N$. Every point of M is contained in two closed sub-supermanifolds of dimension $0|2N$ tangent to $T_\ell M$ and $T_r M$ respectively. They are the Schubert cells we are interested in. The supercommutator between $T_\ell M$ and $T_r M$ defines the Frobenius map

(2) $b: T_\ell M \otimes T_r M \to TM/(T_\ell M + T_r M) = T_0 M$

The following statements contain the essential geometric features of the picture we want to keep in the curved case.

3. Proposition. a) The sum $T_\ell M + T_r M$ in TM is a direct subsheaf in TM of rank $0|4N$.

b) There is a well defined isomorphism $T_0 M = (S^{2|0})^* \otimes (\tilde{S}^{2|0})^*$ making the maps (1) and (2) to coincide. ■

Finally, as in n°1, we must introduce a real structure ρ on $T \oplus T^*$ (in superalgebra $(ab)^\rho = (-1)^{\tilde{a}\tilde{b}} a^\rho b^\rho$; cf. [24] for further details). We shall assume that $T^\rho = T^*$, in this case $(T_\ell M)^\rho = T_r M$, $(S^{2|0})^\rho = \tilde{S}^{2|0}$. One can check that over a ρ-stable big cell in M some natural sections of $T_\ell M$, $T_r M$, $T_0 M$ generate the Poincare superalgebra introduced by physicists (see e.g. [34]).

4. Curved superspace. A complex supermanifold $M^{4|4N}$ with the following structures will be called superspace of N-extended supergravity.

a) Two integrable distributions $T_\ell M, T_r M \subset TM$ or rank $0|2N$ whose sum is direct.

b) Two locally free sheaves S_ℓ, S_r of rank $2|0$, two locally free sheaves $F_\ell; F_r = F_\ell^*$ of rank $0|N$ and structure isomorphisms $T_\ell M = S_\ell^* \otimes F_\ell$, $T_r M = F_r \otimes S_r^*$.

c) A real structure ρ on M such that its real points in M_{rd} form a four-manifold, and extensions of this real structure to $S_\ell \oplus S_r$,

$F_\ell \oplus F_r$ interchanging left subsheaves with right ones.

d) Volume forms $v_{\ell,r} \in \Gamma(M_{\ell,r}, \text{Ber } M_{\ell,r})$ such that $v_\ell^\rho = v_r$. A choice of these volume forms corresponds to the choice of spinor metrics $\varepsilon_\ell, \varepsilon_r$ in n°1.

This data is subjected to one axiom. Set $T_0M = TM/(T_\ell M \oplus T_r M)$. Then the Frobenius map $\varphi: T_\ell M \otimes T_r M \to T_0 M$ coincides with the natural map $S_\ell{}^* \otimes F_\ell \otimes F_r \otimes S_r{}^* \to S_\ell{}^* \otimes S_r{}^*$ under the appropriate identification $T_0M = S_\ell{}^* \otimes S_r{}^*$ as in Proposition 3 b).

5. <u>Lagrangian</u>. Let M be a superspace of N-extended supergravity. Using the data above one can construct a canonical isomorphism:

$$\text{(3)} \qquad \text{Ber } M = [\pi_\ell{}^* \text{ Ber } M_\ell \otimes \pi_r{}^* \text{ Ber } M_r]^{\frac{2-N}{4-N}} \quad (N \neq 4)$$

(From this point on we define Ber M as $\text{Ber}^*(\Omega^1_{odd}M)$).

Hence the volume forms v_ℓ, v_r make it possible to define a section (for $N \neq 4$)

$$\text{(4)} \qquad w = (\pi_\ell{}^* v_\ell \otimes \pi_r{}^* v_r)^{\frac{N-2}{N-4}} \in \Gamma(M, \text{Ber } M)$$

In this way we get for $N = 1$ the correct action of simple supergravity. In the case $N = 2$ the action is certainly wrong since it gives trivial equations of motion. It seems that considering N as a formal parameter and taking the left (or right) part of the coefficient of the Taylor expansion of (4) at $N = 2$ we get an action suggested by E. Sokachev. Anyway, for $N > 1$ one must take into account new constraints which might take the form of integrability of more Schubert cells.

It is also certain that the other types of flag supermanifolds and their curved versions are necessary for a fuller understanding of supergravity and super Yang-Mills equations. For example, in a recent paper by A. Galperin, E. Ivanov, S. Kalytsyn, V. Ogievetsky and E. Sokachev the manifold $F(2|0,2|1,2|2;T)$ implicitly appears which in the curved version can be defined as the projectivized bundle $P(F_\ell) = P(F_r) \to M$.

In the same vein, for the largest physically acceptable case $N = 8$, the 11|32-dimensional flag supermanifold $F(2|0,2|1,2|8,T)$ or its curved version $P(F_\ell) \to M$ seems to be the space considered in the context of the so called dimensional reduction, or the generalized Kaluza-Klein model.

6. Prepotential. To conclude, we give some coordinate calculations which make it possible to identify our geometric picture with that of the article [28]. Set $N = 1$ and choose in M_ℓ a local coordinate system $(x_\ell{}^a, \theta_\ell^\alpha)$. Assume that the following properties are true: functions $(x_\ell^a)_{rd}$ are ρ-stable and functions $(x^a = \frac{1}{2}(x_\ell^a + x_r^a),$ $\theta_\ell^\alpha, \theta_r^{\dot\alpha})$ are local coordinates on M, where $\theta_r^{\dot\alpha} = (\theta_\ell^{\dot\alpha})^\rho$, $x_r^a = (x_\ell^a)^\rho$. Such coordinates (x_ℓ, θ_ℓ) on M_ℓ, (x_r, θ_r) on M_r and $(x, \theta_\ell, \Theta_r)$ on M will be called distinguished ones.

Now we set

$$H^a = \frac{1}{2i}(x_\ell{}^a - x_r{}^a). \tag{5}$$

These four real nilpotent superfunctions on M are called the Ogievetsky-Sokachev prepotential. Working locally and identifying $F_{\ell,r}$ with ΠO_M we can say that the prepotential completely defines the geometry of the superspace, except for the forms $v_{\ell,r}$ which must be given separately :

$$v_{\ell,r} = \Phi^3_{\ell,r}\, D^*(d\theta^\alpha_{\ell,r}\; d\theta^a_{\ell,r}) \tag{6}$$

Some calculations (cf. [26] for details) show that the action (3) can be expressed through (5) and (6) by means of the Wess-Zumino formula

$$w = \frac{1}{8}\,\mathrm{Ber}(E^A_B)\; D^*(d\theta^\alpha_\ell, d\theta_r{}^{\dot\beta}, dr^a) , \tag{7}$$

where E^A_B is the transition matrix between the frames

$\left(\frac{\partial}{\partial x^a}, \frac{\partial}{\partial \theta_\ell{}^\alpha}, \frac{\partial}{\partial \theta_r{}^{\dot\alpha}}\right) = (\partial_a, \partial_\alpha, \partial_{\dot\alpha})$ and $(\frac{i}{2}[\tilde\Delta_\alpha, \tilde\Delta_{\dot\alpha}], \tilde\Delta_\alpha, \tilde\Delta_{\dot\alpha})$.

This last frame can be defined in three steps.

Step 1. $\Delta_\alpha = \partial_\alpha + X^a_\alpha \partial_a$ and $\Delta_{\dot\alpha} = -\partial_{\dot\alpha} - X^a_{\dot\alpha}\partial_a$ are defined as local bases for $T_\ell M$ and $T_r M$ respectively. From this one finds the coefficients

$$X_{\alpha}^{a} = i\left[\left(1 - i\frac{\partial H}{\partial x}\right)^{-1}\right]_{b}^{a} H^{b};\quad X_{\dot\alpha}^{a} = -i\left[\left(1 + i\frac{H}{x}\right)^{-1}\right]_{b}^{a} \partial_{\dot\alpha} H^{b}$$

Step 2. The volume forms $v_{\ell,r}$ define the spinor metrics $E_{\ell,r} \in \mathrm{Ber}\, S_{\ell,r}$:

$$E_{\ell} = (\pi_{\ell}{}^{*}v_{\ell})^{1/3} \otimes (\pi_{r}{}^{*}v_{r})^{-2/3},\quad E_{r} = (\pi_{\ell}{}^{*}v_{\ell})^{-2/3} \otimes (\pi_{r}{}^{*}v_{r})^{1/3}.$$

Step 3. The multiplier F, defining $\tilde\Delta_{\alpha} = F\Delta_{\alpha}$ and $\tilde\Delta_{\dot\alpha} = F^{\rho}\Delta_{\dot\alpha}$, is constructed in such a way, that $D^{*}(\tilde\Delta_{\alpha}) = E_{\ell}$, $D^{*}(\tilde\Delta_{\dot\alpha}) = E_{r}$.

The structure frame $(\frac{i}{2}[\tilde\Delta_{\alpha},\tilde\Delta_{\dot\alpha}],\tilde\Delta_{\alpha},\tilde\Delta_{\dot\alpha})$ can be used to describe the geometry of simple supergravity Cartan style. In this approach it appears as the final product rather than the starting point.

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