Note on the Landweber-Stong elliptic genus

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In algebraic topology one studies genera, which are ring homomorphisms from the oriented bordism ring Ω_*^{SO} to a $\mathbb{Q}$-algebra R (commutative, with 1). To a genus $\varphi: \Omega_*^{SO} \to R$ one associates the following three power series with coefficients in R :

(i) $g(x)$, an odd power series with leading term x, the logarithm of the formal group law of φ (this means that the formal group, whose definition we do not repeat here, equals $g^{-1}(g(x)+g(y))$). It is given explicitly by $g(x) = \sum_{n=0}^{\infty} \varphi(\mathbb{CP}^{2n}) \frac{x^{2n+1}}{2n+1}$ and hence, since the classes of the $\mathbb{CP}^{2n}$ generate $\Omega_*^{SO} \otimes \mathbb{Q}$, determines φ completely.

(ii) $P(u)$, an even power series with leading term 1, the Hirzebruch characteristic power series of φ. This means that if $\mathcal{P}$ denotes the stable $H^*(\cdot;R)$-valued exponential characteristic class on oriented bundles characterized by $\mathcal{P}(\xi) = P(c_1(\xi))$ if ξ is a complex line bundle (regarded as a real 2-plane bundle), then the genus of an arbitrary oriented manifold M is obtained by evaluating $\mathcal{P}(TM)$ on the homology fundamental class of M.

(iii) $F(y)$, a power series with leading term 1, the KO-theory characteristic power series of φ. This means that if $\mathcal{F}$ denotes the stable $KO(\cdot)\otimes R$-valued exponential characteristic class on oriented bundles characterized by $\mathcal{F}(\xi) = F(\xi - [2])$ for ξ as above (this makes sense because $\xi - [2]$ is nilpotent in $KO(B_\xi)\otimes R$, as one sees by applying the complexified Chern character), then the genus of an arbitrary Spin manifold is obtained by evaluating $\mathcal{F}(TM)$ on a certain KO_*-fundamental class of M.

These three power series determine one another by the formulas

$$(1) \qquad \frac{u}{g^{-1}(u)} = P(u) = \frac{u/2}{\sinh u/2} F(e^u + e^{-u} - 2) ,$$

where g^{-1} denotes the inverse power series of g.

Recently, a particular class of genera has come into prominence through the work of Landweber, Stong, Ochanine, Witten and others. These genera are characterized topologically by the property that $\varphi(M)$ vanishes if M is the total space of the complex projective bundle associated to an even-dimensional complex vector bundle over a closed oriented manifold, and numerically by the property that the power series $g'(x)^{-2}$ is a polynomial of degree ≤ 4, i.e., that

(2) $$g(x) = \int_0^x \frac{dt}{\sqrt{1 - 2\delta t^2 + \varepsilon t^4}} \quad \text{for some } \delta, \varepsilon \in R .$$

(The equivalence of these two definitions is due to Ochanine [5].) Since this is an elliptic integral, such φ are called elliptic genera. Landweber and Stong [4] discovered that there is a particular elliptic genus with values in the power series ring $R = \mathbb{Q}[[q]]$ satisfying:

(a) For $r \geq 1$ the coefficient of y^r in $F(y)$ belongs to $q^{2r-1}R$.

(This, or rather the weaker statement that the coefficient of y^r is divisible by q^{r+1} for $r \geq 2$, arises from a certain natural property of the above-mentioned KO-characteristic class $\mathcal{F}$ which we do not formulate here.) Based on numerical computations, they conjectured that condition (a) characterizes the genus in question up to a reparametrization (i.e., up to replacing q by $aq + bq^2 + \ldots$ with $a \neq 0$) and that with a suitable choice of parameter one has

(b) $F(y)$ has coefficients in $\mathbb{Z}[[q]]$.

By what was said in (iii), this means that the genus takes on values in $\mathbb{Z}[[q]]$ for all Spin manifolds. These facts were proved by D. and G. Chudnovsky [2], whose formulas show that with a suitable normalization one also has

(c) The leading term of the coefficient of y^r in $F(y)$ for $r \geq 1$ is $-q^{2r-1}$, and

(d) The genus takes values in the subring $M_*^{\mathbb{Q}}(\Gamma_0(2)) \subset \mathbb{Q}[[q]]$ of modular forms on $\Gamma_0(2)$ with rational Fourier coefficients.

(We recall basic definitions about modular forms below.) In particular, the δ and ε of equation (2) are certain modular forms (of weights 2 and 4); since $M_*^{\mathbb{Q}}(\Gamma_0(2))$ is known to be the free polynomial algebra on δ and ε, it follows that the Landweber-Stong genus is universal for all elliptic genera. This universal, modular form-valued elliptic genus has been the object of considerable interest; it gives rise to new cohomology theories (the "elliptic cohomology" of Landweber, Stong, and Ravenel) and to connections with index theory, string theory, etc. The purpose of this note is to give elementary proofs of a variety of formulas for the power series g, P, and F associated to the Landweber-Stong genus (and in particular, easy proofs of the properties (a)-(d)). These proofs use ideas from the theory of elliptic functions and modular forms but we will prove everything we need from scratch.

THEOREM. Let $R = \mathbb{Q}[[q]]$. Then the following five formulas define the same power series $P(u) \in R[[u]]$:

(3) $$P(u) = 1 - \sum_{\substack{k>0 \\ 2|k}} \frac{G_k^*}{2^{k-2}(k-1)!} u^k ,$$

(4) $$P(u) = \exp\left(\sum_{\substack{k>0 \\ 2|k}} \frac{2\widetilde{G}_k}{k!} u^k \right) ,$$

(5) $$P(u) = \frac{u}{g^{-1}(u)}$$ with g given by (2) ,

(6) $$P(u) = \frac{u/2}{\sinh u/2} \prod_{n=1}^{\infty} \left[\frac{(1-q^n)^2}{(1-q^n e^u)(1-q^n e^{-u})}\right]^{(-1)^n} ,$$

(7) $$P(u) = \frac{u/2}{\sinh u/2} \cdot \left[1 - \sum_{r=1}^{\infty} a_r (e^u + e^{-u} - 2)^r \right] ,$$

where G_k^*, $\widetilde{G}_k$, δ, ε and $a_r \in R$ are defined by

$$G_k^* = G_k^*(q) = \frac{2^{k-1}-1}{2k} B_k + \sum_{n\geq 1} \Big(\sum_{\substack{d|n \\ 2\nmid d}} d^{k-1} \Big) q^n ,$$

$$\widetilde{G}_k = \widetilde{G}_k(q) = -\frac{1}{2k} B_k + \sum_{n\geq 1} \Big(\sum_{d|n} (-1)^{n/d} d^{k-1} \Big) q^n ,$$

$$\delta = -3G_2^* = 3\widetilde{G}_2 = -\frac{1}{8} - 3 \sum_{n\geq 1} \Big(\sum_{\substack{d|n \\ 2\nmid d}} d \Big) q^n ,$$

$$\varepsilon = -\frac{1}{6}(G_4^* + 7\widetilde{G}_4) = \sum_{n\geq 1} \Big(\sum_{\substack{d|n \\ 2\nmid n/d}} d^3 \Big) q^n ,$$

(8) $$a_r = \sum_{m\geq 1} \frac{q^{m(2r-1)}(1+q^{2m})}{(1-q^{2m})^{2r}} = \sum_{n\geq 1} \sum_{\substack{d|n \\ 2\nmid d}} \left[\binom{\frac{1}{2}(d-1)+r}{2r-1} + \binom{\frac{1}{2}(d-3)+r}{2r-1} \right] q^n$$

(here $B_2 = \frac{1}{6}$, $B_4 = -\frac{1}{30}$, ... are Bernoulli numbers and $\sum_{d|n}$ denotes a sum over positive divisors of n). The genus with characteristic power series $P(u)$ satisfies properties (a)-(d).

Each of the five formulas in the theorem describes some aspect of the genus with characteristic power series $P(u)$: (3) and (4) describe the genus in cohomology and make the modularity property (d) clear (since G_k^* and $\widetilde{G}_k$ are the Fourier expansions of well-known Eisenstein series, as recalled below), (5) shows that the genus is elliptic, and (6) and (7) describe the genus in K-theory and (both) make the properties (a)-(c) evident. (To deduce (a) and (c) from (6) one has to split off the terms $n=1$ and $n=2$ from the infinite product.) Formula (6) was given by the Chudnovskys, but with a different proof. It has been generalized by Witten [6] to get other genera whose coefficients are modular forms, and in this form interpreted by him, using ideas from quantum field theory, as the equivariant index formula (Atiyah-Bott-Singer fixed point theorem) for a Dirac operator on the free loop space of a manifold. We shall return to these other genera at the end of the note.

Proof of the theorem. Consider meromorphic functions $\psi: \mathbb{C} \to \mathbb{C}$ satisfying

(9) $$\begin{cases} \psi(u+2\pi i) = -\psi(u), \quad \psi(u+4\pi i\tau) = \psi(u), \quad \psi(-u) = -\psi(u), \\ \psi \text{ has poles only for } u \in L, \quad \psi(u) = \frac{1}{u} + O(1) \text{ as } u \to 0, \end{cases}$$

where τ is in the complex upper half-plane and L denotes the lattice $\mathbb{Z}\cdot 4\pi i\tau + \mathbb{Z}\cdot 2\pi i$. Clearly there can be at most one such function, since the difference of any two would be holomorphic and doubly periodic, hence constant, hence zero because odd. We will give different constructions showing that ψ exists and equals $\frac{1}{u}P(u)$ for $P(u)$ given by any of the five equations (3)-(7), with $q = e^{2\pi i\tau}$.

First, define ψ by the rapidly convergent series

$$(10)\qquad \psi(u) = \sum_{m\in\mathbb{Z}} \frac{1}{2\sinh(\frac{u}{2}+2\pi i m\tau)} = \sum_{m\in\mathbb{Z}} \frac{1}{q^m e^{u/2} - q^{-m}e^{-u/2}} .$$

The properties (9) are immediately checked. Combining the terms m and -m, we find

$$\psi(u) = \frac{1}{e^{u/2}-e^{-u/2}} - (e^{u/2}-e^{-u/2}) \sum_{m=1}^{\infty} \frac{q^m(1+q^{2m})}{(1-q^{2m}e^u)(1-q^{2m}e^{-u})}$$

or, setting $y = e^u + e^{-u} - 2$,

$$u\,\psi(u) = \frac{u/2}{\sinh u/2}\left(1 - \sum_{m=1}^{\infty} \frac{q^m(1+q^{2m})y}{(1-q^{2m})^2 - q^{2m}y}\right) .$$

Expanding the geometric series in y, we find the function $P(u)$ defined by (7), with a_r given by the first formula in (8). The second formula in (8) follows from the first by applying the binomial theorem; either one makes properties (a)-(c) evident.

Next, define $\psi(u)$ as $u^{-1}P(u)$ with $P(u)$ given by the product formula (6). Again it is easy to check that this function satisfies (9) and hence is the same as the one just considered. This proves (6) and gives a second proof of (a)-(c).

The Taylor expansions (3) and (4) of $P(u)$ and $\log P(u)$ are easily obtained from the above two constructions: the first construction gives

$$P(u) = \frac{u/2}{\sinh u/2} - \sum_{m=1}^{\infty}\left(\frac{q^m e^{u/2}}{1-q^{2m}e^u} - \frac{q^m e^{-u/2}}{1-q^{2m}e^{-u}}\right) = \frac{u/2}{\sinh u/2} - \sum_{\substack{m,d\geq 1\\ d\text{ odd}}} (e^{\frac{du}{2}} - e^{-\frac{du}{2}})\, q^{md},$$

which (on substituting the Taylor series of $\frac{u/2}{\sinh u/2}$ and $\sinh\frac{du}{2})$ is seen to be equivalent to (3), and the second gives

$$\log P(u) = \log\frac{u/2}{\sinh u/2} + \sum_{n=1}^{\infty}(-1)^n \sum_{d=1}^{\infty}(e^{du}+e^{-du}-2)\,\frac{q^{nd}}{d} ,$$

which is similarly equivalent to (4) (the Taylor expansion of $\log\frac{u/2}{\sinh u/2}$ can be found by differentiation).

Finally, the "elliptic" property (5) also follows from the axiomatic characterization (9): the properties (9) imply that $\psi(u)^2$ and $\psi'(u)^2$ are even and invariant under translation by L and have poles at $u=0$ with leading terms u^{-2} and u^{-4} as their only singularities (mod L), so ψ'^2 must be a monic quadratic polynomial of ψ^2, i.e., $\psi'^2 = \psi^4 - 2\delta\psi^2 + \varepsilon$ for some $\delta, \varepsilon \in \mathbb{C}$, and then $\frac{1}{\psi(u)} = u + \dots$ can be written as $g^{-1}(u)$ where $g(x) = x + \dots$ is given by the elliptic integral (2). The expansions of δ and ε as functions of $q = e^{2\pi i\tau}$ can be obtained by comparing the coefficients of u^2 and u^4 in $P(u)$ obtained from formulas (3), (4), and (5).

It remains to check property (d), i.e., that the series $P(u)$ defined in the theorem has Taylor coefficients which are modular forms on $\Gamma_0(2)$. We recall that

$\Gamma_0(2)$ is the subgroup of $SL_2(\mathbb{Z})$ consisting of matrices $\binom{a\ b}{c\ d}$ with c even and that for $\Gamma = \Gamma_0(2)$ or $SL_2(\mathbb{Z})$ a modular form of weight k on Γ (k an integer, necessarily even and nonnegative) is a holomorphic function $f: \mathcal{H} \to \mathbb{C}$ ($\mathcal{H}$ = upper half-plane) satisfying $f(\frac{a\tau+b}{c\tau+d}) = (c\tau+d)^k f(\tau)$ for all $\tau \in \mathcal{H}$, $\binom{a\ b}{c\ d} \in \Gamma$ and having a Fourier expansion $\sum a(n)\, q^n$ with $a(n)$ of polynomial growth in n. The $\mathbb{C}$-vector space of such forms is denoted by $M_k(\Gamma)$, the $\mathbb{Q}$-vector space (of the same dimension) of forms with $a(n) \in \mathbb{Q}$ for all n is denoted by $M_k^{\mathbb{Q}}(\Gamma)$, and the graded ring $\bigoplus_k M_k^{\mathbb{Q}}(\Gamma)$ by $M_*^{\mathbb{Q}}(\Gamma)$. For $\Gamma = SL_2(\mathbb{Z})$ this ring is $\mathbb{Q}[G_4, G_6]$, where

$$(11) \qquad G_k = G_k(\tau) = -\frac{B_k}{2k} + \sum_{n=1}^{\infty} \Big(\sum_{d|n} d^{k-1} \Big)\, q^n \qquad (k>0,\ k \text{ even}) .$$

For $k \geq 4$ the function $G_k(\tau)$ is modular of weight k because $\frac{2(2\pi i)^k}{(k-1)!} G_k$ is equal to the absolutely convergent <u>Eisenstein series</u> $\sum{}' (m\tau+n)^{-k}$ (summation over all pairs of integers m, n, not both zero). The function G_2 is "nearly" modular: it satisfies

$$(12) \qquad G_2(\frac{a\tau+b}{c\tau+d}) = (c\tau+d)^2\, G_2(\tau) + \frac{i}{4\pi}\, c(c\tau+d) \quad \text{for } \tau \in \mathcal{H},\ \binom{a\ b}{c\ d} \in SL_2(\mathbb{Z}) .$$

From these facts and the easily checked identities

$$(13) \qquad G_k^*(\tau) = G_k(\tau) - 2^{k-1}\, G_k(2\tau) , \qquad \widetilde{G}_k(\tau) = -G_k(\tau) + 2\, G_k(2\tau) \qquad (k \geq 2)$$

it follows that G_k^* and $\widetilde{G}_k$ are modular forms on $\Gamma_0(2)$ for all k (including $k=2$). Therefore each of the formulas (3), (4), or (5) shows that $P(u)$ has coefficients in $M_*^{\mathbb{Q}}(\Gamma_0(2))$; more precisely, the coefficient of u^k is in $M_k^{\mathbb{Q}}(\Gamma_0(2))$ for each $k \geq 0$. The modularity also follows from (10) because the expansion $\frac{1}{\sinh x} = \sum_{n \in \mathbb{Z}} \frac{(-1)^n}{x+\pi i n}$ gives

$$u\,\psi(u) = u \sum_{m \in \mathbb{Z}} \sum_{n \in \mathbb{Z}} \frac{(-1)^n}{u + 4\pi i m\tau + 2\pi i n} = 1 - \sum_{\substack{k>0 \\ k \text{ even}}} (\frac{u}{4\pi i})^k \sum_{m,n \in \mathbb{Z}}{}' \frac{(-1)^n}{(m\tau + n/2)^k}$$

with the inner sum clearly a modular form of weight k on $\Gamma_0(2)$ (here some care is needed with the conditionally convergent double series), or from the axiomatic characterization (9) by noting that for $\binom{a\ b}{c\ d} \in \Gamma_0(2)$ and ψ satisfying (9) the function $\tilde{\psi}(u) = (c\tau+d)\, \psi((c\tau+d)u)$ satisfies (9) with respect to $\tilde{\tau} = \frac{a\tau+b}{c\tau+d}$.

This completes the proof of the theorem. From a purely modular point of view, there are two surprising aspects of the formulas it contains. First of all, the space of modular forms $M_k(\Gamma_0(2))$ breaks up in a natural way as the direct sum of a space of Eisenstein series (forms whose Fourier coefficients $a(n)$ are sums of powers of divisors of n with congruence conditions) and a space of "cusp forms" (forms satisfying $a(n) = O(n^{k/2})$). The former is spanned by G_k^* and $\widetilde{G}_k$ and hence has dimension 1 for $k=2$ and 2 for $k>2$ (the dimension of the full space M_k is $k/2$). It is quite remarkable that the coefficients of both $P(u)$ and $\log P(u)$ belong to this tiny subspace. The other surprising fact is that, although the Eisenstein series $G_2^*, G_4^*, \ldots$ have rational Fourier coefficients and non-zero constant terms, there is a rational linear combination of $1, G_2^*, \ldots, G_{2r}^*$, namely a_r, which vanishes to order $2r-1$ (i.e. twice as far as one has any right to expect) and is monic with integral coefficients. This, and also the fact that the a_r have much smaller coefficients (i.e., that the G_k^* satisfy congruences to high moduli), are illustrated by the first values:

$$\begin{aligned}
G_2^* &= \tfrac{1}{24} + q + q^2 + 4q^3 + q^4 + 6q^5 + 4q^6 + 8q^7 + q^8 + 13q^9 + \dots \\
G_4^* &= -\tfrac{7}{240} + q + q^2 + 28q^3 + q^4 + 126q^5 + 28q^6 + 344q^7 + q^8 + 757q^9 + \dots \\
G_6^* &= \tfrac{31}{504} + q + q^2 + 244q^3 + q^4 + 3126q^5 + 244q^6 + 16808q^7 + q^8 + 59293q^9 + \dots \\
a_1 &= q + q^2 + 4q^3 + q^4 + 6q^5 + 4q^6 + 8q^7 + q^8 + 13q^9 + \dots \\
a_2 &= q^3 + 5q^5 + q^6 + 14q^7 + 31q^9 + \dots \\
a_3 &= q^5 + 7q^7 + 27q^9 + \dots
\end{aligned}$$

We now turn to the other genera introduced by Witten. The same formal power series calculation as that which showed the equivalence of (4) and (6) gives

$$(14)\qquad \frac{u/2}{\sinh u/2}\prod_{n=1}^{\infty}\frac{(1-q^n)^2}{(1-q^ne^u)(1-q^ne^{-u})} = \exp\Big(\sum_{\substack{k>0\\ 2|k}}\frac{2}{k!}G_k\,u^k\Big)$$

where G_k is defined by (11). Call this power series $P_W(u)$ and the associated genus φ_W (W for "Witten" or "Weierstrass"; terminology due to Peter Landweber). Let us compare this genus and power series with those of Landweber-Stong:

- The left side of (14) shows that the KO-theory characteristic power series of φ_W can be written in the form

$$\prod_{n=1}^{\infty}\Big[1-\frac{q^n}{(1-q^n)^2}y\Big]^{-1} = \sum_{r=0}^{\infty} b_r y^r \qquad (y = e^u - 2 + e^{-u})$$

where b_r belongs to $\mathbb{Z}[[q]]$ and has leading coefficient q^r. Thus the analogues of properties (a)-(c) hold for the Witten genus, and $\varphi_W(M)$ belongs to $\mathbb{Z}[[q]]$ if M is a Spin manifold. The product in (14) is simpler than that in (6). It can be neatly expressed by saying that the associated KO-valued characteristic class $\mathcal{F}_W(\xi)$ is a certain tensor product of sums of symmetric powers of ξ, and in this form has a natural interpretation as an index formula for a Dirac-like operator on the free loop space of M; the corresponding expression for the Landweber-Stong genus also has an interpretation as an index formula for an operator, but this time one which has no finite-dimensional version. (For all this, see Witten's paper [6].)

- The right side of (14) shows that $\varphi_W(M)$ is a modular form on $SL_2(\mathbb{Z})$ if the first Pontryagin class of M vanishes rationally, since then G_2 drops out of the characteristic class $\mathcal{F}_W(TM)$. In this case $\varphi_W(M)$ is simpler than $\varphi_{LS}(M)$ because it is a modular form on the full modular group rather than a congruence subgroup. On the other hand, if $p_1(M)$ does not vanish then $\varphi_W(M)$ is not a modular form at all, but belongs instead to the larger ring

$$\hat{M}_* = \mathbb{Q}[G_2, G_4, G_6] \supset M_* = M_*^{\mathbb{Q}}(SL_2(\mathbb{Z})) = \mathbb{Q}[G_4, G_6]\ .$$

This may not be all bad: the ring $\hat{M}_*$ is also studied in the theory of modular forms and is in many respects nearly as good as M_* (the elements of $\hat{M}_*$ are "almost modular" by (12), and the mod p reductions of $\hat{M}_* \cap \mathbb{Z}[[q]]$ and $M_* \cap \mathbb{Z}[[q]]$ agree for all primes p). In one respect it is even better: it is closed under the differentiation operator $D = q\frac{d}{dq} = \frac{1}{2\pi i}\frac{d}{d\tau}$ (specifically, $DG_2 = \frac{5}{6}G_4 - 2G_2^2$, $DG_4 = \frac{7}{10}G_6 - 8G_2G_4$, $DG_6 = \frac{400}{7}G_4^2 - 12G_2G_6$, and more generally $Df + 2kG_2f \in M_{k+2}$ for $f \in M_k$).

- There is no analogue of the additive formulae (3) and (7) or of the elliptic property (5), so φ_W is not an elliptic genus and does not have a simple description in cohomology. This is because the axiomatic characterization (9) no longer applies: the function $u^{-1}P_W(u)$ is nearly, but not quite, doubly periodic. (It changes by a factor $-q^{1/2}e^u$ under $u \to u + 2\pi i\tau$. The function $\frac{d^2}{du^2}\log u^{-1}P_W(u)$ is periodic, and in fact equals $2G_2$ minus the Weierstrass $\wp$-function for the lattice $\mathbb{Z}\cdot 2\pi i\tau + \mathbb{Z}\cdot 2\pi i$.) This is closely related to the non-modularity of the coefficients of $P_W(u)$ as functions of τ (cf. comments at the end of the note).

Witten discusses one other genus, the one associated to the signature operator. Let $\mathcal{L}(\xi)$ and $\mathcal{L}_n(\xi)$ ($n \geq 1$) be the characteristic classes with characteristic power series $\frac{u}{\tanh u}$ and $\frac{1+q^ne^{2u}}{1-q^ne^{2u}}$, where q is a parameter. The G-signature theorem of Atiyah and Singer says that for a smooth finite-dimensional manifold X with S^1-action, the equivariant signature, an element of the representation ring $R(S^1)\otimes\mathbb{C} \simeq \mathbb{C}[q,q^{-1}]$, is given by

$$(15) \qquad \langle \mathcal{L}(TM)\prod_{n=1}^{\infty}\mathcal{L}_n(\nu_n),\ [M]\rangle \quad ,$$

where $M = X^{S^1}$ is the fixed point set and ν_n the subbundle of the normal bundle of M in X on which S^1 acts via $\begin{pmatrix}\cos n\theta & \sin n\theta\\ -\sin n\theta & \cos n\theta\end{pmatrix}$. (We are being very brief and a little imprecise here.) This equivariant signature is constant, either (cf. [1])

(i) because it is defined in terms of the action of S^1 on the middle cohomology of X and this action is trivial since S^1 is connected, or

(ii) because it is an element of $\mathbb{C}[q,q^{-1}]$ which is regular at both $q=0$ and $q=\infty$.

Witten's idea was to apply (15) to the case when X is the free loop space of a smooth manifold M; then M is the fixed point set and each ν_n is finite-dimensional (and in fact isomorphic to $TM\otimes\mathbb{C}$), so that the formula makes sense even though X itself is infinite-dimensional. On the other hand, neither (i) nor (ii) applies and we get a non-trivial power series in q. Modifying $\mathcal{L}_n$ by dividing its defining characteristic power series by its value at $u=0$ (i.e., looking at the associated stable class), and replacing u by u/2, we find that this power series is $\left(2^{\frac{1}{2}}\prod_n\frac{1+q^n}{1-q^n}\right)^{\dim M}\varphi_S(M)$, where φ_S is the $\mathbb{Q}[[q]]$-valued genus with characteristic power series

$$(16) \qquad P_S(u) = \frac{u/2}{\tanh u/2}\prod_{n=1}^{\infty}\left(\frac{1+q^ne^u}{1-q^ne^u}\cdot\frac{1+q^ne^{-u}}{1-q^ne^{-u}}\right)\Big/\left(\frac{1+q^n}{1-q^n}\right)^2 \quad .$$

For this genus, analogues of all five formulas in the theorem about P(u) hold. The analogue of (6) is (16) itself, and the analogues of (3), (4), (5), and (7) are:

(17) $$P_S(u) = 1 - \sum_{\substack{k>0 \\ 2|k}} \frac{2\,\widetilde{G}_k}{(k-1)!}\, u^k ,$$

(18) $$P_S(u) = \exp\Big(\sum_{\substack{k>0 \\ 2|k}} \frac{4\,G_k^*}{k!}\, u^k \Big) ,$$

(19) $$P_S(u) = u \,/\, g_S^{-1}(u) , \quad \text{where} \quad g_S(x) = \int_0^x (1 - 2\delta_S t^2 + \varepsilon_S t^4)^{-\frac{1}{2}}\, dt \quad \text{with}$$

$$\delta_S = \frac{1}{4} + 6 \sum_{n\geq 1} \Big(\sum_{\substack{d|n \\ 2\nmid d}} d \Big) q^n , \qquad \varepsilon_S = \frac{1}{16} + \sum_{n\geq 1} \Big(\sum_{d|n} (-1)^d d^3 \Big) q^n ,$$

(20) $$P_S(u) = \frac{u/2}{\tanh u/2} \Big(1 - 2 \sum_{r\geq 1} c_r \,(e^u + e^{-u} - 2)^r \Big) \quad \text{with}$$

$$c_r = \sum_{m\geq 1} \frac{(-1)^m q^{mr}}{(1-q^m)^{2r}} = \sum_{n\geq 1} \Big[\sum_{d|n} (-1)^{n/d} \binom{r+d-1}{2r-1} \Big] q^n \in q^r\, \mathbb{Z}[[q]] .$$

Indeed, (18) is obtained directly from (16) by logarithmic differentiation (like the proof of (14) or of the equality of (4) and (6)). But from equation (13) and the modularity property $G_k(\frac{-1}{\tau}) = \tau^k G_k(\tau)$ (respectively (12) for $k=2$) it follows that

(21) $$G_k^*(\tfrac{-1}{2\tau}) = 2^{k-1} \tau^k \widetilde{G}_k(\tau) , \qquad \widetilde{G}_k(\tfrac{-1}{2\tau}) = 2\, \tau^k G_k^*(\tau) ,$$

so (writing $P(\tau;u)$ instead of $P(u)$ for the Landweber-Stong genus to emphasize the dependence on τ, and similarly for P_S) equation (18) says

(22) $$P_S(\tau;u) = P(\tfrac{-1}{2\tau}; \tfrac{u}{\tau}) .$$

In other words, P and P_S are the same function, but expanded at the two cusps 0 and ∞ of $\mathbb{H}/\Gamma_0(2)$ (which are interchanged by $\tau \to -1/2\tau$). Substituting formulas (3) and (5) into (22) and using (21) again gives (17) and (19). Finally, equation (22) leads to an analogue of the property (9) and hence to an analogue of (10), namely

$$u^{-1} P_S(u) = \sum_{m\in\mathbb{Z}} \frac{(-1)^m}{2\tanh(u/2 + \pi i m\tau)} = \frac{1/2}{\tanh u/2} - \sum_{m\geq 1} \frac{(-1)^m (e^u - e^{-u})}{q^m + q^{-m} - e^u - e^{-u}} ,$$

from which (20) easily follows. Equation (20) expresses $P_S(u)$ as $\frac{u/2}{\tanh u/2} G(e^u + e^{-u} - 2)$ with $G(y) \in \mathbb{Z}[[q, qy]]$; such a formula has an interpretation like the one given for equation (1) in (iii) of the introduction, but using a different KO_*-fundamental class (the one associated to the signature operator).

We make a final remark. Throughout this note there has been an interplay between the modularity properties of the various functions with respect to the variable τ and their elliptic properties with respect to the variable u. Functions of two variables having this dual modular/elliptic nature are called Jacobi forms. More precisely, a Jacobi form of weight k and index m is a function $\phi: \mathbb{H}\times\mathbb{C} \to \mathbb{C}$ satisfying

$$\phi(\tfrac{a\tau+b}{c\tau+d}, \tfrac{z}{c\tau+d}) = (c\tau+d)^k\, e^{2\pi i m c z^2/(c\tau+d)}\, \phi(\tau,z)$$

for $\binom{a\ b}{c\ d}$ in $SL_2(\mathbb{Z})$ or a congruence subgroup and

$$\phi(\tau, z+\lambda\tau+\mu) = e^{-2\pi i m(\lambda^2\tau+2\lambda z)}\,\phi(\tau,z)$$

for all (λ,μ) in $\mathbb{Z}^2$ or a sublattice of finite index. (The theory of such forms was developed in [3].) The most important examples of Jacobi forms are theta series. Using the Jacobi triple product identity, we find that $u^{-1}P_W(u)$ can be expressed as a quotient of theta-series,

$$u^{-1}P_W(u) = \Big(\sum_{n>0}\left(\tfrac{-4}{n}\right) n\, q^{n^2/8}\Big)\Big/\Big(\sum_{n\in\mathbb{Z}}\left(\tfrac{-4}{n}\right) q^{n^2/8} e^{nu/2}\Big)$$

(here $(\frac{-4}{n})$ equals 0 for n even, $(-1)^{\frac{1}{2}(n-1)}$ for n odd), and is therefore a Jacobi form with respect to τ and $z=\frac{u}{2\pi i}$, of weight -1 and index $-\frac{1}{2}$. (It is because the index is non-zero that $u^{-1}P_W(u)$ is not quite elliptic in u and that its Taylor coefficients are not quite modular forms in τ.) The other two characteristic power series we have been considering are related to P_W by

$$P(u) = P_W(2\tau;u)^2/P_W(\tau;u), \qquad P_S(u) = P_W(\tau;u)^2/P_W(2\tau;2u),$$

so $u^{-1}P(u)$ and $u^{-1}P_S(u)$ are also Jacobi forms of weight -1, but of index 0. It is interesting to note that in all of the recent occurrences of modular forms in algebraic topology, string theory, and the theory of Kac-Moody algebras, it is in fact Jacobi forms which are entering.

REFERENCES

1. M.F. Atiyah and F. Hirzebruch: Spin-manifolds and group actions. In: Essays on Topology and Related Topics (Mémoires dédiés à Georges de Rham), pp. 18-26. Springer, New York 1970.

2. D.V. Chudnovsky and G.V. Chudnovsky: Elliptic modular functions and elliptic genera. To appear in Topology.

3. M. Eichler and D. Zagier: The Theory of Jacobi Forms. Progress in Math. 55, Birkhäuser, Boston-Basel-Stuttgart 1985.

4. P. Landweber and R.E. Stong: Circle actions on Spin manifolds and characteristic numbers. To appear in Topology.

5. S. Ochanine: Sur les genres multiplicatifs définis par des intégrales elliptiques. To appear in Topology 26 (1987) 143-151

6. E. Witten: The index of the Dirac operator in loop space. To appear in this volume.