A NOTE ON AN INVARIANT OF FINTUSHEL AND STERN

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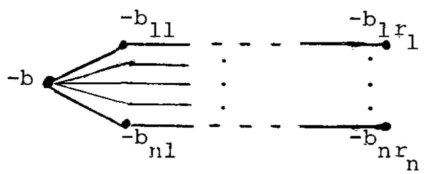
Let $\alpha_1,\ldots,\alpha_n$ be pairwise prime integers with $\alpha_i > 1$ for each i. Let $\Sigma = \Sigma(\alpha_1,\ldots,\alpha_n)$ be the Seifert fibered homology 3-sphere with singular fibers of orders $\alpha_1,\ldots,\alpha_n$. By [N-R] (see also [N1], [N2]), Σ has a unique orientation making it the link of a complex surface singularity $(V(\alpha_1,\ldots,\alpha_n),p)$; we give Σ this orientation. This is also the unique orientation for which Σ bounds a plumbed 4-manifold with negative definite intersection form. A minimal such plumbing (i.e. admitting no (-1)-blow down) is unique. It is given by the following plumbing diagram (which is also the minimal good resolution diagram for the above singularity):



$b \geq 1, \quad b_{ij} \geq 2$

with weights determined by

$$[b_{i1},\ldots,b_{ir_i}] = \frac{\alpha_i}{\beta_i},$$

$$b = \frac{1}{\alpha} + \sum_{i=1}^{n} \frac{\beta_i}{\alpha_i},$$

where $[b_1,\ldots,b_r]$ denotes the continued fraction

$$[b_1,\ldots,b_r] = b_1 - \cfrac{1}{b_2 - \cfrac{\ddots}{\ddots - \cfrac{1}{b_r}}},$$

and

$$\alpha = \alpha_1\alpha_2\cdots\alpha_n,$$

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and the β_i are determined by:

$$0 < \beta_i < \alpha_i, \quad \beta_i \frac{\alpha}{\alpha_i} \equiv -1 \pmod{\alpha_i}.$$

Fintushel and Stern have defined an invariant

$$R = R(\alpha_1,\ldots,\alpha_n) = \frac{2}{\alpha} - 3 + n + \sum_{i=1}^{n} \frac{2}{\alpha_i} \sum_{k=1}^{\alpha_i - 1} \cot\left(\frac{\pi\alpha k}{\alpha_i^2}\right) \cot\left(\frac{\pi k}{\alpha_i}\right) \sin^2\left(\frac{\pi k}{\alpha_i}\right)$$

and shown that if $R > 0$ then Σ cannot bound a $\mathbb{Z}/2$-acyclic 4-manifold ([F-S]; the invariant arises as the index of a certain differential operator).

Proposition. $R = 2b-3$ *with* R *and* b *as above*.

Since b is a positive integer, the result of Fintushel and Stern can thus be reformulated:

Theorem. *If* $\Sigma(\alpha_1,\ldots,\alpha_n)$ *bounds a* $\mathbb{Z}/2$*-acyclic manifold then the "central curve" of the corresponding resolution diagram has self-intersection* $-b = -1$.

Lemma. *If* a *and* b *are coprime integers with* $a > 1$ *then*

$$\frac{1}{a} \sum_{k=1}^{a-1} \cot\left(\frac{\pi k b}{a}\right) \cot\left(\frac{\pi k}{a}\right) \sin^2\left(\frac{\pi k}{a}\right) = \frac{b^*}{a} - \frac{1}{2}$$

where $0 < b^* < a$ *and* $bb^* \equiv -1 \pmod{a}$.

Proof. If $\zeta = e^{2\pi i\alpha}$ then

$$\cot \pi\alpha = i\,\frac{\zeta+1}{\zeta-1}, \quad \sin^2 \pi\alpha = -\frac{1}{4}\,\zeta^{-1}(\zeta-1)^2.$$

Thus the sum in question is

$$\frac{1}{4a} \sum_{\substack{\zeta^a=1 \\ \zeta\neq 1}} \frac{\zeta^b+1}{\zeta^b-1}\,\frac{\zeta+1}{\zeta-1}\,\zeta^{-1}(\zeta-1)^2$$

$$= \frac{1}{4a} \sum_{\substack{\zeta^a=1 \\ \zeta\neq 1}} \left(\frac{\zeta-1}{\zeta^b-1}\right)(\zeta^b+1)(1+\zeta^{-1})$$

$$= \frac{1}{4a} \sum_{\substack{\eta^a=1 \\ \eta\neq 1}} \left(\frac{\eta^\gamma - 1}{\eta - 1}\right) (\eta+1)(1+\eta^{-\gamma}),$$

where we have substituted $\zeta = \eta^\gamma$, $b\gamma \equiv 1 \pmod a$, $0 < \gamma < a$ (so $\gamma = a-b^*$),

$$= \frac{1}{4a} \sum_{\substack{\eta^a=1 \\ \eta\neq 1}} (1+\eta+\ldots+\eta^{\gamma-1})(1+\eta+\eta^{-\gamma}+\eta^{1-\gamma})$$

$$= \frac{1}{4a} \Big[\sum_{\eta^a=1} (1+\eta+\ldots+\eta^{\gamma-1})(1+\eta+\eta^{-\gamma}+\eta^{1-\gamma}) - 4\gamma \Big]$$

$$= \frac{1}{4a}[a(1+1) - 4\gamma] = \frac{b^*}{a} - \frac{1}{2}$$

since

$$\sum_{\eta^a=1} \eta^j = \begin{cases} a, & a \mid j \\ 0, & a \nmid j \,. \end{cases}$$

Applying this lemma to the invariant R, we see

$$R = \frac{2}{\alpha} - 3 + n + 2\sum_{i=1}^{n}\left(\frac{\beta_i}{\alpha_i} - \frac{1}{2}\right)$$

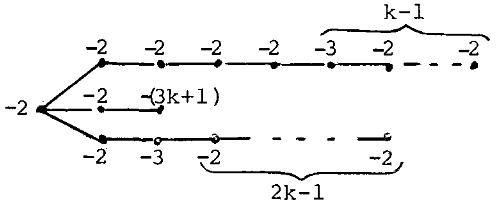
$$= 2\left(\frac{1}{\alpha} + \sum_{i=1}^{n} \frac{\beta_i}{\alpha_i}\right) - 3 = 2b - 3,$$

as claimed.

If M is a 3-dimensional homology sphere which bounds a 4-manifold X with definite intersection form B and if M also bounds a $\mathbb{Z}$-acyclic 4-manifold, then by a result of S. Donaldson [D] the form B must be diagonalizable over $\mathbb{Z}$. Some more recent work of Donaldson which he described at the Special Year indicated that one may be able to replace "$\mathbb{Z}$-acyclic" by "$\mathbb{Z}/2$-acyclic" in this statement. Thus the above theorem, possibly with $\mathbb{Z}/2$-acyclic weakened to $\mathbb{Z}$-acyclic, would be a consequence of Donaldson's work if the answer to the following question is "yes".

<u>Question</u>. <u>If the intersection form for the plumbing described above is diagonalizable over</u> $\mathbb{Z}$, <u>must</u> b <u>equal</u> 1?

For $n = 3$, or $n = 4$ and $b \neq 3$, the answer is "yes", but our proof is clearly not the "right" proof, being too tedious to be worth giving here. More generally one might ask if a negative definite unimodular form over $\mathbb{Z}$ represented by an integrally weighted tree with no weight -1 is necessarily non-diagonalizable. We know no counterexample, although large diagonal summands can exist. For example $\Sigma(6k-1, 6k+1, 6k+2)$ has resolution diagram



and the intersection form is equivalent to $E_8 \oplus (3k-1)\langle -1\rangle$. Similar periodicities abound, the most basic being that $\Sigma(p,q,r+kpq)$ has resolution diagram containing the resolution diagram for $\Sigma(p,q,r)$ and its intersection form is equivalent to the form for $\Sigma(p,q,r)$ plus k diagonal -1's.

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