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CORRECTION TO "THE EICHLER-SELBERG TRACE FORMULA ON $SL_2(\mathbb{Z})$"

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The paper in question is a translation of the author's paper "Trace des opérateurs de Hecke" (Séminaire Delange-Pisot-Poitou, 17^e année, 1975/6, n^o 23) and appeared as an appendix to Part I of Serge Lang's recent book <u>Introduction to Modular Forms</u> (Springer-Verlag, 1976, pp. 44-54; all page and equation numbers below refer to this appendix). Its purpose was to give a self-contained account, in the language of the classical theory of modular forms, of the formula of Selberg and Eichler for the trace of the Hecke operator $T(N)$ acting on $S_k(SL_2(\mathbb{Z}))$. Unfortunately, as several people have pointed out to me, the calculation of the contribution from the hyperbolic matrices with rational fixed points (Case 3 , p.53) is incorrect. The contribution from all such matrices with given determinant u^2 ($u \in \mathbb{Z}$, $u > 0$, $u^2+4N = t^2$ with $t \in \mathbb{Z}$) is given by

$$(1) \qquad \int_F R(z) \frac{dxdy}{y^2} ,$$

where $F = \{z=x+iy \in H \mid |z| \geq 1, |x| \leq \frac{1}{2}\}$ is a fundamental domain for the action of $SL_2(\mathbb{Z})$ on the upper half-plane H and

$$(2) \qquad R(z) = \sum_{\substack{a,b,c \in \mathbb{Z} \\ b^2-4ac = u^2}} \frac{y^k}{(a|z|^2+bx+c-ity)^k} \qquad (z=x+iy \in H).$$

Substituting (2) into (1) and interchanging the summation and integration gives

$$(3) \qquad \int_F R(z) \frac{dxdy}{y^2} = u\, I$$

with

$$(4) \qquad I = \int_H \frac{y^k}{(|z|^2-ity-\frac{1}{4}u^2)^k} \frac{dxdy}{y^2} .$$

This integral is computed by integrating first over x and then over y ,i.e. as

$$(5) \qquad \int_0^\infty \left(\int_{-\infty}^\infty (x^2+y^2-ity-\tfrac{1}{4}u^2)^{-k} dx \right) y^{k-2} dy ,$$

and it is claimed that this leads to the value

$$(6) \qquad (-1)^{(k-2)/2} \frac{2\pi}{k-1} u^{-1} (u+|t|)^{-k+1}$$

for I, which when multiplied by u gives the correct value for (1). However, the computation given contains a sign mistake (the integral (5) is correctly evaluated in the text as $\frac{\pi i^{k-2}}{2(k-1)!} \frac{d^{k-2}}{dt^{k-2}} (-\frac{4}{u} \frac{1}{u+|t|})$, which is the negative of (6)), and in fact the integral (1) is not equal to u times expression (5). The reason is that the expression obtained by substituting (2) into (1) is not absolutely convergent, so that the interchange leading to (3) is not justified; moreover, equation (4) makes no sense until we specify the order of integration over H, since the integral is not absolutely convergent.

The correct procedure is to replace the integral (1) by the limit as $C \to \infty$ of the integral over the compact region $F_C = \{z=x+iy \in F \mid y \geqslant C\}$, in which the convergence of the sum (2) is uniform. Then equations (3) and (4) remain valid if the integration over H in (4) is interpreted as $\lim_{\varepsilon\to 0} \int_{H_\varepsilon}$, where H_ε is obtained from H by removing all points which have imaginary part $> \frac{1}{\varepsilon}$ or which lie in the interior of a circle of radius $\frac{\varepsilon}{c}$ tangent to the real axis at any rational point $\frac{d}{c}$, $(c,d) = 1$. Since the only poles of the integrand in (4) are at $z= \pm\frac{1}{2} u$, we can shrink all of these circles to zero except those tangent to the real axis at $\frac{1}{2}u$ and $-\frac{1}{2}u$. Hence

$$(7) \qquad I = I_1 - \lim_{\varepsilon\to 0} I_{\varepsilon,\frac{1}{2}u} - \lim_{\varepsilon\to 0} I_{\varepsilon,-\frac{1}{2}u} ,$$

where I_1 is the integral (5) and

$$I_{\varepsilon,\pm\frac{1}{2}u} = \int_0^{2\varepsilon} \left(\int_{\pm\frac{1}{2}u-\sqrt{2\varepsilon y-y^2}}^{\pm\frac{1}{2}u+\sqrt{2\varepsilon y-y^2}} (x^2+y^2-ity-\tfrac{1}{4}u^2)^{-k}\, dx \right) y^{k-2}\, dy$$

is the integral over the circle tangent to the real axis at $\pm\frac{1}{2}u$, the integration being carried out first over x and then over y. Making the substitution $x = \pm\frac{1}{2}u + \varepsilon a$, $y = \varepsilon b$, we find

$$I_{\varepsilon,\pm\frac{1}{2}u} = \int_0^2 \int_{-\sqrt{2b-b^2}}^{\sqrt{2b-b^2}} \frac{b^{k-2}}{(\pm ua - itb + \varepsilon(a^2+b^2))^k}\, da\, db ,$$

so

$$\lim_{\varepsilon\to 0} I_{\varepsilon,\pm\frac{1}{2}u} = \int_0^2 \int_{-\sqrt{2b-b^2}}^{\sqrt{2b-b^2}} \frac{b^{k-2}}{(\pm ua - itb)^k}\, da\, db$$

$$= -\frac{1}{k-1} u^{-1} \int_0^2 \left\{ (u\sqrt{2b-b^2}-itb)^{-k+1} + (u\sqrt{2b-b^2}+itb)^{-k+1} \right\} b^{k-2}\, db$$

$$= -\frac{2}{k-1} u^{-1} \int_{-\infty}^{\infty} (uv+it)^{-k+1} \frac{v\, dv}{v^2+1} ,$$

where in the last line we have made the substitution $b = \frac{2}{v^2+1}$. The latter integral can be evaluated easily by contour integration (for example, if $t>0$ then the only pole of the integrand in the upper half plane is at $v=i$) and equals the negative of expression (6). Since also I_1 equals the negative of (6), as stated above, the expressions (7) and (6) are equal.

A second and minor correction is that on page 49, line 3 and page 52, line 26, the 48 should be replaced by 24, and in the middle of page 52 the $\frac{\pi}{6}$ should be $\frac{\pi}{3}$.