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On an approximate identity of Ramanujan

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Abstract. In his second notebook, Ramanujan says that

$$\frac{q}{x+}\,\frac{q^4}{x+}\,\frac{q^8}{x+}\,\frac{q^{12}}{x+}\cdots = 1 - \frac{qx}{1+}\,\frac{q^2}{1-}\,\frac{q^3x}{1+}\,\frac{q^4}{1-}\cdots$$

"nearly" for q and x between 0 and 1. It is shown in what senses this is true. In particular, as $q \to 1$ the difference between the left and right sides is approximately $\exp\{-c(x)/(1-q)\}$ where $c(x)$ is a function expressible in terms of the dilogarithm and which is monotone decreasing with $c(0) = \pi^2/4, c(1) = \pi^2/5$; thus the difference in question is less than $2\cdot 10^{-85}$ for $q = 0{\cdot}99$ and all x between 0 and 1.

Keywords. Approximate identity; asymptotic behaviour; Rogers–Ramanujan identities; dilogarithm function.

1. Introduction

Ramanujan's notebooks abound with assertions of strange identities, a good many of which have remained mysterious until now. I recently learned of one of these from Henri Cohen, to whom it had been shown by Bruce Berndt: on page 289, formula (4) of the Second Notebook, Ramanujan says that

$$1 - \cfrac{qx}{1 + \cfrac{q^2}{1 - \cfrac{q^3x}{1 + \cfrac{q^4}{1 - \cfrac{q^5x}{1 + \ddots}}}}} = \cfrac{q}{x + \cfrac{q^4}{x + \cfrac{q^8}{x + \cfrac{q^{12}}{x + \ddots}}}} \quad \text{“nearly”} \tag{1}$$

for q and x between 0 and 1. At first sight it is not even clear what this means; certainly the two continued fractions do not have the same power series expansions in q or x. However, Cohen did some computations and found that the two sides of (1) are numerically very close if q is near one, as shown in figure 1 or in the following small table giving the two values in question for $x = 0{\cdot}5$ or 1 and $q = 0{\cdot}8$, $0{\cdot}9$ and $0{\cdot}95$:

	$q = 0{\cdot}8$	$q = 0{\cdot}9$	$q = 0{\cdot}95$
$x = 0{\cdot}5$	0·774652...	0·7767340180...	0·77859859961698872648...
	0·774627...	0·7767340194...	0·77859859961698872686...
$x = 1$	0·59124...	0·605146977...	0·611726198935852157...
	0·59086...	0·605146958...	0·611726198935852104...

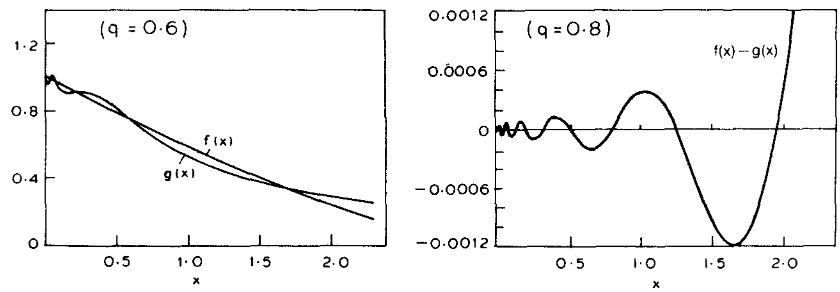

Figure 1. The functions $f(x)$ and $g(x)$ for $q = 0{\cdot}6$, $q = 0{\cdot}8$.

Cohen also showed that the two sides of (1) agree for $x = q$. Moreover, they both satisfy the same functional equation

$$(h(x) + qx)h(q^2x) = 1 \quad (q \text{ fixed}), \tag{2}$$

so that they in fact agree whenever x is equal to an odd power of q (we will give the proofs of these statements below). However, it remained to find a quantitative measure for the difference of the two expressions, in order to explain the extreme degree of nearness seen in the table. This is provided by the following result, whose proof is the object of the present paper.

Theorem. *For $0 < q < 1$ and $x > 0$ denote by $f(x) = f(x; q)$ and $g(x) = g(x; q)$ the left and right sides of (1), respectively. Then:*

(I) *For $x = 1$, we have $f(1) - g(1) = O(\exp\{(\pi^2/5)/\log q\})$. More precisely, setting $Q = \exp\{(\pi^2/5)/\log q\}$ we have*

$$f(1) = q^{\frac{1}{5}}\frac{\sqrt{5}-1}{2}\left(1 + \sqrt{5}Q + \frac{5-\sqrt{5}}{2}Q^2 - \frac{5-3\sqrt{5}}{2}Q^3 - \cdots\right),$$

$$g(1) = q^{\frac{1}{5}}\frac{\sqrt{5}-1}{2}\left(1 - \sqrt{5}Q + \frac{5-\sqrt{5}}{2}Q^2 + \frac{5-3\sqrt{5}}{2}Q^3 - \cdots\right)$$

and in particular $f(1) - g(1) = (5 - \sqrt{5})q^{\frac{1}{5}}Q + O(Q^2)$ as $q \to 1$.

(II) *For $x \to 0$ we have $f(x) - g(x) = O(\exp\{(\pi^2/4)/\log q\})$. More precisely, setting $Q = \exp\{(\pi^2/4)/\log q\}$ and $\theta = (\pi \log x)/(2 \log q)$ we have*

$$f(x) = 1 + O(x),$$

$$g(x) = \frac{1 - 2Q\cos\theta + 2Q^4\cos 2\theta - 2Q^9\cos 3\theta + \cdots}{1 + 2Q\cos\theta + 2Q^4\cos 2\theta + 2Q^9\cos 3\theta + \cdots} + O(x),$$

and in particular $f(x) - g(x) = 4Q\cos(\pi\lambda/2) + O(Q^2)$ as $x \to 0$ through a sequence $x = q^{4n+\lambda}$, $n \to \infty$.

(III) *In general $f(x) - g(x) = \exp\{[c(x) + o(1)]/\log q\}\cos\theta$ for $q \to 1$ with θ as in* (II) *and*

$$c(x) = \frac{\pi^2}{6} + \tfrac{1}{2}Li_2(((1+x^2/4)^{\frac{1}{2}} - x/2)^2)$$

$$+ \tfrac{1}{2}\log^2((1+x^2/4)^{\frac{1}{2}} + x/2) - \log(x)\log((1+x^2/4)^{\frac{1}{2}} + x/2)$$

$$\left(Li_2(t) = \sum_{n=1}^{\infty} \frac{t^n}{n^2} \textit{ the dilogarithm function} \right).$$

The theorem implies, for instance, that for $q = 0{\cdot}99$ the left- and right-hand sides of (1) agree to about 85 decimal digits for x near 1, to about 96 digits for x near $\frac{1}{2}$, ($c(\frac{1}{2}) = 2.218\ldots$), and to about 107 digits for x near 0.

The known values $Li_2(1) = \pi^2/6$ and $Li_2(\frac{1}{2}(3-\sqrt{5})) = (\pi^2/15) - \log^2(\frac{1}{2}(1+\sqrt{5}))$ ([2], 1.4.1) imply $c(0) = \pi^2/4, c(1) = \pi^2/5$, so the formula of part (III) is compatible with the assertions in (I) and (II). A graph of the function $c(x)$ is shown in figure 2. Notice that $c(x)$ becomes negative for x larger than about 6.177, so for x this large the difference between the two sides of (1) becomes exponentially big rather than exponentially small as $q \to 1$.

In the next section of the paper we give some simple transformations of the continued fractions in (1). Sections 3 to 5 contain the proofs of assertions (I), (II), and (III), respectively, and we conclude with some speculations about what Ramanujan himself had in mind when stating (1).

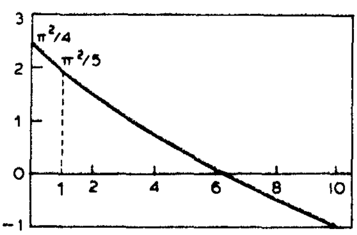

Figure 2. Graph of $c(x)$.

2. Preliminaries

In this section we give various representations of $f(x)$ and $g(x)$ as quotients of infinite sums. The formulae in the first proposition were given by Berndt.

PROPOSITION 1.

Define power series $F(x) = F(x;q) \in \mathbb{Z}[[q,x]]$ and $G(x) = G(x;q) \in \mathbb{Z}[[q,x^{-1}]]$ by

$$F(x) = \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(1-q^2)(1-q^4)\cdots(1-q^{2n})(1-q^3x)(1-q^5x)\cdots(1-q^{2n+1}x)},$$

$$G(x) = \sum_{n=0}^{\infty} \frac{x^{-2n}q^{4n^2}}{(1-q^4)(1-q^8)\cdots(1-q^{4n})}.$$

Then

$$f(x)=(1-qx)\frac{F(q^{-2}x)}{F(x)},\quad g(x)=\frac{q}{x}\frac{G(q^{-2}x)}{G(x)}.$$

The functions $F(x)$ and $G(x)$ satisfy the recursions

$$\frac{1}{1-qx}F(x)=q^{-1}xF(q^{-2}x)+(1-q^{-1}x)F(q^{-4}x),$$

$$G(x)=G(q^{-2}x)+q^4x^{-2}G(q^{-4}x).$$

An immediate corollary is that both $h=f$ and $h=g$ satisfy the recursion (2).

Proof.

(i) For $j\geqslant 0$ define $F_j(x)$ like $F(x)$ but with q^{n^2+n} replaced by $q^{n^2+(2j+1)n}$ in the numerator. Then for $j\geqslant 1$

$$F_{j-1}(x)-F_j(x)=\sum_{n=1}^{\infty}\frac{q^{n^2+(2j-1)n}}{(1-q^2)\cdots(1-q^{2n-2})(1-q^3x)\cdots(1-q^{2n+1}x)}$$
$$=\frac{q^{2j}}{1-q^3x}F_j(q^2x)$$

and

$$F_{j-1}(q^{-2}x)-F_{j-1}(x)$$
$$=qx\sum_{n=1}^{\infty}\frac{q^{n^2+(2j-1)n}}{(1-q^2)\cdots(1-q^{2n-2})(1-qx)(1-q^3x)\cdots(1-q^{2n+1}x)}$$
$$=\frac{q^{2j+1}x}{(1-qx)(1-q^3x)}F_j(q^2x).$$

These two equations give for $j=1$ the recurrence for $F_0=F$ stated in the Proposition, and for $j\geqslant 1$ the recurrence

$$f_{j-1}(x)=1-q^{2j-1}x/(1+q^{2j}/f_j(x))$$

for the quantities $f_j(x)=(1-q^{2j+1}x)F_j(q^{2j-2}x)/F_j(q^{2j}x)$. It now follows by induction that $f_0=f$ as claimed.

(ii) We have

$$G(x)-G(q^{-2}x)=\sum_{n=1}^{\infty}\frac{x^{-2n}q^{4n^2}}{(1-q^4)\cdots(1-q^{4n-4})}$$
$$=\sum_{n=0}^{\infty}\frac{x^{-2(n+1)}q^{4(n+1)^2}}{(1-q^4)\cdots(1-q^{4n})}=q^4x^{-2}G(q^{-4}x).$$

Replacing x by $q^{-2n+2}x$ and rearranging, we find

$$xG(q^{-2n+2}x)/G(q^{-2n}x) = x + \frac{q^{4n}}{xG(q^{-2n}x)/G(q^{-2n-2}x)} \quad (n \geqslant 1)$$

and hence by induction $xG(x)/G(q^{-2}x) = q/g(x)$ as asserted. □

The representation for f is a little more awkward than that for g because $F(x)$ has poles at $x = q^{-3}, q^{-5}, \ldots$. To eliminate these, set

$$\hat{F}(x) = (1 - q^3x)(1 - q^5x)\cdots F(x).$$

Then both the recursion for F and the formula for f become simpler:

$$\hat{F}(x) = q^{-1}x\hat{F}(q^{-2}x) + \hat{F}(q^{-4}x), \quad f(x) = \frac{\hat{F}(q^{-2}x)}{\hat{F}(x)}. \tag{3}$$

Write $\hat{F}(x) = \sum_{n=0}^{\infty} a_n x^n$ where $a_n \in \mathbb{Z}[[q]]$. Comparing coefficients of x^n in the first equation in (3) gives $(1 - q^{-4n})a_n = q^{-2n+1}a_{n-1}$, so by induction

$$a_n = \frac{(-1)^n q^{n^2+2n}}{(1-q^4)(1-q^8)\cdots(1-q^{4n})} a_0.$$

Now writing $H(x)$ for $a_0^{-1}\hat{F}(q^{-2}x)$ (the value of a_0 is irrelevant; it is in fact equal to $\prod_1^{\infty}(1 + q^{2n})$) we have

Proposition 2.

Define $H(x) = H(x;q) \in \mathbb{Z}[[q,x]]$ by

$$H(x) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2} x^n}{(1-q^4)(1-q^8)\cdots(1-q^{4n})}.$$

Then $H(x) = -qxH(q^2x) + H(q^4x)$ and $f(x) = H(x)/H(q^2x)$. □

Of course, we could have proved this directly by an argument similar to the proof of Proposition 1 and avoided introducing F entirely.

3. Asymptotic behaviour for $x = 1$; the Rogers–Ramanujan identities

Define a function $\zeta(z)$ in the upper half-plane $\mathfrak{H} = \{z \in \mathbb{C} \mid \Im(z) > 0\}$ by

$$\zeta(z) = \cfrac{q^{\frac{1}{5}}}{1 + \cfrac{q}{1 + \cfrac{q^2}{1 + \ddots}}} \quad (z \in \mathfrak{H}), \tag{4}$$

where $q=\exp(2\pi iz)$ and q^{λ} for $\lambda\in\mathbb{Q}$ denotes $\exp(2\pi i\lambda z)$. For $x=1$, both continued fractions in (1) can be expressed in terms of ζ:

$$f(1;q)=-q^{\frac{1}{5}}/\zeta(z+\tfrac{5}{2}),\quad g(1;q)=q^{\frac{1}{5}}\zeta(4z)\quad (q=\exp(2\pi iz), z\in\mathfrak{H}). \tag{5}$$

The key to understanding the behaviour of $f(1)$ and $g(1)$ as $q\to 1$ is the fact that $\zeta(z)$ is a modular function, namely

$$\zeta(\gamma\langle z\rangle)=\zeta(z)\quad \forall\gamma\in\Gamma(5),\quad z\in\mathfrak{H},$$

where

$$\Gamma(5)=\left\{\begin{pmatrix} a & b\\ c & d\end{pmatrix}\middle| a,b,c,d\in\mathbb{Z}, ad-bc=1, a\equiv d\equiv 1\,(\mathrm{mod}\,5), b\equiv c\equiv 0(\mathrm{mod}\,5)\right\}$$

and $\gamma\langle z\rangle$ for a 2×2 matrix $\gamma=\begin{pmatrix} a & b\\ c & d\end{pmatrix}$ of determinant 1 denotes the image $(az+b)/(cz+d)$ of z under the associated fractional linear transformation. This fact, well known to Ramanujan, follows from the Roger–Ramanujan identities. Recall that these are the formulae

$$\sum_{n=0}^{\infty}\frac{q^{n^2}}{(1-q)\cdots(1-q^n)}=\prod_{n\equiv\pm 1(5)}(1-q^n)^{-1},$$

$$\sum_{n=0}^{\infty}\frac{q^{n^2+n}}{(1-q)\cdots(1-q^n)}=\prod_{n\equiv\pm 2(5)}(1-q^n)^{-1}$$

([1], Theorem 362 and 363). The left-hand sides of these equations are $G(1;q^{\frac{1}{4}})$ and $G(q^{-\frac{1}{4}};q^{\frac{1}{4}})$, respectively, where $G(x;q)$ is defined as in Proposition 1, so by (5) and that Proposition we have

$$\zeta(z)=q^{-\frac{1}{20}}g(1;q^{\frac{1}{4}})=\frac{q^{\frac{1}{5}}G(q^{-\frac{1}{4}};q^{\frac{1}{4}})}{G(1;q^{\frac{1}{4}})}=q^{\frac{1}{5}}\prod_{n=1}^{\infty}(1-q^n)^{(n/5)}$$

($(n/5)=$ Legendre symbol $=+1,-1,0$ for $n\equiv\pm 1,\pm 2,0\ (\mathrm{mod}\,5)$, respectively). On the other hand, we have

$$\prod_{n\equiv 0,\pm 1(5)}(1-q^n)=\sum_n(-1)^n q^{\frac{1}{2}(5n^2+3n)}$$

$$\prod_{n\equiv 0,\pm 2(5)}(1-q^n)=\sum_n(-1)^n q^{\frac{1}{2}(5n^2+n)}$$

by the Jacobi triple product identity ([1], Theorems 355 and 356), so this can be written

$$\zeta(z)=\frac{\theta_{10,3}(z)}{\theta_{10,1}(z)},\quad \theta_{10,j}(z)=\sum_n(-1)^n q^{(10n+j)^2/40}\quad (j=1,3). \tag{6}$$

This equation makes it clear that $\zeta(z)$ is a modular function, since both theta-series $\theta_{10,3}(z)$ and $\theta_{10,1}(z)$ are modular forms of weight $\frac{1}{2}$. In fact $\zeta(z)$ is a well-known modular function called Klein's icosahedral function. It is a "Hauptmodul" for $\Gamma(5)$, i.e., defines an isomorphism from $\mathfrak{H}/\Gamma(5)\cup\{\text{cusps}\}$ to $\mathbb{P}^1(\mathbb{C})$.

We will not give the proof of the invariance of $\zeta(z)$ under $\Gamma(5)$, since in view of (5) what we actually need are the transformation laws satisfied by $\zeta(z)$ with respect to matrices sending the cusps 0 and $\frac{5}{2}$ to infinity. To obtain these, recall that the Poisson summation formula implies

$$\sum_{n=1}^{\infty}\chi(n)e^{-\pi n^2 t/f}=\frac{G(\chi)}{(ft)^{\frac{1}{2}}}\sum_{n=1}^{\infty}\bar{\chi}(n)e^{-\pi n^2/ft}\quad(\Re(t)>0)\tag{7}$$

for any even primitive Dirichlet character χ of conductor $f>1$, where $G(\chi)$ is the Gauss sum $\sum_{n=1}^{f}\chi(n)\cos(2\pi n/f)$ associated to χ, a complex number of absolute value $\sqrt{f}$. Applying this to the two primitive even characters of conductor 20 gives

$$\theta_{10,1}(it)\pm i\theta_{10,3}(it)=\frac{2\sin(2\pi/5)}{(5t)^{\frac{1}{2}}}\left(1\pm i\frac{\sqrt{5}-1}{2}\right)\left(\theta_{10,1}\left(\frac{i}{t}\right)\mp i\theta_{10,3}\left(\frac{i}{t}\right)\right)$$

and hence from (6)

$$\zeta(it)=\frac{\dfrac{\sqrt{5}-1}{2}-\zeta\left(\dfrac{i}{t}\right)}{1+\dfrac{\sqrt{5}-1}{2}\zeta\left(\dfrac{i}{t}\right)}.$$

Inserting this into the second of equation (5) and letting t tend to 0 through positive real values, we find

$$g(1;q)=q^{\frac{1}{5}}\frac{\dfrac{\sqrt{5}-1}{2}-\zeta\left(\dfrac{-\pi i}{2\log q}\right)}{1+\dfrac{\sqrt{5}-1}{2}\zeta\left(\dfrac{-\pi i}{2\log q}\right)}$$

$$=q^{\frac{1}{5}}\frac{\sqrt{5}-1}{2}\;\frac{1-\frac{1}{2}(\sqrt{5}+1)Q/(1+Q^5/(1+Q^{10}/(1+\cdots)))}{1+\frac{1}{2}(\sqrt{5}-1)Q/(1+Q^5/(1+Q^{10}/(1+\cdots)))}$$

($Q=\exp\{(\pi^2/5)/\log q\}$), giving the second formula in (I) of the Theorem. Similarly (7) applied to the two non-real even primitive characters of conductor 40 gives

$$\theta_{10,1}(\tfrac{1}{2}(5+it))\pm i\theta_{10,3}(\tfrac{1}{2}(5+it))$$

$$=\frac{2\sin(\pi/5)}{\sqrt{5}}(1\mp\tfrac{1}{2}(\sqrt{5}+1)i)(\theta_{10,1}(\tfrac{1}{2}(5+i/t))\mp i\theta_{10,3}(\tfrac{1}{2}(5+i/t)))$$

and hence by (6) and (5)

$$\zeta\left(\frac{5+it}{2}\right)=\frac{\frac{1}{2}(\sqrt{5}+1)-\zeta(\frac{1}{2}(5+i/t))}{-1+\frac{1}{2}(\sqrt{5}+1)\zeta(\frac{1}{2}(5+i/t))},$$

$$f(1;q)=q^{\frac{1}{5}}\frac{\sqrt{5}-1}{2}\;\frac{1+\frac{1}{2}(\sqrt{5}+1)Q/(1-Q^5/(1-Q^{10}/(1-\cdots)))}{1-\frac{1}{2}(\sqrt{5}-1)Q/(1-Q^5/(1-Q^{10}/(1-\cdots)))},$$

giving the first formula in (I) also.

4. Asymptotic behaviour for x small: theta series

Write the function $H(x)$ of Proposition 2 as $H_+(x) + H_-(x)$, where

$$H_\pm(x) = \pm \sum_{\substack{n \geq 0 \\ (-1)^n = \pm 1}} \frac{q^{n^2} x^n}{(1-q^4)\cdots(1-q^{4n})},$$

so that $H_+(x) \in \mathbb{Z}[x^2][[q^4]]$, $H_-(x) \in qx\mathbb{Z}[x^2][[q^4]]$. As a refinement of the recursion $H(x) = -qxH(q^2x) + H(q^4x)$ of Proposition 2 we have

$$\begin{aligned} H_\pm(x) - H_\pm(q^4x) &= \pm \sum_{\substack{n \geq 1 \\ (-1)^n = \pm 1}} \frac{q^{n^2} x^n}{(1-q^4)\cdots(1-q^{4n-4})} \\ &= \pm \sum_{\substack{n \geq 0 \\ (-1)^n = \mp 1}} \frac{q^{(n+1)^2} x^{n+1}}{(1-q^4)\cdots(1-q^{4n})} = -qxH_\mp(q^2x). \end{aligned}$$

An equivalent formulation of this is that the 2×2 matrix

$$\mathscr{H}(x) = \begin{pmatrix} H_+(x) & H_-(x) \\ H_-(q^2x) & H_+(q^2x) \end{pmatrix}$$

satisfies the recursion

$$\mathscr{H}(q^2x) = \begin{pmatrix} 0 & 1 \\ 1 & qx \end{pmatrix} \mathscr{H}(x) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{8}$$

This implies that the determinant of $\mathscr{H}(x)$ is invariant under $x \mapsto q^2x$ and hence, since $\mathscr{H}(x)$ tends to the identity matrix for $x \to 0$ with q fixed, that $\det \mathscr{H}(x) = 1$ identically. Now the content of Proposition 2 can be reformulated as

$$f(x) = \mathscr{H}(x)\langle 1 \rangle, \tag{9}$$

where $\langle\ \rangle$ denotes fractional linear transformation as in § 3. The key to Ramanujan's assertion is that $g(x)$, the second expression in (1), is given by

$$g(x) = \mathscr{H}(x)\langle \lambda(x) \rangle \tag{10}$$

where

$$\lambda(x) = \lambda(x; q) = \frac{\theta_-(x)}{\theta_+(x)}, \quad \theta_\pm(x) = \sum_{\substack{n \in \mathbb{Z} \\ (-1)^n = \pm 1}} q^{n^2} x^n. \tag{11}$$

Indeed, by applying the Poisson summation formula to θ_+ and θ_- as in the last section (this is of course the most classical application of the Poisson summation formula, going back to Jacobi), we obtain

$$\lambda(x) = \frac{1 - 2Q\cos\theta + 2Q^4\cos 2\theta - 2Q^9\cos 3\theta + \cdots}{1 + 2Q\cos\theta + 2Q^4\cos 2\theta + 2Q^9\cos 3\theta + \cdots} \tag{12}$$

where $Q = \exp((\pi^2/4)/\log q)$, $\theta = (\pi/2)(\log x/\log q)$, and since $\mathscr{H}(x) = \mathbf{1}_2 + O(x)$ we immediately deduce from (10) the assertion of part (II) of the Theorem.

Equation (10) follows immediately from Proposition 1 and

PROPOSITION 3.

Denote by $\vec{G}(x)$ and $\vec{\theta}(x)$ the vectors

$$\begin{pmatrix} qx^{-1}G(q^{-2}x) \\ G(x) \end{pmatrix} \quad \textit{and} \quad \begin{pmatrix} \theta_-(x) \\ \theta_+(x) \end{pmatrix},$$

respectively. Then $\vec{G}(x) = c\cdot\mathscr{H}(x)\vec{\theta}(x)$ where c is a scalar independent of x.

(In fact $c = \prod_{n=1}^{\infty}(1-q^{4n})^{-1}$, but we do not need this fact and will omit the proof.)

Proof. The recursion satisfied by $G(x)$ can be written in terms of $\vec{G}(x)$ as

$$\vec{G}(q^2x) = q^{-1}x^{-1}\begin{pmatrix} 0 & 1 \\ 1 & qx \end{pmatrix}\vec{G}(x).$$

Combining this with (8) we see that the vector $\vec{t}(x) = \mathscr{H}(x)^{-1}\vec{G}(x)$ satisfies the recursion

$$\vec{t}(q^2x) = q^{-1}x^{-1}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\vec{t}(x). \tag{13}$$

Write $\vec{t}(x)$ as $\begin{pmatrix} t_-(x) \\ t_+(x) \end{pmatrix}$. Since $G(x)$ is an even power series in x we have

$$\vec{G}(x) = \begin{pmatrix} \text{odd} \\ \text{even} \end{pmatrix}, \quad \mathscr{H}(x) = \begin{pmatrix} \text{even} & \text{odd} \\ \text{odd} & \text{even} \end{pmatrix}$$

(where "odd" and "even" denote even and odd functions of x), so $t_-(x)$ is odd and $t_+(x)$ is even, i.e..

$$t_\pm(x) = \sum_{\substack{n \\ (-1)^n = \pm 1}} t_n x^n$$

for some coefficients $t_n \in \mathbb{Z}[[q]]$. (Note that all coefficients of $\mathscr{H}, \vec{G}, \vec{\theta}$ and $\vec{t}$ are in the ring $\mathbb{Z}[x, x^{-1}][[q]]$ and can be expanded as doubly infinite Laurent series in x with coefficients in $\mathbb{Z}[[q]]$.) The recursion (13) now gives $t_{n+1} = q^{2n+1}t_n$ for all $n \in \mathbb{Z}$, so $t_n = t_0 q^{n^2}$. This proves Proposition 3 with $c = t_0$. $\square$

The fact that the recursion (13) is satisfied by $\vec{\theta} = c^{-1}\vec{t}$ as well as by $\vec{t}$ says that $\theta_\pm(q^2x) = q^{-1}x^{-1}\theta_\mp(x)$. Therefore $\lambda(x)$ is changed to its reciprocal under $x \mapsto q^2x$ and hence is invariant under $x \mapsto q^4x$, which is why $\lambda(x)$ must have a Fourier series expansion in $\theta = (\pi/2)(\log x/\log q)$ (as given explicitly in (12)). The property $\lambda(q^2x) = \lambda(x)^{-1}$ together with the obvious symmetry property $\lambda(x^{-1}) = \lambda(x)$ implies that $\lambda(x) = 1$

for x an odd power of q (this also follows from (12), or from the well-known product expansion of λ, or from (11) since

$$\sum_{n\,\mathrm{odd}} q^{n^2+n} = \sum_{n\,\mathrm{even}} q^{n^2+n}$$

via $n \to -1-n$), so (9) and (10) imply Cohen's result that $f(x)$ and $g(x)$ agree for such x. Finally, instead of using just $\mathscr{H}(x) = \mathbf{1}_2 + O(x)$ we could use the full expansion

$$\mathscr{H}(x) = \begin{pmatrix} 1 + \dfrac{q^4x^2}{(1-q^4)(1-q^8)} + \cdots & -\dfrac{qx}{1-q^4} - \cdots \\ -\dfrac{q^3x}{1-q^4} - \cdots & 1 + \dfrac{q^8x^2}{(1-q^4)(1-q^8)} + \cdots \end{pmatrix}$$

and thus replace the formula in (II) of the Theorem by a full expansion in powers of x for $\theta \pmod{2\pi}$ fixed (i.e. for x tending to 0 through a sequence $q^{4n}x_0, n \to \infty$). In particular, to two terms we have

$$f(x) = 1 - \frac{q}{1+q^2}x + \mathrm{O}(x^2), \quad g(x) = \lambda(x) - \mu(x)x + \mathrm{O}(x^2) \quad \text{for} \quad x \to 0,$$

where $\mu(x)$ is the periodic function

$$\mu(x) = \frac{q}{1-q^4}(1 - q^2\lambda(x)^2) \quad \left(= \frac{q}{1+q^2} + \mathrm{O}(Q) \text{ as } q \to 1 \right).$$

5. Asymptotic behaviour for x arbitrary: the dilogarithm function

In the last two sections we studied the asymptotics of $f(x;q) - g(x;q)$ as $q \to 1$ for $x = 1$ and x near 0. We now study how these asymptotics change as x changes. Write $\varepsilon(x) = \varepsilon(x;q)$ for $f(x) - g(x)$. Subtracting the functional equation (2) with $h = g$ from the same equation with $h = f$, we find

$$\begin{aligned} 0 &= (g(x) + \varepsilon(x) + qx)(g(q^2x) + \varepsilon(q^2x)) - (g(x) + qx)g(q^2x) \\ &= \varepsilon(x)f(q^2x) + \varepsilon(q^2x)g(q^2x)^{-1} \end{aligned}$$

or

$$\varepsilon(x) = -\frac{1}{f(q^2x)g(q^2x)}\varepsilon(q^2x).$$

By induction this gives

$$\varepsilon(x) = \frac{(-1)^n}{f(q^2x)g(q^2x)\cdots f(q^{2n}x)g(q^{2n}x)}\varepsilon(q^{2n}x). \tag{14}$$

As $n \to \infty$ we have

$$q^{2n}x \to 0, \quad f(q^{2n}x) \to 1, \quad g(q^{2n}x) \to \begin{Bmatrix} \lambda(x) & \text{for } n \text{ even} \\ \lambda(q^2x) = \lambda(x)^{-1} & \text{for } n \text{ odd,} \end{Bmatrix}$$

by the results of the last section (specifically, by equations (9) and (10) and the periodicity $\lambda(q^4x) = \lambda(x)$), so letting $n \to \infty$ in (14) gives the closed formula

$$\varepsilon(x) = \left(\prod_{n=1}^{\infty} \frac{1}{f(q^{4n-2}x)g(q^{4n-2}x)f(q^{4n}x)g(q^{4n}x)} \right)(1 - \lambda(x)), \tag{15}$$

where the nth term of the product tends to 1 with exponential rapidity. We use equations (14) and (15) to study the asymptotics of $\varepsilon(x;q)$ as $q \to 1$ with x fixed. (Actually, this is not quite right since as $\log x/\log q$ varies modulo 4 the value of ε will oscillate; thus we should either let q tend to 1 continuously and restrict x to lie in an interval $[q^{-2}x_0, q^2x_0]$ with $\log x/\log q \pmod 4$ constant, or else fix x and let q tend to 1 through a sequence of values for which $\log x/\log q \pmod 4$ is constant.) In fact, since the behaviour of $1 - \lambda(x)$ as $q \to 1$ is completely known—it is asymptotically equal to $4\exp\{(\pi^2/4)/\log q\} \cos[(\pi/2)(\log x/\log q)]$— and since both $\varepsilon(x)$ and $1 - \lambda(x)$ vanish whenever x is an odd power of q, it is more convenient to study the ratio $\varepsilon(x)/(1 - \lambda(x))$, the first factor in (15). It is also natural to consider the $(\log q)$th power of this, i.e. to define a new quantity $A(x;q)$ by

$$\varepsilon(x;q) = (1 - \lambda(x;q))A(x;q)^{1/\log q}, \tag{16}$$

because by parts (I) and (II) of the Theorem we know that $A(x;q)$ has the well-defined limits $e^{-\pi^2/20}$ and 1 as $x \to 1$, $q \to 1$ and $x \to 0$, $q \to 1$, respectively. We want to show that $A(x;q)$ tends to a limit as $q \to 1$ for any x, and to evaluate this limit.

The first thing to notice is that $g(x;q)$ tends to a well-defined limit as $q \to 1$, since the continued fraction on the right of (1) converges at $q = 1$ for all $x > 0$. Call this limit $\gamma(x)$; then $\gamma(x) = 1/(x + \gamma(x))$ and consequently

$$\gamma(x) = (1 + x^2/4)^{\frac{1}{2}} - x/2.$$

The continued fraction on the left of (1) also converges for $q = 1$ if x is sufficiently small (actually, as one easily checks, for $x < 2$), and it must have the same limit $\gamma(x)$ because of the functional equation (2). On the other hand, equation (14) implies

$$A(q^{4n}x;q) = (f(q^2x)g(q^2x)f(q^4x)g(q^4x)\cdots f(q^{4n}x)g(q^{4n}x))^{\log q}A(x;q). \tag{17}$$

Choose a small number δ and let $q \to 1$ and $n \to \infty$ in such a way that $q^{4n} = e^{-\delta}$. Then each factor $f(q^{2i}x)$ and $g(q^{2i}x)$ in (17) equals $\gamma(x) + O(\delta)$ for $q \to 1$, so (17) gives

$$\begin{aligned} A(e^{-\delta}x;q) &= (\gamma(x) + O(\delta))^{4n\log q}A(x;q) = (\gamma(x) + O(\delta))^{-\delta}A(x;q) \\ &= (1 - \delta\gamma(x) + O(\delta^2))A(x;q) \quad (q \to 1). \end{aligned}$$

Together with the fact that $\lim_{x\to 0} A(x;q) = 1$ for all q, this shows that $A(x;q)$ for $q \to 1$ has a limit $A(x)$ which satisfies $A'(x) = (\gamma(x)/x)A(x)$ and $A(0) = 1$, or equivalently

$$A(x) = e^{c(x) - \pi^2/4}, \quad c(x) = \frac{\pi^2}{4} + \int_0^x \frac{1}{t}\log\left[\left(1 + \frac{t^2}{4}\right)^{\frac{1}{2}} - \frac{t}{2}\right]dt. \tag{18}$$

In view of (16) and the known asymptotics of $1 - \lambda(x)$ for $q \to 1$, this proves (III) of the Theorem except for the evaluation of the integral $c(x)$. Setting $t = 2y$ and integrating by parts, we find

$$c(x) = \frac{\pi^2}{4} - \log\left(\frac{x}{2}\right)\log\left[\left(1 + \frac{x^2}{4}\right)^{\frac{1}{2}} + \frac{x}{2}\right] + \int_0^{x/2} \frac{\log y}{(1+y^2)^{\frac{1}{2}}} \mathrm{d}y.$$

The integral is evaluated in [2] (A.3.1.(6)) in terms of the dilogarithm, and this gives the formula asserted in the Theorem (up to the evaluation of the constant, which is fixed by $c(0) = \pi^2/4$, $Li_2(1) = \pi^2/6$). As mentioned in the introduction, the formula $c(1) = \pi^2/5$, which we know to be true by (II) of the Theorem, is equivalent to the special value $Li_2((3 - \sqrt{5})/2) = \pi^2/15 - \log^2((1 + \sqrt{5})/2)$ of the dilogarithm, a value well known to Ramanujan.

6. Final remarks

In this paper we have made Ramanujan's assertion (1) precise in various senses and given proofs of these statements. It is reasonable to ask how much of what we have done Ramanujan actually had in mind. Obviously this is pure speculation. I would guess that he knew (I) of the Theorem, since the function defined by the continued fraction (4) was a favourite of his, and that he knew the magic identity (10), which—since he certainly knew (9) and that $\lambda(x)$ is very close to 1—would suffice to imply (1) in the rough form stated. I do not think that he knew the full asymptotics of $f - g$ for $q \to 1$ as given in (III) of the Theorem, since he was particularly fond of the dilogarithm and of its evaluation at special arguments and would hardly have failed to at least mention the formula in his notebook. As to how he might have discovered (10), I have no idea, since I absolutely do not know where this identity comes from. I myself found it by the stupid method of evaluating the difference $f(x;q) - g(x;q)$ for hundreds of values of x and q and thus discovering numerically that $g(x)$ for x small had the Fourier expansion given in (II); this at least suggested looking at the theta-series $\theta_+(x)$ and $\theta_-(x)$, after which it was not too hard to discover the matrix $\mathscr{H}(x)$ and the identity (10) (the proofs, of course, were easy once the formulae were known, as in all identities of this type). Such extensive numerical computations would be impossible without a computer even for a Ramanujan—but then again a Ramanujan would not (and in this case did not) need them to discover mysterious and beautiful identities which would be hidden to ordinary mortals.

References

[1] Hardy G H and Wright E M, *An introduction to the theory of numbers, 4th ed.* (Oxford: University Press) (1960)

[2] Lewin L, *Polylogarithms and associated functions*, (New York: North Holland) (1981)

Note added in proof

While in India for the Ramanujan centenary celebrations, I learned that a formula equivalent to (10) *is* in fact contained in Ramanujan's "Lost Notebook".