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# Realizability of a Model in Infinite Statistics

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**Abstract.** Following Greenberg and others, we study a space with a collection of operators $a(k)$ satisfying the "$q$-mutator relations" $a(l)a^{\dagger}(k) - qa^{\dagger}(k)a(l) = \delta_{k,l}$ (corresponding for $q = \pm 1$ to classical Bose and Fermi statistics). We show that the $n! \times n!$ matrix $A_n(q)$ representing the scalar products of $n$-particle states is positive definite for all $n$ if $q$ lies between $-1$ and $+1$, so that the commutator relations have a Hilbert space representation in this case (this has also been proved by Fivel and by Bożejko and Speicher). We also give an explicit factorization of $A_n(q)$ as a product of matrices of the form $(1 - q^j T)^{\pm 1}$ with $1 \leqq j \leqq n$ and $T$ a permutation matrix. In particular, $A_n(q)$ is singular if and only if $q^M = 1$ for some integer $M$ of the form $k^2 - k$, $2 \leqq k \leqq n$.

## 1. Introduction

In this paper we study the following object: a Hilbert space $\mathbf{H}$ together with a non-zero distinguished vector $|0\rangle$ (vacuum state) and a collection of operators $a_k$: $\mathbf{H} \to \mathbf{H}$ satisfying the commutation relations ("$q$-mutator relations")

$$a(l)\,a^{\dagger}(k) - q a^{\dagger}(k)\,a(l) = \delta_{k,l} \quad (\forall k, l) \tag{1}$$

and the relations

$$a(k)|0\rangle = 0 \quad (\forall k). \tag{2}$$

Here $q$ is a fixed real number and $a^{\dagger}(l)$ denotes the adjoint of $a(l)$. The statistics based on the commutation relation (1) generalizes classical Bose and Fermi statistics, corresponding to $q = 1$ and $q = -1$, respectively, as well as the intermediate case $q = 0$ suggested by Hegstrom and investigated by Greenberg [1]. The study of the general case was inititated by Polyakov and Biedenharn [2].

Our first main result is a realizability theorem saying that the object just described exists if $-1 < q < 1$. In view of (2), we can think of the $a(k)$ as annihilation operators and the $a^{\dagger}(k)$ as creation operators. As well as the 0-particle state $|0\rangle$, our space must contain the many-particle states obtained by applying combinations of $a(k)$'s and $a^{\dagger}(k)$'s to $|0\rangle$. To prove the realizability of our model it is obviously necessary and sufficient to consider the minimal space containing these vectors. We therefore define for each $q \in \mathbb{R}$ an inner product space $\mathbf{H}(q)$ generated by $|0\rangle$ and its images under polynomials in the operators $a(k)$ and $a^\dagger(k)$, subject to the relations (1) and (2). It has a basis consisting of $n$-particle states

$$\mathbf{x}_\mathbf{k} = a^\dagger(k_1) \cdots a^\dagger(k_n)|0\rangle$$

for each $n \geqq 0$ and each $n$-tuple of indices $\mathbf{k} = (k_1, \ldots, k_n)$, since we can use (1) to write any monomial in the $a(k)$'s and $a^\dagger(k)$'s as a sum of monomials having all the $a(k)$'s on the right and all the $a^\dagger(k)$'s on the left, and the only ones of these which do not annihilate $|0\rangle$ are those consisting of $a^\dagger(k)$'s only (the linear independence is clear). By the same argument, we can use (1) and (2) to calculate each scalar product $(\mathbf{x}_\mathbf{l}, \mathbf{x}_\mathbf{k})$ as a polynomial in $q$, for instance, for $k \neq l$ we have

$$\begin{aligned}(\mathbf{x}_{kl}, \mathbf{x}_{lk}) &= \langle 0|a(l)\,a(k)\,a^\dagger(l)\,a^\dagger(k)|0\rangle = q\langle 0|a(l)\,a^\dagger(l)\,a(k)\,a^\dagger(k)|0\rangle \\ &= q\langle 0|(1+qa^\dagger(l)\,a(l))\,(1+qa^\dagger(k)\,a(k))|0\rangle = q\langle 0|0\rangle = q\,.\end{aligned}$$

(Here $\langle 0|$ denotes the operator $(|0\rangle, \cdot)$ and we have normalized by $\langle 0|0\rangle = 1$.) In particular, for each value of $q$ the infinite matrix $A(q) = \{(x_\mathbf{l}, x_\mathbf{k})\}_{\mathbf{l},\mathbf{k}}$ is well-defined. The condition for the Hilbert space realizability of the $q$-mutator relation (1) is then that $A(q)$ be positive definite, i.e., that $(\mathbf{x}, \mathbf{x}) > 0$ for every non-zero vector $\mathbf{x} \in \mathbf{H}(q)$.

**Theorem 1.** *The matrix $A(q)$ is positive definite for $-1 < q < 1$, so that the $q$-mutator relation (1) has a Hilbert space realization for $q$ in this range.*

It is easy to see that $(x_\mathbf{k}, x_\mathbf{l})$ vanishes unless $\mathbf{k}$ is a permutation of $\mathbf{l}$. Thus the space $\mathbf{H}(q)$ [respectively the matrix $A(q)$] is the direct sum of infinitely many finite-dimensional spaces (respectively matrices) indexed by all *unordered* $n$-tuples $\{k_1, \ldots, k_n\}$, and we only have to show the positive definiteness of these. We will show in Sect. 2 that the general case of this follows from the case when all of the indices $k_i$ are distinct. It is not hard to see (Sect. 2) that

$$(\mathbf{x}_{\pi(1)\ldots\pi(n)}, \mathbf{x}_{1\ldots n}) = q^{I(\pi)} \tag{3}$$

for each permutation $\pi$ in the $n^{\text{th}}$ symmetric group $\mathfrak{S}_n$, where $I(\pi)$ denotes the number of *inversions* of $\pi$, i.e., the number of $i, j \in [1, n]$ for which $i < j$ but $\pi(i) > \pi(j)$. Thus the problem reduces to showing that the $n! \times n!$ matrix $A_n = A_n(q)$ defined by

$$A_n(\pi, \sigma) = q^{I(\sigma^{-1}\pi)} \quad (\pi, \sigma \in \mathfrak{S}_n) \tag{4}$$

is positive definite for $q$ between $-1$ and 1. For this, in turn, it is sufficient by continuity to show that $A_n(q)$ is non-singular in this range, since $A_n(0)$ is the identity matrix and the eigenvalues of $A_n(q)$ vary continuously with $q$ and are real for $q$ real (because $A_n(q)$ is real and symmetric). We will prove the following stronger statement.

**Theorem 2.** *The determinant of the matrix $A_n(q)$ is given by*

$$\det A_n(q) = \prod_{k=1}^{n-1} (1 - q^{k^2+k})^{\frac{n!\,(n-k)}{k^2+k}}\,. \tag{5}$$

*In particular, $A_n(q)$ is non-singular for all complex numbers $q$ except the $N^{\text{th}}$ roots of unity for $N = 2, 6, 12, \ldots, n^2 - n$.*

We will also describe explicitly the inverse of $A_n(q)$. Based on calculations for $n \leqq 5$, we conjecture that

$$A_n(q)^{-1} \overset{?}{\in} \frac{1}{\Delta_n} M_{n!}(\mathbb{Z}[q]), \qquad \Delta_n := \prod_{k=1}^{n-1} (1 - q^{k^2+k}). \tag{6}$$

For instance, for $n = 3$ we have

$$A_3(q) = \begin{bmatrix} 1 & q & q & q^2 & q^2 & q^3 \\ q & 1 & q^2 & q & q^3 & q^2 \\ q & q^2 & 1 & q^3 & q & q^2 \\ q^2 & q & q^3 & 1 & q^2 & q \\ q^2 & q^3 & q & q^2 & 1 & q \\ q^3 & q^2 & q^2 & q & q & 1 \end{bmatrix}, \tag{7}$$

where the rows and columns are indexed by the elements of $\mathfrak{S}_3$ in the order [123], [213], [132], [231], [312], [321] (we use $[j_1 \dots j_n]$ to denote the element $\pi$ of $\mathfrak{S}_n$ defined by $\pi(i) = j_i$). The determinant of this matrix is $(1-q^2)^6(1-q^6)$ and its inverse is

$$A_3(q)^{-1} = \Delta_3^{-1} \begin{bmatrix} 1+q^2 & -q & -q & -q^4 & -q^4 & q^3+q^5 \\ -q & 1+q^2 & -q^4 & -q & q^3+q^5 & -q^4 \\ -q & -q^4 & 1+q^2 & q^3+q^5 & -q & -q^4 \\ -q^4 & -q & q^3+q^5 & 1+q^2 & -q^4 & -q \\ -q^4 & q^3+q^5 & -q & -q^4 & 1+q^2 & -q \\ q^3+q^5 & -q^4 & -q^4 & -q & -q & 1+q^2 \end{bmatrix}. \tag{8}$$

Finally, we remark that the matrix $A_n(q)$ splits as a direct sum of pieces corresponding to the irreducible representations of $\mathfrak{S}_n$, the piece corresponding to a representation $\Pi$ of dimension $d$ being the direct sum of $d$ copies of a $d \times d$ matrix $A_{n,\Pi}(q)$. For the bosonic and fermionic cases $q = 1$ and $q = -1$ all of these matrices are identically zero except for the one corresponding to the one-dimensional trivial or alternating representation, respectively, but for $-1 < q < 1$ Theorem 1 says that every representation of every symmetric group occurs in a non-trivial (indeed, non-degenerate) way. (This is the reason for the term "infinite statistics" used by the physicists.) It would be of interest to calculate the determinants of the matrices $A_{n,\Pi}(q)$, say in terms of the Young diagram corresponding to $\Pi$. By Theorem 2, each of these determinants is a product of cyclotomic polynomials $\Phi_m(q)$ for integers $m$ dividing some $k^2 + k$, $1 \leqq k \leqq n - 1$.

The paper is organized as follows. In Sect. 2 we give some generalities on group determinants and show that Theorem 1 follows from Theorem 2, which is then proved in Sect. 3. In Sect. 4 we give an explicit description of the inverse matrix of $A_n(q)$, while Sect. 5 gives a conjectural formula for the "number operators" in the Hilbert space $\mathbf{H}(q)$.

The author would like to thank O. W. Greenberg who told him about the $q$-mutator relation and suggested the problem of proving the positive definiteness for $-1 < q < 1$. This positive definiteness has been proved independently by Fivel and by Bożejko and Speicher [3]. (However, Fivel apparently asserts that the zeros of $A_n(q)$ are all roots of $q^{2n} = 1$, which contradicts Theorem 2 and is false for all

$n \geqq 4$.) Consequences and related results are discussed in several subsequent papers by Greenberg [4].

## 2. Group Determinants and the Reduction to $A_n(q)$

Let $G$ be a finite group of order $m$ and $\varrho: G \to GL(V)$ a representation of $G$ on a (finite-dimensional) complex vector space $V$. We can extend $\varrho$ to an algebra homomorphism from the group algebra

$$\mathbb{C}[G] = \left\{ \sum_{g \in G} t_g g \mid t_g \in \mathbb{C} \text{ for } g \in G \right\}$$

to the matrix algebra $\operatorname{End}(V)$ by $\varrho(\sum t_g g) = \sum t_g \varrho(g)$. The determinant of $\varrho(\sum t_g g)$ is a polynomial $F_\varrho(\mathbf{t})$ of degree $\dim(V)$ in the $m$ variables $\mathbf{t} = \{t_g\}_{g \in G}$ which is determined by and uniquely determines the isomorphism class of the representation $\varrho$. Thus the entire representation theory of $G$ can be expressed in terms of the "group determinants" $F_\varrho(\mathbf{t})$; this is in fact the way that representation theory was developed in its early years (see for instance Weber's *Lehrbuch der Algebra*, Vol. 2, Chap. 7).

If $V$ is reducible, say $V = V_1 \oplus V_2$, then $F_\varrho(\mathbf{t})$ splits as $F_{\varrho_1}(\mathbf{t}) F_{\varrho_2}(\mathbf{t})$, so the study of group determinants can be reduced to the case of irreducible representations of $G$. At the other extreme, let $(V, R)$ be the (*right*) *regular representation* of $G$, i.e. $V = \mathbb{C}^G$ is the $m$-dimensional vector space of functions $f: G \to \mathbb{C}$ and $\varrho = R$ is given by

$$(R(g)f)(g') = f(g'g) \quad (g, g' \in G).$$

The matrix representation of $R$ with respect to the basis of $\delta$-functions on $G$ is clearly given by

$$R(g)_{g_1, g_2} = \begin{cases} 1 & \text{if } g_1 g = g_2, \\ 0 & \text{otherwise}, \end{cases}$$

so that the group determinant $F_R(\mathbf{t})$ is the determinant of the $m \times m$ matrix $(t_{g_1^{-1} g_2})_{g_1, g_2 \in G}$. It is well known that $R$ contains every irreducible representation $\Pi$ of $G$ with positive multiplicity (equal to $\dim \Pi$). Hence if $F_R(\mathbf{t}) \neq 0$ for some $\mathbf{t} \in \mathbb{C}^m$ then $F_\Pi(\mathbf{t}) \neq 0$ for every irreducible representation $\pi$ and consequently $F_\varrho(\mathbf{t}) \neq 0$ for every representation $\varrho$ of $G$.

Now apply this to $G = \mathfrak{S}_n$, $m = n!$. Formula (4) and the discussion just given say that $A_n = A_n(q)$ is just the matrix representation $R(\alpha_n)$ of the element

$$\alpha_n = \alpha_n(q) = \sum_{\pi \in \mathfrak{S}_n} q^{I(\pi)} \pi \in \mathbb{C}[\mathfrak{S}_n] \tag{9}$$

acting on the regular representation $(\mathbb{C}^{\mathfrak{S}_n}, R)$. Here we are thinking of $q$ as being a complex number; if $q$ is thought of as a variable, then $\alpha_n(q)$ belongs to the group ring $\mathbb{Z}[q][\mathfrak{S}_n]$. We will usually consider $q$ as fixed and omit it from the notation. To prove Theorems 1 and 2, we will forget that $\alpha_n$ is acting on $\mathbb{C}^{\mathfrak{S}_n}$ and simply show that it is invertible in the group algebra if $\prod_{i=1}^{n-1}(1 - q^{k^2+k}) \neq 0$, in which case the inverse of the matrix $A_n$ is simply the matrix $R(\alpha_n^{-1})$.

We now use this point of view to show how the positive definiteness of $A(q) = \{(\mathbf{x}_l, \mathbf{x}_k)\}$ follows from that of the $n! \times n!$ matrices $A_n(q)$ for $n = 1, 2, 3, \ldots$. Equation (1) gives by induction the formula for any indices $l, k_1, \ldots, k_n$ (not necessarily distinct)

$$a(l)\,a^\dagger(k_1)\cdots a^\dagger(k_n)=q^n a^\dagger(k_1)\cdots a^\dagger(k_n)\,a(l)+\sum_{\substack{1\leqq i\leqq n\\ k_i=l}} q^{i-1}a^\dagger(k_1)\cdots\widehat{a^\dagger(k_i)}\cdots a^\dagger(k_n),$$

where the sum runs over those indices $i$ for which $k_i$ equals $l$ and the hat over the $i^{\text{th}}$ term of the product indicates that this term is to be omitted. Combining this with (2) gives

$$a(l)\,a^\dagger(k_1)\cdots a^\dagger(k_n)|0\rangle=\sum_{\substack{1\leqq i\leqq n\\ k_i=l}} q^{i-1}a^\dagger(k_1)\cdots\widehat{a^\dagger(k_i)}\cdots a^\dagger(k_n)|0\rangle.$$

Now induction on $m$ gives a formula for $a(l_m)\cdots a(l_1)\,a^\dagger(k_1)\cdots a^\dagger(k_n)|0\rangle$ as a sum of terms $q^N a^\dagger(k_1)\cdots\widehat{a^\dagger(k_{i_1})}\cdots\widehat{a^\dagger(k_{i_m})}\cdots a^\dagger(k_n)|0\rangle$, where $i_1,\ldots,i_m$ are distinct indices with $k_{i_1},\ldots,k_{i_m}$ equal to $l_1,\ldots,l_m$ in some order, the final result for $m=n$ being

$$a(l_n)\cdots a(l_1)\,a^\dagger(k_1)\cdots a^\dagger(k_n)|0\rangle=\sum_{\substack{1\leqq i_1,\ldots,i_n\leqq n\\ i_1,\ldots,i_n\ \text{distinct}\\ k_{i_1}=l_1,\ldots,k_{i_n}=l_n}} q^{\#\{1\leqq r<s\leqq n,\ i_r>i_s\}}|0\rangle,$$

i.e., in the notation of Sect. 1,

$$(\mathbf{x}_\mathbf{l},\mathbf{x}_\mathbf{k})=\langle 0|a(l_n)\cdots a(l_1)\,a^\dagger(k_1)\cdots a^\dagger(k_n)|0\rangle=\sum_{\substack{\pi\in\mathfrak{S}_n\\ l_i=k_{\pi(i)}\,(i=1,\ldots,n)}} q^{I(\pi)}. \quad (10)$$

This formula includes (4) and also shows that $(\mathbf{x}_\mathbf{l},\mathbf{x}_\mathbf{k})=0$ unless $\mathbf{l}$ and $\mathbf{k}$ are permutations of one another, as already mentioned in Sect. 1, so that $A(q)$ splits up into the matrices $A_{\mathbf{k}_0}$ having as entries the numbers $(\mathbf{x}_\mathbf{l},\mathbf{x}_\mathbf{k})$ for $\mathbf{l}$ and $\mathbf{k}$ ranging over all permutations of a given index set $\mathbf{k}_0$, e.g. for $\mathbf{k}_0=(k,k,l)$ with $k\neq l$

$$A_{\mathbf{k}_0}=\begin{pmatrix}(\mathbf{x}_{kkl},\mathbf{x}_{kkl}) & (\mathbf{x}_{kkl},\mathbf{x}_{klk}) & (\mathbf{x}_{kkl},\mathbf{x}_{lkk})\\ (\mathbf{x}_{klk},\mathbf{x}_{kkl}) & (\mathbf{x}_{klk},\mathbf{x}_{klk}) & (\mathbf{x}_{klk},\mathbf{x}_{lkk})\\ (\mathbf{x}_{lkk},\mathbf{x}_{kkl}) & (\mathbf{x}_{lkk},\mathbf{x}_{klk}) & (\mathbf{x}_{lkk},\mathbf{x}_{lkk})\end{pmatrix}$$

$$=\begin{pmatrix}1+q & q+q^2 & q^2+q^3\\ q+q^2 & 1+q^3 & q+q^2\\ q^2+q^3 & q+q^2 & 1+q\end{pmatrix}.$$

In each such matrix, the rows and columns are indexed by the permutations $\mathbf{k}=\pi\mathbf{k}_0$ of $\mathbf{k}_0$ or equivalently by the left cosets $G/H$, where $G=\mathfrak{S}_n$ and $H$ is the subgroup of permutations of $\mathfrak{S}_n$ fixing $\mathbf{k}_0$. Write $\mathbf{k}=\sigma\mathbf{k}_0$, $\mathbf{l}=\tau\mathbf{k}_0$ with $\sigma,\tau\in\mathfrak{S}_n$; then (10) says that the $(\mathbf{l},\mathbf{k})$ matrix coefficient of $A_{\mathbf{k}_0}$ is equal to

$$\sum_{\substack{\pi\in\mathfrak{S}_n\\ \pi\sigma H=\tau H}} q^{I(\pi)}.$$

But a moment's thought shows that this is simply the $(\tau H,\sigma H)$-matrix coefficient (with respect to the basis of $\delta$-functions) of the element

$$\alpha_n=\sum_{\pi} q^{I(\pi)}\pi$$

on the subspace $V=\mathbb{C}^{G/H}$ of $\mathbb{C}^G$ consisting of functions $f\colon G\to\mathbb{C}$ which satisfy $f(gh)=f(g)$ for all $g\in G$, $h\in H$. This subspace is invariant under the action $R$ of $G$ on $\mathbb{C}^G$, so that $(V,R)$ is a representation of $G$. Hence if $\alpha_n$ is invertible in the group algebra $\mathbb{C}[G]$, then the matrix $A_{\mathrm{k}_0}$ is invertible. This completes the reduction of Theorem 1 to Theorem 2.

## 3. Factorization of $\alpha_n$; Proof of Theorem 2

We first introduce some notations. As in Sect. 1 we denote by $[i_1, i_2, \ldots, i_n]$ the permutation in $\mathfrak{S}_n$ which sends 1 to $i_1$, 2 to $i_2, \ldots, n$ to $i_n$. We identify $\mathfrak{S}_{n-1}$ with the subgroup of $\mathfrak{S}_n$ consisting of permutations fixing $n$. For $1\leqq k\leqq n$ we denote by $T_{k,n}$ the element $[1,\ldots,k-1,n,k,k+1,\ldots,n-1]$ of $\mathfrak{S}_n$, i.e.

$$T_{k,n}(i)=\begin{cases} i & 1\leqq i<k,\\ n & i=k,\\ i-1 & k<i\leqq n,\end{cases}\qquad T_{k,n}^{-1}(i)=\begin{cases} i & 1\leqq i<k,\\ i+1 & k\leqq i<n,\\ k & i=n.\end{cases}$$

Any element $\pi\in\mathfrak{S}_n$ can be represented uniquely as $\sigma T_{k,n}$ with $\sigma\in\mathfrak{S}_{n-1}$ and $1\leqq k\leqq n$ (namely $k=\pi^{-1}(n)$, $\sigma=\pi T_{k,n}^{-1}$), and a short calculation shows that then $I(\pi)$ equals $I(\sigma)+n-k$. Hence

$$\alpha_n=\sum_{\pi\in\mathfrak{S}_n} q^{I(\pi)}\pi=\sum_{\substack{\sigma\in\mathfrak{S}_{n-1}\\ 1\leqq k\leqq n}} q^{I(\sigma T_{k,n})}\sigma T_{n,k}=\Big(\sum_{\sigma\in\mathfrak{S}_{n-1}} q^{I(\sigma)}\sigma\Big)\Big(\sum_{k=1}^{n} q^{n-k}\,T_{k,n}\Big).$$

In other words,

**Proposition 1.** *Define* $\beta_n=\beta_n(q)=\sum_{k=1}^{n} q^{n-k}\,T_{k,n}\in\mathbb{C}[\mathfrak{S}_n]$. *Then* $\alpha_n=\alpha_{n-1}\beta_n$.

Here $\alpha_{n-1}$ is considered as an element of $\mathbb{C}[\mathfrak{S}_n]$ via the inclusion $\mathfrak{S}_{n-1}\subset\mathfrak{S}_n$. In particular, the representation of $\alpha_{n-1}$ in $R_n$, the $n!$-dimensional regular representation of $\mathfrak{S}_n$, consists of $n$ copies of the representation of $\alpha_{n-1}$ in $R_{n-1}$. Thus in terms of the matrices $A_n$ we can rewrite Proposition 1 as $A_n=(A_{n-1}\otimes 1_n)\cdot B_n$, where $A_{n-1}\otimes 1_n$ denotes the $n!\times n!$ block matrix with $n$ copies of $A_{n-1}$ on the diagonal blocks and zeros elsewhere and $B_n=B_n(q)$ has the matrix coefficients

$$B_n(\pi,\sigma)=\begin{cases} q^{n-k} & \text{if } \pi\sigma^{-1}=T_{k,n} \text{ for some } 1\leqq k\leqq n,\\ 0 & \text{otherwise}.\end{cases}$$

In particular, $\det(A_n(q))=\det(A_{n-1}(q))^n\det(B_n(q))$, so by induction on $n$ we have reduced Theorem 2 to the simpler

**Theorem 2′.** $\det(B_n(q))=\prod_{i=1}^{n-1}(1-q^{k^2+k})^{\frac{n!}{k^2+k}}$.

We now make a second reduction by expressing $B_n$ in turn as a product of yet simpler matrices.

**Proposition 2.** *For each $n$ define elements $\gamma_n$, $\delta_n$ in the group algebra $\mathbb{C}[\mathfrak{S}_n]$ by*

$$\gamma_n=(1-q^{n-1}T_{1,n})\,(1-q^{n-2}T_{2,n})\cdots(1-q\,T_{n-1,n}),$$
$$\delta_n=(1-q^{n+1}T_{1,n})\,(1-q^{n}T_{2,n})\cdots(1-q^{2}T_{n,n}).$$

*Then* $\beta_n \gamma_n = \delta_{n-1}$.

*Proof.* Let $\beta_{r,n} = \sum_{k=r}^{n} q^{n-k} T_{k,n}$, so that $\beta_{1,n} = \beta_n$, $\beta_{n,n} = 1$ (note that $T_{n,n} = 1 \in \mathfrak{S}_n$). Using the easily checked commutation relation

$$T_{k,n} T_{r,n} = T_{r,n-1} T_{k+1,n} \quad (r \leqq k \leqq n-1),$$

we find

$$\begin{aligned}\beta_{r,n} \cdot (1 - q^{n-r} T_{r,n}) &= \sum_{k=r+1}^{n} q^{n-k} T_{k,n} + q^{n-r} T_{r,n} \\ &\quad - q^{n-r} T_{r,n} - \sum_{k=r}^{n-1} q^{2n-k-r} T_{k,n} T_{r,n} \\ &= \sum_{k=r+1}^{n} q^{n-k} T_{k,n} - \sum_{k=r+1}^{n} q^{2n-k+1-r} T_{r,n-1} T_{k,n} \\ &= (1 - q^{n-r+1} T_{r,n-1}) \cdot \beta_{r+1,n}\end{aligned}$$

and hence by induction on $r$ (starting with the trivial case $r = 0$)

$$\begin{aligned}&\beta_{1,n}(1 - q^{n-1} T_{1,n}) \cdots (1 - q^{n-r} T_{r,n}) \\ &\quad = (1 - q^n T_{1,n-1}) \cdots (1 - q^{n+1-r} T_{r,n-1}) \beta_{r+1,n}.\end{aligned}$$

The case $r = n - 1$ of this identity is the desired identity. □

To complete the proof of Theorem 2 we need to compute the determinants of the factors in $\gamma_n$ and $\delta_{n-1}$ under the regular representation $R_n$ of $\mathfrak{S}_n$. We use the inclusions $\mathfrak{S}_b \subset \mathfrak{S}_{b+1} \subset \cdots \subset \mathfrak{S}_n$ to define elements $T_{a,b} \in \mathfrak{S}_n$ for all $1 \leqq a \leqq b \leqq n$ (we actually need only the cases $b = n - 1$ and $n$). Its characteristic polynomial is given by:

**Lemma.** *For* $1 \leqq a \leqq b \leqq n$ *the determinant of* $R_n(1 - tT_{a,b})$ *is* $(1 - t^{b-a+1})^{\frac{n!}{b-a+1}}$.

*Proof.* The element $T_{a,b} \in \mathfrak{S}_n$ is a cyclic permutation of the indices $a, a+1, \ldots, b$ and hence has order $b - a + 1$. But if $G$ is an arbitrary finite group of order $m$ and $g \in G$ an element of order $d$, then the characteristic polynomial $\det(1 - tR(g))$ of $g$ under the regular representation is $(1 - t^d)^{m/d}$, because the cycle structure of the permutation of $G$ given by left multiplication by $g^{-1}$ consists of $m/d$ disjoint cycles of length $d$. The lemma follows. □

The proof of Theorem 2 is now immediate: we have

$$\det(R_n(\gamma_n)) = \prod_{k=1}^{n-1} \det(R_n(1 - q^k T_{n-k,n})) = \prod_{k=1}^{n-1} (1 - q^{k(k+1)})^{\frac{n!}{k+1}},$$

$$\det(R_n(\delta_n)) = \prod_{k=1}^{n} \det(R_n(1 - q^{k+1} T_{n-k+1,n})) = \prod_{k=1}^{n} (1 - q^{k(k+1)})^{\frac{n!}{k}},$$

and hence

$$\det(B_n) = \det(R_n(\beta_n)) = \frac{\det(R_n(\delta_{n-1}))}{\det(R_n(\gamma_n))} = \frac{\det(R_{n-1}(\delta_{n-1}))^n}{\det(R_n(\gamma_n))}$$

$$= \prod_{k=1}^{n-1} (1-q^{k(k+1)})^{\frac{n\cdot(n-1)!}{k} - \frac{n!}{k+1}}$$

$$= \prod_{k=1}^{n-1} (1-q^{k(k+1)})^{\frac{n!}{k(k+1)}},$$

which is Theorem 2′; Theorem 2 then follows by induction from this and Proposition 1.

## 4. Formula for $A_n(q)^{-1}$

According to Propositions 1 and 2 we have

$$\beta_n = \delta_{n-1}\gamma_n^{-1},$$

$$\alpha_n = \beta_2 \cdots \beta_n = \delta_1 \gamma_2^{-1} \delta_2 \gamma_3^{-1} \cdots \gamma_{n-1}^{-1} \delta_{n-1} \gamma_n^{-1}$$

and hence

$$\alpha_n^{-1} = \gamma_n \delta_{n-1}^{-1} \gamma_{n-1} \cdots \gamma_2 \delta_1^{-1}.$$

To invert $\alpha_n$, therefore, the first step is to invert $\delta_k$ for each $k$.

**Proposition 3.** *For $\pi \in \mathfrak{S}_n$ define $W(\pi) \in \mathbb{Z}$ by*

$$W(\pi) = \sum_{\substack{1 \leqq i < j \leqq n \\ \pi(i) > \pi(j)}} (1 + (n+1-i)(n+1-j)\delta_{j-1,i})$$

$$= I(\pi) + \sum_{\substack{1 \leqq i \leqq n-1 \\ \pi(i+1) < \pi(i)}} (n+1-i)(n-i)$$

*and set* $\varepsilon_n = \sum_{\pi \in \mathfrak{S}_n} q^{W(\pi)} \pi^{-1} \in \mathfrak{S}_n$. *Then* $\delta_n^{-1} = \frac{1}{\Delta_{n+1}} \varepsilon_n$ *with $\Delta_{n+1}$ as in Eq.* (6).

*Proof.* Denote by $\sigma \mapsto \tilde{\sigma}$ the map $\mathfrak{S}_{n-1} \to \mathfrak{S}_n$ defined by $\tilde{\sigma}(1) = 1$, $\tilde{\sigma}(i) = \sigma(i-1) + 1$ for $i > 1$ (this is a homomorphism since $\tilde{\sigma}$ is just $T_{1,n}^{-1}\sigma T_{1,n}$). Then $\tilde{T}_{a,b} = T_{a+1,b+1}$ for $1 \leqq a < b \leqq n-1$, so $\delta_n = (1 - q^{n+1} T_{1,n}) \tilde{\delta}_{n-1}$. Hence by induction it suffices to show that $\varepsilon_n (1 - q^{n+1} T_{1,n}) = (1 - q^{n^2+n}) \tilde{\varepsilon}_{n-1}$.

For $\pi \in \mathfrak{S}_n$, let $k = \pi^{-1}(1)$ and denote by $\pi'$ the element $T_{1,n}\pi$ of $\mathfrak{S}_n$. Since $\pi'(k) = n$ but $\pi'(i) = \pi(i) - 1$ for all $i \neq k$, all the terms in the definition of $W(\pi)$ and of $W(\pi')$ are the same except those with $i$ or $j$ equal to $k$, so

$$\begin{aligned} W(\pi') - W(\pi) &= \sum_{k<j\leqq n} (1+(n+1-k)(n+1-j)\delta_{j,k+1}) \\ &\quad - \sum_{1\leqq i<k} (1+(n+1-i)(n+1-k)\delta_{i,k-1}) \\ &= n-k+(n+1-k)(n-k)-(k-1) \\ &\quad - \begin{cases} (n+1-k)(n+2-k) & \text{if } k>1 \\ 0 & \text{if } k=1 \end{cases} \\ &= \begin{cases} -n-1 & \text{if } k \neq 1, \\ n^2-1 & \text{if } k=1. \end{cases} \end{aligned}$$

Hence

$$\begin{aligned} \varepsilon_n(1-q^{n+1}T_{1,n}) &= \sum_{\pi\in\mathfrak{S}_n} (q^{W(\pi)} - q^{W(T_{1,n}\pi)+n+1})\pi^{-1} \\ &= \sum_{\substack{\pi\in\mathfrak{S}_n \\ \pi(1)=1}} (q^{W(\pi)} - q^{W(\pi)+n^2+n})\pi^{-1} \\ &= (1-q^{n^2+n}) \sum_{\sigma\in\mathfrak{S}_{n-1}} q^{W(\sigma)}\tilde{\sigma}^{-1} \end{aligned}$$

as desired. □

We next give a formula expressing $\gamma_n$ as a sum rather than a product.

**Proposition 4.** *The element $\gamma_n \in \mathfrak{S}_n$ defined in Proposition 2 is given by*

$$\gamma_n = \sum_{k=1}^{n} \gamma_{n,k}, \qquad \gamma_{n,k} = (-1)^{n-k} \sum_{\pi\in\mathfrak{S}_{n,k}} q^{I(\pi)}\pi^{-1},$$

*where $\mathfrak{S}_{n,k}$ is the subset of $\mathfrak{S}_n$ of cardinality $\binom{n-1}{k-1}$ consisting of those permutations $\pi$ for which $\pi(1) < \cdots < \pi(k) > \cdots > \pi(n)$.*

*Proof.* Multiplying out the terms in the product defining $\gamma_n$, we find

$$\gamma_n = \sum_{s=0}^{n-1} (-1)^s \sum_{1\leqq i_1<\ldots<i_s\leqq n-1} q^{(n-i_1)+\cdots+(n-i_s)} T_{i_1,n}T_{i_2,n}\cdots T_{i_s,n}.$$

The element $\sigma = T_{i_1,n}T_{i_2,n}\cdots T_{i_s,n}$ of $\mathfrak{S}_n$ maps $i_1$ to $n$, $i_2$ to $n-1, \ldots,$ and $i_s$ to $n-s+1$, and maps the rest of $\{1,2,\ldots,n\}$ monotone increasingly to $\{1,2,\ldots,n-s\}$. Moreover, it is easy to check that $(n-i_1)+\cdots+(n-i_s)$ equals $I(\sigma)$. The proposition now follows on setting $\pi = \sigma^{-1}$ and $k = n-s$. □

The explicit formulas for $\delta_n^{-1}$ and $\gamma_n$ just given together with the formula $\alpha_n^{-1} = \gamma_n\delta_{n-1}^{-1}\alpha_{n-1}^{-1}$ give an inductive method to calculate $\alpha_n$ for each $n$. To describe this a little more explicitly, we define another element of $\mathbb{C}[\mathfrak{S}_n]$ by

$$\zeta_n = \varepsilon_n\alpha_n^{-1}$$

with $\varepsilon_n$ as in Proposition 3. We conjecture that $\zeta_n$ has coefficients which are polynomials in $q$. Propositions 1–3 give $\alpha_n^{-1} = \Delta_n^{-1}\gamma_n\zeta_{n-1}$, so this conjecture implies the conjecture in (6). In fact, the two propositions are equivalent. Indeed, for each $\pi\in\mathfrak{S}_{n,k}$ we have $\pi(k) = n$ and hence $\pi = \sigma T_{n,k}$ for some $\sigma\in\mathfrak{S}_{n-1}$, so $\gamma_{n,k}$

equals $T_{n,k}^{-1}\gamma_{n-1,k}^{*}$ with $\gamma_{n-1,k}^{*}\in\mathbb{C}[\mathfrak{S}_{n-1}]$ (in fact $\gamma_{n-1,k}^{*}=\gamma_{n-1,k-1}-\gamma_{n-1,k}$). It follows that if $\pi$ is any element of $\mathfrak{S}_n$, and $\pi=\sigma T_{n,k}(1\leqq k\leqq n,\ \sigma\in\mathfrak{S}_{n-1})$ its canonical decomposition as at the beginning of Sect. 3, then the coefficient of $\pi$ in $\Delta_n\alpha_n^{-1}$ equals the coefficient of $\sigma^{-1}$ in $\gamma_{n-1,k}^{*}\zeta_{n-1}$. In particular, taking $k=n$ we find that the first $(n-1)!$ coefficients in $\Delta_n\alpha_n^{-1}$ are exactly the coefficients of $\zeta_{n-1}$, so that the integrality of $\Delta_n\alpha_n$ implies that of $\zeta_{n-1}$ for each $n$.

We illustrate with numerical examples for $n\leqq 4$. For $n=2$ we have

$$\alpha_2=1+qT_{1,2},\qquad \alpha_2^{-1}=\frac{1}{1-q^2}(1-qT_{1,2}),\qquad \varepsilon_2=1+q^3T_{1,2},$$
$$\zeta_2=\varepsilon_2\alpha_2^{-1}=(1+q^2)-qT_{1,2}.$$

We see that $\zeta_2$ is integral and that its coefficients are the first two coefficients of $\Delta_3\alpha_3^{-1}$, i.e., the first two coefficients of the matrix in (8). The other coefficients of $\alpha_3^{-1}$ are obtained by multiplying $\zeta_2$ by the elements $\gamma_{2,k}^{*}$ for $k=2$ and $k=3$, and we find

$$\begin{aligned}(1-q^2)(1-q^6)\alpha_3^{-1}&=(1-T_{2,3}^{-1}(qT_{1,3}+q^2T_{2,3})+T_{1,3}^{-1}(q^3T_{1,3}T_{2,3}))\zeta_2\\&=(1+q^2)[123]-q[213]-q[132]-q^4[231]\\&\quad-q^4[312]+(q^3+q^5)[321],\end{aligned}$$

giving the remaining coefficients in the first row of the matrix in (8) (the other rows are permutations of the first one). Write this as $\Delta_3\alpha_3^{-1}=\{1+q^2,\,-q,\,-q,\,-q^4,\,-q^4,\,q^3+q^5\}$ in the obvious notation. Using this value of $\alpha_3^{-1}$ and the value $\varepsilon_3=\{1,\ q^7,\ q^3,\ q^4,\ q^8,\ q^{11}\}$ we find $\zeta_3=\{1+2q^2+q^4+2q^6+q^8,\,-q-q^3-q^5-q^7,\,-q-q^7,\,-q^4,\,-q^4,\,q^3+q^5\}$, which is integral as claimed. Now multiplying this by the various components $\gamma_{4,k}(1\leqq k\leqq 4)$, we find

$$\begin{aligned}\alpha_4^{-1}&=(1-q^2)^{-1}(1-q^6)^{-1}(1-q^{12})^{-1}\\&\quad\times\{1+2q^2+q^4+2q^6+q^8,\,-q-q^3-q^5-q^7,\,-q-q^7,\,-q^4,\,-q^4,\,q^3+q^5;\\&\qquad-q-q^3-q^5-q^7,\,q^2+q^4+q^6,\,-q^4,\,-q^9-q^{11},\,0,\,q^{10};\\&\qquad-q^4,\,0,\,q^3+q^5,\,q^{10},\,-q^8-q^{10}-q^{12},\,q^7+q^9+q^{11}+q^{13};\\&\qquad-q^9-q^{11},\,q^{10},\,q^{10},\,q^7+q^{13},\,q^7+q^9+q^{11}+q^{13},\\&\qquad\quad-q^6-2q^8-q^{10}-2q^{12}-q^{14}\},\end{aligned}$$

where the 24 components have been listed in the obvious order (namely, the elements $\sigma\in\mathfrak{S}_3$ in the order above, followed by the elements $T_{3,4}\sigma$ with the same $\sigma$, then the $T_{2,4}\sigma$, then $T_{1,4}\sigma$). This gives the 24 elements of the first row of the matrix $A_4(q)^{-1}$, the other rows of course being permutations of this one. We have also checked the $\zeta_4$ has integral coefficients and thus that (6) holds for $n=5$.

## 5. Number Operators

For each index $k$, the $k^{\text{th}}$ *number operator* is the operator on $\mathbf{H}$ having each vector $\mathbf{x}_k=a^\dagger(k_1)\cdots a^\dagger(k_n)|0\rangle$ as an eigenvector with eigenvalue equal to the number of $i$ with $k_i=k$, so that the eigenspace of $N(k)$ with eigenvalue $r$ is the space spanned by the states containing exactly $r$ particles of type $k$. It is easy to see that this definition is equivalent to the requirements

$$N^\dagger(k) = N(k), \quad N(k)|0\rangle = 0, \quad [N(k), a^\dagger(l)] = \delta_{kl} a^\dagger(l) \quad \text{for all } l \tag{11}$$

(and hence $[N(k), a(l)] = -\delta_{kl} a(l)$ for all $l$).

Consider first the case in which there is only one operator $a(1)$ and its adjoint, i.e., only one kind of particle. In this case $\mathbf{H}(q)$ can be realized explicitly as the space spanned by vectors $e_0 = |0\rangle, e_1, e_2, \ldots$ with

$$a(1)e_n = \sqrt{n\frac{1-q^n}{1-q}}\, e_{n-1} \quad (=0 \text{ for } n=0), \qquad a^\dagger(1)e_n = \sqrt{\frac{1}{n+1}\frac{1-q^{n+1}}{1-q}}\, e_{n+1},$$

since then

$$a(1)\,a^\dagger(1)\,e_n - q\,a^\dagger(1)\,a(1)\,e_n = \frac{1-q^{n+1}}{1-q}\,e_n - q\,\frac{1-q^n}{1-q}\,e_n = e_n,$$

while the number operator $N(1)$ is given by either of the two formulas [5]

$$N(1) = \sum_{n=1}^{\infty} \frac{(1-q)^n}{1-q^n}\, a^\dagger(1)^n\, a(1)^n = \sum_{n=1}^{\infty} \frac{(1-q)^n}{\log(1/q^n)}\, (a^\dagger(1)\,a(1))^n, \tag{12}$$

as one sees either by computing the action of the expressions on the right on the vectors $e_n$ or else by verifying the relations (11) using (1) and (2). The first formula makes sense for all $q$ between $-1$ and 1, the second (which can be rewritten $\frac{1-q^{N(1)}}{1-q} = a^\dagger(1)\,a(1)$) only for $0 < q < 1$. Both reduce to $N(1) = a^\dagger(1)\,a(1)$ in the limit as $q$ tends to 1. For $q = 0$ the first formula reduces to

$$N(1) = \sum_{n=1}^{\infty} a^\dagger(1)^n\, a(1)^n \quad (q=0), \tag{13}$$

which makes sense since only finitely many of the terms act non-trivially on any given state.

In [1], Greenberg showed that the generalization of (13) to the case when there are many indices $k$ is

$$N(k) = \sum_{n=1}^{\infty} \sum_{k_2,\ldots,k_n} a^\dagger(k_n) \cdots a^\dagger(k_2)\, a^\dagger(k)\, a(k)\, a(k_2) \cdots a(k_n) \quad (q=0).$$

We now give a conjectural generalization of this formula to the case of arbitrary $q$ between $-1$ and 1. It is convenient to express the formula for all $N(k)$ simultaneously by giving a formula for the energy operator $\mathscr{E} = \sum_k E_k N(k)$, where the $E_k$ (interpreted as the energy of particle $k$) are scalar coefficients.

**Conjecture.** *The operator $\mathscr{E}$ is given by*

$$\mathscr{E} = \sum_{n=1}^{\infty} \sum_{k_1,\ldots,k_n} \sum_{\pi\in\mathfrak{S}_n} \sum_{i=1}^{n} c_i(q,\pi)\, E_{k_i}\, a^\dagger(k_{\pi(n)}) \cdots a^\dagger(k_{\pi(1)})\, a(k_1) \cdots a(k_n),$$

*where the coefficients $c_i(q,\pi)$ are given by*

$$\sum_{\substack{\pi\in\mathfrak{S}_n\\ 1\le i\le n}} c_i(q,\pi)\,X^{i-1}\pi = \alpha_n^{-1}(1-qXT_{1,2})(1-q^2XT_{1,3})\cdots(1-q^{n-1}XT_{1,n})\in\mathbb{C}[X][\mathfrak{S}_n].$$

This formula gives the correct result up to terms annihilating all 1-, 2-, and 3-particle states, viz:

$$\begin{aligned}
\mathscr{E} &= \sum_k E_k\,a^\dagger(k)\,a(k)\\
&+\frac{1}{1-q^2}\sum_{k,l}\left\{(E_k+q^2E_l)\,a^\dagger(l)\,a^\dagger(k)-q(E_k+E_l)\,a^\dagger(k)\,a^\dagger(l)\right\}a(k)\,a(l)\\
&+\frac{1}{(1-q^2)(1-q^6)}\times\\
&\sum_{k,l,m}\Big\{((1+q^2)E_k+(q^2+q^6)E_l+(q^6+q^8)E_m)\,a^\dagger(m)\,a^\dagger(l)\,a^\dagger(k)\\
&\qquad -q(E_k+E_l+q^6E_m)\,a^\dagger(m)\,a^\dagger(k)\,a^\dagger(l)\\
&\qquad -q(E_k+q^6(E_l+E_m))\,a^\dagger(l)\,a^\dagger(m)\,a^\dagger(k)\\
&\qquad -q^4(E_k+E_l+E_m)\,a^\dagger(k)\,a^\dagger(m)\,a^\dagger(l)\\
&\qquad -q^4(E_k+E_l+E_m)\,a^\dagger(l)\,a^\dagger(k)\,a^\dagger(m)\\
&\qquad +q^3(1+q^2)\,(E_k+E_l+E_m)\,a^\dagger(k)\,a^\dagger(l)\,a^\dagger(m)\Big\}\,a(k)\,a(l)\,a(m)\\
&+\dots
\end{aligned}$$

**Note added in proof.** The conjecture stated in this section has now been proved by Sonia Stanciu (see paper following this one).

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