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Heegner Points and Derivatives of L-Series. II

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Dedicated to Friedrich Hirzebruch

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Introduction

Let $X_0(N)$ be the usual modular curve with complex points $\bar{\mathfrak{H}}/\Gamma_0(N)$ and K an imaginary quadratic field of discriminant D in which the prime factors of N are all split. The theory of complex multiplication produces certain points $x \in X_0(N)$, called Heegner points, which are rational over the Hilbert class field H of K. In our first paper [9] we computed $\langle (x)-(\infty), (x')-(\infty)\rangle_H$ for two such Heegner points x and x' which are conjugate over H, where $\langle\ ,\ \rangle$ denotes the global height pairing on the Jacobian J of $X_0(N)$, in terms of the derivatives at $s=1$ of certain L-series associated to modular forms of weight 2 on $\Gamma_0(N)$. As a consequence one obtains a formula for the global height pairing $\langle y_D, y_D\rangle_K$, where y_D is the Heegner divisor $\mathrm{Tr}_{H/K}((x)-(\infty)) \in J(K)$. The result was that the height of the f-isotypical component of y_D, where f is a normalized newform of weight 2 on $\Gamma_0(N)$, is up to a simple factor equal to the first derivative at $s=1$ of the L-series of f over K. This L-series is the product of $L(f,s)=\sum a(n)n^{-s}$ and $L(f,D,s)=\sum a(n)\left(\frac{D}{n}\right)n^{-s}$, where $f(z) =\sum a(n)e^{2\pi i n z}$, and always vanishes at $s=1$ if a Heegner point of discriminant D exists.

We would like to have information about the position of the Heegner divisor y_D as a vector in the Mordell-Weil group, rather than just its length or the length of its components in the various Hecke eigenspaces. To do this, we will put all of the Heegner divisors in the same group and then compute their height pairings with one another. Let J^* denote the Jacobian of $X_0^*(N)$, the quotient of $X_0(N)$ by the Fricke involution w_N. The action of the non-trivial element of $\mathrm{Gal}(H/K)$ on $\mathrm{Tr}_{H/K}((x))$ is the same as that of w_N, so the image y_D^* of y_D in J^* is defined over $\mathbb{Q}$. Its f-component is non-trivial only if f is a modular form on $\Gamma_0^*(N)$, and this is the case precisely when $L(f,s)$ has a minus sign in its functional equation and hence a zero (of odd order) at $s=1$. The result quoted above then says that the height (now over $\mathbb{Q}$) of the f-component of y_D^* is equal, up to a simple factor, to $L'(f,1)L(f,D,1)$. On the other hand, an important result of Waldspurger expresses $L(f,D,1)$ as a multiple of $c(D)^2$, where $c(D)$ is the $|D|$-th Fourier coefficient of a modular form of weight 3/2 corresponding to f under the Shimura lifting. This leads one to guess that the height pairing of the f-components of $y_{D_0}^*$ and $y_{D_1}^*$ for different discriminants D_0 and D_1 should be related to the product $L'(f,1)c(D_0)c(D_1)$.

In this paper we will establish a result of this nature. Actually, the theory of forms of half-integral weight is adequate to express the result neatly only when N is prime; in general we must use instead the theory of Jacobi forms as developed in [4, 15]. Combining this result with multiplicity one theorems guaranteeing the uniqueness of the lifting to Jacobi forms we will show that *the f-eigencomponents of the Heegner divisors y_D^* all lie on a single line in $(J^*(\mathbb{Q})\otimes\mathbb{R})_f$ and that their positions on that line are given by the Fourier coefficients of a Jacobi form.* The subspace they generate is non-trivial precisely when $L'(f,1)\neq 0$. We will also prove a formula relating the height pairings of the Heegner divisors to integrals of modular forms over certain geodesic cycles on $X_0(N)$ associated to real quadratic fields. Note that the statement about 1-dimensionality is in accordance with the Birch-Swinnerton-Dyer conjecture, which predicts that $(J^*(\mathbb{Q})\otimes\mathbb{R})_f$ has dimension 1 when $L'(f,1)\neq 0$.

We would like to emphasize the strong analogy of this theorem with some previous work of Hirzebruch and Zagier [10], in which the intersection numbers of certain modular curves on a Hilbert modular surface Y were computed and related to the coefficients of a modular form of weight 2. The intersection number, like our height pairing, is expressed as a sum of local terms (which are calculated, in both cases, by counting representations by quadratic forms). The modular form of weight 2 then determines the positions of the curves in the homology group $H_2(Y)$ and hence in $\mathrm{Pic}(Y)$ since Y is simply connected. This fact actually permits one to deduce our theorem from the results of [10] in some special cases, e.g., in the first non-trivial case $N=37$ [23]. It is therefore a great pleasure to dedicate this paper to Hirzebruch, who has taught all three of us so much.

In the remainder of this introduction we will give the precise definitions of the Heegner divisors, Jacobi forms, and integrals over geodesic cycles, and state our main results.

1. Heegner Divisors

Let K be an imaginary quadratic field of discriminant D and class number h, and assume that D is a square modulo $4N$ (or equivalently, that every prime divisor p of N is split or ramified in K, and split if $p^2|N$). Fix a residue class $r \pmod{2N}$ with $r^2 \equiv D \pmod{4N}$. If $\tau \in \mathfrak{H}$ (upper half-plane) is the root of a quadratic equation

$$\begin{gathered} a\tau^2+b\tau+c=0, \quad a,b,c\in\mathbb{Z}, \quad a>0, \quad a\equiv 0 \pmod{N}, \\ b\equiv r \pmod{2N}, \quad b^2-4ac=D \end{gathered} \tag{1}$$

then we know by the theory of complex multiplication that the image of τ in $\mathfrak{H}/\Gamma_0(N)\subset X_0(N)(\mathbb{C})$ is defined over H, the Hilbert class field of K. There are exactly h such images, permuted simply transitively by $\mathrm{Gal}(H/K)$; their sum is thus a divisor $P_{D,r}$ of degree h defined over K. (Actually, if $D=-3$ or -4 we define $P_{D,r}$ as $1/3$ or $1/2$ of this divisor to correct for the presence of extra units.) From a modular point of view, points of $X_0(N)$ correspond to diagrams $E\xrightarrow{\phi}E'$ where E and E' are elliptic curves and ϕ a cyclic N-isogeny, and the points of $P_{D,r}$ correspond to diagrams where E and E' both have complex multiplication by the ring of integers of K and the kernel of ϕ is annihilated by the primitive ideal $\mathfrak{n}=\left(N,\frac{r+\sqrt{D}}{2}\right)$ of norm N. We write $y_{D,r}$ for the divisor $P_{D,r}-h\cdot(\infty)$ of degree 0 on $X_0(N)$ and for its class in the Jacobian J, and $P^*_{D,r}$, $y^*_{D,r}$ for the images of $P_{D,r}$ and $y_{D,r}$ in $X_0^*(N)$ and J^*, respectively; as stated above, the latter are defined over $\mathbb{Q}$. Our goal is a formula for the height pairing $\langle y^*_{D_0,r_0}, y^*_{D_1,r_1}\rangle_{\mathbb{Q}}$ of two such divisors. In the case $D_0=D_1=D$, $r_0=r_1=r$, the value of r is irrelevant for this question, since the group $W\cong(\mathbb{Z}/2\mathbb{Z})^t$ ($t=$ number of prime factors of N) of Atkin-Lehner involutions of $X_0(N)$ permutes the $P_{D,r}$ or $y_{D,r}$ for a given D transitively and since the height pairing on a Jacobian is invariant under automorphisms of the underlying curve. This is why the role of the square-root r of $D \pmod{4N}$ was not stressed in [9]; it becomes important now because for different discriminants there is no canonical compatible choice of square-roots.

2. Jacobi Forms

A Jacobi form of weight k and index N is a function $\phi:\mathfrak{H}\times\mathbb{C}\to\mathbb{C}$ satisfying the transformation law

$$\phi\left(\frac{a\tau+b}{c\tau+d},\frac{z}{c\tau+d}\right)=(c\tau+d)^k e^{2\pi iN\frac{cz^2}{c\tau+d}}\phi(\tau,z) \qquad \forall\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in SL_2(\mathbb{Z})$$

and having a Fourier expansion of the form

$$\phi(\tau, z) = \sum_{\substack{n, r \in \mathbb{Z} \\ r^2 \leqq 4Nn}} c(n, r) q^n \zeta^r \qquad (q = e^{2\pi i\tau}, \zeta = e^{2\pi i z}), \tag{2}$$

where $c(n, r)$ depends only on $r^2 - 4Nn$ and on the residue class of $r \pmod{2N}$. Such functions arise from theta series [$c(n, r)$ is the number of vectors in a $2k$-dimensional lattice having length n and scalar product r with a fixed vector of length N] and Siegel modular forms $\left[\phi \text{ is the coefficient of } e^{2\pi i N\tau'} \text{ in the Fourier expansion of a Siegel modular form } F\begin{pmatrix} \tau & z \\ z & \tau' \end{pmatrix} \text{ of weight } k \text{ on } Sp_4(\mathbb{Z})\right]$. One can define Jacobi cusp forms [require $r^2 < 4Nn$ in (2)], Eisenstein series, a Petersson scalar product, Hecke operators, and new forms [4]. Using a trace formula, it is possible to show [15] that the new part of the space $J_{k+1,N}^{\text{cusp}}$ of Jacobi cusp forms of weight $k+1$ and index N is isomorphic as a Hecke module to the new part of the space $S_{2k}(N)^-$ of cusp forms of weight $2k$ on $\Gamma_0(N)$ with eigenvalue -1 under the involution $f(z) \mapsto (-Nz^2)^{-k} f(-1/Nz)$ (these are the cusp forms whose Hecke L-series have a functional equation with a minus sign under $s \mapsto 2k - s$). In particular, if $f \in S_{2k}(N)^-$ is a normalized newform, then there is a non-zero Jacobi form $\phi = \phi_f \in J_{k+1,N}^{\text{cusp}}$, unique up to a scalar, having the same eigenvalues as f under all Hecke operators T_m, $(m, N) = 1$. We can choose ϕ_f to have real Fourier coefficients (in fact, coefficients in the totally real number field generated by the coefficients of f).

3. *Cycle Integrals*

Let Δ be the discriminant of a real quadratic field of narrow class number h and assume that Δ is a square modulo $4N$ (i.e., make the same assumptions on the splitting behavior of prime factors of N as above for Heegner divisors). Choose a residue class $\varrho \pmod{2N}$ with $\varrho^2 \equiv \Delta \pmod{4N}$. Then the points $z = x + iy \in \mathfrak{H}$ satisfying an equation of the form

$$a|z|^2 + bx + c = 0, \qquad a, b, c \in \mathbb{Z}, \qquad a \equiv 0 \pmod{N},$$
$$b \equiv \varrho \pmod{2N}, \qquad b^2 - 4ac = \Delta$$

[the real quadratic analogue of (1)] form an infinite union of semicircles whose image in $\mathfrak{H}/\Gamma_0(N) \subset X_0(N)(\mathbb{C})$ is a union of h closed geodesics, in 1:1 correspondence with the narrow ideal classes of $\mathbb{Q}(\sqrt{\Delta})$. Each such geodesic is the quotient γ_Q of one of the semicircles by a matrix $M = M_Q \in SL_2(\mathbb{Z})$ corresponding to a unit of the quadratic form $Q(\xi, \eta) = a\xi^2 + b\xi\eta + c\eta^2$. If $f \in S_{2k}(N)$, we can define the *cycle integral*

$$r_{k,N,Q}(f) = \int_{z_0}^{Mz_0} f(z) Q(z, 1)^{k-1} dz \quad (\text{any } z_0 \in \mathfrak{H}).$$

If Δ is a product of two negative discriminants D_0 and D_1, then there is a corresponding genus character χ from the narrow ideal class group of $\mathbb{Q}(\sqrt{\Delta})$ to $\{\pm 1\}$, and summing over the h classes of Q (for fixed Δ and ϱ) with weighting factors $\chi(Q)$ gives a cycle integral which we shall denote $r_{k,N,\Delta,\varrho,D_0}(f)$. For $k=1$ this is just the integral of the differential form $f(z)dz$ over the closed cycle $\gamma(D_0, D_1, \varrho) = \sum_{[Q]} \chi(Q)\gamma_Q$ on $X_0(N)$, which is invariant under w_N and anti-invariant under complex conjugation.

The Results

Because the space $J_{k+1,N}^{\text{cusp}}$ of Jacobi cusp forms is isomorphic as a Hecke module to a subspace of $S_{2k}(N)^-$, the "lifting map"

$$\mathscr{S}_{D_0,r_0}: \phi \to \sum_{m=1}^{\infty} (\text{coefficient of } q^{n_0}\zeta^{r_0} \text{ in } \phi|T_m)e^{2\pi i m\tau}$$

(where n_0, r_0 are integers with $D_0 = r_0^2 - 4Nn_0 < 0$) maps $J_{k+1,N}^{\text{cusp}}$ to $S_{2k}(N)^-$. After a preliminary chapter on quadratic forms and associated orders in quaternion algebras, we will construct in Chap. II the kernel function for $\mathscr{S}_{D_0,r_0}$. It turns out that the Fourier coefficients of $\mathscr{S}_{D_0,r_0}^*$, the adjoint of $\mathscr{S}_{D_0,r_0}$ with respect to the Petersson scalar products in $J_{k+1,N}^{\text{cusp}}$ and $S_{2k}(N)$, are given by the cycle integrals defined above. This leads to the first main result:

Theorem A. *Let $f \in S_{2k}(N)^-$ be a normalized newform, $\phi = \phi_f \in J_{k+1,N}^{\text{cusp}}$ a Jacobi form corresponding to f as above, and $D_i = r_i^2 - 4Nn_i < 0$ $(i=0,1)$ two coprime fundamental discriminants. Then*

$$\frac{1}{\|\phi\|^2} c(n_0, r_0)c(n_1, r_1) \doteq \frac{1}{\|f\|^2} r_{k,N,D_0D_1,r_0r_1,D_0}(f), \tag{3}$$

where $\|\phi\|$ and $\|f\|$ are the norms of ϕ and f in their respective scalar products and $c(n,r)$ denotes the coefficient of $q^n\zeta^r$ in ϕ.

(Here and in the next paragraph, $\doteq$ means equality up to an elementary non-zero factor which depends only on N and k.) Note that (3) makes sense since ϕ is unique up to a non-zero real constant and replacing ϕ by $\lambda\phi$ multiplies both $\|\phi\|^2$ and $c(n_0, r_0)c(n_1, r_1)$ by λ^2.

Next, in Chap. III we will construct a modular form $F \in S_{2k}(N)^-$, depending on the same data k, N, $D_0 < 0$, $D_1 < 0$, and $r_0 r_1 \pmod{2N}$ with $r_i^2 \equiv D_i \pmod{4N}$, by starting with a non-holomorphic Eisenstein series of weight 1 for the Hilbert modular group of $\mathbb{Q}(\sqrt{D_0 D_1})$ and applying to it a differential operator of H. Cohen and a holomorphic projection operator. We prove that the scalar product of F with a normalized newform $f \in S_{2k}(N)^-$ is given by

$$(F, f) \doteq r_{k,N,D_0D_1,r_0r_1,D_0}(f)\; L'(f, k). \tag{4}$$

We also calculate the Fourier coefficients of F. They turn out to be given as a sum of two terms, one of which is a finite integral linear combination of logarithms of prime numbers and the other an infinite sum of Legendre functions. This infinite sum is shown in Chap. IV to be a finite linear combination of values of a certain Green's function at Heegner points. More precisely, for $k > 1$ odd we prove

$$\text{coefficient of } q^m \text{ in } F \quad \doteq (D_0 D_1)^{\frac{k-1}{2}} \sum_{\tau_0, \tau_1} G_{N,k}^*(\tau_0, \tau_1) + \sum_p n(p) \log p; \tag{5}$$

here τ_0 and τ_1 run over the points of the Heegner divisors $P^*_{D_0,r_0}$ and $T_m P^*_{D_1,r_1}$ ($T_m = m^{\text{th}}$ Hecke operator) and $G^*_{N,k}$ is the unique function on $X_0^*(N)^2$ which is an eigenfunction with eigenvalue $k(k-1)$ of the hyperbolic Laplace operator (in each variable) and is bounded except for a logarithmic singularity along the diagonal, while the second sum runs over primes and $n(p)$ is an explicitly given integer which is non-zero only if p divides one of the integers $\frac{D_0 D_1 - r^2}{4N}$, $r \equiv r_0 r_1 \pmod{2N}$, $|r| < \sqrt{D_0 D_1}$. For k even the result is similar but with $G^*_{N,k}$ replaced by a function on $X_0(N)^2$ which is odd with respect to the action of w_N in each variable. For $k=1$ we prove a similar formula for m prime to N, where now $G^*_{N,1}$ is harmonic on $X_0^*(N)^2$ and is bounded except for logarithmic singularities along the diagonal and the axes $X_0^*(N) \times \{\infty\}$, $\{\infty\} \times X_0^*(N)$.

We also show in Chap. IV that the right-hand side of (5) for $k=1$ equals $\langle y^*_{D_0,r_0}, T_m y^*_{D_1,r_1}\rangle$, where $\langle\ ,\ \rangle$ is the canonical height pairing on $J^*(\mathbb{Q})$. [Specifically, the terms $\sum G^*_{N,k}(\tau_0,\tau_1)$ and $n(p)\log p$ are the local height contributions from the places ∞ and p, respectively; they are calculated by counting the number of embeddings of certain Clifford orders into $\begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ N\mathbb{Z} & \mathbb{Z} \end{pmatrix} \subset M_2(\mathbb{Q})$ or into an Eichler order of discriminant N in the quaternion algebra over $\mathbb{Q}$ ramified at p and at ∞.] This leads to the formula

$$\langle (y^*_{D_0,r_0})_f, (y^*_{D_1,r_1})_f \rangle = \frac{(f,F)}{(f,f)} \tag{6}$$

for the height pairings of the f-eigencomponents of $y^*_{D_0,r_0}$ and $y^*_{D_1,r_1}$, where $f \in S_2(N)^- = S_2(\Gamma_0^*(N))$ is a normalized newform and $\langle\ ,\ \rangle$ has been extended to $J^*(\mathbb{Q}) \otimes \mathbb{R}$ by linearity. From (4) and (6), we obtain (putting in the constants)

Theorem B. *Let $D_0, D_1 < 0$ be coprime fundamental discriminants, $D_i \equiv r_i^2 \pmod{4N}$, and $f \in S_2(\Gamma_0^*(N))$ a normalized newform. Then*

$$\langle (y^*_{D_0,r_0})_f, (y^*_{D_1,r_1})_f \rangle = \frac{L'(f,1)}{4\pi \|f\|^2} \int_{\gamma(D_0,D_1,r_0r_1)} f(z)\,dz\,.$$

Combining this with Theorem A gives the identity

$$\langle (y^*_{D_0,r_0})_f, (y^*_{D_1,r_1})_f \rangle = \frac{L'(f,1)}{4\pi \|\phi\|^2} c(n_0,r_0) c(n_1,r_1)\,. \tag{7}$$

On the other hand, the main result of [9] implies that

$$\langle (y^*_{D,r})_f, (y^*_{D,r})_f \rangle = \frac{|D|^{1/2}}{8\pi^2} \frac{L'(f,1)}{\|f\|^2} L(f,D,1) \quad \text{if} \quad (D,2N)=1\,,$$

and by an analogue of Waldspurger's theorem proved in Chap. II this is equivalent to

$$\langle (y^*_{D,r})_f, (y^*_{D,r})_f \rangle = \frac{L'(f,1)}{4\pi \|\phi\|^2} c(n,r)^2 \qquad (D = r^2 - 4Nn)\,. \tag{8}$$

Together, Eqs. (7) and (8) imply that $(y^*_{D_0,r_0})_f$ and $(y^*_{D_1,r_1})_f$ are collinear (Cauchy-Schwarz in the case of equality!), and this gives our main theorem:

Theorem C. *Let $f\in S_2(\Gamma_0^*(N))$ be a normalized newform. Then the subspace of $J^*(\mathbb{Q})\otimes\mathbb{R}$ generated by the f-eigencomponents of all Heegner divisors $(y^*_{D,r})_f$ with $(D,2N)=1$ has dimension 1 if $L'(f,1)\neq 0$ and 0 if $L'(f,1)=0$. More precisely, $(y^*_{D,r})_f = c\left(\frac{r^2-D}{4N}, r\right)y_f$, where $c(n,r)$ is the coefficient of $q^n\zeta^r$ in a Jacobi form $\phi_f\in J_{2,N}$ and $y_f\in(J^*(\mathbb{Q})\otimes\mathbb{R})_f$ is independent of D and r with $\langle y_f,y_f\rangle=L'(f,1)/4\pi\|\phi_f\|^2$.*

We also discuss in Chap. V the interpretation of (5) for $k>1$, the modifications that would be needed in the proof of Theorem C for $(D,2N)\neq 1$, and the relation of Theorems B and C to the conjecture of Birch and Swinnerton-Dyer. The ideal statement of Theorem C, analogous to the main theorem of [10], would be that the formal power series $\sum\limits_{\substack{n,r\in\mathbb{Z}\\ r^2-4Nn<0}} y^*_{r^2-4Nn,r}q^n\zeta^r$ lies in the tensor product $J^*(\mathbb{Q})\otimes J^{\mathrm{cusp}}_{2,N}$, with non-vanishing f-component iff $L'(f,1)\neq 0$.

For the reader's convenience we remark that Chaps. II, III, and IV are essentially independent (all three use parts of Chap. I) and can be read in any order. The results of Chap. II (in particular, Theorem A) are of independent interest in the theory of Jacobi forms.

Notations

The symbols $e^m(x)$ and $e_m(x)$ $(m\in\mathbb{N})$ denote $e^{2\pi imx}$ and $e^{2\pi ix/m}$, respectively. In $e^m(x)$, x is a complex variable, while in $e_m(x)$ it is taken to be in $\mathbb{Z}/m\mathbb{Z}$. We sometimes write $e(x)$ for $e^{2\pi ix}$ $(x\in\mathbb{C})$.

For a function $f:\mathbb{Z}\to\mathbb{C}$ we write $f^{\pm}$ for f symmetrized or antisymmetrized, respectively, i.e., $f^{\pm}(r)=f(r)\pm f(-r)$. By $d\|n$ $(n\in\mathbb{N})$ we mean $d|n$ and $\left(d,\frac{n}{d}\right)=1$. In a sum of the form $\sum\limits_{d|n}$ we understand that the summation is over positive divisors only. The abbreviation "mod" is frequenctly omitted; thus we often write $a\equiv b(n)$ instead of $a\equiv b \pmod n$. The symbols $\sum\limits_{a(c)}$ and $\sum\limits_{\varrho(c)^*}$ denote sums over representatives for all residue classes or all primitive residue classes modulo c, respectively.

By a *discriminant* we mean any non-zero integer $\Delta\equiv 0,1$ (4), by a *fundamental discriminant* either 1 or the discriminant of a quadratic field. Any discriminant Δ can be written uniquely as Δ_0c^2 with Δ_0 fundamental and $c\geq 1$; c is called the *conductor* of Δ. A *prime discriminant* is a fundamental discriminant with exactly one prime factor [i.e., -4, -8, 8, or $(-1)^{(p-1)/2}p$ with p an odd prime]. For Δ a discriminant, $\left(\frac{\Delta}{\cdot}\right)$ is the Kronecker symbol $\left[\text{the totally multiplicative function with } \left(\frac{\Delta}{-1}\right)=\operatorname{sign}(\Delta) \text{ and } \left(\frac{\Delta}{p}\right) \text{ for } p \text{ prime defined as 0 if } p|\Delta,\ +1 \text{ if } p\nmid\Delta \text{ and } \Delta\equiv\text{square } (4p),\ -1 \text{ otherwise}\right]$.

$\mathfrak{H}$ denotes the upper half-plane, $\Gamma(1)$ the full modular group $SL_2(\mathbb{Z})$ and $\Gamma_0(N)$ ($N\in\mathbb{N}$) the subgroup of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $N|c$. For $k>0$ we denote by $S_{2k}(N)$ the space of cusp forms of weight $2k$ on $\Gamma_0(N)$ and by $J_{k,N}$ (resp. $J_{k,N}^{\text{cusp}}$) the space of Jacobi forms (resp. Jacobi cusp forms) of weight k and index N on the full Jacobi modular group $\Gamma(1)^J=SL_2(\mathbb{Z})\ltimes\mathbb{Z}^2$. (For the theory of Jacobi forms we refer to [4].) The Petersson scalar products on these spaces are normalized by

$$(f,g)=\int\limits_{\Gamma_0(N)\backslash\mathfrak{H}} f(\tau)\overline{g(\tau)}v^{2k-2}du\,dv \quad (f,g\in S_{2k}(N)),$$

$$(\phi,\psi)=\int\limits_{\Gamma(1)^J\backslash\mathfrak{H}\times\mathbb{C}} \phi(\tau,z)\overline{\psi(\tau,z)}v^{k-3}e^{-4\pi Ny^2/v}dx\,dy\,du\,dv \quad (\phi,\psi\in J_{k,N}^{\text{cusp}}),$$

where $\tau=u+iv$, $z=x+iy$.

I. Quadratic Forms, Genus Theory, and Clifford Algebras

In this preliminary chapter we collect together some material about quadratic forms which will be used in later chapters. Specifically, Sect. 1 treats the classification under $\Gamma_0(N)$ of binary quadratic forms of the form $ax^2+bxy+cy^2$ with a divisible by N, Sect. 2 discusses the definition of genus characters on such forms, and Sect. 3 treats the Clifford algebras of binary quadratic forms (which are quaternion algebras) and their orders. The reader may want to skip this chapter now and refer to the results later as needed.

1. $\Gamma_0(N)$-Classification of Binary Quadratic Forms

We consider integral binary quadratic forms $[a,b,c](x,y)=ax^2+bxy+cy^2$. As usual the group $\Gamma(1)$ operates on such forms by

$$[a,b,c]\circ\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}(x,y)=[a,b,c](\alpha x+\beta y,\gamma x+\delta y)$$

preserving the discriminant $\Delta=b^2-4ac$ and the greatest common divisor (a,b,c), the number of classes with fixed values of these invariants being finite. We denote by $\mathcal{Q}_\Delta$ and $\mathcal{Q}_\Delta^0$ the set of all quadratic forms of discriminant Δ and the subset of primitive forms (greatest common divisor $=1$), respectively. We are interested in the classification with respect to the subgroup $\Gamma_0(N)$, $N\in\mathbb{N}$. A further invariant in this case is the greatest common divisor of a and N, which we suppose to be N; yet another invariant under this assumption is the value of b modulo $2N$. Thus for an integer $\varrho \bmod 2N$ and a discriminant Δ with $\Delta\equiv\varrho^2 \pmod{4N}$ we set

$$\mathcal{Q}_{N,\Delta,\varrho}=\{[a,b,c]\in\mathcal{Q}_\Delta\,|\,a\equiv 0 \pmod N,\ b\equiv\varrho \pmod{2N}\}.$$

This collection of forms is $\Gamma_0(N)$-invariant and we are interested in describing its orbits under $\Gamma_0(N)$ and (in the next two sections) defining certain $\Gamma_0(N)$-invariant functions (genus characters) on it.

From now on we denote forms in $\mathcal{Q}_{N,\Delta,\varrho}$ by $[aN,b,c]$ instead of $[a,b,c]$. The greatest common divisor of a, b, and c is $\Gamma_0(N)$-invariant, and setting

$$\mathcal{Q}_{N,\Delta,\varrho}^0=\{[aN,b,c]\in\mathcal{Q}_{N,\Delta,\varrho}\,|\,(a,b,c)=1\}$$

we have a $\Gamma_0(N)$-invariant bijection of sets

$$\mathcal{Q}_{N,\Delta,\varrho}=\bigcup_{\ell^2|\Delta}\ \bigcup_{\substack{\lambda(2N)\\ \lambda^2\equiv\Delta/\ell^2(4N)\\ \ell\lambda\equiv\varrho(2N)}}\ell\cdot\mathcal{Q}^0_{N,\Delta/\ell^2,\lambda}. \tag{1}$$

Thus we can reduce to the study of forms $Q\in\mathcal{Q}^0_{N,\Delta,\varrho}$, which we call *$\Gamma_0(N)$-primitive.* Set

$$m=\left(N,\varrho,\frac{\varrho^2-\Delta}{4N}\right); \tag{2}$$

this g.c.d. is well-defined even though ϱ is only an integer modulo $2N$ since replacing ϱ by $\varrho+2N$ replaces $\frac{\varrho^2-\Delta}{4N}$ by $\frac{\varrho^2-\Delta}{4N}+\varrho+N$. For $Q=[aN,b,c]\in\mathcal{Q}^0_{N,\Delta,\varrho}$ we have $(N,b,ac)=m$ and $(a,b,c)=1$, so the two numbers

$$(N,b,a)=m_1 \quad\text{and}\quad (N,b,c)=m_2 \tag{3}$$

are coprime and have product m. Conversely, we have:

Proposition. *Define m by* (2) *and fix a decomposition $m=m_1m_2$ with $m_1,m_2>0$, $(m_1,m_2)=1$. Then there is a 1:1 correspondence between the $\Gamma_0(N)$-equivalence classes of forms $[aN,b,c]\in\mathcal{Q}^0_{N,\Delta,\varrho}$ satisfying* (3) *and the $SL_2(\mathbb{Z})$-equivalence classes of forms in $\mathcal{Q}^0_\Delta$ given by*

$$Q=[aN,b,c]\ \mapsto\ \tilde{Q}=[aN_1,b,cN_2]; \tag{4}$$

here $N_1\cdot N_2$ is any decomposition of N into coprime positive factors satisfying $(m_1,N_2)=(m_2,N_1)=1$. In particular, $|\mathcal{Q}^0_{N,\Delta,\varrho}/\Gamma_0(N)|=2^\nu\cdot|\mathcal{Q}^0_\Delta/SL_2(\mathbb{Z})|$, where ν is the number of prime factors of m.

[Note: $|\mathcal{Q}^0_\Delta/SL_2(\mathbb{Z})|$ equals $h(\Delta)$ for $\Delta>0$, 2 for $\Delta=0$, and $2h(\Delta)$ for $\Delta<0$, where $h(\Delta)$ is the class number of Δ in the standard notation, the factor 2 arising because $\mathcal{Q}^0_\Delta$ for $\Delta<0$ ($\Delta\leq 0$) contains both positive and negative (semi-)definite forms while $h(\Delta)$ counts only the positive ones.]

Proof. This is essentially Lemma 2, p. 64, of [10], but since the proof there was only sketched and the statement somewhat more special (N/m was supposed square-free and prime to m) and not quite correct $\left[\text{the factor } 1+\left(\frac{\Delta}{q}\right) \text{ should be replaced by 0 if } q^2|\Delta\right]$, we give a complete proof here.

First of all, there clearly is a decomposition $N=N_1N_2$ with $(N_1,N_2)=(N_1,m_2)=(N_2,m_1)=1$: we write N as a product of prime powers p^r and include p^r into N_i if $p|m_i$ and into either N_1 or N_2 if $p\nmid m$. The form $\tilde{Q}$ defined by (4) is primitive because of $(a,b,c)=1$, Eq. (3), and the properties of N_1,N_2, so (4) defines a map $\mathcal{Q}^0_{N,\Delta,\varrho,m_1,m_2}\to\mathcal{Q}^0_\Delta$, where $\mathcal{Q}^0_{N,\Delta,\varrho,m_1,m_2}$ is the set of forms $[aN,b,c]\in\mathcal{Q}_0$ satisfying (3). This map induces a map $\mathcal{Q}^0_{N,\Delta,\varrho,m_1,m_2}/\Gamma_0(N)\to\mathcal{Q}^0_\Delta/SL_2(\mathbb{Z})$ because for $M=\begin{pmatrix}\alpha&\beta\\ N\gamma&\delta\end{pmatrix}\in\Gamma_0(N)$ and any Q we have by an easy calculation $\widetilde{Q\circ M}=\tilde{Q}\circ\tilde{M}$ with $\tilde{M}=\begin{pmatrix}\alpha&N_2\beta\\ N_1\gamma&\delta\end{pmatrix}\in SL_2(\mathbb{Z})$. We must show that this induced map is injective and surjective.

Injectivity. Suppose $Q=[aN,b,c]$, $Q'=[a'N,b',c']\in\mathcal{Q}^0_{N,\Delta,\varrho,m_1,m_2}$ with $\tilde{Q}'=\tilde{Q}\circ\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}$ for some $\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\in SL_2(\mathbb{Z})$. We show that $N_1|\gamma$, $N_2|\beta$; then $\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}=\tilde{M}$ for some $M\in\Gamma_0(N)$ and automatically $Q'=Q\circ M$. Written out in full, the relation $\tilde{Q}'=\tilde{Q}\circ\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}$ says

$$\begin{aligned}a'N_1&=aN_1\alpha^2+b\alpha\gamma+cN_2\gamma^2\,,\\ b'&=2aN_1\alpha\beta+b(\alpha\delta+\beta\gamma)+2cN_2\gamma\delta\,,\\ c'N_2&=aN_1\beta^2+b\beta\delta+cN_2\delta^2\,.\end{aligned}$$

Reducing the first equation $(\operatorname{mod} N_1)$ and the second $(\operatorname{mod} 2N_1)$ and noting that $b'\equiv\varrho\equiv b\ (\operatorname{mod} 2N_1)$ and $\alpha\delta-\beta\gamma=1$, we obtain

$$0\equiv\gamma(b\alpha+cN_2\gamma)\ (\operatorname{mod} N_1)\,,\qquad 0\equiv\gamma(b\beta+cN_2\delta)\ (\operatorname{mod} N_1)\,,$$

and these imply $\gamma\equiv 0\ (\operatorname{mod} N_1)$ because the g.c.d. of $b\alpha+cN_2\gamma$ and $b\beta+cN_2\delta$ equals that of b and cN_2 $\left[\text{since } \begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} \text{ has determinant } 1\right]$ and this is prime to N_1 by assumption. The proof that $\beta\equiv 0\ (\operatorname{mod} N_2)$ is exactly similar.

Surjectivity. Let $[\tilde{a},\tilde{b},\tilde{c}]$ be any primitive form of discriminant Δ; we must show that it is $SL_2(\mathbb{Z})$-equivalent to a form $\tilde{Q}$ with $Q\in\mathcal{Q}^0_{N,\Delta,\varrho,m_1,m_2}$. Thus we want $\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\in SL_2(\mathbb{Z})$ such that the numbers

$$\begin{aligned}a&=\tilde{a}\alpha^2+\tilde{b}\alpha\gamma+\tilde{c}\gamma^2\,,\\ b&=2\tilde{a}\alpha\beta+\tilde{b}(\alpha\delta+\beta\gamma)+2\tilde{c}\gamma\delta\,,\\ c&=\tilde{a}\beta^2+\tilde{b}\beta\delta+\tilde{c}\delta^2\end{aligned}$$

satisfy $a\equiv 0\,(N_1)$, $b\equiv\varrho\,(2N)$, $c\equiv 0\,(N_2)$ (then $[a,b,c]=\tilde{Q}$ for $Q=[aN_2,b,c/N_2]$, and Q automatically satisfies (3)). It is easily checked that these congruences hold if

$$\begin{pmatrix}\tilde{a} & \frac{1}{2}(\tilde{b}-\tilde{\varrho})\\ \frac{1}{2}(\tilde{b}+\tilde{\varrho}) & \tilde{c}\end{pmatrix}\begin{pmatrix}\alpha\\ \gamma\end{pmatrix}\equiv\begin{pmatrix}0\\ 0\end{pmatrix}(N_1)\,,\qquad \begin{pmatrix}\tilde{a} & \frac{1}{2}(\tilde{b}+\tilde{\varrho})\\ \frac{1}{2}(\tilde{b}-\tilde{\varrho}) & \tilde{c}\end{pmatrix}\begin{pmatrix}\beta\\ \delta\end{pmatrix}\equiv\begin{pmatrix}0\\ 0\end{pmatrix}(N_2)\,. \quad (5)$$

The first equation is solvable in coprime integers $\alpha,\gamma\ (\operatorname{mod} N_1)$ because $(\tilde{a},\tilde{b},\tilde{c})=1$ and the determinant of the matrix multiplying $\begin{pmatrix}\alpha\\ \gamma\end{pmatrix}$ is $0\ (\operatorname{mod} N_1)$. Similarly the second is solvable in coprime integers $\beta,\delta\ (\operatorname{mod} N_2)$. Since $(N_1,N_2)=1$, we can add to (5) the congruence conditions on $\begin{pmatrix}\alpha\\ \gamma\end{pmatrix}$ $(\operatorname{mod} N_2)$ and $\begin{pmatrix}\beta\\ \delta\end{pmatrix}$ $(\operatorname{mod} N_1)$ required to get $\det\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\equiv 1\ (\operatorname{mod} N)$. Since reduction modulo N from $SL_2(\mathbb{Z})$ to $SL_2(\mathbb{Z}/N\mathbb{Z})$ is surjective, there is a matrix $\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\in SL_2(\mathbb{Z})$ with the needed properties. This completes the proof.

Denote by $\mathcal{Q}_{N,\Delta}$ (resp. $\mathcal{Q}^0_{N,\Delta}$) the set of all [resp. all $\Gamma_0(N)$-primitive] quadratic forms $[aN,b,c]$ of discriminant Δ, so

$$\mathcal{Q}_{N,\Delta}=\bigcup_{\substack{\varrho(2N)\\ \varrho^2\equiv\Delta(4N)}}\mathcal{Q}_{N,\Delta,\varrho}=\bigcup_{\ell^2|\Delta}\ell\cdot\mathcal{Q}^0_{N,\Delta/\ell^2},\qquad \mathcal{Q}^0_{N,\Delta}=\bigcup_{\substack{\varrho(2N)\\ \varrho^2\equiv\Delta(4N)}}\mathcal{Q}^0_{N,\Delta,\varrho}.$$

Observing that 2^ν for a number m with ν prime factors is just the number of squarefree divisors of m, we find from the proposition

$$|\mathcal{Q}^0_{N,\Delta}/\Gamma_0(N)|=|\mathcal{Q}^0_\Delta/SL_2(\mathbb{Z})|\cdot\sum_{\substack{d|N\\ d^2|\Delta\\ d\text{ squarefree}}}n_{\Delta/d^2}(N/d),$$

where $n_\Delta(N)$ denotes the number of square roots $(\operatorname{mod} 2N)$ of $\Delta\,(\operatorname{mod} 4N)$.

We end this section with some remarks on the action of Hecke operators and Atkin-Lehner involutions on quadratic forms. Let N' be a positive integer with $N'\|N$. For $Q\in\mathcal{Q}_{N,\Delta}$ we define

$$Q|W_{N'}=\frac{1}{N'}Q\circ\begin{pmatrix}\alpha N' & \beta\\ \gamma N & \delta N'\end{pmatrix},\quad\text{where}\quad \alpha,\beta,\gamma,\delta\in\mathbb{Z},\qquad \alpha\delta N'-\beta\gamma\frac{N}{N'}=1.\quad(6)$$

Such matrices $\begin{pmatrix}\alpha N' & \beta\\ \gamma N & \delta N'\end{pmatrix}$ exist and any two are both left and right $\Gamma_0(N)$-equivalent, so (6) gives a well-defined map from $\mathcal{Q}_{N,\Delta}/\Gamma_0(N)$ to itself. It is easily checked that these maps are isomorphisms and satisfy the relation $W_{N'}\circ W_{N''}=W_{N'N''/(N',N'')^2}$, so they form a group of order 2^{t-1}, $t=$ number of prime factors of N. Writing out (6), we see that $[aN,b,c]\circ W_{N'}$ has a middle coefficient which is congruent to b modulo $\frac{2N}{N'}$ and to $-b$ modulo $2N'$, so

$$W_{N'}:\mathcal{Q}_{N,\Delta,\varrho}/\Gamma_0(N)\xrightarrow{\sim}\mathcal{Q}_{N,\Delta,\varrho^*}/\Gamma_0(N),\qquad \varrho^*\equiv\begin{cases}\varrho\,(\operatorname{mod} 2N/N'),\\ -\varrho\,(\operatorname{mod} 2N').\end{cases}\quad(7)$$

The Hecke operators T_m $(m\geqq 1)$ are defined, also in analogy with the theory of modular forms, as the one-to-many maps from $\mathcal{Q}_{N,\Delta}/\Gamma_0(N)$ to $\mathcal{Q}_{N,\Delta m^2}/\Gamma_0(N)$ (i.e., homomorphisms between the free abelian groups generated by these sets) sending $[Q]$ to the finite collection (or sum) of all $[Q\circ A]$, where A runs over the set of left $\Gamma_0(N)$-equivalence classes of matrices $\begin{pmatrix}\alpha & \beta\\ \gamma N & \delta\end{pmatrix}$ of determinant m with $\alpha,\beta,\gamma,\delta\in\mathbb{Z}$, α prime to m. Note that ϱ goes to $m\varrho$ under this correspondence. If F is any function on $\bigcup_\Delta\mathcal{Q}_{N,\Delta}/\Gamma_0(N)$ which is homogeneous of degree r [i.e. $F(\ell Q)=\ell^r F(Q)$ for $\ell\in\mathbb{N}$] and we define

$$\mathcal{L}_{N,\Delta,\varrho}(F)=\sum_{Q\in\mathcal{Q}_{N,\Delta,\varrho}/\Gamma_0(N)}F(Q),$$

then for $m=p$ a prime not dividing N we have

$$\mathcal{L}_{N,\Delta,\varrho}(F|T_p)=\mathcal{L}_{N,\Delta p^2,\varrho p}(F)+p^r\left(\frac{\Delta}{p}\right)\mathcal{L}_{N,\Delta,\varrho}(F)+p^{2r+1}\mathcal{L}_{N,\Delta/p^2,\varrho/p}(F);$$

here $F|T_p$ has the obvious meaning, the last term is to be omitted if $p^2\nmid\Delta$, and ϱ/p is the unique solution $\lambda\,(2N)$ of $p\lambda\equiv\varrho\,(2N)$, $\lambda^2\equiv\Delta\,(4N)$. For a proof (with $N=1$ and in a somewhat different context), see [22, pp. 290–292]. By induction on the powers of primes dividing m, one deduces from this the formula

$$\mathscr{L}_{N,\Delta,\varrho}(F|T_m)=\sum_{m=dd'}\left(\frac{\Delta}{d}\right)d^r\mathscr{L}_{N,\Delta d'^2,\varrho d'}(F)$$

for Δ fundamental and m prime to N.

2. The Generalized Genus Character χ_{D_0}

Classical genus theory associates to each discriminant Δ and fundamental discriminant divisor D_0 of Δ [i.e. D_0 is a fundamental discriminant and $\Delta/D_0\equiv 0$ or 1 (mod4)] a $\Gamma(1)$-invariant function χ_{D_0}: $\mathscr{Q}^0_\Delta\to\{\pm 1\}$ by setting $\chi_{D_0}(Q)=\left(\frac{D_0}{n}\right)$ for any integer n prime to D_0 represented by Q; such an n always exists and the value of $\left(\frac{D_0}{n}\right)$ is independent of the choice. The set $\mathscr{Q}^0_\Delta/\Gamma(1)$ has a natural group structure and χ_{D_0} is a homomorphism; conversely, all homomorphisms $\mathscr{Q}^0_\Delta/\Gamma(1)\to\{\pm 1\}$ have the form χ_{D_0} for some discriminant divisor D_0 of Δ, the only relations being $\chi_{D_0}=\chi_{D_1}$ if $\Delta=D_0D_1g^2$ for some $g\in\mathbb{N}$. In this section we give a natural extension of this function to a $\Gamma_0(N)$-invariant function on $\mathscr{Q}_{N,\Delta}$ under the assumption that *both D_0 and Δ/D_0 are squares modulo* $4N$. For $Q\in\mathscr{Q}^0_{N,\Delta}$ we set $\chi_{D_0}(Q)=\chi_{D_0}(\tilde{Q})$ with $\tilde{Q}$ as in Sect. 1; we then extend to non-$\Gamma_0(N)$-primitive forms by $\chi_{D_0}(\ell Q)=\left(\frac{D_0}{\ell}\right)\chi_{D_0}(Q)$ [cf. (1) of Sect. 1], so $\chi_{D_0}(\ell Q)=0$ if $(\ell,D_0)>1$. A different formulation of this definition, and the main properties of the function χ_{D_0}, are given in the following proposition.

Proposition 1. *Let $N\geqq 1$, D_0 a fundamental discriminant and Δ a discriminant divisible by D_0 such that both D_0 and Δ/D_0 are squares modulo $4N$. For $Q=[aN,b,c]\in\mathscr{Q}_{N,\Delta}$ set*

$$\chi_{D_0}(Q)=\begin{cases}\left(\frac{D_0}{n}\right) & \text{if}\quad (a,b,c,D_0)=1\,,\\ 0 & \text{otherwise}\,,\end{cases}\tag{1}$$

where in the first case n is an integer prime to D_0 represented by the form $[aN_1,b,cN_2]$ for some decomposition $N=N_1N_2$, $N_i>0$. Such an n exists and the value of $\left(\frac{D_0}{n}\right)$ is independent of the choice of N_1,N_2, and n. The function χ_{D_0} is $\Gamma_0(N)$-invariant and has the following properties:

P1 (Multiplicativity):

$$\chi_{D_0}([aN,b,c])=\chi_{D_0}([a_1N,b,ca_2])\chi_{D_0}([a_2N,b,ca_1])\quad \text{if}\quad (a_1,a_2)=1\,.$$

P2 (Invariance under the Fricke involution):

$$\chi_{D_0}([aN,b,c])=\chi_{D_0}([cN,-b,a])\,.$$

P3 (Explicit formula):

$$\chi_{D_0}([aN,b,c])=\left(\frac{D_1}{N_1a}\right)\left(\frac{D_2}{N_2c}\right)$$

for any splitting $D_0 = D_1 D_2$ of D_0 into discriminants (necessarily fundamental and coprime) and $N = N_1 N_2$ of N into positive factors such that $(D_1, N_1 a) = (D_2, N_2 c) = 1$, $\chi_{D_0} = 0$ if no such splittings exist.

Proof. The condition $(a, b, c, D_0) = 1$ is equivalent to $(a, c, D_0) = 1$ since $b^2 \equiv 4acN\,(D_0)$. If it is satisfied, we can find N_1 and N_2 satisfying

$$N = N_1 N_2, \qquad N_1, N_2 > 0, \qquad (N_1, c, D_0) = (N_2, a, D_0) = 1; \tag{2}$$

then the g.c.d. of the coefficients of the form $[aN_1, b, cN_2]$ is prime to D_0 and therefore by a well-known theorem this form represents integers prime to D_0. Let n be such an integer and define $\chi_{D_0}(Q)$ by (1). Then $\chi_{D_0}(Q) = \prod_{p|D_0} \left(\frac{p^*}{n}\right)$ where p^* is the prime discriminant divisor of D divisible by p $\left[\text{i.e. } p^* = \left(\frac{-1}{p}\right) p \text{ for } p \neq 2,\ p^* = -4,\ 8 \text{ or } -8 \text{ with } D_0/p^* \equiv 1\,(4) \text{ for } p = 2\right]$. If $p \nmid aN_1$ then $\left(\frac{p^*}{n}\right) = \left(\frac{p^*}{aN}\right)$ since $n = aN_1 x^2 + bxy + cN_2 y^2$ implies $4aN_1 n = (2aN_1 x + by)^2 - \Delta y^2$ and p^* divides Δ. [If p is odd, this equation says that $aN_1 n$ is a square modulo p, and $\left(\frac{p^*}{\cdot}\right)$ is the Legendre symbol. If $p = 2$ one has to distinguish according to the three values of p^* and use that Δ/p^* is 0 or 1 modulo 4.] Similarly $\left(\frac{p^*}{n}\right) = \left(\frac{p^*}{cN}\right)$ if $p \nmid cN_2$. Since $(aN_1, cN_2, D_0) = 1$, each $p|D_0$ satisfies one of these conditions. This shows that each $\left(\frac{p^*}{n}\right)$, and hence also their product $\chi_{D_0}(Q)$, is independent of the choice of n, and also shows that the right-hand side of the "explicit formula" P3 is independent of the splitting $D = D_1 D_2$ [for a given splitting $N = N_1 N_2$ satisfying (2)] and that this formula is true. We still have to check the independence of this splitting; the $\Gamma_0(N)$-invariance then follows as in Sect. 1 (namely changing Q by $M \in \Gamma_0(N)$ changes the form $[aN_1, b, cN_2]$ by $\tilde{M}$), and P1 and P2 are obvious from the explicit formula P3.

The passage from any splitting $N = N_1 N_2$ to any other can be accomplished by moving one prime ℓ at a time, so we can restrict ourselves to such changes. If $\ell \nmid D_0$ then multiplying one N_i by ℓ and dividing the other N_i by ℓ changes $\left(\frac{D_1}{N_1 a}\right) \cdot \left(\frac{D_2}{N_2 c}\right)$ by $\left(\frac{D_1 D_2}{\ell}\right)$, and this is 1 since $D_1 D_2 = D_0 \equiv \text{square} \pmod{4\ell}$ by hypothesis. Assume $\ell | D_0$. If $\ell | ac$, then (2) forces us to include the full power of ℓ dividing N into N_1 or N_2 (depending whether $\ell | a$ or $\ell | c$) and the problem of moving ℓ from one side to the other does not arise. Assume that $\ell \nmid ac$ and, for convenience, $\ell \neq 2$ (the case $\ell = 2$ is similar and will be left to the reader). Since D is fundamental, $\ell^2 \nmid D_0$, so $\ell \| D_0$. Then the fact that D_0 is a square $\pmod{4N}$ implies $\ell^2 \nmid N$. Suppose that $\ell | N_1$; then $\ell \| N_1$ and $\ell \nmid N_2$, and the change we are considering is $N_1, N_2 \to N_1/\ell, N_2 \ell$. The condition $(N_1, D_1) = 1$ implies $\ell \nmid D_1$ and hence $\ell \| D_2$, so as splitting of D_0 for the new splitting $N_1/\ell \cdot N_2 \ell$ of N we can take $D_1 \ell^* \cdot D_2/\ell^*$ (we have already shown that the formula in P3 is independent of the splitting of D_0 for given N_1, N_2). Thus $\left(\frac{D_1}{N_1 a}\right)\left(\frac{D_2}{N_2 c}\right)$ is replaced by $\left(\frac{D_1 \ell^*}{N_1 a/\ell}\right)\left(\frac{D_2/\ell^*}{N_2 c\ell}\right)$. These two products differ by

$$\left(\frac{\ell^*}{N_1 a/\ell}\right)\left(\frac{D_1}{\ell}\right)\left(\frac{\ell^*}{N_2 c}\right)\left(\frac{D_2/\ell^*}{\ell}\right)=\left(\frac{\ell^*}{Nac/\ell}\right)\left(\frac{D_0/\ell^*}{\ell}\right)=\left(\frac{Nac/\ell}{\ell}\right)\left(\frac{-D_0/\ell}{\ell}\right).$$

But $b^2-4Nac=D_0\cdot\Delta/D_0$ implies that $\ell|b$ and (hence) $4Nac/\ell\equiv -D_0/\ell\cdot\Delta/D_0\,(\ell)$, so this equals the Legendre symbol $\left(\frac{\Delta/D_0}{\ell}\right)$, which is 1 because of the hypothesis $\Delta/D_0\equiv$ square $(4N)$. This completes the proof.

We remark that a function like χ_{D_0} (for $N=1$) was defined in [12] and an explicit formula like our P3 proved there (Proposition 6, p. 263).

Finally, we give one further property of our genus characters:

P4 (Invariance under Atkin-Lehner involutions):

$$\chi_{D_0}(Q|W_{N'})=\chi_{D_0}(Q)\quad \textit{for all}\quad Q\in\mathcal{Q}_{N,D},\qquad N'\|N\quad (W_{N'}\text{ as in Sect. 1}).$$

The proof, which is somewhat more complicated than that of P2, will be omitted since this result will not be used in the sequel.

The remainder of Sect. 2 is devoted to the proof of the following technical proposition (needed in Chap. II), which gives a formula for the function $\chi_{D_0}([aN,b,c])$ in terms of Gauss sums. By assumption $D_0\equiv r_0^2\,(4N)$, $\Delta/D_0\equiv r^2\,(4N)$ for some integers r_0 and r; we can always choose them so that their product is congruent to a chosen square root ϱ of Δ (mod $4N$). Then we have:

Proposition 2. *Write $D_0=r_0^2-4Nn_0$, $D=\frac{\Delta}{D_0}=r^2-4Nn$, and suppose $b\equiv r_0 r\,(2N)$. Denote by $F(x,y)$ the second degree polynomial $Nx^2+r_0xy+n_0y^2+rx+sy+n$ with $s=\frac{r_0r-b}{2N}$, and for any $c\geqq 1$ set*

$$\mathcal{F}_c=\mathcal{F}_c(N,r_0,n_0,r,s,n)=\frac{1}{c}\sum_{\lambda(c)^*}\sum_{x,y(c)}e_c(\lambda F(x,y)).$$

Then for any $a\geqq 1$ we have

$$\frac{1}{a}\sum_{d|a}\left(\frac{D_0}{d}\right)\mathcal{F}_{a/d}=\begin{cases}\chi_{D_0}\left(\left[aN,b,\frac{b^2-\Delta}{4Na}\right]\right) & \textit{if}\quad a\,\Big|\,\frac{b^2-\Delta}{4N},\\ 0 & \textit{otherwise}.\end{cases}\tag{3}$$

Remark. Using the standard identity $\sum_{\lambda(c)^*}e_c(\lambda m)=\sum_{d|(c,m)}\mu\left(\frac{c}{d}\right)d$ (Ramanujan sum), we can rewrite $\mathcal{F}_c$ as $\sum_{d|c}\mu\left(\frac{c}{d}\right)\frac{c}{d}N(d)$, where $N(d)$ is the number of solutions in integers x,y (mod d) of $F(x,y)\equiv 0\,(d)$. The proposition is then equivalent to the Dirichlet series identity

$$\sum_{d=1}^{\infty}N(d)d^{-s-1}=L\left(s+1,\left(\frac{D_0}{\cdot}\right)\right)^{-1}\zeta(s)\sum_{a\left|\frac{b^2-\Delta}{4N}\right.}\chi_{D_0}\left(\left[aN,b,\frac{b^2-\Delta}{4Na}\right]\right)a^{-s}.$$

Proof. We can simplify the assertion to be proved by making two reduction steps. First of all, both sides of (3) are multiplicative functions of a (this can be seen easily for the left-hand side, and follows for the right-hand side from P1 above), so we may assume that a is a prime power p^α; we shall treat only the case $p \neq 2$, leaving the case $p=2$ to the reader. Secondly, we may replace the second-degree polynomial F by $F \circ M$ for any $M \in SL_2(\mathbb{Z})$ without affecting the correctness of (3). Indeed, the value of $\mathcal{F}_c$ is unchanged by this since $M\begin{pmatrix} x \\ y \end{pmatrix}$ runs over $(\mathbb{Z}/c\mathbb{Z})^2$ as $\begin{pmatrix} x \\ y \end{pmatrix}$ does. On the other hand, replacing F by $F \circ M$ replaces the quadratic form $Q_0=[N, r_0, n_0]$ by the equivalent form $Q_0 \circ M$ and hence does not change either D_0 or the value of $\chi_{p^* \cdot D_0/p^*}(Q_0)=\chi_{p^* \cdot D_0/p^*}(N)$ for any prime $p|D_0$, while it replaces $\begin{pmatrix} -s \\ r \end{pmatrix}$ by $M^{-1}\begin{pmatrix} -s \\ r \end{pmatrix}$ and hence leaves invariant the quantity

$$C=Q_0(-s, r)+nD_0=\frac{b^2-\Delta}{4N},$$

and the right-hand side of (3) depends only on D_0, a, C and the $\chi_{p^* \cdot D_0/p^*}(N)$ by P3 of Proposition 1. For $c=p^\nu$ with $p \neq 2$ (indeed, for any odd c), we can find $M \in SL_2(\mathbb{Z})$ diagonalizing Q_0 (mod c), so we can assume $r_0 \equiv 0\,(c)$. Then

$$\mathcal{F}_c=\frac{1}{c}\sum_{\lambda(c)^*} e_c(\lambda n) G_c(\lambda N, \lambda r) G_c(\lambda n_0, \lambda s),$$

where $G_c(A, B)$ denotes the one-variable Gauss sum $\sum_{x(c)} e_c(Ax^2+Bx)$.

Case 1: $p \nmid D_0$. Using the formula (easily deducible from the standard case $B=0$)

$$c=p^\nu \quad (p \neq 2),\ p \nmid A \ \Rightarrow\ G_c(A, B)=\sqrt{c}\,\varepsilon(c)\left(\frac{A}{c}\right) e_c(-B^2(4A)^{-1}), \tag{4}$$

where $\varepsilon(c)$ equals 1 or i according as $c \equiv 1\,(4)$ or $c \equiv 3\,(4)$ and $(4A)^{-1}$ denotes the inverse of $4A$ (mod c), we find

$$\begin{aligned}\mathcal{F}_c&=\frac{1}{c}\sum_{\lambda(c)^*} c\,\varepsilon(c)^2\left(\frac{Nn_0}{c}\right) e_c(-\lambda(r^2(4N)^{-1}+s^2(4n_0)^{-1})+\lambda n)\\&=\left(\frac{D_0}{c}\right)\sum_{\lambda(c)^*} e_c(\lambda D_0^{-1} C)=\left(\frac{D_0}{c}\right)\sum_{\lambda(c)^*} e_c(\lambda C),\end{aligned}$$

so for a any power of p

$$\begin{aligned}\sum_{d|a}\left(\frac{D_0}{d}\right)\mathcal{F}_{a/d}&=\left(\frac{D_0}{a}\right)\sum_{d|a}\sum_{\substack{\lambda(a)\\(\lambda, a)=d}} e_a(\lambda C)\\&=\left(\frac{D_0}{a}\right)\sum_{\lambda(a)} e_a(\lambda C)=\begin{cases}\left(\frac{D_0}{a}\right)a & \text{if } a|C,\\ 0 & \text{if } a \nmid C,\end{cases}\end{aligned}$$

in agreement with (3) by P3 of Proposition 1.

Case 2: $p|D_0$. Since $p \neq 2$ and D_0 is fundamental, $p^2 \nmid D_0$, so we can assume [after acting by a suitable element of $SL_2(\mathbb{Z})$] that $p \nmid N$, $p \| n_0$. The sum $G_c(\lambda N, \lambda r)$ can be evaluated again by (2), and the sum $G_c(\lambda n_0, \lambda s)$ by

$$\left.\begin{array}{l} c=p^\nu \\ p \neq 2 \\ p \| A \end{array}\right\} \Rightarrow G_c(A,B)=\begin{cases} 0 & \text{if } p \nmid B, \\ \sqrt{pc}\,\varepsilon(c/p)\left(\dfrac{A/p}{c/p}\right) e_{c/p}(-(B/p)^2(4A/p)^{-1}) & \text{if } p|B. \end{cases}$$

$\left(\textit{Proof.}\right.$ Replacing x by $x+\dfrac{c}{p}$ in the definition of G_c gives $G_c(A,B)=e_p(B)G_c(A,B)$, so $G_c=0$ if $p \nmid B$. If $p|B$, then $G_c(A,B)=G_{c/p}(A/p,B/p)$ and we apply (4).$\left.\right)$ This gives

$$\mathscr{F}_c=\begin{cases} \sqrt{p}\,\varepsilon(p)\left(\dfrac{N}{c}\right)\left(\dfrac{n_0/p}{c/p}\right)\sum\limits_{\lambda(c)}\left(\dfrac{\lambda}{p}\right)e_c(\lambda(D_0/p)^{-1}C/p) & \text{if } p|C, \\ 0 & \text{if } p \nmid C. \end{cases}$$

The inner sum is 0 if $c \nmid C$ (replace λ by $\lambda+p$) and equals $\varepsilon(p)\sqrt{p}\cdot\dfrac{c}{p}\cdot\left(\dfrac{D_0/p}{p}\right)\left(\dfrac{C/c}{p}\right)$ if $c|C$. Hence

$$\mathscr{F}_c=\begin{cases} c\left(\dfrac{D_0/p^*}{c}\right)\left(\dfrac{NC/c}{p}\right) & \text{if } c|C, \\ 0 & \text{if } c \nmid C, \end{cases}$$

where $p^*=(-1)^{\frac{p-1}{2}}p$. By P3 of Proposition 1 this proves (3) in this case also (note that the left-hand side of (3) reduces to the single term $\dfrac{1}{a}\mathscr{F}_a$ for $a=p^\nu$, $p|D_0$).

3. Clifford Algebras and Eichler Orders

Fix an integer $N \geqq 1$. In this section we will consider a primitive integral binary quadratic form $q(x,y)$ which represents only integers which are squares $(\bmod 4N)$. Then q has the form $q=[D_0, 2n, D_1]$ where D_0 and D_1 are squares $(\bmod 4N)$ and $n^2 \equiv D_0 D_1 \pmod{4N}$. We will further assume that D_0 and D_1 are relatively prime and that q is non-degenerate over $\mathbb{Q}$.

The discriminant of q is equal to $4(n^2-D_0D_1)$. By hypothesis, this is divisible by $16N$. We define

$$M=\frac{\operatorname{disc}(q)}{16N}=\frac{n^2-D_0D_1}{4N}.$$

If p is any prime dividing NM, we define $\varepsilon(p)=\pm 1$ as follows: Let D be any integer prime to p which is represented by q (either D_0 or D_1 will always do) and define $\varepsilon(p)=\left(\dfrac{D}{p}\right)$. This is independent of the choice of D and equals $+1$ if $p|N$. For any positive divisor $d=\prod p_i^{a_i}$ of M we let $\varepsilon(d)=\prod \varepsilon(p_i)^{a_i}$ and introduce the Dirichlet series

$$\ell(s)=\sum_{d|M}\varepsilon(d)d^s=\prod_{p^a\|M}\frac{1-\varepsilon(p)^{a+1}p^{(a+1)s}}{1-\varepsilon(p)p^s}. \tag{1}$$

Since $\operatorname{disc}(q) \neq 0$, the form q determines a non-degenerate quadratic space of dimension 2 over $\mathbb{Q}$. We let B be the Clifford algebra of this space. Then B is a quaternion algebra over $\mathbb{Q}$ with basis $\langle 1, e_0, e_1, e_0e_1\rangle$ satisfying the multiplicative relations

$$e_0^2 = D_0, \quad e_1^2 = D_1, \quad e_0e_1 + e_1e_0 = 2n.$$

Proposition 1. 1) *A finite prime p is ramified in B iff $p|M$ and $\varepsilon(p)^{\operatorname{ord}_p(M)} = -1$.*
2) *The infinite place is ramified in B iff D_0, D_1, and M are all negative.*

Proof. Recall that a place v is ramified in B iff $B \otimes \mathbb{Q}_v$ is a division algebra. We may restate this in terms of the ternary quadratic form given by the square on the subspace B^0 of elements of trace 0 in B. This subspace has basis $\langle e_0, e_1, e_0e_1 - n\rangle$ and squaring is given by the formula

$$(xe_0 + ye_1 + z(e_0e_1 - n))^2 = q(x, y) + 4NMz^2.$$

B is ramified at v iff this form does not represent 0 in $\mathbb{Q}_v$. Over $\mathbb{R}$, this requires that q be negative definite, which is equivalent to the condition in (2). Over $\mathbb{Q}_p$, a short calculation using [17, p. 37] shows that one must have $\varepsilon(p)^{\operatorname{ord}_p(M)} = -1$.

Corollary. *The order of $\ell(s)$ at $s=0$ is the number of finite primes p which ramify in B. In particular, $\ell(0) \neq 0$ if $B \cong M_2(\mathbb{Q})$.*

We now introduce the order $S = \mathbb{Z} + \mathbb{Z}\alpha_0 + \mathbb{Z}\alpha_1 + \mathbb{Z}\alpha_0\alpha_1$ of reduced discriminant NM in the quaternion algebra B, where $\alpha_i = (e_i + D_i)/2$. By construction, S contains the quadratic orders $\mathbb{Z}[\alpha_0]$ and $\mathbb{Z}[\alpha_1]$ of discriminants D_0 and D_1, respectively. Recall that an order R of B is an *Eichler order of index N* if for all primes $p \nmid N$ the localization $R_p = R \otimes \mathbb{Z}_p \subseteq B_p = B \otimes \mathbb{Q}_p$ is a maximal order and for all primes $p|N$ there is an isomorphism from B_p to $M_2(\mathbb{Q}_p)$ which maps R_p to the order $\left\{\begin{pmatrix} a & b \\ Nc & d\end{pmatrix} \middle| a, b, c, d \in \mathbb{Z}_p\right\}$. [Recall that $\varepsilon(p) = +1$ for $p|N$, so B_p is isomorphic to $M_2(\mathbb{Q}_p)$ by Proposition 1.]

Proposition 2. *The number $\varrho(S)$ of Eichler orders of index N in B which contain S is given by the formula*

$$\varrho(S) = \prod_{\substack{p|M \\ \varepsilon(p)=1}} (1 + \operatorname{ord}_p(M)) = \sum_{\substack{d|M \\ (d, \operatorname{disc}(B))=1}} \varepsilon(d).$$

Proof. Since a global order R is completely determined by its localizations [17, p. 83], it suffices to calculate, for each prime p, the number of local Eichler orders R_p containing S_p. If $p \nmid NM$, then S_p is maximal in B_p and $R_p = S_p$ is the only choice. If $p|NM$, then $\varepsilon(p)$ is defined. If $\varepsilon(p) = -1$ then $p \nmid N$ and R_p must be a maximal order. If B_p is a division algebra, this maximal order is unique. Even when B_p is a matrix algebra there is a unique maximal order R_p containing S_p, because S_p contains the ring of integers $\mathcal{O}_p$ in the unramified quadratic extension of $\mathbb{Q}_p$, and an argument similar to [7, Sect. 3] shows that $S_p = \mathcal{O}_p + p^{\operatorname{ord}_p(M)/2} R_p$. [Note that $\operatorname{ord}_p(M)$ is even in this case by Proposition 1.] Finally, assume $\varepsilon(p) = 1$. Since p splits in $\mathcal{O}_{D_0}$ or $\mathcal{O}_{D_1}$, S_p contains the ring $\mathcal{O}_p = \mathbb{Z}_p \oplus \mathbb{Z}_p$ and is therefore conjugate to an order of the form $\left\{\begin{pmatrix} a & b \\ NMc & d\end{pmatrix} \middle| a, b, c, d \in \mathbb{Z}_p\right\}$ in $M_2(\mathbb{Q}_p)$ [17, p. 39]. The Eichler orders of index N which contain this order have the form $\left\{\begin{pmatrix} a & p^{-k}b \\ Np^k c & d \end{pmatrix} \middle| a,b,c,d \in \mathbb{Z}_p\right\}$ with $0 \leqq k \leqq \mathrm{ord}_p(M)$. Hence there are $\mathrm{ord}_p(M)+1$ possibilities for R_p. This completes the proof.

We will now use Proposition 2 to determine the number of embeddings of S into certain Eichler orders of index N in B. To do this, we begin with a simple combinatorial observation. Let X and Y be two sets on which a group G acts and Σ a G-stable subset $X \times Y$. For $x \in X$ the stabilizer G_x of x acts on the set $Y_x = \{y \in Y \mid (x,y) \in \Sigma\}$. Similarly for $y \in Y$ the stabilizer G_y of y acts on $X_y \subseteqq X$. Then there are natural bijections

$$\bigcup_{x \in X/G} Y_x/G_x \xleftarrow{\sim} \Sigma/G \xrightarrow{\sim} \bigcup_{y \in Y/G} X_y/G_y,$$

so

$$\sum_{x \in X/G} \mathrm{Card}(Y_x/G_x) = \sum_{y \in Y/G} \mathrm{Card}(X_y/G_y). \tag{2}$$

We apply this to

X = set of all Eichler orders R of index N in B,

Y = set of all algebra homomorphisms $\phi: S \to B$,

$\Sigma = \{(R,\phi) \mid \phi(S) \subseteqq R\}$, $\quad G = B^\times/\mathbb{Q}^\times$ (acting by conjugation).

The set Y/G has a single element, since any embedding ϕ extends to an automorphism of B, which is inner by the Skolem-Noether theorem. Take this element to be the inclusion $S \subset B = S \otimes \mathbb{Q}$; then the stabilizer G_y is trivial and the set X_y is the set of all Eichler orders $R \subset B$ of index N containing S, so the expression on the right of (2) is the number $\varrho(S)$ of Proposition 2. On the other hand, the coset space X/G is finite and represents the set of global conjugacy classes of Eichler orders of index N in B. If x corresponds to the order R, then $G_x = \mathrm{Norm}_{B^\times}(R)/\mathbb{Q}^\times$. Hence (2) gives

$$\begin{aligned}\varrho(S) &= \sum_{R \bmod B^\times/\mathbb{Q}^\times} \mathrm{Card}\{\phi: S \to R\ (\mathrm{mod}\ \mathrm{Norm}_{B^\times}(R)/\mathbb{Q}^\times)\} \\ &= \sum_{R \bmod B^\times/\mathbb{Q}^\times} [\mathrm{Norm}_{B^\times}(R): R^\times \mathbb{Q}^\times]^{-1}\, \mathrm{Card}\{\phi: S \to R\ (\mathrm{mod}\ R^\times/\pm 1)\},\end{aligned}$$

since $\mathrm{Norm}_{B^\times}(R)/\mathbb{Q}^\times$ acts faithfully on the embeddings of S into R, and contains $R^\times\mathbb{Q}^\times/\mathbb{Q}^\times \cong R^\times/\{\pm 1\}$ with finite index.

Let R_1 be a fixed Eichler order of index N in B, and let $\hat{R}_1 = R_1 \otimes \hat{\mathbb{Z}}$ in $\hat{B} = \hat{R} \otimes \mathbb{Q} = R \otimes \hat{\mathbb{Q}}$ $\left(\text{where } \hat{\mathbb{Z}} = \prod_p \mathbb{Z}_p \subset \hat{\mathbb{Q}} = \hat{\mathbb{Z}} \otimes \mathbb{Q}\right)$. Since the global Eichler orders are all locally conjugate, the set of Eichler orders R (mod $B^\times/\mathbb{Q}^\times$) is identified with the double cosets $\mathrm{Norm}_{\hat{B}^\times}(\hat{R}_1)\backslash \hat{B}^\times/B^\times$. If g is an element representing the double coset, the order $R_g = g^{-1}\hat{R}g \cap B$ is well-defined up to conjugacy in B. We have $\mathrm{Norm}_{B^\times}(R_g) = \mathrm{Norm}_{\hat{B}^\times}(g^{-1}\hat{R}_1 g) \cap B^\times$ in $\hat{B}^\times$.

We wish to rewrite our sum over Eichler orders up to conjugacy as a sum over the (possibly larger) double coset space $\hat{R}_1^\times \backslash \hat{B}^\times/B^\times$, which is also finite and indexes the left ideal classes for the order R_1. (This is the classical distinction between "types" and "classes" in the theory of orders.) Each coset g must be taken with multiplicity $1/e_g$, where

$$e_g = \mathrm{Card}\{g' \in \hat{R}^\times \backslash \hat{B}^\times / B^\times \mid g' = g \text{ in } \mathrm{Norm}_{\hat{B}^\times}(\hat{R}_1) \backslash \hat{B}^\times / B^\times\}.$$

An easy calculation shows that

$$e_g = [\mathrm{Norm}_{\hat{B}^\times}(\hat{R}_g) : \hat{R}_g^\times \cdot \hat{\mathbb{Q}}^\times \cdot \mathrm{Norm}_{B^\times}(R_g)].$$

Multiplying our two indices, we find that the above formula for $\varrho(S)$ becomes

$$\varrho(S) = \sum_{g \in \hat{R}_1^\times \backslash \hat{B}^\times / B^\times} [\mathrm{Norm}_{\hat{B}^\times}(\hat{R}_g) : \hat{R}_g^\times \hat{\mathbb{Q}}^\times]^{-1} \, \mathrm{Card}\{\phi : S \to R_g \, (\mathrm{mod}\, R_g^\times / \pm 1)\}.$$

The weighting factor is now independent of g, since the map $n \mapsto g^{-1} n g$ identifies $\mathrm{Norm}_{\hat{B}^\times}(\hat{R}_1)$ with $\mathrm{Norm}_{\hat{B}^\times}(\hat{R}_g)$ and maps $\hat{R}_1^\times$ to $\hat{R}_g^\times$. This factor is calculated locally, and is equal to 2^{s+t}, where s is the number of finite primes ramified in B and t the number of primes dividing N [17, pp. 43–44]. Hence finally

$$\sum_{g \in \hat{R}^\times \backslash \hat{B}^\times / B^\times} \mathrm{Card}\{\phi : S \to R_g \, (\mathrm{mod}\, R_g^\times / \pm 1)\} = 2^{s+t} \varrho(S).$$

The orders R_g appearing in this formula are precisely the h_R right orders of the left ideal classes for the order $R = R_1$. Hence we may state the result in elementary terms:

Proposition 3. *Let R be an Eichler order of index N in B and $R_1, \ldots, R_h$ the right orders of left ideals $I_1, \ldots, I_h$ which represent the distinct ideal classes of R. Then*

$$\sum_{i=1}^{h} \mathrm{Card}\{\phi : S \to R_i \, (\mathrm{mod}\, R_i^\times / \pm 1)\} = 2^{s+t} \varrho(S),$$

where s is the number of finite primes which ramify in B, t the number of primes dividing N, and $\varrho(S)$ is given by Proposition 2.

Let us examine Proposition 3 when $B \cong M_2(\mathbb{Q})$. Here we may take R to be the Eichler order $\left\{ \begin{pmatrix} a & b \\ Nc & d \end{pmatrix} \middle| \, a, b, c, d \in \mathbb{Z} \right\}$ in $M_2(\mathbb{Z})$. This order has class number 1, by the strong approximation theorem for SL_2, so the sum on the left-hand side of Proposition 3 has only one term. The group $R^\times / \pm 1$ is $\bar{\Gamma}_0(N)$, the degree 2 extension of $\Gamma_0(N)$ by $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. Since $\mathrm{disc}\, B = 1$, we have $\varrho(S) = \ell(0)$ by Proposition 2. Hence Proposition 3 may be restated

$$\mathrm{Card}\{\phi : S \to R \, (\mathrm{mod}\, \bar{\Gamma}_0(N))\} = 2^t \ell(0). \tag{3}$$

This formula is now true for *any* S, since both sides vanish when $B = S \otimes \mathbb{Q}$ is not isomorphic to $M_2(\mathbb{Q})$.

We can rewrite (3) in more elementary terms, since specifying an embedding $\phi : S \to R$ is the same as giving the images $E_0 = \phi(e_0)$ and $E_1 = \phi(e_1)$. These are matrices of trace 0 in the suborder $\mathbb{Z} + 2R$ of R [because $E_i \equiv D_i \, (\mathrm{mod}\, 2R)$] and satisfy $E_i^2 = D_i$, $\mathrm{Tr}(E_0 E_1) = 2n$. Hence

$$E_i = \begin{pmatrix} b_i & 2c_i \\ -2a_i N & -b_i \end{pmatrix}$$

with

$$a_i, b_i, c_i \in \mathbb{Z}, \quad b_i^2 - 4a_ic_iN = D_i, \quad b_0b_1 - 2N(a_0c_1 + c_0a_1) = n.$$

Each E_i corresponds to a quadratic form $Q_i = [a_iN, b_i, c_i] \in \mathcal{Q}_{N,D_i}$ in the notation of Sect. 1. Let Δ_N denote the discriminant form on the lattice $\mathcal{Q}_N = \bigcup_D \mathcal{Q}_{N,D}$ of all quadratic forms $[aN, b, c]$ with $a, b, c \in \mathbb{Z}$; then the associated bilinear form B_{Δ_N} with $B_{\Delta_N}(Q, Q) = \Delta_N(Q)$ is given by

$$B_{\Delta_N}([a_0N, b_0, c_0], [a_1N, b_1, c_1]) = b_0b_1 - 2N(a_0c_1 + c_0a_1).$$

Hence (3) can be restated as follows:

Corollary.

$$\begin{aligned}&\tfrac{1}{2}\mathrm{Card}\{(Q_0, Q_1) \in \mathcal{Q}_N^2/\Gamma_0(N) \,|\, \Delta_N(Q_0) = D_0,\ \Delta_N(Q_1) = D_1,\ B_{\Delta_N}(Q_0, Q_1) = n\} \\ &= 2^t \sum_{d \mid \frac{D_0D_1 - n^2}{4N}} \varepsilon(d).\end{aligned}$$

The factor $\frac{1}{2}$ comes from the fact that $\bar{\Gamma}_0(N)/\Gamma_0(N) \cong \mathbb{Z}/2$ acts freely on the set of pairs (Q_0, Q_1) in question, because at least one of D_0, D_1 is odd.

The corollary just stated was proved for $N = 1$ in [8, pp. 211–213], using a more complicated method involving the number theory of the quadratic fields $\mathbb{Q}(\sqrt{D_0}), \mathbb{Q}(\sqrt{D_1}), \mathbb{Q}(\sqrt{D_0D_1})$ and of the biquadratic field $\mathbb{Q}(\sqrt{D_0}, \sqrt{D_1})$. Notice that the left-hand side of the identity counts the $\bar{\Gamma}_0(N)$-equivalence classes of representations of the binary quadratic form $q = [D_0, 2n, D_1]$ by the ternary form Δ_N, since the conditions on Q_0 and Q_1 just say $\Delta_N(\xi Q_0 + \eta Q_1) = q(\xi, \eta)$.

We shall use Proposition 3 in one further case in this paper. Assume that D_0, D_1, and M are all negative, so B is ramified at infinity and $\ell(0) = 0$. Then $\ell'(0) \neq 0$ if and only if B is ramified at a single finite prime p. In this case we have $\ell'(0) = \frac{1}{2}(\mathrm{ord}_p(M) + 1) \cdot \varrho(S) \cdot (-\log p)$. We shall see that the orders R_i occurring in Proposition 3 in this case are just the endomorphism rings of the supersingular points $(\mathrm{mod}\, p)$ of the curve $X_0(N)$, and that the embeddings of the Clifford order S into the orders R_i will be relevant in calculating the local height pairing of Heegner divisors.

II. Liftings of Jacobi Modular Forms

In [15] it was shown that the space $J_{k,N}^{\mathrm{cusp}}$ of Jacobi cusp forms of weight k and index N is isomorphic as a Hecke module to a certain subspace of the space of cusp forms of weight $2k-2$ on $\Gamma_0(N)$, and lifting maps $J_{k,N}^{\mathrm{cusp}} \to S_{2k-2}(N)$ were constructed. The purpose of this chapter is to construct the kernel functions for these liftings. This will lead to several identities relating the Fourier coefficients of a Jacobi-Hecke eigenform to the periods and to the special values of twists of L-series of the corresponding form in $S_{2k-2}(N)$.

1. Kernel Functions for Geodesic Cycle Integrals

Let $N \in \mathbb{N}$, $\varrho \in \mathbb{Z}/2N\mathbb{Z}$, and $\Delta > 0$ be a discriminant satisfying $\Delta \equiv \varrho^2\,(4N)$, and denote by $\mathcal{Q}_{N,\Delta,\varrho}$ the set of binary quadratic forms $Q(x, y) = ax^2 + bxy + cy^2$ with integer coefficients and discriminant Δ satisfying $a \equiv 0\,(N)$, $b \equiv \varrho\,(2N)$ as in Chap. 0, Sect. 1. Let D_0 be a fundamental discriminant dividing Δ such that both D_0 and Δ/D_0 are squares modulo $4N$, and let $\chi_{D_0}: \mathcal{Q}_{N,\Delta,\varrho} \to \{\pm 1, 0\}$ be the function defined in Sect. 2 of Chap. 0 (generalized genus character). The group $\Gamma_0(N)$ acts on $\mathcal{Q}_{N,\Delta,\varrho}$ in the usual way and χ_{D_0} is $\Gamma_0(N)$-invariant. For an integer $k>1$ we define

$$f_{k,N,\Delta,\varrho,D_0}(z) = \sum_{Q \in \mathcal{Q}_{N,\Delta,\varrho}} \frac{\chi_{D_0}(Q)}{Q(z,1)^k} \qquad (z \in \mathfrak{H}).$$

This series converges absolutely and uniformly on compact sets and defines a holomorphic cusp form of weight $2k$ on $\Gamma_0(N)$. For $k=1$ the series no longer converges absolutely but we define

$$f_{1,N,\Delta,\varrho,D_0}(z) = \lim_{s\to 0} f_{1,N,\Delta,\varrho,D_0}(z;s),$$

where

$$f_{k,N,\Delta,\varrho,D_0}(z;s) = \sum_{Q \in \mathcal{Q}_{N,\Delta,\varrho}} \frac{\chi_{D_0}(Q)}{Q(z,1)^k} \frac{\mathrm{Im}(z)^s}{|Q(z,1)|^s} \qquad \left(\mathrm{Re}(s) > \frac{1-k}{2}\right);$$

this is then a holomorphic modular form of weight 2 on $\Gamma_0(N)$ and is a cusp form if $D_0 \neq 1$. It follows from P2 of Proposition 1, Chap. I, Sect. 2, that

$$f_{k,N,\Delta,\varrho,D_0} \in M_{2k}(N)^{\operatorname{sign} D_0}$$

for all $k \geqq 1$, where $M_{2k}(N)^\varepsilon$ $(\varepsilon = \pm 1)$ denotes the $(-1)^k\varepsilon$-eigenspace of W_N on $M_{2k}(N)$ or equivalently, the subspace of cusp forms in $S_{2k}(N)$ with sign ε in the functional equation of their L-series.

The functions $f_{k,N,\Delta,\varrho,D_0}$ were introduced (for $N = D_0 = 1$) in [20, Appendix 2] and have been used several times [14, 13, 12] in connection with the Shimura correspondence between modular forms of weight $2k$ and weight $k+\frac{1}{2}$. Their Fourier coefficients are given by the following proposition.

Proposition 1. *The Fourier expansion of* $f_{k,N,\Delta,\varrho,D_0}(z)$ $(k \geqq 1)$ *is given by*

$$f_{k,D,\Delta,\varrho,D_0}(z) = \sum_{m=0}^{\infty} c_{k,N}^{\pm 1}(m,\Delta,\varrho,D_0) e^{2\pi i m z};$$

here $\pm 1 = (-1)^k \cdot \operatorname{sign} D_0$, $c_{k,N}^{\pm 1}(m,\Delta,\varrho,D_0)$ *is* $c_{k,N}(m,\Delta,\varrho,D_0)$ *symmetrized or antisymmetrized with respect to* ϱ *(see Notations), and*

$$c_{k,N}(0,\Delta,\varrho,D_0) = \begin{cases} -i\pi & \text{if } \; k=1, \; D_0=1, \; \Delta=f^2 \; (f>0), \; \varrho \equiv f(2N), \\ 0 & \text{otherwise}, \end{cases}$$

$$c_{k,N}(m,\Delta,\varrho,D_0) = i^k (\operatorname{sign} D_0)^{-1/2} \frac{(2\pi)^k}{(k-1)!} (m^2/\Delta)^{\frac{k-1}{2}}$$
$$\times \Bigg[|D_0|^{-1/2} \varepsilon_N(m,\Delta,\varrho,D_0) + i^k (\operatorname{sign} D_0)^{1/2} \pi \sqrt{2} (m^2/\Delta)^{1/4}$$
$$\times \sum_{a \geqq 1} (Na)^{-1/2} S_{Na}(m,\Delta,\varrho,D_0) J_{k-\frac{1}{2}}\left(\frac{\pi m \sqrt{\Delta}}{Na}\right)\Bigg]$$

for $m>0$, *where* $(\operatorname{sign} D_0)^{\pm 1/2}=1$ *for* $D_0>0$, $\pm i$ *for* $D_0<0$,

$$\varepsilon_N(m,\Delta,\varrho,D_0)=\begin{cases}\left(\dfrac{D_0}{m/f}\right) & \text{if } \Delta=D_0^2f^2 \quad (f>0), \quad f|m, \quad D_0f\equiv\varrho(2N),\\ 0 & \text{otherwise},\end{cases}$$

$$S_{Na}(m,\Delta,\varrho,D_0)=\sum_{\substack{b(2Na)\\ b\equiv\varrho(2N)\\ b^2\equiv\Delta(4Na)}}\chi_{D_0}\left(\left[aN,b,\frac{b^2-\Delta}{4Na}\right]\right)e_{2Na}(mb)$$

and

$$J_{k-1/2}(t)=\sum_{\nu\geqq 0}(-1)^\nu\frac{(t/2)^{k+2\nu-1/2}}{\nu!\,\Gamma(k+\nu+1/2)}$$

is the Bessel function of order $k-\frac{1}{2}$.

The proof is essentially the same as the one given in [20, pp. 44–45], for $N=D_0=1$ and in [12, pp. 246–250], for non-trivial level and character, and will not be repeated here. Notice that our functions are always cusp forms for $D_0\neq 1$, while the functions in [12] were sometimes non-cuspidal for $k=1$ and N not squarefree [12, pp. 249–250]; this is because our character χ_{D_0} is slightly different from the character ω_D in [12] and because of our assumption that both D_0 and Δ/D_0 are squares modulo $4N$.

The second property we need is the relation to *cycle integrals* (or *geodesic periods*) of modular forms. For $f\in S_{2k}(N)$ and $Q=[a,b,c]\in\mathcal{Q}_{N,\Delta,\varrho}$ set

$$r_{k,N,Q}(f)=\int_{\gamma_Q}f(z)Q(z,1)^{k-1}dz,$$

where γ_Q is the image in $\Gamma_0(N)\backslash\mathfrak{H}$ of the semicircle $a|z|^2+bx+c=0$ $(x=\operatorname{Re}(z))$, oriented from $\dfrac{-b-\sqrt{\Delta}}{2a}$ to $\dfrac{-b+\sqrt{\Delta}}{2a}$, if $a\neq 0$ or of the vertical line $bx+c=0$, oriented from $-c/b$ to $i\infty$ if $b>0$ and from $i\infty$ to $-c/b$ if $b<0$, if $a=0$. It is easily checked that this makes sense [i.e. that the integrand is invariant with respect to the subgroup of $\Gamma_0(N)$ preserving Q] and depends only on the $\Gamma_0(N)$-equivalence class of Q. Cycle integrals of this type were first used by Shintani [14] and were studied in detail (for $N=1$) in [13]. Define

$$r_{k,N,\Delta,\varrho,D_0}(f)=\sum_{Q\in\mathcal{Q}_{N,\Delta,\varrho}/\Gamma_0(N)}\chi_{D_0}(Q)r_{k,N,Q}(f).$$

Then we have

Proposition 2. *For* $f\in S_{2k}(N)^{\operatorname{sign} D_0}$,

$$(f,f_{k,N,\Delta,\varrho,D_0})=\pi\binom{2k-2}{k-1}2^{-2k+2}\Delta^{-k+1/2}r_{k,N,\Delta,\varrho,D_0}(f).$$

For a proof (at least in the case when Δ is not a square) see [13, p. 232] (for $N=1$, $k>1$) or [12, pp. 265–266] (for N arbitrary and $k\geqq 1$). The argument used in these two references in fact gives

$$(f, f_{k,N,\Delta,\varrho,D_0}(\cdot\,;\bar{s})) = \frac{\Gamma(\frac{1}{2})\Gamma\left(k+\frac{s-1}{2}\right)}{\Gamma(k+s/2)}\Delta^{-k+\frac{1}{2}-s/2} r_{k,N,\Delta,\varrho,D_0}(f),$$

an identity which will be used in Chap. III.

2. *Poincaré Series for Jacobi Forms*

For the theory of Jacobi forms we refer to [4]. We recall only that a Jacobi cusp form of weight k and index N has a Fourier expansion of the form

$$\phi(\tau,z) = \sum_{\substack{n,r\in\mathbb{Z}\\ 4nN>r^2}} c(n,r)q^n\zeta^r \qquad (\tau\in\mathfrak{H},\ z\in\mathbb{C},\ q=e^{2\pi i\tau},\ \zeta=e^{2\pi iz}),$$

where $c(n,r)$ depends only on r^2-4nN and on the residue class of $r\ (\mathrm{mod}\ 2N)$ and is $(-1)^k$-symmetric under $r\to -r$, and that there is a non-degenerate scalar product (Petersson product) on the space $J_{k,N}^{\mathrm{cusp}}$ of all such forms. Hence for integers n, r with $r^2<4Nn$ there is a unique function $P_{k,N,(n,r)}\in J_{k,N}^{\mathrm{cusp}}$, depending only on r^2-4Nn and on $r\ (\mathrm{mod}\ 2N)$, such that

$$(\phi, P_{k,N,(n,r)}) = \alpha_{k,N}(4Nn-r^2)^{-k+3/2}\cdot(\text{coefficient of } q^n\zeta^r \text{ in } \phi) \tag{1}$$

for all $\phi\in J_{k,N}^{\mathrm{cusp}}$, where $(\cdot,\cdot)$ is the Petersson product and

$$\alpha_{k,N} = \frac{N^{k-2}\Gamma(k-\frac{3}{2})}{2\pi^{k-3/2}}.$$

Proposition. *The Poincaré series $P_{k,N,(n,r)}$ has the expansion*

$$P_{k,N,(n,r)}(\tau,z) = \sum_{\substack{n',r'\in\mathbb{Z}\\ r'^2<4Nn'}} g_{k,N,(n,r)}^{\pm}(n',r')q^{n'}\zeta^{r'}, \tag{2}$$

where $\pm 1=(-1)^k$, $g_{k,N,(n,r)}^{\pm}(n',r')$ is $g_{k,N,(n,r)}(n',r')$ symmetrized or anti-symmetrized with respect to r' (cf. Notations), and

$$g_{k,N,(n,r)}(n',r') = \delta_N(n,r,n',r') + i^k\pi\sqrt{2}N^{-1/2}(D'/D)^{\frac{k}{2}-\frac{3}{4}}$$
$$\times\sum_{c\geqq 1} H_{N,c}(n,r,n',r')J_{k-\frac{3}{2}}\left(\frac{\pi}{Nc}\sqrt{D'D}\right),$$

where

$$D'=r'^2-4Nn', \qquad D=r^2-4Nn,$$

$$\delta_N(n,r,n',r') = \begin{cases} 1 & \text{if} \quad D'=D, \quad r'\equiv r\,(2N),\\ 0 & \text{otherwise},\end{cases}$$

and

$$H_{N,c}(n,r,n',r') = c^{-3/2}\sum_{\substack{\varrho(c)^*\\ \lambda(c)}} e_c((N\lambda^2+r\lambda+n)\varrho^{-1}+n'\varrho+r'\lambda)e_{2Nc}(rr')$$

is a Kloosterman-type sum.

Proof. Let us first suppose $k>2$. We claim that

$$P_{k,N,(n,r)}(\tau,z)=\sum_{\gamma\in\Gamma(1)^J_\infty\backslash\Gamma(1)^J}(e^{n,r}|_{k,N}\gamma)(\tau,z), \tag{3}$$

where $e^{n,r}=e^{2\pi i(n\tau+rz)}$, the operation $|_{k,N}$ is as in [4], and $\Gamma(1)^J_\infty = \left\{\left(\begin{pmatrix}1 & n\\ 0 & 1\end{pmatrix},(0,\mu)\right)\middle| n,\mu\in\mathbb{Z}\right\}$ is the stabilizer of the function $e^{n,r}$ in the full Jacobi group $\Gamma(1)^J=SL_2(\mathbb{Z})\ltimes\mathbb{Z}^2$. Indeed, the sum on the right of (3) converges absolutely and uniformly on compact sets and clearly defines a function in $J^{\mathrm{cusp}}_{k,N}$. By the usual unfolding argument, we see that its Petersson product with an arbitrary $\phi\in J^{\mathrm{cusp}}_{k,N}$ equals

$$\int_{\Gamma(1)^J_\infty\backslash\mathfrak{H}\times\mathbb{C}}\phi(\tau,z)\overline{e^{n,r}(\tau,z)}v^k e^{-4\pi Ny^2/v}v^{-3}dxdydudv \quad (z=x+iy,\ \tau=u+iv).$$

Putting in the Fourier expansion of ϕ and observing that a fundamental domain for the action of $\Gamma(1)^J_\infty$ on $\mathfrak{H}\times\mathbb{C}$ is $([0,\infty)\times[0,1])\times(\mathbb{R}\times[0,1])$, we find that the integral equals

$$\begin{aligned}&\sum_{n',r'}c(n',r')\int_0^\infty\int_0^1\int_{-\infty}^\infty\int_0^1 e^{2\pi i((n'-n)u+(r'-r)x)}e^{-2\pi((n'+n)v+(r'+r)y)}\\ &\qquad\times v^{k-3}e^{-4\pi Ny^2/v}dudvdxdy\\ &\quad=c(n,r)\int_0^\infty e^{-4\pi nv}v^{k-3}\left(\int_{-\infty}^\infty e^{-4\pi(ry+Ny^2/v)}dy\right)dv.\end{aligned}$$

The inner integral equals $\left(\dfrac{4N}{v}\right)^{-1/2}e^{\pi r^2v/N}$, so the double integral equals $|D|^{-k+\frac{3}{2}}\alpha_{k,N}$. This proves our claim.

We now have to compute the Fourier development of the right-hand side of (3). A set of representatives for $\Gamma(1)^J_\infty\backslash\Gamma(1)^J$ is formed by the pairs $\left(\begin{pmatrix}a & b\\ c & d\end{pmatrix},(\lambda a,\lambda b)\right)$ where $\lambda,c,d\in\mathbb{Z}$ with $(c,d)=1$ and for each c,d we have chosen $a,b\in\mathbb{Z}$ with $ad-bc=1$. Hence

$$\begin{aligned}P_{k,N,(n,r)}(\tau,z)=&\sum_{\substack{c,d,\lambda\in\mathbb{Z}\\(c,d)=1}}(c\tau+d)^{-k}e^N\left(\frac{-cz^2}{c\tau+d}+\lambda^2\frac{a\tau+b}{c\tau+d}+2\lambda\frac{z}{c\tau+d}\right)\\ &\times e^n\left(\frac{a\tau+b}{c\tau+d}\right)e^r\left(\frac{z}{c\tau+d}+\lambda\frac{a\tau+b}{c\tau+d}\right).\end{aligned}\tag{4}$$

We split up the sum into the terms with $c=0$ and those with $c\neq0$. If $c=0$, then $d=\pm1$, so these terms give

$$\sum_{\lambda\in\mathbb{Z}}q^{N\lambda^2+r\lambda+n}(\zeta^{2N\lambda+r}\pm\zeta^{-2N\lambda-r})=\sum_{\substack{n',r'\in\mathbb{Z}\\r'^2<4Nn'}}\delta^{\pm}_N(n,r,n',r')q^{n'}\zeta^{r'}\quad(\pm1=(-1)^k).$$

The terms with $c<0$ give $(-1)^k$ times the contribution of the terms with $c>0$, with z replaced by $-z$ (replace a,b,c,d by their negatives), so we need only consider the terms with $c>0$. Using the identities

$$\frac{a\tau+b}{c\tau+d}=\frac{a}{c}-\frac{1}{c(c\tau+d)},\quad \frac{z}{c\tau+d}+\lambda\frac{a\tau+b}{c\tau+d}=\frac{z-\frac{\lambda}{c}}{c\tau+d}+\lambda\frac{a}{c},$$

$$\lambda^2\frac{a\tau+b}{c\tau+d}+2\lambda\frac{z}{c\tau+d}-\frac{cz^2}{c\tau+d}=-\frac{c\left(z-\frac{\lambda}{c}\right)^2}{c\tau+d}+\lambda^2\frac{a}{c},$$

and replacing d,λ by $d+\alpha c$, $\lambda+\beta c$ with the new d and λ running $(\operatorname{mod} c)^*$ and $(\operatorname{mod} c)$, respectively, and $\alpha,\beta\in\mathbb{Z}$, we obtain for these terms the contribution

$$\sum_{\substack{c>0,\,\alpha,\beta\in\mathbb{Z}\\ d(c)^*,\,\lambda(c)}} c^{-k}\left(\tau+\frac{d}{c}+\alpha\right)^{-k} e^N\left(-\frac{\left(\tau-\frac{\lambda}{c}-\beta\right)^2}{\tau+\frac{d}{c}+\alpha}+\lambda^2\frac{a}{c}\right)$$

$$\times e^n\left(\frac{a}{c}-\frac{1}{c^2\left(\tau+\frac{d}{c}+\alpha\right)}\right) e^r\left(\frac{z-\frac{\lambda}{c}-\beta}{c\left(\tau+\frac{d}{c}+\alpha\right)}+\lambda\frac{a}{c}\right)$$

$$=\sum_{c\geqq 1} c^{-k}\sum_{\substack{d(c)^*\\ \lambda(c)}} e_c((N\lambda^2+r\lambda+n)d^{-1})F_{k,N,c,(n,r)}\left(\tau+\frac{d}{c},\,z-\frac{\lambda}{c}\right)$$

with

$$F_{k,N,c,(n,r)}(\tau,z)=\sum_{\alpha,\beta\in\mathbb{Z}}(\tau+\alpha)^{-k}e^N\left(-\frac{(z-\beta)^2}{\tau+\alpha}\right)e^n\left(\frac{-1}{c^2(\tau+\alpha)}\right)e^r\left(\frac{z-\beta}{c(\tau+\alpha)}\right).$$

The Poisson summation formula gives

$$F_{k,N,c,(n,r)}(\tau,z)=\sum_{n',r'\in\mathbb{Z}}\gamma(n',r')q^{n'}\zeta^{r'}$$

with

$$\gamma(n',r')=\int_{C_1-i\infty}^{C_1+i\infty}\tau^{-k}e(-n'\tau)\int_{C_2-i\infty}^{C_2+i\infty}e\left(-\frac{N}{\tau}z^2+\frac{rz}{c\tau}-\frac{n}{c^2\tau}-r'z\right)dz\,d\tau$$

$$(C_1>0,\ C_2\in\mathbb{R}).$$

We substitute $z\to z+\frac{1}{2N}\left(\frac{r}{c}-r'\tau\right)$. Then the inner integral becomes

$$e\left(\frac{-rr'}{2Nc}\right)e\left(\frac{D'}{4N}\tau+\frac{D}{4Nc}\tau^{-1}\right)e(n'\tau)\int_{C_2'-i\infty}^{C_2'+i\infty}e\left(-\frac{N}{\tau}z^2\right)dz.$$

The latter integral is standard and equals $\left(\frac{\tau}{2iN}\right)^{1/2}$. Hence we find

$$\gamma(n',r')=(2N)^{-1/2}e\left(\frac{-rr'}{2Nc}\right)\int_{C_1-i\infty}^{C_1+i\infty}(\tau/i)^{1/2}\tau^{-k}e\left(\frac{D'}{4N}\tau+\frac{D}{4Nc}\tau^{-1}\right)d\tau.$$

If $D' \geqq 0$ we can deform the path of integration up to $i\infty$, so $\gamma(n', r') = 0$ in this case. For $D' < 0$ we make the substitution $\tau = ic^{-1}(D/D')^{1/2}s$ to get

$$\gamma(n', r') = 2\pi(2N)^{-1/2} e\left(\frac{-rr'}{2Nc}\right) i^{-k} c^{k-\frac{3}{2}} (D'/D)^{\frac{k}{2}-\frac{3}{4}}$$

$$\times \frac{1}{2\pi i} \int_{C_1'-i\infty}^{C_1'+i\infty} s^{-k+1/2} e^{\frac{2\pi}{4Nc}(D'D)^{1/2}(s-s^{-1})} ds \qquad (C_1' > 0).$$

The function $t \mapsto (t/\mu)^{k-\frac{3}{4}} J_{k-\frac{3}{2}}(2\sqrt{\mu t})$ is the inverse Laplace transform of $s \mapsto s^{-k-\frac{1}{2}} e^{-\mu s}$ [1, 29.3.80], so the integral equals $2\pi i J_{k-\frac{3}{2}}\left(\frac{\pi}{Nc}(D'D)^{1/2}\right)$. From this formula (2) (for $k > 2$) follows immediately.

If $k = 2$, then the series in (4) does not converge absolutely. By Hecke's "convergence trick" we define $P_{2,N,(n,r)}(\tau, z; s)$ for $\mathrm{Re}(s) > 0$ as the series in (4) with $(c\tau + d)^{-k}$ replaced by $(c\tau + d)^{-2}|c\tau + d|^{-2s}$. One can then easily compute the Fourier expansion of this function in the same manner as above. From the Fourier expansion one shows immediately that $P_{2,N,(n,r)}(\tau, z; s)$ has a holomorphic continuation to $s = 0$ and that its limiting value as $s \to 0$ is holomorphic in z and has the expansion given by (2) with $k = 2$. Moreover, it is easily checked that the Jacobi cusp form $P_{2,N,(n,r)}$ defined by the property (1) is equal to this limit. The details are standard and will be left to the reader. This completes the proof of the proposition.

3. *Lifting Maps*

Let $r_0 \in \mathbb{Z}$ and D_0 a negative fundamental discriminant with $D_0 = r_0^2 - 4Nn_0$. For $\phi \in J_{k+1,N}^{\mathrm{cusp}}$ we set

$$\mathscr{S}_{D_0,r_0}(\phi)(w) = \sum_{n=1}^{\infty} \left(\sum_{d|n} \left(\frac{D_0}{d}\right) d^{k-1} c\left(\frac{n^2}{d^2} n_0, \frac{n}{d} r_0\right)\right) e^{2\pi i n w} \quad (w \in \mathfrak{H}), \qquad (1)$$

where $c(n, r)$ is the coefficient of $q^n\zeta^r$ in ϕ as in Sect. 2. It was proved in [15] by means of a trace formula that $\mathscr{S}_{D_0,r_0}$ maps $J_{k+1,N}^{\mathrm{cusp}}$ into $S_{2k}(N)^-$, preserves newforms, and commutes with the action of Hecke operators. [The coefficient of $e^{2\pi i n w}$ in (1) is just the coefficient of $q^{n_0}\zeta^{r_0}$ in $\phi|T_n$.] We now prove the main result of this chapter, a formula for the adjoint map of $\mathscr{S}_{D_0,r_0}$ in terms of the cycle integrals of Sect. 1.

Theorem. *For $f \in S_{2k}(N)^-$, the function*

$$\mathscr{S}_{D_0,r_0}^*(f)(\tau, z) = \left(\frac{i}{2N}\right)^{k-1} \sum_{\substack{n, r \in \mathbb{Z} \\ r^2 < 4nN}} r_{k,N,D_0(r^2-4nN),r_0 r,D_0}(f) q^n \zeta^r$$

$$(\tau \in \mathfrak{H},\ z \in \mathbb{C},\ q = e^{2\pi i \tau},\ \zeta = e^{2\pi i z}) \qquad (2)$$

is a Jacobi cusp form of weight $k+1$ and index N. The maps $\mathscr{S}_{D_0,r_0}: J_{k+1,N}^{\mathrm{cusp}} \to S_{2k}(N)^-$ and $\mathscr{S}_{D_0,r_0}^*: S_{2k}(N)^- \to J_{k+1,N}^{\mathrm{cusp}}$ are adjoint maps with respect to the Petersson scalar products, i.e., $(\mathscr{S}_{D_0,r_0}(\phi), f) = (\phi, \mathscr{S}_{D_0,r_0}^*(f))$ for all $f \in S_{2k}(N)^-$ and $\phi \in J_{k+1,N}^{\mathrm{cusp}}$.

Proof. Set

$$\Omega_{k,N,D_0,r_0}(w;\tau,z)$$
$$=c_{k,N,D_0}\sum_{\substack{n,r\in\mathbb{Z}\\ r^2<4nN}}(4nN-r^2)^{k-1/2}f_{k,N,D_0(r^2-4nN),r_0r,D_0}(w)q^n\zeta^r,$$

where $f_{k,N,\Delta,\varrho,D_0}$ is the function defined in Sect. 1 and

$$c_{k,N,D_0}=\frac{(-2i)^{k-1}|D_0|^{k-1/2}}{N^{k-1}\pi\binom{2k-2}{k-1}}.$$

It follows from Proposition 2 of Sect. 1 that

$$\mathscr{S}^*_{D_0,r_0}(f)(\tau,z)=(f,\Omega_{k,N,D_0,r_0}(\cdot\,;-\bar\tau,-\bar z))\quad\forall f\in S_{2k}(N)^-,$$

where $(\ ,\ )$ denotes the Petersson scalar product in $S_{2k}(N)^-$. Hence to prove the first assertion of the theorem it suffices to show that Ω_{k,N,D_0,r_0} is a Jacobi cusp form (of weight $k+1$ and index N) with respect to (τ,z), and to prove the second it suffices to show that Ω is the kernel function for the map $\mathscr{S}_{D_0,r_0}$, i.e., that

$$\mathscr{S}_{D_0,r_0}(\phi)(w)=(\phi,\Omega_{k,N,D_0,r_0}(-\bar w;\cdot,\cdot))\quad\forall\phi\in J^{\mathrm{cusp}}_{k+1,N},$$

where now $(\ ,\ )$ denotes the Petersson scalar product in $J^{\mathrm{cusp}}_{k+1,N}$. In view of the definition of $\mathscr{S}_{D_0,r_0}$, the defining equation of Jacobi-Poincaré series [(1) of Sect. 2], and the fact that $\Omega_{k,N,D_0,r_0}(-\bar\tau,-\bar z;w)=\overline{\Omega_{k,N,D_0,r_0}(\tau,z;-\bar w)}$, this is clearly equivalent to the following basic identity:

$$\Omega_{k,N,D_0,r_0}(w;\tau,z)=c_{k,N,D_0}\frac{i^{k-1}(2\pi)^k}{(k-1)!}$$
$$\times\sum_{m=1}^{\infty}m^{k-1}\left(\sum_{dd'=m}\left(\frac{D_0}{d}\right)d'^kP_{k+1,N,(n_0d'^2,r_0d')}(\tau,z)\right)e^{2\pi imw}.\quad(3)$$

We shall prove this identity by the method of [20], i.e., we expand both sides in a double Fourier series and then compare Fourier coefficients. The Fourier developments of $f_{k,N,\Delta,\varrho,D_0}$ and $P_{k+1,N,(n,r)}$ were given in Proposition 1 of Sect. 1 and the Proposition of Sect. 2, respectively. Inserting them into (3), we see that the identity we have to show is

$$i^{k-1}\frac{(2\pi)^k}{(k-1)!}|D|^{k-1/2}(m^2/D_0D)^{\frac{k-1}{2}}\Bigg(|D_0|^{-1/2}\varepsilon_N^{\pm}(m,DD_0,rr_0,D_0)$$
$$+i^{k+1}\pi\sqrt{2}(m^2/D_0D)^{1/4}$$
$$\times\sum_{a\geqq1}(Na)^{-1/2}S^{\pm}_{Na}(m,DD_0,rr_0,D_0)J_{k-1/2}\left(\frac{\pi m}{Na}\sqrt{D_0D}\right)\Bigg)$$
$$=i^{k-1}\frac{(2\pi)^k}{(k-1)!}m^{k-1}\sum_{d|m}\left(\frac{D_0}{d}\right)(m/d)^k\Bigg(\delta_N^{\pm}\left(\frac{m^2}{d^2}n_0,\frac{m}{d}r_0,n,r\right)$$
$$+i^{k+1}\pi\sqrt{2}N^{-1/2}\left(D\Big/\frac{m^2}{d^2}D_0\right)^{k/2-1/4}$$
$$\times\sum_{c\geqq1}H^{\pm}_{N,c}\left(\frac{m^2}{d^2}n_0,\frac{m}{d}r_0,n,r\right)J_{k-1/2}\left(\frac{\pi}{Nc}\sqrt{\frac{m^2}{d^2}D_0D}\right)\Bigg)$$

or equivalently

$$m^{k-1}(D_0/D)^{k/2}\varepsilon_N^{\pm}(m, DD_0, rr_0, D_0)+i^{k+1}(D/D_0)^{k/2-1/4}m^{k-1/2}\pi\sqrt{2}N^{-1/2}$$
$$\times \sum_{a\geqq 1} a^{-1/2}S_{Na}^{\pm}(m, DD_0, rr_0, D_0)J_{k-1/2}\left(\frac{\pi m}{Na}\sqrt{D_0D}\right)$$
$$=m^{k-1}\sum_{d|m}\left(\frac{D_0}{d}\right)(m/d)^k\delta_N^{\pm}\left(\frac{m^2}{d^2}n_0, \frac{m}{d}r_0, n, r\right)$$
$$+i^{k+1}(D/D_0)^{k/2-1/4}m^{k-1/2}\pi\sqrt{2}N^{-1/2}$$
$$\times\sum_{d|m}\left(\frac{D_0}{d}\right)d^{-1/2}\sum_{c\geqq 1}H_{N,c}^{\pm}\left(\frac{m^2}{d^2}n_0, \frac{m}{d}r_0, n, r\right)J_{k-1/2}\left(\frac{\pi}{Nc}\frac{m}{d}\sqrt{D_0D}\right) \tag{4}$$

for all $m\geqq 1$, $n\geqq 0$, $r\in\mathbb{Z}$ with $D=r^2-4Nn<0$.

We first show that the first terms of both sides of (4) agree. For this it is sufficient to show that

$$(D/D_0)^{k/2}\varepsilon_N(m, DD_0, rr_0, D_0)=\sum_{d|m}\left(\frac{D_0}{d}\right)(m/d)^k\delta_N\left(\frac{m^2}{d^2}n_0, \frac{m}{d}r_0, n, r\right). \tag{5}$$

The left-hand side of (5) is zero unless $D=D_0f^2$ for some $f\in\mathbb{N}$ with $f|m$ and $r_0^2f\equiv r_0r \pmod{2N}$, in which case it equals $\left(\frac{D_0}{m/f}\right)f^k$. By definition, the right-hand side is zero unless $D=D_0f^2$ with $f\in\mathbb{N}$, $f|m$ and $r\equiv r_0f\,(2N)$, in which case (with $d=m/f$) it also equals $\left(\frac{D_0}{m/f}\right)f^k$. Hence we must show that under the condition $D=D_0f^2$ the congruence

$$r\equiv r_0f \pmod{2N} \tag{6}$$

follows from the congruence

$$rr_0\equiv r_0^2f \pmod{2N}. \tag{7}$$

Let $t=(r_0, 2N)$. Since D_0 is fundamental, t is a product of different primes each of which exactly divides N, and from $D=D_0f^2$ it follows that $t|r$, so $r\equiv 0\equiv r_0f\,(t)$. Hence (7) implies (6).

In the second term on the right-hand side of (4) we substitute $cd=a$ to get

$$i^{k+1}(D/D_0)^{k/2-1/4}m^{k-1/2}\pi\sqrt{2}N^{-1/2}$$
$$\times\sum_{a\geqq 1}\sum_{d|(a,m)}\left(\frac{D_0}{d}\right)d^{-1/2}H_{N,a/d}^{\pm}\left(\frac{m^2}{d^2}n_0, \frac{m}{d}r_0, n, r\right)J_{k-1/2}\left(\frac{\pi m}{Na}\sqrt{DD_0}\right).$$

Hence for the proof of (3) it suffices to show the following

Lemma. *For all $m\geqq 1$, $n\geqq 0$, $r\in\mathbb{Z}$ with $D=r^2-4Nn<0$ we have*

$$S_{Na}(m, DD_0, rr_0, D_0)=\sum_{d|(a,m)}\left(\frac{D_0}{d}\right)(a/d)^{1/2}H_{N,a/d}\left(\frac{m^2}{d^2}n_0, \frac{m}{d}r_0, n, r\right). \tag{8}$$

Proof. If we put in the definitions of S_{Na} and $H_{N,c}$ and multiply both sides of (8) with $e_{2Na}(-mr_0r)$, then the identity to be proved becomes

$$\sum_{\substack{b(2aN)\\ b\equiv rr_0(2N)\\ b^2\equiv DD_0(4aN)}} \chi_{D_0}\left(\left[aN,b,\frac{b^2-D_0D}{4Na}\right]\right)e_a\left(\frac{b-r_0r}{2N}m\right)$$

$$=a^{-1}\sum_{d|(a,m)}\left(\frac{D_0}{d}\right)d\sum_{\substack{\varrho(a/d)^*\\ \lambda(a/d)}} e_{a/d}\left(\left(N\lambda^2+\frac{m}{d}r_0\lambda+\frac{m^2}{d^2}n_0\right)\varrho^{-1}+n\varrho+r\lambda\right).$$

As functions of m both sides of this are periodic with period a, so it will be sufficient to show that their Fourier transforms are equal. Hence we must show that for every $h'\in\mathbb{Z}/a\mathbb{Z}$ we have

$$\frac{1}{a}\sum_{\substack{b(2aN)\\ b\equiv rr_0(2N)\\ b^2\equiv DD_0(4aN)}}\sum_{m(a)}\chi_{D_0}\left(\left[aN,b,\frac{b^2-DD_0}{4aN}\right]\right)e_a\left(\left(\frac{b-rr_0}{2N}-h'\right)m\right)$$

$$=\frac{1}{a^2}\sum_{m(a)}\sum_{d|(a,m)}\left(\frac{D_0}{d}\right)d$$

$$\times\sum_{\substack{\varrho(a/d)^*\\ \lambda(a/d)}} e_{a/d}\left(\left(N\lambda^2+\frac{m}{d}r_0\lambda+\frac{m^2}{d^2}n_0\right)\varrho^{-1}+n\varrho+r\lambda-h'\frac{m}{d}\right).$$

Set $h=2Nh'+r_0r$. Then the expression on the left is easily seen to equal $\chi_{D_0}\left(\left[Na,h,\frac{h^2-D_0D}{4Na}\right]\right)$ or 0 according as $h^2\equiv D_0D\,(4Na)$ or not, while for the right-hand side we obtain after replacing m by md and then (λ,m) by $(\varrho\lambda,\varrho m)$ the expression

$$a^{-1}\sum_{d|a}\left(\frac{D_0}{d}\right)\frac{1}{a/d}\sum_{\substack{\varrho(a/d)^*\\ \lambda,m(a/d)}} e_{a/d}(\varrho(N\lambda^2+r_0m\lambda+n_0m^2+r\lambda-h'm+n)).$$

Thus Eq. (8) is equivalent to Proposition 2 of Sect. 2, Chap. I.

This completes the proof of the theorem.

4. *Cycle Integrals and the Coefficients of Jacobi Forms*

In [15] it was proved that the subspaces of newforms in $J^{\mathrm{cusp}}_{k+1,N}$ and $S_{2k}(N)^-$ are isomorphic as Hecke modules. Let $f\in S_{2k}(N)^-$ be a normalized newform and $\phi\in J^{\mathrm{cusp}}_{k+1,N}$ a non-zero Jacobi form having the same Hecke eigenvalues. The following statement is then a formal consequence of the theorem of Sect. 3.

Theorem. *Let $D_0=r^2-4Nn_0<0$ be a fundamental discriminant. Then for all $n,r\in\mathbb{Z}$ with $D=r^2-4Nn<0$ we have*

$$\frac{c(n,r)\overline{c(n_0,r_0)}}{(\phi,\phi)}=\left(\frac{i}{2N}\right)^{k-1}\frac{r_{k,N,DD_0,rr_0,D_0}(f)}{(f,f)},\tag{1}$$

where (,) denotes the Petersson scalar product and $r_{k,N,DD_0,rr_0,D_0}(f)$ is the cycle integral defined in Sect. 1.

Note that the left-hand side of (1) is independent of the choice of ϕ, since replacing ϕ by $\mu\phi$ changes both the numerator and the denominator by $|\mu|^2$. We can always choose ϕ to have real coefficients; then the bar in (1) can be omitted.

We remark that a formula analogous to (1) for Fourier coefficients of modular forms of half-integral weight on $\Gamma_0(4N)$ with N odd and squarefree was given in [12].

Proof. By the strong multiplicity theorem quoted above and the fact that the lifting map $\mathcal{S}_{D_0,r_0}$ commutes with Hecke operators, we know that $\mathcal{S}_{D_0,r_0}(\phi)$ is a multiple of f. Comparing the coefficients of q^1 in these two forms we obtain

$$\mathcal{S}_{D_0,r_0}(\phi)=c(n_0,r_0)f\,. \tag{2}$$

The same multiplicity 1 theorem implies that

$$\mathcal{S}^*_{D_0,r_0}(f)=\lambda\phi$$

for some $\lambda\in\mathbb{C}$. By the theorem of the last section we have

$$\begin{aligned}\lambda c(n,r)&=\text{coefficient of } q^n\zeta^r \text{ in } \mathcal{S}^*_{D_0,r_0}(f)\\&=\left(\frac{i}{2N}\right)^{k-1} r_{k,N,DD_0,rr_0,D_0}(f)\,.\end{aligned}$$

On the other hand,

$$\begin{aligned}\lambda c(n,r)(\phi,\phi)&=c(n,r)(\mathcal{S}^*_{D_0,r_0}(f),\phi)\\&=c(n,r)(f,\mathcal{S}_{D_0,r_0}(\phi))\\&=c(n,r)\overline{c(n_0,r_0)}(f,f)\,,\end{aligned}$$

where in the last line we have used (2). Comparing the two formulas we obtain (1).

For a fundamental discriminant D with $(D,N)=1$ we denote by

$$L(f,D,s)=\sum_{n=1}^{\infty}\left(\frac{D}{n}\right)a_n n^{-s}\qquad(\mathrm{Re}(s)\gg 0)\,,$$

the L-series of $f=\sum a_n q^n$ twisted by the quadratic character $\left(\frac{D}{\cdot}\right)$. Recall that $L(f,D,s)$ has a holomorphic continuation to $\mathbb{C}$ and satisfies the functional equation

$$\begin{aligned}L^*(f,D,s)&:=(2\pi)^{-s}(ND^2)^{s/2}\Gamma(s)L(f,D,s)\\&=(-1)^k\left(\frac{D}{-N}\right)w_f L^*(f,D,2k-s)\,,\end{aligned}$$

where w_f is the eigenvalue of f under the Fricke involution w_N. Since by assumption f lies in the subspace $S_{2k}(N)^-$ we have $w_f=(-1)^k$, so for $D<0$ and D a square modulo $4N$ we have

$$L^*(f,D,s)=L^*(f,D,2k-s)\,.$$

By setting $(n,r)=(n_0,r_0)$ in (1) we obtain as in [12] a refinement of a result of Waldspurger [18] about the value of the twisted L-series at the central point:

Corollary 1. *Let $D=r^2-4Nn<0$ be fundamental with $(D,N)=1$. Then*

$$\frac{|c(n,r)|^2}{(\phi,\phi)}=\frac{(k-1)!}{2^{2k-1}\pi^k N^{k-1}}|D|^{k-1/2}\frac{L(f,D,k)}{(f,f)}. \tag{3}$$

Remark. The power of 2 in the formula in Corollary 4, p. 67 of [4] is given incorrectly.

Proof. By (1), we have

$$\frac{|c(n,r)|^2}{(\phi,\phi)}=\left(\frac{i}{2N}\right)^{k-1} r_{k,N,D^2,r^2,D}(f)/(f,f)$$

with

$$r_{k,N,D^2,r^2,D}(f)=\sum_{Q\in\mathcal{Q}_{N,D^2,r^2}/\Gamma_0(N)}\chi_D(Q)\int_{\gamma_Q} f(w)Q(w,1)^{k-1}dw.$$

A set of $\Gamma_0(N)$-representatives of quadratic forms $Q=[aN,b,c]$ with $b\equiv r^2\,(2N)$ and discriminant D^2 is given by $\{[0,D,\mu]\,|\,\mu\,(\mathrm{mod}\,D)\}$. Hence

$$\begin{aligned} r_{k,N,D^2,r^2,D}(f)&=\sum_{\mu(D)}\left(\frac{D}{\mu}\right)\int_{i\infty}^{-\mu/D} f(w)(Dw+\mu)^{k-1}dw\\ &=-i^kD^{k-1}\int_0^\infty\sum_{\mu(D)}\left(\frac{D}{\mu}\right)f\left(it+\frac{\mu}{|D|}\right)t^{k-1}dt\\ &=i^{-k+1}|D|^{k-1/2}\int_0^\infty\sum_{n=1}^\infty\left(\frac{D}{n}\right)a_ne^{-2\pi nt}t^{k-1}dt\\ &=i^{-k+1}|D|^{k-1/2}(2\pi)^{-k}\Gamma(k)L(f,D,k), \end{aligned}$$

where in the last line we have used analytic continuation. Equation (3) follows from this.

As in [12] we obtain from (3) that $L(f,D,k)\geqq 0$. One can also deduce that (ϕ,ϕ) is an algebraic multiple of one of the periods ω_+,ω_- associated to f.

Finally (as in [12]), by squaring both sides of (1), taking absolute values, and then applying (3) we obtain

Corollary 2. *Let $D_0=r_0^2-4Nn_0<0$, $D=r^2-4Nn<0$ be two negative fundamental discriminants prime to N. Then*

$$(D_0D)^{k-1/2}L(f,D_0,k)L(f,D,k)=\frac{(2\pi)^{2k}}{(k-1)!^2}|r_{k,N,DD_0,rr_0,D_0}(f)|^2. \tag{4}$$

A formula similar to that in Corollary 2 has been independently proved by Waldspurger [19], by different methods and in much greater generality.

From (3) and (4) we also obtain growth estimates for $L(f,D,k)$ and $c(n,r)$, as in [12].

III. A Modular Form Related to $L'(f,k)$

Let $N\geqq 1$ and $\Delta>0$ a discriminant which is a product of two negative discriminants D_0,D_1 which are squares $(\mathrm{mod}\,4N)$, ϱ an integer $(\mathrm{mod}\,2N)$ with

$\varrho^2 \equiv \Delta(4N)$. In Chap. II we constructed for each $k \geq 1$ a function $f_{k,N,\Delta,\varrho,D_0}$ in $S_{2k}(N)^-$ [the space of cusp forms of weight $2k$ on $\Gamma_0(N)$ whose L-series have a minus sign in their functional equation] such that for all $f \in S_{2k}(N)^-$ the scalar product $(f, f_{k,N,\Delta,\varrho,D_0})$ equals, up to a simple factor, the cycle integral $r_{k,N,\Delta,\varrho,D_0}(f)$. In this chapter we will construct another function $F = F_{k,N,\Delta,\varrho,D_0} \in S_{2k}(N)^-$ such that the scalar product (f, F) for $f \in S_{2k}(N)^-$ is (again up to a simple factor) the product of $r_{k,N,\Delta,\varrho,D_0}(f)$ with $L'(f, k)$, the derivative of the L-series of f at the symmetry point of its functional equation. The construction starts with a non-holomorphic weight 1 Eisenstein series for the Hilbert modular group of $\mathbb{Q}(\sqrt{\Delta})$ and makes use of a differential operator of H. Cohen which maps Hilbert modular forms to ordinary modular forms. We also compute the Fourier coefficients of F. They turn out to be the sum of two terms – an infinite sum involving Legendre functions of the second kind and a finite sum involving Legendre functions of the first kind. These expressions will be given an arithmetic interpretation in Chap. IV.

We will suppose that Δ is a fundamental discriminant (i.e., that D_0 and D_1 are fundamental and coprime); this simplifies the calculations considerably.

1. Construction of the Modular Form F

Associated to the discriminant Δ and the decomposition $\Delta = D_0 D_1$ we have the real quadratic field $K = \mathbb{Q}(\sqrt{\Delta})$ of discriminant Δ and the genus character $\chi: C_K \to \{\pm 1\}$ (C_K = narrow ideal class group of K), defined by the property $\chi(\mathfrak{a}) = \left(\frac{D_i}{N\mathfrak{a}}\right)$ if $\mathfrak{a}$ is an integral ideal prime to D_i and $N\mathfrak{a}$ its norm. [This is well-defined since every integral ideal $\mathfrak{a}$ splits as $\mathfrak{a}_0\mathfrak{a}_1$ with $\mathfrak{a}_i$ prime to D_i and since $\left(\frac{D_0}{N\mathfrak{a}}\right) = \left(\frac{D_1}{N\mathfrak{a}}\right)$ if $\mathfrak{a}$ is an ideal prime to Δ.] The residue class $\varrho \pmod{2N}$ with $\varrho^2 \equiv \Delta(4N)$ corresponds to a primitive integral ideal

$$\mathfrak{n} = \mathbb{Z}N + \mathbb{Z}\frac{\varrho + \sqrt{\Delta}}{2} \subset \mathcal{O}_K = \mathbb{Z} + \mathbb{Z}\frac{\varrho + \sqrt{\Delta}}{2}$$

of norm N. For $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 0$ we have the Eisenstein series

$$\begin{aligned} E_s(z, z') &= E_{K,\chi,1,s,\mathfrak{n}}(z, z') \\ &= \sum_{[\mathfrak{a}]} \chi(\mathfrak{a}) N(\mathfrak{a})^{1+2s} \sum_{m,n}{}' \frac{y^s y'^s}{(mz+n)(m'z'+n')|mz+n|^{2s}|m'z'+n'|^{2s}} \\ &\quad (z = x+iy,\ z' = x'+iy' \in \mathfrak{H}), \end{aligned} \tag{1}$$

where the first sum runs over the wide ideal classes of K and the second over all non-zero pairs $(m, n) \in (\mathfrak{n}\mathfrak{a} \times \mathfrak{a})/\mathcal{O}_K^\times$, and the prime on m, n denotes conjugation in $K/\mathbb{Q}$. This function is non-holomorphic in z, z' but transforms like a Hilbert modular form of weight 1 with respect to the congruence subgroup $\Gamma_0(\mathfrak{n})$ of $SL_2(\mathcal{O}_K)$, i.e.,

$$E_s\left(\frac{az+b}{cz+d}, \frac{a'z'+b'}{c'z'+d'}\right) = (cz+d)(c'z'+d')E_s(z, z')$$

for $a, b, d \in \mathcal{O}_K$, $c \in \mathfrak{n}$, $ad - bc = 1$. The Eisenstein series E_s is known to have an analytic continuation in s. Our assumptions on D_0, D_1 and N imply $\chi(\mathfrak{n}) = 1$ [in fact $\chi(\mathfrak{p}) = 1$ for all prime ideals $\mathfrak{p}$ dividing $\mathfrak{n}$], and this in turn implies that E_s vanishes identically at $s = 0$ (cf. [8, p. 214], where the case $N = 1$ is discussed in more detail). We set

$$E(z, z') = \frac{\partial}{\partial s} E_s(z, z')|_{s=0}.$$

This is again a non-holomorphic function of z and z' that transforms under $\Gamma_0(\mathfrak{n})$ like a holomorphic Hilbert modular form of weight 1.

In [2] Cohen showed how to map Hilbert modular forms to ordinary modular forms by first applying a suitable differential operator and then restricting to the diagonal $\mathfrak{H} \subset \mathfrak{H} \times \mathfrak{H}$. A special case of this construction is that the operator

$$(\mathscr{C}_{k-1}\Phi)(z) = \sum_{\ell=0}^{k-1} (-1)^\ell \binom{k-1}{\ell}^2 \frac{\partial^{k-1}\Phi(z, z')}{\partial z^\ell \partial z'^{k-1-\ell}}\bigg|_{z=z'}$$

sends Hilbert modular forms Φ of weight 1 on $\Gamma_0(\mathfrak{n})$ to ordinary modular forms of weight $2k$ on $\Gamma_0(N)$. This is true even if Φ is not holomorphic. Hence the function

$$\tilde{F}(z) = \tilde{F}_{k, N, \Delta, \varrho, D_0}(z) = \frac{1}{(2\pi i)^{k-1}} \frac{\Delta^{k/2}}{8\pi^2} (\mathscr{C}_{k-1}E)(z) \qquad (z \in \mathfrak{H})$$

transforms like a modular form of weight $2k$ on $\Gamma_0(N)$. Finally, we define

$$F(z) = F_{k, N, \Delta, \varrho, D_0}(z) = \pi_{\mathrm{hol}}(\tilde{F}),$$

the holomorphic projection of $\tilde{F}$, i.e., the unique form in $S_{2k}(N)$ such that $(f, F) = (f, \tilde{F})$ for all $f \in S_{2k}(N)$ (cf. [16] or [9]).

It is easily checked that replacing $E_s(z, z')$ by $(Nzz')^{-1} E_s\left(\frac{-1}{Nz}, \frac{-1}{Nz'}\right)$ has the same effect as replacing $\mathfrak{n}$ by $\mathfrak{n}'$ or as interchanging z and z'. This implies that

$$F_{k, N, \Delta, \varrho, D_0}|_{2k} W_N = F_{k, N, \Delta, -\varrho, D_0} = (-1)^{k-1} F_{k, N, \Delta, \varrho, D_0}.$$

Hence F lies in the $(-1)^{k-1}$-eigenspace of W_N, which is $S_{2k}(N)^-$.

2. *Fourier Expansion of F*

The construction described in Sect. 1 was given for $k = N = 1$ in [8, p. 214], and the Fourier development of F was computed there in that case. Of course, since $S_2(1) = \{0\}$, the final step in the argument in [8] was that the expression obtained for the m^{th} Fourier coefficient of F had to be equal to zero for all m, a conclusion which no longer holds here. However, most of the rest of the computation generalizes, so we shall be relatively brief in our presentation and emphasize only those aspects which are new for the case of forms of higher weight and level.

Write the Fourier development of F as $\sum_{m=1}^{\infty} a_m e^{2\pi i m z}$. The formula for a_m is somewhat different according as $k > 1$ or $k = 1$.

Proposition 1. *Suppose $k>1$. Then for all $m \geqq 1$,*

$$a_m=(m\sqrt{\Delta})^{k-1}\left\{\sum_{\substack{|n|<m\sqrt{\Delta}\\ n\equiv m\varrho(2N)}} \sigma'_\chi\left(\left(\frac{n+m\sqrt{\Delta}}{2}\right)\mathfrak{n}^{-1}\right)P_{k-1}\left(\frac{n}{m\sqrt{\Delta}}\right)\right.$$
$$-\sum_{\substack{n>m\sqrt{\Delta}\\ n\equiv m\varrho(2N)}} \sigma_{0,\chi}\left(\left(\frac{n+m\sqrt{\Delta}}{2}\right)\mathfrak{n}^{-1}\right)Q_{k-1}\left(\frac{n}{m\sqrt{\Delta}}\right)$$
$$\left.-\sum_{\substack{n>m\sqrt{\Delta}\\ n\equiv -m\varrho(2N)}} \sigma_{0,\chi}\left(\left(\frac{n+m\sqrt{\Delta}}{2}\right)\mathfrak{n}'^{-1}\right)Q_{k-1}\left(\frac{n}{m\sqrt{\Delta}}\right)\right\},$$

where $\sigma_{0,\chi}(\mathfrak{a})$ and $\sigma'_\chi(\mathfrak{a})$ for $\mathfrak{a}$ an integral ideal of K denote the value and derivative, respectively, of

$$\sigma_{s,\chi}(\mathfrak{a})=\sum_{\mathfrak{b}|\mathfrak{a}} \chi(\mathfrak{b})N(\mathfrak{b})^s$$

at $s=0$ and $P_{k-1}(t)$ and $Q_{k-1}(t)$ are Legendre functions of the first and second kinds.

We recall that $P_{k-1}(t)$ is a polynomial of degree $k-1$ and $Q_{k-1}(t)$ $(t>1)$ a function satisfying

$$Q_{k-1}(t)=\frac{1}{2}P_{k-1}(t)\log\frac{t+1}{t-1}+(\text{polynomial of degree } k-2)$$

and

$$Q_{k-1}(t)=O(t^{-k}) \quad \text{as} \quad t\to\infty;$$

these properties characterize P_{k-1} and Q_{k-1} uniquely up to a scalar multiple, which is fixed by the normalization $P_{k-1}(1)=1$. The first few values are

$$P_0(t)=1,\quad P_1(t)=t,\quad P_2(t)=\tfrac{1}{2}(3t^2-1),\quad P_3(t)=\tfrac{1}{2}(5t^3-3t),$$
$$Q_0(t)=\frac{1}{2}\log\frac{t+1}{t-1},\quad Q_1(t)=\frac{t}{2}\log\frac{t+1}{t-1}-1,\quad Q_2(t)=\frac{3t^2-1}{4}\log\frac{t+1}{t-1}-\frac{3t}{2}.$$

The result for $k=1$ is more complicated because of the different properties of the holomorphic projection operator π_{hol} in this case and because the infinite series in Proposition 1 now diverge. To state it, we need the function $Q_{s-1}(t)$ ($s\in\mathbb{C}$, $\mathrm{Re}(s)>0$) for non-integral s; it is defined by

$$Q_{s-1}(t)=\int_0^\infty (t+\sqrt{t^2-1}\cosh v)^{-s}dv=\frac{\Gamma(s)^2}{2\Gamma(2s)}\left(\frac{1+t}{2}\right)^{-s}F\left(s,s;2s;\frac{2}{1+t}\right),$$

where $F(a,b;c;z)$ is Gauss's hypergeometric function. We also use the notation $\sigma(m)$ for the sum of the positive divisors of $m\in\mathbb{N}$ and $H(D_i)$ $(i=0,1)$ for the value at $s=0$ of the L-series $L\left(s,\left(\frac{D_i}{\cdot}\right)\right)$; this equals $\frac{1}{2}$ or $\frac{1}{3}$ for $D_i=-4$ or -3 and $h(D_i)$, the ideal class number of $\mathbb{Q}(\sqrt{D_i})$, otherwise. With these notations, we can state:

Proposition 2. *Suppose $k=1$. Then for all $m\geqq 1$ prime to N,*

$$a_m = \sum_{\substack{|n|<m\sqrt{\Delta}\\ n\equiv m\varrho(2N)}} \sigma'_\chi\left(\left(\frac{n+m\sqrt{\Delta}}{2}\right)\mathfrak{n}^{-1}\right)$$
$$-\lim_{s\to 1}\left[\sum_{\substack{|n|>m\sqrt{\Delta}\\ n\equiv m\varrho(2N)}} \sigma_{0,\chi}\left(\left(\frac{n+m\sqrt{\Delta}}{2}\right)\mathfrak{n}^{-1}\right) Q_{s-1}\left(\frac{|n|}{m\sqrt{\Delta}}\right)+\frac{\lambda}{s-1}\right]$$
$$+\lambda\left[\frac{1}{2}\log\frac{N}{\Delta}-\sum_{p|N}\frac{\log p}{p+1}+\sigma(m)^{-1}\sum_{d|m} d\log\frac{m}{d^2}-\frac{L'}{L}\left(1,\left(\frac{D_0}{\cdot}\right)\right)\right.$$
$$\left.-\frac{L'}{L}\left(1,\left(\frac{D_1}{\cdot}\right)\right)+2\frac{\zeta'}{\zeta}(2)+2\right],$$

where $\lambda=\dfrac{-12H(D_0)H(D_1)\sigma(m)}{N\prod_{p|N}(1+1/p)}$.

Proof. We first need the Fourier development of $E(z,z')=\frac{\partial}{\partial s}E_s(z,z')|_{s=0}$. This is essentially the same as that given in [8, p. 215], for the case $N=1$, and we simply quote it:

$$E(z,z')=\sum_{\nu\in\mathfrak{n}\mathfrak{d}^{-1}} c(\nu,y,y')e^{2\pi i(\nu z+\nu' z')}$$

with

$$c(\nu,y,y')=\begin{cases} 2L_K(1,\chi)\log(yy')+4C & \text{if } \nu=0,\\ 8\pi^2\Delta^{-1/2}\sigma'_\chi((\nu)\mathfrak{d}\mathfrak{n}^{-1}) & \text{if } \nu\gg 0,\\ -4\pi^2\Delta^{-1/2}\sigma_{0,\chi}((\nu)\mathfrak{d}\mathfrak{n}^{-1})\Phi(|\nu'|y') & \text{if } \nu>0>\nu',\\ -4\pi^2\Delta^{-1/2}\sigma_{0,\chi}((\nu')\mathfrak{d}\mathfrak{n}^{-1})\Phi(|\nu|y) & \text{if } \nu'>0>\nu,\\ 0 & \text{if } \nu\ll 0,\end{cases}$$

where y and y' are the imaginary parts of z and z', $\mathfrak{d}=(\sqrt{\Delta})$ the different of K,

$$L_K(s,\chi)=\sum_{0\neq\mathfrak{a}}\chi(\mathfrak{a})N(\mathfrak{a})^{-s}=L\left(s,\left(\frac{D_0}{\cdot}\right)\right)L\left(s,\left(\frac{D_1}{\cdot}\right)\right)$$

the L-series of χ, C the number

$$C=L'_K(1,\chi)+(\tfrac{1}{2}\log(\Delta N)-\log\pi-\gamma)L_K(1,\chi)$$

($\gamma=$Euler's constant), $\sigma_{0,\chi}$ and σ'_χ the arithmetic functions defined in Proposition 1, and Φ the real-valued function

$$\Phi(t)=\int_1^\infty e^{-4\pi t u}\frac{du}{u}\qquad (t>0)$$

(exponential integral).

We next compute the Fourier coefficients of $\tilde{F}(z)$. To do this we must compute $\mathscr{C}_{k-1}(c(v,y,y')e^{2\pi i(vz+v'z')})$ for all $v \in \mathfrak{n}\mathfrak{d}^{-1}$. For $v=0$ this is given by

$$\mathscr{C}_{k-1}(2L_K(1,\chi)\log yy'+4C)=\begin{cases}4L_K(1,\chi)\log y+4C & \text{if} \quad k=1\,,\\ 2(k-2)!\,(1+(-1)^{k-1})L_K(1,\chi)y^{1-k} & \text{if} \quad k>1\,.\end{cases}$$

For $v \ll 0$ it is 0. If $v \gg 0$ then $c(v,y,y')$ is a constant and

$$\mathscr{C}_{k-1}(e^{2\pi i(vz+v'z')})=(2\pi i)^{k-1}\left(\sum_{\ell=0}^{k-1}(-1)^{\ell}\binom{k-1}{\ell}^2 v^{\ell}v'^{k-1-\ell}\right)e^{2\pi i(v+v')z}.$$

Using the identity

$$\sum_{m=0}^{n}(-1)^{m-n}\binom{n}{m}^2 t^m=(t+1)^n P_n\left(\frac{t-1}{t+1}\right), \tag{1}$$

we can write the expression in brackets as $(v+v')^{k-1}P_{k-1}\left(\dfrac{v'-v}{v'+v}\right)$. If $v>0>v'$, then $c(v,y,y')$ depends only on y'. By Leibniz's rule we have

$$\begin{aligned}\frac{d^r}{dz'^r}(\Phi(|v'|y')e^{2\pi iv'z'})&=\sum_{s=0}^{r}\binom{r}{s}\frac{d^s}{dz'^s}\Phi(|v'|y')\frac{d^{r-s}}{dz'^{r-s}}(e^{2\pi iv'z'})\\ &=(2\pi iv')^r e^{2\pi iv'z'}\int_1^{\infty}e^{-4\pi|v'|y'u}(1-u)^r\frac{du}{u}\end{aligned}$$

for all $r \geqq 0$, so

$$\begin{aligned}&\mathscr{C}_{k-1}(\Phi(|v'|y')e^{2\pi i(vz+v'z')})\\ &=(2\pi i)^{k-1}\sum_{\ell=0}^{k-1}(-1)^{\ell}\binom{k-1}{\ell}^2 v^{\ell}v'^{k-1-\ell}\\ &\quad\times\left(\int_1^{\infty}e^{-4\pi|v'|y'u}(1-u)^{k-1-\ell}\frac{du}{u}\right)e^{2\pi i(v+v')z}\\ &=(2\pi i)^{k-1}e^{2\pi i(v+v')z}\\ &\quad\times\int_{|v'|}^{\infty}(u+v+v')^{k-1}P_{k-1}\left(\frac{u-v+v'}{u+v+v'}\right)e^{-4\pi yu}\frac{du}{u},\end{aligned}$$

where in the last line we have used (1) and replaced u by $|v'|^{-1}u$. The result for $v<0<v'$ is obtained from this by replacing v by v' (so v now runs over $\mathfrak{n}'\mathfrak{d}^{-1}$). Putting this all together, we find

$$\tilde{F}(z)=\sum_{m=-\infty}^{\infty}a_m(y)e^{2\pi imz}$$

with

$$a_m(y) = \begin{cases} \dfrac{\sqrt{\Delta}}{2\pi^2}(L_K(1,\chi)\log y + C) & \text{if} \quad m=0, \quad k=1, \\ \text{constant}\cdot y^{1-k} & \text{if} \quad m=0, \quad k>1, \\ 0 & \text{if} \quad m>0 \end{cases}$$

$$+(m\sqrt{\Delta})^{k-1} \sum_{\substack{\nu \in \mathfrak{n}\mathfrak{d}^{-1} \\ \nu \gg 0 \\ \mathrm{Tr}(\nu)=m}} \sigma'_\chi((\nu)\mathfrak{d}\mathfrak{n}^{-1}) P_{k-1}\left(\frac{\nu'-\nu}{m}\right)$$

$$-\frac{\Delta^{\frac{k-1}{2}}}{2} \left\{ \sum_{\substack{\nu \in \mathfrak{n}\mathfrak{d}^{-1} \\ \nu > 0 > \nu' \\ \mathrm{Tr}(\nu)=m}} \sigma_{0,\chi}((\nu)\mathfrak{d}\mathfrak{n}^{-1}) \right.$$

$$\left. \times \int_{|\nu'|}^{\infty} (u+m)^{k-1} P_{k-1}\left(\frac{u-\nu+\nu'}{u+m}\right) e^{-4\pi y u} \frac{du}{u} + (-1)^{k-1}(\ldots) \right\}, \tag{2}$$

where $(\ldots)$ means the expression obtained from the preceding one by replacing $\mathfrak{n}$ by $\mathfrak{n}'$ at both occurrences.

We now turn to the last step, computing the holomorphic projection $F(z)$ of $\tilde{F}(z)$. We assume first that $k>1$. From (2) and the fact that $\Phi(x)$ or any of its derivatives is bounded by a polynomial in x times $e^{-4\pi x}$ as $x \to \infty$, we see that $F(z) = O(y^{-k+1})$ as $y = \mathrm{Im}(z) \to \infty$, and the same is true at any other cusp since $E_s(z, z')$ has a Fourier development of the same type at all cusps. Hence the hypotheses of Proposition 5.1, p. 288, of [9] are satisfied and $\pi_{\mathrm{hol}}(\tilde{F})$ is given by Sturm's formula

$$\begin{aligned} F(z) &= \sum_{m=1}^{\infty} a_m e^{2\pi i m z} \in S_{2k}(N), \\ a_m &= \frac{(4\pi m)^{2k-1}}{(2k-2)!} \int_0^\infty a_m(y) e^{-4\pi m y} y^{2k-2} dy. \end{aligned} \tag{3}$$

We substitute into this $a_m(y)$ from (2). The first term in (2) is absent for $m>0$, and the second is a constant (i.e., independent of y) and hence unaffected by the holomorphic projection process [taking $a_m(y)=c$ in (3) gives $a_m=c$, too]. The conditions $\mathrm{Tr}(\nu)=m$, $\nu \gg 0$, and $\nu \in \mathfrak{n}\mathfrak{d}^{-1}$ are equivalent to $\nu = \dfrac{n+m\sqrt{\Delta}}{2\sqrt{\Delta}}$ with $|n| < m\sqrt{\Delta}$, $n \equiv m\varrho\,(2N)$, so this term is the same as the first term given in Proposition 1. Applying the same argument to the remaining terms, we see that to prove Proposition 1 we need only prove the integral identity

$$\frac{(4\pi)^{2k-1}}{(2k-2)!} \int_0^\infty e^{-4\pi m y} y^{2k-2} \left(\int_{|\nu'|}^\infty (u+m)^{k-1} P_{k-1}\left(\frac{u-\nu+\nu'}{u+m}\right) e^{-4\pi y u} \frac{du}{u} \right) dy$$

$$= 2(-1)^{k-1} m^{-k} Q_{k-1}\left(\frac{\nu-\nu'}{m}\right)$$

for $v>-v'>0$, $v+v'=m>0$. Interchanging the integrals and performing the integration over y, we can write the left-hand side of this as

$$\int_{|v'|}^{\infty}(u+m)^{-k}P_{k-1}\left(\frac{u-v+v'}{u+m}\right)\frac{du}{u}=(2v)^{1-k}\int_{-1}^{1}(1-x)^{k-1}\frac{P_{k-1}(x)dx}{v-v'+mx}.$$

We therefore need to prove the identity

$$\int_{-1}^{1}(1-x)^{k-1}P_{k-1}(x)\frac{dx}{\lambda+x}=2(-1-\lambda)^{k-1}Q_{k-1}(\lambda)\qquad(\lambda>1).$$

But this is easy: by the defining property of $Q_{k-1}(x)$, we have

$$\begin{aligned}&\int_{-1}^{1}\frac{(1-x)^{k-1}P_{k-1}(x)}{x+\lambda}dx\\&=\int_{-1}^{1}\left[\frac{(1+\lambda)^{k-1}P_{k-1}(-\lambda)}{x+\lambda}+(\text{polynomial in } x,\lambda)\right]dx\\&=(-1-\lambda)^{k-1}P_{k-1}(\lambda)\log\frac{\lambda+1}{\lambda-1}+(\text{polynomial in } \lambda)\\&=2(-1-\lambda)^{k-1}Q_{k-1}(\lambda)+(\text{polynomial in } \lambda),\end{aligned}$$

and since $Q_{k-1}(\lambda)=O(\lambda^{-k})$ and the integral on the left is $O(1/\lambda)$ as $\lambda\to\infty$, the polynomial in the last line must vanish. This completes the proof of Proposition 1.

For $k=1$ the simple formula (3) must be replaced by the following, which is a restatement of Propositions 6.2 and 6.7 of [9].

Holomorphic Projection Lemma. *Let* $\tilde{F}(z)=\sum_{m=-\infty}^{\infty}a_m(y)e^{2\pi imz}$ *be a function on* $\mathfrak{H}$ *which transforms like a holomorphic modular form of weight 2 on* $\Gamma_0(N)$, *and suppose that for every divisor* M *of* N *there are numbers* $A(M), B(M)$ *such that*

$$(cz+d)^{-2}\tilde{F}\left(\frac{az+b}{cz+d}\right)=A(M)\log y+B(M)+O(y^{-\varepsilon})\quad as\quad y=\mathrm{Im}(z)\to\infty\tag{4}$$

with $\varepsilon>0$ *for all* $\begin{pmatrix}a&b\\c&d\end{pmatrix}\in SL_2(\mathbb{Z})$ *with* $(c,N)=M$. *Let* $F(z)=\sum_{m=1}^{\infty}a_me^{2\pi imz}\in S_2(N)$ *be the holomorphic projection of* $\tilde{F}$. *Then*

$$\begin{aligned}a_m=\lim_{s\to1}&\left[4\pi m\int_0^{\infty}a_m(y)e^{-4\pi my}y^{s-1}dy+\frac{24\alpha\sigma(m)}{s-1}\right]\\&-48\alpha\sigma(m)\left[\frac{1}{\sigma(m)}\sum_{d|m}d\log\frac{m}{d}+\sum_{p|N}\frac{\log p}{p^2-1}+\log 2+\frac{1}{2}+\frac{\zeta'}{\zeta}(2)+\frac{\beta}{2\alpha}\right]\end{aligned}$$

for $(m,N)=1$, *where*

$$\alpha=\prod_{p|N}(1-p^{-2})^{-1}\cdot\sum_{M|N}\frac{\mu(M)A(M)}{M^2},$$

$$\beta=\prod_{p|N}(1-p^{-2})^{-1}\cdot\sum_{M|N}\frac{\mu(M)[B(M)-A(M)\log M]}{M^2}$$

($\mu(M)$ = Möbius function).

Applying this in our case, we see from (2) that (4) holds for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, with

$$A(N) = \frac{\sqrt{\Delta}}{2\pi^2} L_K(1,\chi) = \frac{\sqrt{D_0 D_1}}{2\pi^2} L_{D_0}(1) L_{D_1}(1) = \tfrac{1}{2} H(D_0) H(D_1),$$

$$B(N) = \frac{\sqrt{\Delta}}{2\pi^2} C = \tfrac{1}{2} H(D_0) H(D_1) \left[\tfrac{1}{2} \log \Delta N - \log \pi - \gamma + \frac{L'_{D_0}}{L_{D_0}}(1) + \frac{L'_{D_1}}{L_{D_1}}(1) \right].$$

To find the development at other cusps we choose $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ with $(c, N) = M$ and observe that $(cz+d)(cz'+d) E_s\left(\frac{az+b}{cz+d}, \frac{az'+b}{cz'+d} \right)$ is an Eisenstein series defined by the same expression as in (1) of Sect. 1 but with the summation conditions in the inner sum replaced by $m, n \in \mathfrak{a}$, $dm - cn \in \mathfrak{n}\mathfrak{a}$. A standard calculation shows that this Eisenstein series equals

$$\left(\frac{M}{N} \right)^{1+2s} L_K(1+2s,\chi)(yy')^s - \frac{M^{1-2s}}{N} \frac{\pi \Gamma(s+\frac{1}{2})^2}{\sqrt{\Delta}\,\Gamma(s+1)^2} L_K(2s,\chi)(yy')^{-s} + (\text{exp. small})$$

as $y, y' \to \infty$ (the first term comes from the summands with $m=0$, the second from those with $m \neq 0$ upon replacing the sum over n by the corresponding integral). This gives for the function $F(z) = \frac{\sqrt{\Delta}}{8\pi^2} \frac{\partial}{\partial s} E_s(z,z)|_{s=0}$ an expansion like (4) with

$$A(M) = M \frac{A(N)}{N}, \qquad B(M) = M \left(\frac{B(N)}{N} + \frac{A(N)}{N} \log \frac{M}{N} \right)$$

[$A(N)$, $B(N)$ as above]. Hence

$$\alpha = \prod_{p|N} (1-p^{-2})^{-1} \cdot \frac{A(N)}{N} \cdot \sum_{M|N} \frac{\mu(M)}{M} = \frac{\frac{1}{2} H(D_0) H(D_1)}{N \prod_{p|N} (1+p^{-1})},$$

$$\beta = \alpha \left[\tfrac{1}{2} \log \frac{\Delta}{N} - \log \pi - \gamma + \frac{L'_{D_0}}{L_{D_0}}(1) + \frac{L'_{D_1}}{L_{D_1}}(1) + \sum_{p|N} \frac{\log p}{p-1} \right].$$

On the other hand, (2) and the arithmetic considerations given for $k > 1$ show that

$$a_m(y) = a_m^0 - \frac{1}{2} \sum_{\substack{|n| > m\sqrt{\Delta} \\ n \equiv m\varrho (2N)}} \varrho_n \Phi\left(\frac{|n| - m\sqrt{\Delta}}{2\sqrt{\Delta}} y \right)$$

with

$$a_m^0 = \sum_{\substack{n < m\sqrt{\Delta} \\ n \equiv m\varrho(2N)}} \sigma'_\chi\left(\left(\frac{n + m\sqrt{\Delta}}{2} \right) \mathfrak{n}^{-1} \right), \qquad \varrho_n = \sigma_{0,\chi}\left(\left(\frac{n + m\sqrt{\Delta}}{2} \right) \mathfrak{n}^{-1} \right),$$

so

$$4\pi m \int_0^\infty a_m(y) e^{-4\pi m y} y^{s-1} dy$$
$$= (4\pi m)^{1-s} \Gamma(s) a_m^0 - \frac{m^{1-s}}{2} \sum_n \varrho_n \Psi_{s-1}\left(\frac{|n|}{2m\sqrt{\Delta}} - \frac{1}{2}\right),$$

where

$$\Psi_{s-1}(\lambda) = 4\pi \int_0^\infty \Phi(\lambda y) e^{-4\pi y} y^{s-1} dy \qquad (\lambda > 0).$$

The same calculation as in [8, p. 210] shows that

$$\frac{1}{2}\sum_n \varrho_n \Psi_{s-1}\left(\frac{|n|}{2m\sqrt{\Delta}} - \frac{1}{2}\right) = \frac{\Gamma(2s)}{(4\pi)^{s-1}\Gamma(s+1)} \sum_n \varrho_n Q_{s-1}\left(\frac{|n|}{m\sqrt{\Delta}}\right) + O(s-1)$$

as $s \to 1$, so the holomorphic projection lemma gives

$$a_m = a_m^0 - \lim_{s \to 1} \left[\sum_{\substack{|n| < m\sqrt{\Delta} \\ n \equiv m\varrho(2N)}} \varrho_n Q_{s-1}\left(\frac{|n|}{m\sqrt{\Delta}}\right) - \frac{(4\pi m)^{s-1}\Gamma(s+1)}{\Gamma(2s)} \cdot \frac{24\alpha\sigma(m)}{s-1} \right]$$
$$- 24\alpha\sigma(m) \left[\frac{1}{\sigma(m)} \sum_{d|m} d \log\frac{m}{d^2} \right.$$
$$\left. + 2\sum_{p|N} \frac{\log p}{p^2 - 1} + \log 4 + 1 + 2\frac{\zeta'}{\zeta}(2) - \frac{\beta}{\alpha} \right],$$

and this is equivalent to the formula in Proposition 2.

3. *Evaluation of* (F, f)

The object of this section is to calculate the scalar product (F, f) for a newform $f \in S_{2k}(N)$. Since we have seen that F belongs to $S_{2k}(N)^-$, and since $S_{2k}(N)^-$ and $S_{2k}(N)^+$ are orthogonal, we may restrict attention to $f \in S_{2k}(N)^-$. For such an f the L-series $L(f, s)$ vanishes at $s = k$, and its derivative at this point will enter our formula.

Theorem. *Let $f \in S_{2k}(N)^-$ be a normalized newform. Then*

$$(F_{k,N,\Delta,\varrho,D_0}, f) = \frac{i^{k-1}\Gamma(k-\frac{1}{2})}{2^{k+1}\pi^{k+1/2}} r_{k,N,\Delta,\varrho,D_0}(f) L'(f, k),$$

where $r_{k,N,\Delta,\varrho,D_0}(f)$ is the cycle integral defined in Chap. II, Sect. 1.

Proof. By the definition of the holomorphic projection operator, (F, f) equals $(\tilde{F}, f)$. Also, the operators $\left.\frac{\partial}{\partial s}\right|_{s=0}$ and $\mathscr{C}_{k-1}$ commute, since they involve differentiation with respect to different variables. We will therefore first compute $\mathscr{C}_{k-1}$ of the Eisenstein series $E_s(z, z')$, then the scalar product of f with this, and finally the derivative at $s = 0$. For the first step, we use:

Lemma. *Let m, m', n, n' be real numbers, $s \in \mathbb{C}$, and denote by h the function*

$$h(z, z') = \frac{y^s y'^s}{(mz+n)(m'z'+n')|mz+n|^{2s}|m'z'+n'|^{2s}}$$

($z, z' \in \mathfrak{H}$, $y = \mathrm{Im}(z)$, $y' = \mathrm{Im}(z')$). Then

$$(\mathscr{C}_{k-1}h)(z) = \sum_{0 \leq j \leq \frac{k-1}{2}} P_{k,j}(s)(mn'-m'n)^{k-1-2j} \frac{y^{2s-2j}}{Q(z)^k |Q(z)|^{2s-2j}},$$

where $Q(z)$ denotes the quadratic polynomial $(mz+n)(m'z+n')$ and the $P_{k,j}(s)$ are polynomials in s with

$$P_{k,j}(0) = \begin{cases} (k-1)! & \text{if } j = 0, \\ 0 & \text{if } j > 0. \end{cases}$$

Proof of the Lemma. The function h is the product of a function of z and a function of z'; we first calculate the derivatives of these functions individually. By Leibniz's rule

$$\begin{aligned}
&\frac{1}{\ell!}\frac{d^\ell}{dz^\ell}\left(\frac{y^s}{(mz+n)|mz+n|^{2s}}\right) \\
&= \sum_{p+q=\ell} \frac{1}{p!}\frac{d^p}{dz^p}\left(\frac{1}{(mz+n)^{s+1}(m\bar{z}+n)^s}\right)\frac{1}{q!}\frac{d^q}{dz^q}(y^s) \\
&= \sum_{p+q=\ell} \frac{(-1)^p \binom{s+p}{p} m^p}{(mz+n)^{s+p+1}(m\bar{z}+n)^s} \frac{\binom{s}{q} y^{s-q}}{(2i)^q} \\
&= \frac{(2iy)^{-\ell} y^s}{(mz+n)|mz+n|^{2s}} \sum_{p+q=\ell} (-1)^p \binom{s+p}{p}\binom{s}{q}\left(\frac{2imy}{mz+n}\right)^p \\
&= \frac{(2iy)^{-\ell} y^s}{(mz+n)|mz+n|^{2s}} \sum_{j=0}^{\ell} (-1)^{\ell-j}\binom{\ell}{j}\binom{s+j}{j}\left(\frac{m\bar{z}+n}{mz+n}\right)^j,
\end{aligned}$$

where in the last line we have used the identities

$$1 - \frac{2imy}{mz+n} = \frac{m\bar{z}+n}{mz+n}$$

and

$$\sum_{p+q=\ell} (-1)^p \binom{s+p}{p}\binom{s}{q}(1-X)^p = \sum_{j=0}^{\ell} (-1)^{\ell-j}\binom{\ell}{j}\binom{s+j}{j} X^j.$$

Inserting this formula into the definition of the operator $\mathscr{C}_{k-1}$, we find

$$\begin{aligned}
(\mathscr{C}_{k-1}h)(z) &= \frac{(k-1)!^2 (2iy)^{1-k} y^{2s}}{Q(z)|Q(z)|^{2s}} \\
&\times \sum_{\ell+\ell'=k-1} \frac{(-1)^\ell}{\ell!\,\ell'!}\left(\sum_{j=0}^{\ell} (-1)^{\ell-j}\binom{\ell}{j}\binom{s+j}{j}\left(\frac{m\bar{z}+n}{mz+n}\right)^j\right) \\
&\times \left(\sum_{j'=0}^{\ell'} (-1)^{\ell'-j'}\binom{\ell'}{j'}\binom{s+j'}{j'}\left(\frac{m'\bar{z}+n'}{m'z+n'}\right)^{j'}\right).
\end{aligned}$$

For fixed j, j' we have $\sum_{\ell+\ell'=k-1} \frac{(-1)^\ell}{\ell!\ell'!}\binom{\ell}{j}\binom{\ell'}{j'}=0$ unless $j+j'=k-1$, when only the terms $\ell=j$, $\ell'=j'$ contribute. Hence

$$(\mathscr{C}_{k-1}h)(z)=\frac{(k-1)!^2(2iy)^{1-k}y^{2s}}{Q(z)|Q(z)|^{2s}} \times \sum_{\ell+\ell'=k-1}\frac{1}{\ell!\ell'!}\binom{s+\ell}{\ell}\binom{s+\ell'}{\ell'}\left(-\frac{m\bar{z}+n}{mz+n}\right)^\ell\left(\frac{m'\bar{z}+n'}{m'z+n'}\right)^{\ell'}.$$

But clearly

$$(k-1)!^2\sum_{\ell+\ell'=k-1}\frac{1}{\ell!\ell'!}\binom{s+\ell}{\ell}\binom{s+\ell'}{\ell'}X^\ell X'^{\ell'} = \sum_{2j+h=k-1} P_{k,j}(s)(4XX')^j(X+X')^h$$

for some polynomials $P_{k,j}(s)$, because the left-hand side is a symmetric homogeneous polynomial of degree $k-1$ in X and X'. Therefore

$$(\mathscr{C}_{k-1}h)(z)=\frac{(2iy)^{1-k}y^{2s}}{Q(z)|Q(z)|^{2s}} \times \sum_{0\leqq j\leqq\frac{k-1}{2}} P_{k,j}(s)\left(\frac{-4\overline{Q(z)}}{Q(z)}\right)^j\left(\frac{2iy(mn'-m'n)}{Q(z)}\right)^{k-1-2j},$$

which is equivalent to the statement of the lemma. The statement about $P_{k,j}(0)$ is clear from the definition of the $P_{k,j}(s)$. In fact, $P_{k,j}(s) =4^{-j}\binom{2j}{j}\binom{k-1}{2j}\prod_{n=1}^{k-1}(s-j+n)$, but we will not need this formula.

Applying the lemma term by term to the series defining E_s (in the region of absolute convergence), we find

$$(\mathscr{C}_{k-1}E_s)(z)=\sum_{0\leqq j\leqq\frac{k-1}{2}} P_{k,j}(s)\Delta^{\frac{k-1}{2}-j} \times \sum_{[\mathfrak{a}]}\chi(\mathfrak{a})\sum_{m,n} r(m,n)^{k-1-2j}\frac{y^{2s-2j}}{Q_{mn}(z)|Q_{mn}(z)|^{2s-2j}}, \tag{1}$$

where the summations over $[\mathfrak{a}]$ and m, n are the same as in the definition of E_s and

$$r(m,n)=\frac{mn'-m'n}{N(\mathfrak{a})\sqrt{\Delta}}, \qquad Q_{mn}(z)=\frac{(mz+n)(m'z+n')}{N(\mathfrak{a})}.$$

Note that $r(m,n)$ is an integer, equal to 0 if $\mathbb{Z}m+\mathbb{Z}n$ is one-dimensional and to $\pm[\mathfrak{a}:\mathbb{Z}m+\mathbb{Z}n]$ otherwise, and that Q_{mn} is a quadratic polynomial with integer coefficients and discriminant $r(m,n)^2\Delta$. Applying $|_{2k}\gamma$ for $\gamma\in\Gamma_0(N)$ to (1) permutes the m, n in the inner sum, leaving $r(m,n)$ invariant. Therefore we can rewrite (1) as

$$\mathscr{C}_{k-1}E_s=\sum_j P_{k,j}(s)\Delta^{\frac{k-1}{2}-j}\sum_{[\mathfrak{a}]}\sum_{r\in\mathbb{Z}} r^{k-1-2j}\Phi_{k,2s-2j,\mathfrak{a},r}, \tag{2}$$

where

$$\Phi_{k,s,\mathfrak{a},r}(z)=\chi(\mathfrak{a})\sum_{\substack{(m,n)\in(\mathfrak{n}\mathfrak{a}\times\mathfrak{a})/\mathcal{O}_K^\times\\ r(m,n)=r}}{}' \frac{y^s}{Q_{mn}(z)^k|Q_{mn}(z)|^{2s}}$$

is a (non-holomorphic) modular form of weight $2k$ on $\Gamma_0(N)$, and then [because of the absolute convergence for $\mathrm{Re}(s)\gg 0$] compute the scalar product of each term of (2) with f separately. Note that $\Phi_{k,s,\mathfrak{a},r}$ depends on the narrow ideal class of $\mathfrak{a}$ but that $\Phi_{k,s,\mathfrak{a},r}+(-1)^{k-1}\Phi_{k,s,\mathfrak{a},-r}$ [and hence the inner sum in (2)] depends only on the wide ideal class, since replacing $\mathfrak{a}$, m, and n by $\lambda\mathfrak{a}$, λm, and λn with $\lambda\in K^\times$, $N(\lambda)<0$, replaces $\chi(\mathfrak{a})$, $r(m,n)$, and $Q_{mn}(z)$ by their negatives. We now consider the individual terms of (2).

<u>$r=0$.</u> If $r(m,n)=0$, then $\frac{m}{n}\in\mathbb{Q}$, so $m=\lambda c$, $n=\lambda d$ for some coprime integers c,d (unique up to sign) and $\lambda\in\mathfrak{a}$. Hence

$$\Phi_{k,s,\mathfrak{a},0}(z)=\frac{\chi(\mathfrak{a})}{2}\sum_{\substack{c,d\in\mathbb{Z}\\(c,d)=1}}\Bigg(\sum_{\substack{\lambda\in\mathfrak{a}\\ c\lambda\in\mathfrak{n}\mathfrak{a}}}{}' N(\lambda)^{-k}|N(\lambda)|^{-s}\Bigg)\frac{y^s}{(cz+d)^{2k}|cz+d|^{2s}}.$$

The inner sum depends only on the greatest common divisor of c and N, so this is a linear combination of functions

$$\sum_{\substack{c,d\in\mathbb{Z}\\(c,d)=1\\(c,N)=M}}\frac{y^s}{(cz+d)^{2k}|cz+d|^{2s}}\qquad (M|N,\ \mathrm{Re}(s)\gg 0)$$

which are non-holomorphic Eisenstein series of weight $2k$ on $\Gamma_0(N)$. Therefore

$$(f,\Phi_{k,s,\mathfrak{a},0})=0 \tag{3}$$

since f is a cusp form.

<u>$r=\pm 1$.</u> For the reason given above, we have

$$\sum_{[\mathfrak{a}]}(\Phi_{k,s,\mathfrak{a},1}+(-1)^{k-1}\Phi_{k,s,\mathfrak{a},1})=\sum_{[\mathfrak{a}]\in\mathscr{C}_K}\Phi_{k,s,\mathfrak{a},1}$$

where $[\mathfrak{a}]$ on the left-hand side of the equation runs over the wide ideal class group of K and on the right over the narrow ideal class group (which is always twice as big, since the fact that Δ is a product of negative discriminants implies that all units of K have norm $+1$). For $r(m,n)=1$, the quadratic polynomial $Q_{mn}(z)$ has the form aNz^2+bz+c with $a,b,c\in\mathbb{Z}$, $b\equiv\varrho\,(2N)$, $b^2-4Nac=\Delta$, i.e., it equals $Q(z,1)$ for some $Q\in\mathcal{Q}_{N,\Delta,\varrho}$ in the notation of Chap. I. Conversely, every $Q\in\mathcal{Q}_{N,\Delta,\varrho}$ occurs this way exactly once [there is a bijection between $\mathscr{C}_K$ and $\mathcal{O}_{N,\Delta,\varrho}/\Gamma_0(N)$ by the results of Sect. 1 of Chap. I, and the different choices of (m,n) with $r(m,n)=1$ for given $\mathfrak{a}$ correspond to the forms which are $\Gamma_0(N)$-equivalent to a single such form]. Also, $\chi(\mathfrak{a})=\chi_{D_0}(Q)$ under this correspondence. Hence

$$\sum_{\mathfrak{a}}(\Phi_{k,s,\mathfrak{a},1}(z)+(-1)^{k-1}\Phi_{k,s,\mathfrak{a},-1}(z))=f_{k,N,\Delta,\varrho,D_0}(z;s)$$

in the notation of Sect. 1 of Chap. II, and the last equation of that section gives

$$\Big(f, \sum_{[\mathfrak{a}]}(\Phi_{k,\bar{s},\mathfrak{a},1}+(-1)^{k-1}\Phi_{k,\bar{s},\mathfrak{a},-1})\Big) = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(k+\frac{s-1}{2}\right)}{\Gamma\left(k+\frac{s}{2}\right)}\Delta^{-k+\frac{1-s}{2}} r_{k,N,A,\varrho,D_0}(f). \tag{4}$$

$|r|>1$. For a fixed ideal $\mathfrak{a}$, the pairs $(m,n)\in(\mathfrak{n}\mathfrak{a}\times\mathfrak{a})/\mathcal{O}_K^\times$ with $r(m,n)=r$ are obtained from a fixed pair m_0, n_0 with $r(m_0,n_0)=1$ by $(m\,n)=(m_0\,n_0)\gamma$ with

$$\gamma\in R_N^r=\left\{\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in M_2(\mathbb{Z})\,|\,c\equiv 0\,(N),\ ad-bc=r\right\}.$$

The quadratic polynomial $Q_{mn}(z)$ equals $(cz+d)^2Q_0(\gamma z)$ under this correspondence, where $Q_0(z)=Q_{m_0n_0}(z)$, and the action of $\mathcal{O}_K^\times$ on (m,n) corresponds to multiplication of γ on the left by an element of the stabilizer $\Gamma_0(N)_{Q_0}$ of Q_0 in $\Gamma_0(N)$. Hence

$$\Phi_{k,s,\mathfrak{a},r}(z)=r^{-k}|r|^{-s}\sum_{\gamma\in\Gamma_0(N)_{Q_0}\backslash R_N^r}\frac{y^s}{Q_0(z)^k|Q_0(z)|^s}\Bigg|_{2k}\gamma$$

$\left[\text{where } \phi|_{2k}\gamma \text{ for } \gamma=\begin{pmatrix} a & b\\ c & d\end{pmatrix} \text{ of determinant } r \text{ is defined as } r^k(cz+d)^{-2k}\phi(\gamma z)\right]$. Comparing this equation with the same one for $r=1$ [with $R_N^1=\Gamma_0(N)$] gives

$$\Phi_{k,s,\mathfrak{a},\pm r}(z)=r^{-k-s}\sum_{\gamma\in\Gamma_0(N)\backslash R_N^r}\Phi_{k,s,\mathfrak{a},\pm 1}|_{2k}\gamma \qquad (r\geqq 1),$$

a finite sum. The value of (a,N) for $\gamma=\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in R_N^r$ is left $\Gamma_0(N)$-invariant, and the terms with $(a,N)=M>1$ yield an old form. $\Big[$We can write $\gamma=\gamma'\circ\begin{pmatrix} M & 0\\ 0 & 1\end{pmatrix}$ with $\gamma'\in\Gamma_0(N)\backslash R_{N/M}^{r/M}$, so these terms have the form $g(Mz)$ where g is a modular form on $\Gamma_0(N/M)$ obtained as a trace from $\Gamma_0(N)$.$\Big]$ The sum over the terms with $(a,N)=1$ is, by definition of the Hecke operator T_r, just $r^{1-k}\Phi_{k,s,\mathfrak{a},\pm 1}|T_r$. Hence

$$\Phi_{k,s,\mathfrak{a},r}+(-1)^{k-1}\Phi_{k,s,\mathfrak{a},-r}=r^{1-2k-s}(\Phi_{k,s,\mathfrak{a},1}+(-1)^{k-1}\Phi_{k,s,\mathfrak{a},-1})|_{2k}T_r + (\text{old form}) \qquad (r\geqq 1),$$

and therefore, since T_r is self-adjoint with respect to the Petersson product,

$$(f, \Phi_{k,\bar{s},\mathfrak{a},r}+(-1)^{k-1}\Phi_{k,\bar{s},\mathfrak{a},-r})=\frac{a(r)}{r^{2k-1+s}}(f, \Phi_{k,\bar{s},\mathfrak{a},1}+(-1)^{k-1}\Phi_{k,\bar{s},\mathfrak{a},-1}) \tag{5}$$

for $r\geqq 1$, where $a(r)=$ coefficient of q^r in $f=$ eigenvalue of f under T_r. The argument we have given here was used for $N=1$ in [13].

Combining (2)–(6), we find

$$(f, \mathscr{C}_{k-1} E_{\bar{s}}) = \left[\sum_{0 \leq j \leq \frac{k-1}{2}} P_{k,j}(s) \frac{\Gamma(\frac{1}{2})\Gamma(k+s-j-\frac{1}{2})}{\Gamma(k+s-j)} \right] \times \Delta^{-\frac{k}{2}-s} L(f, k+2s) r_{k,N,\Delta,\varrho,D_0}(f). \tag{6}$$

The factor in square brackets is holomorphic at $s=0$ and has the value $\Gamma(\frac{1}{2})\Gamma(k-\frac{1}{2})$ there. On the other hand, $L(f, k+2s)$ vanishes at $s=0$. Hence

$$\left(f, \frac{\partial}{\partial s} \mathscr{C}_{k-1} E_s|_{s=0}\right) = \frac{\partial}{\partial \bar{s}} (f, \mathscr{C}_{k-1} E_s)|_{s=0}$$
$$= 2\Gamma(\tfrac{1}{2})\Gamma(k-\tfrac{1}{2})\Delta^{-k/2} L'(f, k) r_{k,N,\Delta,\varrho,D_0}(f),$$

and this is equivalent to the statement of the theorem.

Using the formula for $P_{k,j}(s)$ stated after the lemma, we can compute the factor in square brackets in (6) and hence rewrite this equation more explicitly:

$$(f, \pi_{\mathrm{hol}}(\mathscr{C}_{k-1} E_{\bar{s}})) = (f, \mathscr{C}_{k-1} E_{\bar{s}})$$
$$= 2^{-2k+2-2s} \binom{2k-2}{k-1} \frac{\Gamma(k+2s)}{\Gamma(s+1)^2} \pi \Delta^{-\frac{k}{2}-s} r_{k,N,\Delta,\varrho,D_0}(f) L(f, k+2s).$$

In other words, one can express the cycle integral of f times the value of $L(f, s)$ for any s, not just its derivative at $s=k$, as the scalar product of f with a modular form whose Fourier coefficients can be calculated in closed form.

IV. Height Pairings of Heegner Divisors

In the introduction we briefly defined Heegner divisors $y_{D,r}^*$ of degree 0 on $X_0(N)$ over $\mathbb{Q}$. These are indexed by negative discriminants D of conductor prime to N and classes $r \pmod{2N}$ with $r^2 \equiv D \pmod{4N}$. Their precise definition will be given in Sect. 1. Our main aim in this chapter is to compute the global height pairing $\langle y_{D_0,r_0}^*, y_{D_1,r_1}^* \rangle$ on the Jacobian $J_0^*(N)$ over $\mathbb{Q}$. In the height computation we shall assume that D_0 and D_1 are relatively prime. The global height depends only on the classes of the divisors $y_{D,r}^*$ in the Jacobian of $X_0^*(N)$, but using Néron's theory we will express the pairing as a sum of local symbols $\langle y_{D_0,r_0}^*, y_{D_1,r_1}^* \rangle_v$, indexed by the places v of $\mathbb{Q}$, which depend on the divisors representing the classes. We will compute the archimedean symbol using Green's functions for the Riemann surface $\mathfrak{H}/\Gamma_0^*(N)$ and the non-archimedean symbols using intersection theory on a regular model for $X_0^*(N)$ over $\mathbb{Z}$.

1. Heegner Divisors

Let D be a negative discriminant which is a square $(\bmod 4N)$ and has conductor prime to N and let r be a class (mod $2N$) with $r^2 \equiv D\,(4N)$. We define a rational divisor $P_{D,r}$ on $X_0(N)$ over $K = \mathbb{Q}(\sqrt{D})$. Recall that the affine points of $X_0(N)$ are given by $x = (\pi: E \to E')$, where E and E' are elliptic curves and π a cyclic N-isogeny.

Let $\mathcal{O}_D=\mathbb{Z}+\mathbb{Z}\frac{r+\sqrt{D}}{2}$ be the ring of discriminant D in K and $\mathfrak{n}$ the primitive ideal $\mathbb{Z}N+\mathbb{Z}\frac{r+\sqrt{D}}{2}$ of index N. The points x in $P_{D,r}$ are those which admit endomorphisms by $\mathcal{O}_D$ [i.e., $\mathcal{O}_D$ maps into $\mathrm{End}(E)$ and $\mathrm{End}(E')$ with the obvious diagram commuting] and such that the kernel of π is annihilated by $\mathfrak{n}$. The point x is counted with multiplicity $1/e$, where e is the order of $\mathrm{Aut}(x)/\pm 1$.

If c denotes the non-trivial automorphism of K over $\mathbb{Q}$, then $(P_{D,r})^c=P_{D,-r}=w_N(P_{D,r})$ [5]. Hence the image $P^*_{D,r}$ of $P_{D,r}$ on the quotient curve $X_0^*(N)=X_0(N)/w_N$ is rational over $\mathbb{Q}$ and depends only on $\pm r \pmod{2N}$. By [5], or by the proposition in Sect. 1 of Chap. I, the degrees of $P_{D,r}$ and $P^*_{D,r}$ are both equal to $H(D)$, the Hurwitz class number. We define the divisor $y^*_{D,r}$ of degree zero on $X_0^*(N)$ over $\mathbb{Q}$ by

$$y^*_{D,r}=P^*_{D,r}-H(D)(\infty^*),$$

where ∞^* is the rational cusp on $X_0^*(N)$ which is the image of the cusp ∞ (or 0) on $X_0(N)$.

Over the complex numbers, $X_0(N)=\bar{\mathfrak{H}}/\Gamma_0(N)$ and $X_0^*(N)=\bar{\mathfrak{H}}/\Gamma_0^*(N)$, where $\bar{\mathfrak{H}}=\mathfrak{H}\cup\mathbb{P}^1(\mathbb{Q})$ and $\Gamma_0^*(N)=\Gamma_0(N)\cup w_N\Gamma_0(N)\subset PSL_2(\mathbb{R})$, $w_N=\frac{1}{\sqrt{N}}\begin{pmatrix}0&-1\\N&0\end{pmatrix}$. [We are assuming that $N>1$, so $\Gamma_0^*(N)$ contains $\Gamma_0(N)$ with index 2.] The divisors $P_{D,r}$ and $P^*_{D,r}$ are given by

$$P_{D,r}=\mathcal{P}_{D,r}/\Gamma_0(N), \qquad P^*_{D,r}=\mathcal{P}^*_{D,r}/\Gamma_0^*(N), \tag{1}$$

where $\mathcal{P}_{D,r}$ and $\mathcal{P}^*_{D,r}$ are the infinite subsets of $\mathfrak{H}$ defined by

$$\begin{aligned}&\mathcal{P}_{D,r}=\{\tau\in\mathfrak{H}\,|\,aN\tau^2+b\tau+c=0 \text{ for some } [aN,b,c]\in\mathcal{Q}_{N,D,r} \text{ with } a>0\},\\ &\mathcal{P}^*_{D,r}=\mathcal{P}_{D,r}+\mathcal{P}_{D,-r}\end{aligned} \tag{2}$$

($\mathcal{Q}_{N,D,r}$ as in Chap. I). These equations are to be interpreted as equalities of divisors with rational coefficients, so a common point of $\mathcal{P}_{D,r}$ and $\mathcal{P}_{D,-r}$ has multiplicity 2 in $\mathcal{P}^*_{D,r}$ and a point of $\mathcal{P}/\Gamma$ represented by $\tau\in\mathcal{P}$ is counted with multiplicity equal to the multiplicity of τ in $\mathcal{P}$ divided by the order of the stabilizer of τ in Γ. The point ∞^* over $\mathbb{C}$, of course, corresponds to the cusp $i\infty$ (or 0) in $\bar{\mathfrak{H}}/\Gamma_0^*(N)$.

Finally, by the formula at the end of Sect. 1 of Chap. I, we see that

$$T_m(P_{D,r})=\sum_{m=dd'}\left(\frac{D}{d'}\right)P_{Dd^2,rd} \tag{3}$$

for D fundamental and $(m,N)=1$. Since T_m commutes with w_N and sends ∞^* to a multiple of ∞^*, the same formula holds for $P^*_{D,r}$ and $y^*_{D,r}$.

2. *Review of Local Symbols*

We review the basic ideas of Néron's theory; for more details see [6]. Let X be a non-singular, complete, geometrically connected curve over the locally compact field k_v. We normalize the valuation map $|\ |_v: k_v^\times\to\mathbb{R}_+^\times$ so that for any Haar measure dx on k_v we have the formula $\alpha^*(dx)=|\alpha|_v\cdot dx$. Let $\mathfrak{a}$ and $\mathfrak{b}$ denote divisors of degree zero on X over k_v with disjoint support; then Néron defines a symbol

$\langle \mathfrak{a}, \mathfrak{b}\rangle_v$ in $\mathbb{R}$ which is bi-additive, symmetric, continuous, and satisfies the property $\langle \sum m_x(x), (f)\rangle_v = \log|\prod f(x)^{m_x}|_v$ when $\mathfrak{b}=(f)$ is principal. These properties characterize the local symbol completely.

When $\mathfrak{a}$ and $\mathfrak{b}$ have the point z (and no other) in common, one can extend Néron's definition by choosing a uniformizing parameter π at z and defining

$$\langle \mathfrak{a}, \mathfrak{b}\rangle_v = \lim_{y\to z}\{\langle \mathfrak{a}_y, \mathfrak{b}\rangle - \operatorname{ord}_z(\mathfrak{a})\operatorname{ord}_z(\mathfrak{b})\log|\pi(y)|_v\},$$

where $\mathfrak{a}_y$ is the divisor obtained from $\mathfrak{a}$ by replacing z by a nearby point y not in the support of $\mathfrak{b}$.

When v is archimedean, one can compute the Néron symbol as follows. Associated to $\mathfrak{b}$ is a Green's function $g_{\mathfrak{b}}$ on the Riemann surface $X(\bar{k}_v)-|\mathfrak{b}|$ which satisfies $\partial\bar{\partial} g_{\mathfrak{b}}=0$ and has logarithmic singularities at the points in $|\mathfrak{b}|$. More precisely, the function $g_{\mathfrak{b}}-\operatorname{ord}_z(\mathfrak{b})\log|\pi|_v$ is regular at every point z, where π is a uniformizing parameter at z. These conditions characterize $g_{\mathfrak{b}}$ up to the addition of a constant, as the difference of any two such functions would be globally harmonic. The local formula for $\mathfrak{a}=\sum m_x(x)$ is then

$$\langle \mathfrak{a}, \mathfrak{b}\rangle_v = \sum m_x g_{\mathfrak{b}}(x).$$

This is well-defined since $\sum m_x = 0$ and satisfies the required properties since if $\mathfrak{b}=(f)$ we could take $g_{\mathfrak{b}}=\log|f|_v$.

If v is a non-archimedean place, let $\mathcal{O}_v$ denote the valuation ring of k_v and q_v the cardinality of the residue field. Let $\mathscr{X}$ be a regular model for X over $\mathcal{O}_v$ and extend the divisors $\mathfrak{a}$ and $\mathfrak{b}$ to divisors A and B of degree zero on $\mathscr{X}$. These extensions are not unique, but if we insist that A have zero intersection with each fibral component of $\mathscr{X}$ over the residue field, then the intersection product $(A\cdot B)$ is well-defined. We have the formula

$$\langle \mathfrak{a}, \mathfrak{b}\rangle_v = -(A\cdot B)\log q_v.$$

Finally, if X, $\mathfrak{a}$, and $\mathfrak{b}$ are defined over the global field k we have $\langle \mathfrak{a}, \mathfrak{b}\rangle_v=0$ for almost all completions k_v and the sum

$$\langle a, b\rangle = \sum_v \langle \mathfrak{a}, \mathfrak{b}\rangle_v$$

depends only on the classes a and b of $\mathfrak{a}, \mathfrak{b}$ in the Jacobian. This is equal to the global height pairing of Néron and Tate. The same decomposition formula into local symbols can be used even when the divisors $\mathfrak{a}$ and $\mathfrak{b}$ representing a and b have non-disjoint support, provided that the uniformizing parameter π at each point of their common support is chosen over k.

We will apply this theory to compute the global height pairing of Heegner divisors on $X_0^*(N)$ as a sum over places v of $\mathbb{Q}$:

$$\langle y^*_{D_0,r_0}, y^*_{D_1,r_1}\rangle = \sum_v \langle y^*_{D_0,r_0}, y^*_{D_1,r_1}\rangle_v.$$

Since the divisors $y^*_{D,r}$ have the cusp ∞^* in their common support, we must fix a uniformizing parameter π at this cusp. We let π denote the Tate parameter q on the family of degenerating elliptic curves near ∞^*. This is defined over $\mathbb{Q}$, and will even give a uniformizing parameter over $\mathbb{Z}$ on the modular regular model $\mathscr{X}^*$. Over $\mathbb{C}$ we have $q=e^{2\pi i z}$ on $X_0^*(N)(\mathbb{C})\cong \bar{\mathfrak{H}}/\Gamma_0^*(N)$, where $z\in\bar{\mathfrak{H}}$ with $\operatorname{Im}(z)$ sufficiently large.

3. The Archimedean Contribution

Let z_0 and z_1 be points of $\mathfrak{H}$ which are not in the same $\Gamma_0^*(N)$-orbit, and $x_0 \neq x_1$ the corresponding points of $X_0^*(N)(\mathbb{C})$. Then the local symbol

$$\langle (x_0)-(\infty^*), (x_1)-(\infty^*)\rangle_G = G_N^*(z_0, z_1) \tag{1}$$

defines a bi-$\Gamma_0^*(N)$-invariant function G_N^* on $\mathfrak{H}\times\mathfrak{H}$ minus the $\Gamma_0^*(N)$-orbit of the diagonal. The archimedean part of the height pairing of y_{D_0,r_0}^* and y_{D_1,r_1}^* is given by

$$\langle y_{D_0,r_0}^*, y_{D_1,r_1}^*\rangle_\infty = \tfrac{1}{2} G_N^*(P_{D_0,r_0}^*, P_{D_1,r_1}^*), \tag{2}$$

where we are using the convention that $f(\mathfrak{a})$ for a divisor $\mathfrak{a}=\sum m_x(x)$ means $\sum m_x f(x)$. The factor $\frac{1}{2}$ arises because $\mathbb{Q}_\infty$ is $\mathbb{R}$, not $\mathbb{C}$.

We need a formula for the function G_N^*. For $s\in\mathbb{C}$ with $\operatorname{Re}(s)>1$, define the point-pair invariant

$$g_s(z_0, z_1) = -2Q_{s-1}\left(1+\frac{|z_0-z_1|^2}{2\operatorname{Im}(z_0)\operatorname{Im}(z_1)}\right) \quad (z_0, z_1\in\mathfrak{H},\ z_0\neq z_1),$$

where $Q_{s-1}(t)$ is the Legendre function of the second kind. The function defined by the convergent sum

$$G_{N,s}(z_0, z_1) = \sum_{\gamma\in\Gamma_0(N)} g_s(z_0, \gamma z_1)$$

is the resolvent kernel function for $\mathfrak{H}/\Gamma_0(N)$. It is bi-$\Gamma_0(N)$-invariant, finite on $(\mathfrak{H}/\Gamma_0(N))^2$ except for a logarithmic singularity along the diagonal, and has eigenvalue $s(s-1)$ for the hyperbolic Laplacian. Another eigenfunction with this eigenvalue is the Eisenstein series of weight 0 at the cusp ∞:

$$E_N(z,s) = \sum_{\gamma\in\left(\begin{smallmatrix}* & * \\ 0 & *\end{smallmatrix}\right)\backslash\Gamma_0(N)} \operatorname{Im}(\gamma z)^s \quad (z\in\mathfrak{H},\ \operatorname{Re}(s)>1),$$

which satisfies $E_N(z,s)=y^s+\phi_N(s)y^{1-s}+O(e^{-y})$ as $y\to\infty$ for a certain meromorphic function $\phi_N(s)$, specified below. Let $G_{N,s}^*$, E_N^*, ϕ_N^* be the corresponding functions when $\Gamma_0(N)$ is replaced by $\Gamma_0^*(N)$. We have

$$G_{N,s}^*(z_0, z_1) = G_{N,s}(z_0, z_1) + G_{N,s}(z_0, w_N z_1), \qquad E_N^*(z,s) = E_N(z,s) + E_N(w_N z, s). \tag{3}$$

Proposition 1. *The function G_N^* defined by* (1) *is given by*

$$G_N^*(z_0, z_1) = \lim_{s\to 1}\left[G_{N,s}^*(z_0, z_1) - \frac{4\pi}{1-2s}(E_N^*(z_0,s)+E_N^*(z_1,s)-\phi_N^*(s))\right]. \tag{4}$$

Proof. The function G_N^* of (1) satisfies

(i) G_N^* is bi-$\Gamma_0^*(N)$-invariant, symmetric, and harmonic in each variable;

(ii) for fixed z_0,

$$G_N^*(z_0, z_1) = e_{z_0}\log|z_0-z_1|^2 + O(1) \quad \text{as} \quad z_1\to z_0,$$
$$G_N^*(z_0, z_1) = 4\pi\operatorname{Im}(z_1) + o(1) \quad \text{as} \quad z_1\to i\infty,$$

and $G_N^*(z_0, z_1)=O(1)$ as z_1 tends to any point of $\mathfrak{H}$ not $\Gamma_0^*(N)$-equivalent to z_0 or to any cusp not $\Gamma_0^*(N)$-equivalent to ∞. [Here e_{z_0} is the order of the stabilizer of z_0 in $\Gamma_0^*(N)$.]

Indeed, from the axioms in Sect. 2 and the choice of the uniformizing parameter $q=e^{2\pi iz}$ at ∞ we see that

$$G_N^*(z_0, z_1)=f(z_1)-\lim_{z\to i\infty}(f(z)+\log|e^{2\pi iz}|_{\mathbb{C}})$$
$$=f(z_1)+\lim_{z\to i\infty}(f(z)-4\pi\,\mathrm{Im}(z)),$$

where f is a Green's function associated to the divisor $(x_0)-(\infty^*)$ on $X_0^*(N)$. This makes the harmonicity and the behavior as $x_1\to$cusp obvious, while for $x_1\to x_0$ we have $f(x_1)=\log|\pi(x_1)|_{\mathbb{C}}+O(1)$ where π is a uniformizing parameter at x_0, and one can take π to be $(z-z_0)^{e_{z_0}}$. The expression on the right-hand side of (4) also satisfies (i) and (ii), by the same arguments as in [9, pp. 239–241] and [8, p. 208]. This shows that the two expressions are equal, since their difference is globally harmonic, hence constant, and vanishes at ∞^*.

Combining (2) and (4) and recalling that $y_{D,r}^*=P_{D,r}^*-H(D)(\infty^*)$ with $P_{D,r}^*$ of degree $H(D)$, we find

$$\langle y_{D_0,r_0}^*, y_{D_1,r_1}^*\rangle_\infty=\tfrac{1}{2}\lim_{s\to 1}\Bigg[G_{N,s}^*(P_{D_0,r_0}^*, P_{D_1,r_1}^*)-\frac{4\pi}{1-2s}(H(D_1)E_N^*(P_{D_0,r_0}^*, s)$$
$$+H(D_0)E_N^*(P_{D_1,r_1}^*, s)-H(D_0)H(D_1)\phi_N^*(s))\Bigg]. \tag{5}$$

We now proceed to evaluate each of the four terms in the square brackets. From (3) and Eqs. (1), (2) and $w_N P_{D,r}=P_{D,-r}$ of Sect. 1 we have

$$G_{N,s}^*(P_{D_0,r_0}^*, P_{D_1,r_1}^*)=G_{N,s}(P_{D_0,r_0}, P_{D_1,r_1})+G_{N,s}(P_{D_0,r_0}, P_{D_1,-r_1}), \tag{6}$$

so for the first term in (5) it suffices to evaluate $G_{N,s}(P_{D_0,r_0}, P_{D_1,r_1})$.

Proposition 2. *For* $n\in\mathbb{Z}$, $n^2\equiv D_0D_1\ (4N)$, *define*

$$\varrho(n)=\sum_{d\,\big|\,\frac{n^2-D_0D_1}{4N}}\varepsilon(d), \tag{7}$$

where ε *is associated to the quadratic form* $[D_0, -2n, D_1]$ *as in Sect. 3 of Chap. I. Then for* $\mathrm{Re}(s)>1$,

$$G_{N,s}(P_{D_0,r_0}, P_{D_1,r_1})=-2\sum_{\substack{n>\sqrt{D_0D_1}\\ n\equiv -r_0r_1(2N)}}\varrho(n)Q_{s-1}\left(\frac{n}{\sqrt{D_0D_1}}\right). \tag{8}$$

Proof. From (1) of Sect. 1 and the definition of $G_{N,s}$ we have

$$G_{N,s}(P_{D_0,r_0}, P_{D_1,r_1})=\sum_{(\tau_0,\tau_1)\in(\mathscr{P}_{D_0,r_0}\times\mathscr{P}_{D_1,r_1})/\Gamma_0(N)^2}G_{N,s}(\tau_0,\tau_1)$$
$$=\sum_{(\tau_0,\tau_1)\in(\mathscr{P}_{D_0,r_0}\times\mathscr{P}_{D_1,r_1})/\Gamma_0(N)}g_s(\tau_0,\tau_1),$$

where in the second equation $\Gamma_0(N)$ acts diagonally on $\mathscr{P}_{D_0,r_0}\times\mathscr{P}_{D_1,r_1}$ and in both equations we are making our usual conventions about multiplicities [i.e., each term $G_{N,s}(\tau_0,\tau_1)$ or $g_s(\tau_0,\tau_1)$ is to be weighted with a factor equal to the reciprocal of the order of the stabilizer of (τ_0,τ_1) in $\Gamma_0(N)^2$ or $\Gamma_0(N)$]. Associated to τ_i $(i=0,1)$ we have the positive definite quadratic form $q_i=[a_iN, b_i, c_i]$ of discriminant D_i, with

$b_i \equiv r_i\,(2N)$, with $q_i(\tau_i, 1)=0$. Then

$$g_s(\tau_0,\tau_1) = -2Q_{s-1}\left(1+\frac{|\tau_0-\tau_1|^2}{2\operatorname{Im}(\tau_0)\operatorname{Im}(\tau_1)}\right) = -2Q_{s-1}\left(\frac{n}{\sqrt{D_0D_1}}\right)$$

with $n=2N(a_0c_1+a_1c_0)-b_0b_1$, i.e., $n=-B_{\Delta_N}(q_0,q_1)$ in the notation of Chap. I, Sect. 3. Clearly $n>\sqrt{D_0D_1}$ and $n\equiv -r_0r_1\,(2N)$. This proves (8) with $\varrho(n)$ replaced by

$$\tilde{\varrho}(n) = \#\{(\tau_0,\tau_1)\in(\mathscr{P}_{D_0,r_0}\times\mathscr{P}_{D_1,r_1})/\Gamma_0(N)\mid B_{\Delta_N}(q_0,q_1)=-n\}$$

or equivalently – since there is a 1:1 correspondence between positive definite forms in $\mathscr{Q}_{N,D,r}$ and their roots in $\mathfrak{H}$ –

$$\tilde{\varrho}(n)=\tfrac{1}{2}\,\#\{(q_0,q_1)\in(\mathscr{Q}_{N,D_0,r_0}\times\mathscr{Q}_{N,D_1,r_1})/\Gamma_0(N)\mid B_{\Delta_N}(q_0,q_1)=-n\}.$$

[The $\frac{1}{2}$ arises because the condition $B_{\Delta_N}(q_0,q_1)=-n$ forces q_0 to both be positive definite or both negative definite, and we want to count only the former.] Let t denote the number of prime factors of N. Then

$$2^t\tilde{\varrho}(n)=\tfrac{1}{2}\,\#\{(q_0,q_1)\in(\mathscr{Q}_{N,D_0}\times\mathscr{Q}_{N,D_1})/\Gamma_0(N)\mid B_{\Delta_N}(q_0,q_1)=-n\}$$

because the group $W\cong(\mathbb{Z}/2\mathbb{Z})^t$ of Atkin-Lehner involutions acts freely on the set of pairs $(r_0,r_1)\,(\mathrm{mod}\,2N)$ with $r_i^2\equiv D_i\,(4N)$ and with a fixed product $r_0r_1\,(\mathrm{mod}\,2N)$, since D_0 and D_1 are coprime. This last expression equals $2^t\varrho(n)$ by the Corollary to Proposition 3 of Chap. I, Sect. 3, so $\tilde{\varrho}(n)=\varrho(n)$. This proves (8).

We now turn to the Eisenstein series $E_N^*(z,s)$. Here we must evaluate, for $D=D_0$ or D_1, the sum

$$\sum_{\tau\in\mathscr{P}_{D,r}^*/\Gamma_0^*(N)} E_N^*(\tau,s) = \sum_{\tau\in\mathscr{P}_{D,r}^*/\Gamma_0(N)} E_N(\tau,s).$$

To do this, we recall that $E_N(z,s)$ can be expressed [9, (2.16)] as

$$E_N(z,s)=N^{-s}\prod_{p\mid N}(1-p^{-2s})^{-1}\sum_{d\mid N}\frac{\mu(d)}{d^s}E\left(\frac{N}{d}z,s\right), \tag{9}$$

where $E(z,s)$ is the Eisenstein series for $SL_2(\mathbb{Z})$. Hence our sum becomes

$$N^{-s}\prod_{p\mid N}(1-p^{-2s})^{-1}\sum_{d\mid N}\frac{\mu(d)}{d^s}\left(\sum_{\tau\in\mathscr{P}_{D,r}/\Gamma_0(N)}E\left(\frac{N}{d}\tau,s\right)+\sum_{\tau\in\mathscr{P}_{D,-r}/\Gamma_0(N)}E\left(\frac{N}{d}\tau,s\right)\right).$$

The association $[Na,b,c]\mapsto\left[\frac{N}{d}a,b,dc\right]$ identifies $\mathscr{P}_{D,r}/\Gamma_0(N)$ with $\mathscr{P}_D/\Gamma_0(1)$ for any divisor d of N, by the proposition of Chap. I, Sect. 1. Hence each of the inner sums is independent of d and equal to $\sum_{\tau\in\mathscr{P}_D/SL_2(\mathbb{Z})} E(\tau,s)$. But this sum is given by the "zeta-function"

$$\sum_{\tau\in\mathscr{P}_D/SL_2(\mathbb{Z})}E(\tau,s)=2^{-s}|D|^{s/2}\zeta(2s)^{-1}\zeta_D(s)=2^{-s}|D|^{s/2}\zeta(2s)^{-1}\zeta(s)L_D(s),$$

where $L_D(s)$ is the holomorphic function of s introduced in [21] with $L_D(1)=\frac{H(D)}{\sqrt{|D|}}$.

[If D is fundamental, then $\zeta_D(s)$ is the Dedekind zeta function of $\mathbb{Q}(\sqrt{D})$ and $L_D(s)=L\left(s,\left(\frac{D}{\cdot}\right)\right)$, while for general D it differs from this by a finite Euler product over prime factors of the conductor.] Since $\sum_{d|N} \mu(d)d^{-s} = \prod_{p|N}(1-p^{-s})$, we find that

$$E_N^*(P_{D,r}^*, s) = \frac{2^{1-s}|D|^{s/2}}{N^s \prod_{p|N}(1+p^{-s})} \cdot \frac{\zeta(s)}{\zeta(2s)} \cdot L_D(s)$$

and consequently

$$E_N^*(P_{D,r}^*, s) = \frac{6}{\pi} \frac{H(D)}{N \prod_{p|N}(1+p^{-1})} \left[\frac{1}{s-1} + \left(\log \frac{|D|^{1/2}}{2N} + \sum_{p|N} \frac{\log p}{p+1} + \gamma \right.\right.$$
$$\left.\left. -2\frac{\zeta'}{\zeta}(2) + \frac{L_D'}{L_D}(1) \right) + O(s-1) \right] \quad \text{as} \quad s \to 1 \tag{10}$$

(γ = Euler's constant).

Finally, we must obtain an expression for $\phi_N^*(s)$, the coefficient of y^{1-s} in the Fourier expansion of $E_N^*(z,s)$ at ∞. But $E_N^*(z,s)$ is given in terms of $E(z,s)$ by (3) and (9). For the Eisenstein series of level 1, we know that

$$E(z,s) = y^s + \frac{\Gamma(\frac{1}{2})\Gamma(s-\frac{1}{2})\zeta(2s-1)}{\Gamma(s)\zeta(2s)} y^{1-s} + O(e^{-y})$$

as $y \to \infty$. Hence

$$\phi_N^*(s) = N^{-s} \prod_{p|N}(1-p^{-2s})^{-1} \frac{\Gamma(\frac{1}{2})\Gamma(s-\frac{1}{2})\zeta(2s-1)}{\Gamma(s)\zeta(2s)} \left[\sum_{d|N} \frac{\mu(d)}{d^s} \left\{ \left(\frac{N}{d}\right)^{1-s} + d^{1-s} \right\} \right]$$
$$= \frac{\Gamma(\frac{1}{2})\Gamma(s-\frac{1}{2})\zeta(2s-1)}{\Gamma(s)\zeta(2s)} \left[N^{1-2s} \prod_{p|N} \frac{1-p^{-1}}{1-p^{-2s}} + N^{-s} \prod_{p|N} \frac{1-p^{-2s+1}}{1-p^{-2s}} \right].$$

For s near 1 this gives

$$\phi_N^*(s) = \frac{6}{\pi} \frac{1}{N \prod_{p|N}(1+p^{-1})} \left[\frac{1}{s-1} + \left(-\frac{3}{2}\log N + \sum_{p|N} \frac{\log p}{p+1} \right.\right.$$
$$\left.\left. -\log 4 + 2\gamma - 2\frac{\zeta'}{\zeta}(2) \right) + O(s-1) \right] \quad \text{as} \quad s \to 1. \tag{11}$$

Combining (5), (6), (8) [applied to both $(D_0, r_0), (D_1, r_1)$ and $(D_0, r_0), (D_1, -r_1)$], (10), and (11), we obtain our final result:

Theorem.

$$\langle y_{D_0,r_0}^*, y_{D_1,r_1}^* \rangle_\infty$$
$$= \lim_{s \to 1} \left[-\sum_{\substack{n^2 > D_0 D_1 \\ n \equiv r_0 r_1 (2N)}} \varrho(n) Q_{s-1}\left(\frac{|n|}{\sqrt{D_0 D_1}} \right) - \frac{\lambda}{s-1} \right]$$
$$+ \lambda \left[\frac{1}{2} \log \frac{N}{D_0 D_1} - \sum_{p|N} \frac{\log p}{p+1} + 2\frac{\zeta'}{\zeta}(2) - \frac{L_{D_0}'}{L_{D_0}}(1) - \frac{L_{D_1}'}{L_{D_1}}(1) + 2 \right]$$

with $\lambda = -12H(D_0)H(D_1)/N \prod_{p|N}\left(1+\frac{1}{p}\right)$ *and* $\varrho(n)$ *as in* (7).

4. The Contributions from Finite Places

We fix a finite prime p of $\mathbb{Q}$. To calculate the local symbol $\langle y^*_{D_0,r_0}, y^*_{D_1,r_1}\rangle_p$ in terms of intersection theory, we need a regular model $\mathscr{X}^*$ for $X_0^*(N)$ over the valuation ring $\mathbb{Z}_p$ of $\mathbb{Q}_p$.

A modular model $\underline{X}$ for $X_0(N)$ over $\mathbb{Z}$ was constructed by Katz and Mazur [11], using ideas of Drinfeld. The scheme $\underline{X}$ is smooth over $\mathbb{Z}[1/N]$ and regular except at closed points $\underline{x}$ in characteristic $p|N$ where $\mathrm{Aut}(\underline{x}) \neq \{\pm 1\}$. We let $\mathscr{X}^*$ denote the minimal desingularization of the quotient of $\underline{X}$ by the Fricke involution w_N. Then $\mathscr{X}^*$ is smooth over $\mathbb{Z}_p$ when $p\nmid N$. When $p|N$ the curve $\mathscr{X}^* \otimes \mathbb{Z}/p$ may have several components.

Let $\underline{P}^*_{D,r}$ be the multi-section of $\mathscr{X}^*$ over $\mathbb{Z}_p$ which extends $P^*_{D,r}$ and $\underline{\infty}^*$ the section corresponding to the cusp ∞^*.

Proposition 1. $\langle y^*_{D_0,r_0}, y^*_{D_1,r_1}\rangle_p = -(\underline{P}^*_{D_0,r_0} \cdot \underline{P}^*_{D_1,r_1})_p \log p$.

Proof. Since the discriminants D_0 and D_1 are relatively prime, we may assume that $p\nmid D_0$. We then claim that the divisor $\underline{P}^*_{D_0,r_0} - H(D_0)\underline{\infty}^*$ has zero intersection with each fibral component of $\mathscr{X}^*$ over $\mathbb{Z}/p$. This is clear when $p\nmid N$ and $\mathscr{X}^*$ is smooth, with a single component over $\mathbb{Z}/p$, so assume that $p|N$. Since $p\nmid D_0$ and $D_0 \equiv r_0^2\,(4N)$, we see that p must split in the imaginary quadratic field $\mathbb{Q}(\sqrt{D_0})$. Let $\mathfrak{p}$ be the unique factor of p which divides $\dfrac{r_0+\sqrt{D_0}}{2}$. Then the points $\underline{x}$ in the divisor $\underline{P}_{D_0,r_0}$ all reduce $(\mathrm{mod}\,\mathfrak{p})$ to the component $\mathscr{F}_{n,0}$ of $\underline{X}\otimes\mathbb{Z}/p$ containing the cusp $\underline{\infty}$ [9, (3.1)] and all reduce $(\mathrm{mod}\,\bar{\mathfrak{p}})$ to the component $\mathscr{F}_{0,n}$ of $\underline{X}\otimes\mathbb{Z}/p$ containing the cusp $\underline{0}$. These cusps, and the corresponding components, are interchanged by w_N and give rise to a single component $\mathscr{F}^*$ containing $\underline{\infty}$ $(\mathrm{mod}\,p)$ on $\underline{X}/w_N$. Since the reduction is ordinary, the points in $\underline{P}^*_{D_0,r_0}$ all reduce to regular points on $\mathscr{F}^*$, so lie in the same component as $\underline{\infty}^*$ in $\mathscr{X}^*$. Hence $\underline{P}^*_{D_0,r_0} - H(D_0)\underline{\infty}^*$ has zero intersection with this special component; it clearly has zero intersection with all others.

From the general theory described in Sect. 2, we have the identity

$$\langle y^*_{D_0,r_0}, y^*_{D_1,r_1}\rangle_p = -((\underline{P}^*_{D_0,r_0} - H(D_0)\underline{\infty}^*)\cdot(\underline{P}^*_{D_1,r_1} - H(D_1)\underline{\infty}^*))_p \log p\,.$$

But the points in $\underline{P}^*_{D,r}$ have no intersection with $\underline{\infty}^*$ $(\mathrm{mod}\,p)$, since they correspond to elliptic curves with complex multiplication and hence have potentially good reduction at all primes. Finally, since q is a uniformizing parameter at $\underline{\infty}^*$ over $\mathbb{Z}_p$, we find that the local symbol is calculated using the convention that $(\underline{\infty}^*\cdot\underline{\infty}^*)_p = 0$ [6]. Hence the intersection product on the right-hand side reduces to $-(\underline{P}^*_{D_0,r_0}\cdot \underline{P}^*_{D_1,r_1})\log p$.

We now turn to the computation of the relevant intersection multiplicities. First we have:

Proposition 2. *If* $p|N$ *then* $(\underline{P}^*_{D_0,r_0}\cdot\underline{P}^*_{D_1,r_1})_p = 0$.

Proof. We may assume $p\nmid D_0$, so the points in $\underline{P}_{D_0,r_0}$ have ordinary reduction on $\underline{X}$. If p is split in $\mathbb{Q}(\sqrt{D_1})$ the same is true for the points in $\underline{P}_{D_1,r_1}$. By Deuring's theory [3], singular points with ordinary reduction and distinct quadratic fields of multiplication reduce to distinct points (mod$p$). If p is not split in $\mathbb{Q}(\sqrt{D_1})$, the points in $\underline{P}_{D_1,r_1}$ have supersingular reduction on $\underline{X}$, so are disjoint from the points in the reduction of $\underline{P}_{D_0,r_0}$, which are ordinary. Hence the intersection number is zero in all cases.

We henceforth assume that $p\nmid N$, so $\underline{X}$ is smooth over $\mathbb{Z}_p$ and $\mathscr{X}^* \cong \underline{X}/w_N$. Denote the projection map by $F: \underline{X} \to \mathscr{X}^*$; then $F_*(\underline{P}_{D,r}) = \underline{P}^*_{D,r}$ and $F^*(\underline{P}^*_{D,r}) = \underline{P}_{D,r} + \underline{P}_{D,-r}$. Since D_0 and D_1 are relatively prime, we may re-order them so $p\nmid D_0$. Fix a square root $\sqrt{D_0}$ of D_0 in the completion W of the maximal unramified extension of $\mathbb{Z}_p$. A Heegner point $x=(\pi: E \to E')$ lies in the divisor P_{D_0,r_0} if $\mathcal{O}_0 = \mathbb{Z} + \mathbb{Z}\frac{D_0+\sqrt{D_0}}{2}$ embeds into $\mathrm{End}(x)$ and $\alpha_0 = \frac{r_0+\sqrt{D_0}}{2} \in \mathrm{End}(x)$ annihilates $\ker\pi$. Here, and in what follows, we will write $\sqrt{D_0}$ for the endomorphism which gives multiplication by our fixed square root of D_0 on the tangent space. The points $\underline{x}$ in $\underline{P}_{D_0,r_0}$ are all rational over W. A point y lies in the divisor $P_{D_1,r_1} + P_{D_1,-r_1}$ if the order $\mathcal{O}_1$ of discriminant D_1 acts on y and the cyclic subgroup of order N in its diagram is annihilated by an element $\alpha_1 \in \mathcal{O}_1$ of trace r_1 (hence also by $-\alpha_1$ of trace $-r_1$). The divisor $\underline{P}_{D_1,r_1} + \underline{P}_{D_1,-r_1}$ is always rational over W; its points $\underline{y}$ are all rational over W iff $p\nmid D_1$. We have

$$(\underline{P}^*_{D_0,r_0} \cdot \underline{P}^*_{D_1,r_1})_p = (\underline{P}_{D_0,r_0} \cdot (\underline{P}_{D_1,r_1} + \underline{P}_{D_1,-r_1}))_W \tag{1}$$

and we now turn to the intersection product on $\underline{X}$ over W.

Assume first that $p\nmid D_1$. Then we may fix a square root $\sqrt{D_1}$ in W. A point $\underline{y} = (\psi: F \to F')$ lies in the divisor $\underline{P}_{D_1,r_1}$ over W if the endomorphism $\alpha_1 = \frac{r_1+\sqrt{D_1}}{2}$ annihilates $\ker\psi$. Suppose that $(\underline{x}\cdot \underline{y})_W > 0$ for $\underline{x} \in \underline{P}_{D_0,r_0}$ and $\underline{y} \in \underline{P}_{D_1,r_1}$. Then our diagrams reduce to the same isogeny $z \equiv (\phi: E \to E') \equiv (\psi: F \to F')$ on $\underline{X} \otimes W/pW$. Write R for the endomorphism ring $\mathrm{End}_{W/pW}(z)$. The reduction of endomorphisms gives injections $\mathrm{End}_W(\underline{x}) \hookrightarrow R$, $\mathrm{End}_W(\underline{y}) \hookrightarrow R$. Since D_0 and D_1 are coprime, R cannot be an order in a quadratic field, so it follows from Deuring's theory that z is a supersingular point on $\underline{X} \otimes W/pW$, that $R \otimes \mathbb{Q}$ is a quaternion algebra over $\mathbb{Q}$ ramified only at p and at infinity, and that R is an Eichler order of index N in this quaternion algebra. Moreover, the embeddings of $\mathrm{End}_W(\underline{x})$ and $\mathrm{End}_W(\underline{y})$ give elements $\sqrt{D_i}$ and $\alpha_i = (r_i + \sqrt{D_i})/2$ $(i=0,1)$ in R satisfying

$$\sqrt{D_0}\sqrt{D_1} + \sqrt{D_1}\sqrt{D_0} = 2n \quad \text{for some} \quad n \in \mathbb{Z}, \quad n \equiv r_0 r_1 \,(\mathrm{mod}\, 2N), \quad n^2 < r_0 r_1 .$$

[Proof. $\sqrt{D_0}\sqrt{D_1} + \sqrt{D_1}\sqrt{D_0} = 4\mathrm{Tr}(\alpha_0\alpha_1) - 2r_0 r_1$ is an even integer $2n$ with $n \equiv r_0 r_1$ (2). The elements α_0 and α_1 lie in the annihilator of $\ker(\phi) = \ker(\psi)$, which is a two-sided ideal $I \subset R$ of index N. Locally at a prime $\ell | N$ we can identify $R \otimes \mathbb{Z}_\ell$ with $\begin{pmatrix} \mathbb{Z}_\ell & \mathbb{Z}_\ell \\ N\mathbb{Z}_\ell & \mathbb{Z}_\ell \end{pmatrix}$ and $I \otimes \mathbb{Z}_\ell$, $\bar{I} \otimes \mathbb{Z}_\ell$ with $\begin{pmatrix} N\mathbb{Z}_\ell & \mathbb{Z}_\ell \\ N\mathbb{Z}_\ell & \mathbb{Z}_\ell \end{pmatrix}$ and $\begin{pmatrix} \mathbb{Z}_\ell & \mathbb{Z}_\ell \\ N\mathbb{Z}_\ell & N\mathbb{Z}_\ell \end{pmatrix}$, respectively. This shows that $\mathrm{Tr}(I\bar{I}) \subseteq N\mathbb{Z}$ and hence $N | \mathrm{Tr}(\alpha_0 \bar{\alpha}_1)$, which is equivalent to the congruence $n \equiv r_0 r_1$ $(2N)$. Finally, $n^2 < D_0 D_1$ since $R \otimes \mathbb{Q}$ is a definite quaternion algebra and $\sqrt{D_0}\sqrt{D_1}$ a non-central element of trace $2n$ and norm $D_0 D_1$.]

Thus we get an embedding of

$$S = S_{[D_0, 2n, D_1]} = \mathbb{Z} + \mathbb{Z}\frac{r_0 + e_0}{2} + \mathbb{Z}\frac{r_1 + e_1}{2} + \mathbb{Z}\frac{r_0 r_1 + r_0 e_1 + r_1 e_0 + e_0 e_1}{4}$$

$$(e_0^2 = D_0,\ e_1^2 = D_1,\ e_0 e_1 + e_1 e_0 = 2n),$$

the Clifford order studied in Sect. 3 of Chap. I, into R by sending e_i to $\sqrt{D_i}$. Actually, the intersection of $\underline{x}$ and $\underline{y}$ gives *two* embeddings of S into R, as we could also have mapped e_0 to $-\sqrt{D_0}$ and e_1 to $-\sqrt{D_1}$. Finally, the intersection number $(\underline{x} \cdot \underline{y})_W$ is given by

$$(\underline{x} \cdot \underline{y})_W = \tfrac{1}{2}(\operatorname{ord}_p(M) + 1), \quad \text{where} \quad M = \frac{D_0 D_1 - 4n^2}{4N},$$

if we assume R has no units besides ± 1. Indeed, by [9, Proposition 6.1] we have

$$(\underline{x} \cdot \underline{y})_W = \tfrac{1}{2} \sum_{i \geq 1} \operatorname{Card} \operatorname{Hom}_{W/p^i W}(\underline{x}, \underline{y})_{\deg 1},$$

and when $|R^\times| = 2$ the i^{th} summand equals 2 for $i \leq k$ and 0 for $i > k$, where k is the largest integer such that $\underline{x} \equiv \underline{y} \pmod{p^k}$. The condition $\underline{x} \equiv \underline{y} \pmod{p^i}$ means that S embeds into $\operatorname{End}_{W/p^i W}(x) = \operatorname{End}_{W/p^i W}(y)$, which is the unique suborder of index p^{2i-2} in R containing the ring of integers in a quadratic field inert at p [7], and this happens if $p^{2i-2} | M$. Hence $k = (\operatorname{ord}_p(M) + 1)/2$, and this proves our claim.

We now consider the converse. Pick a supersingular point z on $\underline{X} \otimes W/pW$ and let $R = \operatorname{End}_{W/pW}(z)$ and $I \subset R$ the ideal annihilating the kernel of the cyclic N-isogeny defined by z. Fix a binary quadratic form $[D_0, 2n, D_1]$ with $n^2 < D_0 D_1$ and $n \equiv r_0 r_1 \pmod{2N}$, let S be the associated Clifford order and $M = (D_0 D_1 - n^2)/4N$. To each embedding ϕ of S into R, normalized by insisting that $\phi(e_0) = \sqrt{D_0}$ on the tangent space, we wish to attach a certain contribution to the intersection pairing of the divisors $\underline{P}_{D_0}$ and $\underline{P}_{D_1}$ on $\underline{X}$ over W. Write $\phi(e_1) = \varepsilon \sqrt{D_1}$ with $\varepsilon = \pm 1$. The ideal I contains the elements $\frac{s_0 + \sqrt{D_0}}{2}$ and $\frac{s_1 + \varepsilon\sqrt{D_1}}{2}$ for some integers s_0, s_1 (well-defined modulo N), and the proof above shows that $s_0 s_1 \equiv n (2N)$. Via the lifting theorem [8, Proposition 2.7], which also holds when the conductor of the order is prime to the characteristic, the given embeddings of $\mathcal{O}_0$ and $\mathcal{O}_1$ into R correspond to points $\underline{x} \in \underline{P}_{D_0, r_0}$, $\underline{y} \in \underline{P}_{D_1, \varepsilon s_1}$ which reduce to $z \pmod{p}$ and are congruent modulo p^k where $k = (\operatorname{ord}_p(M) + 1)/2$.

Since the total intersection is given by the sum over all the supersingular points in the special fibre, which correspond to the right orders R_i of the distinct left ideal classes for a fixed Eichler order of index N in the quaternion algebra $B(p)$ ramified at p and ∞, we find that

$$\sum_{s_0 s_1 \equiv r_0 r_1 (2N)} (\underline{P}_{D_0, s_0} \cdot (\underline{P}_{D_1, s_1} + \underline{P}_{D_1, -s_1}))_W$$

$$= \sum_{R_i} \sum_{\substack{n \equiv r_0 r_1 (2N) \\ n^2 < D_0 D_1}} \frac{1}{2} \# \{S_{[D_0, 2n, D_1]} \to R_i \bmod R_i^\times / \pm 1\} \cdot \frac{\operatorname{ord}_p(M) + 1}{2}.$$

The factor 1/2 in the last sum comes from the fact that we are taking only the normalized embeddings [those with $\phi(e_0)=\sqrt{D_0}$]. The left-hand side is equal to $2^t(\underline{P}_{D_0,r_0}\cdot(\underline{P}_{D_1,r_1}+\underline{P}_{D_1,-r_1}))_W$, where t is the number of prime divisors of N, because the group of Atkin-Lehner involutions has order 2^t and acts freely on the pairs $(s_0,s_1)\,(\mathrm{mod}\,2N)$ with product $r_0r_1\,(\mathrm{mod}\,2N)$. Hence we obtain the formula

$$\begin{aligned}&(\underline{P}_{D_0,r_0}\cdot(\underline{P}_{D_1,r_1}+\underline{P}_{D_1,-r_1}))_W\\&=\frac{1}{2^{t+1}}\sum_{\substack{n\equiv r_0r_1(2N)\\ n^2<D_0D_1}}\sum_{R_i}\#\{S_{[D_0,2n,D_1]}\to R_i \bmod R_i^\times/\pm 1\}\cdot\frac{\mathrm{ord}_p(M)+1}{2}.\end{aligned}\tag{2}$$

We have proved (2) under the assumption that all R_i have only the units ± 1, but in fact the formula as written (i.e., with the embeddings being counted only up to conjugation by $R_i/\pm 1$) is true without this assumption. We omit the proof. We have also been assuming that $p\nmid D_1$, but in fact (2) remains true also if $p|D_1$. We now indicate the changes in the argument which must be made in this case. If p does not divide the conductor of D_1, then the points in $\underline{P}_{D_1,r_1}$ are pointwise rational over the ramified quadratic extension $W[\sqrt{D_1}]$ and are conjugate, over W, to the points in the divisor $\underline{P}_{D_1,-r_1}$. Hence the left-hand side of (2) is equal to $(\underline{P}_{D_0,r_0}\cdot\underline{P}_{D_1,r_1})_\pi$, where π is a prime in $W[\sqrt{D_1}]$. Again, such an intersection gives an embedding of $S_{[D_0,2n,D_1]}$ into an Eichler order R in $B(p)$; in this case $\mathrm{ord}_p(M)$ $=\mathrm{ord}_p\left(\dfrac{D_0D_1-4n^2}{4N}\right)=1$ as $p^2|n^2$ so the relevant points $\underline{x}$ and $\underline{y}$ in these divisors are congruent $(\mathrm{mod}\,\pi)$ but not $(\mathrm{mod}\,\pi^2)$. Here, however, when we consider the converse, an embedding $S\to R$ always lifts to an intersection in $\underline{P}_{D_0,s_0}\cdot\underline{P}_{D_1,s_1}$ with $s_0s_1\equiv r_0r_1\,(\mathrm{mod}\,2N)$, as $\varepsilon\sqrt{D_1}\equiv\sqrt{D_1}\equiv 0\,(\mathrm{mod}\,\pi)$. Hence (2) continues to hold in this case. If, on the other hand, p divides the conductor of D_1, then p is prime to N and $D_1=p^{2s}D_1'$, $r_1\equiv p^s r_1'\,(\mathrm{mod}\,2N)$ with $s\geqq 1$ and the conductor of D_1' prime to p. As before, an intersection of $\underline{x}\in\underline{P}_{D_0,r_0}$ and $\underline{y}\in\underline{P}_{D_1,r_1}$ gives rise to an embedding of the Clifford algebra $S_{[D_0,2n,D_1]}$ with $n\equiv r_0r_1\,(2N)$ into an Eichler order of index N in $B(p)$. If $M=(D_0D_1-4n^2)/4N$, then $\mathrm{ord}_p(M)$ is odd. Hence $n=p^sn'$, $M=p^{2s}M'$ and any embedding of $S_{[D_0,2n,D_1]}$ extends to an embedding of the larger order $S_{[D_0,2n',D_1']}$. Our previous arguments show that embeddings of the larger order contribute to the intersection pairing of $\underline{P}_{D_0,r_0}$ with $\underline{P}_{D_1',r_1'}+\underline{P}_{D_1',-r_1'}$, with multiplicity $\frac{1}{2}(\mathrm{ord}_p(M')+1)$. On the other hand, the divisor $\underline{P}_{D_1,r_1}$ has the form $\underline{P}_{D_1',r_1'}+\sum(\underline{y})$ where the points $\underline{y}$ are locally quasi-canonical liftings of levels $1\leqq r\leqq s$ of points $\underline{y}'\in\underline{P}_{D_1',r_1'}$ [7, Sect. 5]. Since each quasi-canonical lifting $\underline{y}$ is congruent to $\underline{y}'$ with multiplicity 1, the total contribution of an embedding of $S_{[D_0,2n,D_1]}$ to our intersection pairing is $\frac{1}{2}(\mathrm{ord}_p(M')+1)+s=\frac{1}{2}(\mathrm{ord}_p(M)+1)$ as before.

We now combine Proposition 1 with formulas (1) and (2) to obtain the following formula for the local height pairing:

$$\begin{aligned}\langle y^*_{D_0,r_0},y^*_{D_1,r_1}\rangle_p=&\sum_{\substack{n\equiv r_0r_1(2N)\\ n^2<D_0D_1}}2^{-t-1}\sum_{R_i}\#\{\phi:S\to R_i \bmod R_i^\times/\pm 1\}\\&\times\frac{\mathrm{ord}_p(M)+1}{2}\cdot(-\log p).\end{aligned}$$

We can simplify this formula by applying the results of Sect. 3 of Chap. I, which count the number of embeddings of S, the Clifford order of $[D_0, 2n, D_1]$, into Eichler orders in $S\otimes\mathbb{Q}$ in terms of the Dirichlet series $\ell(s)=\sum_{d|M}\varepsilon(d)d^s$. Since we are assuming that $D_0, D_1<0$ and $n^2<D_0D_1$, the algebra $B=S\otimes\mathbb{Q}$ is definite and hence ramifies at at least one finite place. Hence $\ell(0)=0$ and the derivative $\ell'(0)=\sum_d\varepsilon(d)\log d$ is non-zero precisely when B is isomorphic to some $B(p)$. In this case, we showed in Proposition 3 of Chap. I, Sect. 3, that

$$\ell'(0)=2^{-t-1}\sum_{R_i}\#\{S\to R_i^\times \bmod R_i/\pm 1\}\cdot\tfrac{1}{2}(\operatorname{ord}_p(M)+1)\cdot(-\log p).$$

Comparing this with our formula for the local height, we obtain our final result:

Theorem. *The total non-archimedean contribution to the height pairing of the Heegner divisors $y^*_{D_0,r_0}$ and $y^*_{D_1,r_1}$ is given by*

$$\sum_{p\text{ prime}}\langle y^*_{D_0,r_0}, y^*_{D_1,r_1}\rangle_p=\sum_{\substack{n\equiv r_0r_1(2N)\\ n^2<D_0D_1}}\left(\sum_{d\,\middle|\,\frac{D_0D_1-n^2}{4N}}\varepsilon(d)\log d\right),$$

where ε is the (± 1)-valued function associated to the quadratic form $[D_0,\cdot, D_1]$.

V. Heights and L-Series

In this chapter we combine the results of Chaps. II–IV to obtain the principal results as stated in the introduction. The final section discusses the form these results take for elliptic curves parametrized by modular forms.

1. Hecke Operators and the Main Identity

Let D_0 and D_1 be coprime negative fundamental discriminants with $D_i\equiv r_i^2 \pmod{4N}$ and $F=F_{k,N,D_0D_1,r_0r_1,D_0}=\sum a_m q^m$ the cusp form in $S_{2k}(N)^-$ constructed in Chap. III. In this section we combine the results of Chaps. III and IV to relate the coefficients a_m to height pairings and to values of Green's functions for $X_0(N)$. Our main interest is in the case $k=1$.

Theorem 1. *Suppose $k=1$. Then for $m\geq 1$ prime to N, the Fourier coefficient a_m of F is equal to the global height pairing $\langle y^*_{D_0,r_0}, T_m y^*_{D_1,r_1}\rangle$, where T_m denotes the m^{th} Hecke operator on $X_0^*(N)$.*

Proof. We will compute the non-archimedean and archimedean parts of the height pairing separately. In particular, we will show that the finite part is given by

$$\sum_p\langle y^*_{D_0,r_0}, T_m y^*_{D_1,r_1}\rangle_p=\sum_{\substack{n\equiv m\varrho(2N)\\ n^2<m^2\Delta}}\sigma'_\chi\left(\left(\frac{n+m\sqrt{\Delta}}{2}\right)\mathfrak{n}^{-1}\right), \tag{1}$$

where Δ and ϱ denote D_0D_1 and $r_0r_1 \pmod{2N}$ and $\chi, \mathfrak{n}, \sigma'_\chi$ are as in Chap. III (χ the genus character of $K=\mathbb{Q}(\sqrt{\Delta})$ corresponding to $\Delta=D_0\cdot D_1$, $\mathfrak{n}$ the primitive ideal

$\mathbb{Z}N+\mathbb{Z}\frac{\varrho+\sqrt{\Delta}}{2}$ of norm N in K, $\sigma'_\chi(\mathfrak{a})$ for an integral ideal $\mathfrak{a}$ of K the derivative at $s=0$ of $\sigma_{s,\chi}(\mathfrak{a})=\sum_{\mathfrak{b}|\mathfrak{a}}\chi(\mathfrak{b})N(\mathfrak{b})^s\Big)$. The expression on the right of (1) is the first term in the formula for a_m given in Proposition 2 of Chap. III, Sect. 2. The archimedean part of the height pairing will then be shown to give the remaining two terms of that proposition (i.e., the infinite sum of Legendre functions and the term with the factor λ).

The first observation is that

$$\langle y^*_{D_0,r_0}, T_m y^*_{D_1,r_1}\rangle_v=\langle T_{m_0}y^*_{D_0,r_0}, T_{m_1}y^*_{D_1,r_1}\rangle_v$$

for any place v and any decomposition $m=m_0m_1$ with m_0 and m_1 coprime, since $T_m=T_{m_0}T_{m_1}$ and T_{m_0} is self-adjoint with respect to the local height pairing. We choose m_0 and m_1 such that m_0 is prime to D_1 and m_1 to D_0. (This is possible since D_0 and D_1 are assumed coprime; the splitting is not unique and some of our intermediate formulas will depend on it, but the final result will not.) By formula (3) of Sect. 1 of Chap. IV and the remark following it, we deduce that, for any place v of $\mathbb{Q}$,

$$\langle y^*_{D_0,r_0}, T_m y^*_{D_1,r_1}\rangle_v=\sum_{\substack{m_0=d_0d'_0\\ m_1=d_1d'_1}}\left(\frac{D_0}{d'_0}\right)\left(\frac{D_1}{d'_1}\right)\langle y^*_{D_0d_0^2,r_0d_0}, y^*_{D_1d_1^2,r_1d_1}\rangle_v\,. \tag{2}$$

The point is that the discriminants $D_0d_0^2$ and $D_1d_1^2$ occurring on the right-hand side of (2) are coprime and have conductor prime to N, so we can apply the results of Chap. IV.

For integers M all of whose prime factors p satisfy $\left(\frac{\Delta m^2}{p}\right)\neq-1$ we define $\ell(M,s)=\sum_{d|M}\varepsilon(d)d^s$, where ε is the multiplicative function on such integers given on primes p with $\left(\frac{\Delta m^2}{p}\right)\neq-1$ by $\varepsilon(p)=\left(\frac{D_i}{p}\right)$ if $p\nmid D_im_i$. [This depends on our chosen decomposition of m in the case when $p|m$ and $\left(\frac{\Delta}{p}\right)=-1$.] Combining (2) with the theorem of Sect. 4 of Chap. IV, we find

$$\begin{aligned}\sum_p\langle y^*_{D_0,r_0}, T_m y^*_{D_1,r_1}\rangle_p&=\sum_{\substack{m_0=d_0d'_0\\ m_1=d_1d'_1}}\left(\frac{D_0}{d'_0}\right)\left(\frac{D_1}{d'_1}\right)\sum_{\substack{x\equiv\varrho d_0d_1(2N)\\ x^2<\Delta d_0^2d_1^2}}\ell'\left(\frac{\Delta d_0^2d_1^2-x^2}{4N},0\right)\\&=\sum_{\substack{n\equiv m\varrho(2N)\\ n^2<\Delta m^2}}\sum_{\substack{d'_0|(n,m_0)\\ d'_1|(n,m_1)}}\left(\frac{D_0}{d'_0}\right)\left(\frac{D_1}{d'_1}\right)\ell'\left(\frac{\Delta m^2-n^2}{4Nd_0'^2d_1'^2},0\right),\end{aligned}$$

where in the second line we have set $n=d'_0d'_1x$ and interchanged the order of summation. This is equivalent to (1) by virtue of the following lemma.

Lemma. *Fix $n\equiv m\varrho\,(2N)$ and define $\ell(k,s)$ for $k|M=\frac{m^2\Delta-n^2}{4N}$ as above. Then*

$$\sum_{\substack{d_0|(n,m_0)\\ d_1|(n,m_1)}}\left(\frac{D_0}{d_0}\right)\left(\frac{D_1}{d_1}\right)(d_0d_1)^s\ell\left(\frac{M}{d_0^2d_1^2},s\right)=\sigma_{s,\chi}\left(\left(\frac{n+m\sqrt{\Delta}}{2}\right)\mathfrak{n}^{-1}\right). \tag{3}$$

If $|n|<m\sqrt{\Delta}$ *then each term in this formula vanishes for* $s=0$ *and*

$$\sum_{\substack{d_0|(n,m_0)\\ d_1|(n,m_1)}} \left(\frac{D_0}{d_0}\right)\left(\frac{D_1}{d_1}\right)\ell'\left(\frac{M}{d_0^2 d_1^2},0\right)=\sigma'_\chi\left(\left(\frac{n+m\sqrt{\Delta}}{2}\right)\mathfrak{n}^{-1}\right).$$

Proof. The second statement follows from the first by differentiating at $s=0$, since if M is positive then $\varepsilon(M)=-1$ and hence $\ell(M/d^2,0)=0$ for all $d^2|M$. We therefore need only prove (3). Write $\mathfrak{a}$ for the ideal $\left(\frac{n+m\sqrt{\Delta}}{2}\right)\mathfrak{n}^{-1}$, with norm $N(\mathfrak{a})=M$. Both sides of (3) are clearly given by Euler products extending over primes p dividing M, so we need only prove that the Euler factors for such p are the same. Write p^ν for the largest power of p dividing $\mathfrak{a}$; since $(N/\mathfrak{n}),m)=1$, this is just the power of p dividing the g.c.d. of m and $(n+m\Delta)/2$. Also write $p^{2\nu+\delta}$ for the exact power of p dividing M and ε $(=\pm 1)$ for $\varepsilon(p)$. We distinguish three cases, according to the splitting behavior of p in $K=\mathbb{Q}(\sqrt{\Delta})$.

Case 1. $\left(\frac{\Delta}{p}\right)=-1$. Here $(p)=\mathfrak{p}$ is inert in K, so we have $\mathfrak{p}^\nu\|\mathfrak{a}$ and $\delta=0$. Since m_0 and m_1 are coprime, we have (after possible renumbering) $p\nmid m_0$, $p^\nu|m_1$, $\varepsilon(p)=\left(\frac{D_0}{p}\right)$. [Recall that this is the one case when the definition of $\varepsilon(p)$ depends on the splitting $m=m_0m_1$.] Hence $\left(\frac{D_1}{p}\right)=-\varepsilon$. The p-Euler factor on the right of (3) is just $1+p^{2s}+p^{4s}+\ldots+p^{2\nu s}$ [since $\mathfrak{a}$ is divisible by $1,\mathfrak{p},\mathfrak{p}^2,\ldots,\mathfrak{p}^\nu$ and $\mathfrak{p}=(p)$ is a principal ideal with $\chi=1$ and norm p^2], while that on the left is

$$\sum_{j=0}^{\nu}\left(\frac{D_1}{p^j}\right)p^{js}\ell(p^{2\nu-2j},s)=\sum_{j=0}^{\nu}(-\varepsilon p^s)^j(1+\varepsilon p^s+\varepsilon^2p^{2s}+\ldots+(\varepsilon p^s)^{2\nu-2j});$$

and using the identity

$$\sum_{j=0}^{\nu}x^j(1+x+\ldots+x^{2\nu-2j})=\sum_{j=0}^{\nu}\frac{x^j+x^{2\nu+1-2j}}{1+x}=\frac{1}{1+x}\sum_{j=0}^{2\nu+1}x^j=\frac{1-x^{2\nu+2}}{1-x^2},$$

we see that these agree. (Notice that the final expression involves only powers of x^2, which is why the choice of $\varepsilon=\pm 1$, which was arbitrary in this case, does not matter.)

Case 2. $\left(\frac{\Delta}{p}\right)=0$. Here $(p)=\mathfrak{p}^2$ is ramified in K, so we have $\mathfrak{p}^{2\nu+\delta}\|\mathfrak{a}$ and $\delta=0$ or 1. Since D_0 and D_1 are coprime, p divides exactly one, say D_0; then $p\nmid D_1$ and $\left(\frac{D_1}{p}\right)=\varepsilon$. In the left-hand side of (3) only the terms with $p\nmid d_0d_1$ contribute $\left[\text{since } m_1 \text{ is prime to } D_0,\ p\nmid d_1, \text{ and the terms with } p|d_0 \text{ give zero since } \left(\frac{D_0}{p}\right)=0\right]$, so the p-Euler factor is just $\ell(p^{2\nu+\delta},s)=\sum_{j=0}^{2\nu+\delta}(\varepsilon p^s)^j$. The p-factor on the right is the same since $\mathfrak{p}^{2\nu+\delta}\|\mathfrak{a}$, $\chi(\mathfrak{p})=\left(\frac{D_1}{p}\right)=\varepsilon$, and $N(\mathfrak{p})=p$.

Case 3. $\left(\frac{\Delta}{p}\right)=+1$. Here $(p)=\mathfrak{p}\mathfrak{p}'$ splits in K and we have (possibly after renaming $\mathfrak{p}$ and $\mathfrak{p}'$) $\mathfrak{p}^{\nu+\delta}\|\mathfrak{a}$, $\mathfrak{p}'^{\nu}\|\mathfrak{a}$. Also $\left(\frac{D_0}{p}\right)=\left(\frac{D_1}{p}\right)=\varepsilon$, so $\chi(\mathfrak{p})=\chi(\mathfrak{p}')=\varepsilon$, and, of course, $N(\mathfrak{p})=N(\mathfrak{p}')=p$. Therefore the p-Euler factor on the right-hand side of (3) is

$$(1+\varepsilon p^s+\varepsilon^2 p^{2s}+\ldots+(\varepsilon p^s)^\nu)(1+\varepsilon p^s+\varepsilon^2 p^{2s}+\ldots+(\varepsilon p^s)^{\nu+\delta})$$

and that on the left is

$$\sum_{j=0}^{\nu}(\varepsilon p^s)^j(1+\varepsilon p^s+\varepsilon^2 p^{2s}+\ldots+(\varepsilon p^s)^{2\nu+\delta-2j}).$$

The equality of these two expressions follows from a calculation like the one for $\left(\frac{\Delta}{p}\right)=-1$.

This completes the proof of the lemma.

We now turn to the archimedean part of the height pairing. In Sect. 3 of Chap. IV we wrote $\langle y^*_{D_0,r_0}, y^*_{D_1,r_1}\rangle_\infty$ as the sum of two terms

$$C_1=\tfrac{1}{2}(C_1^+ + C_1^-),\qquad C_1^\pm=\lim_{s\to 1}\left[G_{N,s}(P_{D_0,r_0},P_{D_1,\pm r_1})-\frac{\lambda}{s-1}\right] \tag{4}$$

and

$$C_2=\lim_{s\to 1}\left[\frac{-2\pi}{1-2s}(H(D_0)E^*_N(P^*_{D_1,r_1})+H(D_1)E^*_N(P^*_{D_0,r_0})\right.$$
$$\left.-H(D_0)H(D_1)\phi^*_N(s))+\frac{\lambda}{s-1}\right], \tag{5}$$

where $\lambda=-12H(D_0)H(D_1)/N\prod_{p|N}(1+p^{-1})$; evaluating them gave the two terms $\lim_{s\to 1}[\ldots]$ and $\lambda[\ldots]$ in the theorem of that section. Now applying T_m to $y^*_{D_1,r_1}$ multiplies $E^*_N(P^*_{D_1,r_1})$ by $m^{1-s}\sum_{d|m}d^{2s-1}$ (since E^*_N is an eigenfunction of T_m with this eigenvalue) and multiplies the other three terms in the square brackets in (5) by $\sum_{d|m}d=\sigma(m)$ (since 1 is an eigenfunction of T_m with this eigenvalue). Hence it replaces C_2 by $\sigma(m)C_2+\lambda\sum_{d|m}\log\frac{m}{d^2}$, which, in view of the formula for C_2, is equal to the final term $\lambda[\ldots]$ in the formula for a_m given in Sect. 2 of Chap. III. On the other hand, the effect on (4) of applying T_m to $P^*_{D_1,r_1}$ is to replace $C_1^\pm$ by $\lim_{s\to 1}\left[G_{N,s,m}(P_{D_0,r_0},P_{D_1,\pm r_1})-\frac{\sigma(m)\lambda}{s-1}\right]$, where $G_{N,s,m}(z_0,z_1)=G_{N,s}(z_0,T_m z_1)$. Therefore Theorem 1 will follow if we show

$$G_{N,s,m}(P_{D_0,r_0},P_{D_1,r_1})=-2\sum_{\substack{n>m\sqrt{\Delta}\\ n\equiv -m\varrho(2N)}}\sigma_{0,\chi}\left(\left(\frac{n-m\sqrt{\Delta}}{2}\right)\mathfrak{n}^{-1}\right)Q_{s-1}\left(\frac{n}{m\sqrt{\Delta}}\right). \tag{6}$$

But this follows, by the same argument as used above for the finite places, from Proposition 2 of Sect. 3, Chap. IV [the $\varrho(n)$ occurring there is just

$\ell\left(\frac{D_0D_1-n^2}{4N}, 0\right)$, so we can apply our Lemma with $s=0\Big]$ if we observe that the analogue of (2) holds with $\langle\ ,\ \rangle_v$ replaced by $G_{N,s}$, which also satisfies $G_{N,s}(z_0, T_m z_1) = G_{N,s}(T_{m_0} z_0, T_{m_1} z_1)$. This completes the proof of Theorem 1.

Substituting Eq. (6) with $s=k$ into Proposition 1 of Sect. 2, Chap. III, we obtain the analogue of Theorem 1 for higher weights:

Theorem 2. *Let $k>1$ and $F=F_{k,N,\Delta,\varrho,D_0}=\sum_{m\geqq 1} a_m q^m$ the modular form constructed in Chap. III. Then for $m\geqq 1$,*

$$a_m = \sum_{\substack{|n|<m\sqrt{\Delta} \\ n\equiv m\varrho(2N)}} (m\sqrt{\Delta})^{k-1} P_{k-1}\left(\frac{n}{m\sqrt{\Delta}}\right) \sigma'_\chi\left(\left(\frac{n+m\sqrt{\Delta}}{2}\right)\mathfrak{n}^{-1}\right)$$
$$+\tfrac{1}{2}\left[G_{N,k,m}(P_{D_0,r_0}, P_{D_1,r_1}) + (-1)^{k-1} G_{N,k,m}(P_{D_0,r_0}, P_{D_1,-r_1})\right].$$

Because $(m\sqrt{\Delta})^{k-1}P_{k-1}\left(\frac{n}{m\sqrt{\Delta}}\right)$ is an integer (P_{k-1} is a polynomial of degree and parity $k-1$ with integral coefficients) and $\sigma'_\chi(\mathfrak{a})$ for an ideal $\mathfrak{a}$ with $\chi(\mathfrak{a})=-1$ is an integral linear combination of logarithms of prime numbers (in fact, always a nonpositive integer multiple of the logarithm of a single prime number), we can write this as

$$a_m = \tfrac{1}{2} G^{\pm}_{N,k,m}(P_{D_0,r_0}, P_{D_1,r_1}) + \sum_p n(p)\log p, \tag{7}$$

where $\pm 1 = (-1)^{k-1}$ and signifies (± 1)-symmetrization with respect to r_1, the sum runs over primes p and is finite $\left(\text{indeed, contains only primes } < \frac{m^2\Delta}{4N}\right)$, and $n(p)\in\mathbb{Z}$. This formula suggests two problems:

1. Interpret the right-hand side of (7) as a higher weight height pairing defined on Heegner sections of a certain local coefficient system over $X_0(N)$.

2. If $\{\lambda_m\}_{m\geqq 1}$ is a finite collection of integers such that $\sum_m \lambda_m a(m)=0$ for all cusp forms $\sum a(m)q^m \in S_{2k}(N)^-$, then (7) implies that

$$\sum_m \lambda_m G^{\pm}_{N,k,m}(P_{D_0,r_0}, P_{D_1,r_1}) = \sum_{\tau_0\in P_{D_0,r_0}} \sum_{\tau_1\in P_{D_1,r_1}} \sum_m \lambda_m G^{\pm}_{N,k,m}(\tau_0,\tau_1)$$

is the logarithm of a rational number; show that each summand $\sum_m \lambda_m G^{\pm}_{N,k,m}(\tau_0,\tau_1)$ is the logarithm of an algebraic number.

The analogues of both these questions for the case $D_0=D_1$, $r_0=r_1$ were discussed in the last section of [9], so we say nothing further about them here. Evidence for the algebraicity conjecture *2.* will be presented in a later paper.

2. Consequences

In this section we prove Theorems B and C of the Introduction and discuss some extensions.

Let $D_i = r_i^2 - 4n_i N < 0$ $(i=0,1)$ be coprime fundamental discriminants and write y_i^* for $y_{D_i,r_i}^* \in J^*(\mathbb{Q})$, where $J^* = \mathrm{Jac}(X_0^*(N))$. The power series

$$G(z) = \sum_{m=1}^{\infty} \langle y_0^*, T_m y_1^* \rangle q^m \qquad (q = e^{2\pi i z})$$

defines a modular form in $S_2(\Gamma_0^*(N)) = S_2(N)^-$ by the formal argument given on p. 306 of [9]. Theorem 1 of Sect. 1 says that the coefficients of q^m in F and G agree for all m prime to N. In particular, $F-G$ is an oldform, so $(G,f)=(F,f)$ if f is a normalized newform in $S_2(N)^-$. The argument on p. 308 of [9] shows that (G,f) equals (f,f) times the height pairing $\langle (y_0^*)_f, (y_1^*)_f \rangle$, where $(y_i^*)_f$ is the f-eigencomponent of y_i^* on $J^*(\mathbb{Q}) \otimes \mathbb{R}$ [obtainable as $\sum \alpha_m T_m y_i^*$ where $\sum \alpha_m T_m$ is a finite linear combination of Hecke operators which is the identity on f and 0 on its orthogonal complement in $S_2(N)^-$]. Hence

$$\langle (y_0^*)_f, (y_1^*)_f \rangle = \frac{1}{\|f\|^2} (F,f),$$

and substituting for (F,f) the expression

$$(F,f) = \frac{1}{4\pi} L'(f,1) r_{2,N,\Delta,\varrho,D_0}(f) = \frac{1}{4\pi} L'(f,1) \int_{\gamma(D_0,D_1,\varrho)} f(z)\,dz$$

obtained in Sect. 3 of Chap. III we obtain

$$\langle (y_0^*)_f, (y_1^*)_f \rangle = \frac{L'(f,1)}{4\pi \|f\|^2} \int_{\gamma(D_0,D_1,\varrho)} f(z)\,dz\,, \tag{1}$$

which is Theorem B. Here Δ and ϱ denote $D_0 D_1$ and $r_0 r_1 \pmod{2N}$ as usual, and $\gamma(D_0, D_1, \varrho)$ is the cycle $\sum_{Q \in \mathcal{Q}_{N,\Delta,\varrho}/\Gamma_0(N)} \chi_{D_0}(Q) \gamma_Q$ on $X_0(N)$ as in the Introduction and in Sect. 1 of Chap. II.

Next we observe that (1) remains true when $D_0 = D_1$, $r_0 = r_1$, at least if D_0 is prime to $2N$. Indeed, in that case we have

$$\int_{\gamma(D_0,D_0,\varrho)} f(z)\,dz = \frac{|D_0|^{1/2}}{2\pi} L(f, D_0, 1)$$

$\left[\text{where } L(f, D_0, s) \text{ denotes the twist of } L(f,s) \text{ by } \left(\frac{D_0}{\cdot}\right)\right]$ by the proof of Corollary 1 in Sect. 4 of Chap. II, and

$$\langle (y_0^*)_f, (y_0^*)_f \rangle = \frac{|D_0|^{1/2}}{8\pi^2 \|f\|^2} L'(f,1) L(f, D_0, 1)$$

by the main theorem of [9]. Now applying the theorem of Chap. II, Sect. 4, to the pairs (D_0, D_1), (D_0, D_0), and (D_1, D_1), we obtain

$$\langle (y_0^*)_f, (y_1^*)_f \rangle = c_0 c_1 L, \qquad \langle (y_0^*)_f, (y_0^*)_f \rangle = c_0^2 L, \qquad \langle (y_1^*)_f, (y_1^*)_f \rangle = c_1^2 L$$

where $c_i = c(n_i, r_i)$ is the coefficient of $q^{n_i} \zeta^{r_i}$ in a non-zero Jacobi form $\phi = \phi_f \in J_{2,N}$ with real coefficients having the same Hecke eigenvalues as f and $L = \dfrac{L'(f,1)}{4\pi \|\phi\|^2}$.

These formulas imply that $c_0(y_1^*)_f - c_1(y_0^*)_f$ has height 0 and hence vanishes, since the height pairing on $J^*(\mathbb{Q})\otimes\mathbb{R}$ is positive definite. Hence $(y_0^*)_f$ and $(y_1^*)_f$ are collinear. To obtain Theorem C we need the same statement when D_0 and D_1 are not necessarily coprime. But this follows, since by Lemma 3.2 of [15] we can choose a fundamental discriminant $D_2 = r_2^2 - 4n_2N$ prime to D_0, D_1, and $2N$ with $c_2 = c(n_2, r_2) \neq 0$, and then $(y_0^*)_f = \frac{c_0}{c_2}(y_2^*)_f$, $(y_1^*)_f = \frac{c_1}{c_2}(y_2^*)_f$ by the special case of coprime discriminants. We deduce that $c_i^{-1}(y_i^*)_f$ is independent of i (for $c_i \neq 0$), i.e., that there exists a vector $y_f \in (J^*(\mathbb{Q})\otimes\mathbb{R})_f$ with height $\langle y_f, y_f\rangle = L$ such that

$$(y_{D,r}^*)_f = c(n,r)\,y_f \qquad (r^2 - 4nN = D) \tag{2}$$

for all fundamental discriminants D prime to $2N$. This is Theorem C.

We observe that (2) remains true for D not fundamental (but still prime to $2N$). Indeed, start with (2) for D fundamental and multiply both sides by $a(m)$, the coefficient of q^m in f, for some m prime to N. Since $a(m)$ is also the eigenvalue of f under T_m, the left-hand side is $(T_m y_{D,r}^*)_f$ and similarly the right-hand side is y_f times the coefficient of $q^n\zeta^r$ in $\phi|T_m$. Equation (3) of Chap. IV, Sect. 1, expresses $T_m y_{D,r}^*$ as the sum of $y_{Dm^2,rm}^*$ and a linear combination of $y_{Dd^2,rd}^*$ with $d<m$, and the action of T_m on Jacobi forms in $J_{2,N}$ is given by exactly the same formula [4, Sect. 4]. It follows by induction that (2) is true with D, n, r replaced by Dm^2, nm^2, and rm. But it is easily checked that any R with $R^2 \equiv Dm^2 \pmod{4N}$ is congruent to $rm \pmod{2N}$ for some r with $r^2 \equiv D \pmod{4N}$, so this proves (2) for general discriminants D prime to $2N$.

It is also doubtless true that (2) remains true when D is not prime to $2N$ (at least if the conductor of D is prime to N). One could prove this either by proving the main theorem in [9] without the restriction $(D, 2N) = 1$ or by proving Theorem B of this paper without the restriction $(D_0, D_1) = 1$. We made this restriction to keep the paper from being even longer. To remove it, one would have to make the following changes:

Chapter II: No change [the assumption $(D_0, D_1) = 1$ was not made here].

Chapter III: The definition of $E_s(z, z')$ must be modified by replacing

$$\sum_{[\mathfrak{a}]} \chi(\mathfrak{a}) \sum'_{(m,n)\in(\mathfrak{n}\mathfrak{a}\times\mathfrak{a})/\mathcal{O}_K^\times} \cdots$$

by

$$\sum_{[Q]\in\mathcal{Q}_{N,\Delta,\varrho}/\Gamma_0(N)} \chi_{D_0}(Q) \sum'_{(m,n)\in(L_Q\times L'_Q)/\Gamma_0(N)_Q} \cdots,$$

where L_Q and L'_Q for $Q = [aN, b, c]$ denote the $\mathbb{Z}$-lattices $\mathbb{Z}aN + \mathbb{Z}\frac{b+\sqrt{\Delta}}{2}$ and $\mathbb{Z}a + \mathbb{Z}\frac{b+\sqrt{\Delta}}{2}$. Then the definition of F and the calculation of (F, f) in Sect. 3 are unchanged, as is the analytical part of the computation of the Fourier coefficients of F in Sect. 2, but the arithmetic functions $\sigma_{0,\chi}\left(\left(\frac{n+m\sqrt{\Delta}}{2}\right)\mathfrak{n}^{-1}\right)$ and $\sigma'_\chi\left(\left(\frac{n+m\sqrt{\Delta}}{2}\right)\mathfrak{n}^{-1}\right)$ would have to be recomputed.

Chapter IV: Here, too, the general setup and the analytic part of the computation (Green's functions) are unchanged, but the arithmetic part would have to be generalized. In particular, the results of Sect. 3 of Chap. I concerning the number of embeddings of Clifford orders the Eichler orders would have to be extended to the case when the quadratic form $q=[D_0, 2n, D_1]$ is not primitive.

If one knew (2) for all D and r, one could restate it in the form

$$\sum_{\substack{n,r\in\mathbb{Z}\\ r^2<4Nn}} (y^*_{r^2-4Nn,r})_f q^n\zeta^r = \phi_f(\tau,z)\, y_f. \tag{3}$$

Summing this over all new forms would say that the power series $\sum_{n,r} (y^*_{r^2-4Nn,r})_{\mathrm{new}} q^n\zeta^r$ belongs to $J^*(\mathbb{Q})^{\mathrm{new}}\otimes J_{2,N}^{\mathrm{cusp,\,new}}$. Presumably the same relation would hold for the old parts by induction on N; this would then give the "ideal statement" formulated at the end of the Introduction.

3. *Relations with the Conjecture of Birch and Swinnerton-Dyer*

Let $f\in S_2(\Gamma_0^*(N))$ be a normalized eigenform, $\phi_f=\sum c(n,r)q^n\zeta^r\in J_{2,N}$ a non-zero Jacobi form with real coefficients corresponding to it. By the results of Sect. 2, there are elements

$$y_f\in(J^*(\mathbb{Q})\otimes\mathbb{R})_f,\qquad e_f\in H_1(X_0^*(N)(\mathbb{C});\mathbb{R})_f^-$$

[where the minus on H_1 indicates the eigenvalue -1 for the involution induced by complex conjugation on $X_0^*(N)(\mathbb{C})$] such that

$$(y^*_{D,r})_f=c(n,r)y_f \tag{1}$$

for any fundamental discriminant $D=r^2-4nN<0$ prime to $2N$ and

$$[\gamma(D_0,D_1,r_0r_1)]_f=c(n_0,r_0)c(n_1,r_1)e_f$$

for any fundamental discriminants $D_i=r_i^2-4n_iN<0$, where $[\gamma(D_0,D_1,\varrho)]$ is the homology class represented by the cycle $\gamma(D_0,D_1,\varrho)$. The last statement is true without the assumption $(D_0,D_1)=1$ or $(D_0D_1,2N)=1$ (it was proved in Chap. II in full generality), but to make sense of it we must check that $\gamma(D_0,D_1,\varrho)$ is a closed cycle and hence really represents a class in $H_1(X_0^*(N);\mathbb{Z})$. This is obvious if $D_0\neq D_1$ [each γ_Q, $Q\in\mathcal{Q}_{N,\Lambda,\varrho}$, is a closed geodesic in $X_0(N)$], but remains true also if $D_0=D_1$. Indeed, for $D_0=D_1=D$ the endpoints of the semicircle γ_Q for $Q=[aN,b,c]$ are the cusps $\frac{-b\pm D}{2aN}$, and these are $\Gamma_0(N)$-equivalent to the endpoints of $\gamma_{\bar{Q}}$, where $\bar{Q}=[-aN,b,-c]$, with the same induced orientation. Hence $\gamma_Q-\gamma_{\bar{Q}}$ is a closed oriented cycle in $X_0(N)$, and $\gamma(D_0,D_1,\varrho)$ is a sum of such because $\chi_D(Q)=-\chi_D(\bar{Q})$ for every Q. $\Big($Interestingly, each γ_Q defines a closed cycle in $X_0^*(N)$ if D is prime to N, because the endpoints $\frac{-b\pm D}{2aN}$ are always interchanged by w_N in that case.$\Big)$

The vector space $H_1(X_0^*(N)(\mathbb{C});\mathbb{R})_f^-$ is one-dimensional, as is the space $(J^*(\mathbb{Q})\otimes\mathbb{R})_f$ if the Birch-Swinnerton-Dyer conjecture is correct and $L'(f,1)\neq0$. In this case y_f and e_f are generators and Eq. (1) of Sect. 2 gives the relation

$$\langle y_f, y_f \rangle = \frac{L'(f,1)}{4\pi \|f\|^2} \int_{e_f} f(z)dz \tag{2}$$

between them. If ϕ_f is chosen to have coefficients in K_f, the totally real number field generated by the Fourier coefficients of f, then y_f and e_f belong to $(J^*(\mathbb{Q}) \otimes K_f)_f$ and $H_1(X_0^*(N); K_f)_f^-$, which are 1-dimensional over K_f. The differential $\omega_f = f(z)dz$ is also defined over K_f [on the canonical model of $X_0^*(N)$ over $\mathbb{Q}$], and $\|\omega_f\|^2 := \int_{X_0^*(N)(\mathbb{C})} |\omega_f|^2$ equals $4\pi^2 \|f\|^2$. Hence (2) can be written

$$L'(f,1) = \frac{\|\omega_f\|^2}{\frac{1}{2i} \int_{e_f} \omega_f} \cdot \langle y_f, y_f \rangle, \tag{3}$$

in which the first factor is a K_f-multiple of the "+"-period of f and the second, assuming that $\operatorname{rk} J^*(\mathbb{Q})_f = 1$, a K_f-multiple of the f-part of the regulator of the Jacobian.

Equation (3) is an identity of the same sort as that predicted by the Birch-Swinnerton-Dyer conjecture. We now make the comparison between the two more explicit in the case when f corresponds to an elliptic factor E of J^* over $\mathbb{Q}$. This is the case exactly when $K_f = \mathbb{Q}$; then ϕ_f can be chosen uniquely up to sign by requiring that the coefficients $c(n,r)$ are integers with no common factor. [By using the action of Hecke operators, we see that this is equivalent to making the same assumption on the $c(n,r)$ with $r^2 - 4Nn$ fundamental.] Changing the choice of sign replaces y_f by $-y_f$ and has no effect on e_f. Let $p: X_0^*(N) \to E$ be a non-trivial map defined over $\mathbb{Q}$ and taking the cusp ∞ to $0 \in E$. The image of e_f in $H_1(E, \mathbb{Q})^-$ lies in $H_1(E, \mathbb{Z})^- \approx \mathbb{Z}$, because $c(n_0, r_0)c(n_1, r_1)$ times it is the integral class $p_*[\gamma(D_0, D_1, r_0 r_1)]$ and the integers $c(n_0, r_0)c(n_1, r_1)$ have no common factor (take $n_0 = n_1$, $r_0 = r_1$). Let n denote the index of the subgroup it generates. Similarly, let $P \in E(\mathbb{Q}) \otimes \mathbb{Q}$ be the image of y_f under p_*. [If we knew (1) for all D, then an argument like that just used would show that P belongs to $E(\mathbb{Q})$. Since we haven't proved this fact and don't know that the coefficients $c(n,r)$ for r prime to N have no common factor, we cannot exclude the possibility of a denominator.] Finally, let ω_E be a Neron differential on E, normalized so that $p^* \omega_E = c\omega_f$ with $c > 0$ (c is automatically an integer), and $\Omega_+ = \int_{E(\mathbb{R})} |\omega_E|$ the real period of E. If $\hat{h}$ denotes the canonical height function on $E(\mathbb{Q}) \otimes \mathbb{Q}$, then we have the relation $\hat{h}(P) = \deg(p) \langle y_f, y_f \rangle$. Moreover, $\|\omega_E\|^2 := \int_{E(\mathbb{C})} |\omega_E|^2$ is equal to $\frac{c^2}{\deg(p)} \|\omega_f\|^2$ and is also equal to Ω_+ times the integral of $\frac{1}{2i}\omega_E$ over a generator of $H_1(E; \mathbb{Z})^-$. Hence (3) becomes:

Proposition. *With the above notations,* $L'(E/\mathbb{Q}, 1) = \frac{1}{cn} \Omega_+ \hat{h}(P)$.

Suppose $L'(E/\mathbb{Q},1)\neq 0$. Then the Birch-Swinnerton-Dyer conjecture predicts that $\operatorname{rk} E(\mathbb{Q})=1$ and that

$$L'(E/\mathbb{Q},1)=\frac{|Ш|m}{[E(\mathbb{Q}):\mathbb{Z}P]^2}\,\Omega_+\hat{h}(P),$$

where Ш is the (conjecturally finite) Shafarevich-Tate group and $m=\prod_{p\mid N}[E(\mathbb{Q}_p):E^0(\mathbb{Q}_p)]$. Hence we are led to

Conjecture. *If* $L'(E/\mathbb{Q},1)\neq 0$ *then* $[E(\mathbb{Q}):\mathbb{Z}P]^2=c\cdot n\cdot m\cdot|Ш|$.

The numbers c, n, and m are easily determined in any given case and typically involve only a few small primes, so the essence of this conjecture is that the index of the canonical 1-dimensional subgroup of the Mordell-Weil group which we have constructed using Heegner divisors is roughly the square-root of $|Ш|$. On the other hand, since Ш, if finite, has square order, our conjecture also predicts that, when $L'(E/\mathbb{Q},1)\neq 0$, the integer $c\cdot n\cdot m$ is a perfect square.

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