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Rotation Numbers of Products of Circle Homeomorphisms

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Introduction

Let $H = \mathrm{Homeo}^+(S^1)$ be the group of orientation preserving homeomorphisms of the circle. Our main question is the following: for given numbers γ_1 and γ_2, what are the possible rotation numbers of a product $q_1 q_2$ of elements q_1, q_2 of H with rotation numbers γ_1, γ_2? What if some or all of q_1, q_2, $q_1 q_2$ are required to be conjugate to rotations? Our original motivation was the question of which Seifert fibered 3-manifolds admit transverse foliations, which we discuss in Sect. 7. The answers turn out to be much more subtle than we originally expected.

We can make the question more precise by working in the universal covering group $\tilde{H}$ of H. This is the group of homeomorphisms of $\mathbb{R}$ which lift from an orientation preserving homeomorphism of $S^1 = \mathbb{R}/\mathbb{Z}$, that is $\tilde{H} = \{Q : \mathbb{R} \to \mathbb{R} \mid Q$ monotonically increasing, $Q(r+1) = Q(r) + 1$ for all $r \in \mathbb{R}\}$. ($\tilde{H}$ is simply connected since it is a convex subset of $\mathbb{R}^{\mathbb{R}}$.) For $\gamma \in \mathbb{R}$ define $\mathrm{sh}(\gamma) \in \tilde{H}$ by

$$\mathrm{sh}(\gamma)(x) = x + \gamma, \qquad x \in \mathbb{R}.$$

The center of $\tilde{H}$ is $Z = \{\mathrm{sh}(n) \mid n \in \mathbb{Z}\}$ and $\tilde{H}/Z = H$.

For $Q \in \tilde{H}$ the (Poincaré) *rotation number* of Q is defined as

$$\mathrm{Rot}(Q) = \lim_{n \to \infty} \frac{1}{n} [Q^n(x) - x], \qquad x \in \mathbb{R},$$

where Q^n means $Q \circ \ldots \circ Q$ (n times). As is well known, this limit exist and is independent of x. The rotation number of an element $q \in H$ is defined as $\mathrm{rot}(q) = \mathrm{Rot}(Q) \pmod{\mathbb{Z}} \in \mathbb{R}/\mathbb{Z}$ where Q is a lift of q, but we will never actually use this. Rotation number is a conjugacy invariant. Clearly $\mathrm{Rot}(\mathrm{sh}(\gamma)) = \gamma$. However, not every element of rotation number γ is conjugate to $\mathrm{sh}(\gamma)$.

Our question above is the case $n = 3$ of the:

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Problem. *Given* $\gamma_1, \ldots, \gamma_n \in \mathbb{R}$, $(n \geqq 3)$, *and* $b \in \mathbb{Z}$, *when do there exist* $Q_i \in \tilde{H}$ *with* $Q_1 \ldots Q_n = \mathrm{sh}(b)$ *and* $\mathrm{Rot}(Q_i) = \gamma_i$ *for* $i = 1, \ldots, n$? *What if we require* Q_i *to be conjugate to* $\mathrm{sh}(\gamma_i)$ *for* i *in some specified subset* $J \subseteqq \{1, \ldots, n\}$?

We require $n \geqq 3$ because the problem is trivial for $n = 2$ (the answer is $\gamma_1 + \gamma_2 = b$). In principle the answer for $n > 3$ can be inductively deduced from the answer for $n = 3$, but, as we shall see, there is some nonobvious arithmetic involved.

We shall say $(J; b; \gamma_1, \ldots, \gamma_n)$ is *realizable* if the above problem has a positive answer. We can ask the same question in any subgroup of $\tilde{H}$, for instance the subgroups $C^r\tilde{H}$, $1 \leqq r \leqq \omega$, of smooth or analytic diffeomorphisms. We then speak of *realizability in this subgroup*.

Our results described below offer very strong evidence for the following

Conjecture 1. $(J; b; \gamma_1, \ldots, \gamma_n)$ *is realizable in* $\tilde{H}$ *if and only if it is realizable in some* 3-*dimensional Lie subgroup of* $C^\omega\tilde{H}$.

Such a Lie subgroup is conjugate to some $\tilde{G}_n$, $n = 1, 2, 3, \ldots$, described as follows. Let $G_1 = \mathrm{PSL}(2, \mathbb{R})$. Then G_1 acts by projective homeomorphisms on $\mathbb{R}P^1 = S^1$, so $G_1 \subset H$, and hence $\tilde{G}_1 = \widetilde{\mathrm{PSL}}(2, \mathbb{R}) \subset \tilde{H}$. Let G_n be the n-fold cyclic cover of G_1, which also acts on S^1 (by lifting the action of G_1 to the n-fold cover of S^1 which is also a circle) so $G_n \subset H$, and $\tilde{G}_n \subset \tilde{H}$. Note that $\tilde{G}_n \cong \tilde{G}_1$; in fact $\tilde{G}_n$ is

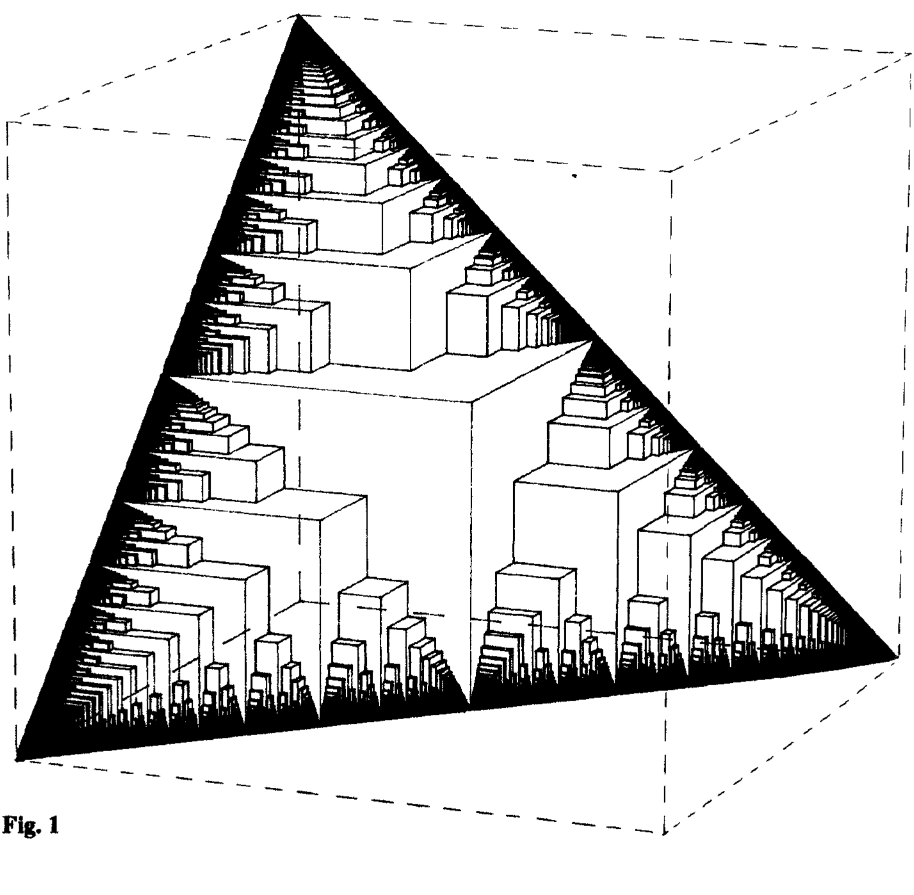

Fig. 1

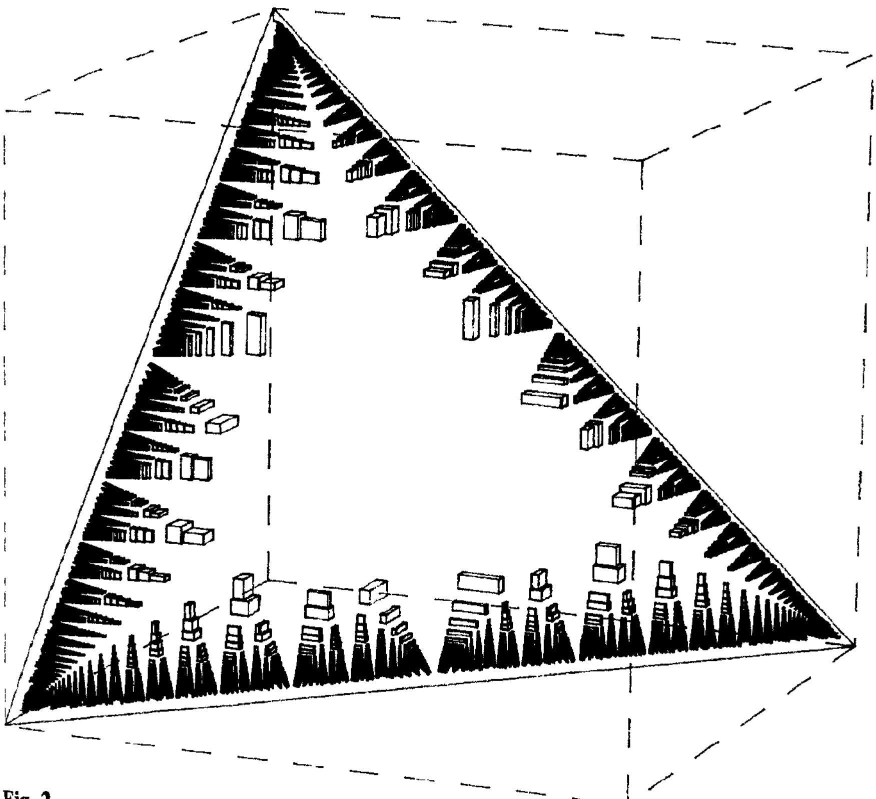

Fig. 2

conjugate to $\tilde{G}_1$ in the group of *all* homeomorphisms of $\mathbb{R}$ by $\tilde{G}_n = P_n^{-1}\tilde{G}_1 P_n$ where $P_n: \mathbb{R} \to \mathbb{R}$ is $P_n(x) = nx$.

We shall show that conjecture 1 holds if it holds for $n=3$. As evidence for the conjecture when $n=3$ we offer the following pictures which we discuss in more detail later. We describe below how Conjecture 1 can easily be reduced to the case $b=1$ and $0<\gamma_i<1$. The set of $(\gamma_1, \gamma_2, \gamma_3) \in (0,1)^3$ for which $(\phi; 1; \gamma_1, \gamma_2, \gamma_3)$ is realizable in some $\tilde{G}_k$ is the subset $\bar{R}_3 \subset (0,1)^3$ of points on and below the piecewise linear surface pictured in Fig. 1. We cannot yet decide realizability for points in the union S of the open stair-step regions of Fig. 2. All points outside $\bar{R}_3 \cup S$ (pictured in Fig. 3) are non-realizable, even in $\tilde{H}$.

For realizability of $(J; 1; \gamma_1, \gamma_2, \gamma_3)$, $J \subsetneq \{1,2,3\}$, the pictures are the same except that faces of $\bar{R}_3$ (respectively S) perpendicular to the i-th coordinate axis should be removed (resp. added) for each $i \in J$.

The volume of the "unsolved" region S is approximately 0.0010547 while the volume of $\bar{R}_3$ is

$$\tfrac{25}{8} + 3\zeta(2) + 3\zeta(3) - \frac{6\zeta(2)\zeta(3)}{\zeta(5)} = 0.224649208402\ldots$$

Thus we could say our conjecture is about 99.9% proved and at least 99.5% true!

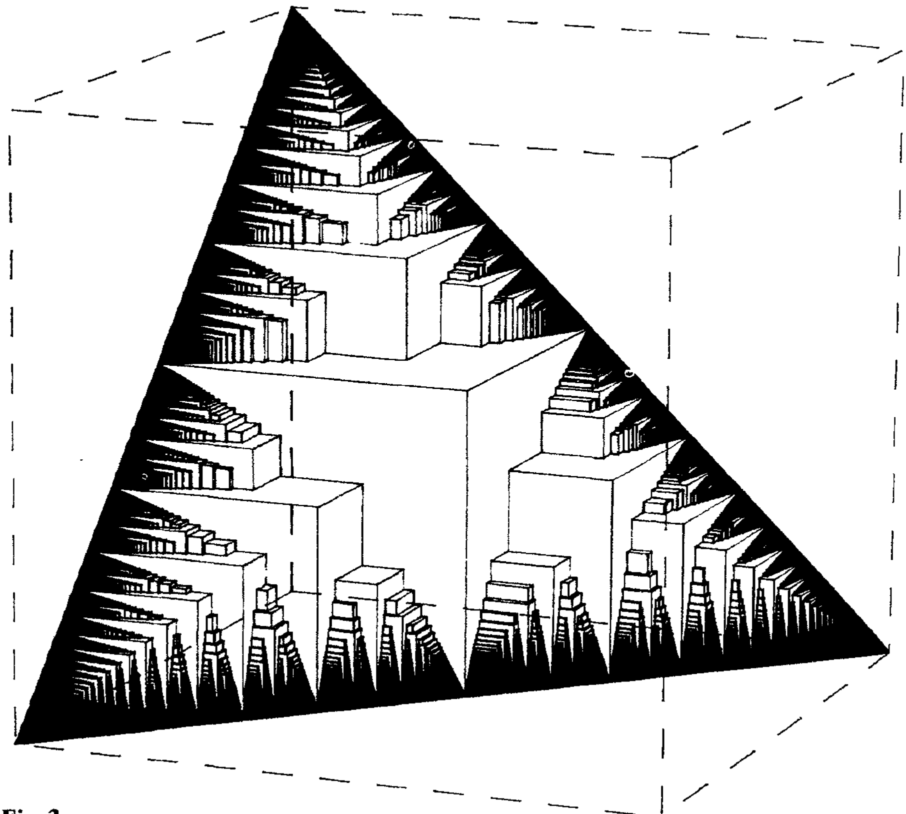

Fig. 3

Don Zagier did the above volume computation (see Appendix). We are grateful to him for many useful discussions and for help in drawing the pictures.

1. Statement of Results

By adjusting the Q_i by central elements and adjusting b accordingly we can assume that $0 \leqq \gamma_i < 1$ for $i = 1, \ldots, n$. There is clearly also no loss in assuming $\gamma_i \neq 0$ for $i \in J$. We make these normalizing assumptions from now on.

In case $\gamma_i = 0$ for some i, our problem is solved (implicitely) in [EHN] (see Sect. 2):

Theorem 1. *Suppose the number g of vanishing γ_i is positive. Then $(J; b; \gamma_1, \ldots, \gamma_n)$ is realizable if and only if $2 - g \leqq b \leqq n - 2$. It is then even realizable in $\tilde{G}_1$.*

We may now assume

$$0 < \gamma_i < 1 \quad \text{for} \quad i = 1, \ldots, n$$

Under this assumption we have ([EHN], see also Theorem 3.1):

Theorem 2. *If* $(J;b;\gamma_1,\ldots,\gamma_n)$ *is realizable in* $\tilde{H}$ *then* $1\leqq b\leqq n-1$. *If* $2\leqq b\leqq n-2$ *then* $(J;b;\gamma_1,\ldots,\gamma_n)$ *is realizable in* $\tilde{G}_1$.

Thus only the cases $b=1$ or $b=n-1$ remain open. If $b=n-1$ we may replace each γ_i by $1-\gamma_i$ which replaces $b=n-1$ by $b=1$. Thus until further notice we will assume:

$$b=1, \quad 0<\gamma_i<1 \quad \text{for} \quad i=1,\ldots,n.$$

Theorem 3. $(J;1;\gamma_1,\ldots,\gamma_n)$ *is realizable in some* $\tilde{G}_k$ *if and only if the following condition* $(*)$ *holds:*

$$\left.\begin{array}{l}\text{For some integers } 0<a<m \text{ with } \gcd(a,m)=1 \text{ and some permu-}\\ \text{tation } \left(\dfrac{a_1}{m},\ldots,\dfrac{a_n}{m}\right) \text{ of } \left(\dfrac{a}{m},\dfrac{m-a}{m},\dfrac{1}{m},\ldots,\dfrac{1}{m}\right), \text{ we have:}\\ \qquad\qquad \gamma_i\leqq\dfrac{a_i}{m} \quad \text{for} \quad i\notin J,\\ \qquad\qquad \gamma_i<\dfrac{a_i}{m} \quad \text{for} \quad i\in J.\end{array}\right\} \quad (*)$$

See Fig 1 for the case $n=3$.

In view of these results, Conjecture 1 is equivalent to:

Conjecture 2. $(J;1;\gamma_1,\ldots,\gamma_n)$ *is realizable in* $\tilde{H}$ *if and only if condition* $(*)$ *holds.*

We shall show

Theorem 4. *Conjectures 1 and 2 hold for all n if they hold for* $n=3$.

The non-realizability results which give Fig. 2 are rather ad hoc. They are described in Sects. 4 and 5. Given the naivety of the approach used in Sect. 4, we found it remarkable that it leads to results so close to a definitive answer. Section 6 describes our pictures in more detail. Section 7 describes the application to transverse foliations of Seifert manifolds.

2. Basic Material

We first note the fact (which has already been used implicitly) that the realizability of $(J;b;\gamma_1,\ldots,\gamma_n)$ is invariant under permutations of the index set $\{1,\ldots,n\}$. Indeed, since $Q_1Q_2=Q_2(Q_2^{-1}Q_1Q_2)$, a simple induction shows that one can permute the factors in a product $Q_1\circ\ldots\circ Q_n$ without changing the value of the product if one replaces each factory by a conjugate as necessary.

Another fact that we shall use with little comment in the following is that if $Q\in\tilde{G}_m$ has $\mathrm{Rot}(Q)=\gamma\notin\frac{1}{m}\mathbb{Z}$, then Q is conjugate to $\mathrm{sh}(\gamma)$. This follows from the fact that any element of $\mathrm{PSL}(2,\mathbb{R})$ either has a fixed point on S^1 or is conjugate to a rotation.

We shall use Knuth's "floor" and "ceiling" notation $\lfloor\gamma\rfloor$, $\lceil\gamma\rceil$ for"integer part of γ" and "least integer not smaller than γ". Moreover $\{\gamma\}$ will denote fractional part of γ.

For $Q \in \tilde{H}$ we denote

$$\underline{m}(Q) = \min\{Q(x) - x | x \in \mathbb{R}\}$$
$$\bar{m}(Q) = \max(Q(x) - x | x \in \mathbb{R}\}.$$

Clearly

$$0 \leqq \bar{m}(Q) - \underline{m}(Q) < 1. \tag{2.1}$$

If the interval $[\underline{m}(Q), \bar{m}(Q)]$ contains an integer β then there exists a point x with $Q(x) - x = \beta$, so $Q^n(x) = x + n\beta$ by induction, so $\mathrm{Rot}(Q) = \beta$. Hence

$$\mathrm{Rot}(Q) \notin \mathbb{Z} \Rightarrow \lfloor \mathrm{Rot}(Q) \rfloor < \underline{m}(Q) \leqq \mathrm{Rot}(Q) \leqq \bar{m}(Q) < \lceil \mathrm{Rot}(Q) \rceil. \tag{2.2}$$

Moreover, for any elements $Q_1, \ldots, Q_n$ of $\tilde{H}$

$$\begin{aligned} \bar{m}(Q_1 \cdots Q_n) &\geqq \bar{m}(Q_1) + \sum_{i=2}^{n} \underline{m}(Q_i) \\ \underline{m}(Q_1 \cdots Q_n) &\leqq \underline{m}(Q_1) + \sum_{i=2}^{n} m(Q_i). \end{aligned} \tag{2.3}$$

For these and other elementary properties of $\underline{m}$ and $\bar{m}$ see Lemma 2.1 of [EHN].

Proof of Theorem 1. The necessity of the condition $2 - g \leqq b \leqq n - 2$ in Theorem 1 is immediate from the inequalities (2.3) above. To see sufficiency, suppose $0 < \gamma_i < 1$ for $i = 1, \ldots, n - g$ and $\gamma_i = 0$ for $i = n - g + 1, \ldots, n$ with $g > 0$. Assume $2 - g \leqq b \leqq n - 2$. These inequalities imply that there exists an $r \in \mathbb{R}$ satisfying

$$0 < r < n - g \quad (\text{or} \quad r = 0 \quad \text{if} \quad n = g) \tag{$*$}$$

$$b - r \in (1 - g, g - 1) \cup \{0\}. \tag{$**$}$$

By Theorem 2.5 of [EHN] inequalities ($*$) are equivalent to the existence of conjugates Q_i of $\mathrm{sh}(\gamma_i)$ for $i = 1, \ldots, n - g$ with $r \in [\underline{m}(Q_1 \ldots Q_{n-g}), \bar{m}(Q_1 \ldots Q_{n-g})]$. Put $X = (Q_1 \ldots Q_{n-g})^{-1} \mathrm{sh}(b)$, so $b - r \in [\underline{m}(X), \bar{m}(X)]$. Then by Theorem 2.3 of [EHN], the fact that $[\underline{m}(X), \bar{m}(X)] \cap ((1 - g, g - 1) \cup \{0\})$ is nonempty is equivalent to the existences of $Q_{n-g+1}, \ldots, Q_n$ of rotation number 0 with $X = Q_{n-g+1} \cdots Q_n$. This proves Theorem 1; by Theorem 4.1 of [EHN] this proof can also be carried out in $\tilde{G}_1 = \widetilde{\mathrm{PSL}}(2, \mathbb{R})$.

The following lemma will be needed in Sect. 4.

Lemma 2.1. *Suppose $Q \in \tilde{H}$ has $\mathrm{Rot}(Q) = \gamma$. Then for any $\varepsilon > 0$ there exists a conjugate Q' of Q with*

$$\gamma - \varepsilon \leqq \underline{m}(Q') \leqq \bar{m}(Q') \leqq \gamma + \varepsilon.$$

Proof. We first observe that for any $x \in \mathbb{R}$

$$\left| \frac{1}{n}(Q^n(x) - x) - \gamma \right| < \frac{1}{n}.$$

Indeed, this follows from the inequalities $\underline{m}(Q^n) \leqq \mathrm{Rot}(Q^n) = n\gamma \leqq \bar{m}Q^n$ and $\bar{m}(Q^n) - \underline{m}(Q^n) < 1$.

To prove the lemma choose $n>\frac{1}{\varepsilon}$ and put

$$P(x)=\frac{1}{n}\sum_{i=0}^{n-1} Q^i(x), \qquad x\in\mathbb{R}.$$

Since $\tilde{H}$ is a convex subset of $\mathbb{R}^{\mathbb{R}}$ we have $P\in\tilde{H}$. Moreover, $Q'=PQP^{-1}$ satisfies the lemma. Indeed, $PQ(x)-P(x)=\frac{1}{n}(Q^n(x)-x)$, so putting $y=P(x)$ we have $PQP^{-1}(y)-y=\frac{1}{n}(Q^n(x)-x)$ which is within $\frac{1}{n}$ of γ. Since y is arbitrary this proves the lemma.

3. Realizability in $\tilde{G}_m$

In [JN] the following is proved

Theorem 3.1. *$(J;b;\gamma_1,\dots,\gamma_n)$ with $0<\gamma_i<1$ for $i=1,\dots,n$ is realizable in $\tilde{G}_1=\widetilde{\mathrm{PSL}}(2,\mathbb{R})$ if and only if one of the following conditions is true*
(a) $2\leqq b\leqq n-2$,
(b) $b=1$ *and* $\sum_{i=1}^{n}\gamma_i\leqq 1$,
(c) $b=n-1$ *and* $\sum_{i=1}^{n}\gamma_i\geqq n-1$.

Note that an element of $\tilde{G}_1$ of non-integral rotation number is always conjugate to a shift, so J plays no role in Theorem 3.1.

Define for $J\subset\{1,\dots,n\}$ and m a positive integer:

$$T=\left\{(\gamma_1,\dots,\gamma_n)\in(0,1)^n\,\middle|\,\sum_1^n\gamma_i\leqq 1\right\},$$

$$\begin{aligned}R_n(m;J)=\{(\gamma_1,\dots,\gamma_n)\in(0,1)^n\,|\,\exists a_1,\dots,a_n\in\mathbb{Z}, a_1+\dots+a_n\\ =m+n-2, \gamma_i\leqq a_i/m \text{ for } i\notin J, \gamma_i<a_i/m \text{ for } i\in J\}\end{aligned}$$

Proposition 3.2. *Suppose $(\gamma_1,\dots,\gamma_n)\in(0,1)^3-T$, that is, $(J;1;\gamma_1,\dots,\gamma_n)$ is not realizable in $\tilde{G}_1$. Then $(J;1;\gamma_1,\dots,\gamma_n)$ is realizable in $\tilde{G}_m$ if and only if $(\gamma_1,\dots,\gamma_n)\in R_n(m;J)$.*

Proof. Let

$$a_i=\begin{cases}\lfloor m\gamma_i\rfloor+1 & \text{for} \quad i\in J,\\ \lceil m\gamma_i\rceil & \text{for} \quad i\notin J.\end{cases}$$

Then $(\gamma_1,\dots,\gamma_n)\in R(m;J)$ if and only if $\sum a_i\leqq n+m-2$. By permuting indices we may assume $m\gamma_i\notin\mathbb{Z}$ for $i=1,\dots,k$ and $m\gamma_i\in\mathbb{Z}$ for $i=k+1,\dots,n$. We shall first consider the case that $\{k+1,\dots,n\}\subset J$, so

$$a_i=\lfloor m\gamma_i\rfloor+1 \quad \text{for all} \quad i.$$

$(J; 1; \gamma_1, \ldots, \gamma_n)$ is realizable in $\tilde{G}_m$ if and only if $(J; m; m\gamma_1, \ldots, m\gamma_n)$ is realizable in $\tilde{G}_1$. We renormalize to get

$$\left(J_0; m - \sum_1^n \lfloor m\gamma_i \rfloor; \{m\gamma_1\}, \ldots, \{m\gamma_k\}\right)$$

with $J_0 = J - \{k+1, \ldots, n\}$. Thus we must show

$$\left(J_0; m - \sum_1^n \lfloor m\gamma_i \rfloor; \{m\gamma_1\}, \ldots, \{m\gamma_k\}\right)$$

satisfies one of the conditions a), b), or c) of Theorem 3.1 if and only if $\sum a_i \leqq m + n - 2$.

If it satisfies b) of Theorem 3.1 then $m \sum_1^n \gamma_i = \sum_1^n \lfloor m\gamma_i \rfloor + \sum_1^k \{m\gamma_i\} = -\left(m - \sum_1^n \lfloor m\gamma_i \rfloor\right) + \sum_1^k \{m\gamma_i\} + m \leqq m$, so $(\gamma_1, \ldots, \gamma_n) \in T$, contradicting assumption. Cases a) and c) imply $2 \leqq m - \sum_1^n \lfloor m\gamma_i \rfloor$ and hence $\sum_1^n a_i = \sum_1^n \lfloor m\gamma_i \rfloor + n \leqq m + n - 2$, as desired.

Conversely, assume $\sum_1^n a_i \leqq m + n - 2$. Since $\sum_1^n \gamma_i > 1$ we get $m + n - 2 \geqq \sum_1^n a_i = \sum_1^k \lfloor m\gamma_i \rfloor + \sum_{k+1}^n m\gamma_i + n > \sum_1^n m\gamma_i + n - k > m + n - k$, so $k > 2$. Moreover $\sum_1^n \lfloor m\gamma_i \rfloor = \sum_1^n a_i - n \leqq m - 2$, so $m - \sum_1^n \lfloor m\gamma_i \rfloor \geqq 2$. Also $\sum_1^n \lfloor m\gamma_i \rfloor > \sum_1^n m\gamma_i - k \geqq m - k$, so $m - \sum_1^n \lfloor m\gamma_i \rfloor \leqq k - 1$. Thus by part a) of Theorem 3.1 we are done unless $m - \sum_1^n \lfloor m\gamma_i \rfloor = k - 1$. But in this case $\sum_1^k \{m\gamma_i\} = \sum_1^n m\gamma_i - \sum_1^n \lfloor m\gamma_i \rfloor > m - (m - k + 1) = k - 1$, so we are done by part c) of Theorem 3.1.

Now suppose $\{k+1, \ldots, n\} \notin J$, in fact suppose $\{k+1, \ldots, n\} - J$ has g elements. Then we must show that

$$\left(m - \sum_1^n \lfloor m\gamma_i \rfloor, \{m\gamma_i\}, \ldots, \{m\gamma_k\}, 0, \ldots, 0\right)$$

(g zeros) is realizable in $\tilde{G}_1$ if and only if $\sum_1^n a_i \leqq m + n - 2$. By Theorem 1 we have realizability if and only if

$$2 - g \leqq m - \sum_1^n \lfloor m\gamma_i \rfloor \leqq g + k - 2\,. \tag{$*$}$$

But $a_i = \lfloor m\gamma_i \rfloor$ for $i \in \{k+1, \ldots, n\} - J$ and $a_i = \lfloor m\gamma_i \rfloor + 1$ otherwise, so $\sum_1^n a_i = \sum_1^n \lfloor m\gamma_i \rfloor + n - g$. The left hand inequality in ($*$) is thus equivalent to $\sum_1^n a_i \leqq m + n - 2$ while the right hand inequality in ($*$) holds automatically $\left(\text{since } m - \sum_1^n \lfloor m\gamma_i \rfloor \leqq m - \sum_1^n m\gamma_i + n - 1 < n - 1\right)$. The proof is complete.

Define

$$R_n(J) = \bigcup_{m=2}^{\infty} R_n(m; J).$$

Corollary 3.3. *Assume* $0<\gamma_i<1$ *for* $i=1, \ldots, n$. *Then* $(J; 1; \gamma_1, \ldots, \gamma_n)$ *is realizable in some* $\tilde{G}_m$ *if and only if* $(\gamma_1, \ldots, \gamma_n) \subset R_n(J)$.

Proof. This is immediate from the above Proposition once we know that $T \subset R_n(J)$, which is part of the following Proposition.

Proposition 3.4. *Denote* $R_n = R_n(\{1, \ldots, n\})$ *and let* $\overline{R_n}$ *be its closure in* $(0, 1)^n$. *Then*
(i) R_n *is open.* $R_n \subseteqq R_n(J) \subseteqq \overline{R_n}$ *for all* J.
(ii) $T \subset R_n$.
(iii) $R_n(\phi) = \overline{R_n}$.

Proof. Put $R_n(m) = R_n(m; \{1, \ldots, n\})$. Then $R_n(m)$ is open and $R_n = \bigcup_{m=2}^{\infty} R_n(m)$ and $R_n(m, J) \subseteqq \overline{R_n}$, so (i) follows. For (ii) suppose $(\gamma_1, \ldots, \gamma_n) \in T$. If $\sum_1^n \gamma_i < 1$ then we can find rational $\frac{a_i}{m}$ with $0<\gamma_i<\frac{a_i}{m}$ for $i=1, \ldots, n$ and $\sum_1^n \frac{a_i}{m}<1$, so $(\gamma_1, \ldots, \gamma_n) \in R_n(m) \subset R_n$. If $\sum_1^n \gamma_i = 1$ and the γ_i are all rational then we can write $(\gamma_1, \ldots, \gamma_n) = \frac{1}{k}(a_1, \ldots, a_n)$ for some $a_1, \ldots, a_n, k \in \mathbb{Z}$. Put $m = k-1$. Then $\gamma_i < \frac{a_i}{m}$ for each i and $\sum_1^n a_i = k \leqq m+n-2$ so $(\gamma_1, \ldots, \gamma_n) \in R_n(m) \subset R_n$. Finally, if $\sum_1^n \gamma_i = 1$ and some γ_i is irrational then at least two γ_i's are rationally independent, say γ_1 and γ_2. The set of $(m\gamma_1, m\gamma_2) \pmod{\mathbb{Z} \oplus \mathbb{Z}}$, m a positive integer, is dense in $(\mathbb{R}/\mathbb{Z})^2$, so we can find m with $\frac{1}{2} < \{m\gamma_1\} < 1$ and $\frac{1}{2} < \{m\gamma_2\} < 1$. Then $\lfloor m\gamma_i + 1 \rfloor - m\gamma_i < \frac{1}{2}$ for $i=1, 2$, so, putting $a_i = \lfloor m\gamma_i + 1 \rfloor$ for $i=1, \ldots, n$, we have $\sum_1^n a_i = \sum_1^n \lfloor m\gamma_i + 1 \rfloor < \sum_1^n m\gamma_i + n - 1 = m+n-1$, so $\sum_1^n a_i \leqq m+n-2$, so $(\gamma_1, \ldots, \gamma_n) \in R_n(m)$.

To see statement (iii) of the Proposition, suppose $(\gamma_1^{(j)}, \ldots, \gamma_n^{(j)})$, $j=1, 2, \ldots$, is a sequence of points in R_n which approaches a limit $x \in (0, 1)^n$. If there exists m such that $(\gamma_1^{(j)}, \ldots, \gamma_n^{(j)}) \in R_n(m; \phi)$ for all sufficiently large j then the limit is in $R_n(m; \phi)$ since $R_n(m; \phi)$ is closed. Otherwise $\sum_1^n \gamma_i^{(j)} \leqq 1 + \frac{n-1}{m_j}$ where $m_j \to \infty$, so $x \in T$, whence $x \in R_n$ by part (ii). Thus $\overline{R_n} \subseteqq R_n(\phi)$, so $\overline{R_n} = R_n(\phi)$ by part (i).

Define

$$S_n(J) = \{(\gamma_1, \ldots, \gamma_n) \in (0, 1)^n \mid \text{condition } (*) \text{ holds}\}$$

where $(*)$ is the condition:

$$\left.\begin{array}{l}\text{For some integers } 0<a<m \text{ and some permutation } \left(\dfrac{a_i}{m},\ldots,\dfrac{a_n}{m}\right)\\ \text{of } \left(\dfrac{a}{m},\dfrac{m-a}{m},\dfrac{1}{m},\ldots,\dfrac{1}{m}\right) \text{ we have}\\ \qquad\qquad \gamma_i \leqq \dfrac{a_i}{m} \quad \text{for} \quad i\notin J\,,\\ \qquad\qquad \gamma_i < \dfrac{a_i}{m} \quad \text{for} \quad i\in J\,.\end{array}\right\}\quad (*)$$

Note that if $(*)$ holds with given a and m, then it also holds with $a'=a/gcd(a,m)$ and $m'=m/gcd(a,m)$. Thus we could require $gcd(a,m)=1$ in $(*)$, which is condition $(*)$ of Theorem 3.

Clearly $S_n(J)\subseteq R_n(J)$. To prove Theorem 3 we must show $S_n(J)=R_n(J)$. Before we do so we define another subset of $(0,1)^n$ relevant to Theorem 4.

For any subset $J\subseteq\{1,\ldots,n\}$ let

$$J_1=J\cap\{1,\ldots,n-2\}$$

$$J_2=\left\{\begin{array}{c}\phi\\ \{2\}\\ \{3\}\\ \{2,3\}\end{array}\right\} \text{according as } J\cap\{n-1,n\}=\left\{\begin{array}{c}\phi\\ \{n-1\}\\ \{n\}\\ \{n-1,n\}\end{array}\right\}.$$

Define inductively

$$T_3(J)=R_3(J)$$

$$T_n(J)=\{(\gamma_1,\ldots,\gamma_n)\in(0,1)^n \mid \exists x, (\gamma_1,\ldots,\gamma_{n-2},x)\in T_{n-1}(J_1)$$
$$\text{and } (1-x,\gamma_{n-1},\gamma_{n-2})\in T_3(J_2)\}\,.$$

If, in this inductive definition, we replaced $T_3(J)$ by the (conjecturally equal) set

$$\{(\gamma_1,\gamma_2,\gamma_3)\mid(\mathrm{J};1;\gamma_1,\gamma_2,\gamma_3) \text{ is realizable in } \tilde{H}\}\,,$$

then we would have $T_n(J)=\{(\gamma_1,\ldots,\gamma_n)\mid(J;1;\gamma_1,\ldots,\gamma_n)$ is realizable in $\tilde{H}\}$ for all n and J. This is because $Q_1\ldots Q_n=\mathrm{sh}(1)$ in $\tilde{H}$ if and only if $Q_1\ldots Q_{n-2}X=\mathrm{sh}(1)$ and $(\mathrm{sh}(1)X^{-1})Q_{n-1}Q_n=\mathrm{sh}(1)$ with $X=Q_{n-1}Q_n$. Thus Theorem 4 of the Introduction is proved if we show always $T_n(J)=R_n(J)$. Thus to complete the proofs of Theorems 3 and 4 we must show:

Lemma 3.5. *$S_n(J)=R_n(J)=T_n(J)$ for $n\geqq 3$ and any J.*

Proof. Clearly $S_n(J)\subseteq R_n(J)\subseteq T_n(J)$, so we must show $T_n(J)\subseteq S_n(J)$. We do this by induction. Our original proof of the induction start, $n=3$, is superceded by the following much simpler proof due to Don Zagier.

For $n=3$ the statement to be proved is $R_3(J)\subseteq S_3(J)$. It is sufficient to show that the external points $\left(\frac{a}{m},\frac{b}{m},\frac{c}{m}\right)$, $m=a+b+c-1$, of $R_3(\phi)$ are in $S_3(\phi)$. Without loss of generality $a\leqq b\leqq c$. Moreover, we may assume $2\leqq a$ since if $a=1$ we are done. We consider two cases:

Case 1. $m \leqq 2a+2b-2$. Then $\frac{c}{m} \leqq \frac{1}{2}$, so $\left(\frac{a}{m}, \frac{b}{m}, \frac{c}{m}\right) \leqq \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$.

Case 2. $m \geqq 2a+2b-1$. Put $k = \left\lfloor \frac{b}{a} \right\rfloor$ and let $I = \bigcup_{l=2k+1}^{\infty} I_l$ where $I_l = \left[\frac{lb}{k}, \frac{l(a+b-1)}{k}\right]$. These intervals overlap since $l \geqq 2k+1 \Rightarrow \frac{l(b+a-1)}{k} - \frac{(l+1)b}{k} = \frac{l(a-1)-b}{k} \geqq \frac{(2k+1)(a-1)-(ka+a-1)}{k} = a-2 \geqq 0$. Hence $I = \left[\frac{(2k+1)b}{k}, \infty\right)$. But $\frac{(2k+1)b}{k} \leqq 2b + \frac{ka+a-1}{k} \leqq 2b+2a-1 \leqq m$, so $m \in I$, that is $m \in I_l$ for some $l > 2k$. Then $\left(\frac{a}{m}, \frac{b}{m}, \frac{c}{m}\right) \leqq \left(\frac{1}{l}, \frac{k}{l}, \frac{l-k}{l}\right)$.

We shall prove the induction step just for $J = \phi$. It will be clear that the same proof works for any J if appropriate inequalities are replaced by strict inequalities. Assume therefore that $T_k(J) = S_k(J)$ for $k = 3, \ldots, n-1$ and any J. Let $(\gamma_1, \ldots, \gamma_n) \in T_n(\phi)$. Hence there exists $x \in (0, 1)$ such that $(\gamma_1, \ldots, \gamma_{n-2}, x) \in T_{n-1}(\phi)$ and $(1-x, \gamma_{n-1}, \gamma_n) \in T_3(\phi)$. By the induction hypothesis we can find $a, m \in \mathbb{Z}$ and a permutation $(a_1, \ldots, a_{n-1})$ of $(a, m-a, 1, \ldots, 1)$ such that $\gamma_i \leqq \frac{a_i}{m}$ for $i = 1, \ldots, n-2$ and $x \leqq \frac{a_{n-1}}{m}$. We can also find $b, l \in \mathbb{Z}$ and a permutation (b_1, b_2, b_3) of $(b, l-b, 1)$ such that $1-x \leqq \frac{b_1}{l}$, $\gamma_{n-1} \leqq \frac{b_2}{l}$, $\gamma_n \leqq \frac{b_3}{l}$. Note that $1 - \frac{b_1}{l} \leqq x \leqq \frac{a_{n-1}}{m}$, whence

$$1 \leqq \frac{b_1}{l} + \frac{a_{n-1}}{m}. \tag{$*$}$$

Since $\sum_1^{n-1} a_i = m+n-3$, this is equivalent to

$$\sum_{i=1}^{n-2} \frac{a_i}{m} - \frac{n-3}{m} \leqq \frac{b_1}{l}. \tag{$**$}$$

We consider four cases:

Case 1. $a_{n-1} = 1$ and $b_1 = 1$. In this case $(*)$ forces $l = m = 2$, so $\gamma_i \leqq \frac{1}{2}$ for all i, so $(\gamma_1, \ldots, \gamma_n) \in R_n(2; \phi) \subset R_n(\phi)$.

Case 2. $a_{n-1} = 1$ and $b_1 > 1$. Then without loss of generality $b_2 = 1$. By $(*)$ we have $l \geqq m$. In addition, since $b_1 = l - b_3$, inequality $(*)$ says $1 \leqq \frac{l-b_3}{l} + \frac{1}{m}$ and hence $\frac{1}{m} \geqq \frac{b_3}{l}$. Thus

$$\left(\frac{a_1}{m}, \ldots, \frac{a_{n-2}}{m}, \frac{b_2}{l}, \frac{b_3}{l}\right) \leqq \left(\frac{a_1}{m}, \ldots, \frac{a_{n-2}}{m}, \frac{1}{m}, \frac{1}{m}\right)$$

(component-wise inequality) so $(\gamma_1, \ldots, \gamma_n) \in R_n(m; \phi) \subset R_n(\phi)$.

Case 3. $a_{n-1}>1$ and $b_1=1$. Then without loss of generality $a_1=\ldots=a_{n-3}=1$. By (*) we have $m\geqq l$. Since $\sum_1^{n-2}\frac{a_i}{m}-\frac{n-3}{m}=\frac{a_{n-2}}{m}$, by (**) we have $\frac{a_{n-2}}{m}\leqq\frac{1}{l}$. Then

$$\left(\frac{a_1}{m},\ldots,\frac{a_{n-2}}{m},\frac{b_2}{l},\frac{b_3}{l}\right)\leqq\left(\frac{1}{l},\ldots,\frac{1}{l},\frac{b_2}{l},\frac{b_3}{l}\right).$$

Case 4. $a_{n-1}>1$ and $b_1>1$. We again assume $a_1=\ldots=a_{n-3}=1$ and $b_2=1$. By (**) we have $\frac{a_{n-2}}{m}\leqq\frac{b_1}{l}$. If $l\leqq m$ then

$$\left(\frac{a_1}{m},\ldots,\frac{a_{n-3}}{m},\frac{a_{n-2}}{m},\frac{b_2}{m},\frac{b_3}{m}\right)\leqq\left(\frac{1}{l},\ldots,\frac{1}{l},\frac{b_1}{l},\frac{1}{l},\frac{b_3}{l}\right).$$

If $l\geqq m$ then

$$\left(\frac{a_1}{m},\ldots,\frac{a_{n-3}}{m},\frac{a_{n-2}}{m},\frac{b_2}{m},\frac{b_3}{m}\right)\leqq\left(\frac{1}{m},\ldots,\frac{1}{m},\frac{a_{n-2}}{m},\frac{1}{m},\frac{m-a_{n-2}}{m}\right).$$

The last inequality $\frac{b_3}{l}\leqq\frac{m-a_{n-2}}{m}$ follows from $\frac{a_{n-2}}{m}\leqq\frac{b_1}{l}$ and the fact that $b_3=l-b_1$.

4. Nonrealizability

Theorem 4.1 below is our main nonrealizability result. It leaves undecided for $n=3$ only the region S of Fig. 2 (Sect. 1) together with six additional boxes, namely the following box of volume 0.001736... and all boxes obtained by permuting the coordinates:

$$\{(\gamma_1,\gamma_2,\gamma_3)|\tfrac{1}{5}\leqq\gamma_1\leqq\tfrac{1}{4},\tfrac{1}{3}\leqq\gamma_2\leqq\tfrac{3}{8},\tfrac{1}{2}\leqq\gamma_3\leqq\tfrac{7}{12}\}.$$

In the next section we give a special argument to eliminate these boxes.

Theorem 4.1. *Suppose* $\left(\frac{\beta_1}{\alpha_1},\frac{\beta_2}{\alpha_2},\frac{\beta_3}{\alpha_3}\right)\in\mathbb{Q}^3$ *with* $0<\frac{\beta_i}{\alpha_i}<1$ *for each i and*

$$\frac{\beta_1}{\alpha_1}=1-\left(\frac{\beta_2}{\alpha_2}+\frac{\beta_3}{\alpha_2(\alpha_3-1)}\right)$$

Then $(J;1;\gamma_1,\ldots,\gamma_n)$ is not realizable in $\tilde{H}$ if, after some permutation of indices, we have $\gamma_i>\frac{\beta_i}{\alpha_i}$ for $i\notin J\cap\{1,2,3\}$ and $\gamma_i\geqq\frac{\beta_i}{\alpha_i}$ for $i\in J\cap\{1,2,3\}$.

Remark. Exchanging indices 2 and 3 gives a weaker version of the theorem unless $\frac{\beta_2}{\alpha_2-1}\leqq\frac{\beta_3}{\alpha_3-1}$.

Proof. Suppose $\gamma_i>\frac{\beta_i}{\alpha_i}$ for $i=1,2,3$ and suppose we have $Q_i\in\tilde{H}$ with $Q_1\ldots Q_n=\mathrm{sh}(1)$ and $\mathrm{Rot}(Q_i)=\gamma_i$ for $i=1,\ldots,n$. Choose any $\varepsilon>0$. By Lemma 2.1 we may assume, by conjugating if necessary, that $\underline{m}(Q_1) \geqq \gamma_1 - \varepsilon$. Then

$$\bar{m}(Q_2 \ldots Q_n) = \bar{m}(Q_1^{-1}\,\mathrm{sh}(1)) \leqq 1 - \gamma_1 + \varepsilon. \tag{4.1}$$

Write $\delta = 1 - \gamma_1 + \varepsilon$ and $m_i = \underline{m}(Q_i)$ and $M_i = \bar{m}(Q_i)$ for $i = 2, 3$. Then (4.1) implies

$$\left.\begin{aligned} m_2 + M_3 &\leqq \delta \\ M_2 + m_3 &\leqq \delta. \end{aligned}\right\} \tag{4.2}$$

On the other hand, the facts that $\mathrm{Rot}(Q_i) > \dfrac{\beta_i}{\alpha_i}$ for $i = 2, 3$ imply $\mathrm{Rot}(Q_i^{\alpha_i}) > \beta_i$ and hence, by inequalities (2.2) and (2.3),

$$\left.\begin{aligned} \beta_2 &\leqq m_2 + (\alpha_2 - 1) M_2 \\ \beta_3 &\leqq m_3 + (\alpha_3 - 1) M_3 \end{aligned}\right\} \tag{4.3}$$

(actually with strict inequality, but we don't need this). Combining (4.2) and (4.3) gives

$$\begin{aligned} \beta_2 - \delta &\leqq -M_3 + (\alpha_2 - 1) M_2 \\ \beta_3 - \delta &\leqq -M_2 + (\alpha_3 - 1) M_3. \end{aligned}$$

Adding $(\alpha_3 - 1)$ times the first inequality to the second gives

$$\beta_3 - \delta + (\alpha_3 - 1)(\beta_2 - \delta) \leqq ((\alpha_3 - 1)(\alpha_2 - 1) - 1)\,\mathrm{M}_2$$

On the other hand (4.2) implies $M_2 < \delta$, so

$$\beta_3 - \delta + (\alpha_3 - 1)(\beta_2 - \delta) < ((\alpha_3 - 1)(\alpha_2 - 1) - 1)\delta,$$

or

$$\alpha_3 \beta_2 + \beta_3 - \beta_2 - \alpha_3 \delta < (\alpha_2 \alpha_3 - \alpha_2 - \alpha_3)\delta,$$

so

$$\delta > \frac{\alpha_3 \beta_2 + \beta_3 - \beta_2}{\alpha_2 \alpha_3 - \alpha_2} = \frac{\beta_2}{\alpha_2} + \frac{\beta_3}{\alpha_2(\alpha_3 - 1)}.$$

This is equivalent to

$$\gamma_1 - \varepsilon < 1 - \left(\frac{\beta_2}{\alpha_2} + \frac{\beta_3}{\alpha_2(\alpha_3 - 1)}\right).$$

Since this holds for every $\varepsilon > 0$ we have

$$\gamma_1 \leqq 1 - \left(\frac{\beta_2}{\alpha_2} + \frac{\beta_3}{\alpha_2(\alpha_3 - 1)}\right)$$

which contradicts the assumption that $\gamma_1 > \dfrac{\beta_1}{\alpha_1}$.

If either of the $Q_i, i = 2, 3$, is conjugate to $\mathrm{sh}(\gamma_i)$ with $\gamma_i \geqq \dfrac{\beta_i}{\alpha_i}$ then we still get inequality (4.3) so the proof goes through with no change. If Q_1 is conjugate to $\mathrm{sh}(\gamma_1)$ with $\gamma_1 \geqq \dfrac{\beta_1}{\alpha_1}$ then by conjugating if necessary we may assume $Q_1 = \mathrm{sh}(\gamma_1)$.

Then

$$\bar{m}(Q_2 \dots Q_n) = \bar{m}(Q_1^{-1}\,\mathrm{sh}(1)) = 1 - \gamma_1 .$$

By letting $\delta = 1 - \gamma_1$, inequality (4.2) remains valid, so as before we conclude $\delta > \frac{\beta_2}{\alpha_2} + \frac{\beta_3}{\alpha_2(\alpha_3 - 1)}$. This implies $\frac{\beta_1}{\alpha_1} \leqq \gamma_1 < 1 - \left(\frac{\beta_2}{\alpha_2} + \frac{\beta_3}{\alpha_2(\alpha_3 - 1)}\right)$ giving the desired contradiction in this case.

Using a standard linear programming argument one can show that Theorem 4.1 is the best result that one can extract out of inequalities (4.2) and (4.3).

5. A Special Nonrealizability Result

Theorem 5.1. *Let* $(\alpha_1, \alpha_2, \alpha_3) = (2, 3, 5)$ *and let* J *be any subset of* $\{1, 2, 3\}$. *If* $(J; 1; \gamma_1, \gamma_2, \gamma_3)$ *satisfies*

$$\frac{1}{\alpha_i} \leqq \gamma_i < 1 \quad \textit{for} \quad i \in J ,$$

$$\frac{1}{\alpha_i} < \gamma_i < 1 \quad \textit{for} \quad i \notin J ,$$

then $(J; 1; \gamma_1, \gamma_2, \gamma_3)$ *is not realizable.*

Proof. Suppose $Q_1 Q_2 Q_3 = \mathrm{sh}(1)$ is a realization. By assumption $Q_i \sim \mathrm{sh}(\gamma_i)$ with $\gamma_i \geqq \frac{1}{\alpha_i}$ if $i \in J$ and $\mathrm{Rot}(Q_i) = \gamma_i > \frac{1}{\alpha_i}$ if $i \notin J$. Either way we have $\underline{m}(Q_i^{\alpha_i}) \geqq 1$. We shall show this leads to a contradiction.

We write $Q_i^{\alpha_i} = \mathrm{sh}(1) R_i^{-1}$, so $\bar{m}(R_i) \leqq 0$ for $i = 1, 2, 3$. In Fig. 4 we have drawn the Cayley graph of the icosahedral group on the surface of a dodecahedron. Consider the following sequence of equations in $\tilde{H}$ which correspond to a sequence of expanding polygonal paths on the Cayley graph:

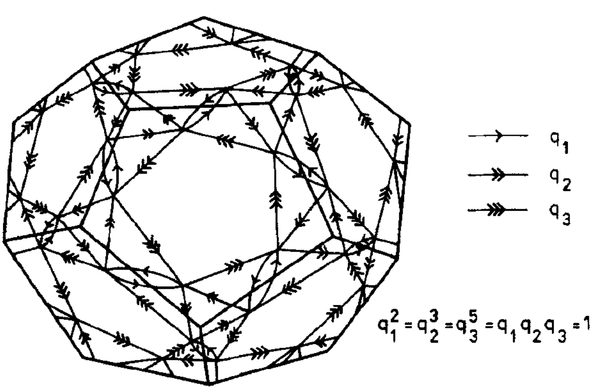

Fig. 4

$$\begin{aligned}
\mathrm{id} &= Q_3^5\,\mathrm{sh}(-1) R_3 = Q_3^4 Q_2^{-1} Q_1^{-1} R_3 = Q_3^4 Q_2^{-1} Q_1\,\mathrm{sh}(-1) R_1 R_3 \\
&= Q_3^4 Q_2^2 Q_1\,\mathrm{sh}(-2) R_2^{(1)} R_1 R_3 = Q_3^4 Q_1^{-1} Q_3^{-1} Q_2 Q_1\,\mathrm{sh}(-1) R_2^{(1)} R_1 R_3 \\
&= Q_3^4 Q_1 Q_3^{-1} Q_2 Q_1\,\mathrm{sh}(-2) R_1^{(1)} R_2^{(1)} R_1 R_3 = \dots ,
\end{aligned}$$

where $R_i^{(j)}$ are suitable conjugates of R_i. Once the path in question has expanded over the dodecahedron and contracted back to a trivial path via the back side of the dodecahedron we have a relation

$$\mathrm{id} = \mathrm{sh}(k)R$$

with $k \in \mathbb{Z}$ and R a product of conjugates of the R_i. We claim $k = -2$; since $\bar{m}(R) \leqq 0$ this gives $0 = \bar{m}(\mathrm{id}) = \bar{m}(\mathrm{sh}(-2)R) \leqq -2$, which is the desired contradiction.

As we deform the path as above, the relation $Q_3^5 = \mathrm{sh}(1)R_3^{-1}$ is used 12 times (once for each of the 12 faces of the dodecahedron), the relation $Q_2^3 = \mathrm{sh}(1)R_2^{-1}$ is used 20 times (once for each vertex of the dodecahedron) and $Q_1^2 = \mathrm{sh}(1)R_1^{-1}$ is used 30 times (once for each edge of the dodecahedron). This contributes $-(12+20+30) = -62$ to k. The relation $Q_1Q_2Q_3 = \mathrm{sh}(1)$ is used 60 times, contributing 60 to k, so $k = -62 + 60 = -2$, as claimed.

6. Description of the Figures

In this section we describe the numerical ingredients that go into drawing $\bar{R}_3$ and S in Figs. 1 through 3. We shall not give detailed proofs. The only proof that is not easy to complete is that the domain S of Fig. 2 represents the best our results will give, namely, that no further part of S is excluded by Theorems 4.1 and 5.1. This proof is tedious and did not seem worth including.

We shall need the following notation. Let a, b be integers satisfying

$$a \geqq 2, \quad b \geqq 2, \quad gcd(a,b) = 1 \tag{6.1}$$

and define integers a^* and b^* by

$$\left.\begin{aligned} aa^* &\equiv 1 \,(\mathrm{mod}\, b) \quad 0 < a^* < a \\ bb^* &\equiv 1 \,(\mathrm{mod}\, a) \quad 0 < b^* < b\,. \end{aligned}\right\} \tag{6.2}$$

Define

$$\left.\begin{aligned} A &= b + b^* - a^*, \quad & B &= a + a^* - b^* \\ A^* &= a^*, & B^* &= b^*\,. \end{aligned}\right\} \tag{6.3}$$

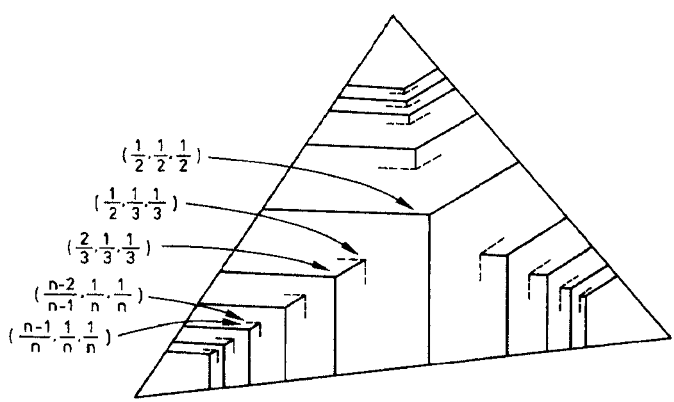

Fig. 5

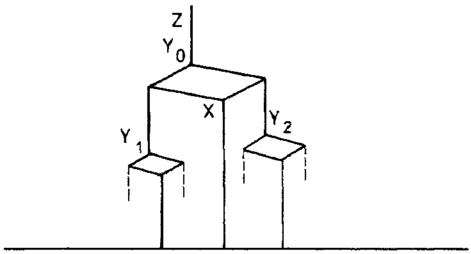

Fig. 6

Lemma 6.1. *The correspondence* $(a, b, a^*, b^*) \to (A, B, A^*, B^*)$ *is a bijective involution on the set of 4-tuples satisfying* (6.1) *and* (6.2). *Moreover* $(a+b, b, a^*, A)$ *and* $(a, a+b, B, b^*)$ *also satisfy* (6.1) *and* (6.2).

The easiest part of Fig. 1 is explicated in Fig. 5 above. A typical part of the remaining detail of Fig. 1 is pictured in Fig. 6 with coordinates given by the following formulae (notation as above).

$$X = \left(\frac{a+a^*-b^*}{a+b}, \frac{b+b^*-a^*}{a+b}, \frac{1}{a+b}\right) = \left(\frac{B}{A+B}, \frac{A}{A+B}, \frac{1}{A+B}\right),$$

$$Y_0 = \left(\frac{a-b^*}{a}, \frac{b-a^*}{b}, \frac{1}{a+b}\right),$$

$$Y_1 = \left(\frac{(a+b)-A}{a+b}, \frac{b-a^*}{b}, \frac{1}{(a+b)+b}\right),$$

$$Y_2 = \left(\frac{a-b^*}{a}, \frac{(a+b)-B}{a+b}, \frac{1}{a+(a+b)}\right).$$

$$Z = \left(\frac{a-b^*}{a}, \frac{b-a^*}{b}, \frac{1}{\max(a, b)}\right)$$

For future reference Fig. 7 repeats a portion of Fig. 6 with some lengths shown.

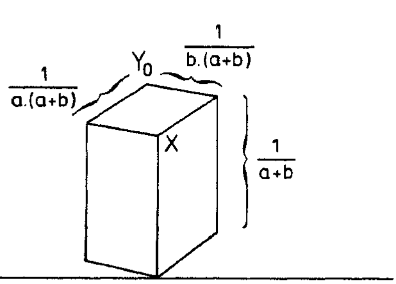

Fig. 7

The point $Y_0 = \left(\frac{a-b^*}{a}, \frac{b-a^*}{b}, \frac{1}{a+b}\right)$, and any point with a permutation of these coordinates, is at a "dimple" of $\bar{R}_3$. In each such "dimple" there is a connected domain $S(Y_0)$ described below in which realizability is not excluded by Theorem 4.1. However if Y_0 has coodinates a permutation of $(\frac{1}{2}, \frac{1}{3}, \frac{1}{5})$ then $S(Y_0)$ is excluded by

Theorem 5.1. Thus S of Fig. 2 is the union of the remaining $S(Y_0)$'s. By permuting coordinates if necessary, we may assume Y_0 has coordinates as above and $a>b$. Then $S(Y_0)$ is shown in Fig. 8 with coordinates given by the following formulae (the figure is schematic; vertical scale is compressed):

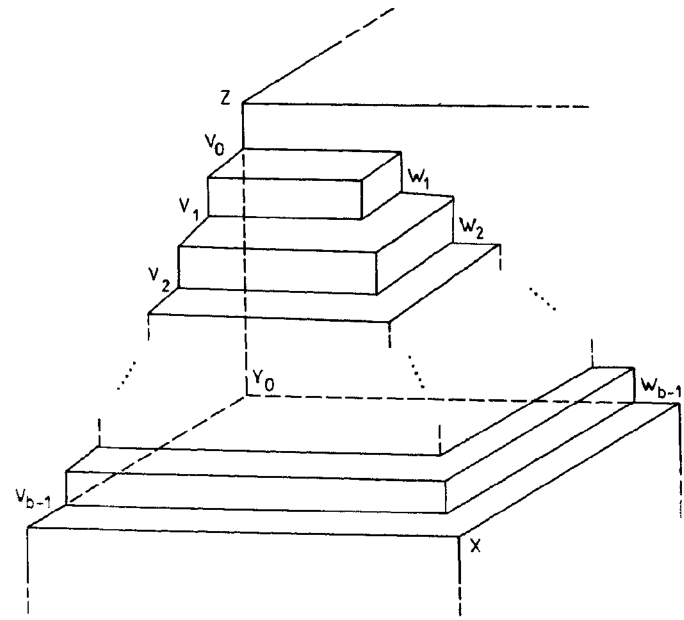

Fig. 8

$$V_i = \left(\frac{a-b^*}{a} + \frac{i}{ab(a+i)}, \frac{b-a^*}{b}, \frac{1}{a+1+i}\right) \quad (a>b)$$

$$W_i = \left(\frac{a-b^*}{a}, \frac{b-a^*}{b} + \frac{a-b+i}{ab(a+i)}, \frac{1}{a+1+i}\right) \quad (a>b).$$

A trivial calculation shows that $\left(\frac{\beta_1}{\alpha_1}, \frac{\beta_2}{\alpha_2}, \frac{\beta_3}{\alpha_3}\right) = \mathrm{V}_i$ and $\left(\frac{\beta_2}{\alpha_2}, \frac{\beta_1}{\alpha_1}, \frac{\beta_3}{\alpha_3}\right) = \mathrm{W}_i$ (note the transposition of indices) are tuples satisfying the premiss of Theorem 4.1.

7. Transverse Foliations

The problem of which Seifert fibered 3-manifolds M admit a transverse foliation (that is a codimension one foliation transverse to all the fibers of the Seifert fibration) was solved in [EHN] except in the case that M is orientable and Seifert fibered over the 2-sphere. The key to the solution was the following theorem.

Theorem. *M has a transverse foliation if and only if there exists a homomorphism $\phi: \pi_1(M) \to \tilde{H}$ with $\phi(Z) = \mathrm{sh}(1)$, where $Z \in \pi_1(M)$ is represented by a generic fiber of M.*

If M has Seifert invariant $(g=0; b; (\alpha_1, \beta_1), \ldots, (\alpha_n, \beta_n))$ with $0 < \frac{\beta_i}{\alpha_i} < 1$ then this theorem says that M admits a transverse foliation if and only if $\left(\{1, \ldots, n\}; b; \frac{\beta_1}{\alpha_1}, \ldots, \frac{\beta_n}{\alpha_n}\right)$ is "realizable" in the sense of this paper.

In the case left open by [EHN], namely $b=1$, we therefore have conjecturally that a transverse foliation exists if and only if there exist integers $0<a<m$ with $gcd(a,m)=1$ and a permutation $\left(\frac{a_1}{m}, \ldots, \frac{a_n}{m}\right)$ of $\left(\frac{a}{m}, \frac{m-a}{m}, \frac{1}{m}, \ldots, \frac{1}{m}\right)$ with $0 < \frac{\beta_i}{\alpha_i} < \frac{a_i}{m}$ for $i=1, \ldots, n$. Moreover, the foliation can then be chosen to be analytic (in fact, even transversely projective), as in the previously solved cases. As we have pointed out, this paper proves this conjecture in 99.9% of all cases. It also contradicts the result announced in [G], and shows that the answer is considerably more subtle than earlier results (see also [M, S, W] might have led one to expect.

Appendix: Volume of R_3

by Don Zagier

From the description of R_3 in Sect. 6 we see

$$\mathrm{Vol}(R_3) = V_0 + V_1 + V_2$$

where

$$V_0 = \tfrac{1}{2} \cdot \tfrac{1}{2} \cdot \tfrac{1}{2} = \tfrac{1}{8}$$

is the volume of the central cube,

$$V_1 = 3 \sum_{n=3}^{\infty} \frac{1}{n} \cdot \frac{1}{n} \cdot \left(\frac{n-1}{n} - \frac{n-2}{n-1}\right) = 3 \sum_{n=3}^{\infty} \frac{1}{n^3(n-1)}$$

is the volume of the part indicated in Fig. 5, and

$$V_2 = 6 \sum_{\substack{2 \leq a < b \\ (a,b)=1}} \frac{1}{a(a+b)} \cdot \frac{1}{b(a+b)} \cdot \frac{1}{a+b}$$

is the sum of volumes of the rectangular blocks of Fig. 7. Now

$$V_1 = 3\left(\sum_{k \geq 1} \frac{1}{k(k+1)^3} - \frac{1}{8}\right)$$

and

$$\begin{aligned} V_2 &= 6 \sum_{\substack{2 \leq a \leq b \\ (a,b)=1}} \frac{1}{ab(a+b)^3} = 3 \sum_{\substack{a,b \geq 2 \\ (a,b)=1}} \frac{1}{ab(a+b)^3} \\ &= 3\left(\sum_{\substack{a,b>0 \\ (a,b)=1}} \frac{1}{ab(a+b)^3} - 2\sum_{k \geq 1} \frac{1}{k(k+1)^3} + \frac{1}{8}\right). \end{aligned}$$

Thus

$$\mathrm{Vol}(R_3)=3S_2-3S_1+\tfrac{1}{8},$$

where

$$S_1=\sum_{k\geq 1}\frac{1}{k(k+1)^3}$$

and

$$S_2=\sum_{\substack{a,b>0\\(a,b)=1}}\frac{1}{ab(a+b)^3}.$$

To compute S_1 we note that

$$\frac{1}{k(k+1)^3}=\frac{1}{k}-\frac{1}{k+1}-\frac{1}{(k+1)^2}-\frac{1}{(k+1)^3}$$

whence

$$\begin{aligned}S_1&=\sum_{k\geq 1}\left(\frac{1}{k}-\frac{1}{k+1}\right)-\sum_{k\geq 1}\frac{1}{(k+1)^2}-\sum_{k\geq 1}\frac{1}{(k+1)^3}\\&=1-(\zeta(2)-1)-(\zeta(3)-1)\\&=3-\zeta(2)-\zeta(3).\end{aligned}$$

For S_2 we use

$$\frac{1}{kl(k+l)^3}=\frac{1}{k^4l}-\frac{1}{k^4(k+l)}-\frac{1}{k^3(k+l)^2}-\frac{1}{k^2(k+l)^3}.$$

Thus

$$\begin{aligned}\zeta(5)S_2&=\sum_{\substack{d,a,b>0\\(a,b)=1}}\frac{1}{d^5ab(a+b)^3}\\&=\sum_{k,l>0}\frac{1}{kl(k+l)^3}\\&=\sum_{k,l>0}\left(\frac{1}{k^4l}-\frac{1}{k^4(k+l)}\right)-\sum_{k,l>0}\left(\frac{1}{k^3(k+l)^2}+\frac{1}{k^2(k+l)^3}\right)\\&=\sum_{k>0}\frac{1}{k^4}\left(1+\frac{1}{2}+\dots+\frac{1}{k}\right)-\sum_{n\neq m}\frac{1}{n^2m^3}\\&=\sum_{k\geq l>0}\frac{1}{k^4l}-\sum_{n,m>0}\frac{1}{n^2m^3}+\sum_{n>0}\frac{1}{n^5}\\&=S_3-\zeta(2)\zeta(3)+\zeta(5),\end{aligned}$$

where

$$S_3=\sum_{k\geq l>0}\frac{1}{k^4l}=\sum_{l,m>0}\frac{1}{(l+m)^4l}+\sum\frac{1}{l^5}.$$

But

$$2S_3 = \sum_{k,l>0}\left(\frac{1}{(l+m)^4 l} + \frac{1}{(l+m)^4 m}\right) + 2\zeta(5)$$
$$= \sum_{k,l>0}\frac{1}{(l+m)^3 lm} + 2\zeta(5)$$
$$= \zeta(5)S_2 + 2\zeta(5).$$

Combining the last two formulae, one finds

$$S_2 = 4 - \frac{2\zeta(2)\zeta(3)}{\zeta(5)},$$

and hence

$$\mathrm{Vol}(R_3) = \tfrac{25}{8} + 3\zeta(2) + 3\zeta(3) - \frac{6\zeta(2)\zeta(3)}{\zeta(5)}.$$

References

[EHN] Eisenbud, D., Hirsch, U., Neumann, W.: Transverse foliations of Seifert bundles and self homeomorphisms of the circle. Comment. Math. Helv. **56**, 638–660 (1981)

[G] Carmen Gazolas, M. del: Fibrés de Seifert: classification et existence de feuilletages. C.R. Acad. Sci. Paris **295**, 677–679 (1982)

[JN] Jankins, M., Neumann, W.D.: Homomorphisms of Fuchsian groups to PSL(2, R). Comment. Math. Helv. (to appear)

[M] Milnor, J.: On the existence of a connection with curvature zero. Comment. Math. Helv. **32**, 215–223 (1957–1958)

[S] Sullivan, S.: A generalization of Milnor's inequalities concerning affine foliations and affine manifolds. Comment. Math. Helv. **51**, 183–184 (1976)

[W] Wood, J.W.: Bundles with totally disconnected structure group. Comment. Math. Helv. **46**, 257–273 (1971)

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