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On the Zeros of the Weierstrass $\wp$-Function

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The Weierstrass $\wp$-function, defined for $\tau \in \mathfrak{H}$ (upper half-plane) and $z \in \mathbb{C}$ by

$$\wp(z,\tau) = \frac{1}{z^2} + \sum_{\substack{\omega \in \mathbb{Z}+\mathbb{Z}\tau \\ \omega \neq 0}} \left(\frac{1}{(z+\omega)^2} - \frac{1}{\omega^2} \right),$$

is the basic and most famous function of elliptic function theory. As is well known, $\wp(z,\tau)$ is for fixed τ doubly periodic in z and takes on each value in $\mathbb{C} \cup \{\infty\}$ exactly twice (counting multiplicity) as z ranges over $\mathbb{C}/\mathbb{Z}+\mathbb{Z}\tau$. In particular, since $\wp(z,\tau)$ is an even function of z, there is for each $\tau \in \mathfrak{H}$ a number $z_0(\tau)$, well-defined up to sign and translation by $\mathbb{Z}+\mathbb{Z}\tau$, such that

$$\wp(z,\tau) = 0 \Leftrightarrow z \equiv \pm z_0(\tau) \,(\operatorname{mod} \mathbb{Z}+\mathbb{Z}\tau).$$

The purpose of this note is to prove the following explicit formula for $z_0(\tau)$ which, despite the long history of the function $\wp$, seems not to have been noticed earlier; this formula arose out of the authors' investigation of "Jacobi forms" $\left[\text{functions on } \mathbb{C} \times \mathfrak{H} \text{ satisfying a transformation law of a certain kind under the transformations } (z,\tau) \to \left(\frac{z+m+n\tau}{c\tau+d}, \frac{a\tau+b}{c\tau+d} \right) \text{ with } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}),\ m, n \in \mathbb{Z}\right]$ and is a special case of more general results, but in view of the special interest attaching to the $\wp$-function it seemed worthwhile to publish it separately.

Theorem. *The zeros of* $\wp(z,\tau)$ $(\tau \in \mathfrak{H},\ z \in \mathbb{C})$ *are given by*

$$z = m + \frac{1}{2} + n\tau \pm \left(\frac{\log(5+2\sqrt{6})}{2\pi i} + 144\pi i \sqrt{6} \int_{\tau}^{i\infty} (t-\tau) \frac{\Delta(t)}{E_6(t)^{3/2}}\, dt \right)$$

$(m, n \in \mathbb{Z})$, *where* $E_6(t)$ *and* $\Delta(t)$ $(t \in \mathfrak{H})$ *denote the normalized Eisenstein series of weight* 6 *and unique normalized cusp form of weight* 12 *on* $\mathrm{SL}_2(\mathbb{Z})$, *respectively, and the integral is to be taken over the vertical line* $t = \tau + i\mathbb{R}_+$ *in* $\mathfrak{H}$.

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For $\tau = i$ we know a priori that $\wp(z,\tau)$ vanishes at the point $z = \frac{1+i}{2}$ $\left[\text{because } \left(\frac{\partial \wp}{\partial z}\right)^2 = 4\wp^3 - g_2(\tau)\wp - g_3(\tau) \text{ and } g_3(i) = 0, \text{ so the zero of } \wp(z,i) \text{ is a double one and hence a 2-division point; that it is } \frac{1+i}{2} \text{ rather than } \frac{1}{2} \text{ or } \frac{i}{2} \text{ is easily verified}\right]$. Comparing this with the result of the theorem yields the integral identity

$$\int_1^\infty \frac{\Delta(it)}{E_6(it)^{3/2}}(t-1)\,dt = \frac{\pi - \log(5+2\sqrt{6})}{288\pi^2\sqrt{6}}.$$

Expanding $\Delta/E_6^{3/2}$ in a Fourier series and integrating term by term, we obtain the following amusing corollary, in which all mention of the $\wp$-function (or, for that matter, of modular forms) has been suppressed:

Corollary. *Define integers A_n $(n \geqq 1)$ by the formal power series expansion*

$$\sum_{n=1}^\infty A_n q^n = \frac{q \prod_{n=1}^\infty (1-q^n)^{24}}{\left(1 - 504 \sum_{n=1}^\infty \sum_{d|n} d^5 q^n\right)^{3/2}}$$

$(A_1 = 1,\ A_2 = 732,\ A_3 = 483336, \ldots)$. *Then*

$$\sum_{n=1}^\infty \frac{A_n}{n^2} e^{-2\pi n} = \frac{\pi - \log(5+2\sqrt{6})}{72\sqrt{6}}.$$

We remark that the series in the corollary converges very slowly since (as is not hard to show) A_n satisfies the asymptotic formula

$$A_n \sim C n^{1/2} e^{2\pi n} \qquad (C = 1/216\sqrt{2\pi}).$$

First Proof (Modular Forms)

We begin by observing that the function $\wp$ satisfies the transformation law

$$\wp\left(\frac{z}{c\tau+d}, \frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^2\,\wp(z,\tau) \qquad \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})\right),$$

as one easily checks from the definition. It follows that the function $z_0(\tau)$ satisfies the transformation equation

$$\frac{z_0(\tau)}{c\tau+d} \equiv \pm z_0\left(\frac{a\tau+b}{c\tau+d}\right) \qquad \left(\operatorname{mod} \mathbb{Z} + \mathbb{Z}\frac{a\tau+b}{c\tau+d}\right),$$

i.e. $z_0(\tau)$ is a (many-valued) modular form of weight -1 on $\mathrm{SL}_2(\mathbb{Z})$. Now the function $z_0(\tau)$ has infinitely many branches, but (except for sign) these differ from each other by the *linear* functions $\tau \to m + n\tau$ $(m, n \in \mathbb{Z})$, so the second derivative $z_0''(\tau)$

is well-defined up to sign and the function $z_0''(\tau)^2$ is single-valued. Writing the transformation law of z_0 as

$$z_0(\tau)=m+n\tau\pm(c\tau+d)z_0\left(\frac{a\tau+b}{c\tau+d}\right)$$

and differentiating twice we obtain

$$z_0''(\tau)=\pm(c\tau+d)^{-3}z_0''\left(\frac{a\tau+b}{c\tau+d}\right),$$

i.e. the function $z_0''(\tau)^2$ transforms under the action of $\mathrm{SL}_2(\mathbb{Z})$ like a modular form of weight 6.

At first sight it appears that z_0, and hence $z_0''^2$, is a holomorphic function of τ, since it (or each branch of it) is locally bounded and is defined implicitly by the vanishing of a meromorphic function in two variables. However, since z_0 is many-valued we also have to worry about ramification or coalescing of different branches. The branches $z_0(\tau)+m+n\tau$ $(m,n\in\mathbb{Z})$ can never meet, since $m+n\tau\neq 0$ for $(m,n)\neq(0,0)$, but the branches $z_0(\tau)$ and $-z_0(\tau)+m+n\tau$ can meet. This happens when $\wp(\tau,z)$ has a zero at one of the three 2-division points

$$z\equiv\frac{1}{2},\frac{\tau}{2},\frac{1+\tau}{2}\ (\mathrm{mod}\,\mathbb{Z}+\mathbb{Z}\tau)$$

[the point $z=0$ also satisfies $z\equiv -z\ (\mathrm{mod}\,\mathbb{Z}+\mathbb{Z}\tau)$ but cannot be a zero of $\wp$ since it is always a pole], i.e. whenever the function $\wp\left(\frac{1}{2},\tau\right)\wp\left(\frac{\tau}{2},\tau\right)\wp\left(\frac{1+\tau}{2},\tau\right)$ vanishes at τ. But this function is easily checked to be a holomorphic modular form of weight 6 on $\mathrm{SL}_2(\mathbb{Z})$ and hence a multiple (in fact $-1/864$) of $E_6(\tau)$. Thus

$$\begin{aligned}\text{two branches of } z_0(\tau) \text{ meet} &\Leftrightarrow \wp(z,\tau) \text{ has a double zero}\\ &\Leftrightarrow E_6(\tau)=0.\end{aligned}$$

[This can also be seen from the equation $\wp'(z)^2=4\wp(z)^3-g_2p(z)-g_3$, since g_3 is a multiple of E_6. Also, as is well-known, $E_6(\tau)$ vanishes precisely when τ is $\mathrm{SL}_2(\mathbb{Z})$-equivalent to i, but we shall not need this fact.] At such a point τ_0, exactly two branches of $z_0(\tau)$ coincide [since the function $\wp(\cdot,\tau):\mathbb{C}/\mathbb{Z}+\mathbb{Z}\tau\to\mathbb{C}\cup\{\infty\}$ is exactly 2-to-1], so each branch of z_0 has the local form

$$z_0(\tau)=\pm\sum_{r=0}^{\infty}c_r(\tau-\tau_0)^{r+\frac{1}{2}}+\frac{m}{2}+\frac{n}{2}\tau\quad(\tau\to\tau_0)$$

with $c_0\neq 0$; differentiating twice and squaring we get

$$z_0''(\tau)^2=\frac{c_0^2}{16}(\tau-\tau_0)^{-3}-\frac{3}{8}c_0c_1(\tau-\tau_0)^{-2}+\dots.$$

Thus $z_0''(\tau)^2$ is meromorphic with a triple pole at each τ with $E_6(\tau)=0$, so the function $E_6(\tau)^3z_0''(\tau)^2$, which transforms like a modular form of weight $3\times 6+2\times 3=24$, is holomorphic in the whole upper half-plane. In a moment we shall see that it is also holomorphic at the cusp of $\mathfrak{H}/\mathrm{SL}_2(\mathbb{Z})$ and in fact has a zero of order 2 there, so it must be a constant multiple of $\Delta(\tau)^2$, i.e.

$$\begin{aligned} z_0''(\tau) &= \pm C_0 \frac{\Delta(\tau)}{E_6(\tau)^{3/2}} \quad (\tau \in \mathfrak{H}) \\ &= \pm C_0 (e^{2\pi i\tau} + 732 e^{4\pi i\tau} + \ldots) \quad (\mathrm{Im}(\tau) > 1) \end{aligned}$$

for some $C_0 \neq 0$; integrating twice, we obtain the final formula

$$\begin{aligned} z_0(\tau) &= C_1 + C_2\tau \pm C_0 \int_\tau^{i\infty} \frac{\Delta(t)}{E_6(t)^{3/2}} (t-\tau)\,dt \quad (\tau \in \mathfrak{H}) \\ &= C_1 + C_2\tau \mp \frac{C_0}{4\pi^2} (e^{2\pi i\tau} + 183 e^{4\pi i\tau} + \ldots) \quad (\mathrm{Im}(\tau) > 1) \end{aligned}$$

for (each branch of) $z_0(\tau)$.

To prove the last assertion and determine the values of the constants of integration C_0, C_1, and C_2, we investigate the asymptotic behaviour of $z_0(\tau)$ as $\tau \to i\infty$. Computing the Fourier development of $\wp(z,\tau)$ in the same way as one calculates the Fourier development of Eisenstein series, one finds

$$\begin{aligned} (2\pi i)^{-2} \wp(z,\tau) = {} & \frac{1}{\zeta - 2 + \zeta^{-1}} + \frac{1}{12} + (\zeta - 2 + \zeta^{-1}) q \\ & + (2\zeta^2 + \zeta - 6 + \zeta^{-1} + 2\zeta^{-2}) q^2 + \ldots, \end{aligned}$$

where we have set $q = e^{2\pi i\tau}$ and $\zeta = e^{2\pi i z}$ [the coefficient of q^n for $n > 0$ is $\sum_{d|n} d(\zeta^d - 2 + \zeta^{-d})$, but we shall not need more terms than those given]. The ambiguity $z \to \pm z + m + n\tau$ $(m, n \in \mathbb{Z})$ in z corresponds to the ambiguity $\zeta \to q^n \zeta^{\pm 1}$ $(n \in \mathbb{Z})$ in ζ. To ask whether some branch of $z_0(\tau)$ has a finite limit as $\tau \to i\infty$ is equivalent to asking whether there is some finite value of ζ making the above expansion vanish for $q = 0$. Clearly, such a value is given by $\zeta - 2 + \zeta^{-1} = -12$ or $\zeta = -\varepsilon^{\pm 1}$, where $\varepsilon = 5 + 2\sqrt{6}$ is the fundamental unit of $\mathbb{Q}(\sqrt{6})$. In terms of $z = \frac{1}{2\pi i} \log \zeta$ this corresponds to

$$z = m + \frac{1}{2} \pm \frac{1}{2\pi i} \log \varepsilon \quad (m \in \mathbb{Z}).$$

Thus $z_0(\tau)$ has a branch tending to each of these values of z as $\tau \to i\infty$. To find the Fourier expansion of this branch, we write

$$z_0(\tau) = m + \frac{1}{2} \pm \frac{1}{2\pi i} (\log \varepsilon + A e^{2\pi i\tau} + B e^{4\pi i\tau} + \ldots)$$

with as yet unknown coefficients $A, B, \ldots$, substitute the corresponding value

$$\zeta = e^{2\pi i z_0(\tau)} = -\varepsilon^{\pm 1} \left(1 + Aq + \left(\frac{A^2}{2} + B\right) q^2 + \ldots\right)^{\pm 1}$$

into the above expansion for $\wp(z,\tau)$, and successively equate the coefficients of each power of q to 0. This gives the values

$$A=\pm 72\sqrt{6}, \quad B=\pm 13176\sqrt{6}=183A, \ldots;$$

comparing this expansion of $z_0(\tau)$ with the one given above we obtain

$$C_0=\pm 144\pi i\sqrt{6}, \quad C_1=m+\frac{1}{2}\pm\frac{1}{2\pi i}\log\varepsilon, \quad C_2=n \quad (m,n\in\mathbb{Z})$$

and hence the assertion of the theorem.

We observe that the same method can be applied to find the solutions of any equation of the form

$$\wp(z,\tau)=\phi(\tau)$$

where ϕ is a modular form of weight 2 [of course, since there are no holomorphic modular forms of weight 2 on $\mathrm{SL}_2(\mathbb{Z})$ we must take $\phi(\tau)$ to be either a meromorphic modular form or else modular on a subgroup Γ of $\mathrm{SL}_2(\mathbb{Z})$]: Again the solutions are of the form

$$z\equiv \pm z_\phi(\tau) \quad (\operatorname{mod} \mathbb{Z}+\mathbb{Z}\tau),$$

where z_ϕ transforms up to sign and translation by $\mathbb{Z}+\mathbb{Z}\tau$ like a modular form of weight -1 and hence $z_\phi''(\tau)^2$ like a modular form of weight 6 on Γ. By considering the ramification points of $z_\phi(\tau)$, i.e. the points where $z_\phi(\tau)$ is a 2-division point on $\mathbb{C}/\mathbb{Z}+\mathbb{Z}\tau$, one sees that

$$z_\phi''(\tau)=\pm\psi(\tau)/(4\phi(\tau)^3-g_2(\tau)\phi(\tau)-g_3(\tau))^{3/2},$$

where $\psi(\tau)$ is a modular form of weight 12 on Γ which is holomorphic wherever ϕ is. By substituting the Fourier expansions of $\wp$ and ϕ into the defining equation for z_ϕ we can compute as many terms as desired of the Fourier development of ψ and hence identify this function entirely, after which z_ϕ is obtained by two-fold integration.

Second Proof (Elliptic Integrals)

Following a suggestion of Hirzebruch, we can also prove the formulas for z_0 and z_ϕ by using elliptic integrals rather than modular forms. Indeed, from the formula for the derivative of $\wp(z)$ we obtain

$$dz=\frac{d\wp(z)}{\wp'(z)}=\frac{d\wp(z)}{\sqrt{4\wp(z)^3-g_2\wp(z)-g_3}}$$

and hence

$$z_0(\tau)=\int_{\wp=\infty}^{\wp=0} dz=\pm\int_0^\infty \frac{dX}{\sqrt{4X^3-g_2X-g_3}}.$$

Here

$$g_2 = 60 \times 2\zeta(4) E_4 = \frac{4\pi^4}{3} E_4, \qquad g_3 = 140 \times 2\zeta(6) E_6 = \frac{8\pi^6}{27} E_6,$$

where

$$E_4 = 1 + 240 \sum_{n \geqq 1} \sigma_3(n) q^n, \qquad E_6 = 1 - 504 \sum_{n \geqq 1} \sigma_5(n) q^n$$

($\sigma_\nu(n) = \sum_{d|n} d^\nu, q = e^{2\pi i \tau}$) are the normalized Eisenstein series of weight 4 and 6; the change of variables $X = \frac{\pi^2}{3} t$ gives

$$z_0(\tau) = \pm \frac{\sqrt{3}}{2\pi} \int_0^\infty \frac{dt}{\sqrt{t^3 - 3E_4 t - 2E_6}}.$$

We now compute the second derivative of this by differentiating with respect to τ under the integral sign. The derivatives of E_4 and E_6 are given by

$$\frac{1}{2\pi i} \frac{d}{d\tau} E_4 = \frac{1}{3}(E_2 E_4 - E_6), \qquad \frac{1}{2\pi i} \frac{d}{d\tau} E_6 = \frac{1}{2}(E_2 E_6 - E_4^2),$$

where $E_2 = 1 - 24 \sum_{n \geqq 1} \sigma_1(n) q^n$ (which is not a modular form). Hence

$$\frac{1}{2\pi i} \frac{\partial}{\partial \tau} \frac{1}{(t^3 - 3E_4 t - 2E_6)^{1/2}} = \frac{1}{2} \frac{(E_2 E_4 - E_6)t + E_2 E_6 - E_4^2}{(t^3 - 3E_4 t - 2E_6)^{3/2}}.$$

To differentiate again we also need the derivative of E_2, which is given by

$$\frac{1}{2\pi i} \frac{d}{d\tau} E_2 = \frac{1}{12}(E_2^2 - E_4);$$

we then find after some computation

$$\begin{aligned}
&\frac{1}{(2\pi i)^2} \frac{\partial^2}{\partial \tau^2} \frac{1}{(t^3 - 3E_4 t - 2E_6)^{1/2}} \\
&\quad = (t^3 - 3E_4 t - 2E_6)^{-5/2} \Big\{ \frac{5}{24}(E_2^2 E_4 + E_4^2 - 2E_2 E_6) t^4 \\
&\qquad + \frac{7}{24}(E_2^2 E_6 + E_4 E_6 - 2E_2 E_4^2) t^3 + \frac{1}{8}(E_2^2 E_4^2 - 2E_2 E_4 E_6 + 6E_6^2 - 5E_4^3) t^2 \\
&\qquad + \frac{1}{24}(5E_2^2 E_4 E_6 + 6E_2 E_4^3 - 16 E_2 E_6^2 + 5E_4^2 E_6) t \\
&\qquad + \frac{1}{12}(2E_2^2 E_6^2 - 4E_2 E_4^2 E_6 + 9E_4^4 - 7E_4 E_6^2) \Big\} \\
&\quad = \frac{\partial}{\partial t} \frac{At^2 + Bt + C}{(t^3 - 3E_4 t - 2E_6)^{3/2}},
\end{aligned}$$

where

$$A=-\frac{1}{12}(E_2^2E_4+E_4^2-2E_2E_6)=-\frac{3}{5}\cdot\frac{1}{(2\pi i)^2}\frac{\partial^2 E_4}{\partial\tau^2}=-144\sum_{n\geq 1}n^2\sigma_3(n)q^n,$$

$$B=-\frac{1}{12}(E_2^2E_6+E_4E_6-2E_2E_4^2)=-\frac{2}{7}\cdot\frac{1}{(2\pi i)^2}\frac{\partial^2 E_6}{\partial\tau^2}=144\sum_{n\geq 1}n^2\sigma_5(n)q^n,$$

$$C=+\frac{1}{6}(E_4^3-E_6^2)=288\Delta=288q\prod_{n\geq 1}(1-q^n)^{24}.$$

Integrating from $t=0$ to $t=\infty$ gives

$$\frac{-1}{4\pi^2}z_0''(\tau)=\frac{\sqrt{3}}{2\pi}\frac{At^2+Bt+C}{(t^3-3E_4t-2E_6)^{3/2}}\bigg|_0^\infty=\pm\frac{36i\sqrt{6}}{\pi}\frac{\Delta(\tau)}{E_6(\tau)^{3/2}},$$

in accordance with the formula obtained earlier. To complete the proof we must still compute $z_0(\tau)$ for $\tau\to i\infty$. But at infinity the discriminant Δ of the cubic $t^3-3E_4t-2E_6$ tends to 0, so the cubic degenerates into the product of a linear factor and the square of a linear factor and the elliptic integral defining z_0 becomes elementary. More precisely, for $\tau\to i\infty$ we have $E_4\to 1$, $E_6\to 1$ and hence

$$\begin{aligned}z_0(i\infty)&=\pm\frac{\sqrt{3}}{2\pi}\int_0^\infty\frac{dt}{\sqrt{t^3-3t-2}}\\&=\pm\frac{\sqrt{3}}{2\pi}\left(\int_0^2+\int_2^\infty\right)\frac{dt}{(t+1)\sqrt{t-2}};\end{aligned}$$

making the substitution $t=2-3x^2$ in the first integral and $t=2+3x^2$ in the second we find

$$\begin{aligned}z_0(i\infty)&=\pm\frac{1}{\pi i}\int_0^{\sqrt{2/3}}\frac{dx}{1-x^2}\pm\frac{1}{\pi}\int_0^\infty\frac{dx}{1+x^2}\\&=\pm\frac{1}{2\pi i}\log\frac{1+x}{1-x}\bigg|_0^{\sqrt{2/3}}\pm\frac{1}{\pi}\arctan x\bigg|_0^\infty\\&=\pm\frac{1}{2\pi i}\log(5+2\sqrt{6})\pm\frac{1}{2},\end{aligned}$$

and combining this with the result already obtained for $z_0''(\tau)$ we recover the formula of the theorem.

The same method applies to the solution $z=z_\phi(\tau)$ of the equation $\wp(z,\tau)=\phi(\tau)$, where $\phi(\tau)$ is any meromorphic function of τ: we have

$$z_\phi(\tau)=\frac{\sqrt{3}}{2\pi}\int_{\frac{3}{\pi^2}\phi(\tau)}^{\infty}\frac{dt}{\sqrt{t^3-3E_4t-2E_6}}$$

(with the indeterminacy coming from the choice of square root and of path of integration) and from this we can determine $z_\phi(i\infty)$ and $z_\phi''(\tau)$ in the same way as in the special case $\phi=0$. Namely, if $\phi(\tau)$ has a limit λ as $\mathrm{Im}(\tau)\to\infty$, then

$$z_\phi(i\infty)=\frac{\sqrt{3}}{2\pi}\int_{\frac{3\lambda}{\pi^2}}^{\infty}\frac{dt}{(t+1)\sqrt{t-2}}=\frac{1}{2}-\frac{1}{\pi}\arctan\left(\sqrt{\frac{\lambda}{\pi^2}-\frac{2}{3}}\right)$$
$$=\frac{1}{2}\pm\frac{1}{2\pi i}\log\frac{1+\sqrt{\frac{2}{3}-\frac{\lambda}{\pi^2}}}{1-\sqrt{\frac{2}{3}-\frac{\lambda}{\pi^2}}}.$$

As to the second derivative, we find

$$z'_\phi(\tau)=-\frac{\sqrt{3}}{2\pi}\cdot\frac{3}{\pi^2}\phi'(\tau)\left(\frac{27}{\pi^6}\phi(\tau)^3-\frac{9}{\pi^2}E_4\phi(\tau)-2E_6\right)^{-1/2}$$
$$+\frac{\sqrt{3}}{2\pi}\int_{\frac{3}{\pi^2}\phi(\tau)}^{\infty}\frac{\partial}{\partial\tau}[(t^3-3E_4t-2E_6)^{-1/2}]dt,$$

$$z''_\phi(\tau)=-\frac{3\sqrt{3}}{2\pi^3}\frac{d}{d\tau}\left(\frac{\phi'(\tau)}{\sqrt{\frac{27}{\pi^6}\phi^3-\frac{9}{\pi^2}E_4\phi-2E_6}}\right)$$
$$-\frac{3\sqrt{3}}{2\pi^3}\phi'(\tau)\frac{\partial}{\partial\tau}[(t^3-3E_4t-2E_6)^{-1/2}]\Big|_{t=\frac{3}{\pi^2}\phi(\tau)}$$
$$+\frac{\sqrt{3}}{2\pi}\int_{\frac{3}{\pi^2}\phi(\tau)}^{\infty}\frac{\partial^2}{\partial\tau^2}[(t^3-3E_4t-2E_6)^{-1/2}]dt.$$

Using the formulas given earlier for $\frac{d}{d\tau}E_4$, $\frac{d}{d\tau}E_6$, and $\frac{\partial^2}{\partial\tau^2}[(t^3-3E_4t-2E_6)^{-1/2}]$ we find after some computation the result

$$z''_\phi(\tau)=\pm\frac{1}{72\pi^2}(4\phi^3-g_2\phi-g_3)^{-3/2}\{-2(4\phi^3-g_2\phi-g_3)\phi^{**}$$
$$+(12\phi^2-g_2)\phi^{*2}+(36g_3\phi+2g_2^2)\phi^*$$
$$+12g_2\phi^4+3g_2^2\phi^2+6g_2g_3\phi-g_2^3+27g_3^2\},$$

where we have set

$$\phi^*=12\pi^2\left(\frac{1}{2\pi i}\frac{\partial}{\partial\tau}-\frac{1}{6}E_2\right)\phi,\qquad \phi^{**}=12\pi^2\left(\frac{1}{2\pi i}\frac{\partial}{\partial\tau}-\frac{1}{3}E_2\right)\phi^*.$$

This formula holds for any meromorphic function $\phi(\tau)$; if ϕ is a (meromorphic) modular form of weight 2 on a subgroup Γ of $\mathrm{SL}_2(\mathbb{Z})$ then, as is well-known, ϕ^* and ϕ^{**} are modular forms on Γ of weight 4 and weight 6, respectively, so we have obtained an explicit formula of the form

$$z''_\phi(\tau)=\frac{\psi(\tau)}{(4\phi(\tau)^3-g_2(\tau)\phi(\tau)-g_3(\tau))^{3/2}},$$

where ψ is a (meromorphic) modular form of weight 12 on Γ and is a cusp form if ϕ is a holomorphic modular form. As a further corollary we observe that the equation $\wp(z,\tau)=\phi(\tau)$ for the special function

$$\phi(\tau)=\wp(a\tau+b,\tau) \quad (a,b\in\mathbb{C}, \text{ not both } \in\mathbb{Z})$$

$\left[\text{which is a modular form of weight 2 on } \Gamma(N) \text{ if } a,b\in\frac{1}{N}\mathbb{Z}\right]$ has the special solution $z=a\tau+b$ with $z''(\tau)=0$. Hence the form ψ must vanish identically in this case and we have obtained a non-linear second order differential equation satisfied by all the functions $\wp(a\tau+b,\tau)$.